Frequency dependent mass and stiffness matrices of bar and beam elements and their equivalency with the dynamic stiffness matrix

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Abstract

Starting from the solutions of the governing differential equations of motion in free vibration, the frequency dependent mass and stiffness matrices of bar and beam elements have been derived in this paper, but importantly, their equivalency with the corresponding dynamic stiffness matrix is established. In sharp contrast to series solutions, reported in the literature, explicit expressions for each term of the frequency dependent mass and stiffness matrices of bar and beam elements are generated in concise form through the application of symbolic computation and their relationship with the single dynamic stiffness matrix (which contains both the mass and stiffness properties) for each of the two element types is highlighted. The theory is demonstrated by numerical results. By splitting the dynamic stiffness matrix into frequency dependent mass and stiffness matrices and at the same time retaining the exactness of results, the investigation paves the way for future research to overcome the difficulty to include damping in the dynamic stiffness research which has not been possible earlier. Furthermore, the frequency dependent mass and stiffness matrices derived in this paper permit the application of the Wittrick-Williams algorithm to compute with certainty the exact natural frequencies of structures comprising bar and beam elements.

Key words: Dynamic stiffness method, free vibration analysis, Wittrick-Williams algorithm, Frequency dependent mass and stiffness matrices

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1. Introduction

The concept of frequency dependent mass and stiffness matrices for bars and beams was first introduced by Przemieniecki [1] more than half a century ago which he later included in his classical text [2]. Przemieniecki ingeniously generated the elements of the frequency dependent mass and stiffness matrices for bars and beams using exact shape functions obtained from the solution of the governing differential equations in free vibration. To this end, he successfully developed the expressions for the elements of both mass and stiffness matrices for bar elements in explicit analytical (algebraic) form, but for beam elements, the task appeared to be too difficult. Consequently Przemieniecki [2] resorted to series expansions of the elements of the mass and stiffness matrices as a function of the square of the frequency. He demonstrated how the natural frequencies of bar and beam elements can be obtained accurately with the help of the frequency dependent mass and stiffness matrices. Essentially, he truncated the frequency-dependent series of the mass and stiffness matrices for bars and beams using just two terms in his expansion and he showed that even with two terms in the series, his method gives much better accuracy for natural frequencies than the conventional finite element method (FEM) which of course, uses frequency-independent mass and stiffness matrices. Przemieniecki's work [1, 2] is of great significance, particularly for modal analysis in the high frequency range which is often required as an important prerequisite in the statistical energy analysis (SEA) method [3-6] because the modal density of structures in the high frequency range is usually very high [7, 8]. The need for high frequency vibration analysis of classical structures like bars and beams with great accuracy is particularly significant when carrying out energy flow analysis [9, 10]. For such applications, the traditional FEM which is generally most effective within low and perhaps medium frequency range, may give inaccurate or unreliable results.

Przemieniecki's original idea of developing frequency dependent mass and stiffness matrices was picked up in later years by a few investigators [11-15]. For instance, Paz and Dung [11] carried out a theoretical investigation using a power series expansion for a beam element by including the effect of an axial force in their theory so that both free vibration and buckling analyses can be carried out. They did not present any numerical results but provided the expressions for the elements of the stiffness, mass and geometric matrices of beams using first and second order terms. Downs [12] used dynamic discretization method and generated an equivalent mass matrix of a beam in ascending power of the square of the frequency and formulated the deformation function in power series in a similar manner. He demonstrated the application of his method by numerical results for natural frequencies, using eight segments in his dynamic discretization model. Melosh and Smith [13] applied frequency-dependent mass and stiffness matrices of a bar element to investigate the free vibration characteristics of trusses. Their investigation was principally focused on bar elements and their assemblies, and they did not include beam elements in their analysis. By contrast, Fergusson and Pilkey [14] considered frequency-dependent mass matrices of structural elements solely from a theoretical standpoint and their theory was by and large general. They essentially showed that the Taylor series expansion of consistent mass matrix with appropriate stiffness matrix leads to the dynamic matrix, and acceptable accuracy of results can be obtained by truncating the frequency dependent shape function of a given order. Several years later, the method of frequency-dependent mass and stiffness matrices for bar and beam elements based on power series expansion of frequencies (originally proposed by Przemieniecki [1, 2]) was further extended by Dumont and de Oliveira [15] when they solved the response problem of plane frames. The authors calculated the dynamic response for arbitrarily applied loads at nodes. They also calculated the response by imposing arbitrary initial displacements at some chosen nodes.

In all the above investigations including Przemieniecki's original contribution [1, 2], the frequency-dependent mass and stiffness matrices for beam elements were developed in series form and no one appears to have made any attempt to derive explicit exact expressions for each of the elements of the frequency-dependent mass and stiffness matrices. The reason for this may be attributed to the fact that the level of complexity in such derivation is quite substantial in that the algebra becomes unwieldy. However, with the advent of symbolic computing [16-19], it appears that the problem can be overcome. One of the purposes of this paper is to take full advantage of symbolic computation to generate explicit expressions for the elements of the frequency-dependent mass and stiffness matrices of bar and beam elements in an exact sense, without resorting to power series expansion which unavoidably leads to approximation resulting from truncation errors.

At this point, it would be instructive to introduce the readers (who are not familiar with the subject) to the concept of the dynamic stiffness matrix of a structural element. The concept was pioneered by Kolousek [20]. In simple terms, the dynamic stiffness matrix of a structural element is a single frequency-dependent matrix which contains both the mass and stiffness properties of the element and the matrix is derived from the exact shape function obtained from the exact solution of the governing differential equations of motion of the structural element undergoing free natural vibration. The dynamic stiffness matrix of a structural element constitutes the fundamental basis of the dynamic stiffness method (DSM) [21-25] in free vibration analysis for which the Wittrick-Williams algorithm [26] is generally used as solution technique. The assembly procedure to form the overall dynamic stiffness of the final structure before applying the Wittrick-Williams algorithm, is similar to that of the finite element method (FEM), but of course, unlike the FEM for which separate mass and stiffness matrices are assembled, DSM needs only one matrix, called the dynamic stiffness matrix, to assemble. The publications of Williams and Wittrick [27], Akesson [28], Williams and Howson [29] and Howson et al. [30] provide useful information about the application of the DSM in the context of bar and beam elements in frameworks. In a relatively recent publication, Naprstek and Fischer [31] presented a formulation for static and dynamic analysis of beam assembles using a differential system on an oriented graph. Their formulation was sufficiently general, but they considered the differential operators to be linear and symmetrical. It is interesting to note that their method had all the essential features of the DSM. In a follow-up paper [32], they gave a different, but important perspective of the DSM by introducing polynomial and hyperbolic approximations for the dynamic stiffness elements of bar and beam members.

A secondary, but important purpose of this paper is to show that the frequency dependent mass and stiffness matrices of bar and beam elements derived in this paper can be related to their corresponding dynamic stiffness matrices. The equivalency between the two systems of matrices has far reaching consequences in structural dynamics because it extends the applicability of DSM in a wider context to enable free vibration and response analyses with the inclusion of damping which have not been possible before.

Clearly, the dynamic behaviour of a structure is predominantly influenced by its mass and stiffness properties. In this respect, the estimation of mass properties of a structure is relatively easy and is generally non-controversial whereas the estimation of its stiffness properties is significantly harder, and often gives rise to difficulty. The frequency-dependent stiffness matrix used in the existing DSM does not distinguish between the mass and stiffness properties of a structure because it treats both properties combinedly in a unitary manner, rather than considering them separately. Thus, in the existing DSM, it is impossible to investigate the independent effects of the mass and stiffness properties of a structure when predicting its overall dynamic behaviour. On the contrary, the current investigation based on separate developments of frequency-dependent mass and stiffness matrices of a structure (and yet retaining the exactness of the analysis like the DSM), offers flexibility to the designers to manipulate the mass and stiffness properties of the structure to produce desirable dynamic effects.

Furthermore, the proposed method overcomes the limitation of the FEM in that the number of eigenvalues that can be meaningfully extracted in FEM is restricted to the size of the mass and stiffness matrices with obviously, the higher order eigenvalues becoming considerably less accurate. In the proposed method, the accuracy as well as the number of eigenvalues that can be computed are not compromised in any way, and the results are always exact and independent of the number of elements used in the analysis. For instance, by using a single element, any number of eigenvalues to any desired accuracy can be computed using the proposed method, which of course, is impossible in FEM.

