Review of the Dynamic Stiffness Method for Free Vibration Analysis of Beams

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Resume:

After receiving his Bachelor's and Master's Degree in Mechanical Engineering from the University of Calcutta and the Indian Institute of Technology, Kharagpur, respectively, Ranjan Banerjee joined the Structural Engineering Division of the Indian Space Research Organisation, Trivandrum in 1971 and worked there for four years, first as a Structural Engineer and then as a Senior Structural Engineer. Later in the year 1975 he was awarded a Commonwealth Scholarship to study for a PhD degree at Cranfield University where he conducted research within the technical areas of structural dynamics and aeroelasticity and received his PhD in 1978. He then joined the Structural Engineering Division of the University of Cardiff and worked there for six years first as a Research Associate and then as a Senior Research Associate to investigate the free vibration and buckling characteristics of space structures using the dynamic stiffness method. He joined City, University of London in 1985 as a Lecturer in Aircraft Structures and he was promoted to Senior Lecturer and Reader in 1994 and 1998 respectively. In March 2003 he was promoted to a Personal Chair in Structural Dynamics. To date Professor Banerjee has published 115 journal papers and 100 conference papers from his research. In recognition of his contributions, he was awarded the degree of Doctor of Science (DSc) by City, University of London in 2016.

Abstract:

The application of the dynamic stiffness method (DSM) for free vibration analysis of beams is surveyed in this paper. The historical development of the DSM that has taken place in several stages is discussed in detail with reference to the free vibration problems of beams. In particular the suitability of the DSM in solving the free vibration problems of beams through the application of the well-known Wittrick-Williams algorithm as solution technique is highlighted. The literature concerning homogeneous isotropic metallic beams for which the DSM is well established, is reviewed first and then with the rapid and on-going emergence of advanced composite materials, the development of the DSM in solving the free vibration problems of anisotropic beams is discussed. The free vibration analysis of functionally graded beams using the DSM is also highlighted. The survey covers the DSM application for free vibration analysis of a wide range of beams, including sandwich beams, rotating beams, twisted beams, moving beams, bending-torsion coupled beams, amongst others. Some aspects of the contributions made by the author and his research team are also highlighted. Finally, the future potential of the DSM in solving complex engineering problems is projected.

Keywords:

