Exactly solvable time-dependent non-Hermitian quantum systems from point transformations

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Abstract: We demonstrate that complex point transformations can be used to construct non-Hermitian first integrals, time-dependent Dyson maps and metric operators for non-Hermitian quantum systems. Initially we identify a point transformation as a map from an exactly solvable time-independent system to an explicitly time-dependent non-Hermitian Hamiltonian system. Subsequently we employ the point transformation to construct the non-Hermitian time-dependent invariant for the latter system. Exploiting the fact that this invariant is pseudo-Hermitian, we construct a corresponding Dyson map as the adjoint action from a non-Hermitian to a Hermitian invariant, thus obtaining solutions to the time-dependent Dyson and time-dependent quasi-Hermiticity equation together with solutions to the corresponding time-dependent Schrödinger equation.

1. Introduction

Point transformations are time-dependent canonical transformations used in classical mechanics for a long time [1]. In this context they are designed to extend standard transformations of the configuration coordinates to the entire phase space of a system. In the 1950s they were utilized for the first time by DeWitt in quantum mechanics [2,3] in trying to settle the ambiguity problem of operator ordering. This problem always emerges in the quantization process of a theory when one seeks the quantum analogues for classical expressions involving at least two factors whose mutual Poisson bracket does not vanish.

In addition to solving fundamental conceptual issues in quantum mechanics, point transformations have also been used to map simple exactly solvable models to more complicated systems, including their solutions [4], thus obtaining nontrivial information about the latter. Exploiting the fact that point transformations preserve conserved quantities [1], Zelaya and Rosas-Ortiz [5] recently showed that they may be employed to compute time-dependent invariants or first integrals for Hermitian Hamiltonian systems. Here we demonstrate that when complexifying these transformations they may also be used to construct time-dependent invariants for non-Hermitian systems. The explicit knowledge of these conserved quantities then allows to aid the construction of time-dependent Dyson maps, and therefore metric operators, by finding a similarity transformation. Proceeding in this manner one has simplified the original problem of defining meaningful inner products as one has circumvented solving the more complicated time-dependent Dyson equation or time-dependent quasi-Hermiticity equation. Technically one has therefore reduced the problem to finding the adjoint action that maps a non-Hermitian invariant to a Hermitian one, in analogy to the familiar problem for Hamiltonians with the difference that the map may become explicitly time-dependent.

Our manuscript is organized as follows: In section 2 we outline the general scheme in form of a four-step method leading to solutions of the time-dependent Schrödinger equation for an explicitly time-dependent non-Hermitian Hamiltonian including a metric operator that ensures unitary time-evolution between the obtained states. In section 3 we carry out the first step in our procedure and set up a point transformation for various Hermitian and non-Hermitian reference Hamiltonians leaving a number of functions free that will be fixed in the next step when specifying a concrete non-Hermitian target Hamiltonian. In section 4 we take this target Hamiltonian to be the time-dependent Swanson model. In the next steps we construct an invariant for this model and subsequently a Dyson map and metric operator. In section 5 we carry out the same steps for another non-Hermitian target Hamiltonian, a time-dependent harmonic oscillator with complex linear term. Our conclusions are stated in section 6.

2. Invariants and Dyson maps from point transformations

Our starting problem is having to make sense of a non-Hermitian explicitly time-dependent Hamiltonian $H(x,t) \neq H^{\dagger}(x,t)$ satisfying the time-dependent Schrödinger equation (TDSE)

$$H(x,t)\phi(x,t) = i\hbar\partial_t\phi(x,t). \tag{2.1}$$

Unlike as for Hermitian Hamiltonians we do not only have to solve equation (2.1) for the wavefunction $\phi(x,t)$, but we also have to find a suitable time-dependent metric operator $\rho(t)$ for these solutions to become physically meaningful in a well-defined inner product $\langle \cdot | \cdot \rangle_{\rho(t)} := \langle \cdot | \rho(t) \cdot \rangle$ [6–15], similarly to the time-independent scenario [16–18]. In principle one has to solve for this purpose the time-dependent Dyson equation or the time-dependent quasi-Hermiticity equation

$$h(t) = \eta(t)H(t)\eta^{-1}(t) + i\hbar\partial_t\eta(t)\eta(t)^{-1}, \quad H^{\dagger}(t) = \rho(t)H(t)\rho^{-1}(t) + i\hbar\partial_t\rho(t)\rho^{-1}(t), \tag{2.2}$$

respectively, for $\eta(t)$ or $\rho(t) = \eta^{\dagger}(t)\eta(t)$. Here $h(t) = h^{\dagger}(t)$ is the time-dependent Hermitian counterpart to $H(t)$. While in many cases this is achievable, the construction of $\eta(t)$ and $\rho(t)$ is technically involved as demonstrated in [9–15, 19].

The main purpose of this paper is to present an alternative approach to finding $\rho(t)$ and $\eta(t)$ by exploiting *point transformations* and first integrals. As a starting point one assumes that there exists an exactly solvable time-independent reference Hamiltonian $H_0(\chi)$

satisfying the one-dimensional TDSE

$$H_0(\chi)\psi(\chi,\tau) = i\hbar\partial_\tau\psi(\chi,\tau), \tag{2.3}$$

with χ denoting the coordinate and τ the time in this system. One may then relate (2.3) to the first TDSE (2.1) by means of a complex point transformation

$$\Gamma : H_0\text{-TDSE} \to H\text{-TDSE}, \qquad [\chi,\tau,\psi(\chi,\tau)] \mapsto [x,t,\phi(x,t)]\,. \tag{2.4}$$

Here ψ and ϕ are understood to be implicit functions of (χ,τ) and (x,t), respectively, defined by the equations (2.1) and (2.3). The variables χ, τ, ψ are treated in general as functions of the independent variables x, t, ϕ as

$$\chi = P(x,t,\phi), \qquad \tau = Q(x,t,\phi), \qquad \psi = R(x,t,\phi). \tag{2.5}$$

In practice one may relax some of the (x,t,ϕ)-dependences of the functions P, Q, R or is even forced to do so for concrete systems. We refer here to H_0 and H, as *reference* and *target Hamiltonians* respectively, not to be confused with their corresponding Hermitian counterparts h_0 and h in case they are non-Hermitian.

Having identified the point transformation Γ on the level of the TDSEs one may subsequently apply it exclusively to the time-independent Hamiltonian $H_0(\chi)$ as

$$\Gamma : H_0(\chi) \to I_H(x,t). \tag{2.6}$$

Since real point transformations preserve conserved quantities [1], and $I_H(x,t)$ acquired a time-dependence via the point transformation Γ, it is suggestive to assume that also complex point transformations have this property and that $I_H(x,t)$ is actually the time-dependent conserved *Lewis-Riesenfeld invariant* [20] for the non-Hermitian time-dependent Hamiltonian $H(x,t)$ in (2.1) satisfying

$$i\hbar\frac{dI_H}{dt} = i\hbar\partial_t I_H + [I_H, H] = 0. \tag{2.7}$$

Since H is non-Hermitian, also its first integral, the invariant I_H, must be non-Hermitian, which is evident from (2.7).
As argued successfully in [13, 21–23] one can map this non-Hermitian invariant I_H to a Hermitian invariant I_h by means of a time-dependent similarity transformation $\eta(t)$ as

$$\eta(t)I_H(t)\eta^{-1}(t) = I_h(t). \tag{2.8}$$

Remarkably the map $\eta(t)$ is indeed the Dyson map solving the first equation in (2.2) and the Hermitian operator I_h is the Lewis-Riesenfeld invariant for the Hermitian time-dependent Hamiltonian $h(t)$, identified in (2.2), satisfying

$$i\hbar\frac{dI_h}{dt} = i\hbar\partial_t I_h + [I_h, h] = 0. \tag{2.9}$$

The metric operator is the simply obtained as $\rho(t) = \eta^\dagger(t)\eta(t)$.