2. Problem statement and theoretical formulation

2.1 Fundamental preliminaries

Within the framework of a given displacement field, the mass (**m**) and stiffness (**k**) matrices of a structural element can be generally derived from the consideration of the kinetic and potential energies. In the usual notation, **m** and **k** are given by [2, 33-35]

$$\mathbf{m} = \int_V \rho\, \mathbf{N}^{\mathrm{T}}\, \mathbf{N}\, dv \tag{1}$$

and

$$\mathbf{k} = \int_V \mathbf{B}^{\mathrm{T}}\, \mathbf{D}\, \mathbf{B}\, dv \tag{2}$$

where ρ is the density of material for the structural element, **N** is the shape function which relates the displacements within the element to nodal displacements (Note that the boundary value of the shape function is either zero or one.), **B** is the matrix which relates the strains within the element to nodal displacements, **D** is the matrix which defines the constitutive law, i.e., the stress-strain relationship and the integration is carried out throughout the entire volume of the element. It should be noted that for one dimensional element such as a bar or a beam, the **D** matrix is effectively a matrix with only one term, i.e., it is a 1×1 matrix, containing simply the Young's modulus E of the material.

For a uniform and prismatic bar or beam element, the volume integral of Eqs. (1) and (2) reduces to a single integral along the length coordinate [2, 33-35]. Thus, if *X*-axis is chosen to be the (centroidal) axis of the bar or the beam, Eqs. (1) and (2) become

$$\mathbf{m} = \rho A \int_0^L \mathbf{N}^{\mathrm{T}} \, \mathbf{N} \, dx \tag{3}$$

and for a bar element,

$$\mathbf{k} = EA \int_0^L \mathbf{B}^{\mathrm{T}} \, \mathbf{B} \; dx \tag{4}$$

whereas for a beam element,

$$\mathbf{k} = EI \int_0^L \mathbf{B}^{\mathrm{T}} \, \mathbf{B} \; dx \tag{5}$$

where *A* and *I* are the area and second moment of area of the bar or beam cross-section, respectively.

In the finite element formulation, the shape function **N** for a bar or a beam is assumed to be a function which depends on the length coordinate *x*, but importantly, the shape function is assumed to be frequency independent. For instance, **N** is assumed to be a linear function of *x* for a bar whereas it is assumed to be cubic function in *x* for a beam if there is no distributed load on the beam. In the derivation which follows, the shape function **N** will be formulated from the exact solution of the governing differential equations of motion of a bar and a beam undergoing free natural vibration. Clearly, **N** will be frequency dependent in such cases.

In a rectangular Cartesian coordinate system, Fig. 1 shows a bar or a beam element of length *L* with the centroidal axis coinciding with the *X*-axis of the coordinate system. The origin is taken to be at the left-hand end which will be considered as node 1 whereas the right-hand end at a distance *L* will be considered as node 2 in the subsequent text.

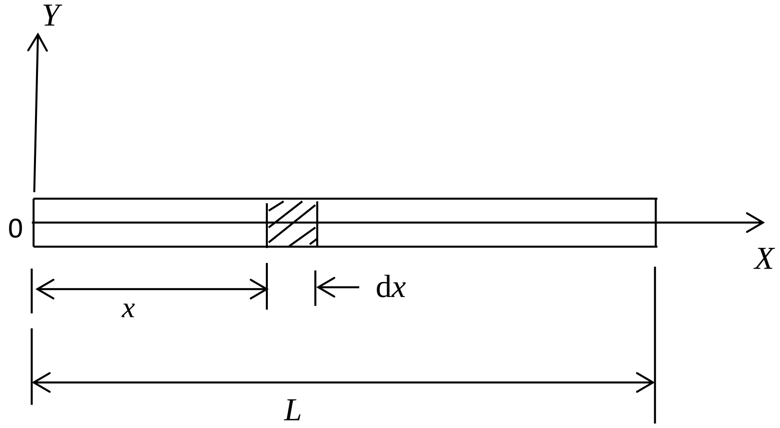

Fig. 1. Coordinate system and notation for a bar or a beam element.

2.2 Frequency dependent mass and stiffness matrices for a bar element

Referring to Fig. 1, the governing differential equation of a bar in axial or longitudinal vibration can be obtained by considering the equilibrium of an elemental length *dx* or by energy formulation using Lagrange's equation or Hamilton's principle. In the usual notation, the governing differential equation of the bar in free vibration is given by [2, 33, 36]

$$EA\frac{\partial^2 u}{\partial x^2} - \rho A\frac{\partial^2 u}{\partial t^2} = 0 \qquad (6)$$

where u is the axial displacement of a point at a distance x, ρ is the density of the bar material, A is the cross-sectional area of the bar so that ρA represents the mass per unit length, E is the Young's modulus of the bar material so that EA represents the axial or extensional rigidity of the bar.

If harmonic oscillation is assumed,

$$u(x,t) = U(x)e^{i\omega t} \qquad (7)$$

where U is the amplitude of axial displacement in longitudinal free vibration and ω is the angular or circular frequency of oscillation.

Substituting Eq. (7) into Eq. (6) and introducing the non-dimensional length $\xi = \frac{x}{L}$ gives

$$\frac{d^2U}{d\xi^2} + \frac{\omega^2}{c^2}U = 0 \qquad (8)$$

where

$$c^2 = \frac{EA}{\rho AL^2} \qquad (9)$$

The general solution of the differential equation, Eq. (8) is given by

$$U(\xi) = A_1 \sin\frac{\omega}{c}\xi + A_2 \cos\frac{\omega}{c}\xi \qquad (10)$$

where A_1 and A_2 are arbitrary constants of integrations.

Referring to Fig. 2, in which the left-hand and right-hand nodes of the bar are 1 and 2 located at $\xi = 0$ and 1, i.e. $x = 0$ and $x = L$, with nodal displacements U_1 and U_2, respectively, the constants A_1 and A_2 in Eq. (10) can now be eliminated to relate the displacement $U(\xi)$ within the bar element to the nodal displacements U_1 and U_2. This essentially refers to the concept of the shape faction which in this case is exact and frequency dependent, unlike the case with the traditional finite element method. Thus, the relationship between $U(\xi)$ and U_1 and U_2 can be shown as

$$\{U(\xi)\} = [N_1 \quad N_2]\begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} \tag{11}$$

where the frequency dependent shape functions are given by

$$N_1 = \cos\frac{\omega\xi}{c} - \sin\frac{\omega\xi}{c}\cot\frac{\omega}{c};\; N_2 = \sin\frac{\omega\xi}{c}\operatorname{cosec}\frac{\omega}{c} \tag{12}$$

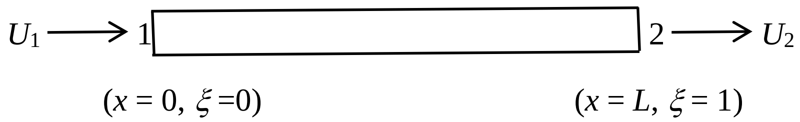

Fig. 2. End conditions for the displacements of a bar in axial motion

The strain $\varepsilon_x(x) = \frac{dU}{dx} = \frac{1}{L}\frac{dU}{d\xi}$ within the bar element can now be related to the nodal displacement U_1 and U_2 so that the **B** matrix of Eqs. (2) and (4) for the bar can be formulated. The strain $\varepsilon_x(x)$ is related to nodal displacements as follows.

$$\{\varepsilon_x(x)\} = \frac{1}{L}\begin{bmatrix} \frac{dN_1}{d\xi} & \frac{dN_2}{d\xi} \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} = \mathbf{B}\begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} \tag{13}$$

where the **B** matrix relating the strain within the bar to its nodal displacements is given by

$$\begin{aligned} \mathbf{B} &= \frac{1}{L}\begin{bmatrix} \frac{dN_1}{d\xi} & \frac{dN_2}{d\xi} \end{bmatrix} \\ &= \frac{\omega}{cL}\begin{bmatrix} -\left(\sin\frac{\omega\xi}{c} + \cot\frac{\omega}{c}\cos\frac{\omega\xi}{c}\right) & \left(\operatorname{cosec}\frac{\omega}{c}\cos\frac{\omega\xi}{c}\right) \end{bmatrix} \end{aligned} \tag{14}$$