Dynamic stiffness method, Wittrick-Williams algorithm, Free vibration analysis

1.0 Introduction

The foundation of the dynamic stiffness method (DSM) was laid down by Kolousek [1-2] who introduced for the first time in the early 1940s the frequency dependent dynamic stiffness coefficients for a Bernoulli-Euler beam derived from its free vibrational response. Later, the coefficients became known as Kolousek functions in the literature. Kolousek's earlier research was subsequently included in a text book [3]. The DSM has undergone ground-breaking changes since its inception and now there are alternative forms and derivatives of the method known as continuous element method (CEM) and spectral element method (SEM). The original concept given by Kolousek enabled researchers to develop a relationship between the amplitudes of forces and displacements at the nodes of a freely vibrating structural element by means of its dynamic stiffness (DS) matrix. Essentially, the basic building block of the DSM is the DS matrix of an individual structural element which can be transformed from its local coordinate axes and assembled to form the overall dynamic stiffness matrix of the final structure in a datum or global coordinate system. At this point it should be noted that there are many similarities between the DSM with the traditional finite element method (FEM) when solving the free vibration problems of structures. Nevertheless, there are some major differences too between the two methods. The FEM is an approximate method based on chosen or assumed shape functions of the displacement field. The mass and stiffness matrices of a structural element in the FEM are derived from these shape functions which are quite obviously not exact. The displacements within the element are related to the nodal displacements with the help of the shape functions. The displacement field within an element is normally chosen or assumed as polynomial functions in terms of some arbitrary constants which are eventually eliminated through the substitution of nodal displacements to generate the shape functions relating the displacements within the element to nodal displacements. When applying the energy formulation, the potential (or strain) energy and the kinetic energy of the element are worked out which leads to the stiffness and mass matrices of the element, respectively. The element mass and stiffness matrices of all individual members in a structure are assembled in the FEM using standard procedure to form the overall stiffness (**[K]**) and mass (**[M]**) matrices of the final structure. In solving the free vibration problem, the linear eigenvalue problem of the type $\left[[\mathbf{K}] - \lambda[\mathbf{M}]\right]\{\boldsymbol{\Delta}\} = \mathbf{0}$ where $\{\boldsymbol{\Delta}\}$ is the vector of nodal displacements is usually formulated and the square roots of λ values (which are eigenvalues) give the natural frequencies of the structure. The DSM works somehow in a slightly different manner. First of all, the shape functions in DSM are not chosen or assumed, but they are obtained from the solution of the governing differential equations of motion of the element in its free vibratory motion. Therefore, unlike the FEM the shape functions in DSM are frequency dependent and as they come from the exact solution of the governing differential equations of the element if free vibration, they can be justifiably regarded as exact because there are no assumptions made en route to describe the displacement field. If there are any perceived assumptions, they are within the limits of the governing differential equations of motion. By using these so-called exact shape functions which are essentially solutions of the free vibratory motion of the element, the dynamic stiffness matrix is developed by applying the boundary conditions of the amplitudes of the harmonically varying forces and displacements at the nodes of the elements in algebraic form. This process yields a single frequency dependent element matrix called the dynamic stiffness matrix which relates the amplitudes of nodal forces and displacements. The dynamic stiffness matrix derived in this way contains both the mass and stiffness properties of the element as functions of the structural parameters as well as the frequency. The assembly procedure in the DSM is very similar to that of the FEM, but a single dynamic stiffness element matrix is assembled for each structural element instead of separate mass and stiffness matrices to form the overall frequency-dependent dynamic stiffness matrix $\mathbf{[K_D]}$ of the final structure. The eigenvalue problem is then formulated as $\mathbf{[K_D]\{\Delta\}}=0$ where $\mathbf{\{\Delta\}}$ is the vector of the amplitudes of the nodal displacements. The extract of the eigenvalues follows next. Here, a significant difference with the FEM arises. The formulation $\mathbf{[K_D]\{\Delta\}}=0$ leads to a transcendental (nonlinear) eigenvalue problem in sharp contrast to the linear eigenvalue problem encountered in the FEM. The most suitable technique to extract the eigenvalues in the DSM is to apply the algorithm of Wittrick and Williams, known as Wittrick-Williams algorithm in the literature which has been highlighted in literally hundreds of papers. The algorithm is robust because it monitors the Sturm sequence property of the dynamic stiffness matrix in such a way that no natural frequency of the structure is missed. The Wittrick-Williams algorithm has become a crucial tool for free vibration analysis of structures using the DSM. The algorithm which will be discussed later can also be applied to solve (elastic) buckling problems, but the emphasis in this paper is on the solution of free vibration problems. Following the pioneering contributions of Kolousek [1-3], Williams and Wittrick [4] made noteworthy contribution for free vibration analysis of skeletal structures and, of course, very importantly, they developed what later known as the Wittrick-Williams algorithm [5-6]. Since then the DSM has continued to enjoy a sustained period of developments [7-85] and certainly, it has now reached a high degree of maturity. This paper is planned to give a general account of these continuing developments of the DSM when solving the free vibration problems of beams made of both isotropic and anisotropic materials, including the contributions made by the author and his co-authors. The paper is structured as follows. Following the introduction of this section (Section-1), Section-2 gives an account of the historical developments of the DSM for beams made of both isotropic and anisotropic (composite) materials. The contributions made by the author and his co-authors towards the DSM development to address the beam vibration problems are given in Section-3 and Section-4 deals with a brief description of the Wittrick-Williams algorithm. The scope for future work in developing the DSM further is highlighted in Section-5. Finally, Section-6 concludes the paper with some remarks.

