The Structure of Equilibria in Trading Networks with Frictions

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Abstract

Several structural results for the set of competitive equilibria in trading networks with frictions are established: The lattice theorem, the rural hospitals theorem, the existence of side-optimal equilibria, and a group-incentive-compatibility result hold with imperfectly transferable utility and in the presence of frictions. While our results are developed in a trading network model, they also imply analogous (and new) results for exchange economies with combinatorial demand and for two-sided matching markets with transfers. *JEL-classification:*C78, D47, D52, L14

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1 Introduction

The assumption of transferable utility is pervasive in models of matching markets, exchange economies with indivisible goods, trading networks, and in mechanism

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design. The transferable utility assumption can simplify the analysis considerably since it allows us to exploit the duality between optimal allocations and supporting equilibrium prices. While the assumption of transferable utility simplifies the analysis, it is often empirically problematic. Wealth effects are present in marriage and labor markets so that matching models with transferable utility are unrealistic for these applications. Even if wealth effects are absent, transaction frictions such as those induced by taxation, subsidies, or transaction costs, make a transferable utility model inapplicable. This has motivated researchers to explore how results for matching markets with transfers (Demange and Gale, 1985; Legros and Newman, 2007; Nöldeke and Samuelson, 2018; Galichon et al., 2019) and for trading networks (Fleiner et al., 2019; Hatfield et al., 2021) can be generalized beyond transferable utility.

In this paper, we contribute to this discussion and study wealth effects and frictions[1] in the context of trading networks (Hatfield et al., 2013). Trading networks with bilateral contracts model complex supply chains in an industry where firms are engaged in upstream as well as downstream contracts. They generalize two-sided matching markets, in the sense that they replace a bipartite graph of potential relations, by an arbitrary graph where each edge represents a potential trade. We show that important structural results for trading networks do not depend on the assumption of transferable utility, and establish several results about the set of competitive equilibria under minimal assumptions on utility functions. Our results apply even in the case of wealth effects, in the presence of frictions, and if constraints make the execution of certain combinations of trades infeasible.

Our results can be summarized as follow: For a model of trading networks with frictions (Fleiner et al., 2019) and under the assumptions of full substitutability (Sun and Yang, 2006; Ostrovsky, 2008; Hatfield et al., 2013) and the laws of aggregate demand and supply (Hatfield and Kominers, 2012; Hatfield et al., 2021), we show that

- the set of competitive equilibria is a sublattice of the price space (first part of Theorem 1),

- a generalized "rural hospitals theorem" holds: the difference between the number of signed downstream and the number of signed upstream contracts

[1] We use the term "frictions" for any situation where utility is not necessarily a function of the sum of transfers received, but a function of the entire vector of transfers. If frictions are present, it thus not only matters how much a firm receives in total transfers, but also through which trades it receives the transfers. This can e.g. be the case if different trades involve different transaction costs. The results in our paper apply to both wealth effects and frictions.

is the same for each firm in each equilibrium (second part of Theorem 1),

- assuming additionally "bounded willingness to pay" (Fleiner et al., 2019),[2] there is an equilibrium that is most preferred by terminal sellers and an equilibrium that is most preferred by terminal buyers (Theorem 2),
- a mechanism that selects buyer-optimal equilibria is group-strategy-proof for terminal buyers on the domain of unit-demand utility functions and similarly a mechanism that selects seller-optimal equilibria is group-strategy-proof for terminal sellers on the domain of unit-supply utility functions (Theorem 3).

While our results are established for trading networks, the results are already new for many-to-one matching markets and for exchange economies with combinatorial demand which are special cases of our model. For matching markets, similar results were so far only known (a) under transferable utility, (b) for models without transfers and with strict preferences, or (c) for one-to-one matching markets with imperfectly transferable utility. For exchange economies with indivisible goods, analogous results were so far only known for (a) quasi-linear utility, or for the case of (b) unit demand.

Working with imperfectly-transferable utility requires us to develop fundamentally new techniques: Similar results without the transferable utility assumption were so far only known for one-to-one matching markets (Demange and Gale, 1985) and the proof techniques developed in this context (in particular the "decomposition lemma") do not adapt to more general settings. On the other hand, techniques from transferable utility models do not generalize to our model: Hatfield et al. (2013) use the efficiency of competitive equilibria and the submodularity of the indirect utility function to establish the lattice property. For our model with frictions, competitive equilibria can fail to be efficient. Moreover, full substitutability implies only the weaker notion of quasi-submodularity (Hatfield et al., 2020). More subtly, as we will discuss below, with wealth effects or frictions there are several non-equivalent definitions of full substitutability and these definitions are not distinguishable through conditions on the indirect utility function alone. Thus, an approach as in Hatfield et al. (2013) that characterizes competitive equilibria through the indirect utility function and uses properties of that function under full substitutability does not generalize. Likewise, the network flow approach to trading networks (Candogan et al., 2021) obtains structural results on the set of equilibria through the duality between optimal allocations and supporting prices.

[2]Alternatively, the result also holds with "bounding compensating variations" instead of bounded willingness to pay, as we show in the full working paper version (Schlegel, 2020).

Since efficiency fails in our setting this duality approach does not generalize. Finally, techniques from trading networks without transfers (Ostrovsky, 2008) that rely on Tarski's fixpoint theorem do not apply to the our model. With transfers, the issue of tie-breaking arises that is not present in models without transfers and with strict preferences. While versions of Tarski's fixpoint theorem for correspondences are known (Zhou, 1994), none of them work under sufficiently weak assumptions to be useful in our environment.

Since existing techniques do not work for our setting, we introduce a new approach to establish structural results for the set of competitive equilibria. The approach can be characterized as a tie-breaking approach: we show that for each finite set of (equilibrium) price vectors and each firm a single-valued selection from the demand correspondence can be made such that the properties of full substitutability and the laws of aggregate demand and supply are satisfied by the selection, and moreover, relevant trades that are demanded in the supporting equilibrium allocations are demanded in the selection. The assumption of the Laws of Aggregate Demand and Supply is necessary for our tie-breaking argument, and our result does not hold under Full Substitutability alone (see Example 3).

We also make a more technical contributions to the literature on trading networks with imperfectly transferable utility and clarify issues related to the definition of Full Substitutability: For the transferable utility model, there are various equivalent definitions of Full Substitutability (Hatfield et al., 2019). The equivalence, however, breaks down if we go beyond transferable utility, and, for our results, it matters which of the full substitutability notions is used. More specifically, it matters how full substitutability restricts the demand at price vectors at which the demand is multi-valued. We consider weak notions of Full Substitutability and the Laws of Aggregate Demand and Supply that only restrict the demand at price vectors where the demand is single-valued and stronger versions that also restrict it at prices where the demand is multi-valued. The notions are equivalent for transferable utility, but not in general. The set of competitive equilibria is a lattice only under the strong versions of Full Substitutability (see Example 1) and the rural hospitals theorem requires the strong versions of the law of aggregate demand and supply (see Example 2). Our group-strategy-proofness result, however, holds also under the weaker notions (Corollary 1). Thus, the exact definition of Full Substitutability matters in the model with frictions.[3]

[3]Related issues occur in Hatfield et al. (2021) where the stronger Monotone Substitutability property is needed that restricts the choice in circumstances where the choice correspondence is multi-valued.

We proceed as follows: In Section 2, we introduce the model and discuss different versions of the Full Substitutability conditions and their relation to each other. In Section 3, we prove our main results: the lattice structure of the set of competitive equilibria, the generalized rural hospitals theorem, the existence of extremal equilibria, and group-incentive compatibility for terminal buyers. In Section 4, we apply our main results to two-sided matching markets, and to exchange economies with indivisible goods.

1.1 Related Literature

The literature on trading networks has its origins in the literature on matching markets with transfers. In a seminal paper, Kelso and Crawford (1982) show that, under the assumption of gross substitutability, competitive equilibria with personalized prices exist and are equivalent to core allocations in a many-to-one labor market matching model. The construction is by an approximation argument where the existence in the continuum is obtained from the existence of an equilibrium in a discrete markets with smaller and smaller price increments. Different versions of a strategy-proofness result for a many-to-one matching model with continuous transfers were established by Hatfield et al. (2014); Schlegel (2018); Jagadeesan et al. (2018). Subsequent to Kelso and Crawford (1982), the question of existence of equilibria has been studied in the context of exchange economies with indivisibilities. See for example Gul and Stacchetti (1999) and the recent contribution of Baldwin and Klemperer (2019).

Trading networks with bilateral contracts and continuous transfers were introduced by Hatfield et al. (2013). Under the assumption of transferable utility and full substitutability they establish many results that we generalize to the case of general utility functions. The notion of full substitutability has been studied in detail by Hatfield et al. (2019) who show the equivalence of various different definitions of full substitutability. The existence result of Hatfield et al. (2013) is proved via a reduction to the existence result of Kelso and Crawford (1982). An alternative approach is via a submodular version of a network flow problem (Candogan et al., 2021).

The work of Hatfield et al. (2013) builds on the work of Ostrovsky (2008) on trading networks without transfers that generalizes matching models with contracts (Hatfield and Milgrom, 2005; Fleiner, 2003; Roth, 1984) beyond two-sided markets. The matching model with contracts in turn originates in the discrete version of the model of Kelso and Crawford (1982). Hatfield and Kominers (2012) and

Fleiner et al. (2016) provide additional results for the discrete trading networks model, which in many ways are parallel to the results we obtain in the continuous model. Importantly, results for the model without continuous transfers rely on the assumption of strict preferences.

All the above mentioned work for the continuous models make the assumption of transferable utility.[4] There are few papers that deal with wealth effects, frictions or constraints and that are particularly close to our work: In a classical paper, Demange and Gale (1985) establish several structural results about the core (or equivalently the set of competitive equilibria) for a one-to-one matching model with continuous transfers. In particular, they show that the core has a lattice structure and an agent that is unmatched in one core allocation receives his reservation utility in each core allocation (the result is often called the rural hospitals theorem in the literature on discrete matching markets). Moreover, they show that the mechanism that selects an extreme point of the bounded lattice is strategy-proof for one side of the market. Importantly, these results are established without assuming transferable utility. They only require that utility is increasing, continuous in transfers and satisfies a full range assumption. We generalize this work to trading networks and to situations in which utility does not satisfy the full range assumption.

In recent work, Fleiner et al. (2019) study trading networks with frictions. Their work is in many regards complementary to our work. In particular, Fleiner et al. (2019) establish the existence of a competitive equilibrium under the assumption of Full Substitutability and mild regularity conditions. Moreover, they study the efficiency of competitive equilibria and provide conditions under which equilibria correspond to allocations satisfying different related cooperative solution concepts. We derive our results for competitive equilibria. However, by the equivalence result of Fleiner et al. (2019) analogous results also would hold for "trail-stable" allocations. All results of Hatfield et al. (2013), except for the maximal domain result (Theorem 7) are generalized to the model with frictions, either in our work or by Fleiner et al. (2019). Table 1 summarizes results for trading networks with frictions.

Kojima et al. (2020b) introduce constraints in the job matching model of Kelso and Crawford (1982) and characterize constraints that leave the gross substitutes condition invariant. The model with constraints is a special case of the model in

[4] Note however that the existence proof of Kelso and Crawford (1982) is actually more general and applies as long as preferences are continuous, monotonic and unbounded in transfers for each bundle.

Table 1: Sufficient conditions for results for trading networks with frictions.

Result (Theorem*)	**Source**	**FS**	**LADS**	**BCV**	**BWP**	**NF**
Existence of Equil. (1)	Fleiner et al. (2019)	x		x		
1st Welfare Theorem (2)	Fleiner et al. (2019)					x
Rural Hospitals (3)	Theorem 1.(ii)	x	x			
Lattice (4)	Theorem 1.(i)	x	x			
Side Optimality (4)	Theorem 2	x	x		x	
Equil.⇒ Stable (5)	Fleiner et al. (2019)					x
Stable ⇒ Equil. (6)	Fleiner et al. (2019)	x		x		
Stable ⇔ Group-Stable (8,9)	Fleiner et al. (2019)	x		x		x
Trail-Stable ⇔ Equil.	Fleiner et al. (2019)	x			x	
Chain-Stable ⇔ Stable	Hatfield et al. (2021)	x	x			
Group-Strategy-Proofness	Theorem 3	x	x		x	

Notation:
Theorem* Corresponding theorem in Hatfield et al. (2013) under transferable utility. The existence of a side-optimal equilibrium additionally assumes finite valuations.
FS stands for *Full Substitutability*,
LADS stands for the *Laws of Aggregate Demand and Supply*,
BCV stands for *Bounded Compensating Variations*,
BWP stands for *Bounded Willingness to Pay*, and
NF stands for *No Frictions*.

the current paper so that we obtain as a corollary of our results a version of a lattice and of the rural hospital theorems for their model of job matching under constraints. In a spin-off paper, Kojima et al. (2020a) study comparative statics for their model and also prove versions of the lattice result and the rural hospitals theorem. These results have been obtained independently and contemporaneously with the results in the current paper.[5]

2 Model

The model follows Hatfield et al. (2013), and the extensions of Fleiner et al. (2019) and Hatfield et al. (2021). We consider a finite set of **firms** F and a finite set of **trades** Ω. Each trade $\omega \in \Omega$ is associated with a buyer $b(\omega) \in F$ and a seller $s(\omega) \in F$ with $b(\omega) \neq s(\omega)$. For a set of trades $\Psi \subseteq \Omega$ and firm $f \in F$ we define the set of **downstream trades** for f by $\Psi_{f\rightarrow} := \{\omega \in \Psi : s(\omega) = f\}$ and the set of **upstream trades** by $\Psi_{\rightarrow f} := \{\omega \in \Omega : b(x) = f\}$. Moreover, we let $\Psi_f := \Psi_{f\rightarrow} \cup \Psi_{\rightarrow f}$. A firm $f \in F$ such that $\Omega_{f\rightarrow} = \emptyset$ is called a **terminal buyer** and a firm such that $\Omega_{\rightarrow f} = \emptyset$ is called a **terminal buyer**. Note that terminal buyers and/or terminal buyers do not need to exist. A **contract** is a

[5]Weaker versions of these results were obtained prior to that in Schlegel (2018).

pair $(\omega, p_\omega) \in \Omega \times \mathbb{R}$, where p_ω is the price attached to the trade ω.

An **allocation** is a pair (Ψ, p) consisting of a set of trades $\Psi \subseteq \Omega$ and a price vector $p \in \mathbb{R}^\Psi$. We denote the set of allocations by $\mathcal{A}$ and we let $\mathcal{A}_f := \{(\Psi_f, (p_\omega)_{\omega \in \Psi_f}) : (\Psi, p) \in \mathcal{A}\}$. An **arrangement** is a pair $[\Psi, p] \in 2^\Omega \times \mathbb{R}^\Omega$. In contrast to an allocation the price vector also contains prices for unrealized trades.

Each firm has a utility function $u^f : \mathcal{A}_f \to \mathbb{R} \cup \{-\infty\}$. For notational convenience we extend u^f to $2^\Omega \times \mathbb{R}^\Omega$ by defining for $\Psi \subseteq \Omega$ and $p \in \mathbb{R}^\Omega$, the utility $u^f(\Psi, p) := u^f(\Psi_f, (p_\omega)_{\omega \in \Psi_f})$. We allow the utility function to take on a value of $-\infty$ in which case the combination of trades is infeasible for the firm.[6] We require that

- if a bundle is infeasible under some prices, then it is infeasible under all prices: if $u^f(\Psi, p) = -\infty$ for $p \in \mathbb{R}^{\Omega_f}$ then $u^f(\Psi, p') = -\infty$ for each $p' \in \mathbb{R}^{\Omega_f}$,
- at least one bundle of trades is feasible: there is a $\Psi \subseteq \Omega_f$ such that $u^f(\Psi, \cdot) > -\infty$.

Moreover, we make the following assumptions on utility functions:

- **Continuity**: For $\Psi \subseteq \Omega_f$ with $u^f(\Psi, \cdot) > -\infty$ the function $u^f(\Psi, \cdot)$ is continuous on $\mathbb{R}^\Psi$.
- **Monotonicity**: For $\Psi \subseteq \Omega_f$ with $u^f(\Psi, \cdot) > -\infty$ and $p, p' \in \mathbb{R}^\Psi$ with $p' \neq p$:
  (i) If $p'_\omega = p_\omega$ for $\omega \in \Psi_{f\to}$ and $p_\omega \leq p'_\omega$ for $\omega \in \Psi_{\to f}$, then $u^f(\Psi, p) > u^f(\Psi, p')$.
  (ii) If $p'_\omega = p_\omega$ for $\omega \in \Psi_{\to f}$ and $p_\omega \geq p'_\omega$ for $\omega \in \Psi_{f\to}$, then $u^f(\Psi, p) > u^f(\Psi, p')$.

Thus, utility is continuous in prices and firms strictly prefer higher sell prices to lower sell prices and lower buy prices to higher buy prices.

We allow utility for a set of trades Ψ to be different for prices $p, p' \in \mathbb{R}^\Psi$, even if the transfers received are the same for both price vectors, i.e. even if $\sum_{\omega \in \Psi_{f\to}} p_\omega - \sum_{\omega \in \Psi_{\to f}} p_\omega = \sum_{\omega \in \Psi_{f\to}} p'_\omega - \sum_{\omega \in \Psi_{\to f}} p'_\omega$. This can e.g. be the case if different trades involve different transaction costs. If utility only depends on

[6] Infeasibilities can for example arise through technological constraints, if producing and selling an output good requires the firm to buy certain input goods. In that case executing a downstream alone without executing related upstream trades is infeasible. Alternatively, infeasibilities can also arise through institutional constraints that restrict, for example, such as in Kojima et al. (2020b), the number of trades that a firm is allowed to execute.

the set of trades and the transfers received, we have a special case of our model: We say that u^f satisfies **no frictions** (Fleiner et al., 2019) if there is a function $\tilde{u}^f : 2^{\Omega_f} \times \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ such that

$$u^f(\Psi, p) = \tilde{u}^f \left(\Psi, \sum_{\omega \in \Psi_{f\to}} p_\omega - \sum_{\omega \in \Psi_{\to f}} p_\omega \right).$$

A utility function without frictions has **full range** if for each $\Psi \subseteq \Omega_f$, $\Psi \neq \emptyset$, $\tilde{u}^f(\Psi, \cdot)$ is a surjective function onto $\mathbb{R}$. It is **quasi-linear** if there is a valuation function $v^f : 2^{\Omega_f} \to \mathbb{R} \cup \{-\infty\}$ such that

$$\tilde{u}^f(\Psi, t) = v^f(\Psi) + t.$$

A utility function u^f induces an **indirect utility function** $v^f : \mathbb{R}^\Omega \to \mathbb{R}$ by

$$v^f(p) := \max_{\Psi \subseteq \Omega_f} u^f(\Psi, p),$$

and a **demand correspondence** $D^f : \mathbb{R}^\Omega \rightrightarrows 2^\Omega$ by:

$$D^f(p) := \text{argmax}_{\Psi \subseteq \Omega_f} u^f(\Psi, p).$$

Continuity of the utility function implies (e.g. by Berge's maximum theorem) that the demand correspondence is upper hemi-continuous. Monotonicity of the utility function implies that price vectors where the demand is single-valued are dense in price space. We will repeatedly use these facts to generate a single-valued selection from the demand-correspondence that inherits its good properties (such as full substitutability or the laws of aggregate demand and supply) by perturbing the price vector such that it becomes single-valued, see in particular Lemma 3. The proof is straightforward and hence omitted.

