Dynamic Stiffness Matrix of a Conical Bar using Rayleigh-Love Theory with Applications

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Abstract

Based on Rayleigh-Love theory, the dynamic stiffness matrix of a conical bar in longitudinal vibration is developed for the investigation of free vibration and response characteristics of such bars and their assemblies. First the governing differential equation of motion in free longitudinal vibration of a conical bar using Rayleigh-Love theory which accounts for the inertia effects due to transverse or lateral deformations is derived by applying Hamilton's principle. Next, for harmonic oscillation, the governing differential equation is recast in the form of Legendre's equation, providing a series solution connected by integration constants. The expressions for the amplitudes of displacements and forces are then obtained by means of the series solution. Finally, the frequency dependent dynamic stiffness matrix is formulated by relating the amplitudes of forces to those of the corresponding displacements at the ends of the conical bar and thereby eliminating the integration constants. As an established solution technique, the Wittrick-Williams algorithm is applied to the resulting dynamic stiffness matrix when computing the natural frequencies and mode shapes of some illustrative examples. The theory is also applied to investigate the response of a cantilever Rayleigh-Love conical bar with a harmonically varying load applied at the tip. The results computed from Rayleigh-Love model based dynamic stiffness theory are compared and contrasted with those computed from conventional classical theory with significant conclusions drawn.

Keywords: Rayleigh-Love theory, dynamic stiffness method, free vibration, response, conical bar, Wittrick-Williams algorithm.

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1. Introduction

Free longitudinal vibration analysis of a uniform bar using classical theory is a straightforward task which can be found in standard texts [1-3]. By using the essential basis of this theory, several investigators have analysed the free vibration behaviour of non-uniform bars [4-10]. The underlying concept described in these publications covering both uniform and non-uniform bars is solely based on longitudinal displacement and clearly the theory does not account for the transverse or lateral deformations of the bar arising from the Poisson's ratio effects. It was Lord Rayleigh [11] who first recognised the significance of the transverse or lateral inertia on the free longitudinal vibration of a bar when he advanced the classical theory. Many years later, Love [12] shed further lights on Rayleigh's theory and the bar model which includes the effects of transverse or lateral strain in the formulation of the free longitudinal vibration, is now known as Rayleigh-Love bar. However, the research on the free longitudinal vibration of Rayleigh-Love bars is mostly confined to uniform (prismatic) bars [13-16]. In a recent publication [17], the Rayleigh-Love theory for a uniform bar was further extended to include the free vibration analysis of frameworks. This was achieved by developing the dynamic stiffness method. A literature survey shows that apparently there has been only one or two attempts to study the free vibration behaviour of conical Rayleigh-Love bars [18-19]. This research is inspired by its applications in the design of foundation for which conical bars are used as idealised structures as highlighted by Meek [20-21]. It is clear from Meek's investigation that a truncated cone model to represent homogeneous soil is a satisfactory engineering idealisation to provide solutions for the foundation-dynamics and other related problems. Based on this premise, the investigation carried out in the current paper provides considerable insights into the free longitudinal vibration behaviour of conical Rayleigh-Love bars and their assemblies. This is accomplished by developing the dynamic stiffness matrix of a conical Rayleigh-Love bar from first principle. The investigation is carried out in following steps. First the governing differential equation of motion in free longitudinal vibration of a conical Rayleigh-love bar is derived by applying Hamilton's principle which required the expressions for its kinetic and potential energies. For harmonic oscillation and by making a carefully thought-out substitution, the governing differential equation is transformed into the form of the well-known Legendre's equation [22] which eventually became amenable to a series solution. In this way, the general solution of the governing differential equation is obtained in series form connected by arbitrary integration constants. The expressions for the amplitudes of axial displacement and axial force are also obtained in terms of the series solution. The dynamic stiffness matrix is then formulated by relating the amplitudes of the forces to those of the corresponding displacements at the ends of the conical Rayleigh-Love bar and thereby eliminating the arbitrary integration constants. Finally, the Wittrick-Williams algorithm [23] is applied to the ensuing dynamic stiffness matrix when computing the natural frequencies and mode shapes of conical Rayleigh-Love bars and their assemblies.

