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Agreement dynamics on small-world networks

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Abstract. – In this paper we analyze the effect of a non-trivial topology on the dynamics of the so-called Naming Game, a recently introduced model which addresses the issue of how shared conventions emerge spontaneously in a population of agents. We consider in particular the small-world topology and study the convergence towards the global agreement as a function of the population size N as well as of the parameter p which sets the rate of rewiring leading to the small-world network. As long as $p \gg 1/N$ there exists a crossover time scaling as N/p^2 which separates an early one-dimensional-like dynamics from a late stage mean-field-like behavior. At the beginning of the process, the local quasi one-dimensional topology induces a coarsening dynamics which allows for a minimization of the cognitive effort (memory) required to the agents. In the late stages, on the other hand, the mean-field like topology leads to a speed up of the convergence process with respect to the one-dimensional case.

The recent past has witnessed an important development of the activities of statistical physicists in the area of social sciences (for a recent collection of papers see [1]). Indeed, statistical physics is the natural field to study how global complex properties can emerge from purely local rules. Social interactions have thus been described and studied by statistical physics models and tools, in particular models of opinion formation in which agents update their internal state, or opinion, through an interaction with its neighbors. An interesting point concerns whether and how a population of agents converge towards a common and shared state (consensus) without external global coordination [2]. Early studies have mostly dealt with agents either able to interact with all the other agents (mean-field case), or sitting on the nodes of regular lattices. Such situations, although not always realistic, have the advantage to be accessible to the usual methods of statistical mechanics.

Even more recently however, the growing field of complex networks [3–5] has allowed to obtain a better knowledge of social networks [6], and in particular to show that the topology of the network on which agents interact is not regular. A natural step has then been to consider various models embedded on more realistic networks and to study the influence of various complex topologies on the corresponding dynamical behavior.

In particular, social networks are typically "small-worlds" in which, on the one hand, the average distance between two agents is small [7], growing only logarithmically with the

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network's size, and, on the other hand, many triangles are present, unlike totally random networks. In order to reconcile both properties, Watts and Strogatz have introduced the now famous small-world network model [8] which allows to interpolate between regular low-dimensional lattices and random networks, by introducing a certain amount of random long-range connections into an initially regular network.

Subsequently, a number of papers have focused on the influence of these long-range "shortcuts" on the behavior of various models defined on the network: from the Ising model [9] to the spreading of epidemics [10], or the evolution of random walks [11]. Dynamics of models inspired by social sciences are no exception, such as the Voter model [12, 13] or Axelrod's model of culture dissemination [14, 15].

In this letter, we consider the effect of a small-world topology on the so-called Naming Game model, which was inspired by the field of semiotic dynamics, a new emerging area focusing on the development of shared communication systems (languages) among a population of agents. Such a process can indeed be considered as arising through self-organization out of local interactions. The language is then seen as constantly reshaped by its users in order to maximize communicative success and expressive power while minimizing the cognitive effort [16, 17]. In addition, there are recent developments in Information Technology in which new forms of semiotic dynamics begin to appear. One example are social tagging sites (such as del.icio.us or www.flickr.com), through which tens of thousands of web users share information by tagging items like pictures or web-sites and thus develop "folksonomies" [18, 19]. In this context, simplified models or "Language Games" have been defined and studied in the theoretical community [16, 17]. As for opinion formation models, it is interesting to understand if a common state for all agents can be reached and, in the positive answer case, how the system converges towards such a state.

The model. – The original model [20] is related to an artificial intelligence experiment called Talking Heads [21], in which embodied software agents observe a set of objects through digital cameras, assign them randomly chosen names and communicate these names to each other. Since different agents can invent different names for the same object, the final emergence of a common dictionary for all the agents is not granted from the start. However, it turns out that such a consensus is in fact experimentally reached. In order to try to capture the essential relevant features of such a dynamics, Baronchelli et al. [22] have proposed a minimal model of Naming Game that reproduces the phenomenology of the experiments, despite the agents of the model are far from the complicate software effectively used as "Talking Heads". Such a model is however amenable to both analytical and extensive numerical treatment [22, 23], allowing for a better understanding of the mechanisms at work.