Lemma 1. *For a continuous and monotonic utility function u^f,*

(i) the induced demand D^f is upper hemi-continuous, i.e. for each $p \in \mathbb{R}^{\Omega_f}$ there is an $\epsilon > 0$ such for any $q \in \mathbb{R}^{\Omega_f}$ with $\|p - q\| < \epsilon$ (where $\|\cdot\|$ denotes the Euclidean norm) we have $D^f(q) \subseteq D^f(p)$,

(ii) the set of price vectors such that the induced demand is single-valued is dense in $\mathbb{R}^{\Omega_f}$, i.e. for each $\epsilon > 0$ and $p \in \mathbb{R}^{\Omega_f}$ there is a $q \in \mathbb{R}^{\Omega_f}$ with $\|p - q\| < \epsilon$ such that $|D^f(q)| = 1$.

2.1 Full Substitutability

Our results rely on a full substitutability assumption on utility functions. Informally, the condition requires that a firm sees upstream (downstream) trades as substitutes to each other, and upstream and downstream trades as complements to each other. Hatfield et al. (2019) show that for transferable utility various ways of defining full substitutability are equivalent, and hence one can work with either of the definitions discussed in their paper. Going beyond transferable utility makes issues more subtle: Not all equivalence results of Hatfield et al. (2019) generalize and it matters which of the full substitutability conditions are used. More specifically, it matters how the full substitutes condition is defined in instances where indifferences matter, i.e. when the demand is multi-valued.

We will proceed as follows: First, we introduce our main definition of full substitutability which restrict the demand both at price vectors where the demand is single-valued and where it is multi-valued. Second, we introduce a weaker version of full substitutability that only restricts the demand at price vectors where the demand is single-valued. We provide an example that shows that the single-valued version of full substitutability is strictly weaker than the multi-valued version. We later show, using this example, that the single-valued full substitutability condition is not sufficient for establishing the lattice and the rural hospitals theorem. Importantly, the difference between the single-valued and multi-valued version of full substitutability only matters for the "cross-side conditions" on firms' demand functions. In particular, the notions are equivalent for a two-sided market and the results for two-sided markets (see Section 4.1) hold under the single-valued notion of full substitutability. Third, we show that the multi-valued and the single-valued versions are, however, closely related in the sense that for each demand correspondence satisfying single-valued full substitutability, a selection from the demand correspondence exists that satisfies multi-valued full substitutability and can be rationalized by a utility function inducing the same indirect utility. In particular, this will allow us, later on (Corollary 1), to obtain a group-strategy-proofness result using the single-valued full substitutability notion.

2.1.1 Multi-Valued Full Substitutability

The following notion of full substitutability is due to Hatfield et al. (2019).[7] Precursors of the full substitutability notion were introduced for exchange economies (Sun and Yang, 2006) and for trading networks without transfers (Ostrovsky, 2008). Full substitutability can be further decomposed in a same-side substitutability and a cross-side complementarity notion.

Expansion Same-Side Substitutability (SSS): For $p, p' \in \mathbb{R}^{\Omega_f}$ and each $\Psi' \in D^f(p')$ there exists a $\Psi \in D^f(p)$ such that if $p_\omega = p'_\omega$ for $\omega \in \Omega_{f\to}$ and $p_\omega \leq p'_\omega$ for $\omega \in \Omega_{\to f}$, then

$$\{\omega \in \Psi_{\to f} : p_\omega = p'_\omega\} \subseteq \Psi'_{\to f},$$

and if $p_\omega = p'_\omega$ for $\omega \in \Omega_{\to f}$ and $p_\omega \geq p'_\omega$ for $\omega \in \Omega_{f\to}$, then

$$\{\omega \in \Psi_{f\to} : p_\omega = p'_\omega\} \subseteq \Psi'_{f\to}.$$

Expansion Cross-Side Complementarity (CSC): For $p, p' \in \mathbb{R}^{\Omega_f}$ and each $\Psi' \in D^f(p')$ there exists a $\Psi \in D^f(p)$ such if $p_\omega = p'_\omega$ for $\omega \in \Omega_{f\to}$ and $p_\omega \leq p'_\omega$ for $\omega \in \Omega_{\to f}$, then

$$\Psi'_{f\to} \subseteq \Psi_{f\to},$$

and if $p_\omega = p'_\omega$ for $\omega \in \Omega_{\to f}$ and $p_\omega \geq p'_\omega$ for $\omega \in \Omega_{f\to}$, then

$$\Psi'_{\to f} \subseteq \Psi_{\to f}.$$

Expansion Full Substitutability (FS): The demand of firm f satisfies Expansion Full Substitutability if it satisfies Expansion Same-Side Substitutability and Expansion Cross-Side Complementarity.

Next we introduce monotonicity properties called the Law of Aggregate Demand respectively the Law of Aggregate Supply. Under quasi-linear utility and

[7] Hatfield et al., 2019 call this version of full substitutability the "demand-language expansion" version of full substitutability (cf. Definition A.3 in Hatfield et al., 2019). Throughout the paper, we use "demand language" definitions of full substitubility that restrict the demand correspondence induced by the utility function. The demand language definitions are generally weaker than the corresponding "choice language" definitions that restrict the choice correspondence induced by the utility function. Consequently, all of our result would also hold under the corresponding "choice language" notions of full substitutability. Alternative multi-valued definitions of full substitutability are discussed in the full working paper version (Schlegel, 2020).

without frictions, the two properties are implied by full substitutability. However, in general they are independent of full substitutability.

Expansion Law of Aggregate Demand (LAD): For $p, p' \in \mathbb{R}^{\Omega_f}$ and each $\Psi' \in D^f(p')$ there exists a $\Psi \in D^f(p)$ such that if $p_\omega = p'_\omega$ for $\omega \in \Omega_{f\to}$ and $p_\omega \leq p'_\omega$ for $\omega \in \Omega_{\to f}$, then

$$|\Psi_{\to f}| - |\Psi_{f\to}| \geq |\Psi'_{\to f}| - |\Psi'_{f\to}|.$$

Expansion Law of Aggregate Supply (LAS): For $p, p' \in \mathbb{R}^{\Omega_f}$ and each $\Psi' \in D^f(p')$ there exists a $\Psi \in D^f(p)$ such that if $p_\omega = p'_\omega$ for $\omega \in \Omega_{\to f}$ and $p_\omega \geq p'_\omega$ for $\omega \in \Omega_{f\to}$, then

$$|\Psi_{f\to}| - |\Psi_{\to f}| \geq |\Psi'_{f\to}| - |\Psi'_{\to f}|.$$

Remark 1. Hatfield et al. (2021) introduce the notion of *Monotone Substitutability* which requires that FS, LAD and LAS hold jointly for the same bundles of trades. Hatfield et al. (2021) do not assume continuity of utility functions for their main result, and in that case, monotone substitutability is generally a stronger property than the combination of FS, LAD and LAS. With continuity, however, monotone substitutability is equivalent to the combination of FS, LAD and LAS. The proof of the equivalence is straightforward. See the full working paper version (Schlegel, 2020).

Alternatively, the combination of FS, LAD and LAS for continuous and monotonic utility functions can also be formulated as a generalized *single-improvement property* as we show in the full working paper version (Schlegel, 2020). Many of our results indirectly rely on this observation. □

2.1.2 Single-Valued Full Substitutability

Next we introduce a weaker notion of full substitutability where the condition only needs to hold at price vectors where the demand is single-valued. The weaker notion of full substitutability together with weaker notions of the law of aggregate demand/supply will be sufficient for our main incentive compatibility result (Corollary 1 in Section 3.3), but not for the other main results in the paper.

Single-Valued Full Substitutability (weak FS): For $p, p' \in \mathbb{R}^{\Omega_f}$ such that $D^f(p) = \{\Psi\}$ and $D^f(p') = \{\Psi'\}$, if $p_\omega = p'_\omega$ for $\omega \in \Omega_{f\to}$ and $p_\omega \leq p'_\omega$ for

$\omega \in \Omega_{\to f}$, then

$$\{\omega \in \Psi_{\to f} : p_\omega = p'_\omega\} \subseteq \Psi'_{\to f} \text{ and } \Psi'_{f\to} \subseteq \Psi_{f\to},$$

and if $p_\omega = p'_\omega$ for $\omega \in \Omega_{\to f}$ and $p_\omega \geq p'_\omega$ for $\omega \in \Omega_{f\to}$, then

$$\{\omega \in \Psi_{f\to} : p_\omega = p'_\omega\} \subseteq \Psi'_{f\to} \text{ and } \Psi'_{\to f} \subseteq \Psi_{\to f}.$$

The notions of **Single-Valued Same Side Substitutability** and **Single-Valued Cross Side Complementarity** are defined analogously.

Remark 2. The single-valued and the expansion notions of same sided substitutability are equivalent. Thus, for two-sided markets the two notions of full substitutability are equivalent. See Appendix A of the full working paper version (Schlegel, 2020) for a discussion of this and related facts. □

We also define single-valued versions of the laws of aggregate demand and supply.

Single-Valued Law of Aggregate Demand (Weak LAD): For $p, p' \in \mathbb{R}^{\Omega_f}$ such that $D^f(p) = \{\Psi\}$ and $D^f(p') = \{\Psi'\}$, if $p_\omega = p'_\omega$ for $\omega \in \Omega_{f\to}$ and $p_\omega \leq p'_\omega$ for $\omega \in \Omega_{\to f}$, then

$$|\Psi_{\to f}| - |\Psi_{f\to}| \geq |\Psi'_{\to f}| - |\Psi'_{f\to}|.$$

Single-Valued Law of Aggregate Supply (Weak LAS): For $p, p' \in \mathbb{R}^{\Omega_f}$ such that $D^f(p) = \{\Psi\}$ and $D^f(p') = \{\Psi'\}$, if $p_\omega = p'_\omega$ for $\omega \in \Omega_{\to f}$ and $p_\omega \geq p'_\omega$ for $\omega \in \Omega_{f\to}$, then

$$|\Psi_{f\to}| - |\Psi_{\to f}| \geq |\Psi'_{f\to}| - |\Psi'_{\to f}|.$$

The following example shows that the two notions of full substitutability that we have defined can differ for trading networks with frictions. In Section 3.1, we will use the example to show that under weak FS the lattice result in our paper does not necessarily hold.

Example 1. Consider four trades $\Omega = \{\alpha^1, \alpha^2, \beta^1, \beta^2\}$ with $f = b(\alpha^1) = b(\alpha^2) =$

$s(\beta^1) = s(\beta^2)$. We let

$$\begin{aligned}
&u^f(\emptyset) = 0,\\
&u^f(\{\alpha^i, \beta^j\}, p_{\alpha^i}, p_{\beta^j}) = 2 - p_{\alpha^i} + p_{\beta^j}, \text{ for } i, j, = 1, 2,\\
&u^f(\{\alpha^1, \alpha^2, \beta^1, \beta^2\}, p) = 4 - \exp\left(\frac{p_{\alpha^1} + p_{\alpha^2}}{2} - 1\right) - \exp\left(1 - \frac{p_{\beta^1} + p_{\beta^2}}{2}\right).
\end{aligned}$$

We let $u^f(\Psi, p) = -\infty$ for any other $\Psi \subseteq \Omega$. Observe that

$$D^f(1, 1, 1, 1) = \{\{\alpha^1, \beta^1\}, \{\alpha^1, \beta^2\}, \{\alpha^2, \beta^1\}, \{\alpha^2, \beta^2\}, \{\alpha^1, \alpha^2, \beta^1, \beta^2\}\}$$

but

$$D^f(0, 1, 1, 1) = \{\{\alpha^1, \beta^1\}, \{\alpha^1, \beta^2\}\}.$$

As $\{\alpha^1, \alpha^2, \beta^1, \beta^2\} \in D^f(1, 1, 1, 1)$, FS would require that there is a $\Psi \in D^f(0, 1, 1, 1)$ with $\{\beta^1, \beta^2\} \subseteq \Psi$. Hence FS is not satisfied. As the demand at $(0, 1, 1, 1)$ and $(1, 1, 1, 1)$ is multi-valued, Weak FS does not impose any structure here. More generally, note that the bundle $\{\alpha^1, \alpha^2, \beta^1, \beta^2\}$ is only demanded at prices $(1, 1, 1, 1)$ so that if we replace u^f by the utility function $\tilde{u}^f$ such that

$$\begin{aligned}
&\tilde{u}^f(\{\alpha^1, \alpha^2, \beta^1, \beta^2\}, \cdot) = -\infty\\
&\tilde{u}^f(\Psi, \cdot) = u^f(\Psi, \cdot) \text{ for } \Psi \neq \{\alpha^1, \alpha^2, \beta^1, \beta^2\},
\end{aligned}$$

only the demand at prices $(1, 1, 1, 1)$ changes. One readily checks that $\tilde{u}^f$ satisfies FS. Hence u^f satisfies Weak FS. Note, moreover, that u^f satisfied LAD and LAS. □

Similarly, weak LAD/LAS is strictly weaker than LAD/LAS.

Example 2. Consider two trades $\Omega = \{\omega_1, \omega_2\}$ with $f = b(\omega_1) = b(\omega_2)$. We let $u^f(\emptyset) = 0$, $u^f(\{\omega_i\}, p_{\omega_i}) = 3 - p_{\omega_i}$ for $i = 1, 2$, and

$$u^f(\{\omega_1, \omega_2\}, p) = \begin{cases} 4 - p_{\omega_1} - p_{\omega_2} & \text{if } p_{\omega_1} + p_{\omega_2} \leq 2,\\ 2 - \frac{(p_{\omega_1} + p_{\omega_2})^2 - 4}{12} & \text{if } 4 \geq p_{\omega_1} + p_{\omega_2} > 2,\\ 5 - p_{\omega_1} - p_{\omega_2} & \text{else.} \end{cases}$$

See Figure 1 for a geometric representation of the demand in the example. The induced demand violates LAD at $p' = (2, 2)$ and, e.g., $p = (1, 2)$ since $\Psi' = \{\omega_1, \omega_2\} \in D^f(p')$ but $D^f(p) = \{\{\omega_1\}\}$. One readily checks that $\tilde{u}^f$ satisfies FS and weak LAD. □

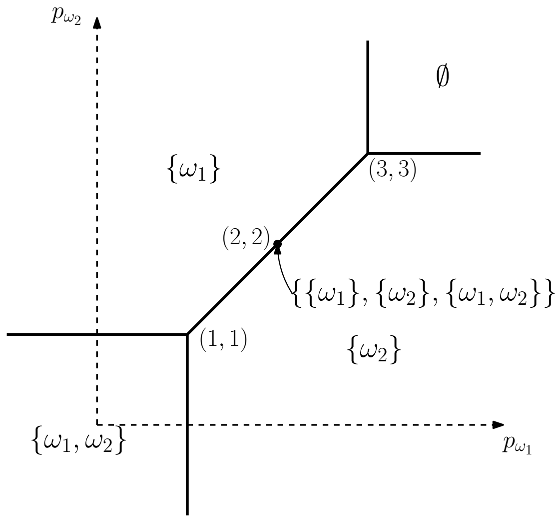

Figure 1: A representation of the demand of firm f from Example 2 in price space. Black lines indicate prices at which f is indifferent between several bundles. At prices $(2,2)$ the bundle $\{\omega_1,\omega_2\}$ is demanded in addition to bundle $\{\omega_1\}$ and $\{\omega_2\}$. Since $\{\omega_1,\omega_2\}$ is demanded at $(2,2)$ but at no other price vector in a neighborhood, NIB is violated.

In Example 1, the bundle $\{\alpha^1,\alpha^2,\beta^1,\beta^2\}$ is only demanded at prices $(1,1,1,1)$, but not at any price vector in the neighborhood. Similarly, in Example 2, the bundle $\{\omega_1,\omega_2\}$ is demanded at $(2,2)$ but at no other price vector in the neighborhood. One can show that if there are no such "isolated" bundles, i.e. bundles that are demanded at a price vector but nowhere in the neighborhood of it, then weak FS and FS are equivalent and weak LAD/LAS and LAD/LAS are equivalent. Formally this requirement is the following:

No Isolated Bundles (NIB): For each $p \in \mathbb{R}^{\Omega_f}$, $\Psi \in D^f(p)$ and $\epsilon > 0$ there is a $q \in \mathbb{R}^{\Omega_f}$ with $\|p-q\| < \epsilon$ and $D^f(q) = \{\Psi\}$.

Conversely if demand satisfies FS (CSC is sufficient here), LAD and LAS then it satisfies NIB. The equivalence between weak FS, weak LAD/LAS and NIB on one side and FS and LAD/LAS on the other side will be crucial for subsequent proofs:

Lemma 2. *Let u^f be a continuous and monotonic utility function inducing a demand D^f.*

(i) If D^f satisfies NIB and weak FS then it satisfies FS.

(ii) If D^f satisfies NIB and weak LAD/LAS, then it satisfies LAD/LAS.

(iii) If D^f satisfying CSC, LAD and LAS, then it satisfies NIB.

The converse of the first part of the lemma is not true in general. The demand in Example 2 satisfies FS but not NIB.

Next we show that for each utility function satisfying weak FS, weak LAD and weak LAS, we can construct a demand correspondence that satisfies FS, LAD and LAS by removing isolated bundles from the original demand. The resulting demand can be rationalized by a continuous and monotonic utility function. Put differently, we show that the two notions of Full Substitutability are almost equivalent in the following sense: for each utility function u^f for which the induced demand D^f satisfies weak FS, weak LAD, and weak LAS, there is a corresponding utility function $\tilde{u}^f$ for which the induced demand $\tilde{D}^f$ satisfies FS, LAD, and LAS, and such that $\tilde{D}^f$ selects from D^f. The utility function can be chosen such that the induced indirect utility is the same.

Proposition 1. *Let u^f satisfy weak FS, weak LAD, and weak LAS. Then there is a utility function $\tilde{u}^f$ that satisfies FS, LAD, and LAS such that the induced indirect utility functions are the same*

$$v^f(p) = \tilde{v}^f(p) \text{ for each } p \in \mathbb{R}^{\Omega_f},$$

and the induced demand is a selection from the original demand,

$$\tilde{D}^f(p) \subseteq D^f(p) \text{ for each } p \in \mathbb{R}^{\Omega_f}.$$

3 Results

3.1 The Lattice Theorem and the Rural Hospitals Theorem

As our first main result we establish that equilibrium prices in trading networks form a lattice and that (modulo indifferences) for each firm the difference between the number of signed upstream and downstream contracts is the same in each equilibrium. The join and meet are the coordinate-wise maximum and minimum of the two price vectors under consideration, i.e. the lattice is a sublattice of $\mathbb{R}^{\Omega}$ with the usual partial order. These results extend results established by Hatfield et al. (2013) for the case of transferable utility, and by Ostrovsky (2008); Hatfield and Kominers (2012); Fleiner et al. (2016) for the case without transfers and with strict preferences (for the solution concepts of chain-stability, stability resp. trail-stability).

In the following, a **competitive equilibrium** for utility profile $u = (u^f)_{f \in F}$ is an arrangement $[\Psi, p] \in 2^\Omega \times \mathbb{R}^\Omega$ such that for each $f \in F$ and the demand D^f induced by u^f we have $\Psi_f \in D^f(p)$. We call $(\Psi, (p_\omega)_{\omega \in \Psi})$ the **equilibrium allocation** induced by $[\Psi, p]$. We denote the set of equilibrium price vectors for u by $\mathcal{E}(u)$ and define for each price vector $p \in \mathbb{R}^\Omega$ the (possibly empty) set $\mathcal{E}(u, p) := \{\Psi \subseteq \Omega : \Psi_f \in D^f(p) \text{ for each } f \in F\}$ of sets of trades that support p as a competitive equilibrium under u.

