Generalized voter-like models on heterogeneous networks

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1 Introduction

The study of dynamical ordering phenomena and consensus formation in initially disordered populations is a central problem in the statistical physics approach to social and natural sciences, whether the broad idea of consensus is referred to opinions, voting intentions, language conventions, social habits or inherited genetic information. Indeed, people tend to align their opinions [1], segregated populations gradually lose their genetic diversity [2], and different social groups spontaneously develop their own seemingly arbitrary traits of dress or jargon [3]. Understanding how global order can emerge in these situations, self-organized by purely local interactions, represents an important theoretical and practical problem. Our ability to grasp these issues has been mainly driven by the analysis of simple statistical models, which capture the essential ingredients of a copying/invasion local dynamics. The simplest of those copying/invasion processes are the voter model [4] and the Moran process [5], which focus principally on social [1] and evolutionary [6] dynamics, respectively.

These two models describe a population as a set of agents, each one carrying a state (opinion, trait, genome) represented by a binary variable $\sigma = \pm 1$. At each time step, an ordered pair of adjacent agents (i, j) is selected at random. In the voter model, as a paradigm of copying processes, the system is updated as $\sigma_i := \sigma_j$, the first agent copying the state of its neighbor. The voter model can thus be conceived a very simplistic model of opinion formation in society in which individuals select their beliefs by the (admittedly not very realistic) procedure of just imitating one of their neighbors. On the other hand, in the Moran process it is the neighboring agent the one who copies the state of the first agent, $\sigma_j := \sigma_i$, or, from another perspective, the state of the first one "invades" the neighbor in contact. The Moran process represents thus a simple approximation to the evolutionary dynamics of a haploid population, constrained to have a fixed size [2].

In finite, initially disordered systems, and in the absence of bulk noise (i.e. agents spontaneously changing their state [1]), stochastic copying/invasion dynamics lead to a uniform state with all individuals sharing the same value σ, the so-called consensus. Being the two final states symmetrical in principle, the dynamical evolution towards either state will only depend on the initial configuration. In order to characterize how the consensus is reached, two quantities are usually consider. The exit probability $E(x)$ is defined as the probability that the final state corresponds to all agents in the state $+1$ when starting from a homogeneous initial condition with a fraction x of agents in state $+1$. Accordingly, the consensus time $T_N(x)$ is defined as the average time required to reach consensus, independently of its value, in a system of N agents.

Voter-like processes were originally considered in regular topologies. In this case, the voter model remarkably is one of the few stochastic non-equilibrium models that can be exactly solved in any number of dimensions [7, 8]. As it turns out, its symmetries imply the ensemble conservation of magnetization $m = \sum_i \sigma_i/N$, which in turn implies that the exit probability takes the linear form $E(x) = x$. On the other hand, the consensus time can be seen to scale with the number N of agents as $T_N \sim N^2$ in $d = 1$, $T_N \sim N \ln(N)$ in $d = 2$ and $T_N \sim N$ for $d > 2$ [9].

The results of the voter and other copying/invasion processes in the context of social and evolutionary dynamics acquire a larger relevance when they are considered in systems endowed with realistic, non-trivial topologies. In fact, the strong heterogeneity of social and ecological substrates is better encoded in terms of a complex network [10], rather than by a homogeneous $d-$dimensional lattice. Therefore, a large theoretical effort has been recently devoted to uncover the effects of a complex topology on the behavior of the voter model, as well as general dynamical processes [11, 12]. At the most basic level, while voter and Moran models are equivalent at the mean-field level for regular topologies, if the connection pattern is given by a complex network they behave differently, since the order in which the interacting agents (i, j) are selected becomes relevant [13, 14]. Additionally, the heterogeneity in the connection pattern, as measured by the degree distribution $P(k)$, defined as the probability that a randomly chosen agent is connected to k other agents (has degree k) plays a relevant role in the scaling of the consensus time, especially for scale-free networks [15] with a degree distribution scaling as a power-law, $P(k) \sim k^{-\gamma}$ [13, 16, 14]. A theoretical understanding of the behavior of the voter model on heterogeneous networks was finally put forward by Redner and coworkers [17, 18, 19], who showed that for the voter model the consensus time scale with the system size as $T_N \sim N\langle k\rangle^2/\langle k^2\rangle$, i.e. inversely proportional to the second moment of the degree distribution $\langle k^2 \rangle = \sum_k k^2 P(k)$. In scale-free networks with degree exponent $\gamma < 3$, the second moment $\langle k^2 \rangle$ diverges and becomes size dependent, implying thus a sublinear growth of the consensus time, as previously observed in numerical simulations [13, 16], and contrarily to the homogeneous mean-field expectation.