2.0 Dynamic Stiffness Formulations for Beams

The derivation of the dynamic stiffness matrix of a structural element follows a methodical and systematic procedure. There are essentially four main steps needed to derive the dynamic stiffness matrix of a structural element. First, the governing differential equation of motion of the structural element in free vibration must be derived by using either Newton's law or Lagrange's equation or Hamilton's principle. Hamilton's principle is preferred because it gives natural boundary conditions which are essential in dynamic stiffness formulation. In the second step, the differential equation needs to be solved in an exact sense in terms of some arbitrary constants. In this step, it is necessary to obtain all expressions for displacements and forces in explicit algebraic form in terms of the integration constants. In the third step, boundary conditions for displacements and forces at the nodes of the element are applied algebraically. Thus, if **{δ}** is the displacement vector comprising the amplitudes of nodal displacements and **{f}** is the force vector comprising the amplitudes of the nodal forces of the element, then the applications of the boundary conditions for displacements and forces will give the matrix relationships **{δ}=[A]{C}** and **{f}=[B]{C}**, respectively where **{C}** is the unknown constant vector and matrices **[A]** and **[B]** are frequency-dependent square matrices already known from the element mass and stiffness properties and other structural parameters of the element. In the fourth and final step, the constant vector **{C}** is eliminated from the two matrix relationships shown above to give $\mathbf{\{f\}=[k_D]\{\delta\}}$ where $\mathbf{[k_D] = [B][A]^{-1}}$ is the required frequency dependent dynamic stiffness matrix. The complete dynamic stiffness formulation process can be automated by taking advantage of symbolic computation wherever possible. In essence upon elimination of the constants from the solution of the governing differential equation of motion of the element undergoing free vibration, the dynamic stiffness matrix $\mathbf{[k_D]}$ of the element is obtained, relating the amplitudes of forces to those of the displacements at the nodes of the element.

2.1 Metallic Beams

Inspired by Kolousek's pioneering work [1-3], Williams and Wittrick developed their algorithm [5, 6] and applied it to investigate the free vibration behaviour of skeletal structures using dynamic stiffness matrix of a Bernoulli-Euler beam [4, 7]. Successive further developments followed in that the dynamic stiffness matrices of an axially loaded Timoshenko beam [8, 9] were published. Fortran based computer programs using the DSM developed by Akesson [10] and Williams and Howson [11] to carry out the free vibration analysis of plane frames became available. These earlier works in DSM led to the development of the computer program BUNVIS-RG [21] which can handle space frame structures very efficiently when investigating their free vibration and buckling behaviour. BUNVIS-RG has many useful features such as sub-structuring, options for the inclusion of spring/lumped mass, eccentric connections, tapered members and several others. It is significant to note that exact dynamic stiffness matrices for linearly tapered beams developed by Banerjee and Williams using Bessel functions [19, 20] were coded in BUNVIS-RG [21].

The next breakthrough in the DSM development came in the 1980s when the dynamic stiffness matrix of a bending-torsion coupled beam [13, 17, 18, 25, 28] was published. This development is of considerable importance because of its applications in aeronautical engineering, particularly in aeroelastic research. A high aspect ratio aircraft wing such as that of a transport aircraft wing can be modelled as an assembly of bending-torsion coupled beams when carrying out modal and flutter analyses. The development of the DSM for bending-torsion coupled beam was significantly enhanced in later years by including the effects of shear deformation and rotatory inertia [30], axial load [31], as well as the inclusion of the warping and axial constraint effects [46]. In a parallel investigation the dynamic stiffness matrix of an axially loaded coupled Timoshenko beam was developed [38]. Subsequently, the dynamic stiffness matrices of tapered beams [19, 20, 26], rotating beams [51, 55, 66, 78, 83], twisted beams [54, 60], sandwich beams [59, 64, 70], spinning beams [61], moving beams [69], cracked beams [75, 84] and functional graded beams [79, 85] were published. Following the development of the unified formulation conceived by Carrera, generally known as CUF (Carrera Unified Formulation) which captures the cross-sectional deformation of a beam in a three-dimensional sense, the DSM in conjunction with CUF was applied [80] to give natural frequencies and mode shapes of a beam which was no longer considered as one-dimensional, but it deformed in its cross-sectional plane.

2.2 Composite Beams

The progressive growth of advanced composite materials during the past decades, particularly the fibre reinforced plastic materials has been phenomenal which fuelled the DSM development to enter into a new and exciting phase, particularly from an aeroelastic standpoint. The dynamic stiffness matrix of a simple flat laminated composite beam which exhibits material coupling between bending and torsional motions due to the fibre orientation was developed [42] which was further enhanced to include the effects of shear deformation and rotatory inertia [43, 45] as well as the additional effect of an axial load [49]. In these developments, only the material coupling arising from the anisotropic nature of fibrous composites was considered. Further enhancement of the dynamic stiffness formulation to include the effect of geometric coupling which occurs due to the geometrical configuration of the cross-section took place [73] soon after. This development is important for aeronautical applications, for example, the centroid and shear centre in an aircraft wing are generally non-coincident giving rise to geometric coupling. This research was exploited to advantage when investigating the aeroelastic optimisation of composite wings [47].