Theorem 1. *Let u be a utility profile such that for each firm the induced demand satisfies FS, LAD and LAS.*

(i) **Lattice Theorem***: Let $p, p' \in \mathcal{E}(u)$ be equilibrium prices. Then $\bar{p}, \underline{p} \in \mathbb{R}^\Omega$ defined by*

$$\bar{p}_\omega := \max\{p_\omega, p'_\omega\}, \quad \underline{p}_\omega := \min\{p_\omega, p'_\omega\},$$

are equilibrium prices.

(ii) **Rural Hospitals Theorem***: Let $p, p' \in \mathcal{E}(u)$ be equilibrium prices. For each $\Psi \in \mathcal{E}(u, p)$ there exists a $\Psi' \in \mathcal{E}(u, p')$ such that for each $f \in F$ we have $|\Psi_{\to f}| - |\Psi_{f \to}| = |\Psi'_{\to f}| - |\Psi'_{f \to}|$.*

The proof and all subsequent proofs of this section are in Appendix B. However, for the moment we give a sketch of the proof strategy and comment on the challenges when generalizing results beyond transferable utility. Let $p, p' \in \mathcal{E}(u)$ be two equilibrium price vectors and consider the pairwise maximum $\bar{p} \in \mathbb{R}^\Omega$ (a dual argument works for the pairwise minimum $\underline{p}$). Suppose for the moment that the demand for each firm is single-valued at p and at p', i.e. each firm f has unique optimal bundles of trades Ψ_f and Ψ'_f at the equilibrium prices p and p'. The argument proceeds in two steps the first of which relies on the FS condition and the second of which relies on LAD/LAS:

No excess supply: For each $f \in F$ let $\bar{\Psi}_f \in D^f(\bar{p})$ be demanded at $\bar{p}$. FS applied at p and $\bar{p}$ resp. at p' and $\bar{p}$ implies that

$$\bar{\Psi}_{f\to} \subseteq \{\omega \in \Psi_{f\to} : p_\omega \geq p'_\omega\} \cup \{\omega \in \Psi'_{f\to} : p'_\omega > p_\omega\} \tag{1}$$

$$\{\omega \in \Psi_{\to f} : p_\omega \geq p'_\omega\} \cup \{\omega \in \Psi'_{\to f} : p'_\omega > p_\omega\} \subseteq \bar{\Psi}_{\to f}. \tag{2}$$

Taking the union of (1) and of (2) over all firms shows that there is no excess supply of trades at $\bar{p}$:

$$\bigcup_{f \in F} \bar{\Psi}_{f\to} \subseteq \{\omega \in \Psi : p_\omega \geq p'_\omega\} \cup \{\omega \in \Psi' : p'_\omega > p_\omega\} \subseteq \bigcup_{f \in F} \bar{\Psi}_{\to f},$$

No excess demand: LAD/LAS imply that for each $f \in F$,

$$|\Psi_{\to f}| - |\Psi_{f\to}| \geq |\bar{\Psi}_{\to f}| - |\bar{\Psi}_{f\to}|. \tag{3}$$

Summing inequality (3) for all firms we obtain

$$0 = \sum_{f\in F} |\Psi_{\to f}| - |\Psi_{f\to}| \geq \sum_{f\in F} |\bar{\Psi}_{\to f}| - |\bar{\Psi}_{f\to}| \Rightarrow |\bigcup_{f\in F} \bar{\Psi}_{f\to}| \geq |\bigcup_{f\in F} \bar{\Psi}_{\to f}|.$$

Combining this inequality with the previous observation that there is no excess supply shows that there is no excess demand and the market clears,

$$\bigcup_{f\in F} \bar{\Psi}_{f\to} = \bigcup_{f\in F} \bar{\Psi}_{\to f},$$

and thus $\bar{p} \in \mathcal{E}(u)$.

Suppose now that we want to generalize the argument to the case of multi-valued demand at the equilibrium prices. A natural idea is to use a perturbation argument: Continuity and monotonicity of utility in transfers allows us (see Lemma 1) to perturb price vectors to obtain a single-valued selection from the demand correspondence at prices p, p' and $\bar{p}$.

Lemma 3. *Let u^f be a utility function inducing a demand correspondence D^f satisfying weak FS, weak LAD and weak LAS. Let $P \subseteq \mathbb{R}^{\Omega_f}$ be finite. Then there is a (single-valued) demand function $\tilde{D}^f : P \to 2^{\Omega_f}$ that selects from D^f, i.e. $\tilde{D}^f(p) \in D^f(p)$ for $p \in P$ and satisfies FS, LAD and LAS.*

Once perturbed, the argument above could be applied to the perturbed price vectors. However, this line of argument has a flaw: There is no guarantee that the trades demanded at the perturbed prices support an equilibrium,[8] since not every collection of demanded trades at an equilibrium price vector support these prices as an equilibrium. While a naive perturbation argument fails to work, we can use a more intricate perturbation argument. We perturb prices for each firm individually. Importantly, we can rely on the observation (Lemma 2) that for each firm f there are prices q (in general different for different firms) close to p where the equilibrium set of trades Ψ_f is the unique demanded bundle of trades. This

[8]This is related to the observation that the set of competitive equilibrium price vectors for our model can fail to be connected. See the example in Roth and Sotomayor (1988) for the case of one-to-one matching with transfers which is a special case of our model. In contrast to this, for the transferable utility case it is easy to show that the set of competitive equilibrium price vectors is convex and thus, in particular, connected.

allows to show that for each firm f there is a $\bar{\Psi}_f \in D^f(\bar{p})$ that satisfies (1), (2) and (3) simultaneously (and a $\underline{\Psi}_f \in D^f(\underline{p})$ that satisfies dual properties for the pairwise minimum $\underline{p}$). This is the content of the following lemma that is a main ingredient in the proof of the theorem.

Lemma 4. *Let u^f be a utility function inducing a demand correspondence D^f satisfying FS, LAD and LAS. Let $p, p' \in \mathbb{R}^{\Omega_f}$ and define $\bar{p}, \underline{p} \in \mathbb{R}^{\Omega_f}$ by*

$$\bar{p}_\omega := \max\{p_\omega, p'_\omega\}, \quad \underline{p}_\omega := \min\{p_\omega, p'_\omega\}.$$

Let $\Psi \in D^f(p)$ and $\Psi' \in D^f(p')$.

(i) There is a $\bar{\Psi} \in D^f(\bar{p})$ with

$$\{\omega \in \Psi_{\to f} : p_\omega \geq p'_\omega\} \cup \{\omega \in \Psi'_{\to f} : p'_\omega > p_\omega\} \subseteq \bar{\Psi}_{\to f},$$
$$\bar{\Psi}_{f\to} \subseteq \{\omega \in \Psi_{f\to} : p_\omega \geq p'_\omega\} \cup \{\omega \in \Psi'_{f\to} : p'_\omega > p_\omega\}.$$

(ii) There is a $\underline{\Psi} \in D^f(\underline{p})$ with

$$\underline{\Psi}_{\to f} \subseteq \{\omega \in \Psi_{\to f} : p'_\omega \geq p_\omega\} \cup \{\omega \in \Psi'_{\to f} : p_\omega > p'_\omega\},$$
$$\{\omega \in \Psi_{f\to} : p'_\omega \geq p_\omega\} \cup \{\omega \in \Psi'_{f\to} : p_\omega > p'_\omega\} \subseteq \underline{\Psi}_{f\to}.$$

(iii) $\bar{\Psi}$ and $\underline{\Psi}$ can be chosen such that

$$|\underline{\Psi}_{\to f}| - |\underline{\Psi}_{f\to}| \geq |\Psi_{\to f}| - |\Psi_{f\to}| \geq |\bar{\Psi}_{\to f}| - |\bar{\Psi}_{f\to}|.$$

With the lemma the proof of the theorem can be carried out as described before. The lemma and the first part of the theorem fail to hold if we replace FS by weak FS, as the following example shows.

Example 1 (cont.). Consider the set of trades $\Omega = \{\alpha^1, \alpha^2, \beta^1, \beta^2\}$ and firm f with the utility function u^f as defined in Example 1. The induced demand D^f satisfies weak FS as previously shown. Moreover, for each $p \in \mathbb{R}^{\Omega_f}$ and $\Psi \in D^f(p)$ we have $|\Psi_{f\to}| = |\Psi_{\to f}|$. Thus D^f satisfies LAD and LAS. Consider four additional firms s^1, s^2, b^1, b^2 with $s^1 = s(\alpha^1), s^2 = s(\alpha^2), b^1 = b(\beta^1)$ and $b^2 = b(\beta^2)$. Define utility functions for the additional firms as follows: For $i = 1,2$ define

$$u^{s^i}(\{\alpha^i\}, p_{\alpha^i}) = p_{\alpha^i},$$
$$u^{b^i}(\{\beta^i\}, p_{\beta^i}) = 2 - p_{\beta^i},$$
$$u^{s^i}(\emptyset) = u^{b^i}(\emptyset) = 0.$$

Observe that the equilibria for u are $[\Omega, (1,1,1,1)]$ and $[\{\alpha^i, \beta^j\}, (0,0,2,2)]$ for $i, j = 1,2$. In particular, the vector $(1,1,2,2)$ is not an equilibrium price vector, since $D^{s^1}(1,1,2,2) = \{\{\alpha^1\}\}$ and $D^{s^2}(1,1,2,2) = \{\{\alpha^2\}\}$ but $D^f(1,1,2,2) = \{\{\alpha^1, \beta^1\}, \{\alpha^1, \beta^2\}, \{\alpha^2, \beta^1\}, \{\alpha^2, \beta^2\}\}$. □

Similarly, the second part of the theorem fails if LAD (LAS) is replaced by weak LAD (weak LAS) as the following example shows.

Example 2 (cont.). Consider the set of trades $\Omega = \{\omega_1, \omega_2\}$ and firm f with the utility function u^f as defined in Example 2. As observed before, D^f satisfies FS, weak LAD, (and, trivially, LAS), but not LAD. Consider a second firm f' with $f' = s(\omega_1) = s(\omega_2)$ with utility function $u^{f'}$ defined by

$$u^{f'}(\{\omega_i\}, p_{\omega_i}) = p_{\omega_i}, \qquad \text{for } i = 1,2,$$
$$u^{f'}(\{\omega_1, \omega_2\}, p) = p_{\omega_1} + p_{\omega_2} - 1.5,$$
$$u^{f'}(\emptyset) = 0.$$

The induced demand satisfies FS and LAS (and, trivially, LAD). The set of equilibrium vectors is $\mathcal{E}(u) = \{p : 1 \leq p_{\omega_1} = p_{\omega_2} \leq 1.5\} \cup \{(2,2)\}$. Each $p \in \mathcal{E}(u) \setminus \{(2,2)\}$ is supported by $\{\omega_1\}$ and by $\{\omega_2\}$. The equilibrium prices $(2,2)$ are supported by $\{\omega_1, \omega_2\}$. An analogous example can be constructed to show that LAS and not just weak LAS is necessary for the Rural Hospitals Theorem. □

It is well-known that the theorem fails to hold without FS, even for transferable utility. The following example shows that the first part of the theorem fails without LAD. More generally, the example shows that without LAD the set of equilibria can even fail to be a lattice with respect to the (weaker) partial ordering induced by terminal sellers' preferences. The example relies on the previous logic highlighted in the discussion of Theorem 1: the FS condition can be used to show that there is no excess supply of trades at the pair-wise maximum of two equilibrium price vectors. However, without the LAD there can still be strict excess demand of trades at the pairwise maximum (or at price vectors dominating it).

Example 3. Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$. Let $b(\omega_i) = f$ for $i = 1,2,3$ and $s(\omega_i) \neq s(\omega_j)$

for $i \neq j$. We let $u^{s(\omega_i)}(\omega_i, p_{\omega_i}) = p_{\omega_i}$, $u^{s(\omega_i)}(\emptyset) = 0$ for $i = 1, 2$ and $u^{s(\omega_3)}(\omega_3, p_{\omega_3}) = -\infty$, $u^{s(b_3)}(\emptyset) = 0$. We define u^f by

$$\begin{aligned} &u^f(\emptyset) = 0, \\ &u^f(\{\omega_1, \omega_2\}, (p_{\omega_1}, p_{\omega_2})) = 2 - p_{\omega_1} - p_{\omega_2}, \\ &u^f(\{\omega_1, \omega_2, \omega_3\}, p) = 1 - \frac{1}{1 + \exp(-(p_{\omega_1} + p_{\omega_2} + p_{\omega_3}))}, \end{aligned}$$

and $u^f(\Psi, \cdot) = -\infty$ else.

Consider the price vectors $p = (0, 1, 0)$ and $p' = (1, 0, 0)$. Note that $\{\omega_1, \omega_2\} \in D^f(p)$ and $\{\omega_1, \omega_2\} \in D^f(p')$. Moreover, we have $D^{s(\omega_1)}(p) = \{\{\omega_1\}\} = D^{s(\omega_1)}(p')$, $D^{s(\omega_2)}(p) = \{\{\omega_2\}\} = D^{s(\omega_2)}(p')$ and $D^{s(\omega_3)}(p) = \{\emptyset\} = D^{s(\omega_3)}(p')$. Thus p and p' are equilibrium price vectors. Suppose there is a $\bar{p}$ that each terminal seller weakly prefers to p and p', i.e. $v^{s(\omega_1)}(\bar{p}) \geq \max\{v^{s(\omega_1)}(p), v^{s(\omega_1)}(p')\} = 1$, $v^{s(\omega_2)}(\bar{p}) \geq \max\{v^{s(\omega_2)}(p), v^{s(\omega_2)}(p')\} = 1$, and $v^{s(\omega_3)}(\bar{p}) \geq v^{s(\omega_3)}(p) = v^{s(\omega_3)}(p') = u^{s(\omega_3)}(\emptyset) = 0$. Thus $\bar{p}_{\omega_1} \geq 1$ and $\bar{p}_{\omega_2} \geq 1$. But then $D^f(\bar{p}) = \{\{\omega_1, \omega_2, \omega_3\}\}$. Moreover, $D^{s(\omega_3)}(\bar{p}) = \{\emptyset\}$. Thus, there is no such equilibrium price vector $\bar{p}$.

To check that u^f satisfies FS, first note that for each $p \in \mathbb{R}^{\Omega_f}$, we have $u^f(\{\omega_1, \omega_2, \omega_3\}, p) > 0 = u^f(\emptyset)$. Thus, at each $p \in \mathbb{R}^{\Omega_f}$ we have $D^f(p) \subseteq \{\{\omega_1, \omega\}, \{\omega_1, \omega_2, \omega_3\}\}$ and the only possible FS violation could occur for $p \leq p'$ with $p'_{\omega_3} = p_{\omega_3}$ and $\{\omega_1, \omega_2, \omega_3\} \in D^f(p)$. However, if $u^f(\{\omega_1, \omega_2, \omega_3\}, p) \geq u^f(\{\omega_1, \omega_2\}, p)$, then, as $u^f(\{\omega_1, \omega_2, \omega_3\}, p) - u^f(\{\omega_1, \omega_2\}, p)$ is increasing in p_{ω_1} and in p_{ω_2} for each p_{ω_3}, we have $u^f(\{\omega_1, \omega_2, \omega_3\}, p') \geq u^f(\{\omega_1, \omega_2\}, p')$. Thus, FS holds.[9] □

3.2 Extremal equilibria

So far we have not considered whether competitive equilibria exist in our model and, in principle, the lattice in Theorem 1 could be empty. Next we show that under the additional assumption of bounded willingness to pay (we follow the terminology of Fleiner et al., 2019), side-optimal equilibria exist, i.e. there exist an equilibrium that is a most preferred equilibrium for all terminal buyers and an equilibrium that is a most preferred equilibrium for all terminal sellers.

Bounded willingness to pay (BWP): The utility function u^f satisfies bounded willingness to pay if there exists a $K \geq 0$ such that for all $p \in \mathbb{R}^{\Omega_f}$ and $\Psi \in D^f(p)$ if $\omega \in \Psi_{\to f}$ then $p_\omega < K$ and if $\omega \in \Psi_{f \to}$ then $p_\omega > -K$.

[9] The example violates the BWP and the BCV conditions that we consider in Section 3.2.

The condition rules out for example the case that for a trade the seller would never sell under any price and the buyer would buy under any price. BWP guarantees that equilibrium prices of trades realized in equilibrium are bounded. It follows straightforwardly from the continuity of utility functions that equilibrium prices of trades realized in equilibrium form a closed set. Thus, the set of equilibrium prices of trades realized in equilibrium is compact. The existence of side-optimal equilibria follows straightforwardly from this:

Theorem 2 (**Existence of Extremal Equilibria**)**.** *Under the assumption of BWP, FS, LAD, LAS, there exists a seller-optimal equilibrium, i.e. a* $\bar{p} \in \mathcal{E}(u)$ *such that for each terminal seller* $f \in F$*:*

$$v^f(\bar{p}) \geq v^f(p) \text{ for each } p \in \mathcal{E}(u),$$

and a buyer-optimal equilibrium, i.e. a $\underline{p} \in \mathcal{E}(u)$ *such that for each terminal buyer* $f \in F$*:*

$$v^f(\underline{p}) \geq v^f(p) \text{ for each } p \in \mathcal{E}(u).$$

Remark 3. Under the assumptions of weak FS and BWP, Fleiner et al. (2019) establish that equilibrium allocations are equivalent to trail-stable allocations.[10] Thus, under BWP, FS, LAD, LAS there is a seller-optimal trail-stable allocation and a buyer-optimal trail-stable allocation. In the case of no frictions, BWP is implied by requiring, that $u^f(\emptyset) > -\infty$ and utility functions have full range. □

Fleiner et al. (2019) also introduce an alternative regularity condition, called bounded compensating variations (BCV), which guarantees that utility of individually rational allocations is bounded for all agents.

Bounded compensating variations: The utility function of firm f satisfies bounded compensating variations if for each $\Psi \subseteq \Omega$ we have

$$\inf_{p \in \mathbb{R}^{\Psi} : u^f(\Psi, p) > u^f(\emptyset)} \left(\sum_{\omega \in \Psi_{f\rightarrow}} p_\omega - \sum_{\omega \in \Psi_{\rightarrow f}} p_\omega \right) > -\infty.$$

[10] Fleiner et al. (2019) use the choice-language definition of weak FS. They show that the choice-language definition is equivalent to the demand-language definition when the price space is amended by infinite prices. Under BWP it is easy to see that the equivalence between the choice-language and the demand-language versions of weak FS and of FS also holds on the standard price space $\mathbb{R}^\Omega$.