In summary, we have proposed a four step method that leads not only to the solutions of the TDSE (2.1), but also to explicit expressions for the Dyson maps and metric operators. The first step consists of selecting a suitable time-independent reference Hamiltonian $H_0(\chi)$, Hermitian or non-Hermitian, and point transform it's corresponding TDSE (2.3). In the second step we fix the free parameters by matching the transformed TDSE with a TDSE (2.1) for a non-Hermitian target Hamiltonian $H(t)$, hence identifying the point transformation Γ by means of (2.4). In the third step we obtain the invariant $I_H(t)$ by acting with Γ on the time-independent reference Hamiltonian $H_0(\chi)$ and in the fourth step we construct the Dyson map η as a similarity transformation by means of (2.8). In case the TDSE for $H_0(\chi)$ is solvable we obtain by construction also the solutions to the original TDSE for $H(x,t)$. Here our main focus is, however, on the construction of I_H, I_h, η and ρ. Let us now demonstrate how this four step strategy is carried out for some concrete time-dependent non-Hermitian Hamiltonian.

3. Point transforming the reference Hamiltonian

One of the simplest choices for an exactly solvable reference Hamiltonian $H_0(\chi)$ is the time-independent Hermitian harmonic oscillator

$$H_0(\chi) = \frac{P^2}{2m} + \frac{1}{2} m\omega^2 \chi^2, \qquad m, \omega \in \mathbb{R}. \tag{3.1}$$

First we identify the point transformation of $H_0(\chi)$ in general terms. Expressing the momentum operator P in the position representation $P = -i\hbar\partial_\chi$, we act with the point transformation Γ on the TDSE (2.3). Simplifying the general functional dependence as stated in (2.5) to

$$\chi = \chi(x,t), \qquad \tau = \tau(t), \qquad \psi = A(x,t)\phi(x,t), \tag{3.2}$$

we convert all terms in the TDSE from the (χ, τ, ψ) to the (x, t, ϕ)-variables, hence obtaining the point transformed differential equation

$$i\hbar\phi_t + \frac{\hbar^2}{2m}\frac{\tau_t}{\chi_x^2}\phi_{xx} + B_0(x,t)\phi_x - V_0(x,t)\phi = 0, \tag{3.3}$$

with

$$B_0(x,t) = -i\hbar\frac{\chi_t}{\chi_x} + \frac{\hbar^2}{2m}\frac{\tau_t}{\chi_x^2}\left(2\frac{A_x}{A} - \frac{\chi_{xx}}{\chi_x}\right), \tag{3.4}$$

$$V_0(x,t) = \frac{1}{2} m\tau_t\chi^2\omega^2 - i\hbar\left(\frac{A_t}{A} - \frac{A_x\chi_t}{A\chi_x}\right) - \frac{\hbar^2}{2m}\frac{\tau_t}{\chi_x^2}\left(\frac{A_{xx}}{A} - \frac{A_x\chi_{xx}}{A\chi_x}\right). \tag{3.5}$$

This form of equation (3.3) agrees with the previously derived expression in [5], where also more details of the computation can be found. However, we allow for a major difference by admitting the potential V_0 of the target Hamiltonian to be complex. The first two assumptions in (3.2) on the functional dependence when compared to the most general dependence $\chi(x,t,\phi)$, $\tau(x,t,\phi)$ are made for convenience to simplify the calculation. The last factorization property of ψ in (3.2) is already made in anticipation on the structure of the target differential equation. Since the TDSE is a linear equation in the fields it does not contain a ϕ_x^2 term so that $\psi_{\phi\phi} = 0$. Hence the linear dependence of ψ on ϕ.

The reference Hamiltonian is a choice and in order to allow for comparison we shall explore here some further simple options

$$H_0^{(1)}(\chi) = \frac{P^2}{2m} \tag{3.6}$$

$$H_0^{(2)}(\chi) = H_0(\chi) + a\chi, \qquad a \in \mathbb{R}, \tag{3.7}$$

$$H_0^{(3)}(\chi) = H_0(\chi) + ib\chi, \qquad b \in \mathbb{R}, \tag{3.8}$$

$$H_0^{(4)}(\chi) = H_0(\chi) + a\{\chi, P\}. \tag{3.9}$$

We note that the reference Hamiltonian does not have to be Hermitian. Then the general form of the point transformed differential equation (3.3) associated with each these reference Hamiltonians remains the same, yet the explicit forms of $B_0(x,t)$ (3.4) and $V_0(x,t)$ (3.5) differ. For the choices (3.6)-(3.9) we obtain

$$B_1(x,t) = B_0(x,t), \qquad V_1(x,t) = V_0(x,t) - \frac{1}{2} m\omega^2\chi^2\tau_t, \tag{3.10}$$

$$B_2(x,t) = B_0(x,t), \qquad V_2(x,t) = V_0(x,t) + a\chi\tau_t, \tag{3.11}$$

$$B_3(x,t) = B_0(x,t), \qquad V_3(x,t) = V_0(x,t) + ib\chi\tau_t, \tag{3.12}$$

$$B_4(x,t) = B_0(x,t) + \frac{2ia\hbar\chi\tau_t}{\chi_x}, \qquad V_4(x,t) = V_0(x,t) - \frac{2ia\chi\hbar A_x\tau_t}{A\chi_x} - ia\hbar\tau_t. \tag{3.13}$$

In order to proceed to the second step in the procedure we need to select a specific target Hamiltonian.

4. The time-dependent Swanson model as target Hamiltonian

As a concrete example for a target Hamiltonian we consider here a prototype non-Hermitian Hamiltonian system, the time-dependent version of the Swanson Hamiltonian [24]. In terms of bosonic creation a and annihilation operators $a^\dagger$, its time-dependent version is usually written in the form

$$\tilde{H}_S(t) = \omega(t)\left(a^\dagger a + 1/2\right) + \tilde{\alpha}(t)a^2 + \tilde{\beta}(t)\left(a^\dagger\right)^2, \tag{4.1}$$

which is clearly non-Hermitian when $\tilde{\alpha} \neq \tilde{\beta}^*$. Dyson maps for the time-independent and time-dependent version were found in [25, 26] and [19], respectively. In order to apply the point transformations it is more convenient to convert the Hamiltonian into coordinate and momentum variables x, p, which is easily achieved. Using the standard representations $a = (x + ip)/2$ and $a^\dagger = (x - ip)/2$ we obtain

$$\tilde{H}_S(t) = \frac{1}{2}\left[\omega(t) - \tilde{\alpha}(t) - \tilde{\beta}(t)\right]p^2 + \frac{1}{2}\left[\omega(t) + \tilde{\alpha}(t) + \tilde{\beta}(t)\right]x^2 + \frac{i}{2}\left[\tilde{\alpha}(t) - \tilde{\beta}(t)\right]\{x,p\} + \frac{\omega(t)}{2}. \tag{4.2}$$