Now with the help of Eqs. (3), (4), (12) and (14), the frequency dependent mass and stiffness matrices of the bar in axial or longitudinal vibration can be derived as follows:

$$\mathbf{m^a}(\omega) = \rho A \int_0^L \begin{bmatrix} N_1 \\ N_2 \end{bmatrix}[N_1 \quad N_2]\, dx = \rho A L \int_0^1 \begin{bmatrix} N_1^2 & N_1 N_2 \\ N_1 N_2 & N_2^2 \end{bmatrix} d\xi = \begin{bmatrix} m_{11}^a & m_{12}^a \\ m_{12}^a & m_{22}^a \end{bmatrix} \tag{15}$$

Substituting the frequency dependent shape functions N_1 and N_2 from Eq. (12) into Eq. (15) and carrying out the integration gives the explicit algebraic expressions for the frequency dependent mass matrix of the bar in axial or longitudinal vibration as follows. (Note that all necessary algebraic manipulation required here and elsewhere in this paper was greatly assisted by symbolic computation [16-19].)

$$m_{11}^a(\omega) = m_{22}^a(\omega) = \frac{\rho ALc}{2\omega}\operatorname{cosec}\frac{\omega}{c}\left(\frac{\omega}{c}\operatorname{cosec}\frac{\omega}{c} - \cos\frac{\omega}{c}\right);$$

$$m_{12}^a(\omega) = \frac{\rho ALc}{2\omega}\operatorname{cosec}\frac{\omega}{c}\left(1 - \frac{\omega}{c}\cot\frac{\omega}{c}\right) \tag{16}$$

Similarly, the frequency dependent stiffness matrix of the bar in axial or longitudinal vibration can be derived with the help of Eqs. (4) and (14) to give

$$\mathbf{k^a}(\omega) = EA\int_0^L \mathbf{B}^{\mathrm{T}}\mathbf{B}\,dx = EAL\int_0^1 \mathbf{B}^{\mathrm{T}}\mathbf{B}\,d\xi = \begin{bmatrix} k_{11}^a & k_{12}^a \\ k_{21}^a & k_{22}^a \end{bmatrix} \tag{17}$$

where

$$k_{11}^a(\omega) = k_{22}^a(\omega) = \frac{EA\omega}{2Lc}\operatorname{cosec}\frac{\omega}{c}\left(\frac{\omega}{c}\operatorname{cosec}\frac{\omega}{c} + \cos\frac{\omega}{c}\right);$$

$$k_{12}^a(\omega) = k_{21}^a(\omega) = -\frac{EA\omega}{2Lc}\operatorname{cosec}\frac{\omega}{c}\left(1 + \frac{\omega}{c}\cot\frac{\omega}{c}\right) \tag{18}$$

The above frequency dependent mass and stiffness matrices $\mathbf{m^a}(\omega)$ and $\mathbf{k^a}(\omega)$ in axial motion can be related to the corresponding dynamic stiffness matrix $\mathbf{k_D^a}(\omega)$ of the bar as follows.

$$\mathbf{k^a}(\omega) - \omega^2\mathbf{m^a}(\omega) = \mathbf{k_D^a}(\omega) \tag{19}$$

where $\mathbf{k_D^a}(\omega)$ is given by [29]

$$\mathbf{k_D^a}(\omega) = \begin{bmatrix} k_{11D}^a & k_{12D}^a \\ k_{21D}^a & k_{22D}^a \end{bmatrix} \tag{20}$$

with

$$k_{11D}^a = k_{22D}^a = \frac{EA}{L}\frac{\omega}{c}\cot\frac{\omega}{c}; \quad k_{12D}^a = k_{21D}^a = -\frac{EA}{L}\frac{\omega}{c}\operatorname{cosec}\frac{\omega}{c} \tag{21}$$

In the context of the relationship between the frequency-dependent mass and stiffness matrices with the dynamic stiffness matrix as shown in Eq. (19), it is worth noting that Richard and Leung [37] gave a theorem and its proof that the partial derivative of the dynamic stiffness matrix with respect to the square of the frequency gives the frequency dependent mass matrix.

Przemieniecki [2] gave the frequency-dependent mass and stiffness matrices for a bar using only two terms and obtained the frequency equation in quadratic form (quadratic in terms of the square of the circular frequency, i.e., quadratic in ω^2). His frequency-dependent mass and stiffness matrices are given by (see pages 283 and 337 of [2]).

$$\mathbf{m}_{\mathbf{P}}^{\mathbf{a}}(\omega) = \frac{\rho AL}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} + \frac{2\rho AL}{45}\left(\frac{\omega}{c}\right)^2 \begin{bmatrix} 1 & \frac{7}{8} \\ \frac{7}{8} & 1 \end{bmatrix} \tag{22}$$

and

$$\mathbf{k}_{\mathbf{P}}^{\mathbf{a}}(\omega) = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{EA}{45L}\left(\frac{\omega}{c}\right)^4 \begin{bmatrix} 1 & \frac{7}{8} \\ \frac{7}{8} & 1 \end{bmatrix} \tag{23}$$

Based on Eqs. (22) and (23), Przemieniecki [2] calculated the first ten natural frequencies of a cantilever bar using a varying number of elements in the analysis (see his Table 12.5 on page 338). His results were better than the conventional finite element method, but understandably his results were approximate and dependent on the number of elements, unlike the present analysis in which a single element can be used to determine any number of natural frequencies within any desired accuracy.

2.2 Frequency dependent mass and stiffness matrices for a beam element

The procedure for the derivation of the frequency-dependent mass and stiffness matrices for a beam element undergoing free bending (or flexural) vibration is similar, but somehow more complicated than that of a bar element given above. The level of complexity increases because of the increase in the degrees of freedom when the shape function involves both bending displacement and bending rotation. Nevertheless, the approach which starts from the solution of the governing differential equation of motion of the element in free vibration and employing the shape function to obtain the frequency-dependent mass and stiffness matrices is similar.

The governing differential equation of motion of a beam (see Fig. 1) exhibiting free bending or flexural vibration in the usual notation is given by [2, 33, 36]

$$EI\frac{\partial^4 w}{\partial x^4} + \rho A\frac{\partial^2 w}{\partial t^2} = 0 \tag{24}$$

As was the case with axial vibration, harmonic oscillation is assumed so that $w = We^{i\omega t}$and the non-dimensional length parameter $\xi = x/L$ is introduced in Eq. (24) to give

$$\frac{d^4W}{d\xi^4} - \beta^4 W = 0 \qquad (25)$$

where W is the amplitude of the bending or flexural vibration and β is given by

$$\beta = \sqrt[4]{\frac{\rho A L^4 \omega^2}{EI}} \qquad (26)$$

The solution of the governing differential equation Eq. (25) is given by [33, 36]

$$W = A_1 \cosh\beta\xi + A_2 \sinh\beta\xi + A_3 \cos\beta\xi + A_4 \sin\beta\xi \qquad (27)$$

where $A_1 - A_4$ are arbitrary constants of integration.

The slope or bending rotation Θ is given by

$$\Theta = \frac{dW}{dx} = \frac{1}{L}\frac{dW}{d\xi} = \frac{\beta}{L}(A_1 \sinh\beta\xi + A_2 \cosh\beta\xi - A_3 \sin\beta\xi + A_4 \cos\beta\xi) \qquad (28)$$

Referring to Fig. 3, the end conditions (or boundary conditions) for W and Θ at nodes 1 and 2 can now be applied to Eqs. (27) and (28) to eliminate the constants A_1- A_4. The end conditions are: at node 1 (x =0, i.e. $\xi = 0$), $W = W_1$ and $\Theta = \Theta_1$ and at node 2 ($x = L$, i.e. $\xi = 1$), $W = W_2$ and $\Theta = \Theta_2$. In this way, the shape function which relates displacement within the element in terms of the nodal displacements can be obtained, particularly by taking advantage of symbolic computation [16-19].