Remark 4. The previous result also holds if BWP is replaced by BCV. See the full working paper version (Schlegel, 2020) for the proof:

Theorem (Existence of Extremal Equilibria with BCV, Schlegel 2020). *Under the assumption of BCV, FS, LAD, LAS, there exists a seller-optimal equilibrium and a buyer-optimal equilibrium.*

3.3 Strategic Considerations

The existence of buyer-optimal equilibria established in Theorem 2, allows us to obtain a group-incentive compatibility result.[11] In the following, a domain of utility profiles is a set $\mathcal{U} = \bigtimes_{f\in F}\mathcal{U}_f$ where $\mathcal{U}_f$ is a set of (continuous and monotonic) utility functions for firm f. A **mechanism** is a function $\mathcal{M} : \mathcal{U} \to \mathcal{A}$. A mechanism is **(weakly) group-strategy-proof** for a set of workers $F' \subseteq F$ on the domain $\mathcal{U}' \subseteq \mathcal{U}$ if for each $u, \tilde{u} \in \mathcal{U}'$ with $\tilde{u}^{-F'} = u^{-F'}$, there exist a $f \in F'$ with

$$u^f(\mathcal{M}(u)) \geq u^f(\mathcal{M}(\tilde{u})).$$

Theorem 2 allows us to define a class of focal mechanisms on the domain of utility profiles satisfying BWP, FS, LAD and LAS: a **buyer-optimal mechanism** maps to each utility profile a buyer-optimal equilibrium allocation.

To obtain a group-strategy-proofness results for terminal buyers for buyer-optimal mechanisms, we have to restrict the domain. In the following a **unit demand** utility function is a u^f such that for the induced demand D^f at each $p \in \mathbb{R}^{\Omega_f}$ and $\Psi \in D^f(p)$ we have $|\Psi_{\to f}| \leq 1$. For the case without transfers and with strict preferences, analogous results are proved by Hatfield and Kominers (2012) (for the case of acyclic networks and the solution concept of stability) and Fleiner et al. (2016) (for arbitrary networks and the solution concept of trail-stability).

Theorem 3 (Group-Strategy-Proofness). *Each buyer-optimal mechanism is group-strategy-proof for terminal buyers on the domain of utility profiles such that terminal buyers' utility functions satisfy Unit Demand and BWP and all other firms' utility functions satisfy BWP, FS, LAD and LAS.*

In view of Proposition 1, we can extend the construction to profiles satisfying BWP, weak (!) FS, weak LAD and weak LAS. For each such profile u there exists a corresponding profile $\tilde{u}$ satisfying BWP, FS, LAD and LAS such that the indirect

[11] In the following we talk about incentives for terminal buyers. A completely analogous result also holds for terminal sellers.

utility functions are the same for both profiles. The mechanism that assigns to each profile u a buyer-optimal equilibrium allocation under a corresponding profile $\tilde{u}$ is group-strategy-proof for terminal buyers (since for terminal buyers (and terminal sellers) the weak FS and the FS condition coincide), and the assigned allocations are equilibrium allocations under u as well.

Corollary 1 (Group-Strategy-Proofness under weak FS). *On the domain of utility profiles such that terminal buyers' utility functions satisfy Unit Demand and BWP and all other firms' utility functions satisfy BWP, weak FS, weak LAD and weak LAS, there exists a group-strategy-proof mechanisms for terminal buyers that implements a competitive equilibrium.*

Remark 5. As noted in Remark 4, the existence of extremal equilibria can alternatively be proved with BWP replaced by BCV. Analogously, we can obtain a group-strategy-proofness result with BWP replaced by BCV. The proof remains unchanged in that case. □

4 Applications

4.1 Two-sided Matching Markets

The results in the previous sections immediately apply to two-sided matching markets. In this case, the results generalize previously known results for two-sided matching markets in two directions: we provide a lattice result, a rural hospitals theorem and a group-strategy-proofness result for markets with a) wealth effects and frictions for both sides of the market b) the possibility that it is infeasible for a hospital to hire certain groups of doctors. As remarked in Section 2.1, the weak version of Full Substitutability is sufficient to obtain the results for two-sided markets.

Instead of a set of firms, the economy now consists of a finite set of **hospitals** H and a finite set of **doctors** D. Each hospital h has a utility function $u^h : \{(D', p) : D' \subseteq D, p \in \mathbb{R}^{D'}\} \to \mathbb{R} \cup \{-\infty\}$ that assigns to each $D' \subseteq D$ and price vector $p \in \mathbb{R}^{D'}$ a utility level. We extend u^h to $2^D \times \mathbb{R}^D$ by letting $u^h(D', p) := u^h(D', (p_d)_{d \in D'})$. We allow the utility function to take on a value of $-\infty$ to indicate that it is infeasible for the hospital to hire a particular group of doctors. This allows us for example to incorporate institutional constraints such as the "generalized interval constraints" characterized by Kojima et al. (2020b) which specify a lower and an upper bound on the number of doctors a hospital can hire. We assume that $u^h(D',p) = -\infty$ implies $u^h(D',p') = -\infty$ for each $p' \in \mathbb{R}^{D'}$. We assume that there is at least one group of doctors $D' \subseteq D$ that is feasible to hire, i.e. such that $u^h(D',\cdot) > -\infty$. Moreover, we require that for $u^h(D',\cdot) \neq -\infty$, the utility function $u^h(D',\cdot)$ is continuous and strictly decreasing in prices. The utility function induces a demand correspondence $D^h : \mathbb{R}^D \rightrightarrows 2^D$ by $D^h(p) := \text{argmax}_{D' \subseteq D} u^h(D',p)$. We assume that doctors are gross substitutes for hospitals. We only need to require the condition for price vectors where the demand is single-valued.

Weak Gross Substitutability: For $p, p' \in \mathbb{R}^D$ with $p \leq p'$, $D^h(p) = \{D'\}$ and $D^h(p') = \{D''\}$ we have $\{d \in D' : p'_d = p_x\} \subseteq D''$.

Moreover, we require the law of aggregate demand.

Law of Aggregate Demand: For $p, p' \in \mathbb{R}^D$ with $p \leq p'$ and each $D' \in D^f(p')$ there is a $\tilde{D} \in D^f(p)$ with $|\tilde{D}| \geq |D'|$.

Each doctor d has a utility function $u^d : H \times \mathbb{R} \cup \{\emptyset\} \to \mathbb{R}$ that is strictly increasing and continuous in its second argument. We extend u^d to $H \cup \{\emptyset\} \times \mathbb{R}^H$ by letting $u^d(h,p) := u^d(h, p_{hd})$ and $u^d(\emptyset, p) := u^d(\emptyset)$.

A **matching** is a function $\mu : H \times D \to 2^D \cup H$ with $\mu(h) \subseteq D$ for each $h \in H$ and $\mu(d) \in H \cup \{\emptyset\}$ for each $d \in D$ such that $d \in \mu(h)$ if and only if $h = \mu(d)$. A **competitive equilibrium** (μ, p) is a pair consisting of a matching μ, and a price vector $p \in \mathbb{R}^{H \times D}$ such that for each $h \in H$ and $p_h := (p_{hd})_{d \in D}$ we have $\mu(h) \in D^h(p_h)$ and for each $d \in D$ and $p_d = (p_{hd})_{h \in H}$ we have $u^d(\mu(d), p_d) = \max_{h \in H \cup \{\emptyset\}} u^d(h, p_d)$. The following is an immediate consequence of Theorems 1, 2 and 3 and generalizes results of Hatfield et al. (2013, 2014) for the transferable utility model.[12]

Corollary 2. *For each matching market such that doctors are weak gross substitutes for hospitals and the law of aggregate demand holds the following is true:*

(i) Let $p, p' \in \mathbb{R}^{H \times D}$ be equilibrium prices. Then $\bar{p}, \underline{p} \in \mathbb{R}^{H \times D}$ defined by

$$\bar{p}_{hd} = \max\{p_{hd}, p'_{hd}\}, \quad \underline{p}_{hd} = \min\{p_{hd}, p'_{hd}\}$$

[12]It is important that we use the "multi-valued" version of the Law of Aggregate Demand. Otherwise, Example 3 demonstrates that the lattice result can fail. An example similar to Example 2 but with two instead of one seller demonstrates that the rural hospitals theorem can fail.

are equilibrium prices.

(ii) *Let $p, p' \in \mathbb{R}^{H \times D}$ be equilibrium prices. For each matching μ supporting p as an equilibrium (μ, p) there is a matching μ' supporting p' as an equilibrium (μ', p') such that*

   (a) *a doctor is unemployed in μ if and only if he is unemployed in μ', i.e. $\mu(d) = \emptyset \Leftrightarrow \mu'(d) = \emptyset$, for each $d \in D$,*

   (b) *each hospital hires the same number of doctors in μ and μ', i.e. $|\mu(h)| = |\mu'(h)|$ for each $h \in H$.*

(iii) *If utility functions satisfy, moreover, BWP, then there exists a worker-optimal equilibrium allocation and a hospital-optimal equilibrium allocation.*

(iv) *The worker-optimal mechanism is group-strategy-proof for workers on the domain of utility profiles such that workers' utility functions satisfy Unit Supply and BWP and hospitals' utility functions satisfy BWP, weak GS, and LAD.*

Proof. We can construct a corresponding trading network with $\Omega = H \cup D$ and $\tilde{u}^h(\Psi, p) = u^h(\{d : (h, d) \in \Psi\}, p)$ for $\Psi \subseteq \Omega_f$ and $\tilde{u}^d(\{(h, d)\}, p_{hd}) = u^d(h, p_{hd})$, $\tilde{u}^d(\emptyset) = u^d(\emptyset)$ and $\tilde{u}^d(\Psi, \cdot) = -\infty$ if $\Psi \subseteq \Omega_f$ with $|\Psi| > 1$. The weak gross substitutes condition then corresponds to the weak SSS condition. Weak SSS is equivalent to SSS as shown in Appendix D of Fleiner et al. (2019) (SSS corresponds to the conjunction of the two properties that Fleiner et al. (2019) call "Increasing Price Full Substitutability for Sales" and "Decreasing Price Full Substitutability for Purchases"). Since the market is two-sided, SSS and FS are equivalent. The corollary follows from Theorems 1, 2 and 3. □

4.2 Exchange economies with uniform pricing

Next, we apply the model to the exchange of indivisible objects. The result extend results of Gul and Stacchetti (1999) and Hatfield et al. (2013) (see the discussion in their Section IV.B) to imperfectly transferable utility. As in Gul and Stacchetti (1999), we maintain the assumption that the market is cleared through transfers of a perfectly divisible good and there is no constraint on the amount of the divisible good an agent can consume. Moreover, negative quantities of the divisible good can be consumed. However, we do not assume that utility in the divisible good is quasi-linear. Similar assumption are standard in the object allocation literature with general preferences, see for example Morimoto and Serizawa (2015).

In the following, we let X be a finite set of heterogeneous indivisible **objects**. From now on, we use the term agents in lieu of firms. Agents have utility functions over bundles of objects and transfers, $\tilde{u}^f : 2^X \times \mathbb{R} \to \mathbb{R}$ such that for each $Y \subseteq X$, $\tilde{u}^f(Y, \cdot)$ is continuous, strictly increasing and has full range,[13] and for each $t \in \mathbb{R}$ and $Y \subseteq Y' \subseteq X$, we have $\tilde{u}^f(Y,t) \leq \tilde{u}^f(Y',t)$. Each agent f is endowed with a bundle of objects $X_f \subseteq X$ such that $X_f \cap X_{f'} = \emptyset$ for $f \neq f'$ and $\bigcup_{f\in F} X_f = X$. An **exchange economy** is a pair $(\tilde{u}, (X_f)_{f\in F})$ of utility functions and endowments for each agent. We define for each $f \in F$ a **demand** correspondence $\tilde{D}^f : \mathbb{R}^X_+ \times 2^X \rightrightarrows 2^X$ by

$$\tilde{D}^f(p, X_f) := \text{argmax}_{Y\subseteq X} \tilde{u}^f \left(Y, \sum_{x\in X_f \setminus Y} p_x - \sum_{x\in Y\setminus X_f} p_x \right).$$

Remark 6. In contrast to quasi-linear utility, demand can depend on the endowment, i.e. in general $\tilde{D}^f(p, X_f) \neq \tilde{D}^f(p, \tilde{X}_f)$ for $X_f \neq \tilde{X}_f$. □

We assume that objects are gross substitutes for agents.[14]

Gross Substitutability (GS): For $p, p' \in \mathbb{R}^X_+$ with $p \leq p'$, if $p'_x = p_x$ for $x \in X_f$, then for each $Y' \in \tilde{D}^f(p', X_f)$ there exists a $Y \in \tilde{D}^f(p, X_f)$ such that $\{x \in Y : p'_x = p_x\} \subseteq Y'$, and if $p'_x = p_x$ for $x \in X \setminus X_f$, then for each $Y \in \tilde{D}^f(p, X_f)$ there exists a $Y' \in \tilde{D}^f(p', X_f)$, such that $\{x \in Y : p'_x = p_x\} \subseteq Y'$.

Moreover, we assume the law of aggregate demand:

Law of Aggregate Demand (LAD): For $p, p' \in \mathbb{R}^X_+$ with $p \leq p'$, if $p'_x = p_x$ for $x \in X_f$, then for each $Y' \in \tilde{D}^f(p', X_f)$ there exists a $Y \in \tilde{D}^f(p, X_f)$, and if $p'_x = p_x$ for $x \in X \setminus X_f$, then for each $Y \in \tilde{D}^f(p, X_f)$ there exists a $Y' \in \tilde{D}^f(p', X_f)$, such that $|Y| \geq |Y'|$.

Remark 7. We assume that there is only one copy of each object. More generally, we can extend the model to multiple units of the same object by creating identical copies of objects. In this case we can use the strong substitutes condition (Baldwin and Klemperer, 2019) that requires that objects are gross substitutes for agents if each of the identical copies of an object is treated as a separate object. The law

[13]This assumption is only necessary for the existence of side-optimal allocations and otherwise redundant.

[14]As in Section 3.1 and in contrast to Section 4.1 we need a multi-valued version of gross substitutability to obtain corresponding results for exchange economies. This is because gross substitutability between a good that an agent owns and one that he does not own corresponds to cross-side complementarity in a trading network.

of aggregate demand can be generalized in an analogous way. One can show that under the assumption of strong substitutes and the generalized law of aggregate demand, an equilibrium with uniform prices (identical copies of the same good have the same price) exists whenever an equilibrium with non-uniform prices (identical copies of the same good can have different prices) exists. All subsequent results generalize to this setting. □

An **allocation of objects** is a partition $Y = (Y_f)_{f\in F}$ with $Y_f \subseteq X$ and $Y_f \cap Y_{f'} = \emptyset$ for $f \neq f'$. A **competitive equilibrium** of the exchange economy $(\tilde{u}, (X_f)_{f\in F})$ is a pair $[Y, p]$ where Y is an allocation of objects and $p \in \mathbb{R}^X_+$ such that for each $f \in F$ we have $Y_f \in \tilde{D}^f(p, X_f)$.

For each exchange economy $(\tilde{u}, (X_f)_{f\in F})$, a corresponding trading network can be defined as follows: The set of trades is

$$\Omega := \{(x, f_1, f_2) \in X \times F \times F : x \in X_{f_1}, f_2 \neq f_1\}$$

where for $\omega = (x, f_1, f_2) \in \Omega$ we have $s(\omega) = f_1 \neq f_2 = b(\omega)$. We write $x(\omega)$ for the object involved in trade ω. For $\Psi \subseteq \Omega_f$ and $p \in \mathbb{R}^{\Omega_f}$ define

$$X_f(\Psi) := \{x(\omega) : \omega \in \Psi_{\to f}\} \cup X_f \setminus \{x(\omega) : \omega \in \Psi_{f\to}\}, \quad p_f(\Psi) := \sum_{\omega\in\Psi_{f\to}} p_\omega - \sum_{\omega\in\Psi_{\to f}} p_\omega.$$

Utility functions are induced by utility functions over bundles of objects and transfers; for $\Psi \subseteq \Omega_f$ and $p \in \mathbb{R}^{\Omega_f}_+$ we let

$$u^f(\Psi, p) = \begin{cases} \tilde{u}^f(X_f(\Psi), p_f(\Psi)), & \text{if } \{x(\omega) : \omega \in \Psi_{f\to}\} \subseteq X_f \text{ and } x(\omega) \neq x(\omega') \\ & \text{for } \omega, \omega' \in \Psi \text{ with } \omega \neq \omega', \\ -\infty, & \text{else.} \end{cases}$$

To apply the results from the previous sections, we also extend the utility functions to negative prices; for $\Psi \subseteq \Omega_f$ and $p \in \mathbb{R}^\Psi \setminus \mathbb{R}^\Psi_+$ we let

$$u^f(\Psi, p) := u^f(\Psi, (\max\{p_\omega, 0\})_{\omega\in\Psi}) + \sum_{\omega\in\Psi_{f\to}} \min\{p_\omega, 0\} - \sum_{\omega\in\Psi_{\to f}} \min\{p_\omega, 0\}.$$

Remark 8. Extending utility for negative prices in this way implies (see the proof of Lemma 5) that the induced demand D^f satisfies FS on $\mathbb{R}^{\Omega_f}$ whenever it satisfies FS on $\mathbb{R}^{\Omega_f}_+$. Moreover, is easy to see that for each $\Psi \subseteq \Omega_f$ with $u^f(\Psi, \cdot) > -\infty$, u^f is continuous (as u^f is continuous on $\mathbb{R}^\Psi_+$, and min and max are continuous) and monotonic on $\mathbb{R}^\Psi$. This will allow us to apply the results from previous sections.

Equilibrium prices in the trading network are non-negative by our assumption that $\tilde{u}^f(Y,t) \leq \tilde{u}^f(Y',t)$ for $t \in \mathbb{R}$ and $Y \subseteq Y' \subseteq X$: Let $p \in \mathbb{R}^\Omega$ and define $\Omega^- := \{\omega \in \Omega : p_\omega < 0\}$. First note that for $\Psi \in D^f(p)$ we have $\Psi \cap \Omega^-_{f\to} = \emptyset$: Define $p^+ \in \mathbb{R}^\Omega$ by $p^+_\omega := \max\{p_\omega, 0\}$ for $\omega \in \Omega_{f\to}$ and $p^+_\omega := p_\omega$ else. Note that $X^f(\Psi) \subseteq X^f(\Psi \setminus \Omega^-_{f\to})$. Thus, by monotonicity

$$u^f(\Psi,p) \leq u^f(\Psi,p^+) \leq u^f(\Psi \setminus \Omega^-_{f\to}, p^+) = u^f(\Psi \setminus \Omega^-_{f\to}, p).$$

As $\Psi \in D^f(p)$, all inequalities hold with equality, in particular, $u^f(\Psi,p) = u^f(\Psi,p^+)$ and therefore, by monotonicity, $\Psi \cap \Omega^-_{f\to} = \emptyset$. By a similar argument, if $\Psi \in D^f(p)$ and $\Omega^-_{\to f} \neq \emptyset$, then $\Psi \cap \Omega^-_{\to f} \neq \emptyset$. Thus, for $p \in \mathbb{R}^\Omega \setminus \mathbb{R}^\Omega_+$ there is excess demand and for each $p \in \mathcal{E}(u)$, we have $p_\omega \geq 0$ for each $\omega \in \Omega$. □

The gross substitutes condition for $\tilde{u}^f$ corresponds to the full substitutability condition for u^f and the law of aggregate demand for $\tilde{u}^f$ implies the laws of aggregate demand and supply for u^f.

Lemma 5. *If $\tilde{u}^f$ satisfies GS, then u^f satisfies FS. If $\tilde{u}^f$ satisfies LAD, then u^f satisfies LAD and LAS.*

In general, different trades involving the same object can be priced differently. In the following, we call $p \in \mathcal{E}(u)$ a **competitive equilibrium** of the trading network **with uniform pricing**, if for $\omega, \omega' \in \Omega$, with $x(\omega) = x(\omega')$ we have $p_{\omega'} = p_\omega$. Trades in the same object are perfect substitutes to each other for the seller of the object, and he will sell the object to a buyer who is offering the highest price. Thus, we can always construct an equilibrium with uniform pricing from an equilibrium with non-uniform pricing by setting the price of the non-realized trades to the highest price for the involved object over all trades in the trading network. Similarly, a competitive equilibrium in the exchange economy, induces a competitive equilibrium with uniform pricing in the trading network. The following theorem can be interpreted as a generalization of Theorem 10 of Hatfield et al. (2013).