Expressing the time-dependent function $\tilde{\alpha}(t)$, $\tilde{\beta}(t)$, $\omega(t)$ in terms of new time-dependent functions $\alpha(t)$, $\Omega(t)$ and $M(t)$ as

$$\tilde{\alpha} = \frac{M\Omega^2}{4} - \frac{1}{4M} + \alpha, \quad \tilde{\beta} = \frac{M\Omega^2}{4} - \frac{1}{4M} - \alpha, \quad \omega = \frac{M\Omega^2}{2} + \frac{1}{2M}, \tag{4.3}$$

the Hamiltonian is converted into the simpler form

$$H_S(x,t) := \tilde{H}_S(t) - \frac{\omega(t)}{2} = \frac{p^2}{2M(t)} + \frac{M(t)}{2}\Omega^2(t)x^2 + i\alpha(t)\{x,p\}, \quad M,\Omega \in \mathbb{R},\ \alpha \in \mathbb{C}, \tag{4.4}$$

which is evidently still non-Hermitian for $\alpha \neq 0$. The Swanson Hamiltonian is $\mathcal{PT}$-symmetric for $\mathcal{PT}$: $x \to -x$, $p \to p$, $i \to -i$ and all time-dependent coefficient functions transforming as $\mathcal{PT}$: $M, \Omega, \alpha \to M, \Omega, \alpha$. Since $\alpha = \alpha_r + i\alpha_i$ is complex, this requires requires $\mathcal{PT}$: $\alpha_r \to \alpha_r$, $\alpha_i \to -\alpha_i$. We notice here that the option $\alpha \in \mathbb{C}$, rather than $\alpha \in \mathbb{R}$, does not exist in the time-independent case when one wishes to maintain the $\mathcal{PT}$-symmetry of the Hamiltonian.

We will explore here two versions of this target Hamiltonian. In one we keep the mass time-independent by setting the time-dependent coefficient in the kinetic energy term to a constant, $M(t) \to m$, and in the other option we take the mass term to be generically time-dependent [27, 28]. Let us now identify the point transformation Γ according to (2.4) for the specified pairs of Hamiltonians.

4.1 Point transformations Γ_i^S from $H_0^i(\chi)$ to $H_S(x,t)$

4.1.1 Point transformation $\Gamma_0^S : H_0(\chi) \to H_S(x,t)$, time-independent mass

Having specified the target Hamiltonian as $H_S(x,t)$, with m time-independent, we express the corresponding TDSE (2.3) in the position representation as

$$i\hbar\phi_t + \frac{\hbar^2}{2m}\phi_{xx} - 2\hbar\alpha(t)x\phi_x - \hbar\alpha(t)\phi - \frac{1}{2}m\Omega(t)x^2\phi = 0. \tag{4.5}$$

Taking $H_0(\chi)$ as reference Hamiltonian, the direct comparison with equation (3.3) leads to the three constraints

$$\frac{\tau_t}{\chi_x^2} = 1, \quad B_0(x,t) = -2\hbar\alpha(t)x, \quad V_0(x,t) = \frac{1}{2}m\Omega(t)x^2 + \hbar\alpha(t). \tag{4.6}$$

Apart from being a complex equation, the first constraint in (4.6) is the same as the one found in [5], where it was solved by

$$\tau(t) = \int^t \frac{ds}{\sigma^2(s)}, \qquad \text{and} \qquad \chi(x,t) = \frac{x+\gamma(t)}{\sigma(t)}, \tag{4.7}$$

but now with $\gamma(t)$ and $\sigma(t)$ potentially being complex functions. Using these expressions in the second constraint in (4.6) yields the equation

$$i\frac{\hbar}{m}\frac{A_x}{A} + \gamma_t + 2i\alpha x - (x+\gamma)\frac{\sigma_t}{\sigma} = 0, \tag{4.8}$$

which may be solved by

$$A(x,t) = \exp\left\{\frac{im}{\hbar}\left[\left(\gamma_t - \gamma\frac{\sigma_t}{\sigma}\right)tx + \left(it\alpha - \frac{\sigma_t}{2\sigma}\right)x^2 + \delta(t)\right]\right\}, \tag{4.9}$$

where $\delta(t)$ is a complex valued function corresponding to the integration constant in the x integration. Proceeding with these expressions to the third constraint in (4.6) yields

$$\begin{aligned} &-i\hbar\frac{\sigma_t}{2\sigma} - \frac{m}{2}\left(2\gamma\gamma_t\frac{\sigma_t}{\sigma} + \gamma_t^2 + \gamma^2\frac{\sigma_t^2}{\sigma^2} - \frac{\omega^2\gamma^2}{\sigma^4} - 2m\delta_t\right) \\ &\quad + \frac{m\gamma}{\sigma}\left[\sigma_{tt} - \frac{\gamma_{tt}}{\gamma}\sigma - \frac{\omega^2}{\sigma^3}\right]x + \frac{m}{2\sigma}\left[\sigma_{tt} - \left(2i\alpha_t - 4\alpha^2 - \Omega\right)\sigma - \frac{\omega^2}{\sigma^3}\right]x^2 = 0. \end{aligned} \tag{4.10}$$

The x-independent term in (4.10) vanishes for

$$\delta(t) = \frac{\gamma}{2\sigma}\left(\sigma\gamma_t - \gamma\sigma_t\right) - \frac{i\hbar}{2m}\log\sigma. \tag{4.11}$$

Furthermore, we recognize that the square brackets of the coefficient functions for the x and x^2 dependent terms amount both to the ubiquitous nonlinear Ermakov-Pinney equation [29, 30] with the constraint

$$\kappa(t) := \frac{\gamma_{tt}}{\gamma} = 2i\alpha_t - 4\alpha^2 - \Omega. \tag{4.12}$$

The general solution to this version of the Ermakov-Pinney (EP) equation, as given by the coefficient functions, can be constructed in terms of the two fundamental solutions $u(t)$ and $v(t)$ to the equations $\ddot{u} + \kappa(t)u = 0$, $\ddot{v} + \kappa(t)v = 0$ as

$$\sigma(t) = \left(Au^2 + Bv^2 + 2Cuv\right)^{1/2}, \tag{4.13}$$

where the constants A, B, C are constrained as $C^2 = AB - \omega^2/W$ with Wronskian $W = u\dot{v} - v\dot{u}$. Given that $\kappa(t)$ is now complex, the time τ and the coordinate χ inevitably become complex, unless we take $\alpha_t = 0$. As we see from (4.3) the latter option still keeps all the coefficients time-dependent although in a somewhat more restricted form.

4.1.2 Point transformation $\hat{\Gamma}_0^S : H_0(\chi) \rightarrow H_S(x,t)$, time-dependent mass

Let us now switch on the time-dependence in the mass so that we have to compare the transformed equation (3.3) with

$$i\hbar\phi_t + \frac{\hbar^2}{2M(t)}\phi_{xx} - 2\hbar\alpha(t)x\phi_x - \hbar\alpha(t)\phi - \frac{1}{2}M(t)\Omega^2(t)x^2\phi = 0, \tag{4.14}$$

instead of (4.5). The direct comparison then changes the three constraints (4.6) into

$$\frac{\tau_t}{m\chi_x^2} = \frac{1}{M(t)}, \quad B(x,t) = -2\hbar\alpha(t)x, \quad V(x,t) = \frac{1}{2}M(t)\Omega^2(t)x^2 + \hbar\alpha(t). \tag{4.15}$$