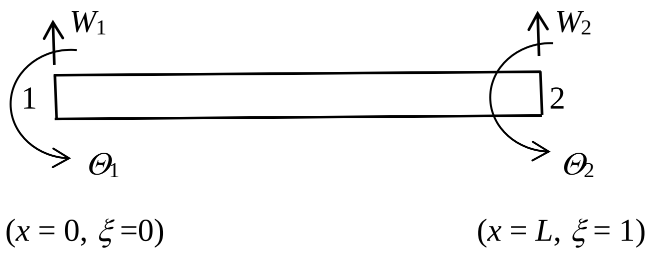

Fig. 3. End conditions for the displacements of a beam in flexural motion.

The task of eliminating the constants $A_1 - A_4$ from Eq. (27) and expressing the displacement W in terms of the nodal displacements to generate the frequency dependent shape functions was not easy, but this was greatly assisted by the application of symbolic computation [16-19]. The following relationship was obtained after expending some effort.

$$\{W(\xi)\} = [N_1 \quad N_2 \quad N_3 \quad N_4]\begin{Bmatrix} W_1 \\ \Theta_1 \\ W_2 \\ \Theta_2 \end{Bmatrix} \qquad (29)$$

where the shape functions N_1, N_2, N_3 and N_4 are given by

$$N_1 = (\mu_1 \cosh\beta\xi - \mu_4 \sinh\beta\xi + \mu_2 \cos\beta\xi + \mu_4 \sin\beta\xi) \tag{30}$$

$$N_2 = \frac{L}{\beta}(\mu_3 \cosh\beta\xi + \mu_2 \sinh\beta\xi - \mu_3 \cos\beta\xi + \mu_1 \sin\beta\xi) \tag{31}$$

$$N_3 = (-\mu_5 \cosh\beta\xi + \mu_7 \sinh\beta\xi + \mu_5 \cos\beta\xi - \mu_7 \sin\beta\xi) \tag{32}$$

$$N_4 = \frac{L}{\beta}(\mu_6 \cosh\beta\xi - \mu_5 \sinh\beta\xi - \mu_6 \cos\beta\xi + \mu_5 \sin\beta\xi) \tag{33}$$

with

$$\mu_1 = (\cosh\beta\cos\beta + \sinh\beta\sin\beta - 1)/\Delta \tag{34}$$

$$\mu_2 = (\cosh\beta\cos\beta - \sinh\beta\sin\beta - 1)/\Delta \tag{35}$$

$$\mu_3 = (\cosh\beta\sin\beta - \sinh\beta\cos\beta)/\Delta \tag{36}$$

$$\mu_4 = (\cosh\beta\sin\beta + \sinh\beta\cos\beta)/\Delta \tag{37}$$

$$\mu_5 = (\cosh\beta - \cos\beta)/\Delta \tag{38}$$

$$\mu_6 = (\sinh\beta - \sin\beta)/\Delta \tag{39}$$

$$\mu_7 = (\sinh\beta + \sin\beta)/\Delta \tag{40}$$

$$\Delta = 2(\cosh\beta\cos\beta - 1) \tag{41}$$

The frequency-dependent mass matrix of the beam can now be derived with the help of Eq. (3) and the shape functions given by Eqs. (30)-(33) as follows.

$$\mathbf{m^b}(\omega) = \rho A \int_0^L \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} [N_1 \quad N_2 \quad N_3 \quad N_4]\, dx = \rho A L \int_0^1 \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} [N_1 \quad N_2 \quad N_3 \quad N_4]\, d\xi \tag{42}$$

Substituting the shape functions N_1, N_2, N_3 and N_4 from Eqs. (30)-(33) into Eq. (42) and performing the integration and after extensive algebraic manipulation using REDUCE [16], the 4×4 frequency-dependent mass matrix was obtained in the following form.

$$\mathbf{m^b}(\omega) = \begin{bmatrix} m_{11}^b & m_{12}^b & m_{13}^b & m_{14}^b \\ m_{12}^b & m_{22}^b & m_{23}^b & m_{24}^b \\ m_{13}^b & m_{23}^b & m_{33}^b & m_{34}^b \\ m_{14}^b & m_{24}^b & m_{34}^b & m_{44}^b \end{bmatrix} \tag{43}$$

The 4×4 frequency-dependent stiffness matrix follows from Eq. (5) in a similar manner to give

$$\mathbf{k^b}(\omega) = EI\int_0^L \mathbf{B}^{\mathrm{T}}\mathbf{B}\,dx = EIL\int_0^1 \mathbf{B}^{\mathrm{T}}\mathbf{B}\mathrm{d}\xi = \begin{bmatrix} k_{11}^b & k_{12}^b & k_{13}^b & k_{14}^b \\ k_{12}^b & k_{22}^b & k_{23}^b & k_{24}^b \\ k_{13}^b & k_{23}^b & k_{33}^b & k_{34}^b \\ k_{14}^b & k_{24}^b & k_{34}^b & k_{44}^b \end{bmatrix} \tag{44}$$

where the **B** matrix in terms of the shape functions N_1, N_2, N_3 and N_4 is given by

$$\mathbf{B} = \frac{1}{L^2}\begin{bmatrix} \frac{d^2N_1}{d\xi^2} & \frac{d^2N_2}{d\xi^2} & \frac{d^2N_3}{d\xi^2} & \frac{d^2N_4}{d\xi^2} \end{bmatrix} \tag{45}$$

By extensive algebraic manipulation with the help of symbolic computation [16-19], the individual elements of the frequency dependent mass and stiffness matrices $\mathbf{m^b}$ and $\mathbf{k^b}$ were generated in explicit analytical form. This was indeed a formidable task, the details of which are not reported here for brevity, but the final expressions of the mass and stiffness elements are obtained in surprisingly concise form. The six independent elements of the frequency-dependent $\mathbf{m^b}$ and $\mathbf{k^b}$ matrices are given below.

$$m_{11}^b(\omega) = m_{33}^b(\omega) = \frac{\rho AL}{2\beta}(\nu_1\mu_5^2 - 2\nu_2\mu_5\mu_7 + \nu_3\mu_7^2) \tag{46}$$

$$m_{22}^b(\omega) = m_{44}^b(\omega) = \frac{\rho AL^3}{2\beta^3}\left(\nu_3\mu_5^2 - 2\nu_2\mu_5\mu_6 + \nu_1\mu_6^2\right) \tag{47}$$

$$m_{12}^b(\omega) = -m_{34}^b(\omega) = \frac{\rho AL^2}{2\beta^2}\{\nu_1\mu_5\mu_6 + \nu_3\mu_5\mu_7 - \nu_2(\mu_5^2 + \mu_6\mu_7)\} \tag{48}$$

$$m_{13}^b(\omega) = \frac{\rho AL}{2\beta}(\nu_2\mu_4\mu_5 + \nu_7\mu_2\mu_5 + \nu_9\mu_2\mu_7 + \nu_{11}\mu_1\mu_7 - \nu_5\mu_1\mu_5 - \nu_3\mu_4\mu_7) \tag{49}$$

$$m_{14}^b(\omega) = -m_{23}^b(\omega) = \frac{\rho AL^2}{2\beta^2}(\nu_3\mu_4\mu_5 + \nu_5\mu_1\mu_6 - \nu_2\mu_4\mu_6 - \nu_7\mu_2\mu_6 - \nu_9\mu_2\mu_5 - \nu_{11}\mu_1\mu_5) \tag{50}$$

$$m_{24}^b(\omega) = \frac{\rho AL^3}{2\beta^3}(\nu_1\mu_3\mu_6 + \nu_4\mu_2\mu_6 + \nu_8\mu_1\mu_5 - \nu_2\mu_3\mu_5 - \nu_6\mu_1\mu_6 - \nu_{10}\mu_2\mu_5) \tag{51}$$

and

$$k_{11}^b(\omega) = k_{33}^b(\omega) = \frac{EI\beta^3}{2L^3}(\nu_1\mu_5^2 - 2\nu_2\mu_5\mu_7 + \nu_3\mu_7^2) \tag{52}$$

$$k_{22}^b(\omega) = k_{44}^b(\omega) = \frac{EI\beta}{2L}\left(\nu_3\mu_5^2 - 2\nu_2\mu_5\mu_6 + \nu_1\mu_6^2\right) \tag{53}$$