Proposition 2. *(i) If $p \in \mathbb{R}^\Omega_+$ are equilibrium prices in the trading network induced by an exchange economy, then $(\max_{\omega\in\Omega, x=x(\omega)} p_\omega)_{x\in X} \in \mathbb{R}^X_+$ are equilibrium prices in the exchange economy.*

(ii) If $p \in \mathbb{R}^X_+$ are equilibrium prices in an exchange economy, then $(p_{x(\omega)})_{\omega\in\Omega} \in \mathbb{R}^\Omega_+$ are equilibrium prices in the trading network induced by the exchange economy.

Proof. Let $[\Psi, p]$ be an equilibrium in the induced trading network. Let $q := (\max_{\omega\in\Omega, x=x(\omega)} p_\omega)_{x\in X}$ and consider the allocation $[(X_f(\Psi))_{f\in F}, q]$ in the exchange economy. By construction, we have $p_\omega \leq q_{x(\omega)}$ for each $\omega \notin \Psi$ and $p_\omega = q_{x(\omega)}$ for $\omega \in \Psi$. Thus

$$\Psi_f \in D^f(p) \Rightarrow X_f(\Psi) \in \tilde{D}^f(q, X_f)$$

and $[(X_f(\Psi))_{f\in F}, q]$ is an equilibrium of the exchange economy.

For the second part, let $[Y, p]$ be an equilibrium of the exchange economy. Define $q := (p_{x(\omega)})_{\omega\in\Omega}$ and consider the set of trades $\Psi \subseteq \Omega$ defined by

$$\Psi := \{\omega \in \Omega : x(\omega) \in Y_{b(\omega)} \cap X_{s(\omega)}\}.$$

By construction, we have

$$Y_f \in \tilde{D}^f(p, X_f) \Rightarrow \Psi_f \in D^f(q).$$

Therefore $[\Psi, q]$ is an equilibrium of the induced trading network. □

Proposition 2 and the previous results for trading networks imply the following:

Corollary 3. *Let $(\tilde{u}, (X_f)_{f\in F})$ be an exchange economy such that objects are gross substitutes for agents and the law of aggregate demand holds.*

(i) **Lattice Theorem***: Let $p, p' \in \mathbb{R}^X_+$ be equilibrium prices. Then the price vectors $\bar{p}, \underline{p} \in \mathbb{R}^X_+$ defined by*

$$\bar{p}_x := \max\{p_x, p'_x\}, \quad \underline{p}_x := \min\{p_x, p'_x\},$$

are equilibrium equilibrium prices.

(ii) **Rural Hospitals Theorem***: Let p, p' be equilibrium prices. For each equilibrium $[Y, p]$ there exists an assignment Y' such that for each $f \in F$ $|Y_f| = |Y'_f|$, i.e. f consumes the same number of objects in Y and Y'.*

(iii) **Existence of Extremal equilibria***: There exist equilibrium price vectors $\bar{p}, \underline{p} \in \mathbb{R}^X_+$, such that for each equilibrium price vector $p \in \mathbb{R}^X_+$ and $x \in X$ we have*

$$\underline{p}_x \leq p_x \leq \bar{p}_x.$$

Remark 9. Throughout this section, we have made the assumption that utility depends on the total amount of the divisible good, but not on how transfers of the divisible good are obtained through different trades. For the induced trading network this means that utility satisfies the no frictions assumption. Frictions for individual trades in the trading network can lead to non-uniform pricing. Suppose for example that an agent is endowed with an object and faces different transactions costs depending on whom he is selling the object to. In this case, he might have an incentive to sell the object to a buyer who is offering a lower price, if transaction costs with this buyer are lower than with other buyers who offer a higher price. Thus, Proposition 2 can fail to hold in the presence of frictions. A slightly more general version of the theorem can be obtained, where it is assumed that utility is symmetric in transfers from different trades with the same objects, but transfers from trades with different objects can enter the utility asymmetrically. In this case, trades in different objects can contain different frictions, however, trades of the same objects are perfect substitutes for each other. □

A Proofs for Section 2.1

A.1 Proof of Lemma 2

Proof. First we show the first and second part of the lemma. Let $p, p' \in \mathbb{R}^{\Omega_f}$ such that $p_\omega = p'_\omega$ for $\omega \in \Omega_{f\to}$ and $p_\omega \leq p'_\omega$ for $\omega \in \Omega_{\to f}$. Let $\Psi' \in D^f(p')$. By upper hemi-continuity there exists an $\epsilon > 0$ such that for $\|p - q\| < \epsilon$ we have $D^f(q) \subseteq D^f(p)$. By NIB, there is a q' with $\|q' - p'\| < \epsilon/2$ and $D^f(q') = \{\Psi'\}$. Let $q := p + q' - p'$. By construction $\|q - p\| = \|q' - p'\| < \epsilon/2 < \epsilon$ and therefore $D^f(q) \subseteq D^f(p)$. By upper hemi-continuity there exists an $\epsilon' > 0$ such that for r' with $\|r' - q'\| < \epsilon'$ we have $D^f(r') = \{\Psi'\} = D^f(q')$. We may choose $\epsilon' < \epsilon/2$. By the second part of Lemma 1, there exists a $\tilde{p} \in \mathbb{R}^{\Omega_f}$ with $\|\tilde{p} - q\| < \epsilon'$ such that demand is single-valued, $D^f(\tilde{p}) = \{\Psi\}$ for a $\Psi \subseteq \Omega_f$. As, $\|\tilde{p} - p\| \leq \|\tilde{p} - q\| + \|p - q\| < \epsilon' + \epsilon/2 < \epsilon$, we have $\Psi \in D^f(p)$. Let $\tilde{p}' := q' + \tilde{p} - q$. As $\|\tilde{p}' - q'\| = \|\tilde{p} - q\| < \epsilon'$, we have $D^f(\tilde{p}') = \{\Psi'\}$. By construction, we have $\tilde{p}' = q' + \tilde{p} - q = q' + \tilde{p} - (p + q' - p') = p' + \tilde{p} - p$ and $\tilde{p} = p + \tilde{p} - p$. Since $p_\omega = p'_\omega$ for $\omega \in \Omega_{f\to}$ and $p_\omega \leq p'_\omega$ for $\omega \in \Omega_{\to f}$, this implies $\tilde{p}_\omega = \tilde{p}'_\omega$ for $\omega \in \Omega_{f\to}$ and $\tilde{p}_\omega \leq \tilde{p}'_\omega$ for $\omega \in \Omega_{\to f}$. By weak FS applied to the vectors $\tilde{p}$ and $\tilde{p}'$ and the fact that demand at both price vectors is single-valued with $D^f(\tilde{p}) = \{\Psi\}$ and $D^f(\tilde{p}') = \{\Psi'\}$ we obtain

$$\{\omega \in \Psi_{\to f} : p_\omega = p'_\omega\} \subseteq \Psi'_{\to f}, \quad \Psi'_{f\to} \subseteq \Psi_{f\to}.$$

If, moreover, weak LAD holds, then

$$|\Psi_{\to f}| - |\Psi_{f\to}| \geq |\Psi'_{\to f}| - |\Psi'_{f\to}|.$$

A completely analogous argument shows the second part of the FS, resp. the LAS.

To show the third part of the lemma, let $p \in \mathbb{R}^{\Omega_f}$ and $\Psi \in D^f(p)$. We show that for each $\epsilon > 0$ there is a $q \in \mathbb{R}^{\Omega_f}$ with $\|p - q\| < \epsilon$ such that $D^f(q) = \{\Psi\}$. Let $\epsilon > 0$. First, consider a vector $\tilde{\epsilon} \in \mathbb{R}^{\Omega_f}$ with $\|\tilde{\epsilon}\| < \epsilon$ such that $\tilde{\epsilon}_\omega > 0$ for $\omega \in \Omega_{\to f} \setminus \Psi$, $\tilde{\epsilon}_\omega < 0$ for $\omega \in \Omega_{f\to} \setminus \Psi$, and $\tilde{\epsilon}_\omega = 0$ for $\omega \in \Psi$. By monotonicity of u^f, for each $\Xi \subseteq \Omega_f$ with $\Xi \nsubseteq \Psi$ we have $u^f(\Xi, p + \tilde{\epsilon}) < u^f(\Xi, p)$, and we have $u^f(\Psi, p + \tilde{\epsilon}) = u^f(\Psi, p)$. Thus, $D^f(p + \tilde{\epsilon}) \subseteq 2^\Psi$ and $\Psi \in D^f(p + \tilde{\epsilon})$. By upper hemi-continuity, there is a $\epsilon' > 0$ such that for $q \in \mathbb{R}^{\Omega_f}$ with $\|q - (p + \tilde{\epsilon})\| < \epsilon'$ we have $D^f(q) \subseteq D^f(p + \tilde{\epsilon})$. We may choose $\epsilon' < \epsilon - \|\tilde{\epsilon}\|$. By the second part of Lemma 1, there is a $q \in \mathbb{R}^{\Omega_f}$ with $\|q - (p + \epsilon)\| < \epsilon'$ such that demand is single-valued, $|D^f(q)| = 1$, and we may choose it such that $q_\omega \leq p_\omega + \tilde{\epsilon}_\omega$ for $\omega \in \Omega_{\to f}$ and $q_\omega \geq p_\omega + \tilde{\epsilon}_\omega$ for $\omega \in \Omega_{f\to}$. We show that for the unique $\Xi \subseteq \Omega_f$ with $D^f(q) = \{\Xi\}$ we have $\Xi_{f\to} = \Psi_{f\to}$. An analogous argument shows that $\Xi_{\to f} = \Psi_{\to f}$.

Let $r \in \mathbb{R}^{\Omega_f}$ with $r_\omega = q_\omega$ for $\omega \in \Omega_{\to f}$ and $r_\omega = p_\omega + \tilde{\epsilon}_\omega$ for $\omega \in \Omega_{f\to}$. By the first part of the CSC condition applied to vectors $p + \tilde{\epsilon}$ (in the role of p') and r (in the role of p), there is a $\Phi \in D^f(r)$ such that $\Psi_{f\to} \subseteq \Phi_{f\to}$. Since $\|r - (p + \tilde{\epsilon})\| \leq \|q - (p + \tilde{\epsilon})\| < \epsilon'$, we have $D^f(r) \subseteq D^f(p + \tilde{\epsilon}) \subseteq 2^\Psi$. Thus, $\Phi \subseteq \Psi$ and, by the previous observation that $\Psi_{f\to} \subseteq \Phi_{f\to}$, we have $\Phi_{f\to} = \Psi_{f\to}$. By LAS applied to prices r (in the role of p') and q (in the role of p) we have

$$|\Xi_{f\to}| - |\Xi_{\to f}| \geq |\Phi_{f\to}| - |\Phi_{\to f}|. \tag{4}$$

Since $\|q - (p + \tilde{\epsilon})\| < \epsilon'$, we have $D^f(q) \subseteq D^f(p + \tilde{\epsilon}) \subseteq 2^\Psi$. Thus, $\Xi \subseteq \Psi$ and, by the previous observation that $\Psi_{f\to} = \Phi_{f\to}$, we have $|\Xi_{f\to}| \leq |\Psi_{f\to}| = |\Phi_{f\to}|$. Together with Inequality (4) this implies $|\Xi_{\to f}| \leq |\Phi_{\to f}|$. By the second part of the CSC condition applied to prices r (in the role of p') and q (in the role of p) we have $\Phi_{\to f} \subseteq \Xi_{\to f}$. Together with the previous inequality this implies $\Phi_{\to f} = \Xi_{\to f}$. Furthermore, together with Inequality (4), this implies $|\Xi_{f\to}| \geq |\Phi_{f\to}|$, and, as $\Phi_{f\to} = \Psi_{f\to}$, we have $|\Xi_{f\to}| \geq |\Psi_{f\to}|$. As observed previously, $\Xi \subseteq \Psi$. Together with the previous inequality this implies $\Xi_{f\to} = \Psi_{f\to}$. □

A.2 Proof of Proposition 1

The proof uses the following lemma which will also be useful subsequently.

Lemma A.1. *Let u^f be a continuous and monotonic utility function inducing a demand correspondence D^f. For each $p, p' \in \mathbb{R}^{\Omega_f}$ and $\Psi' \subseteq \Omega_f$ with $p_\omega \leq p'_\omega$ for $\omega \in \Omega_{f\to} \setminus \Psi'_{f\to}$, $p_\omega \geq p'_\omega$ for $\omega \in \Psi'_{f\to}$, $p_\omega \geq p'_\omega$ for $\omega \in \Omega_{\to f} \setminus \Psi'_{\to f}$ and $p_\omega \leq p'_\omega$ for $\omega \in \Psi'_{\to f}$:*

(i) If D^f satisfies weak FS, weak LAD and weak LAS, then $D^f(p') = \{\Psi'\}$ implies $\Psi' \in D^f(p)$.

(ii) If D^f satisfies FS, LAD and LAS, then $\Psi' \in D^f(p')$ implies $\Psi' \in D^f(p)$.

Proof. We first prove the first part. Let D^f satisfy weak FS, weak LAD and weak LAS and $D^f(p') = \{\Psi'\}$. By monotonicity of u^f it is without loss of generality to assume that $p'_\omega = p_\omega$ for $\omega \in \Omega_f \setminus \Psi'$ (replacing p'_ω with p_ω for $\omega \in \Omega_f \setminus \Psi'$ does not change the utility for trades Ψ' while it weakly decreases the utility for any other set of trades). By upper hemi-continuity of D^f, it suffices to show that for each $\epsilon > 0$ there is a $q \in \mathbb{R}^{\Omega_f}$ with $\|p - q\| < \epsilon$ such that $\Psi' \in D^f(q)$. Let $\epsilon > 0$. By upper hemi-continuity of D^f there is a $\epsilon' > 0$ such that for $q' \in \mathbb{R}^{\Omega_f}$ with $\|p' - q'\| < \epsilon'$ we have $D^f(q') = \{\Psi'\} = D^f(p')$. Define $\tilde{p} \in \mathbb{R}^{\Omega_f}$ such that

$$\tilde{p}_\omega := \begin{cases} p'_\omega, & \text{if } \omega \in \Omega_{f\to} \\ p_\omega, & \text{if } \omega \in \Omega_{\to f}. \end{cases}$$

By the second part of Lemma 1, there is a $r \in \mathbb{R}^{\Omega_f}$ with $\|r - \tilde{p}\| < \min\{\frac{\epsilon}{2}, \frac{\epsilon'}{2}\}$ and a $\tilde{\Psi} \subseteq \Omega_f$ such that $D^f(r) = \{\tilde{\Psi}\}$. By upper hemi-continuity of D^f there is a $\tilde{\epsilon} > 0$ such that for $\|\tilde{q} - r\| < \tilde{\epsilon}$ we have $D^f(\tilde{q}) = \{\tilde{\Psi}\} = D^f(r)$. We may choose $\tilde{\epsilon} < \min\{\frac{\epsilon}{2}, \frac{\epsilon'}{2}\}$. By the second part of Lemma 1, there is a $q \in \mathbb{R}^{\Omega_f}$ with $\|(p + (r - \tilde{p})) - q\| < \tilde{\epsilon}$ and a $\Psi \subseteq \Omega_f$ such that $D^f(q) = \{\Psi\}$. Let $\tilde{q} := \tilde{p} + q - p$ and $q' := p' + q - p$. By construction we have $\|q' - p'\| = \|q - p\| \leq \|(p + (r - \tilde{p})) - q\| + \|r - \tilde{p}\| < \tilde{\epsilon} + \min\{\frac{\epsilon}{2}, \frac{\epsilon'}{2}\} < \min\{\epsilon, \epsilon'\}$. Thus $D^f(q') = \{\Psi'\}$. By construction, we have $\|r - \tilde{q}\| = \|r - (\tilde{p} + q - p)\| < \tilde{\epsilon}$ and therefore $D^f(\tilde{q}) = \{\tilde{\Psi}\} = D^f(r)$. Applying the weak CSC condition to vectors $\tilde{q}$ and q' (note that by construction we have $q'_\omega = \tilde{q}_\omega$ for $\omega \in \Omega_{f\to}$) we have $\Psi'_{f\to} \subseteq \tilde{\Psi}_{f\to}$. Applying the weak SSS condition to vectors $\tilde{q}$ and q' (recall that we have WLOG assumed that $p'_\omega = p_\omega$ for $\omega \in \Omega_f \setminus \Psi'$ and therefore have $p'_\omega = \tilde{p}_\omega = p_\omega$ and $q'_\omega = \tilde{q}_\omega = q_\omega$ for $\omega \in \Omega_f \setminus \Psi'$), we have $\tilde{\Psi}_{\to f} \subseteq \Psi'_{\to f}$. Applying the weak LAD to vectors $\tilde{q}$ and q' it follows that $\tilde{\Psi} = \Psi'$.

Applying the weak CSC, condition to vectors q and $\tilde{q}$ (note that by construction we have $q_\omega = \tilde{q}_\omega$ for $\omega \in \Omega_{\to f}$) we have $\tilde{\Psi}_{\to f} \subseteq \Psi_{\to f}$. Applying the weak SSS condition to vectors q and $\tilde{q}$ (recall that we have $q'_\omega = \tilde{q}_\omega = q_\omega$ for $\omega \in \Omega_f \setminus \Psi' =$

$\Omega_f \setminus \tilde{\Psi}$) we, moreover, have $\Psi_{f\to} \subseteq \tilde{\Psi}_{f\to}$. Applying the weak LAS to vectors q and $\tilde{q}$ it follows that $\tilde{\Psi} = \Psi$.

The second part of the lemma follows from the first as follows: By Lemma 2, there is for each $\epsilon > 0$ a $q' \in \mathbb{R}^{\Omega_f}$ with $\|p' - q'\| < \epsilon$ such that $D^f(q') = \{\Psi'\}$. By the first part of the lemma applied to vectors q' and $q := p + q' - p'$ we have $\Psi' \in D^f(q)$. Thus for each $\epsilon > 0$ there is a $q \in \mathbb{R}^{\Omega_f}$ with $\|p - q\| < \epsilon$ and $\Psi' \in D^f(q)$. Thus, by upper hemi-continuity, $\Psi' \in D^f(p)$.

□

Proof. We first define the demand $\tilde{D}^f$ and show that it is a selection from D^f. Then we rationalize it by a continuous and monotonic utility function that induces the same indirect utility. Afterwards we show that it satisfies FS, LAD and LAS.

For each $\Psi \subseteq \Omega_f$, consider the (possibly empty) set of price vectors p such that Ψ is the unique demanded bundle at p:

$$P_\Psi := \{p \in \mathbb{R}^{\Omega_f} : D^f(p) = \{\Psi\}\}.$$

Let $\bar{P}_\Psi$ be the (topological) closure of P_Ψ. We let

$$\tilde{D}^f(p) := \{\Psi \subseteq \Omega_f : p \in \bar{P}_\Psi\}.$$

By upper hemi-continuity of D^f, for each $\Psi \subseteq \Omega_f$ and $p \in \bar{P}_\Psi$ we have $\Psi \in D^f(p)$. Thus $\tilde{D}^f(p) \subseteq D^f(p)$ for each $p \in \mathbb{R}^{\Omega_f}$.