Thus, also the first constraint in (4.15) differs now from the one found in [5] as a result of the introduction of an explicit time-dependent mass. As we show next, this change from a time-independent to a time-dependent mass permits us to keep the time τ and the coordinate χ to be real for more generic time-dependent coefficient functions. Taking a general form for the mass as

$$M(t) = m\sigma(t)^n, \tag{4.16}$$

allows us to easily to distinguish between the time-independent and time-dependent cases, with the former recovered for $n = 0$. The first constraint in (4.15) is now solved by

$$\tau(t) = \int^t \sigma(y)^r dy \qquad \text{and} \qquad \chi(x,t) = \frac{x+\gamma(t)}{\sigma(t)^s}, \tag{4.17}$$

where we identify $n = -r - 2s$. Using these expressions in the second constraint in (4.15) yields the equation

$$\sigma^{r+2s}\frac{h}{m}\frac{A_x}{A} - i\gamma_t + is(x+\gamma)\frac{\sigma_t}{\sigma} + 2\alpha x = 0, \tag{4.18}$$

which may be solved by

$$A(x,t) = \exp\left\{\frac{im\sigma^{-1-r-2s}}{\hbar}\left[(\sigma\gamma_t - s\gamma\sigma_t)\,x + \left(i\alpha\sigma - \frac{1}{2}s\sigma_t\right)x^2 + \delta(t)\right]\right\}, \tag{4.19}$$

where $\delta(t)$ is a complex valued function corresponding to the integration constant in the x integration. Proceeding with these expressions to the third constraint in (4.15) yields

$$\begin{aligned}
&-i\hbar\frac{q\sigma^{1+r+2s}\sigma_t}{2} + \frac{m}{2}\left[2(1+r+2s)\delta\sigma_t - 2\sigma\delta_t + \sigma^2\gamma_t^2 - 2s\gamma\sigma\gamma_t\sigma_t + \gamma^2\left(s^2\sigma_t^2 - \omega^2\sigma^{2+2s}\right)\right]\\
&-m\left\{\gamma\omega^2\sigma^{2r+2} - \sigma[r+2s]\gamma_t\sigma_t + \gamma s\left[(r+s+1)\sigma_t^2 - \sigma\sigma_{tt}\right] + \sigma^2\gamma_{tt}\right\}x+\\
&\frac{1}{2}m\left\{2i\alpha\sigma[r+2s]\sigma_t - 2i\sigma^2\alpha_t - s[r+s+1]\sigma_t^2 + \sigma\left[\sigma\left(4\alpha^2 - \omega^2\sigma^{2r} + \Omega^2\right) + s\sigma_{tt}\right]\right\}x^2 = 0.
\end{aligned} \tag{4.20}$$

The x-independent term in (4.20) vanishes for

$$\delta(t) = \frac{\gamma}{2}(\sigma\gamma_t - s\gamma\sigma_t) + \sigma^{1+r+2s}\left(c_1 - \frac{is\hbar}{2m}\log\sigma\right), \tag{4.21}$$

where c_1 is a constant. The term proportional to x^2 in (4.20) is a non-linear second order differential equation in σ. To ensure that σ is real, hence our space-time is real, we set the imaginary term to be equal to zero

$$\alpha_r\left[(r+2s)\sigma_t - 4\sigma\alpha_i\right] - \sigma(\alpha_r)_t = 0. \tag{4.22}$$

This equation is satisfied for

$$\alpha_i = \frac{1}{4}\partial_t \ln\left(\frac{\sigma^{r+2s}}{\alpha_r}\right). \tag{4.23}$$

We notice from here that since $\alpha_i \propto \partial_t$ it does indeed transform as $\alpha_i \to -\alpha_i$ under $\mathcal{PT}$ as is required for H_S to be $\mathcal{PT}$-symmetric. The terms proportional to x^2 and x vanish for

$$\sigma_{tt} = \sigma\left[\frac{2\alpha_r\left(2\Omega^2\alpha_r + 8\alpha_r^3 + (\alpha_r)_{tt}\right) - 3(\alpha_r)_t^2}{2r\alpha_r^2}\right] + \frac{\left(\frac{r}{2}+1\right)\sigma_t^2}{\sigma} - \frac{2\omega^2\sigma^{2r+1}}{r}, \tag{4.24}$$

and

$$\gamma_{tt} = \frac{\gamma}{2r}\left(\frac{s\left(16\alpha_r^4 - 3\left(\alpha_r\right)_t^2 + 2\alpha_r(\alpha_r)_{tt}\right)}{\alpha_r^2} + 4s\Omega^2 - \frac{(r+2s)\left(2\omega^2\sigma^{2r+2} + rs\sigma_t^2\right)}{\sigma^2}\right) + \frac{(r+2s)\gamma_t\sigma_t}{\sigma} \tag{4.25}$$

respectively. These equations can be reduced to solvable ones for specific choices of r, s, α_i, α_r and γ. We discuss now some special choices.

$\alpha_i = 0$

Setting now $\alpha_i = 0$, we can solve directly for α_r in (4.23), obtaining

$$\alpha_r = c_2\sigma^{r+2s}. \tag{4.26}$$

Taking the mass to be time-independent and hence α to be time-independent by setting $r = -2s$ and $s = 1$, equations (4.24) and (4.25) reduce to

$$\sigma_{tt} = -4c^2\sigma + \frac{\omega^2}{\sigma^3} - \sigma\Omega^2 \qquad \text{and} \qquad \gamma_{tt} = -\gamma\left(4c^2 + \Omega^2\right), \tag{4.27}$$

respectively. Both of these equations are solvable, with the first being the nonlinear Ermakov-Pinney equation [29, 30]. Another interesting choice is to take $r = -s - 1$ with $s = -1$, in doing so we end up with

$$\sigma_{tt} = \frac{4c^2}{\sigma^3} - \sigma\omega^2 + \sigma\Omega^2 \qquad \text{and} \qquad \gamma_{tt} = -\gamma\left(\frac{4c^2}{\sigma^4} + \Omega^2\right) - \frac{2\gamma_t\sigma_t}{\sigma} \tag{4.28}$$

where again the first equation is a version of the nonlinear EP equation. However, now the Ermakov-Pinney equation is real without any restrictions on $\alpha(t)$, so that also the time τ and the coordinate χ are real. The second equation is a damped harmonic oscillator equation, which we may solve or simply take the integration constant γ to be zero.

$\gamma = 0$

Instead, setting $\gamma = 0$ and parametrizing

$$\alpha_r = \sigma^{-2-r}, \tag{4.29}$$

reduces equation (4.24) to

$$\sigma_{tt} = \frac{-\omega^2\sigma^{2r+1} + 4\sigma^{-2r-3} + \sigma\Omega^2}{r+1}, \tag{4.30}$$

with α now being being genuinely complex

$$\alpha = \alpha_r - i\frac{r+s+1}{2}\partial_t \ln(\sigma). \tag{4.31}$$

Choosing $r = 0$ or $r = -2$ results in equation (4.30) being the respective EP equations given by

$$\sigma_{tt} = \frac{4}{\sigma^3} + \sigma\left(\Omega^2 - \omega^2\right), \quad \text{or} \quad \sigma_{tt} = \frac{\omega^2}{\sigma^3} - \sigma\left(\Omega^2 + 4\right). \tag{4.32}$$

As we have taken $\gamma = 0$ we do not need to select a concrete value for s.