$$k_{12}^b(\omega) = -k_{34}^b(\omega) = \frac{EI\beta^2}{2L^2}\{\nu_1\mu_5\mu_6 + \nu_3\mu_5\mu_7 - \nu_2(\mu_5^2 + \mu_6\mu_7)\} \tag{54}$$

$$k_{13}^b(\omega) = \frac{EI\beta^3}{2L^3}(\nu_2\mu_4\mu_5 + \nu_7\mu_2\mu_5 - \nu_6\mu_2\mu_7 + \nu_4\mu_1\mu_7 - \nu_5\mu_1\mu_5 - \nu_3\mu_4\mu_7) \tag{55}$$

$$k_{14}^b(\omega) = -k_{23}^b(\omega) = \frac{EI\beta^2}{2L^2}(\nu_3\mu_4\mu_5 + \nu_5\mu_1\mu_6 - \nu_2\mu_4\mu_6 - \nu_7\mu_2\mu_6 + \nu_6\mu_2\mu_5 - \nu_4\mu_1\mu_5) \tag{56}$$

$$k_{24}^b(\omega) = \frac{EI\beta}{2L}(\nu_1\mu_3\mu_6 + \nu_{11}\mu_2\mu_6 + \nu_8\mu_1\mu_5 + \nu_9\mu_1\mu_6 - \nu_2\mu_3\mu_5 - \nu_{10}\mu_2\mu_5) \tag{57}$$

where

$$\nu_1 = C_2 + 2jC_4 + 2jC_5 + C_8 + 2\beta \tag{58}$$

$$\nu_2 = C_1 + 2jC_6 - C_7 \tag{59}$$

$$\nu_3 = C_2 + 2jC_4 - 2jC_5 - C_8 \tag{60}$$

$$\nu_4 = C_1 - C_3 + jC_6 \tag{61}$$

$$\nu_5 = C_2 + jC_4 + jC_5 + \beta \tag{62}$$

$$\nu_6 = C_3 + jC_6 - C_7 \tag{63}$$

$$\nu_7 = jC_4 + jC_5 + C_8 + \beta \tag{64}$$

$$\nu_8 = jC_4 - jC_5 - C_8 + \beta \tag{65}$$

$$\nu_9 = C_3 - jC_6 + C_7 - 2 \tag{66}$$

$$\nu_{10} = C_2 + jC_4 - jC_5 - \beta \tag{67}$$

$$\nu_{11} = C_1 + C_3 + jC_6 - 2 \tag{68}$$

with $j = 1$ for the elements of $\mathbf{k^b}$ and $j = -1$ for the elements of $\mathbf{m^b}$ and $C_1 - C_8$ are given by

$$C_1 = \cosh^2\beta; \;\; C_2 = \cosh\beta\sinh\beta; \;\; C_3 = \cosh\beta\cos\beta; \;\; C_4 = \cosh\beta\sin\beta \tag{69}$$

$$C_5 = \sinh\beta\cos\beta; \;\; C_6 = \sinh\beta\sin\beta; \;\; C_7 = \cos^2\beta; \;\; C_8 = \cos\beta\sin\beta \tag{70}$$

The above frequency-dependent mass and stiffness matrices $\mathbf{m^b}(\omega)$ and $\mathbf{k^b}(\omega)$ in bending or flexural motion can be related to the corresponding dynamic stiffness matrix $\mathbf{k_D^b}(\omega)$ of the beam as follows.

$$\mathbf{k^b}(\omega) - \omega^2\mathbf{m^b}(\omega) = \mathbf{k_D^b}(\omega) \tag{71}$$

where $\mathbf{k_D^b}(\omega)$ is given by [29]

$$\mathbf{k_D^b}(\omega) = \begin{bmatrix} k_{11D}^b & k_{12D}^b & k_{13D}^b & k_{14D}^b \\ k_{21D}^b & k_{22D}^b & k_{23D}^b & k_{24D}^b \\ k_{31D}^b & k_{32D}^b & k_{33D}^b & k_{34D}^b \\ k_{41D}^b & k_{42D}^b & k_{43D}^b & k_{44D}^b \end{bmatrix} \tag{72}$$

where

$$k_{11D}^b = k_{33D}^b = \frac{EI\beta^3}{L^3}(C_4 + C_5)/(1 - C_3) \quad (73)$$

$$k_{22D}^b = k_{44D}^b = \frac{EI\beta}{L}(C_4 - C_5)/(1 - C_3) \quad (74)$$

$$k_{12D}^b = k_{21D}^b = -k_{34D}^b = -k_{43D}^b = \frac{EI\beta^2}{L^2}C_6/(1 - C_3) \quad (75)$$

$$k_{13D}^b = k_{31D}^b = -\frac{EI\beta^3}{L^3}(C_9 + C_{10})/(1 - C_3) \quad (76)$$

$$k_{14D}^b = k_{41D}^b = -k_{23D}^b = -k_{32D}^b = \frac{EI\beta^2}{L^2}(C_{12} - C_{11})/(1 - C_3) \quad (77)$$

$$k_{24D}^b = k_{42D}^b = \frac{EI\beta}{L}(C_{10} - C_9)/(1 - C_3) \quad (78)$$

with

$$C_9 = \sin\beta;\ \ C_{10} = \sinh\beta;\ \ C_{11} = \cos\beta;\ \ C_{12} = \cosh\beta \quad (79)$$

As was the case with a bar, Przemienicki [2] obtained the frequency-dependent mass and stiffness matrices for a beam using only two terms (see Eqs. 10.140-10.145 on pages 286-287 of [2]), but he did not report any results for the natural frequencies of a beam using his method. His expression for the frequency-dependent mass and stiffness matrices are given below. (Note that there is a typographical error in Eq. 10.144 on page 287 of Przemieniecki's book [2]. The second diagonal term within $\mathbf{k_0}$ matrix should be $4L^2$, not $4L$.)

$$\mathbf{m^b}(\omega) = \frac{\rho AL}{420}\begin{bmatrix} 156 & 22L & 54 & -13L \\ 22L & 4L^2 & 13L & -3L^2 \\ 54 & 13L & 156 & -22L \\ -13L & -3L^2 & -22L & 4L^2 \end{bmatrix} +$$

$$\beta^4\rho AL\begin{bmatrix} 0.729746 & 0.153233L & 0.659142 & -0.144386L \\ 0.153233L & 0.0325248L^2 & 0.144386L & -0.0314082L^2 \\ 0.659142 & 0.144386L & 0.729746 & -0.153233L \\ -0.144386L & -0.0314082L^2 & -0.153233L & 0.0325248L^2 \end{bmatrix} \times 10^{-3} \quad (80)$$

and

$$\mathbf{k^b}(\omega) = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} +$$

$$\beta^8\frac{EI}{L^3}\begin{bmatrix} 0.364872 & 0.0766162L & 0.329571 & -0.0721933L \\ 0.0766162L & 0.0162624L^2 & 0.0721933L & -0.0157041L^2 \\ 0.329571 & 0.0721933L & 0.364872 & -0.0766162L \\ -0.0721933L & -0.0157041L^2 & -0.0766162L & 0.0162624L^2 \end{bmatrix} \times 10^{-3} \quad (81)$$

Although Prezemieniecki's frequency-dependent mass and stiffness matrices shown above are approximate, unlike the ones derived from the current theory (which gives exact expressions for the elements of the matrices), it should be recognised that Prezemieniecki's matrix expressions give more accurate results than the FEM [34-35].

In the limiting case when the frequency parameter $\beta = \sqrt[4]{\frac{\rho A L^4 \omega^2}{EI}}$ tends to zero, the frequency-dependent mass and stiffness matrices derived above, become the corresponding frequency-independent mass and stiffness matrices encountered in the FEM. However, it should be noted that in the explicit mass and stiffness element expressions given above, the stiffness parameter (β) must not be literally zero, but for computational purposes, a negligibly small value of the frequency (ω) which defines the frequency parameter (β), say ω = 0.0001 rad/s, can be used for practical and realistic problems to capture the degenerate cases of the FEM mass and stiffness matrices, and thus avoiding any possible numerical overflow or ill-conditioning.