The model considers N identical individuals (or agents) which observe the same object and try to communicate its name one to the other. Each agent is endowed with an internal inventory or memory in which it can store an a priori unlimited number of different names or opinions. Initially, each agent has an empty inventory. The dynamics proceeds as follows: at each time step, two individuals are chosen at random for a pairwise interaction (or "communication"). One of these agents acts as the "speaker" and the other one as the "hearer". If the speaker does not know a name for the object (its inventory is empty), it invents a new name and records it. Else, if it already knows one or more synonyms (stored in the inventory), it chooses one of them randomly. The invented or selected word is then transmitted to the hearer. If the hearer already has this term in its memory, the interaction is a success, and both agents retain that term as the right one, canceling all the other terms in their inventories; otherwise, the interaction is a failure, and the new name is included in the inventory of the hearer, without any cancellation.

The way in which agents may interact with each other is determined by the topology of the underlying contact network. The mean-field case corresponds to a fully connected network, in which all agents are in mutual contact. In this case, studied in [22], each agent rarely interacts twice with the same partner, so that the system initially accumulates a large number ($\mathcal{O}(N/2)$) of different names (synonyms) for the object, invented by different agents (speakers) and $\mathcal{O}(N^{3/2})$ total words in the whole population. Interestingly however, this profusion of different names leads in the end to an asymptotic absorbing state in which all the agents share the same name.

As a second step towards the understanding of the model from a statistical physics point of view, we have considered in [23] the case of agents sitting on the nodes of a regular lattice in dimension d. In this case, each agent is connected to a finite number of neighbors ($2d$) so that it may possess only a finite number of different words in its inventory at any given time. As a result, the total amount of memory used by the whole system grows as N instead of $N^{3/2}$. Local consensus appears at very early stages of the evolution, since neighboring agents tend to share the same unique word. The dynamics then proceeds through the coarsening of such clusters of agents sharing a common name; the interfaces between clusters are composed by agents who still have more than one possible name, and diffuse randomly. Because of this particular coarsening process, the average cluster size grows as $\sqrt{t/N}$, and the time to convergence corresponds to the time needed for one cluster to reach the system size, i.e. a time $N^{1+2/d}$ for $d \leq 4$. In one dimension in particular, the convergence is thus dramatically slowed down from $\mathcal{O}(N^{3/2})$ to $\mathcal{O}(N^3)$.

In the following, we investigate the effect of long-range connections which link agents that are far from each other on the regular lattice. We use the small-world model of Watts and Strogatz: starting from a one-dimensional lattice of N sites, with periodic boundary conditions (i.e. a ring), each vertex being connected to its $2m$ nearest neighbors, a stochastic rewiring procedure is applied. The vertices are visited one after the other, and each link connecting a vertex to one of its m nearest neighbors in the clockwise sense is left in place with probability $1-p$, and with probability p is reconnected to a randomly chosen other vertex. For $p = 0$ the network retains a purely one-dimensional topology, while the random network structure is approached as p goes to 1. At small but finite p ($1/N \ll p \ll 1$), a small-world structure with short distances between nodes, together with a large clustering, is obtained.

Global picture: Expected behavior. – For $p = 0$, it has been shown in [23] that the dynamics proceeds by a slow coarsening of clusters of agents sharing the same state or word. At small p, the short-cuts are typically far from each other, with a typical distance $1/p$ between short-cuts so that the early dynamics is not affected and proceeds as in dimension 1. In particular, at very short times many new words are invented since the success rate is small. After a time of order N, each agent has played typically once, and therefore $\mathcal{O}(N)$ different words have been invented: the number of different words reaches a peak which scales as N. Since the number of neighbors of each site is bounded (the degree distribution decreases exponentially [9]), each agent has access only to a finite number of different words, so that the average memory per agent used remains finite, as in finite dimensions and in contrast with the mean-field case. The interaction of neighboring agents first leads to the usual coarsening phenomena as long as the clusters are typically one-dimensional, i.e. as long as the typical cluster size is smaller than $1/p$. However, as the average cluster size reaches the typical distance between two short-cuts $\sim 1/p$, a crossover phenomena is bound to take place; since the cluster size grows as $\sqrt{t/N}$ [23], this corresponds to a crossover time $t_{cross} = \mathcal{O}(N/p^2)$. For times much larger than this crossover, one expects that the dynamics is dominated by the existence of short-cuts and enters a mean-field like behavior. The convergence time is thus