When discussing the role of heterogeneity in social or evolutionary contexts, a relevant question is whether the complexity of the substrate alone is able to encode the heterogeneity of a realistic dynamical process of social or environmental relevance. The common objection to extremely simple models is that in reality individuals behave and relate to their peers in different ways, i.e. they are *heterogeneous* both in the way in which they interact with others and in the way in which they react to these interaction. For example, in a social context, it could be the case that some agents are more reluctant to change their opinion (*zealots*), while some agents can assign different importance to the opinion of their different neighbors (i.e. close friends can be more trusted than casual acquaintances). Lately, different variations of voter-like models have been put forward, in an effort to take into account the intrinsic variability of agents and their individual propensity to interact with peers [18, 20, 21, 22, 23].

In this chapter, we describe a generalization of the voter model on complex networks that encompasses different sources of degree-related heterogeneity and that is amenable to direct analytical solution by applying the standard methods of heterogeneous mean-field theory [11, 12]. Our formalism allows for a compact description of previously proposed heterogeneous voter-like models, and represents a basic framework within which we can rationalize the effects of heterogeneity in voter-like models, as well as implement novel sources of heterogeneity, not previously considered in the literature.

2 A generalized heterogeneous voter-like model on networks

We consider a generalized heterogeneous voter-like model, given as a stochastic process on networks, and defined by the following rules [24]:

- Each vertex i has associated a given *fitness* f_i.
- A source vertex i is selected at random, with a probability $f_i / \sum_j f_j$, i.e., proportional to its fitness f_i.
- A nearest neighbor j of i is then selected at random.
- With probability $Q(i, j)$, i copies the state of vertex j with. Otherwise, nothing happens.

The fitness function f_i affects the probability that the given node i is chosen to initiate the opinion-update process at a given time t [18]. In the case of copying dynamics, it measures the propensity of a given node to change its state. Once the individual i is chosen and a neighbor j is selected, the probability $Q(i,j)$ measures how probable the actual update is, introducing a weight in the adjacency relation between the two individuals [23]. We can easily check that the standard voter model is recovered by setting $f_i = Q(i,j) = 1$, while the Moran process corresponds to $f_i = 1$ and $Q(i,j) = k_i/k_j$. Other variations of copying/invasion dynamics can be recovered by the appropriate selection of the f_i and $Q(i,j)$ functions.

Recently, a generalized formalism for the class of copying/invasion voter-like models on networks has been proposed [25, 26] in which the process is identified by the copying rate C_{ij}, encoding the full structure of the contact network and the stochastic update rules, and that in our case takes the form

$$C_{ij} = \frac{f_i}{\sum_p f_p} \frac{a_{ij}}{k_i} Q(i,j), \tag{1}$$

where a_{ij} is the adjacency matrix of the network, taking value 1 if vertices i and j are connected by an edge, and zero otherwise. Within this formalism, exact results can be obtained, but at the expense of computing the spectral properties of the copying rate matrix C_{ij}, a highly non-trivial task unless the matrix C_{ij} has a relatively simple form.

Here we follow a different path, applying to our model the technique of heterogeneous mean-field theory, which leads to simple estimates for central properties such as the exit probability and the consensus time in a rather economical way. While this technique is known to be not exact in several instances, it still nevertheless able to account with a reasonable accuracy for the results of direct numerical simulations of the model.