It is worth noting that although Kolousek [20] did not split the frequency-dependent dynamic stiffness matrices of bar and beam elements into corresponding frequency-dependent mass and stiffness matrices unlike the present paper, he nevertheless, expanded the dynamic stiffness matrices in terms of the frequency parameter, using many more terms in the series than the only two terms used by Przemieniecki [1, 2].

The frequency-dependent mass and stiffness matrices of the bar and beam elements derived above using the present theory can now be combined by superimposing Figs. 2 and 3 to give Fig. 4 and by using the relationship of Eqs. (15)-(18) and Eqs. (43) and (44) to enable the free vibration analysis of frameworks to be made. Referring to the displacement arrangements of Fig. 4, the frequency dependent mass and stiffness matrices with both axial and bending displacements can be expressed as

$$\mathbf{m}(\omega) = \begin{bmatrix} m_{11}^a & 0 & 0 & m_{12}^a & 0 & 0 \\ 0 & m_{11}^b & m_{12}^b & 0 & m_{13}^b & m_{14}^b \\ 0 & m_{12}^b & m_{22}^b & 0 & m_{23}^b & m_{24}^b \\ m_{12}^a & 0 & 0 & m_{22}^a & 0 & 0 \\ 0 & m_{13}^b & m_{23}^b & 0 & m_{11}^b & -m_{12}^b \\ 0 & m_{14}^b & m_{24}^b & 0 & -m_{12}^b & m_{22}^b \end{bmatrix} \tag{82}$$

and

$$\mathbf{k}(\omega) = \begin{bmatrix} k_{11}^a & 0 & 0 & k_{12}^a & 0 & 0 \\ 0 & k_{11}^b & k_{12}^b & 0 & k_{13}^b & k_{14}^b \\ 0 & k_{12}^b & k_{22}^b & 0 & k_{23}^b & k_{24}^b \\ k_{12}^a & 0 & 0 & k_{22}^a & 0 & 0 \\ 0 & k_{13}^b & k_{23}^b & 0 & k_{11}^b & -k_{12}^b \\ 0 & k_{14}^b & k_{24}^b & 0 & -k_{12}^b & k_{22}^b \end{bmatrix} \tag{83}$$

where the elements of $\mathbf{m}(\omega)$ and $\mathbf{k}(\omega)$ have already been defined in Eq. (16), and Eqs. (46) to (51) and Eq. (18) and Eqs. (52) to (57), respectively.

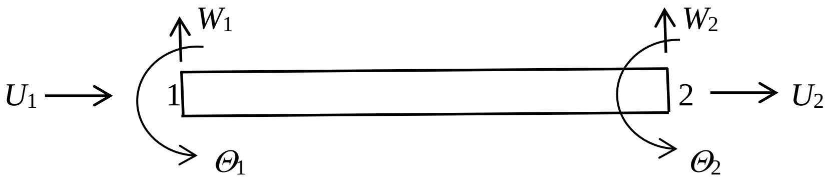

Fig. 4. Beam element showing axial displacement, bending displacement, and bending rotation.

3. Application of the theory

The equivalency of the frequency-dependent mass and stiffness matrices of Eqs. (82) and (83) with the dynamic stiffness matrix enables one to formulate the dynamic stiffness $\mathbf{k_D}(\boldsymbol{\omega})$ of a beam element which includes both axial and bending deformations, as follows

$$\mathbf{k_D}(\omega) = \mathbf{k}(\omega) - \omega^2 \mathbf{m}(\omega) \tag{84}$$

Now the frequency-dependent mass and stiffness matrices $\mathbf{k}(\omega)$ and $\mathbf{m}(\omega)$ of all individual elements in a structure can be assembled using the conventional transformation technique as generally employed in the FEM. Thus, the overall frequency dependent master mass and stiffness matrices $\mathbf{K}(\omega)$ and $\mathbf{M}(\omega)$ and hence the overall master dynamic stiffness matrix $\mathbf{K_D}(\omega)$ of the final structure can be obtained, as follows.

$$\mathbf{K_D}(\omega) = \mathbf{K}(\omega) - \omega^2 \mathbf{M}(\omega) \tag{85}$$

It is now possible to apply the well-established algorithm of Wittricks and Williams [26] to $\mathbf{K_D}(\omega)$ to determine the natural frequencies and the subsequent mode shapes of the structure. The Wittrick-Williams algorithm has been given widespread coverage in the literature and there are hundreds of papers on its application, see for example [21-25]. Therefore, the details of the working principles of the algorithm are not repeated here, but it should be noted that the algorithm gives with absolute certainty the number of natural frequencies that exists below an arbitrarily chosen trial frequency rather than calculating them directly. As successive trial frequencies can be chosen, one can establish the upper and lower bounds of any natural frequency to any desired accuracy. This simple feature of the algorithm is generally applied to compute the natural frequencies and then recover the mode shapes of a structure.

4. Results and discussion

The first set of results was computed for a cantilever bar exhibiting axial or longitudinal vibration using the present theory (which utilises exact frequency-dependent mass and stiffness matrices and applies the Wittrick-Williams algorithm), Przemieniecki's approximate theory based on quadratic frequency-dependent mass and stiffness matrices [2] and the conventional finite element theory [34, 35] which uses frequency-independent mass and stiffness matrices. The present theory gave exact results which were confirmed by the exact solution obtained from the solution of the governing differential equation [33, 36]. It should be noted that the results from the present theory are independent of the number of elements used in the analysis and the exact results can be obtained by using even a single element. This contrasts with Przemieniecki's method [2] and the traditional FEM [34, 35] for which the results are dependent on the number of elements used in the analysis.

The first ten axial or longitudinal natural frequencies $\omega_i^a(i = 1, 2, 3, \dots 10)$ of the cantilever bar computed using Przemieniecki's theory [2] and the finite element theory [34, 35] were factored by the corresponding exact natural frequencies $\omega_i^e(i = 1, 2, 3, \dots 10)$ computed from the current theory. The ratios of the natural frequencies $\frac{\omega_i^a}{\omega_i^e}$ $(i = 1, 2, 3, \dots .10)$ are shown in Table 1 for different number of elements used in Przemieniecki's theory and the FEM. The ratios for the latter are shown in the parenthesis. It can be observed that Przemienicki's theory which retains only two terms in the frequency-dependent mass and stiffness matrices yields much better results than those from the FEM, particularly with increasing number of elements. For instance, when using 10 elements (N = 10), Przemieniecki's theory gives around 5% error in the seventh natural frequency whereas the corresponding error when using the FEM is around 16% when compared with the results computed using the present theory. As expected, the error is much larger in the FEM in relation to Przemieniecki's method, but the error is expected to grow in Przemieniecki's method when computing higher order natural frequencies accurately. Thus, for free vibration analysis in the high frequency range, as demanded by the statistical energy method [3-6], Przemieniecki's theory [2] may not be adequate, let alone the FEM, when the usefulness of the present theory will become apparent.

Comparing results with the present (exact) theory, Fig. 5 shows percentage errors incurred in the fifth natural frequency of the cantilever bar when using the FEM [34, 35] and Przemieniecki's method [2] for different values of the number of elements (N) used in the analysis. Although the error decreases with increasing number of elements, it is expected that in the high frequency range, the error may become unacceptably large.