When setting $r = -2$ we do not need to choose a concrete form for α_r, as in this case equation (4.24) reduces to the Ermakov-Pinney equation

$$\sigma_{tt} = \frac{\omega^2}{\sigma^3} - f(t)\sigma, \quad \text{with} \quad f = 4\alpha_r^2 - \frac{3\left(\alpha_r\right)_t^2}{4\alpha_r^2} + \frac{(\alpha_r)_{tt}}{2\alpha_i} + \Omega^2. \tag{4.33}$$

$\underline{\gamma \neq \mathbf{0}}$

When $\gamma \neq 0$, we still have the same parametrization of α_r and choices for r as in the previous section, but we now have to restrict s so that equation (4.24) becomes solvable. For instance, when $r = -2$ and $s = 1$, we obtain

$$\gamma_{tt} = -\gamma(4 + \Omega^2), \tag{4.34}$$

which is solvable.

4.1.3 Point transformations $\hat{\Gamma}^S_{1,2,4} : H_0^{(1,2,4)}(\chi) \to H_S(x,t)$, time-dependent mass

Let us next explore the point transformations that result when changing the reference Hamiltonian, but keeping the target Hamiltonian to be $H_S(x,t)$ with time-dependent mass. Considering now the second constraint in (4.6) together with (3.10)-(3.13) we can identify the fields $A_i(x,t)$ for the reference Hamiltonians (3.6)-(3.9). Solving the constraints we find

$$A_1(x,t) = A_2(x,t) = A(x,t), \qquad A_4(x,t) = A(x,t)\exp\left[\frac{am\sigma^{-2s}}{i\hbar}\left(2\gamma x + x^2\right)\right], \tag{4.35}$$

such that the $A_i(x,t)$ are identical for the same $B_i(x,t)$. Solving next the third constraint in (4.6) for (3.6)-(3.9) we notice that we always require (4.23) to hold in order to ensure that space-time remains real. In contrast, the other time-dependent functional coefficient δ and the constraining equations for σ and γ vary for each reference Hamiltonians. We obtain

$$H_0^{(1)}: \quad \delta_0^{(1)} = \delta, \quad \sigma_{tt}^{(1)} = \sigma_{tt} + \frac{2\omega^2\sigma^{1+2r}}{r}, \quad \gamma_{tt}^{(1)} = \gamma_{tt} + \frac{(r+2s)\omega^2\gamma\sigma^{2r}}{r},$$

$$H_0^{(2)}: \quad \delta_0^{(2)} = \delta - \sigma^{1+r+2s}\frac{a}{2m}\int^t \gamma\sigma^{r-s}, \quad \sigma^{(2)} = \sigma, \quad \gamma_{tt}^{(2)} = \gamma_{tt} - \frac{a\sigma^{2r+s}}{m},$$

$$H_0^{(4)}: \quad \delta_0^{(4)} = \delta + 2a\sigma^{1+r+2s}\int^t \gamma\sigma^{-1-2s}(s\gamma\sigma_t - \sigma\gamma_t), \quad \sigma_{tt}^{(4)} = \sigma_{tt} + \frac{8a^2\sigma^{1+2r}}{r},$$

$$\gamma_{tt}^{(4)} = \gamma_{tt} + \frac{4a^2(r+2s)\gamma\sigma^{2r}}{r}.$$

Here we understand that σ_{tt} and γ_{tt} are to be replaced by the right hand sides of equations (4.24) and (4.25), respectively.

4.2 Non-Hermitian invariants from Γ_i^S

Having constructed the various point transformations Γ_i^j that relate the TDSEs (2.1) and (2.3) for $H^j(x,t)$ and $H_0^i(\chi)$, respectively, we proceed to the third step in our scheme and employ the point transformations now to act on $H_0^i(\chi)$ exclusively, as specified in (2.6). In this way we obtain directly the invariant I_H for the non-Hermitian Hamiltonian H.

4.2.1 Non-Hermitian invariant from Γ_0^S, time-independent mass

Acting with Γ_0^S, as constructed in section 4.1.1, on $H_0(\chi)$ we obtain the invariant

$$\begin{aligned} I_H(x,t) = &\frac{\sigma^2}{2m}p^2 + m\left(\frac{\gamma\omega^2}{\sigma^2} + 2i\alpha(\sigma^2\gamma_t - \gamma\sigma\sigma_t) - \sigma\sigma_t\gamma_t + \gamma\sigma_t^2\right)x + \sigma\left(\sigma\gamma_t - \gamma\sigma_t\right)p \\ &+\frac{1}{2}\sigma\left[2i\alpha\sigma - \sigma_t\right]\{x,p\} + \frac{m}{2}\left[(\sigma_t - 2i\alpha\sigma)^2 + \frac{\omega^2}{\sigma^2}\right]x^2 \\ &+\frac{m}{2}\left(\frac{\gamma^2\omega^2}{\sigma^2} + \gamma^2\sigma_t^2 + \sigma^2\gamma_t^2 - 2\gamma\gamma_t\sigma\sigma_t\right). \end{aligned} \tag{4.36}$$

We verified that the expression for I_H in (4.36) does indeed satisfy the Lewis-Riesenfeld equation (2.7). Thus $I_H(x,t)$ is the non-Hermitian invariant or first integral for the non-Hermitian Hamiltonian $H(x,t)$. We stress that the invariant has been obtained by a direct calculation and did not involve any assumption or guess work on the general form of the invariant, which one usually has to make when solving (2.7) directly.

4.2.2 Non-Hermitian invariant from $\hat{\Gamma}_0^S$, time-dependent mass

Similarly acting with $\hat{\Gamma}_0^S$, as constructed in section 4.1.2, on $H_0(\chi)$ we obtain the invariant

$$\begin{aligned} \hat{I}_H(x,t) = &\frac{\sigma^{2s}}{2m}p^2 + \left(\sigma^{-r}\gamma_t - \gamma s\sigma^{-r-1}\sigma_t\right)p + \frac{4i\sigma\alpha_r^2 + r\alpha_r\sigma_t - \sigma\alpha_{rt}}{4\alpha_r\sigma^{r+1}}\{x,p\} \\ &+\frac{4m\omega^2\alpha_r^2\sigma^{2r+2} - m\left(4\sigma\alpha_r^2 - ir\alpha_r\sigma_t + i\sigma\alpha_{rt}\right)^2}{8\alpha_r^2\sigma^{2(r+s+1)}}x^2 \\ &+\frac{2\gamma m\omega^2\alpha_r\sigma^{2r+2} + m\left(\sigma\gamma_t - \gamma s\sigma_t\right)\left(4i\sigma\alpha_r^2 + r\alpha_r\sigma_t - \sigma\alpha_{rt}\right)}{2\alpha_r\sigma^{2(r+s+1)}}x \\ &+\frac{1}{2}m\sigma^{-2(r+s+1)}\left[\gamma^2\omega^2\sigma^{2r+2} + \left(\sigma\gamma_t - \gamma s\sigma_t\right)^2\right] \end{aligned} \tag{4.37}$$

Once more we convince ourselves that $\hat{I}_H(x,t)$ does indeed satisfy (2.7).