In the interest of those readers who are not familiar with the dynamic stiffness method (DSM), but maybe accustomed with the traditional finite element method (FEM), it would be instructive to give a brief description of the DSM, its working procedure and its similarities and differences with the FEM.

The free vibration problems of structures are generally solved by applying the conventional finite element method (FEM) which is well-recognised as a universal tool in solid mechanics as well as in other disciplines. Understandably the FEM has a massive volume of literature which is far too extensive to report. However, it should be recognised that the FEM is an approximate method based on chosen or assumed shape functions and the method requires discretisation of the structure into several elements, each acting as a building block. For free vibration problems, the element stiffness and mass matrices are assembled in FEM to form the overall stiffness and mass matrices of the final structure which generally leads to a linear eigenvalue formulation, yielding the natural frequencies and mode shapes of the structure. Furthermore, in the FEM, the number of natural frequencies that can be computed is somehow restricted by the order of the mass and stiffness matrices and the inaccuracy grows when computing the higher order natural frequencies and mode shapes. To circumvent this problem, there is an alternative to the FEM when solving the free vibration problem, which is not restrictive in this respect, but at the same time, the method always provides accurate results regardless of the order of the natural frequencies and mode shapes. The alternative method proposed here is the dynamic stiffness method (DSM) which is robust and yet accurate, but importantly, it can be used in a wider context in the same way as the FEM when analysing the free vibration behaviour of complex structures. The DSM, though different from the FEM, has many common features with the FEM, particularly when assembling the structural properties of individual elements. However, it should be noted that some major differences exist between the DSM and the FEM. For instance, the former is not affected by the number of elements used in the analysis and it always provides exact results whereas the latter is obviously mesh dependent. The accuracy of results in FEM depends very much on the number of elements used in the analysis whereas in the DSM, the results are independent of the number of elements. For example, one single structural element can be considered in the DSM to compute any number of natural frequencies without losing any accuracy. Obviously, this is not possible in the FEM. The exactness of the result in the DSM comes from the fact that the shape function used to derive the element dynamic stiffness matrix of a structural element is based on the exact solution of the governing differential equation of motion of the element executing free natural vibration. The element dynamic stiffness matrix is frequency dependent, comprising both the mass and stiffness properties of the element. This is in sharp contrast to the FEM for which the mass and stiffness matrices are always separate and frequency independent. A systematic procedure to formulate the dynamic stiffness matrix of a structural element is given in [24]. It should be noted that the DSM has been successfully applied to plate and shell structures and the volume of the literature is quite large, but a carefully selected sample of papers published in the past five years in these areas are appended to this paper [25-29]. In all cases, the overall frequency dependent dynamic stiffness matrix of the final structure is assembled from the dynamic stiffness matrices of all individual elements in the structure. The resulting overall dynamic stiffness matrix leads to a non-linear eigenvalue problem, usually handled by the well-established algorithm of Wittrick and William [23] when extracting the natural frequencies and mode shapes of the final structure. The DSM which is often called an *exact* method is indeed suitable for free vibration analysis in all frequency ranges.

2. Theory

Figure 1 shows in a right-handed Cartesian co-ordinate system, a linearly tapered bar of length L and of solid circular cross-section (i.e. it is a conical bar) with the X-axis coinciding with the axis of the bar. The bar tapers with a taper ratio c such that and the diameter $d(x)$, area $A(x)$ and the polar second moment of area $I(x)$ of the cross-section at a distance x from the origin are given by

$$d(x) = d_g\left(1 - c\frac{x}{L}\right); \qquad A(x) = A_g\left(1 - c\frac{x}{L}\right)^2; \qquad I(x) = I_g\left(1 - c\frac{x}{L}\right)^4 \tag{1}$$

where d_g, A_g and I_g are respectively, the diameter, area and the polar second moment of area at the thick end g of the bar which is taken to be the origin.