3 Heterogeneous mean-field analysis

The analytical treatment of voter-like models on complex networks is made possible by the heterogeneous mean-field (HMF) approach, which has traditionally provided a powerful analysis tool for dynamical processes on heterogeneous substrates [12, 11]. Two main assumptions are made: (*i*) Vertices are grouped into degree classes, that is, all vertices in the same class share the same degree and the same dynamical properties; (*ii*) The real (*quenched*) network is coarse-grained into an *annealed* one [11], which disregards the specific connection pattern and postulates that the class of degree k is connected to the class of degree k' with conditional probability $P(k'|k)$ [27]. In general, the HMF approach allows for simple analytic solutions. In the case of voter-like models it has proved remarkably powerful in estimating the quantities of interest, showing reasonable agreement with numerical simulations in real quenched networks [17, 18, 19, 20, 23].

Following the standard HMF procedure, we work with the degree-class average of the fitness function and microscopic copying rate. Averages are taken over the set of vertices with a given fixed degree, i.e.

$$f_i \to \frac{1}{NP(k)} \sum_{i \in k} f_i \equiv f_k, \tag{2}$$

$$Q(i,j) \to \frac{1}{NP(k)} \frac{1}{NP(k')} \sum_{i \in k} \sum_{i \in k'} Q(i,j) \equiv Q(k,k'), \tag{3}$$

where $i \in k$ denotes a sum over the degree class k and $P(k)$ is the network's degree distribution. Thus, f_k represents the fitness of individuals of degree k, assumed to be the same, depending only on degree, for all of them, while $Q(k,k')$ represents the probability that a vertex of degree k copies the state of a vertex of degree k'. Analogously, the contact pattern is transformed according to the method described in Ref. [28], obtaining

$$\frac{a_{ij}}{k_i} \to \frac{[NP(k)]^{-1} \sum_{i \in k} \sum_{j \in k'} a_{ij}}{[NP(k)]^{-1} \sum_{i \in k} \sum_r a_{ir}} \equiv P(k'|k). \tag{4}$$

Disregarding the *microscopic* details of the actual contact pattern, our generalized voter model is thus defined in terms of the *mesoscopic* copying rate

$$C(k,k') \equiv \frac{f(k)}{\langle f(k) \rangle} P(k'|k) Q(k,k'). \tag{5}$$

Here and in the following me adopt the convention $\langle \cdot \rangle = \sum_k P(k)(\cdot)$ In order to provide a quantitative measure of the ordering process, we shall consider the time evolution of the fraction of vertices of degree k in the state $+1$, x_k. Transition rates for x_k will be given by the probability $\Pi(k;\sigma)$ that a spin in state σ at a vertex of degree k flips its value to $-\sigma$ [17, 19, 23] in a microscopic time step. It is easy to show that, from the definition of the generalized voter model, such probabilities can be written as

$$\Pi(k;+1) = x_k P(k) \sum_{k'} (1 - x_{k'}) C(k,k') \tag{6}$$

$$\Pi(k;-1) = (1 - x_k) P(k) \sum_{k'} x_{k'} C(k,k'), \tag{7}$$

thus leading to the rate equation [23]

$$\dot{x}_k = \frac{\Pi(k;-1) - \Pi(k;+1)}{P(k)} \equiv \sum_{k'} C(k,k')(x_{k'} - x_k). \tag{8}$$

Given the very broad definition of the interaction rate $Q(k,k')$, a solution to the problem at hand cannot be easily provided in closed form, unless further assumptions are made. We well thus make the following additional assumptions:

(i) Dynamics proceed on uncorrelated networks, i.e. [29]

$$P(k'|k) = \frac{k' P(k')}{\langle k \rangle}; \tag{9}$$

(ii) The interaction rate can be factorized as

$$Q(k,k') = a(k)b(k')s(k,k'), \tag{10}$$

where $s(k,k')$ is any symmetric function of k and k'.

This simplified form encompasses a broad range of voter-like dynamical processes, including most previously proposed models and a variety of novel applications, with the remarkable advantage of being promptly solvable, as it will become clear in the rest of this Chapter.