Claim 1. *For each $\Psi \subseteq \Omega_f$, let $P^{pr}_\Psi := \{(p_\omega)_{\omega\in\Psi} : p \in P_\Psi\}$ be the projection of P_Ψ to $\mathbb{R}^\Psi$ and $\overline{P^{pr}_\Psi}$ its (topological) closure. Then there is a continuous and monotonic utility function $\tilde{u}^f(\Psi, \cdot)$ such that*

$$\tilde{u}^f(\Psi, p) = u^f(\Psi, p), \qquad \text{if } p \in \overline{P^{pr}_\Psi}, \tag{5}$$

$$\tilde{u}^f(\Psi, p) < u^f(\Psi, p), \qquad \text{if } p \notin \overline{P^{pr}_\Psi}. \tag{6}$$

Proof of Claim 1. Denote for each $p \in \mathbb{R}^\Psi$ by $d(p, P^{pr}_\Psi) := \inf_{q\in P^{pr}_\Psi} \|p - q\|$ the distance from p to P^{pr}_Ψ. We define

$$\tilde{u}^f(\Psi, p) = \begin{cases} u^f(\Psi, p) - d(p, P^{pr}_\Psi), & \text{if } P_\Psi \neq \emptyset, \\ -\infty, & \text{if } P_\Psi = \emptyset. \end{cases}$$

Suppose $P_\Psi \neq \emptyset$. The function $\tilde{u}^f(\Psi, \cdot)$ is continuous since $u^f(\Psi, \cdot)$ is continuous and the distance to a set in $\mathbb{R}^\Psi$ is continuous. Moreover, $d(p, P^{pr}_\Psi) \geq 0$ with equality if and only if $p \in \overline{P_\Psi^{pr}}$. Thus (5) and (6) hold. It remain to show that $\tilde{u}^f(\Psi, \cdot)$ is monotonic. Let $p, p' \in \mathbb{R}^\Psi$ with $p \neq p'$ such that $p'_\omega = p_\omega$ for $\omega \in \Psi_{f\to}$ and $p_\omega \leq p'_\omega$ for $\omega \in \Psi_{\to f}$ (an analogous argument works for downstream trades). For each $\epsilon > 0$ there is $q' \in P_\Psi^{pr}$ such that $|\|p' - q'\| - d(p', P_\Psi^{pr})| < \epsilon$. Let $r' \in P_\psi$ such that $r'|_\Psi = q'$. Let $q := p - (p' - q')$ and define $r \in \mathbb{R}^{\Omega_f}$ by $r_\omega = q_\omega$ for $\omega \in \Psi$ and $r_\omega = r'_\omega$ for $\omega \notin \Psi$. Since $r' \in P_\Psi$ we have $D^f(r') = \{\Psi\}$ and thus, by the first part of Lemma A.1, we have $\Psi \in D^f(r)$. More generally, by upper hemi-continuity of D^f there is a $\epsilon' > 0$ such that for each $s' \in \mathbb{R}^{\Omega_f}$ with $\|s' - r'\| < \epsilon'$ we have $D^f(s') = \{\Psi\}$. Thus, by the first part of Lemma A.1, for each $s \in \mathbb{R}^{\Omega_f}$ with $\|s - r\| < \epsilon'$ we have $\Psi \in D^f(s)$. By the second part of Lemma 1, this implies that for each $\tilde{\epsilon} > 0$ there is a $s \in P_\Psi$ with $\|s - r\| < \tilde{\epsilon}$. Therefore $q = r|_\Psi \in (\bar{P}_\Psi)^{pr} \subseteq \overline{P_\Psi^{pr}}$. Thus, $d(p, P_\Psi^{pr}) = d(p, \overline{P_\Psi^{pr}}) \leq \|p - q\| = \|p' - q'\| < d(p', P_\Psi^{pr}) + \epsilon$. Since this holds for any $\epsilon > 0$ we have $d(p, P_\Psi^{pr}) \leq d(p', P_\Psi^{pr})$. Thus, $\tilde{u}^f(\Psi, p) > \tilde{u}^f(\Psi, p')$. □

Claim 2 implies that $\tilde{D}^f$ can be rationalized by a continuous and monotonic utility function that induces the same indirect utility: By the second part of Lemma 1, for each $p \in \mathbb{R}^{\Omega_f}$ there is a $\Psi \in D^f(p)$ with $p \in \bar{P}_\Psi$. Since $p \in \bar{P}_\Psi$ we have $p|_\Psi \in (\bar{P}_\Psi)^{pr} \subseteq \overline{P_\Psi^{pr}}$ and therefore $\tilde{v}^f(p) = \tilde{u}^f(\Psi, p) = u^f(\Psi, p) = v^f(p)$. Next we show that $\tilde{u}^f$ rationalizes $\tilde{D}^f$ by showing that $\tilde{u}^f(\Psi, p) = \tilde{v}^f(p)$ for $\Psi \in \tilde{D}^f(p)$ and $\tilde{u}^f(\Psi, p) < \tilde{v}^f(p)$ for $\Psi \notin \tilde{D}^f(p)$. Let $\Psi \subseteq \Omega_f$. If $\Psi \in \tilde{D}^f(p)$, then $p \in \bar{P}_\Psi$ and $\Psi \in D^f(p)$. Since $p \in \bar{P}_\Psi$ we have $p|_\Psi \in (\bar{P}_\Psi)^{pr} \subseteq \overline{P_\Psi^{pr}}$ and thus $\tilde{u}^f(\Psi, p) = u^f(\Psi, p) = v^f(p) = \tilde{v}^f(p)$. If $\Psi \notin \tilde{D}^f(p)$ and $\Psi \notin D^f(p)$, then $\tilde{u}^f(\Psi, p) \leq u^f(\Psi, p) < v^f(p) = \tilde{v}^f(p)$. If $\Psi \in D^f(p) \setminus \tilde{D}^f(p)$, then we show $p|_\Psi \notin \overline{P_\Psi^{pr}}$ and thus $\tilde{u}^f(\Psi, p) < u^f(\Psi, p) = v^f(p) = \tilde{v}^f(p)$: By definition of $\tilde{D}^f$ we have $p \notin \bar{P}_\Psi$. Thus, there is a $\epsilon > 0$ such that for each $q \in \mathbb{R}^{\Omega_f}$ with $\|p - q\| < \epsilon$ we have $q \notin \bar{P}_\Psi$. Let $q \in \mathbb{R}^{\Omega_f}$ such that $\|p - q\| < \epsilon$ and $p_\omega = q_\omega$ for $\omega \in \Psi$, $p_\omega > q_\omega$ for $\omega \in \Omega_{f\to} \setminus \Psi$ and $p_\omega < q_\omega$ for $\omega \in \Omega_{\to f} \setminus \Psi$. Since $\Psi \in D^f(p)$ and u^f is monotonic, we have $D^f(q) \subseteq 2^\Psi$. By Lemma 1, we can find a $\epsilon - \|p - q\| > \tilde{\epsilon} > 0$ such that for $\|r - q\| < \tilde{\epsilon}$ we have $D^f(r) \subseteq D^f(q) \subseteq 2^\Psi$. Now suppose for the sake of contradiction that $p|_\Psi = q|_\Psi \in \overline{P_\Psi^{pr}}$. Then, there is a $r \in P_\Psi$ such that $\|p|_\Psi - r|_\Psi\| < \tilde{\epsilon}$. Since $r \in P_\Psi$, we have $D^f(r) = \{\Psi\}$ and thus, in particular, $u^f(\Psi, r) > u^f(\tilde{\Psi}, r)$ for each $\tilde{\Psi} \subsetneq \Psi$. Now define $\tilde{r} \in \mathbb{R}^{\Omega_f}$ by $\tilde{r}_\omega = r_\omega$ for $\omega \in \Psi$ and $\tilde{r}_\omega = q_\omega$ for $\omega \notin \Psi$. By construction, we have $\|\tilde{r} - q\| = \|r|_\Psi - p|_\Psi\| < \tilde{\epsilon}$. Thus $D^f(\tilde{r}) \subseteq 2^\Psi$. Moreover, $u^f(\Psi, \tilde{r}) = u^f(\Psi, r) > u^f(\tilde{\Psi}, r) = u^f(\tilde{\Psi}, \tilde{r})$ for each $\tilde{\Psi} \subsetneq \Psi$. Thus $D^f(\tilde{r}) = \{\Psi\}$ and $\tilde{r} \in P_\Psi$. However, $\|p - \tilde{r}\| \leq \|p - q\| + \|q - \tilde{r}\| < \|p - q\| + \tilde{\epsilon} < \epsilon$ and therefore $\tilde{r} \notin \bar{P}_\Psi$, a contradiction.

Next we show that that $\tilde{D}^f$ satisfies FS, LAD and LAS. By Lemma 2 it suffices to show that $\tilde{D}^f$ satisfies NIB, weak FS, weak LAD and weak LAS. Let $\Psi \subseteq \Omega_f$. Since $\bar{P}_\Psi$ is the closure of P_Ψ there is for each $p \in \bar{P}_\Psi$ and $\epsilon > 0$ a $q \in P_\Psi$ with $\|p - q\| < \epsilon$. By definition of P_Ψ and $\tilde{D}_f$ we have $\tilde{D}^f(q) = D^f(q) = \{\Psi\}$. Thus $\tilde{D}^f$ satisfies NIB. For the other properties, recall that D^f satisfies weak FS, weak LAD and weak LAS. Thus, it suffices to show that for each $p \in \mathbb{R}^{\Omega_f}$ we have $|\tilde{D}^f(p)| = 1$ if and only if $|D^f(p)| = 1$. Let $\Psi \subseteq \Omega_f$ with $p \in \bar{P}_\Psi$. If $p \in P_\Psi$, then $\tilde{D}^f(p) = D^f(p) = \{\Psi\}$. If $p \in \bar{P}_\Psi \setminus P_\Psi$, then p is on the boundary of P_Ψ and for each $\epsilon > 0$ there is a $q \in \mathbb{R}^{\Omega_f} \setminus \bar{P}_\Psi$ with $\|p - q\| < \epsilon$. By the second part of Lemma 1, we may choose q such that $|D^f(q)| = 1$. Since Ω is finite, this implies that there is a $\tilde{\Psi} \neq \Psi$ such that for each $\epsilon > 0$ there is a $q \in \mathbb{R}^{\Omega_f} \setminus \bar{P}_\Psi$ with $\|p - q\| < \epsilon$ and $D^f(q) = \{\tilde{\Psi}\}$. Thus $p \in \bar{P}_{\tilde{\Psi}}$ for $\tilde{\Psi} \neq \Psi$. Hence $|\tilde{D}^f(p)| > 1$. Thus, $|\tilde{D}^f(p)| = 1$ if and only if $|D^f(p)| = 1$ as desired.

□

B Proofs for Section 3

Proof of Lemma 3

Proof. By Lemma 1, there exists an $\epsilon_0 > 0$ such that for each $p \in P$ and every q with $\|q - p\| < \epsilon_0$ we have $D^f(q) \subseteq D^f(p)$. Let $P = \{p^1, \ldots, p^n\}$. By Lemma 1, there is a $\epsilon^1 \in \mathbb{R}^{\Omega_f}$ with $\|\epsilon^1\| < \epsilon_0$ such that $|D^f(p^1 + \epsilon^1)| = 1$ and $\Psi \in D^f(p^1)$ for the unique $\Psi \in D^f(p^1 + \epsilon^1)$. Consider $P^1 := \{p^1 + \epsilon^1, \ldots, p^n + \epsilon^1\}$. For each $i = 1, \ldots, n$ we have $D^f(p^i + \epsilon^1) \subseteq D^f(p^i)$. By Lemma 1, there exists an $\epsilon_1 > 0$ such that for each $p \in P^1$ and every q with $\|q - p\| < \epsilon_1$ we have $D^f(q) \subseteq D^f(p)$. By Lemma 1, there is a $\epsilon^2 \in \mathbb{R}^{\Omega_f}$ with $\|\epsilon^2\| < \epsilon_1$ such that $|D^f(p^2 + \epsilon^1 + \epsilon^2)| = 1$ and $\Psi \in D^f(p^2 + \epsilon^2)$ for the unique $\Psi \in D^f(p^2 + \epsilon^1)$. Next consider $P^2 := \{p^1 + \epsilon^1 + \epsilon^2, \ldots, p^n + \epsilon^1 + \epsilon^2\}$. For each $i = 1, \ldots, n$ we have $D^f(p^i + \epsilon^1 + \epsilon^2) \subseteq D^f(p^i + \epsilon^1) \subseteq D^f(p^i)$ and so on. Iterating in this way, we obtain $\epsilon^1, \ldots, \epsilon^n$ such that for each $i = 1, \ldots, n$, we have $|D^f(p^i + \sum_{j=1}^n \epsilon^j)| = 1$ and $\Psi^i \in D^f(p^i)$ for the unique $\Psi^i \in D^f(p^i + \sum_{j=1}^n \epsilon^j) \subseteq D^f(p^i)$. We define $\tilde{D}^f(p^i) = \Psi^i$. By construction $\tilde{D}^f(p^i) \in D^f(p^i)$. Moreover, as all price vectors are translated by the same vector $\sum_{j=1}^n \epsilon^j$, FS, LAD and LAS follow from weak FS, weak LAD and weak LAS for D^f. □

Proof of Lemma 4

Proof. By Lemma 1, there exists an $\epsilon > 0$ such that for each $q \in \{p, p', \bar{p}, \underline{p}\}$ and every $\tilde{q}$ with $\|\tilde{q} - q\| < \epsilon$ we have $D^f(\tilde{q}) \subseteq D^f(q)$. Let $\epsilon_0 > 0$ such that $\epsilon_0 < \min_{\omega \in \Omega_f : p'_\omega \neq p_\omega} |p'_\omega - p_\omega|$ and $\epsilon_0 \sqrt{|\Omega_f|} < \epsilon$. Define $\epsilon' \in \mathbb{R}^{\Omega_f}$ by

$$\epsilon'_\omega = \begin{cases} \epsilon_0, & \text{if } \omega \in \Psi'_{f\to} \text{ and } p'_\omega \neq p_\omega, \\ -\epsilon_0, & \text{if } \omega \in \Omega_{f\to} \setminus \Psi' \text{ and } p'_\omega \neq p_\omega, \\ -\epsilon_0, & \text{if } \omega \in \Psi'_{\to f} \text{ and } p'_\omega \neq p_\omega, \\ \epsilon_0, & \text{if } \omega \in \Omega_{\to f} \setminus \Psi' \text{ and } p'_\omega \neq p_\omega, \\ 0, & \text{if } p'_\omega = p_\omega. \end{cases}$$

Note that by construction we have $\|\epsilon'\| = \sqrt{\epsilon_0^2 |\{\omega \in \Omega_f : p_\omega \neq p'_\omega\}|} \leq \epsilon_0 \sqrt{|\Omega_f|} < \epsilon$ and thus $D^f(p' + \epsilon') \subseteq D^f(p')$. First we prove the following claim.

Claim 2. *For each $\Xi \in D^f(p' + \epsilon')$ we have $\{\omega \in \Psi' : p'_\omega \neq p_\omega\} \subseteq \Xi$ and $\{\omega \notin \Psi' : p'_\omega \neq p_\omega\} \cap \Xi = \emptyset$.*

Proof. First we show that for each $\Xi \in D^f(p' + \epsilon')$ we have $\{\omega \in \Psi' : p'_\omega \neq p_\omega\} \subseteq \Xi$. Suppose not, and there is a $\Xi \in D^f(p' + \epsilon')$ and a $\tilde{\omega} \in \{\omega \in \Psi' : p'_\omega \neq p_\omega\} \setminus \Xi$. Let $\tilde{p} \in \mathbb{R}^{\Omega_f}$ with $\tilde{p}_{\tilde{\omega}} = p'_{\tilde{\omega}}$ and $\tilde{p}_\omega = p'_\omega + \epsilon'_\omega$ for $\omega \neq \tilde{\omega}$. By the second part of Lemma A.1, we have $\Psi' \in D^f(\tilde{p})$. Thus, by monotonicity, we have $u^f(\Xi, p' + \epsilon') = u^f(\Xi, \tilde{p}) \leq u^f(\Psi', \tilde{p}) < u^f(\Psi', p' + \epsilon')$ contradicting the assumption that $\Xi \in D^f(p' + \epsilon')$.

Next we show that for each $\Xi \in D^f(p' + \epsilon')$ we have $\{\omega \notin \Psi' : p'_\omega \neq p_\omega\} \cap \Xi = \emptyset$. Suppose not, and there is a $\Xi \in D^f(p' + \epsilon')$ and a $\tilde{\omega} \in \{\omega \notin \Psi' : p'_\omega \neq p_\omega\} \cap \Xi$. Let $\tilde{p} \in \mathbb{R}^{\Omega_f}$ with $\tilde{p}_{\tilde{\omega}} = p'_{\tilde{\omega}}$ and $\tilde{p}_\omega = p'_\omega + \epsilon'_\omega$ for $\omega \neq \tilde{\omega}$. By the second part of Lemma A.1, we have $\Psi' \in D^f(\tilde{p})$. Thus, by monotonicity, we have $u^f(\Xi, p' + \epsilon') < u^f(\Xi, \tilde{p}) \leq u^f(\Psi', \tilde{p}) = u^f(\Psi', p' + \epsilon')$ contradicting the assumption that $\Xi \in D^f(p' + \epsilon')$. □

By Lemma 1, there exists $\epsilon_1 > 0$ such that for every q' with $\|q' - (p' + \epsilon')\| < \epsilon_1$ we have $D^f(q') \subseteq D^f(p' + \epsilon')$. We may choose $\epsilon_1 < \epsilon - \|\epsilon'\|$. By the third part of Lemma 2, there is a $q \in \mathbb{R}^{\Omega_f}$ with $\|p - q\| < \epsilon_1$ such that $D^f(q) = \{\Psi\}$. Define $q' := p' + \epsilon' + (q - p)$. Define $\bar{q}$ as the pairwise maximum of q and q', i.e. $\bar{q}_\omega = \max\{q_\omega, q'_\omega\}$, and $\underline{q}$ as the pairwise minimum of q and q', i.e. $\underline{q}_\omega = \min\{q_\omega, q'_\omega\}$.