Clearly, the diameter at the thin-end h on the right-hand side (see Fig. 1) is given by $d_h = d_g(1 - c)$ so that the taper ratio c must lie within the range $0 \le c \le 1$. Thus, $c = 0$ represents a uniform bar whereas when $c = 1$, the bar tapers to a point at $x = L/c$ from the origin, which is the limiting case that cannot be achieved in practice.

Incorporating the Rayleigh-Love theory [17-19], the expressions for kinetic (T) and potential (V) energies of the conical bar of Fig. 1 can be written as

$$T = \frac{1}{2}\int_0^L \{\rho A(x)\dot{u}^2 + \rho\nu^2 I(x)(\dot{u}')^2\}\, dx; \qquad V = \frac{1}{2}\int_0^L EA(x)(u')^2 dx \tag{2}$$

where u is the displacement of a point on the axis of the bar at a distance x from the origin O, ρ is the density, E is the Young's modulus and ν is the Poisson's ratio of the bar material. An overdot and a prime represent differentiation with respect to time t and x, respectively. Hamilton's principle is now applied to derive the governing differential equation of the Rayleigh-Love conical bar undergoing free natural vibration.

Hamilton's principle states

$$\delta \int_{t_1}^{t_2} (T - V)dt = 0 \tag{3}$$

where δ is the variational operator and t_1 and t_2 are the time intervals of the dynamic trajectory.

Substituting T and V from Eqs. (2) and the relationships of Eq. (1) into Eq. (3), making use of the variational operator δ and finally performing integrations by parts, the governing differential equation of motion in free vibration of the conical Rayleigh-Love bar and the associated natural boundary condition which gives the expression for the axial force are obtained as follows.

Governing Differential Equation:

$$EA_g\left(1 - c\frac{x}{L}\right)^2 u'' - 2\frac{c}{L}EA_g\left(1 - c\frac{x}{L}\right)u' + \rho I_g \nu^2\left(1 - c\frac{x}{L}\right)^4 \ddot{u}'' - 4\frac{c}{L}\rho I_g \nu^2\left(1 - c\frac{x}{L}\right)^3 \ddot{u}' - \rho A_g\left(1 - c\frac{x}{L}\right)^2 \ddot{u} = 0 \tag{4}$$

Natural Boundary condition giving the expression for axial force (f)*:*

$$f(x,t) = -\left\{EA_g\left(1 - c\frac{x}{L}\right)^2 u' + \rho I_g \nu^2\left(1 - c\frac{x}{L}\right)^4 \ddot{u}'\right\} \tag{5}$$

If harmonic oscillation is assumed, then

$$u(x,t) = U(x)e^{i\omega t} \tag{6}$$

where ω is the angular or circular frequency, and $U(x)$ are the amplitudes of u.

Substitution of Eq. (6) into Eq. (4) removes the time-dependent part of the partial differential equation by replacing $\ddot{u}$ terms by $-\omega^2 U e^{i\omega t}$ and then by cancelling the $e^{i\omega t}$ term throughout will result into an ordinary differential equation in U with x as the independent variable. However, a further simplification of the differential Eq. (4) is made by replacing the independent variable x by ξ where

$$\xi = 1 - c\frac{x}{L} \tag{7}$$

In this way, the governing differential equation Eq. (4) is transformed after some effort to give.

$$\xi(1 - C_1\xi^2)U'' + 2(1 - 2C_1\xi^2)U' + C_2\xi U = 0 \tag{8}$$

where

$$C_1 = \frac{\rho I_g \nu^2 \omega^2}{EA_g};\ C_2 = \frac{\rho\omega^2 L^2}{Ec^2} \tag{9}$$

and a prime now denotes differentiation with respect to ξ.