In order to provide a general solution in the most compact notation possible, we rewrite the rate equation as

$$\dot{x}_k = \sum_{k'} P(k') \Gamma(k,k')(x_{k'} - x_k), \tag{11}$$

where we have defined $\Gamma(k,k') = u(k)v(k')s(k,k')$ and

$$u(k) = \frac{a(k)f(k)}{\langle f(k) \rangle}, \qquad v(k') = \frac{b(k')k'}{\langle k \rangle}. \tag{12}$$

We analyze the behavior of the linear process at hand in the canonical way, by first determining the inherent conservation laws [17, 18, 19]. We define a generic integral of motion $\omega[x_k(t)]$ for Eq. (11), such that $d\omega/dt = 0$. By definition of total time derivative

$$\frac{d\omega}{dt} = \nabla_{\mathbf{x}} \omega \cdot \dot{\mathbf{x}} = \sum_k \frac{\partial \omega}{\partial x_k} \dot{x}_k = 0. \tag{13}$$

In analogy with previous results [17, 18, 19], we look for conserved quantities that are linear in x_k imposing $\partial\omega/\partial x_k = z_k$ independent of x_k, so that conserved quantities will be given by

$$\omega = \mathbf{z} \cdot \mathbf{x} = \sum_k z_k x_k, \tag{14}$$

where z_k is any solution of $\sum_k z_k \dot{x}_k = 0$ and $\dot{x}_k$ is given by Eq. (11). It is easy to prove that $z_k \propto P(k)v(k)/u(k)$ always satisfies the above condition, so that a conserved quantity is found up to multiplicative factors and additive constants. Upon imposing $\sum_k z_k = 1$ as one of the possible normalization conditions, the conserved quantity becomes

$$\omega = \mathbf{z} \cdot \mathbf{x} = \frac{\langle v(k)/u(k)\, x_k \rangle}{\langle v(k)/u(k) \rangle}. \tag{15}$$

In analogy with the simplest definition of the voter model [17] the conserved quantity bears all the information required to calculate the exit probability E, which we previously introduced as the probability that the final state corresponds to all spins in the state +1. In the final state with all +1 spins we have $\omega = 1$, while $\omega = 0$ is the other possible final state (all −1 spins). Conservation of ω implies then $\omega = E \cdot 1 + [1 - E] \cdot 0$, hence

$$E = \omega = \frac{\langle v(k)/u(k)\, x_k \rangle}{\langle v(k)/u(k) \rangle}. \tag{16}$$

Starting from a homogeneous initial condition, with a given density x of randomly chosen vertices in the state +1, we obtain, since $\omega = x$,

$$E_h(x) = x, \tag{17}$$

completely independent of the defining functions a, b, and s, and taking the same form as for the standard voter model [1]. On the other hand, if the initial condition corresponds to a single seed, that is an individual +1 spin in a vertex of degree k,

$$\omega = E_1(k) = \frac{v(k)/u(k)}{N\langle v(k)/u(k) \rangle}, \tag{18}$$

which does not depend on the functional form of the symmetric interaction term $s(k, k')$.

Eq. (11) predicts that the set of variables of x_k rapidly converge to a steady state. It is easy to see that any choice of x_k that is constant in k is a solution to the steady state condition $\dot{x}_k = 0$. This solution is unique and does not depend on initial conditions if the square matrix $P(k')\Gamma(k, k')$ is irreducible and primitive (it certainly is when working with positive rates, which we will do in the following) [30]. If we call the steady state x^∞, then it is easy to prove that

$$\omega = \sum_{k'} z_{k'} x_{k'} = x^\infty, \tag{19}$$

that is the steady state value for x_k equals the conserved quantity. Such result is well-known in simpler formulations of the voter model and becomes crucial in the computation of the consensus time, even in our generalized case.