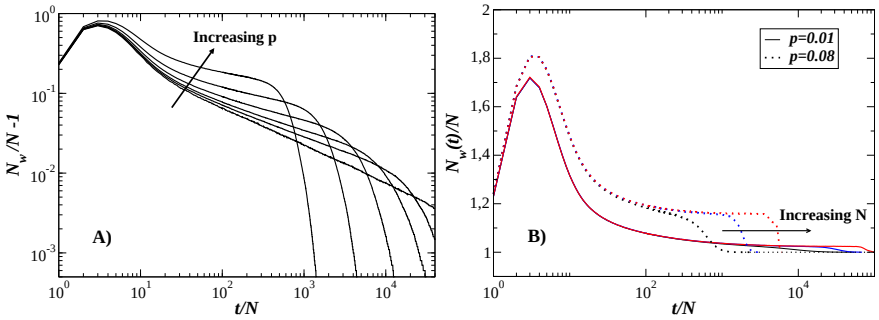

Fig. 1 – A) Average number of words per agent in the system, N_w/N as a function of the rescaled time t/N, for small-world networks with $\langle k\rangle = 8$ and $N = 10^3$ nodes, for various values of p. The curve for $p = 0$ is shown for reference, as well as $p = 5.10^{-3}$, $p = 10^{-2}$, $p = 2.10^{-2}$, $p = 4.10^{-2}$, $p = 8.10^{-2}$, from bottom to top on the left part of the curves. B) (Color Online) N_w/N for $p = 10^{-2}$ and $p = 8.10^{-2}$ and increasing system sizes: $N = 10^3$, $N = 10^4$, $N = 10^5$. Larger system sizes yield larger plateau lengths.

expected to scale as $N^{3/2}$ and not as N^3. In order for this picture to be possible, the crossover time N/p^2 needs to be much larger than 1, and much smaller than the consensus time for the one-dimensional case N^3; these two conditions read $p \gg 1/N$, which is indeed the necessary condition to obtain a small-world network.

It is therefore expected that the small-world topology allows to combine advantages from both finite-dimensional lattices and mean-field networks: on the one hand, only a finite memory per node is needed, in opposition to the $\mathcal{O}(N^{1/2})$ in mean-field; on the other hand the convergence time is expected to be much shorter than in finite dimensions.

Numerical study. – Various quantities of interest can be monitored in numerical studies of the Naming Game model in order to verify and quantify the qualitative expected picture. Among the most relevant ones are the average number of words in the agents inventory, $N_w(t)$, which corresponds to the average memory used, and the total number of distinct words in the system, $N_d(t)$.

Figure 1 displays the evolution of the average number of words per agent as a function of time, for a small-world network with average degree $\langle k\rangle = 8$, and various values of the rewiring probability p and size N. While $N_w(t)$ in all cases decays to N (Fig. 1A), after an initial peak whose height is proportional to N (Fig. 1B), the way in which this convergence is obtained depends on the parameters. At fixed N, for $p = 0$ a power-law behavior $N_w/N - 1 \propto 1/\sqrt{t}$ is observed due to the one-dimensional coarsening process [23]. As soon as $p \gg 1/N$ however, deviations are observed and get stronger as p is increased: the decrease of N_w is first slowed down after the peak, but leads in the end to an exponential convergence. The intermediate slowing down and the faster convergence are both enhanced as p increases. On the other hand, a system size increase at fixed p corresponds, as shown in Fig. 1B, to a slower convergence *even on the rescaled time* t/N, with a longer and longer plateau at almost constant average used memory.