The fundamental basis for the development of the dynamic stiffness matrix of a structural element stems from its governing differential equation of motion in free vibration. The quality of the dynamic stiffness matrix depends primarily on the differential equation itself which can be derived using standard techniques such as Newton’s second law, Lagrange’s equation or Hamilton’s principle, as discussed earlier. A few illustrative examples of the differential equations derived for different types of composite beam elements of uniform cross-section are discussed next. The degree of complexity of the composite beam models is identified by their respective governing differential equations in free vibration. The procedure followed to describe the equations given in the sections below is justifiably designed to lead the reader from easy to hard, i.e. from simpler to complex composite beam elements. The dynamic stiffness formulation which proceeds from the solution of the governing differential equations is similar for all cases and therefore, not elaborated, but attention is focused on the governing differential equations from which the DSM basically originates.

2.2.1 Materially Coupled Composite Beams

Figure 1 shows a composite beam made from a laminate with a given lay-up or stacking sequence. This is the simplest example of a composite beam which maybe thought of as a strip cut from a laminated composite plate. This flat composite beam will exhibit material coupling between the bending and torsional motions because of the effects of ply orientation. Such a composite beam is referred to as Materially Coupled Composite Beam (MCCB) in this paper, which is characterised by its bending stiffness EI, torsional stiffness GJ and importantly, the bending-torsional material coupling rigidity K which is of great significance but is non-existent in metallic beams. Note that K which depends on ply orientation can be exploited to advantage to produce desirable dynamic or aeroelastic effects. There have been several attempts by investigators to obtain theoretical and experimental values of EI, GJ and K for composite beams of different cross-sections. The governing differential equation of the MCCB in free vibration is given by [42]

$$EIh'''' + K\psi''' + m\ddot{h} = 0 \tag{1}$$

$$GJ\psi'''' + Kh''' - I_\alpha \ddot{\psi} = 0 \tag{2}$$

where m is the mass per unit length, I_α is the polar mass moment of inertia per unit length about the Y axis of the beam, h is the bending displacement in the Z direction, ψ is torsional rotation about the Y axis, with primes and dots denote differentiation with respect to position and time, respectively. Equations (1) and (2) have been used in the literature for free vibration and flutter analysis of composite wings and they form the fundamental basis for the development of the dynamic stiffness matrix of an MCCB.

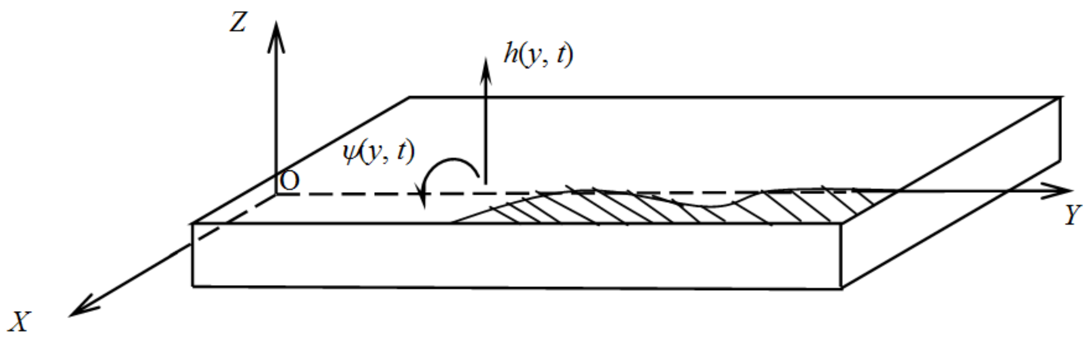

Figure 1 The co-ordinate system and notation for a materially coupled composite beam (MCCB)