As we noted above, the convergence to the steady state distribution occurs on very short time scales. As soon as the steady state is reached, stochastic fluctuations become relevant and the systems begins to fluctuate diffusively around this value, until consensus is reached in one of the two symmetric states. Such fluctuations are integral to finite systems and occur at long time scales, making this time-scale separation possible in large enough systems. In the light of such considerations, the average consensus time $T_N(\mathbf{x})$ for a system in a generic steady state $\mathbf{x}$ can be derived extending the well known recursive method to our general case [19]. At a given time t, $T_N(\mathbf{x})$ must equal the average consensus time at time $t + \Delta t$ plus the elapsed time $\Delta t = 1/N$ that is, in our notation,

$$T_N(\mathbf{x}) = \bar{\Pi}\, T_N(\mathbf{x}) + \sum_{k,s} \Pi(k; s) T_N(\mathbf{x} + \mathbf{\Delta x}^{(k)}) + \Delta t, \tag{20}$$

where $\bar{\Pi} = 1 - \sum_{k,s} \Pi(k;s)$ is the probability that no state change occurs, while the sum is the weighted average over possible state-updates $\mathbf{x} \to \mathbf{x} + \mathbf{\Delta x}^{(k)}$. The variation $\mathbf{\Delta x}^{(k)}$ is a vector whose all components are zero except for the k-th, which equals the update-unit $\Delta_k = [NP(k)]^{-1}$. Expanding to second order in Δ_k, taking $x_k = \omega$ as the initial state and changing variables such that $\partial/\partial x_k = z_k \partial/\partial\omega$ we obtain the backward Kolmogorov equation

$$-1 = \frac{\mathbf{z}^T \Gamma \mathbf{z}}{N} \omega(1-\omega) \frac{\partial^2 T_N}{\partial \omega^2} \tag{21}$$

leading to

$$T_N = -N_{\text{eff}} [\omega \ln \omega + (1-\omega) \ln(1-\omega)] \tag{22}$$

where we have defined the effective system size $N_{\text{eff}} = N / \sum_{k,k'} z_k \Gamma(k,k') z_{k'}$, which, in the case of generalized voter dynamics, Eq. (12), becomes

$$N_{\text{eff}} = N \frac{\langle f(k) \rangle \langle k \rangle \left\langle \frac{k b(k)}{f(k) a(k)} \right\rangle^2}{\left\langle \left\langle s(k,k') k b(k) \frac{[k' b(k')]^2}{f(k') a(k')} \right\rangle \right\rangle}, \tag{23}$$

where we have defined $\langle\langle \cdot \rangle\rangle = \sum_{kk'} P(k) P(k') (\cdot)$.

4 Particular cases

With the formalism developed above, we can easily recover several of the variations of the voter model proposed in the past. Let us look at some of them in following section.

4.1 Standard voter model and Moran process

The standard voter model and Moran process can be recovered by setting $a(k) = b(k) = f(k) = s(k,k') = 1$ and $a(k) = k$, $b(k) = k^{-1}$ and $f(k) = s(k,k') = 1$, respectively. In this case the known results [19, 17] are recovered. Thus, for the voter model, the conserved quantity is

$$\omega = \sum_{k'} \frac{k' P(k')}{\langle k \rangle} x_{k'}(t), \tag{24}$$

the exit probability starting from a single +1 vertex is

$$E_1(k) = \frac{k}{N \langle k \rangle}, \tag{25}$$

and the consensus time takes the form

$$T_N(\omega) = -N \frac{\langle k \rangle^2}{\langle k^2 \rangle} [\omega \ln \omega + (1-\omega) \ln(1-\omega)]. \tag{26}$$

On the other hand, for the Moran process we have

$$\omega = \frac{1}{\langle k^{-1}\rangle} \sum_k \frac{P(k)}{k} x_k \tag{27}$$

$$E_1(k) = \frac{1}{k} \frac{1}{N\langle k^{-1}\rangle} \tag{28}$$

$$T_N(\omega) = -N\langle k\rangle\langle k^{-1}\rangle \times [\omega \ln(\omega) + (1-\omega)\ln(1-\omega)]\,. \tag{29}$$