4.2.3 Non-Hermitian invariant from $\hat{\Gamma}^S_{1,2,4}$, time-dependent mass

The action of $\hat{\Gamma}^S_{1,2,4}$ from section 4.1.3 on $H_0^{(1,2,4)}(\chi)$ yields the invariants

$$\begin{aligned} I_H^{(1)}(x,t) = {} & \frac{\sigma^{2s}}{2m}p^2 + \left(\sigma^{-r}\gamma_t - \gamma s\sigma^{-r-1}\sigma_t\right)p + \frac{4i\sigma\alpha_r^2 + r\alpha_r\sigma_t - \sigma\alpha_{rt}}{4\alpha_r\sigma^{r+1}}\{x,p\} \\ & - \frac{m\left(4\sigma\alpha_r^2 - ir\alpha_r\sigma_t + i\sigma\alpha_{rt}\right)^2}{8\alpha_r^2\sigma^{2(r+s+1)}}x^2 \\ & + \frac{m\left(\sigma\gamma_t - \gamma s\sigma_t\right)\left(4i\sigma\alpha_r^2 + r\alpha_r\sigma_t - \sigma\alpha_{rt}\right)}{2\alpha_r\sigma^{2(r+s+1)}}x \\ & + \frac{1}{2}m\sigma^{-2(r+s+1)}\left(\sigma\gamma_t - \gamma s\sigma_t\right)^2, \end{aligned} \tag{4.38}$$

$$\begin{aligned} I_H^{(2)}(x,t) = {} & \frac{\sigma^{2s}}{2m}p^2 + \left(\sigma^{-r}\gamma_t - \gamma s\sigma^{-r-1}\sigma_t\right)p + \frac{\left(4i\sigma\alpha_r^2 + r\alpha_r\sigma_t - \sigma\alpha_{rt}\right)}{4\alpha_r\sigma^{r+1}}\{x,p\} \\ & + \frac{4m\omega^2\alpha_r^2\sigma^{2r+2} - m\left(4\sigma\alpha_r^2 - ir\alpha_r\sigma_t + i\sigma\alpha_{rt}\right)^2}{8\alpha_r^2\sigma^{2(r+s+1)}}x^2 \\ & + \frac{2a\alpha_r\sigma^{2r+s+2} + 2\gamma m\omega^2\alpha_r\sigma^{2r+2} + m\left(\sigma\gamma_t - \gamma s\sigma_t\right)\left(4i\sigma\alpha_r^2 + r\alpha_r\sigma_t - \sigma\alpha_{rt}\right)}{2\alpha_r\sigma^{2(r+s+1)}}x \\ & + \frac{1}{2\sigma^{2(r+s+1)}}\gamma\sigma^{2r+2}\left(2a\sigma^s + \gamma m\omega^2\right) + m\left(\sigma\gamma_t - \gamma s\sigma_t\right)^2 \end{aligned} \tag{4.39}$$

and

$$\begin{aligned} I_H^{(4)}(x,t) = {} & \frac{\sigma^{2s}}{2m}p^2 + \left(\sigma^{-r}\gamma_t - \gamma s\sigma^{-r-1}\sigma_t\right)p + \frac{4i\sigma\alpha_r^2 + r\alpha_r\sigma_t - \sigma\alpha_{rt}}{4\alpha_r\sigma^{r+1}}\{x,p\} \\ & + \frac{-4m\left(4a^2 - \omega^2\right)\alpha_r^2\sigma^{2r+2} - m\left(4\sigma\alpha_r^2 - ir\alpha_r\sigma_t + i\sigma\alpha_{rt}\right)^2}{8\alpha_r^2\sigma^{2(r+s+1)}}x^2 \\ & + \frac{-2\gamma m\left(4a^2 - \omega^2\right)\alpha_r\sigma^{2r+2} + m\left(\sigma\gamma_t - \gamma s\sigma_t\right)\left(4i\sigma\alpha_r^2 + r\alpha_r\sigma_t - \sigma\alpha_{rt}\right)}{2\alpha_r\sigma^{2(r+s+1)}}x \\ & + \frac{1}{2\sigma^{2(r+s+1)}}m\left[\left(\sigma\gamma_t - \gamma s\sigma_t\right)^2 - \gamma^2\left(4a^2 - \omega^2\right)\sigma^{2r+2}\right] \end{aligned} \tag{4.40}$$

Let us now compare the invariants obtained. First of all we notice that all our invariants can be brought into the form

$$I_H = a_r p^2 + b_r p + \left(c_r + ic_i\right)\{x,p\} + \left(d_r + id_i\right)x^2 + \left(e_r + ie_i\right)x + f_r, \tag{4.41}$$

where we abbreviated the complex time-dependent coefficient functions in I_H and separate them into real and imaginary parts by denoting $x = x_r + ix_i$ with $x_r, x_i \in \mathbb{R}$, $x \in \{a,b,c,d,e,f\}$. When written in this form we notice a very peculiar property that for all of our invariants the time-dependent coefficient functions are related to each other as

$$\frac{e_i}{2b_r} = \frac{d_i}{4c_r} = \frac{c_i}{2a_r} = \alpha_r m\sigma^{-r-2s}. \tag{4.42}$$

As we will see in the next subsection this property is responsible for the fact that all invariants lead to same Dyson map. Notice that when using the conventions as in (4.41) for the Hamiltonian $H_S(x,t)$ and employing the same parameterization for $M(t)$ and $\alpha(t)$, the last relation also holds for the coefficients in the Hamiltonian. We also note that if we were to take $a \to ia$ in $H_0^{(4)}(\chi)$, with $c_2 \to -c_2$, the associated invariant would still posses the same properties as a only appears squared in it. When comparing the expressions for the invariants $I_H^{(i)}$ one needs to keep in mind that the constraining equations also change with i.

4.3 Dyson maps and metric operators

We may now carry out the last step in our scheme and construct a Dyson map by acting adjointly on the invariants I_H. We can verify that the Dyson map constructed in [19] does indeed map I_H to a Hermitian invariant. Alternatively, when utilizing the property (4.42) we also find a time-independent Dyson map

$$\eta = \exp\left(-\alpha_r m \sigma^{-r-2s} x^2\right), \tag{4.43}$$

with the associated time-dependent Hermitian invariant

$$I_h = a_r p^2 + b_r p + c_r\{x,p\} + \left(d_r + 4m^2 a_r \alpha_r^2 \sigma^{-2r-4s}\right) x^2 + e_r x + f_r. \tag{4.44}$$

The corresponding Hermitian Hamiltonian is computed to be

$$h = \frac{\sigma^{r+2s}}{2m} p^2 + \left(2m\alpha_r^2 \sigma^{-r-2s} + \frac{1}{2} m \sigma^{-r-2s} \Omega^2\right) x^2 + \frac{1}{4}\partial_t \ln\left(\frac{\sigma^{r+2s}}{\alpha_r}\right)\{x,p\}, \tag{4.45}$$

which is an extended version of the time-dependent harmonic oscillator with time-dependent mass. For the special choice $\alpha_r = \sigma^{r+2s}$ the coefficient function $\alpha(t)$ becomes real, the Dyson map becomes time-independent and h reduces to the time-dependent harmonic oscillator.

5. The time-dependent harmonic oscillator with complex linear term as target Hamiltonian

To further illustrate the method and demonstrate the importance of the choice of $H_0(\chi)$ we consider next the time-dependent harmonic oscillator with a time-dependent complex linear term

$$H_{CL}(x,t) = \frac{p^2}{2M(t)} + \frac{1}{2} M(t)\Omega^2(t)x^2 + i\beta(t)x, \qquad M,\Omega,\beta \in \mathbb{R}, \tag{5.1}$$

which has been studied previously in [21, 31]. As a reference Hamiltonian we take now $H_0^{(3)}(\chi)$ as defined in (3.8). We have also considered $H_0(\chi)$ as a reference Hamiltonian which leads to a point transformation that renders space-time to be complex.