2.2.2 Materially Coupled Composite Timoshenko Beam

The MCCB model given by Equations (1) and (2) can be substantially enhanced by taking into account the effects of shear deformation and rotatory inertia so that a Materially Coupled Composite Timoshenko Beam (MCCTB) model can be realised for which the governing differential equations of motion in free vibration are given by [45]

$$EI\theta'' + kAG(h' - \theta) + K\psi'' - \rho I\ddot{\theta} = 0 \tag{3}$$

$$kAG(h'' - \theta') - m\ddot{h} = 0 \tag{4}$$

$$GJ\psi'' + K\theta'' - I_\alpha \ddot{\psi} = 0 \tag{5}$$

where ρ is the density of the composite material, I is the second moment of area of the beam cross-section about the X axis (see Figure 1), kAG is the shear rigidity of the composite beam, h is the bending displacement, ψ is torsional rotation, and θ is the angle of rotation, in radians, of the cross-section about the X axis due to bending alone. All other beam parameters have already been defined before. Equations (3)-(5) are the prerequisites to develop the dynamic stiffness matrix of an MCCTB.

2.2.3 Axially Loaded Materially Coupled Composite Timoshenko Beam

The MCCTB model given above can now be further improved by taking into account an axial load (P) applied through the centroid of the cross-section to give a model which can be described as an Axially Loaded Materially Coupled Composite Timoshenko Beam (ALMCCTB). The governing differential equations in free vibration for this model which has a constant compressive axial load P (Note that P can be positive or negative so that tension is included.) are given by [49]

$$EI\theta'' + kAG(h' - \theta) + K\psi'' - \rho I\ddot{\theta} = 0 \tag{6}$$

$$kAG(h'' - \theta') - Ph'' - m\ddot{h} = 0 \tag{7}$$

$$GJ\psi'' + K\theta'' - P(I_\alpha / m)\psi'' - I_\alpha \ddot{\psi} = 0 \tag{8}$$

The three composite beam models MCCB, MCCTB and ALMCCTB described above are useful and maybe satisfactory on many occasions when the coupling between the bending and torsional deformations arises solely from the anisotropic nature of fibrous composites i.e. from the ply orientation. It should be noted that in all of the above three composite beam models, the bending-torsion coupling will be generally prevalent under the application of both static and dynamic loads.

However, there is another type of bending-torsion coupling that can occur which is generally termed as geometric coupling. This coupling is different and independent of the material coupling and it arises from the geometry of the cross-section. This type of coupling can occur both in metallic and composite beams. The free vibration characteristics of metallic beams exhibiting geometric coupling has been investigated by many authors [13, 17, 18] including the present author [25]. The origin of this coupling is based on geometric consideration when the centroid (or mass centre) and the shear centre of a beam cross-section (which are unique points on the cross-section) are non-coincident, i.e. they have different positions on the cross-section. The locus of centroids along the length of the beam is known as the mass axis of the beam whereas the locus of shear centres is known as the elastic or flexural axis of the beam. These two axes (approximated by straight lines) are determined solely by consideration of the cross-sectional geometry of the beam. For many practical structures such as aircraft wings and helicopter blades, they do not generally coincide. The distance between the centroid and the shear centre (or the distance between the elastic and mass axes), usually denoted by x_α introduces inertial coupling during the dynamic deformation. A composite wing with asymmetric cross-section can exhibit both material and geometric coupling between bending and torsional deformations. The basic model for this beam can be termed as Materially and Geometrically Coupled Composite Beam (MGCCB) which is discussed in the next section.

2.2.4 Materially and Geometrically Coupled Composite Beam

In a right handed Cartesian coordinate system Figure 2 shows a Materially and Geometrically Coupled Composite Beam (MGCCB) which is that of a uniform composite wing. The governing differential equations of motion are given by [73]

$$EIh'''' + K\psi''' + m\ddot{h} - mx_\alpha \ddot{\psi} = 0 \tag{9}$$

$$GJ\psi'' + Kh''' - I_\alpha \ddot{\psi} + mx_\alpha \ddot{h} = 0 \tag{10}$$

where x_α is the distance between the mass and elastic axes as shown in Figure 2 and the rest of the parameters have already been defined earlier.

The above equations can be solved in an exact manner and the boundary conditions can be imposed to derive the dynamic stiffness matrix of an MGCCB model in order to investigate its free vibration characteristics.