As mentioned previously, a crossover phenomenon is expected when the one-dimensional clusters reach sizes of order $1/p$, i.e. at a time of order N/p^2. By definition, in the interior of each cluster, sites have only one word in memory, while the sites with more than one word are localized at the interfaces between clusters, whose number is then of order Np. The average excess memory per site (with respect to global consensus) is thus of order p, so that one

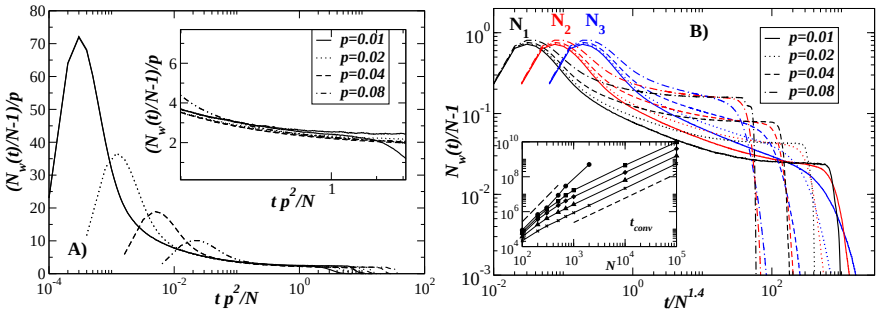

Fig. 2 – A) Rescaled curves of the average number of words per agent in the system, in order to show the collapse around the crossover time N/p^2. For each value of p, two values of the system size ($N = 10^4$ and $N = 10^5$) are displayed. The curves for different sizes are perfectly superimposed before the convergence. B) (Color online) Convergence at large times, shown by the drop of $N_w/N-1$ to 0: the time is rescaled by $N^{1.4}$. For each p, three different sizes ($N_1 = 10^3$ for the left peak, curves in black, $N_2 = 10^4$ for the middle peak, curves in blue, and $N_3 = 10^5$, right peak, curves in red) are shown. On the $N^{1.4}$ scale, the convergence becomes more and more abrupt as N increases. The inset displays the convergence time as a function of size for $p = 0$ (bullets), $p = 0.01$ (squares), $p = 0.02$ (diamonds), $p = 0.04$ (triangles), $p = 0.08$ (crosses); the dashed lines are proportional to N^3 and $N^{1.4}$.

expects $N_w/N - 1 = p\mathcal{G}(tp^2/N)$. Figure 2A indeed shows that the data of $(N_w/N - 1)/p$ for various values of p and N indeed collapse when tp^2/N is of order 1. On the other hand, Fig. 2B indicates that the convergence towards consensus is reached on a timescale of order N^a, with $a \approx 1.4 \pm 0.1$, close to the mean-field case $N^{3/2}$ and in strong contrast with the N^3 behavior of purely one-dimensional systems([1]). Moreover, as also observed in mean-field [22], the transition to the final consensus becomes more and more abrupt as the system size increases.

While the observation of Figs. 1B and 2B could convey the impression that, after the memory peak, the system tends to reach a stationary state whose length increases with N, the analysis of the evolution of the number of *distinct* words instead displays a continuous decrease (see Fig. 3): during this apparent plateau therefore, the system is still evolving continuously towards consensus by elimination of redundant words. Figure 3A points out that this decrease is longer for larger system sizes, while Fig. 3B shows that curves for various system sizes and values of p collapse when correctly rescaled around the crossover time N/p^2.

The combination of the results concerning average used memory and number of distinct words correspond to a picture in which clusters of agents sharing a common unique word compete during the time lapse between the peak and the final consensus. It is thus interesting to measure how the average cluster size evolves with time and how it depends on the rewiring probability p. Figure 4 allows to compare the cluster size $\langle s \rangle$ evolution for the one-dimensional case and for finite p. At $p = 0$, a pure coarsening law $\langle s \rangle \propto \sqrt{t}$ is observed. As p increases, deviations are observed when time reaches the crossover p^2/N, at a cluster size $1/p$, as was expected from the intuitive picture previously developed (Fig. 4 shows the collapse of the curves of $\langle s \rangle p$ vs. tp^2/N for tp^2/N of order 1).

Interestingly, the first deviation from the $\sqrt{t}$ law corresponds to a *slowing down* of the cluster growth, correspondingly with the slowing down observed in Fig. 1A. Because of long-range links, indeed, the clusters are locally more stable, due to the presence of an effective

([1])We also observe that the time to convergence scales as $p^{-1.4\pm.1}$; this is consistent with the fact that for p of order $1/N$ one should recover an essentially one-dimensional behaviour with convergence times of order N^3.