As a result of the harmonic oscillation assumption and the change of variable from x to ξ (see Eq. (7)), the amplitude $F(\xi)$ of the force $f(x, t)$ in Eq. (5) becomes

$$F(\xi) = \frac{EA_g c}{L}\xi^2(1 - C_1\xi^2)U' \tag{10}$$

We seek solution of Eq. (8) in the following form

$$U = \frac{W(\xi\sqrt{C_1})}{\xi} = W\,(\zeta)/\xi = \sqrt{C_1}\frac{W(\zeta)}{\zeta} \tag{11}$$

where W is a function of $\xi\sqrt{C_1}$ which we define as ζ.

By substituting Eq. (11) into Eq. (8) and making some mathematical manipulation, the following governing differential equation in the form of Legendre's differential equation is obtained.

$$(1 - \zeta^2)W''(\zeta) - 2\zeta W'(\zeta) + \mu(\mu + 1)W(\zeta) = 0 \tag{12}$$

where

$$\mu = -\frac{1}{2} + \sqrt{\frac{9}{4} + \frac{C_2}{C_1}} \tag{13}$$

Note that the sign in front of the square root of Eq. (13) whether plus or minus does not really matter when seeking the solution of Eq. (12) because it will not make any difference to the coefficient $\mu(\mu + 1)$ of $W(\zeta)$.

The solution of the Legendre's equation (Eq. (12)) can now be obtained in series form which can be found in many advanced books of mathematics, see for example [22]. In this way, the solution $W(\zeta)$ of the second order ordinary differential equation Eq. (12) can be expressed in terms of two series described by functions, say $\alpha(\zeta)$ and $\beta(\zeta)$, connected by two arbitrary integration constants A_1 and A_2. Thus

$$W(\zeta) = A_1\alpha(\zeta) + A_2\beta(\zeta) \tag{14}$$

The functions $\alpha(\zeta)$ and $\beta(\zeta)$ can be expressed as [22]

$$\alpha(\zeta) = 1 - \mu(\mu + 1)\frac{\zeta^2}{2!} + (\mu - 2)(\mu + 1)(\mu + 3)\frac{\zeta^4}{4!} - \cdots \tag{15}$$

$$\beta(\zeta) = \zeta - (\mu - 1)(\mu + 2)\frac{\zeta^3}{3!} + (\mu - 3)(\mu - 1)(\mu + 2)(\mu + 4)\frac{\zeta^5}{5!} - \cdots \tag{16}$$

The coefficients a_n (n = 0, 1, 2, 3…..) of ζ^n in Eqs. (15) and (16) can be obtained by substituting a_0 = 1 and a_1 = 0 for $\alpha(\zeta)$ and a_0 = 0 and a_1 = 1 for $\beta(\zeta)$ in the following recurrence relationship [22].

$$a_{n+2} = \frac{\{n(n+1)-\mu(\mu+1)\}}{(n+1)(n+2)} a_n \tag{17}$$

With the help of Eq. (11), $U(\zeta)$ can now be expressed as

$$U(\zeta) = \frac{W(\zeta)}{\zeta}\sqrt{C_1} = \frac{\sqrt{C_1}}{\zeta}\{A_1\alpha(\zeta) + A_2\beta(\zeta)\} = \sqrt{C_1}\{A_1\phi(\zeta) + A_2\psi(\zeta)\} \tag{18}$$

where

$$\phi(\zeta) = \frac{\alpha(\zeta)}{\zeta}, \quad \psi(\zeta) = \frac{\beta(\zeta)}{\zeta} \tag{19}$$

The expression for axial force in Eq. (10) in terms of the new variable ζ now becomes

$$F(\zeta) = \frac{EA_g c}{L}\zeta^2(1-\zeta^2)\{A_1\phi'(\zeta) + A_2\psi'(\zeta)\} \tag{20}$$

Now referring to Fig. 2, the boundary conditions for the amplitudes of displacements and forces for the harmonically vibrating conical Rayleigh-Love bar can be applied as follows.