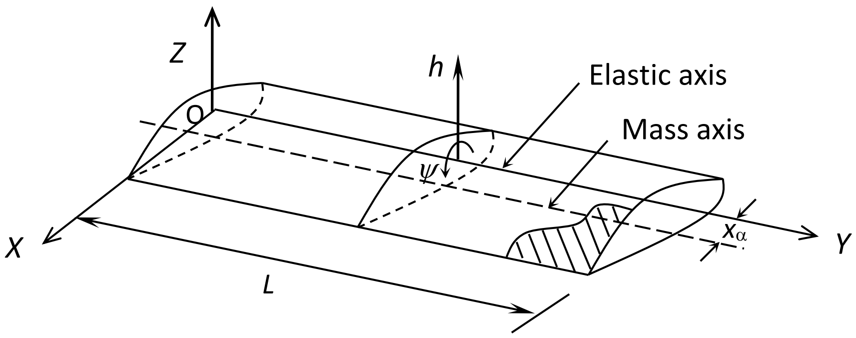

Figure 2 The co-ordinate system and notation for a geometrically and materially coupled composite beam (GMCCB)

2.2.5 Materially and Geometrically Coupled Composite Timoshenko Beam

The effects of shear deformation and rotatory inertia can be included in the above MGCCB model to realise a Materially and Geometrically Coupled Composite Timoshenko Beam (MGCCTB). This topic does not appear to have received a wide coverage in the literature. The governing differential equations of motion for its free vibratory motion have been derived by the author and they are given by

$$EI\theta'' + kAG(h' - \theta) + K\psi'' + \rho I\ddot{\theta} = 0 \tag{11}$$

$$kAG(h'' - \theta') - m(\ddot{h} - x_\alpha \ddot{\psi}) = 0 \tag{12}$$

$$GJ\psi'' + K\theta'' - I_\alpha \ddot{\psi} + mx_\alpha \ddot{h} = 0 \tag{13}$$

The parameters used in Equations (11)-(13) have all been defined earlier and now the formulation of the dynamic stiffness matrix of the MGCCTB can proceed in the usual way.

2.2.6 Axially Loaded Materially and Geometrically Coupled Composite Timoshenko Beam

An axial load (P) can be additionally applied to the MGCCTB model to provide a model for an Axially Loaded Materially and Geometrically Coupled Composite Beam (ALMGCCB). The governing differential equations of motion in free vibration (P positive when compressive) obtained by the author are given by

$$-m\ddot{h} + mx_\alpha \ddot{\psi} - Ph'' + Px_\alpha\psi'' + kAG(h'' - \theta') = 0 \tag{14}$$

$$mx_\alpha\ddot{h} - I_\alpha \ddot{\psi} + Px_\alpha h'' - P(I_\alpha/m)\psi'' + K\theta'' + GJ\psi'' = 0 \tag{15}$$

$$-\rho I\ddot{\theta} + EI\theta'' + K\psi'' + kAG(h' - \theta) = 0 \tag{16}$$

The dynamic stiffness matrix can now be established using the procedure described above.

2.2.7 Dynamic Stiffness Development for Composite Beams Using Carrera Unified Formulation

In a recent publication, the dynamic stiffness matrix of a composite beam has been developed by Pegani et al [82] by using Carrera unified formulation (known as CUF in the literature), which captures the cross-sectional deformations of the beam when it is undergoing free vibration. Within the framework of CUF, a three dimensional displacement field is chosen in the form of Taylor series expansion of the generalised coordinates when developing the dynamic stiffness matrix. For a detailed understanding of the application of CUF in dynamic stiffness formulation, see Refs [80-82].

3.0 Contributions from the Author and his Co-authors

The author and his co-authors have been developing the dynamic stiffness matrices for beams made of both isotropic and anisotropic materials for more than 30 years, which included tapered beams [19, 20], twisted beams [54, 60], bending-torsion [25, 28, 30, 31, 38, 46] and extension-torsion coupled beams [37], sandwich beams [59, 64, 70], rotating beams [51, 55, 66, 78, 83], moving beams [69], spinning beams [61, 65], cracked beams [75], functionally graded beams [79, 85] and beams carrying single and multi-degree of freedom systems [58, 77].