Table 1. The first ten natural frequency ratios $\frac{\omega_i^a}{\omega_i^e}$ $(i = 1, 2, 3, \ldots .10)$ for a cantilever bar in axial or longitudinal vibration using Przemieniecki's method [2] and conventional finite element method [34, 35] for a range of number of elements (*N*) used in the analysis.

i	Natural frequency ratio $\frac{\omega_i^a}{\omega_i^e}$				
	N = 2	*N* = 4	*N* = 6	*N* = 8	*N* = 10
1	1.00146 (1.02586)	1.00009 (1.00644)	1.00002 (1.00286)	1.00001 (1.00161)	1.000002 (1.00103)
2	1.06874 (1.19458)	1.00674 (1.05832)	1.00146 (1.02586)	1.00048 (1.01451)	1.00019 (1.00928)
3		1.04053 (1.15348)	1.00992 (1.07191)	1.00341 (1.04048)	1.00146 (1.02586)
4		1.08296 (1.19145)	1.03233 (1.13649)	1.01184 (1.07918)	1.00522 (1.05080)
5			1.06874 (1.19458)	1.02861 (1.12791)	1.01311 (1.08369)
6			1.07458 (1.17279)	1.05437 (1.17693)	1.02650 (1.12278)
7				1.08042 (1.20139)	1.04588 (1.16329)
8				1.06603 (1.15942)	1.06874 (1.19458)
9					1.08363 (1.19781)
10					1.05928 (1.15006)

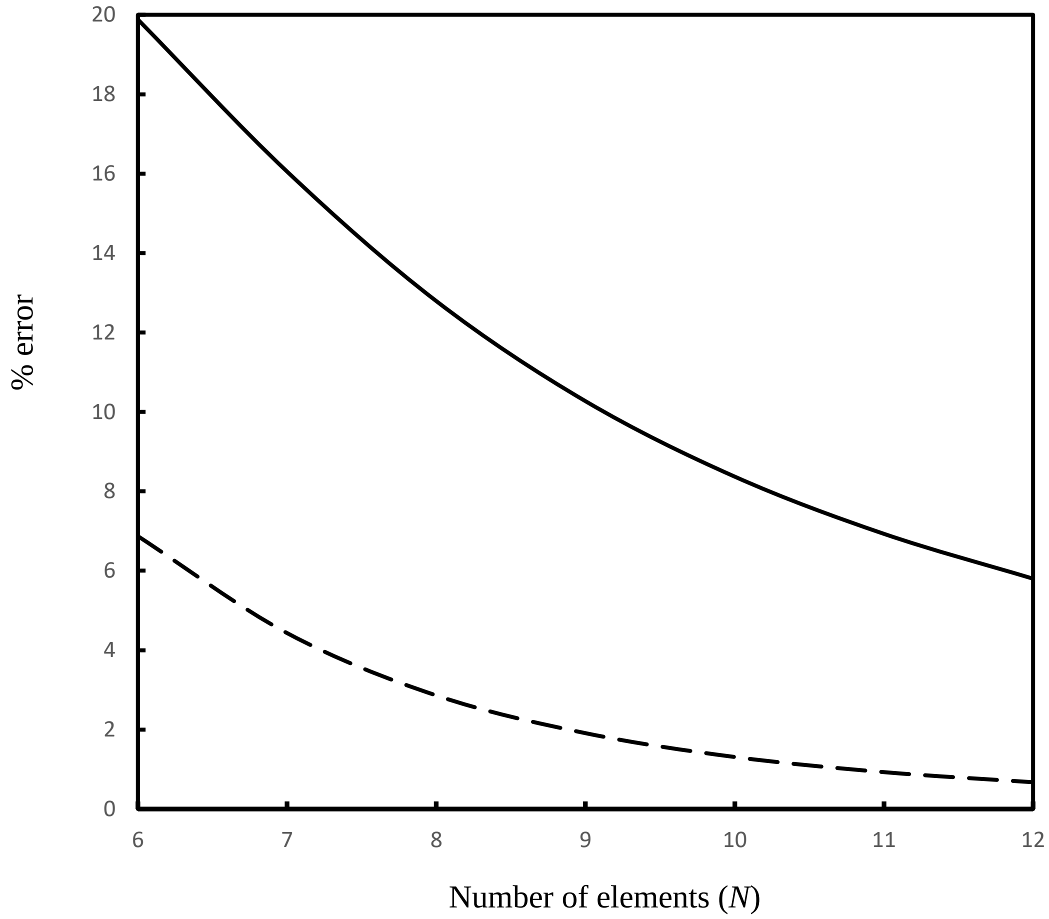

Fig. 5. The effect of the number of elements on the percentage error incurred in the fifth longitudinal (axial) natural frequency of a cantilever bar using the finite element method [34, 35] and Przemieniecki's method [2] when compared to the present (exact) method.

———————— Finite element method; ---------- Przemieniecki's method.

The next set of results were computed for the first fifteen natural frequencies of a cantilever beam in bending or flexural vibration (ω_i^b; $i = 1, 2, 3, \ldots .15$) using Przemieniecki method [2] and FEM [34, 35] with N number of elements where N was taken to be 2, 4, 6, 8 and 10. Using only one element the same number of natural frequencies were computed using the present theory (ω_i^e; $i = 1, 2, 3, \ldots .15$) which gave exact results. The ratios of the natural frequencies $\frac{\omega_i^b}{\omega_i^e}$ $(i = 1, 2, 3, \ldots .15)$ are shown in Table 2 for different values of the number of elements (N) used in Przemieniecki's method and FEM. The ratios for the latter are shown in the parenthesis.

Table 2. The first fifteen natural frequency ratios $\frac{\omega_i^a}{\omega_i^e}$ $(i = 1, 2, 3, \ldots .15)$ for a cantilever beam in bending of flexural vibration using Przemieniecki's method [2] and FEM [34, 35] for a range of number of elements (*N*) used in the analysis.

Frequency No (*i*)	Natural frequency ratio $\frac{\omega_i^b}{\omega_i^e}$				
	N = 2	*N* = 4	*N* = 6	*N* = 8	*N* = 10
1	1.00000 (1.00048)	1.00000 (1.00003)	1.00000 (1.00000)	1.00000 (1.00000)	1.00000 (1.00000)
2	1.00043 (1.00849)	1.00000 (1.00123)	1.00000 (1.00025)	1.00000 (1.00008)	1.00000 (1.00003)
3	1.06309 (1.21816)	1.00021 (1.00774)	1.00001 (1.00183)	1.00000 (1.00061)	1.00000 (1.00025
4	1.50349 (1.80430)	1.00128 (1.01452)	1.00014 (1.00644)	1.00002 (1.00224)	1.00000 (1.00095)
5		1.03017 (1.14149)	1.00083 (1.01503)	1.00011 (1.00579)	1.00002 (1.00252)
6		1.08012 (1.22721)	1.00172 (1.01666)	1.00048 (1.01189)	1.00009 (1.00540)
7		1.18967 (1.39295)	1.02320 (1.12238)	1.00148 (1.01987)	1.00032 (1.00997)
8		1.44339 (1.71670)	1.04365 (1.15794)	1.00197 (1.01767)	1.00091 (1.01630)
9			1.08602 (1.23191)	1.02026 (1.11394)	1.00204 (1.02308)
10			1.15155 (1.3399)	1.03133 (1.13114)	1.00213 (1.01317)
11			1.21888 (1.43617)	1.05409 (1.17456)	1.01866 (1.10931)
12			1.38746 (1.65086)	1.08877 (1.23421)	1.02541 (1.11724)
13				1.13538 (1.30792)	1.03993 (1.14633)
14				1.18611 (1.38576)	1.06135 (1.18616)
15				1.21052 (1.43148)	1.09032 (1.23553)

Clearly Przemieniecki's method though approximate unlike the present method, gives much better results than the FEM. For instance, the error incurred using 8 elements in Przemieniecki's method is around 3% in the tenth natural frequency whereas the corresponding error in FEM using the same number of elements is around 13%. In comparison with the current exact method, Fig. 6 shows the percentage error in the fifth natural frequency of the cantilever beam as a function of the number of elements used in the analysis when applying Przemieniecki's method and FEM, respectively.