The difference between voter and Moran dynamics is quite evident here. By looking the expressions for the conserved quantities, Eq. (24) states that for the voter model densities are weighted with a factor $k/\langle k\rangle$ which compensates the tendency of small degree nodes to change their state, whereas in the Moran process the exact opposite occurs, being the density in Eq. (27) balanced by the factor $k^{-1}/\langle k^{-1}\rangle$. This fact translates in the different forms of the exit probability, Eqs. (25) and (28). For the voter model, a single *mutant* opinion can spread to the whole system more easily when it first starts in a vertex of large degree, due to the fact that large degree vertices are copied from with larger probability. On the other hand, a *mutant* in the Moran process is able to spread faster if it starts on a low degree vertex, owing to the corresponding fact that low degree vertices are invaded with low probability.

4.2 Voter model on weighted networks

The extension of voter model to weighted networks [23] is motivated by those situations in which the strength of a relation can play a role in the process of opinion formation. In this sense, weights would reflect the evidence that the opinion of a given individual can be more easily influenced by a close friend rather than by a casual acquaintance. In a weighted network, the voter model is defined as follows: At each time step a vertex i is selected randomly with uniform probability; then one among the nearest neighbors of i, namely j, is chosen with a probability proportional to the weight $w_{ij} \geq 0$ of the edge joining i and j. That is, the probability of choosing the neighbor j is

$$P_{ij} = \frac{w_{ij}}{\sum_p w_{ip}}. \tag{30}$$

Vertex i is finally updated by copying the state of vertex j. If the weights depend on the degree of the edge's endpoints, $w_{ij} = g(k_i, k_j)$, with $g(k, k')$ a symmetric multiplicative function, i.e. $g(k, k') = g_s(k)g_s(k')$ [23], voter dynamics is recovered by setting $s(k, k') = a(k) = f(k) = 1$ and $b(k) = g_s(k)\langle k\rangle/\langle k g_s(k)\rangle$, which leads to an invasion exit probability

$$E_1(k) = \frac{k g_s(k)}{N\langle k g_s(k)\rangle}, \tag{31}$$

and a consensus time

$$T_N(\omega) = -N \frac{\langle k g_s(k)\rangle^2}{\langle k^2 g_s(k)^2\rangle} \left[\omega \ln \omega + (1-\omega)\ln(1-\omega)\right], \qquad (32)$$

with the conserved quantity

$$\omega = \sum_{k'} \frac{k' g_s(k') P(k')}{\langle k g_s(k)\rangle} x_{k'}(t). \qquad (33)$$

In order to provide an example of weighted voter dynamics, we can consider the special case of weights scaling as a power law of the degree, $g_s(k) = k^\theta$ on a scale-free network with degree distribution of the form $P(k) \sim k^{-\gamma}$. The consensus time starting from homogeneous initial conditions, $x_k(0) = 1/2$ takes the form

$$T_N(1/2) = N \ln(2) \frac{\langle k^{1+\theta}\rangle^2}{\langle k^{2+2\theta}\rangle}. \qquad (34)$$

From this expression, we can obtain different scalings with the network size N, depending on the characteristic exponents γ and θ. Considering only $\gamma > 2$ and using the scaling behavior of the network upper cutoff $k_c \sim N^{1/2}$ for $\gamma < 3$ and $k_c \sim N^{1/(\gamma-1)}$ for $\gamma > 3$ [31], we are led to different regions of behavior for $T_N(1/2)$:

1. If $\theta > \gamma - 2$ both $\langle k^{1+\theta}\rangle$ and $\langle k^{2+2\theta}\rangle$ diverge. In particular, $\langle k^{1+\theta}\rangle \sim k_c^{2+\theta-\gamma}$ and $\langle k^{2+2\theta}\rangle \sim k_c^{3+2\theta-\gamma}$. Thus

$$T_N \sim N k_c^{1-\gamma}. \qquad (35)$$

   If $\gamma < 3$, $k_c \sim N^{1/2}$, and $T_N \sim N^{(3-\gamma)/2}$. If $\gamma > 3$, then $k_c \sim N^{1/(\gamma-1)}$, and $T_N \sim$ const.
2. If $\gamma - 2 > \theta > (\gamma-3)/2$, then $\langle k^{1+\theta}\rangle$ converges and $\langle k^{2+2\theta}\rangle$ diverges. Thus