Displacements:

At $x = 0$ ($\xi = 1$ and $\zeta = \sqrt{C_1}$), $U = U_1$; At $x=L$ ($\xi = 1-c$ and $\zeta = (1-c)\sqrt{C_1}$), $U = U_2$ (21)

Forces:

At $x = 0$ ($\xi = 1$ and $\zeta = \sqrt{C_1}$), $F = F_1$; At $x=L$ ($\xi = 1-c$ and $\zeta = (1-c)\sqrt{C_1}$), $F = -F_2$ (22)

Substituting Eq. (21) into Eq. (18) gives

$$U_1 = \sqrt{C_1}\{A_1\phi(\sqrt{C_1}) + A_2\psi(\sqrt{C_1})\} \tag{23}$$

$$U_2 = \sqrt{C_1}\left[A_1\phi\{(1-c)\sqrt{C_1}\} + A_2\left\{\psi\{(1-c)\sqrt{C_1}\}\right\}\right] \tag{24}$$

Eqs (23) and (24) can be written in matrix forms as follows.

$$\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \tag{25}$$

or

$$\mathbf{U = Q\ A} \tag{26}$$

where

$$\left.\begin{aligned} Q_{11} &= \sqrt{C_1}\phi(\sqrt{C_1}) = \alpha(\sqrt{C_1}) \\ Q_{12} &= \sqrt{C_1}\psi(\sqrt{C_1}) = \beta(\sqrt{C_1}) \\ Q_{21} &= \sqrt{C_1}\phi\{(1-c)\sqrt{C_1}\} = \frac{1}{(1-c)}\alpha\{(1-c)\sqrt{C_1}\} \\ Q_{22} &= \sqrt{C_1}\psi\{(1-c)\sqrt{C_1}\} = \frac{1}{(1-c)}\beta\{(1-c)\sqrt{C_1}\} \end{aligned}\right\} \tag{27}$$

Substituting Eq. (22) into Eq. (20) gives

$$F_1 = \frac{EA_g c}{L}C_1(1-C_1)\{A_1\phi'(\sqrt{C_1}) + A_2\psi'(\sqrt{C_1})\} \tag{28}$$

$$F_2 = \frac{-EA_g c}{L}[C_1(1-c)^2\{1-C_1(1-c)^2\}]\left[A_1\phi'\left\{(1-c)\sqrt{C_1}\right\} + A_2\psi'\left\{(1-c)\sqrt{C_1}\right\}\right] \tag{29}$$

Eqs. (28) and (29) can be written in matrix form as follows

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \tag{30}$$

or

$$\mathbf{F = R\ A} \tag{31}$$

where

$$\left.\begin{aligned} R_{11} &= \frac{EA_g c}{L} C_1(1-C_1)\phi'(C_1) \\ R_{12} &= \frac{EA_g c}{L} C_1(1-C_1)\psi'(C_1) \\ R_{21} &= -\frac{EA_g c}{L}[C_1(1-c)^2\{1-C_1(1-c)^2\}]\phi'\left\{(1-c)\sqrt{C_1}\right\} \\ R_{22} &= -\frac{EA_g c}{L}[C_1(1-c)^2\{1-C_1(1-c)^2\}]\psi'\left\{(1-c)\sqrt{C_1}\right\} \end{aligned}\right\} \tag{32}$$

The constants vector **A** comprising A_1 and A_2 can now be eliminated from Eqs. (26) and (31) to give the frequency dependent dynamic stiffness matrix of an axially vibrating Rayleigh-Love cone relating amplitudes of the forces to those of the displacements at the ends, as follows:

$$\mathbf{F = K\ U} \tag{33}$$

or

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \tag{34}$$

where

$$\mathbf{K = R\ Q^{-1}} \tag{35}$$

is the required dynamic stiffness matrix.