5.1 Point transformation Γ_3^{CL} from $H_0^{(3)}(\chi)$ to $H_{CL}(x,t)$

We have already identified the equations for $B_3(x,t)$ and $V_3(x,t)$ for the reference Hamiltonian $H_0^{(3)}(\chi)$ in (3.12). Comparing now with the time-dependent Schrödinger equation for the target Hamiltonian (2.3) in the position representation we find the three constraints

$$\frac{\tau_t}{m\chi_x^2} = \frac{1}{M(t)}, \qquad B(x,t) = 0, \qquad V(x,t) = \frac{1}{2}M(t)\Omega^2(t)x^2 + i\beta(t)x. \tag{5.2}$$

The first constraint in (5.2) is solved in the same way as in section 4.1.2, i.e. by equations (4.17), together with (4.16). Substituting these expressions into the second constraint in (5.2) and then solving for the field $A(x,t)$ yields

$$A(x,t) = \exp\left\{\frac{im\sigma^{-1-r-2s}}{\hbar}\left[(\sigma\gamma_t - s\gamma\sigma_t)\,x - \frac{1}{2}s\sigma_t x^2 + \delta(t)\right]\right\}, \tag{5.3}$$

where $\delta(t)$ is a complex time-dependent function associated with the integration carried out in x. Next we use all of our determined expressions in the third constraint in (5.2), obtaining

$$\begin{aligned} 0 = &-m\left[\omega^2\sigma^{2r+2} + s(r+s+1)\sigma_t^2 - \sigma\left(s\sigma_{tt} + \sigma\Omega^2\right)\right]x^2 + 2i\sigma^{r+2}\left(\beta\sigma^{2s} - b\sigma^{r+s}\right)x \\ &+ 2m\left[\sigma(r+2s)\gamma_t\sigma_t + \gamma s\left(\sigma\sigma_{tt} - (r+s+1)\sigma_t^2\right) - \sigma^2\gamma_{tt} - \gamma\omega^2\sigma^{2r+2}\right]x - ihs\sigma_t\sigma^{r+2s+1} \\ &+ m\left\{2\sigma_t\left[\delta(r+2s+1) - \gamma s\sigma\gamma_t\right] + \gamma^2 s^2\sigma_t^2 + \sigma\left[\sigma\gamma_t^2 - 2\delta_t\right]\right\} - \gamma\sigma^{2r+2}\left(\gamma m\omega^2 + 2ib\sigma^s\right). \end{aligned} \tag{5.4}$$

Firstly we notice that the x-dependent term in (5.4) contains an imaginary term which would result in space-time becoming complex. However, when setting

$$\beta = b\sigma^{r-s}, \tag{5.5}$$

the imaginary term vanishes and space-time remains real. Secondly we find that the x-independent terms in (5.4) vanishes for

$$\delta(t) = \frac{\gamma}{2}(\sigma\gamma_t - s\gamma\sigma_t) + \sigma^{1+r+2s}\left(c_1 - \frac{is\hbar}{2m}\log\sigma - i\int^t \frac{b\gamma\sigma^{r-s}}{m}\right). \tag{5.6}$$

Finally, the remaining terms proportional to x^2 and x result in the two second order auxiliary differential equations

$$\sigma_{tt} = \frac{\omega^2\sigma^{2r+2} - \sigma^2\Omega^2}{s\sigma} + \frac{(r+s+1)\sigma_t^2}{\sigma} \qquad \text{and} \qquad \gamma_{tt} = \frac{(r+2s)\gamma_t\sigma_t}{\sigma} - \gamma\Omega^2, \tag{5.7}$$

respectively. As discussed in the previous section there are different choices of r and s for which these equations reduce into versions with known solutions. As before, we shall not select concrete values for r and s so we keep the derivation of the invariant and subsequent Dyson map as general as possible.

5.2 Non-Hermitian invariant from Γ_3^{CL}

Acting with Γ_3^{CL}, as constructed in the previous section on $H_0^{(3)}(\chi)$ we obtain the invariant

$$\begin{aligned}I_H(x,t) &= \frac{\sigma^{2s}}{2m}p^2 + (\sigma^{-r}\gamma_t - \gamma s\sigma^{-r-1}\sigma_t)p - \frac{1}{2}s\sigma^{-r-1}\sigma_t\{x,p\} \\ &+ \frac{1}{2}m\sigma^{-2(r+s+1)}\left(\omega^2\sigma^{2r+2} + s^2\sigma_t^2\right)x^2 \\ &\sigma^{-2(r+s+1)}\left[ms\sigma_t\left(\gamma s\sigma_t - \sigma\gamma_t\right) + \sigma^{2r+2}\left(\gamma m\omega^2 + ib\sigma^s\right)\right]x \\ &+ \frac{1}{2}\sigma^{-2(r+s+1)}\left[m\left(\sigma\gamma_t - \gamma s\sigma_t\right)^2 + \gamma\sigma^{2r+2}\left(\gamma m\omega^2 + 2ib\sigma^s\right)\right].\end{aligned} \tag{5.8}$$

We have verified that this expression does indeed satisfy the Lewis-Riesenfeld equation (2.7).

5.3 Time-dependent Dyson map and metric operator

To determine the time-dependent Dyson map associated with the non-Hermitian invariant (5.8) we use the following abbreviated version of the invariant

$$I_H = a_r p^2 + b_r p + c_r\{x,p\} + d_r x^2 + (e_r + ie_i)x + f_r + if_i, \tag{5.9}$$

using the same conventions as in (4.41).

Making now the general Ansatz for the Dyson map

$$\eta(t) = e^{\epsilon(t)p}e^{\lambda(t)x}, \qquad \epsilon,\lambda\in\mathbb{R}, \tag{5.10}$$

we compute the adjoint action of the Dyson map on all the operators that appear in the non-Hermitian invariant. We find that (5.10) maps $I_H(x,t)$ indeed to a Hermitian counterpart when the following constraints are satisfied

$$\epsilon = \frac{a_r f_i}{a_r e_r - b_r c_r}, \qquad \lambda = \frac{c_r\epsilon}{a_r}, \qquad e_i = \frac{2(c_r^2 - a_r d_r)f_i}{b_r c_r - a_r e_r}. \tag{5.11}$$

The time-dependent functions from above do indeed satisfy these equations and when using the explicit expressions for the coefficient functions from (5.8) the time-dependent Dyson map results to

$$\eta(t) = \exp\left(\frac{b\sigma^s}{m\omega^2}p\right)\exp\left(-\frac{bs\sigma^{-1-r-s}\sigma_t}{\omega^2}x\right), \tag{5.12}$$

with σ to be determined by the auxiliary equation (5.7). The corresponding Hermitian invariant is computed to

$$\begin{aligned}I_h(x,t) &= \frac{\sigma^{2s}}{2m}p^2 + (\sigma^{-r}\gamma_t - \gamma s\sigma^{-r-1}\sigma_t)p - \frac{1}{2}s\sigma^{-r-1}\sigma_t\{x,p\} \\ &+ \frac{1}{2}\frac{m}{\sigma^{2(r+s+1)}}\left(\omega^2\sigma^{2r+2} + s^2\sigma_t^2\right)x^2 + \frac{m}{\sigma^{2(r+s+1)}}\left[\gamma\omega^2\sigma^{2r+2} + s\sigma_t\left(\gamma s\sigma_t - \sigma\gamma_t\right)\right]x \\ &+ \frac{b^2 + \gamma^2 m^2\omega^4\sigma^{-2s}}{2m\omega^2}\frac{+m\left(\gamma^2 s^2\sigma_t^2 + \sigma^2\gamma_t^2\right)}{2\sigma^{2(r+s+1)}}.\end{aligned} \tag{5.13}$$

Finally we use the Dyson map (5.10) in the time-dependent Dyson equation (2.2) to compute the corresponding Hermitian Hamiltonian as

$$h(t) = \frac{\sigma^{r+2s}}{2m}p^2 + \frac{1}{2}m\sigma^{-r-2s}\Omega^2 x^2 + \frac{b^2\sigma^{-r-2}\left(\sigma^2\Omega^2 - s^2\sigma_t^2\right)}{2m\omega^4}, \tag{5.14}$$

which is a time-dependent harmonic oscillator with a time-dependent free term.