By construction, we have $q_\omega < q'_\omega$ if and only if $p_\omega < p'_\omega$, $q_\omega > q'_\omega$ if and only if $p_\omega > p'_\omega$, and $q_\omega = q'_\omega$ if and only if $p_\omega = p'_\omega$. Moreover, we have $D^f(q') \subseteq$

$D^f(p'+\epsilon') \subseteq D^f(p')$, we have $\|\bar{q}-\bar{p}\| \leq \|\epsilon'\| + \epsilon_1 < \epsilon$ and thus $D^f(\bar{q}) \subseteq D^f(\bar{p})$, and we have $\|\underline{q}-\underline{p}\| \leq \|\epsilon'\| + \epsilon_1 < \epsilon$ and thus $D^f(\underline{q}) \subseteq D^f(\underline{p})$. Let $P := \{\tilde{q} \in \mathbb{R}^{\Omega_f} : \tilde{q}_\omega \in \{q_\omega, q'_\omega\}$ for all $\omega \in \Omega_f\}$. By Lemma 3, there is a single-valued selection $\tilde{D}^f : P \to 2^{\Omega_f}$ from D^f satisfying FS, LAD and LAS. Let $\bar{\Psi} := \tilde{D}^f(\bar{q})$, $\underline{\Psi} := \tilde{D}^f(\underline{q})$ and $\Psi'' := \tilde{D}^f(q')$. As $D^f(q) = \{\Psi\}$, we have $\tilde{D}^f(q) = \Psi$. First we show that

$$\{\omega \in \Psi_{\to f} : p_\omega \geq p'_\omega\} \cup \{\omega \in \Psi'_{\to f} : p'_\omega > p_\omega\} \subseteq \bar{\Psi}_{\to f}.$$

For $\tilde{q} \in P$ such that $\tilde{q}_\omega = \bar{q}_\omega$ for $\omega \in \Omega_{\to f}$ and $\tilde{q}_\omega = q_\omega$ for $\omega \in \Omega_{f\to}$ let $\tilde{\Psi} := \tilde{D}^f(\tilde{q})$. By CSC for $\tilde{D}^f$, we have $\tilde{\Psi}_{\to f} \subseteq \bar{\Psi}_{\to f}$. By SSS for $\tilde{D}^f$, we have $\{\omega \in \Psi_{\to f} : q_\omega = \bar{q}_\omega \geq q'_\omega\} \subseteq \tilde{\Psi}_{\to f}$ and therefore $\{\omega \in \Psi_{\to f} : q_\omega = \bar{q}_\omega \geq q'_\omega\} \subseteq \bar{\Psi}_{\to f}$. Similarly, for $\tilde{q} \in P$ such that $\tilde{q}_\omega = \bar{q}_\omega$ for $\omega \in \Omega_{\to f}$ and $\tilde{q}_\omega = q'_\omega$ for $\omega \in \Omega_{f\to}$ let $\tilde{\Psi} := \tilde{D}^f(\tilde{q})$. By CSC for $\tilde{D}^f$, we have $\tilde{\Psi}_{\to f} \subseteq \bar{\Psi}_{\to f}$. By SSS for $\tilde{D}^f$, we have $\{\omega \in \Psi''_{\to f} : q'_\omega = \bar{q}_\omega \geq q_\omega\} \subseteq \tilde{\Psi}_{\to f}$ and therefore $\{\omega \in \Psi''_{\to f} : q'_\omega > q_\omega\} \subseteq \{\omega \in \Psi''_{\to f} : q'_\omega = \bar{q}_\omega \geq q_\omega\} \subseteq \bar{\Psi}_{\to f}$. Moreover, by Claim 2 and as $\Psi'' \in D^f(q') \in D^f(p'+\epsilon')$, we have $\{\omega \in \Psi' : p'_\omega \neq p_\omega\} \subseteq \Psi''$ and $\{\omega \notin \Psi' : p'_\omega \neq p_\omega\} \cap \Psi'' = \emptyset$. Therefore

$$\begin{aligned}&\{\omega \in \Psi_{\to f} : p_\omega \geq p'_\omega\} \cup \{\omega \in \Psi'_{\to f} : p'_\omega > p_\omega\} \\ \subseteq &\{\omega \in \Psi_{\to f} : q_\omega \geq q'_\omega\} \cup \{\omega \in \Psi''_{\to f} : q'_\omega > q_\omega\} \subseteq \bar{\Psi}_{\to f}.\end{aligned}$$

Next we show that

$$\bar{\Psi}_{f\to} \subseteq \{\omega \in \Psi_{f\to} : p_\omega \geq p'_\omega\} \cup \{\omega \in \Psi'_{f\to} : p'_\omega > p_\omega\}.$$

Let $\bar{\omega} \in \bar{\Psi}_{f\to}$. We consider two cases. Either $\bar{p}_{\bar{\omega}} = p_{\bar{\omega}}$ or $\bar{p}_{\bar{\omega}} = p'_{\bar{\omega}} > p_{\bar{\omega}}$. In the first case, consider $\tilde{q} \in P$ with $\tilde{q}_\omega = \bar{q}_\omega$ for $\omega \in \Omega_{\to f}$ and $\tilde{q}_\omega = q_\omega$ for $\omega \in \Omega_{f\to}$. Let $\tilde{\Psi} := \tilde{D}^f(\tilde{q})$. By SSS of $\tilde{D}^f$, we have $\bar{\omega} \in \tilde{\Psi}_{f\to}$. By CSC of $\tilde{D}^f$, we have $\tilde{\Psi}_{f\to} \subseteq \Psi_{f\to}$ and hence $\bar{\omega} \in \Psi_{f\to}$. Similarly, if $\bar{p}_{\bar{\omega}} = p'_{\bar{\omega}} > p_{\bar{\omega}}$, consider $\tilde{q} \in P$ with $\tilde{q}_\omega = \bar{q}_\omega$ for $\omega \in \Omega_{\to f}$ and $\tilde{q}_\omega = q'_\omega$ for $\omega \in \Omega_{f\to}$. Let $\tilde{\Psi} := \tilde{D}^f(\tilde{q})$. By SSS of $\tilde{D}^f$, we have $\bar{\omega} \in \tilde{\Psi}_{f\to}$. By CSC of $\tilde{D}^f$, we have $\tilde{\Psi}_{f\to} \subseteq \Psi''_{f\to}$ and hence $\bar{\omega} \in \Psi''_{f\to}$. Since $\{\omega \notin \Psi' : p'_\omega \neq p_\omega\} \cap \Psi'' = \emptyset$ and $p'_{\bar{\omega}} \neq p_{\bar{\omega}}$ this implies $\bar{\omega} \in \Psi'_{f\to}$.

Finally, let $\tilde{q} \in P$ such that $\tilde{q}_\omega = \bar{q}_\omega$ for $\omega \in \Omega_{\to f}$ and $\tilde{q}_\omega = q_\omega$ for $\omega \in \Omega_{f\to}$ and $\tilde{\Psi} := \tilde{D}^f(\tilde{q})$. By LAD for $\tilde{D}^f$ at q and $\tilde{q}$ and LAS for $\tilde{D}^f$ at $\tilde{q}$ and $\bar{q}$ we have

$$|\Psi_{\to f}| - |\Psi_{f\to}| \geq |\tilde{\Psi}_{\to f}| - |\tilde{\Psi}_{f\to}| \geq |\bar{\Psi}_{\to f}| - |\bar{\Psi}_{f\to}|.$$

A completely dual proof shows that $\underline{\Psi}$ has the desired properties. □

Proof of Theorem 1

Proof. Let $\Psi \in \mathcal{E}(u,p)$ and $\Psi' \in \mathcal{E}(u,p')$. First we show that for the pairwise minimum $\underline{p} \in \mathcal{E}(u)$. For each firm $f \in F$ there is by Lemma 4 applied to $\Psi_f \in D^f(p)$ and $\Psi'_f \in D^f(p')$ a $\underline{\Psi}_f \in D^f(\underline{p})$ with

$$\underline{\Psi}_{\to f} \subseteq \{\omega \in \Psi_{\to f} : p'_\omega \geq p_\omega\} \cup \{\omega \in \Psi'_{\to f} : p_\omega > p'_\omega\}, \tag{7}$$

$$\{\omega \in \Psi_{f\to} : p'_\omega \geq p_\omega\} \cup \{\omega \in \Psi'_{f\to} : p_\omega > p'_\omega\} \subseteq \underline{\Psi}_{f\to} \tag{8}$$

and

$$|\underline{\Psi}_{\to f}| - |\underline{\Psi}_{f\to}| \geq |\Psi_{\to f}| - |\Psi_{f\to}|. \tag{9}$$

Taking the union over all firms of (7) and (8) we have

$$\bigcup_{f\in F} \underline{\Psi}_{\to f} \subseteq \{\omega \in \Psi : p'_\omega \geq p_\omega\} \cup \{\omega \in \Psi' : p_\omega > p'_\omega\} \subseteq \bigcup_{f\in F} \underline{\Psi}_{f\to}, \tag{10}$$

and summing inequality (9) over all firms

$$\sum_{f\in F}(|\underline{\Psi}_{\to f}| - |\underline{\Psi}_{f\to}|) \geq \sum_{f\in F}(|\Psi_{\to f}| - |\Psi_{f\to}|) = 0. \tag{11}$$

This implies $\sum_{f\in F} |\underline{\Psi}_{\to f}| \geq \sum_{f\in F} |\underline{\Psi}_{f\to}|$ which together with (10) implies $\bigcup_{f\in F} \underline{\Psi}_{f\to} = \bigcup_{f\in F} \underline{\Psi}_{\to f} =: \underline{\Psi}$ and $[\underline{\Psi}, \underline{p}]$ is an equilibrium. Moreover, since $\bigcup_{f\in F} \underline{\Psi}_{f\to} = \bigcup_{f\in F} \underline{\Psi}_{\to f}$, the left hand side of Inequality (11) is also equal to 0 and the inequality holds with equality. This implies that for each $f \in F$, Inequality (9) holds with equality as well and we have

$$|\Psi_{\to f}| - |\Psi_{f\to}| = |\underline{\Psi}_{\to f}| - |\underline{\Psi}_{f\to}|.$$

A completely dual argument shows that there is a $\bar{\Psi} \in \mathcal{E}(u, \bar{p})$ with

$$|\Psi_{\to f}| - |\Psi_{f\to}| = |\bar{\Psi}_{\to f}| - |\bar{\Psi}_{f\to}|.$$

The same argument as before with $\bar{\Psi}$ in the role of Ψ, and $\bar{p}$ in the role of p establishes (note that the pairwise minimum of $\bar{p}$ and p' is again p') that there is a $\Xi \subseteq \Omega$ such that $[\Xi, p']$ is an equilibrium and for each $f \in F$ we have

$$|\overline{\Psi}_{\to f}| - |\overline{\Psi}_{f\to}| = |\Xi_{\to f}| - |\Xi_{f\to}|.$$

Since

$$|\Psi_{\to f}| - |\Psi_{f\to}| = |\overline{\Psi}_{\to f}| - |\overline{\Psi}_{f\to}|,$$

this concludes the proof. $\square$

Proof of Theorem 2

Proof. Following an idea of Kelso and Crawford (1982), we can characterize competitive equilibria by a zero-surplus condition. Define a surplus function $Z : \mathbb{R}^\Omega \to \mathbb{R}$ by

$$Z(p) := \min_{\Psi\subseteq\Omega} \max_{f\in F} \max_{\Psi'\subseteq\Omega_f} u^f(\Psi', p) - u^f(\Psi, p).$$

By definition, for each $f \in F$, we have $D^f(p) = \mathrm{argmax}_{\Psi'\subseteq\Omega_f} u^f(\Psi', p)$. Thus for each arrangement $[\Psi, p]$, we have $\max_{f\in F} \max_{\Psi'\subseteq\Omega_f} u^f(\Psi', p) - u^f(\Psi, p) \geq 0$ with equality if and only if $\Psi \in \mathcal{E}(u, p)$. Thus $p \in \mathcal{E}(u)$ if and only if $Z(p) = 0$. The surplus function is continuous, as $u^f(\Psi', p) - u^f(\Psi, p)$ is continuous in p and the maximum resp. minimum of finitely many continuous functions is continuous. Thus $\mathcal{E}(u)$ is a closed set, as it is the pre-image of the closed set $\{0\}$ under the continuous function Z.

By BWP, there is a $K > 0$ such that for all $f \in F$, $p \in \mathbb{R}^{\Omega_f}$ and $\Psi \in D^f(p)$ if $\omega \in \Psi_{\to f}$ then $p_\omega < K$ and if $\omega \in \Psi_{f\to}$ then $p_\omega > -K$. Let $\mathcal{E}'(u) := \mathcal{E}(u) \cap [-K, K]^\Omega$. By BWP, for each $p \in \mathcal{E}(u)$, the vector $p' \in \mathbb{R}^\Omega$ defined by $p'_\omega = p_\omega$ for $-K < p_\omega < K$, $p'_\omega = K$ for $p_\omega > K$, and $p'_\omega = -K$ for $p_\omega < -K$ is an equilibrium price vector $p' \in \mathcal{E}'(u)$ with $v^f(p') = v^f(p)$ for each $f \in F$. By Corollary 2 in Fleiner et al. (2019) (as indicated in Footnote 9, under BWP the choice-language version of weak FS used by Fleiner et al. (2019) is equivalent to the demand-language version), $\mathcal{E}(u)$ is non-empty and hence $\mathcal{E}'(u)$ is non-empty. As $\mathcal{E}(u)$ is closed, $\mathcal{E}'(u)$ is compact. From Theorem 1, and observing that the pairwise maximum (minimum) of two vectors in $[-K, K]^\Omega$ is an element of $[-K, K]^\Omega$, we conclude that $\mathcal{E}'(u)$ is a non-empty, compact sublattice of $\mathbb{R}^\Omega$. This implies that $\mathcal{E}'(u)$ has a maximal element $\bar{p}$ and a minimal element $\underline{p}$. By monotonicity and the previous observation that for each $p \in \mathcal{E}(u)$ there is a $p' \in \mathcal{E}'(u)$ with $v^f(p') = v^f(p)$ for each $f \in F$, for each terminal seller f and $p \in \mathcal{E}(u)$ we have $v^f(\bar{p}) \geq v^f(p)$. Thus $\bar{p}$ is a terminal seller optimal equilibrium. Similarly, $\underline{p}$ is a terminal buyer optimal equilibrium $\underline{p}$ under u. $\square$

Proof of Theorem 3

Proof. Let $F' \subseteq F$ be the set of terminal buyers. Let $\mathcal{U} = \bigtimes_{f \in F} \mathcal{U}_f$ where for $f \in F'$ the set $\mathcal{U}_f$ is the set of unit demand and BWP utility functions and for each $f \in F \setminus F'$ the set $\mathcal{U}_f$ is the set of BWP, FS, LAD and LAD utility functions. In the following for $\tilde{u}^f, \hat{u}^f \in \mathcal{U}_f$ etc. we denote the induced demand by $\tilde{D}^f, \hat{D}^f$ etc.

Let $\mathcal{M} : \mathcal{U} \to \mathcal{A}$ be a buyer-optimal mechanism. First we establish that $\mathcal{M}$ is immune to truncation strategies.

Claim 3. *Let $f \in F'$. Let $u, \tilde{u} \in \mathcal{U}$ with $\tilde{u}^{-f} = u^{-f}$ and let $[\Psi, p]$ be a buyer-optimal equilibrium under u. If $\Psi_f \neq \emptyset$, $\tilde{u}^f(\omega, \cdot) = u^f(\omega, \cdot)$ for each $\omega \in \Omega_{\to f}$ and $\tilde{u}^f(\emptyset) > \tilde{u}^f(\Psi, p)$, then for each equilibrium $[\tilde{\Psi}, \tilde{p}]$ under $\tilde{u}$, we have $\tilde{\Psi}_f = \emptyset$.*

Proof. Suppose not. Then $\tilde{\Psi}_f \neq \emptyset$. Let $\tilde{\Psi}_f = \{\tilde{\omega}\}$. Note that also $\{\tilde{\omega}\} \in D^f(\tilde{p})$. Thus $[\tilde{\Psi}, \tilde{p}]$ is an equilibrium under u. But since

$$u^f(\tilde{\omega}, \tilde{p}_{\tilde{\omega}}) = \tilde{u}^f(\tilde{\omega}, \tilde{p}_{\tilde{\omega}}) \geq \tilde{u}^f(\emptyset) > \tilde{u}^f(\Psi, p) = u^f(\Psi, p)$$

this contradicts the buyer optimality of $[\Psi, p]$. □

Second we establish that $\mathcal{M}$ is immune to certain strategies where a single terminal buyer changes the utility function for one trade so that it becomes more attractive relative to the other trades. The claim can be interpreted as an adaption of Lemma 1 of Hatfield and Kojima (2009) to the setting with transfers.

Claim 4. *Let $f \in F'$. Let $u, \hat{u} \in \mathcal{U}$ with $\hat{u}^{-f} = u^{-f}$ such that there is a $\hat{\omega} \in \Omega_{\to f}$ with $\hat{u}^f(\omega, \cdot) = u^f(\omega, \cdot)$ for $\omega \neq \hat{\omega}$ and $\hat{u}^f(\emptyset) = u^f(\emptyset)$. Let $[\bar{\Psi}, \bar{p}]$ be a buyer-optimal equilibrium under u. If for all $p_{\hat{\omega}} \in \mathbb{R}$, we have*

$$u^f(\hat{\omega}, p_{\hat{\omega}}) \leq u^f(\bar{\Psi}, \bar{p}) \Rightarrow \hat{u}^f(\hat{\omega}, p_{\hat{\omega}}) = u^f(\hat{\omega}, p_{\hat{\omega}}),$$
$$u^f(\hat{\omega}, p_{\hat{\omega}}) \geq u^f(\bar{\Psi}, \bar{p}) \Rightarrow \hat{u}^f(\hat{\omega}, p_{\hat{\omega}}) \geq u^f(\hat{\omega}, p_{\hat{\omega}}),$$

then $[\bar{\Psi}, \bar{p}]$ is a buyer-optimal equilibrium under $\hat{u}$.

Proof. Let $[\hat{\Psi}, \hat{p}]$ be a buyer-optimal equilibrium under $\hat{u}$. If $u^f(\hat{\omega}, \hat{p}_{\hat{\omega}}) \leq u^f(\bar{\Psi}, \bar{p})$, then we have $D^f(\hat{p}) = \hat{D}^f(\hat{p})$ and $[\hat{\Psi}, \hat{p}]$ is an equilibrium under u. Moreover, $\hat{u}^f(\hat{\omega}, \bar{p}_{\hat{\omega}}) = u^f(\hat{\omega}, \bar{p}_{\hat{\omega}})$, and therefore $[\bar{\Psi}, \bar{p}]$ is an equilibrium under $\hat{u}$. By buyer-optimality of $[\bar{\Psi}, \bar{p}]$ under u, we have $\hat{u}^{f'}(\hat{\Psi}, \hat{p}) = u^{f'}(\hat{\Psi}, \hat{p}) \leq u^{f'}(\bar{\Psi}, \bar{p}) = \hat{u}^{f'}(\bar{\Psi}, \bar{p})$ for each $f' \in F'$. Thus, $[\bar{\Psi}, \bar{p}]$ is a buyer-optimal equilibrium under $\hat{u}$. It remains to consider the case that $u^f(\hat{\omega}, \hat{p}_{\hat{\omega}}) > u^f(\bar{\Psi}, \bar{p})$. In this case, consider the two sub-cases that $\hat{\Psi}_f = \{\hat{\omega}\}$ or $\hat{\Psi}_f \neq \{\hat{\omega}\}$.

If $\hat{\Psi}_f \neq \{\hat{\omega}\}$, we can show that $[\hat{\Psi}, \hat{p}]$ is an equilibrium under u. Suppose not. Then, as $\hat{\Psi}_f \notin D^f(\hat{p})$ and $u^f(\omega, \hat{p}_\omega) = \hat{u}^f(\omega, \hat{p}_\omega)$ for $\omega \neq \hat{\omega}$, we have $u^f(\hat{\omega}, \hat{p}_{\hat{\omega}}) > u^f(\hat{\Psi}, \hat{p})$. Thus $\hat{u}^f(\hat{\omega}, \hat{p}_{\hat{\omega}}) \geq u^f(\hat{\omega}, \hat{p}_{\hat{\omega}}) > u^f(\hat{\Psi}, \hat{p}) = \hat{u}^f(\hat{\Psi}, \hat{p})$ and therefore $\hat{\Psi}_f \notin \hat{D}^f(\hat{p})$. This contradicts the assumption that $[\hat{\Psi}, \hat{p}]$ is an equilibrium under $\hat{u}$. Thus, $[\hat{\Psi}, \hat{p}]$ is an equilibrium under u and by the same reasoning as above, $[\bar{\Psi}, \bar{p}]$ is a buyer-optimal equilibrium under $\hat{u}$.