With the help of Eqs. (25-26) and (30-31) and performing the matrix inversion and matrix multiplication steps of Eq. (35), the elements of the dynamic stiffness matrix **K** are given as follows.

$$\left.\begin{aligned} K_{11} &= (R_{11}Q_{22} - R_{12}Q_{12})/\Delta \\ K_{12} &= K_{21} = (R_{12}Q_{11} - R_{11}Q_{12})/\Delta \\ K_{22} &= (R_{22}Q_{11} - R_{21}Q_{12})/\Delta \end{aligned}\right\} \tag{36}$$

where

$$\Delta = Q_{11}Q_{22} - Q_{21}Q_{12} \tag{37}$$

and the elements $Q_{11}, Q_{12}, Q_{21}, Q_{22}$ and $R_{11}, R_{12}, R_{21}, R_{22}$ have already been defined in Eqs. (27) and (32), respectively.

3. Applications of the Dynamic Stiffness Matrix

The dynamic stiffness matrix for a conical bar developed above by using the Rayleigh-love model corresponds to axial stiffnesses only which can be combined with the flexural dynamic stiffness matrices of either a tapered [30] or a uniform [31-32] beam based on classical theories, which will enable the free vibration analysis of frameworks consisting of them. The procedure to obtain the dynamic matrix of a beam element in 2D (plane frame) requiring three degrees of freedom at each node is described in [31] whereas that of a 3D beam element (space frame) requiring six degrees of freedom can be found in [32]. Once the individual dynamic stiffness matrices of a structure are assembled to form the overall dynamic stiffness matrix, the computation of the natural frequencies follows from the application of the Wittrick-Williams algorithm [23] which has featured many times in the literature. Basically, the algorithm gives the number of natural frequencies of a structure that lie below a chosen trial frequency. This enables the computation of any natural frequency to any desired accuracy because successive trial frequencies can be chosen by the user to bracket a particular natural frequency. Once the required natural frequency is computed with the help of the Wittrick-Williams algorithm, the corresponding mode shape is recovered by using standard procedure wherein a nodal displacement of the structure is generally given an arbitrarily chosen value and then computing the rest of the nodal displacements in terms of the chosen one. The details of the algorithm are not repeated here, but interested investigators can look up in the literature and can trace back the original paper of Wittrick and Williams [23] for further insight.

4. Numerical Results and Discussion

4.1 Free vibration analysis

The theory developed above is first applied for free vibration analysis of three illustrative examples of conical Rayleigh-Love bar and then for response analysis, see section 4.2. The first example is for a fixed-fixed conical Rayleigh-Love bar for which some comparative results based on generalised hypergeometric series are available in the literature [19]. The data used in the analysis were extracted from [19] and are given below.

$$E = 70\times10^9 \text{ Pa}, \rho = 2700 \text{ kg/m}^3, \nu = 0.33, d_g = 0.30 \text{ m}, c = 2/3 \text{ and } L = 1 \text{ m}$$

The first five natural frequencies of the conical bar computed using the present theory are shown in Table 1 alongside the ones reported in [19]. The close agreement between the two sets of results is evident. The differences in the first five natural frequencies are 0.04%, 0.40%, 0.98%, 1.74% and 2.93%, respectively.

The second example is that of a cantilever conical Rayleigh-Love bar for which the first five natural frequencies and mode shapes are computed using the present theory for representative values of the taper ratios (c) and the length to thick-end radius ratio L/r_g of the conical bar. (Note that r_g is not the radius of gyration of the bar cross section, but it is the radius of the conical bar at the thick end.) Table

2 shows the results in non-dimensional form, given by λ_i $(i = 1,\ 2, 3, \ldots 5)$, where $\lambda_i = \omega_i\sqrt{\rho/(EL^2)}$, ω_i being the natural frequency in rad/s, alongside the results computed using the classical DSM theory for tapered bar [30, 32] which does not include the effects of transverse or lateral inertia (and hence the results from the classical theory are independent of the L/r_g ratio). The differences in the natural frequencies are pronounced for lower values of L/r_g and higher order natural frequencies, as expected. For instance, when L/r_g= 2 and taper ratio c = 0.5, the differences in the first five natural frequencies are 0.34, 6.8, 19, 36 and 58%, respectively. The natural frequencies computed using the present theory are lower than the ones computed using the classical theory, as expected. Representative mode shapes for c = 0.5 and L/r_g= 2 using both the present theory and the classical theory [30, 32] are shown in Fig. 3. The mode shapes corresponding to the first three natural frequencies are virtually unaltered or negligible, but the differences in the mode shapes, particularly for the fourth and fifth natural frequencies are significantly pronounced.