6. Conclusions

We have demonstrated that point transformations can be utilized to construct non-Hermitian invariants for non-Hermitian Hamiltonians. In turn these invariants may then be used to construct Dyson maps simply in form of similarity transformations, which automatically satisfy the time-dependent Dyson equation (2.2). Thus we have bypassed solving this more complicated equation directly. When starting from an exactly solvable reference Hamiltonian the scheme yields also the solution for the TDSE of the target Hamiltonian. By construction the solutions only form an orthonormal system when equipped with a metric operator that is obtained trivially from the constructed Dyson map. We have shown that several different reference Hamiltonians may lead to the same Dyson map.

It would be interesting to explore the scheme further by starting with more complicated choices of the solvable reference Hamiltonian. However, the scheme is of course not limited to exactly solvable models and we could also start with a non-exactly solvable model as a reference system. In such a setting the scheme would still yield an exact invariant and an exact metric operator. Approximated wavefunctions could then be obtained by using the procedure proposed in [32]. Another interesting challenge is to extend the scheme to higher dimensional systems.

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References

[1] W.-H. Steeb, *Invertible point transformations and nonlinear differential equations*, Singapore, World Scientific, 1993.

[2] B. S. DeWitt, Point transformations in quantum mechanics, Phys. Rev. **85**(4), 653 (1952).

[3] K. B. Wolf, Point transformations in quantum mechanics, Revista Mexicana de Física **22**(1), 45–74 (1973).

[4] V. Aldaya, F. Cossío, J. Guerrero, and F. F. López-Ruiz, The quantum Arnold transformation, J. of Phys. A: Math. and Theor. **44**(6), 065302 (2011).

[5] K. Zelaya and O. Rosas-Ortiz, Quantum nonstationary oscillators: Invariants, dynamical algebras and coherent states via point transformations, Physica Scripta **95**(6), 064004 (2020).

[6] C. Figueira de Morisson Faria and A. Fring, Time evolution of non-Hermitian Hamiltonian systems, J. Phys. **A39**, 9269–9289 (2006).

[7] A. Mostafazadeh, Time-dependent pseudo-Hermitian Hamiltonians defining a unitary quantum system and uniqueness of the metric operator, Physics Letters B **650**(2), 208–212 (2007).

[8] M. Znojil, Time-dependent version of crypto-Hermitian quantum theory, Physical Review D **78**(8), 085003 (2008).

[9] H. Bíla, Adiabatic time-dependent metrics in PT-symmetric quantum theories, arXiv preprint arXiv:0902.0474 (2009).

[10] J. Gong and Q.-H. Wang, Time-dependent PT-symmetric quantum mechanics, J. Phys. A: Math. and Theor. **46**(48), 485302 (2013).

[11] A. Fring and M. H. Y. Moussa, Unitary quantum evolution for time-dependent quasi-Hermitian systems with nonobservable Hamiltonians, Phys. Rev. A **93**(4), 042114 (2016).

[12] A. Fring and T. Frith, Exact analytical solutions for time-dependent Hermitian Hamiltonian systems from static unobservable non-Hermitian Hamiltonians, Phys. Rev. A **95**, 010102(R) (2017).

[13] M. Maamache, O. K. Djeghiour, N. Mana, and W. Koussa, Pseudo-invariants theory and real phases for systems with non-Hermitian time-dependent Hamiltonians, The European Physical Journal Plus **132**(9), 383 (2017).

[14] A. Mostafazadeh, Energy observable for a quantum system with a dynamical Hilbert space and a global geometric extension of quantum theory, Phys. Rev. D **98**(4), 046022 (2018).

[15] A. Fring and R. Tenney, Spectrally equivalent time-dependent double wells and unstable anharmonic oscillators, Phys. Lett. A, 126530 (2020).

[16] C. M. Bender and S. Boettcher, Real Spectra in Non-Hermitian Hamiltonians Having PT Symmetry, Phys. Rev. Lett. **80**, 5243–5246 (1998).

[17] A. Mostafazadeh, Pseudo-Hermitian Representation of Quantum Mechanics, Int. J. Geom. Meth. Mod. Phys. **7**, 1191–1306 (2010).

[18] C. M. Bender, P. E. Dorey, C. Dunning, A. Fring, D. W. Hook, H. F. Jones, S. Kuzhel, G. Levai, and R. Tateo, PT Symmetry: In Quantum and Classical Physics, (World Scientific, Singapore) (2019).

[19] A. Fring and M. H. Y. Moussa, Non-Hermitian Swanson model with a time-dependent metric, Phys. Rev. A **94**(4), 042128 (2016).

[20] H. Lewis and W. Riesenfeld, An Exact quantum theory of the time dependent harmonic oscillator and of a charged particle time dependent electromagnetic field, J. Math. Phys. **10**, 1458–1473 (1969).

[21] B. Khantoul, A. Bounames, and M. Maamache, On the invariant method for the time-dependent non-Hermitian Hamiltonians, The European Physical Journal Plus **132**(6), 258 (2017).

[22] A. Fring and T. Frith, Solvable two-dimensional time-dependent non-Hermitian quantum systems with infinite dimensional Hilbert space in the broken PT-regime, J. of Phys. A: Math. and Theor. **51**(26), 265301 (2018).

[23] J. Cen, A. Fring, and T. Frith, Time-dependent Darboux (supersymmetric) transformations for non-Hermitian quantum systems, J. of Phys. A: Math. and Theor. **52**(11), 115302 (2019).

[24] M. S. Swanson, Transition elements for a non-Hermitian quadratic Hamiltonian, J. Math. Phys. **45**, 585–601 (2004).

[25] D. P. Musumbu, H. B. Geyer, and W. D. Heiss, Choice of a metric for the non-Hermitian oscillator, J. Phys. **A40**, F75–F80 (2007).

[26] F. Bagarello and A. Fring, From pseudo-bosons to pseudo-Hermiticity via multiple generalized Bogoliubov transformations, Int. J. of Mod. Phys. B **31**(12), 1750085 (2017).

[27] I. A. Pedrosa, Exact wave functions of a harmonic oscillator with time-dependent mass and frequency, Phys. Rev. A **55**(4), 3219 (1997).

[28] K. Zelaya and V. Hussin, Point transformations: exact solutions of the quantum time-dependent mass nonstationary oscillator, arXiv preprint arXiv:2002.10748 (2020).

[29] V. Ermakov, Transformation of differential equations,, Univ. Izv. Kiev. **20**, 1–19 (1880).

[30] E. Pinney, The nonlinear differential equation $y'' + p(x)y + c/y^3 = 0$, Proc. Amer. Math. Soc. **1**, 681(1) (1950).

[31] A. de Souza Dutra, M. Hott, and V. dos Santos, Non-Hermitian time-dependent quantum systems with real energies, Europhys. Lett. **71**, 166–171 (2005).

[32] A. Fring and R. Tenney, Time-independent approximations for time-dependent optical potentials, The European Physical Journal Plus **135**(2), 163 (2020).