3.1 Isotropic Beams

The dynamic stiffness research carried out by the author and his co-authors is briefly summarized as follows. Reference [19] gives explicit expressions for dynamic stiffness elements of tapered beams for which the governing differential equations in free vibration were solved using Bessel functions. The corresponding (static) stiffness coefficients for buckling analysis were reported in [20]. One of the most significant progresses made was the development of the dynamic stiffness matrix of a bending-torsion coupled beam [25] based on which the code [28] was developed to analyse an aircraft wing. Further work on the dynamic stiffness development of bending-torsion coupled beams involved the inclusion of the important effects of shear deformation and rotatory inertia [30] and an axial load [31]. These efforts culminated in the development of a unified dynamic stiffness theory for a coupled beam, which combined the effects of shear deformation, rotatory inertia and an axial load in a unitary manner [38]. Reference [48] illustrates the general dynamic stiffness development procedure in its entirety. This comprehensive approach was later used to develop the dynamic stiffness matrix of twisted beams using Bernoulli-Euler [54] and Timoshenko [60] theories. The applications of this research include compressor and helicopter blades, amongst others. In order to improve the accuracy and computational efficiency of the structural analysis of sandwich beams, the dynamic stiffness properties of a range of sandwich beams using different theories were developed [59, 64] and importantly the theories were validated by experimental results [70]. These investigations were significant and some of the most interesting modal deformations were captured by using a combination of light and heavy materials such as rubber, aluminium, steel and lead in the core and face materials, respectively. References [58, 77] report the dynamic stiffness development of beams coupled with spring-mass systems which has several practical applications. For instance, the prediction of human-structure interactions and also for the solution of frequency attenuation problems, this research is relevant.

A programme of research was initiated by the author and his research team to formulate the dynamic stiffness matrices of rotating structural elements. Their research was published for centrifugally stiffened beams [51, 55] using Bernoulli-Euler and Timoshenko beam theories, respectively, accounting for an outboard force at the free end, making the applications sufficiently general. The results reported in these papers demonstrated the effects of rotational speed, hub radius and other beam parameters including slenderness ratios on the dynamic behaviour of rotating beams. Subsequent research led to further developments of the dynamic stiffness method for free vibration analysis of rotating tapered beams [66, 78]. The types of taper considered covered a majority of practical cross-sections. The variations of natural frequencies and mode shapes in tapered beams reported in these papers would enable designers to make some engineering judgment as to the suitability of distributing strength and stiffness and hence saving mass and accommodating aesthetic considerations. The centrifugal force in a rotating beam induces tension which increases the stiffness properties and hence has a stabilizing effect, whereas for a spinning beam the effect can be counter-productive as the advancing and retreating modes can be very different. The latter causes instability and there is a critical spinning speed at which the natural frequency tends to zero and the beam becomes unstable. This stimulated an in-depth research towards the development of the dynamic stiffness matrices for spinning beams [61]. In a similar but different context, the instability of beams in dynamic motion was captured again when the dynamic stiffness matrix of a moving beam was formulated [69]. Practical applications of the dynamic stiffness theories for moving beams include chain drives, belt drives and robotics amongst others. These investigations are significant because there will be always a critical moving speed at which a moving beam can become unstable.

3.1 Anisotropic Beams

Advanced composite materials which are anisotropic by their very nature have continued to make headways in structural analysis and design. In this respect, the dynamic stiffness development has kept pace with these development.

Reference [37] reports the extensional-torsional vibration behaviour of a composite beam using the dynamic stiffness method. As a result of ply orientation composite structures exhibit directional properties and the paper demonstrates these effects on the natural frequencies of extension-torsion coupled composite beams. As in all cases with dynamic stiffness formulation, higher natural frequencies and mode shapes can be computed from the theory without the need for further discretisation of the structure and, importantly, without any loss of accuracy in the analysis. This, in sharp contrast to finite element and other approximate methods, is significant, particularly from a computational standpoint because research in areas such as aeroelastic tailoring is generally computer intensive.

It has long been recognised that the effects of shear deformation and rotatory inertia, which are relatively less important for metallic structures, may have significant effects on the free vibration characteristics of composite structures which generally have very low shear moduli. A number of investigations on composite beams using the dynamic stiffness method were carried out to examine the effects of shear deformation, rotatory inertia and axial load [45, 49]. Further research was instigated on the dynamic stiffness development of composite beams including spinning beams [65] and aircraft wings [73]. Although the application of advanced composite materials is overwhelmingly promising, there are however some potential problems, particularly associated with the delamination of composite laminates. Recently developed functionally graded materials for which the properties vary continuously using a predetermined formula have no such problems and thus the dynamic stiffness matrices of functionally graded beams using Bernoulli-Euler [79] and Timoshenko [85] theories were formulated.