The final example for the free vibration analysis is that of a stepped conical bar comprising three individual components each modelled as a conical Rayleigh bar with segment lengths L_1 = 2.5m, L_2 = 1m and L_3 = 0.5m, and the diameter of each segment at the thick end d_1 = 2m, d_2 = 1.5m and d_3 = 0.5m, respectively as shown in Fig. 4. The taper ratio c for the segments are taken as 1/4, 2/3, 1/2, respectively, as indicated in the figure. The material properties used are those of aluminium with Young's modulus E = 70 GP, density ρ = 2700 kg/m^3 and the Poisson's ratio ν = 0.3. Given the practical applications related to the design of foundations [20-21], cantilever boundary condition is appropriately chosen and applied at the thickest end of the stepped conical bar, see Fig. 4. The first five natural frequencies of the problem using the current theory are shown in Table 3 together the results computed using the classical DSM theory [30, 32]. The differences in the two sets of results are 0.59%, 2.22%, 7.22, 12,47% and 13.52% in the first five natural frequencies, respectively, as can be seen in Table 3.

4.2 *Response analysis*

As the dynamic stiffness method relates the amplitudes of forces and displacements of a vibrating structure, advantage is taken thereof to carry out the response analysis of a Rayleigh-Love conical bar. Figure 5 shows a cantilever conical bar with its thick end fixed and a harmonically applied load $P = P_0 e^{i\Omega t}$ applied at its free end. For illustrative purposes, the response at the tip showing the variation of the non-dimensional tip amplitude $\frac{U_T}{U_1}$ against the non-dimensional frequency ratio $\frac{\Omega}{\Omega_1}$ is shown in Figure 6, where U_1 and Ω_1 are defined as follows,

$$U_1 = \frac{P_0 L(1-c)}{EA_g c}; \qquad \Omega_1 = \frac{\pi}{2L}\sqrt{\frac{EA_g}{\rho A_g}} = \frac{\pi}{2L}\sqrt{\frac{E}{\rho}} \tag{38}$$

Note that U_1 is the static displacement at the tip of the cantilevered cone shown in Figure 5 which is worked out using the exact static stiffness matrix given in [33] whereas Ω_1 represents the fundamental natural frequency of a uniform bar in longitudinal vibration.

5. Conclusions

The dynamic stiffness matrix of a conical bar in free longitudinal vibration is developed by using the Rayleigh-Love theory which accounts for the inertia effects arising from the transverse or lateral strains due to Poisson's ratio effects. The governing differential equation of motion which forms the fundamental basis of the dynamic stiffness formulation is derived by using Hamilton's principle, which by significant mathematical manipulation is transformed into the form of Legendre's equation. The differential equation is eventually solved in terms of series solution and the dynamic stiffness is formulated by relating the amplitudes of the forces to those of the displacements of the harmonically vibrating conical Rayleigh-Love bar. The resulting dynamic stiffness matrix is finally operated by the Wittrick-Williams algorithm to compute the natural frequencies and mode shapes of three illustrative examples of varying degrees of complexities. Some of the results are validated against published results. The theory developed is particularly helpful when carrying out the free vibration analysis of conical bars and their assemblies and it can be combined with the dynamic stiffness matrices of other structural elements. The investigation is particularly useful for free vibration analysis in the high frequency range when the traditional finite element method can become inaccurate and unreliable.

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