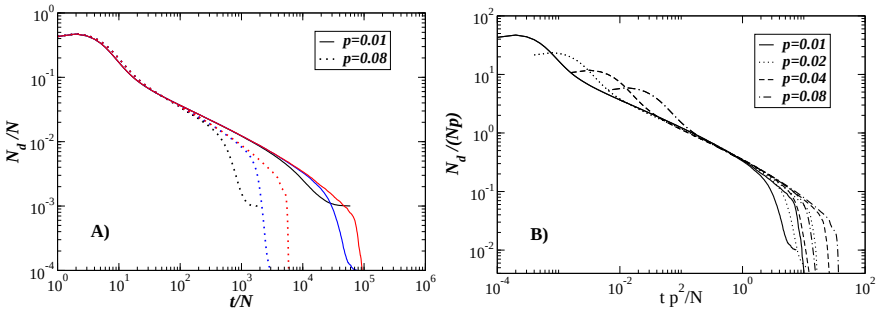

Fig. 3 – A) (Color online) Number of different words in the system as a function of time for $\langle k \rangle = 8$, $p = 10^{-2}$ and $p = 8.10^{-2}$ and increasing sizes (from left to right): $N = 10^3$ (black), $N = 10^4$ (blue), $N = 10^5$ (red). B) Same data rescaled in order to collapse the curves around the crossover time N/p^2. As in Fig. 2A), two values of the system size ($N = 10^4$ and $N = 10^5$) are displayed for each p.

'pinning' of interfaces near a shortcut. This effect is reminiscent of what happens for the Ising model on small-world networks [25] where, at low temperature, the local field transmitted by the shortcuts delay the passage of interfaces. Unlike Ising's zero temperature limit, however, the present dynamics only slows down and is never blocked into disordered configurations.

Strikingly, the final abrupt jump towards a unique cluster of size N starts earlier and from smaller average cluster size as p is increased. Although not intuitive, this behavior can be explained as follows. As p increases, these clusters are smaller and separated by more and more sites which have more than one word in memory (hence a larger value of N_w/N as p increases) and are more and more correlated. The sudden convergence to global consensus is thus obtained through a final fast agreement process between these sites.

Conclusion and perspectives. – In summary this paper has explored how a non trivial interaction topology could affect the dynamical approach to the emergence of shared conventions in populations of agents. We have shown how starting from a trivial one-dimensional like structure, the addition of a finite number of long-range links ($p \gg 1/N$) leads to a strong change in the dynamics, namely a drastic reduction of the memory required to the agents and a significant acceleration of the convergence process which passes from a N^3 dependence,

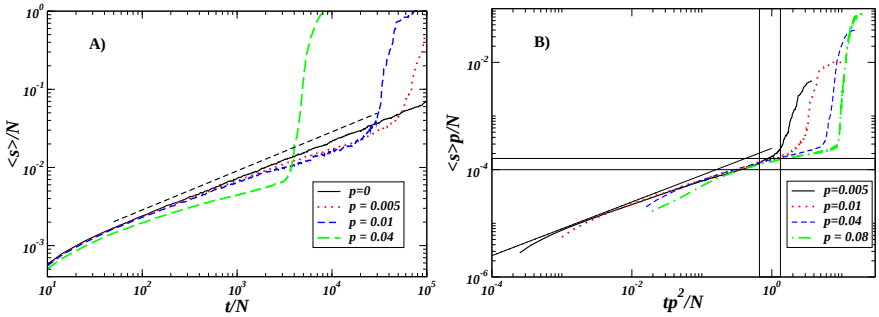

Fig. 4 – (Color online) A) Evolution of the cluster size for $N = 10^4$, various values of p. At increasing p the curves depart more and more from the $t^{1/2}$ behavior through a slowing down of the cluster growth, which however leads to a faster convergence. B) Same curves rescaled around the crossover region.

typical of the one-dimensional case to a $N^{1.4}$ scaling, close to the mean-field one. The overall dynamics occurs in two stages separated by a crossover time scaling as N/p^2. These results open in our view the way to a systematic investigation of this semiotic dynamics on different classes of complex networks where the heterogeneity of the nodes and the correlation between node degrees could play a major role [24].

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