4. The Wittrick-Williams Algorithm

An accurate and reliable method of calculating the natural frequencies and mode shapes of a structure using the dynamic stiffness method is to apply the well-known algorithm of Wittrick and Williams [5, 6] which has featured in numerous papers [15, 35]. Before applying the algorithm the dynamic stiffness matrices of all individual elements in a structure are to be assembled to form the overall dynamic stiffness matrix $\boldsymbol{K}_f$ of the final (complete) structure, which may, of course, consist of a single element. The algorithm monitors the Sturm sequence condition of $\boldsymbol{K}_f$ in such a way that there is no possibility of missing a frequency (or mode) of the structure. This is, of course, not possible in the conventional finite element method. The algorithm (unlike its proof) is very simple to use. However, the procedure is briefly summarised as follows.

Suppose that ω denotes the circular (or angular) frequency of a vibrating structure. Then according to the Wittrick-Williams algorithm [5], j, the number of natural frequencies passed, as ω is increased from zero to ω^*, is given by

$$j = j_0 + \mathrm{s}\{\boldsymbol{K}_f\} \tag{17}$$

where $\boldsymbol{K}_f$, the overall dynamic stiffness matrix of the final structure whose elements all depend on ω, is evaluated at $\omega = \omega^*$; $s\{\boldsymbol{K}_f\}$ is the number of negative elements on the leading diagonal of $\boldsymbol{K}_f^{\Delta}$, $\boldsymbol{K}_f^{\Delta}$ is the upper triangular matrix obtained by applying the usual form of Gauss elimination to $\boldsymbol{K}_f$, and j_0 is the number of natural frequencies of the structure still lying between $\omega = 0$ and $\omega = \omega^*$ when the displacement components to which $\boldsymbol{K}_f$ corresponds are all zeros. (Note that the structure can still have natural frequencies when all its nodes are clamped, because exact member equations allow each individual member to displace between nodes with an infinite number of degrees of freedom, and hence infinite number of natural frequencies between nodes.) Thus

$$j_0 = \sum j_m \tag{18}$$

where j_m is the number of natural frequencies between $\omega = 0$ and $\omega = \omega^*$ for a component member with its ends fully clamped, while the summation extends over all members of the structure. With the knowledge of Equations (17) and (18), it is now possible to ascertain how many natural frequencies of a structure lie below an arbitrarily chosen trial frequency. This simple feature of the algorithm (coupled with the fact that successive trial frequencies can be chosen by the user to bracket a natural frequency) can be used to converge on any required natural frequency to any desired (or specified) accuracy.

5. Scope for Future Work

The literature clearly reveals that the DSM is now sufficiently matured and it is possible to develop general purpose computer programs combining bar, beam, plate and shell elements. (It should be noted that significant strides have already been made in developing the DSM for plates and shell which have not been elucidated in this paper.) Such programs will be much more accurate and computationally efficient than any commercially based FEM software. Computer programs based on the DSM to analyse two and three dimensional skeletal structures that can be modelled by beam dynamic stiffness elements are readily available [21]. It is also possible to combine DSM code with FEM code and the application areas of such joint DSM and FEM code will include aerospace, civil, automotive, ship-building and other areas of engineering. This will satisfy the specific needs of the industry and each application area can be considered on its intrinsic merit. The development of refined dynamic stiffness elements using piezo-electric and functionally graded materials offers considerable future scopes for the DSM, but importantly, the inclusion of damping will play a major role in future DSM developments. So far, the DSM has been predominantly applied to solve the free vibration problem in the absence of damping and in this respect, the response analysis using the DSM with the inclusion of damping will be a challenge, constituting an important area of future activity. The DSM can also be explored for nano structures such as single and multi-walled carbon nano-tubes.

6. Conclusions

The literature concerning the solution of the free vibration problems of beams using the DSM has been reviewed in some detail. Many of the major advances made in developing the dynamic stiffness matrices of isotropic and anisotropic (composite) structural elements are highlighted. The author's own perspective on the current status of the DSM and the scope for its future development are projected in the light of the advancements made to date. The potential possibility of combing the DSM and FEM software is recognised and the prospect for developing the DSM further using piezoelectric and nano materials is also accentuated.

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