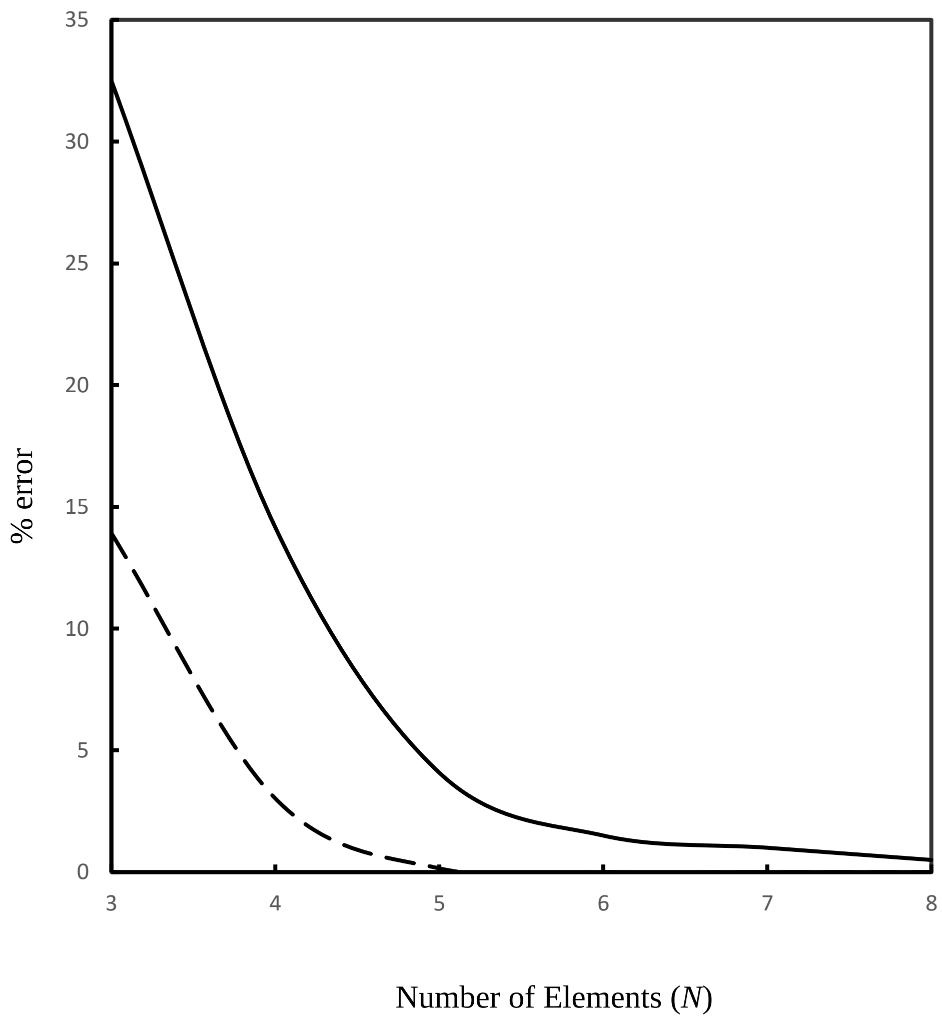

Fig. 6. The effect of the number of elements on the percentage error incurred in the fifth bending natural frequency of a cantilever beam using the finite element method [34, 35] and Przemieniecki's method [2] when compared to the present (exact) method.

———————— Finite element method; − − − − − − − − − − Przemieniecki's method.

The final set of results was obtained for a three-step beam shown in Fig. 7 for which three degrees of freedom (U, W and Θ) are allowed at each node. Each of the three components AB, BC and CD of the beam has hollow circular cross-section with outer diameters d_1, d_2 and d_3, thicknesses t_1, t_2 and t_3 and lengths L_1, L_2 and L_3, respectively.

The data used in the analysis are as follows:

$d_1 = 0.25$ m, $d_2 = 0.20$ m, $d_3 = 0.30$ m, $t_1 = t_2 = t_3 = 0.01$ m, $L_1 = L_2 = L_3 = 1$ m

The material used is steel with Young's modulus $E = 200$ GPa and density $\rho = 7850$ kg/m^3

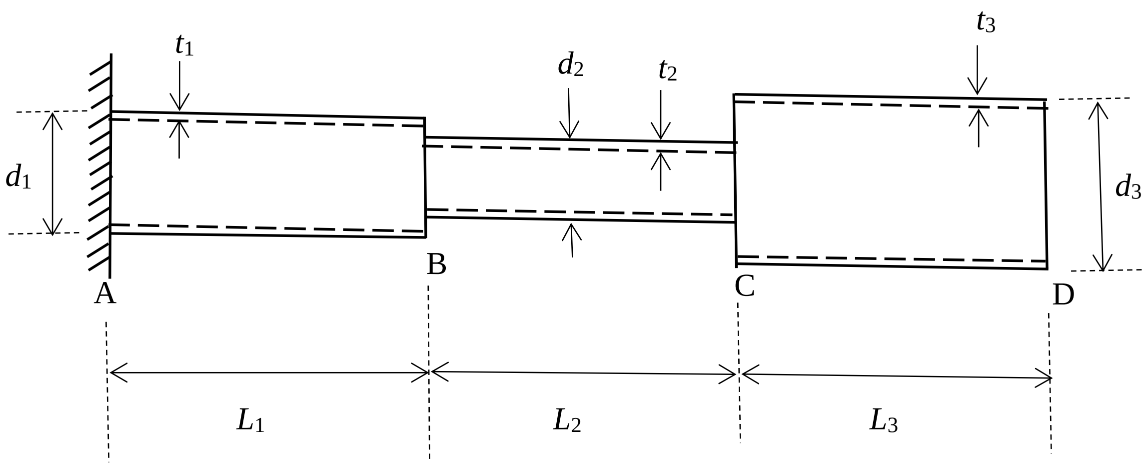

Fig. 7. A three-step beam with hollow circular cross-section and with cantilever boundary condition.

Two types of idealisation were used in the analysis when using FEM [34, 35] and Przemieniecki's method [2]. In the first case, each of the three components AB, BC and CD of the beam is idealised as one element so that the total number of elements N is 3 whereas in the second case, each of the three components AB, BC and CD is split into two elements of equal length so that the total number of elements N becomes 6. Exact results were computed applying the present theory and by using one element for each component of the beam, i.e., $N = 3$.

The first five natural frequencies of the three-step beam using N = 3 and N =6 when using Przemieniecki's method [2] and FEM [34, 35] are shown in Table 3 alongside the results computed using the present (exact) theory. The axial and bending natural frequencies are identified by letters A and B as shown in the table. The percentage errors incurred when using Przemieniecki's method and FEM are shown in the parenthesis. Clearly, the 6-element idealisation predicts the first five natural frequencies accurately by both FEM and Przemieniecki's method, as shown in the table. However, when 3-element idealisation is used, the error in FEM can be quite large in comparison with Przemieniecki's method. For example, the error in fifth natural frequency for this idealisation in FEM is 21.8% whereas in Przemieniecki's method it is around (6.24%).

Table 3. Natural frequencies of a three-step cantilever beam using finite element method, Przemieniecki's method and the current method.

i	ω_i (rad/s)	Mode type	Finite element [34, 35] and Przemieniecki's [2] methods				Present method (exact)
			3-Element Idealisation (N= 3)		6-Element Idealisation (N= 6)		
			FEM	Przemieniecki	FEM	Przemieniecki	
1	ω_1	B	144.42 (0.03%)	144.38 (0.00%)	144.40 (0.02)	144.38 (0.00%)	144.38
2	ω_2	B	916.04 (0.36%)	912.70 (0.00%)	912.98 (0.03%)	912.65 (0.00)	912.65
3	ω_3	A	2494.6 (1.00%)	2470.5 (0.02%)	2476.1 (0.25%)	2469.9 (0.00)	2469.9
4	ω_4	B	2814.8 (1.12%)	2785.1 (0.05%)	2788.8 (0.18%)	2783.1 (0.02%)	2783.7
5	ω_5	B	7069.6 (21.8%)	6168.6 (6.24%)	5848.9 (0.73%)	5814.9 (0.15%)	5806.5

5. Conclusions

The frequency dependent mass and stiffness matrices for bar and beam elements are derived in explicit algebraic forms using the exact shape functions obtained from the exact solutions of the governing differential equations in free vibration. The application of symbolic computation has broadened the scope of the investigation and made the derivation possible. The equivalency of the frequency dependent mass and stiffness matrices with the corresponding dynamic stiffness matrix is established which enabled the application of the Wittrick-Williams algorithm to be made when computing the natural frequencies of structures in a robust and accurate manner. The natural frequencies from the present theory are compared with those obtained from the conventional finite element theory, and Przemieniecki's theory which used approximate expressions for the frequency-dependent mass and stiffness matrices. The accuracy of the finite element method against Przemieniecki's method and the current method is assessed using illustrative numerical examples. The theory developed paves the way to account for the damping effect in free vibration and response analyses when using the dynamic stiffness method, which hitherto had not been possible. It is in the context of free vibration analysis of bars, beams and their assembles in the high frequency range as required in the statistical energy analysis method, the theory developed in this paper is expected to be most effective and useful.

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