If $\hat{\Psi}_f = \{\hat{\omega}\}$, consider the utility function $\tilde{u}^f$ obtained from u^f by truncating as follows: $\tilde{u}^f(\omega, \cdot) = u^f(\omega, \cdot)$ for all $\omega \in \Omega_{\to f}$ and $u^f(\bar{\Psi}, \bar{p}) < \tilde{u}^f(\emptyset) < u^f(\hat{\omega}, \hat{p}_{\hat{\omega}})$. By Claim 3, for each equilibrium $[\Psi, p]$ under $\tilde{u} := (\tilde{u}^f, u^{-f})$ we have $\Psi_f = \emptyset$. Define the utility function $\tilde{u}^f_*$ by $\tilde{u}^f_*(\hat{\omega}, \cdot) = \tilde{u}^f(\hat{\omega}, \cdot) = u^f(\hat{\omega}, \cdot)$, by $\tilde{u}^f_*(\omega, \cdot) = -\infty$ for each $\omega \neq \hat{\omega}$, and $\tilde{u}^f_*(\emptyset) = \tilde{u}^f(\emptyset)$. As for each equilibrium $[\Psi, p]$ under $\tilde{u}$ we have $\Psi_f = \emptyset$, we have $\mathcal{E}(\tilde{u}) \subseteq \mathcal{E}(\tilde{u}_*)$ for $\tilde{u}_* := (\tilde{u}^f_*, u^{-f})$, and in particular, there is an equilibrium $[\tilde{\Psi}, \tilde{p}]$ under $\tilde{u}_*$ with $\tilde{\Psi}_f = \emptyset$. Observe however that $\tilde{u}^f_*(\hat{\omega}, \hat{p}_{\hat{\omega}}) = \tilde{u}^f(\hat{\omega}, \hat{p}_{\hat{\omega}}) = u(\hat{\omega}, \hat{p}_{\hat{\omega}}) > \tilde{u}^f_*(\emptyset)$. Thus $\tilde{D}^f_*(\hat{p}) = \{\{\hat{\omega}\}\}$ and $[\hat{\Psi}, \hat{p}]$ is an equilibrium under $\tilde{u}_*$ with $\tilde{u}^f_*(\hat{\Psi}, \hat{p}) > \tilde{u}^f_*(\emptyset)$. This contradicts the rural hospitals theorem (the second part of Theorem 1). □

With the claim, we can prove the result. Suppose there are profiles $u, \tilde{u} \in \mathcal{U}$ such that $\tilde{u}^{-F'} = u^{-F'}$ and for each $f \in F'$, we have $u^f(\mathcal{M}(\tilde{u})) > u^f(\mathcal{M}(u))$. Let $\mathcal{M}(u) = (\bar{\Psi}, \bar{p})$ and $\mathcal{M}(\tilde{u}) = (\tilde{\Psi}, \tilde{p})$.

We define for each $f \in F'$, a $\hat{u}^f \in \mathcal{U}_f$ as follows: Note that $\tilde{\Psi}_f \neq \emptyset$ as $u^f(\tilde{\Psi}, \tilde{p}) > u^f(\bar{\Psi}, \bar{p}) \geq u^f(\emptyset)$. Let $\tilde{\omega} \in \tilde{\Psi}$ be the unique trade in $\tilde{\Psi}$ such that $b(\tilde{\omega}) = f$. We let $\hat{u}^f(\omega, \cdot) = u^f(\omega, \cdot)$ for $\omega \neq \tilde{\omega}$ and we let $\hat{u}^f(\emptyset) = u^f(\emptyset)$. To construct $\hat{u}^f(\tilde{\omega}, \cdot)$ we proceed as follows: Define $\hat{u}^f(\tilde{\omega}, p_{\tilde{\omega}}) := u^f(\tilde{\omega}, p_{\tilde{\omega}})$ for each $p_{\tilde{\omega}} \in \mathbb{R}$ with $u^f(\tilde{\omega}, p_{\tilde{\omega}}) \leq u^f(\bar{\Psi}, \bar{p})$. Define

$$\hat{u}^f(\tilde{\omega}, \tilde{p}_{\tilde{\omega}}) := \max_{\omega \in \Omega_{\to f}} u^f(\omega, \tilde{p}_\omega).$$

Note that

$$\hat{u}^f(\tilde{\omega}, \tilde{p}_{\tilde{\omega}}) \geq u^f(\tilde{\omega}, \tilde{p}_{\tilde{\omega}}) > u^f(\bar{\Psi}, \bar{p}) = \hat{u}^f(\bar{\Psi}, \bar{p}).$$

For prices $p_{\tilde{\omega}} \neq \tilde{p}_{\tilde{\omega}}$ with $u^f(\tilde{\omega}, p_{\tilde{\omega}}) \geq u^f(\bar{\Psi}, \bar{p})$, we can choose any continuous and monotonic extension such that $\hat{u}^f(\tilde{\omega}, p_{\tilde{\omega}}) \geq u^f(\tilde{\omega}, p_{\tilde{\omega}})$. By Claim 4, $[\bar{\Psi}, \bar{p}]$ is a buyer-optimal equilibrium for $(\hat{u}^f, u^{-f})$. Iterating for all $f \in F'$, $[\bar{\Psi}, \bar{p}]$ is a buyer-optimal equilibrium under $\hat{u} := (\hat{u}^{F'}, u^{-F})$. Note however that by construction of $\hat{u}$, for each $f \in F'$ we have $\tilde{\Psi}_f \in \hat{D}^f(\tilde{p})$. Thus $[\tilde{\Psi}, \tilde{p}]$ is an equilibrium under $\hat{u}$ with $\hat{u}^f(\tilde{\Psi}, \tilde{p}) > \hat{u}^f(\bar{\Psi}, \bar{p})$ for each $f \in F'$. This contradicts the buyer-optimality of $[\bar{\Psi}, \bar{p}]$ under $(\hat{u}^{F'}, u^{-F})$. □

C Proofs for Section 4

Proof of Lemma 5

Proof. First we show the result for non-negative prices. Let $p, p' \in \mathbb{R}_+^{\Omega_f}$. Define $q, q' \in \mathbb{R}_+^X$ by

$$q_x := \begin{cases} \min_{\omega \in \Omega_{\to f}: x(\omega)=x} p_\omega, & \text{for } x \notin X_f, \\ \max_{\omega \in \Omega_{f\to}: x(\omega)=x} p_\omega, & \text{for } x \in X_f, \end{cases} \qquad q'_x := \begin{cases} \min_{\omega \in \Omega_{\to f}: x(\omega)=x} p'_\omega, & \text{for } x \notin X_f, \\ \max_{\omega \in \Omega_{f\to}: x(\omega)=x} p'_\omega, & \text{for } x \in X_f. \end{cases}$$

By construction we have

$$\begin{aligned} D^f(p) &= \{\Psi \subseteq \Omega_f : X_f(\Psi) \in \tilde{D}^f(q), p_\omega = q_{x(\omega)} \text{ for } \omega \in \Psi\}, \\ D^f(p') &= \{\Psi' \subseteq \Omega_f : X_f(\Psi') \in \tilde{D}^f(q'), p'_\omega = q'_{x(\omega)} \text{ for } \omega \in \Psi'\}. \end{aligned}$$

If $p_\omega = p'_\omega$ for $\omega \in \Omega_{f\to}$ and $p_\omega \le p'_\omega$ for $\omega \in \Omega_{\to f}$, then for $\Psi' \in D^f(p')$ there is, by gross substitutability a $Y \in \tilde{D}^f(q)$ with $\{x \in Y : q'_x = q_x\} \subseteq X_f(\Psi')$. Thus, if $x \in Y \setminus X_f$ and $q'_x = q_x$, then $x \in X_f(\Psi')$, and if $x \in X_f \setminus X_f(\Psi')$, then, as $q'_x = q_x$, we have $x \in X_f \setminus Y$. Therefore there is a $\Psi \in D^f(p)$ with

$$\{\omega \in \Psi_{\to f} : p'_\omega = q'_{x(\omega)} = q_{x(\omega)} = p_\omega\} \subseteq \Psi'_{\to f}, \quad \Psi'_{f\to} \subseteq \Psi_{f\to}.$$

Similarly, by the law of aggregate demand, there is a $Y \in \tilde{D}^f(q)$ such that $|Y| \ge |X_f(\Psi')|$. Then there is a $\Psi \in D^f(p)$ with $Y = X_f(\Psi)$. But then

$$\begin{aligned} &|\Psi_{\to f}| - |\Psi_{f\to}| = |Y \setminus X_f| - |X_f \setminus Y| = |Y| - |X_f| \\ \ge &|X_f(\Psi')| - |X_f| = |X_f(\Psi') \setminus X_f| - |X_f \setminus X_f(\Psi')| = |\Psi'_{\to f}| - |\Psi_{f\to}|. \end{aligned}$$

An analogous argument shows that D^f satisfies the second part of the SSS condition, the second part of the CSC condition, and LAS.

Next we establish FS and LAD/LAS on $\mathbb{R}^{\Omega_f}$. Let $p, p' \in \mathbb{R}^{\Omega_f}$ and define $q, q' \in \mathbb{R}^X$ as previously. Moreover, define $p^0 := (\max\{p_\omega, 0\}_{\omega\in\Omega}) \in \mathbb{R}^{\Omega_f}$ and $(p')^0 := (\max\{p'_\omega, 0\}_{\omega\in\Omega}) \in \mathbb{R}^{\Omega_f}$. By construction of u^f and the assumption that $\tilde{u}^f(Y,t) \le \tilde{u}^f(Y',t)$ for $Y \subseteq Y'$, we have

$$\begin{aligned} D^f(p) &= \{\Psi \in D^f(p^0) : \{x \in X : q_x < 0\} \subseteq X_f(\Psi), p_\omega = q_{x(\omega)} \text{ for } \omega \in \Psi\}, \\ D^f(p') &= \{\Psi' \in D^f((p')^0) : \{x \in X : q'_x < 0\} \subseteq X_f(\Psi'), p'_\omega = q'_{x(\omega)} \text{ for } \omega \in \Psi'\}. \end{aligned}$$

In particular, for $\Psi' \in D^f(p')$ we have $\Psi' \in D^f((p')^0)$ and by FS for non-negative prices, there is a $\tilde{\Psi} \in D^f(p^0)$ with

$$\{\omega \in \tilde{\Psi}_{\to f} : (p')^0_\omega = p^0_\omega\} \subseteq \Psi'_{\to f}, \quad \Psi'_{f\to} \subseteq \tilde{\Psi}_{f\to}.$$

We can find a $\Psi \in D^f(p)$, such that $\{\omega \in \tilde{\Psi} : q_{x(\omega)} = q'_{x(\omega)}\} \subseteq \Psi$. Now let $\omega \in \Psi_{\to f}$ and $p'_\omega = p_\omega$. Then $q'_{x(\omega)} = p'_\omega = p_\omega = q_{x(\omega)}$ and $\omega \in \tilde{\Psi}$. Moreover, $(p')^0_\omega = p^0_\omega$ and therefore $\omega \in \Psi'_{\to f}$. Similarly, for all $\omega \in \Omega_{f\to}$ we have $p'_\omega = p_\omega$. If $p_\omega = p'_\omega < q'_{x(\omega)} = q_{x(\omega)}$ then $\omega \notin \Psi$ and $\omega \notin \Psi'$. If $p_\omega = p'_\omega = q'_{x(\omega)} = q_{x(\omega)}$, then $\omega \notin \Psi_{\to f}$ implies $\omega \notin \tilde{\Psi}_{\to f}$. Moreover, $(p')^0_\omega = p^0_\omega$ and therefore $\omega \notin \Psi'_{\to f}$.

To establish LAD, let $\Psi' \in D^f(p')$. Since $\Psi' \in D^f((p')^0)$ and by LAD for non-negative price vectors, there is a $\tilde{\Psi} \in D^f(p^0)$ and hence a $\Psi \in D^f(p)$ with $X_f(\Psi) = X_f(\tilde{\Psi}) \cup \{x \in X : q_x < 0\}$ such that

$$|\Psi_{\to f}| - |\Psi_{f\to}| = |X_f(\Psi) \setminus X_f| - |X_f \setminus X_f(\Psi)| \geq |\tilde{\Psi}_{\to f}| - |\tilde{\Psi}_{f\to}| \geq |\Psi'_{\to f}| - |\Psi'_{f\to}|.$$

An analogous argument shows that D^f satisfies the second part of the SSS condition, the second part of the CSC condition, and LAS. □

Proof of Corollary 3

Proof. For the first part, consider price vectors in the induces trading network $q, q' \in \mathbb{R}^\Omega_+$ defined by $q_\omega := p_{x(\omega)}$ and $q'_\omega := p'_{x(\omega)}$ for each $\omega \in \Omega$. By Proposition 2, q and q' are equilibrium prices in the induced trading network. By Lemma 5, utility functions in the induced trading network satisfy FS, LAD and LAS. Thus, by Theorem 1, price vectors $\bar{q}, \underline{q} \in \mathbb{R}^\Omega_+$ with

$$\bar{q}_\omega = \max\{q_\omega, q'_\omega\}, \quad \underline{q}_\omega = \min\{q_\omega, q'_\omega\},$$

are equilibrium prices in the trading network. By construction of q and q', for each $\omega, \omega' \in \Omega$ with $x(\omega) = x(\omega')$ we have $q_\omega = p_{x(\omega)} = q_{\omega'}$ and $q'_\omega = p'_{x(\omega)} = q'_{\omega'}$. Therefore,

$$\bar{p}_x = \max_{\omega \in \Omega, x = x(\omega)} \bar{q}_\omega \quad \text{and} \quad \underline{p}_x = \max_{\omega \in \Omega, x = x(\omega)} \underline{q}_\omega,$$

and, by Proposition 2, $\bar{p}$ and $\underline{p}$ are equilibrium price vectors.

For the second part, define q and q' as before and let

$$\Psi := \{\omega \in \Omega : x(\omega) \in Y_{b(\omega)} \cap X_{s(\omega)}\}.$$

As shown in the proof of Proposition 2, $[\Psi, q]$ is an equilibrium of the trading network. By the second part of Theorem 1, there is a $\Psi' \subseteq \Omega$ such that $[\Psi', q']$ is an equilibrium of the trading network with

$$|\Psi_{\to f}| - |\Psi_{f\to}| = |\Psi'_{\to f}| - |\Psi'_{f\to}|.$$

Let $Y' = (Y'_f)_{f\in F}$ with $Y'_f := X_f(\Psi')$. As shown in the proof of Proposition 2, $[Y', p']$ is an equilibrium of the exchange economy. Moreover,

$$\begin{aligned}|Y_f| &= |Y_f \setminus X_f| - |X_f \setminus Y_f| + |X_f| = |\Psi_{\to f}| - |\Psi_{f\to}| + |X_f| \\ &= |\Psi'_{\to f}| - |\Psi'_{f\to}| + |X_f| = |Y'_f \setminus X_f| - |X_f \setminus Y'_f| + |X_f| = |Y'_f|.\end{aligned}$$

For the third part, we first show that the set of equilibrium price vectors in the induced trading network, $\mathcal{E}(u)$ is compact. The same argument as in the proof of Theorem 2 establishes that the surplus function $Z : \mathbb{R}^\Omega_+ \to \mathbb{R}$ is continuous and hence $\mathcal{E}(u) \subseteq \mathbb{R}^\Omega_+$ is closed. To show that $\mathcal{E}(u)$ is bounded, note that by the full range assumption there exists a $K > 0$ such that for each $f \in F$ and $Y \subseteq X$ we have $\tilde{u}^f(Y, -K) < \tilde{u}^f(X_f, 0)$. For each equilibrium $[\Psi, p]$ in the trading network and each $f \in F$, we have

$$u^f(\Psi, p) = \tilde{u}^f(X_f(\Psi), p_f(\Psi)) \geq \tilde{u}^f(X_f, 0) = u^f(\emptyset),$$

and therefore by monotonicity of utility in transfers $p_f(\Psi) > -K$. Moreover, $\sum_{f\in F} p_f(\Psi) = 0$. Thus, $p_f(\Psi) < |F| \cdot K$ for each $f \in F$. By the full range assumption, there is a $\tilde{K} > 0$ such that for each $f \in F$ and $Y \subseteq X$, we have $\tilde{u}^f(\emptyset, \tilde{K}) > \tilde{u}^f(Y, |F| \cdot K)$. Note that for each equilibrium $[\Psi, p]$ of the trading network, each $f \in F$ and each $\Psi' \subseteq \Omega_f$ with $X_f(\Psi') = \emptyset$, we have

$$\tilde{u}^f(\emptyset, \sum_{\omega\in\Psi'} p_\omega) = u^f(\Psi', p) \leq u^f(\Psi, p) = \tilde{u}^f(X_f(\Psi), p_f(\Psi)) < \tilde{u}^f(X_f(\Psi), |F|{\cdot}K) < \tilde{u}^f(\emptyset, \tilde{K}).$$

Thus $\sum_{\omega\in\Psi'} p_\omega < \tilde{K}$ and, as $p_\omega \geq 0$ for each $\omega \in \Omega$, we have $0 \leq p_\omega < \tilde{K}$ for each $\omega \in \Psi'$. Now note that for each $\omega \in \Omega$, there exists a $\Psi' \subseteq \Omega_{s(\omega)}$ with $X(\Psi') = \emptyset$ and $\omega \in \Psi'$. Thus for each $\omega \in \Omega$ we have $0 \leq p_\omega < \tilde{K}$. Thus $\mathcal{E}(u)$ is compact and by Propositions 2 non-empty. Moreover, by Theorem 1, $\mathcal{E}(u)$ is a sublattice of $\mathbb{R}^\Omega$. Since $\mathcal{E}(u)$ is a non-empty, compact sublattice of $\mathbb{R}^\Omega$, there exist $\bar{p}, \underline{p} \in \mathcal{E}(u)$ such that for each $p \in \mathcal{E}(u)$ we have $\underline{p}_\omega \leq p_\omega \leq \bar{p}_\omega$ for each $\omega \in \Omega$. By the first part of Proposition 2, the vectors $\underline{q}, \bar{q} \in \mathbb{R}_+^X$ defined by

$$\underline{q}_x := \max_{\omega \in \Omega, x = x(\omega)} \underline{p}_\omega, \quad \bar{q}_x := \max_{\omega \in \Omega, x = x(\omega)} \bar{p}_\omega$$

are equilibrium price vectors in the exchange economy. Now let $q \in \mathbb{R}_+^X$ be an equilibrium price vector in the exchange economy. By the second part of Proposition 2, the price vector $p \in \mathbb{R}_+^\Omega$ defined by $p_\omega := p_{x(\omega)}$ for each $\omega \in \Omega$, is in $\mathcal{E}(u)$. Let $x \in X$. Let $\omega \in \Omega$ with $x = x(\omega)$ and $\underline{q}_x = \underline{p}_\omega$. Then $\underline{q}_x = \underline{p}_\omega \le p_\omega = q_x$. Similarly, let $\omega \in \Omega$ with $x = x(\omega)$ and $\bar{q}_x = \bar{p}_\omega$. Then $\bar{q}_x = \bar{p}_\omega \ge p_\omega = p_x$. Thus $\bar{q}, \underline{q}$ are the desired price vectors. □

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