$$T_N \sim N k_c^{\gamma-2\theta-3}. \qquad (36)$$

   If $\gamma < 3$, $T_N \sim N^{(\gamma-2\theta-1)/2}$; if $\gamma > 3$, then $T_N \sim N^{2(\gamma-\theta-2)/(\gamma-1)}$.
3. If $\theta < (\gamma-3)/2$, then both $\langle k^{1+\theta}\rangle$ and $\langle k^{2+2\theta}\rangle$ converge, and we have

$$T_N \sim N. \qquad (37)$$

These scaling relations contain several interesting aspects. Among these, it is worth highlighting that, in region 1, and for $\gamma > 3$, the analytical results predict a constant scaling for the consensus time, which therefore does *not* depend on the population size. As a consequence, in the thermodynamic limit the ordering process is instantaneous. However, this turns out to be true only at the mean field level, i.e. on annealed networks. Simulating the process on quenched networks produces in fact different results, with a clear dependence of the consensus time on N [23]. Henceforth, this is a typical case showing the limits of the mean field approach in predicting the behavior of phenomena occurring on true finite networks with fixed connections and disorder. Another interesting feature concerns the special value $\gamma = 3$. When $\theta > 0$, this value separates distinct scaling behavior, while as $\theta < 0$ it ceases to be a frontier, the scaling of the consensus being linear in N on both sides of it.

5 A practical example: variable opinion strengths

The strength of the proposed generalized formalism resides in the possibility of deriving simple HMF solutions to new variations of the standard voter-like models, which were not studied in the past and whose solution would have been too hard to compute with more exact techniques. As an application of our formalism, we consider the case of voter dynamics in a society in which certain individuals are more likely to align their opinions with those of their neighbors than others [18]. In particular, the propensity of a certain individual to change opinion will depend on the strength of his/her social ties, that is on his/her number of neighbors. We can accomplish such a description in our formalism, by encoding this dependence in the fitness function $f(k)$ and assuming $a(k) = b(k) = s(k,k') = 1$ for the sake of simplicity. The HMF solution to such a problem will then be easy to derive. Following the steps illustrated in the previous sections, the conserved quantity reads

$$\omega = \sum_k P(k) \frac{k/f(k)}{\langle k/f(k) \rangle} x_k, \tag{38}$$

the exit probability for an individual seed in state +1

$$E_1 = \frac{k/f(k)}{N\langle k/f(k) \rangle}, \tag{39}$$

and the consensus time starting from a homogeneous state ω

$$T_N(\omega) = -N \frac{\langle f(k) \rangle \langle k/f(k) \rangle^2}{\langle k^2/f(k) \rangle} [\omega \ln \omega + (1-\omega) \ln(1-\omega)]. \tag{40}$$

We can provide a practical example by choosing $f(k) = k^\alpha$. If $\alpha < 0$, less connected individuals are more likely to change opinion, and connectedness can be interpreted as a measure of social self-assurance. If $\alpha > 0$, more connected individuals appear more vulnerable to opinion variability. Interestingly, the standard voter model is recovered for $\alpha = 0$, where no agent heterogeneity is postulated. The conserved quantity assumes the simple form $\omega = \langle k^{1-\alpha} x_k \rangle / \langle k^{1-\alpha} \rangle$ and the consensus time starting from an initial condition $\omega = 1/2$ is then

$$T_N(1/2) = N \ln(2) \frac{\langle k^\alpha \rangle \langle k^{1-\alpha} \rangle^2}{\langle k^{2-\alpha} \rangle}. \tag{41}$$

The size scaling behavior of the consensus time can be derived from Eq. (41), following the procedure devised in the previous section for the voter model on weighted networks. The results are illustrated in compact form in Fig. 1, where a phase diagram for the variables γ and α is shown. The phase diagram conveys great insight into the role played by the parameter α. Focusing on the region with $\gamma > 2$, which is of greater interest in the study of dynamical

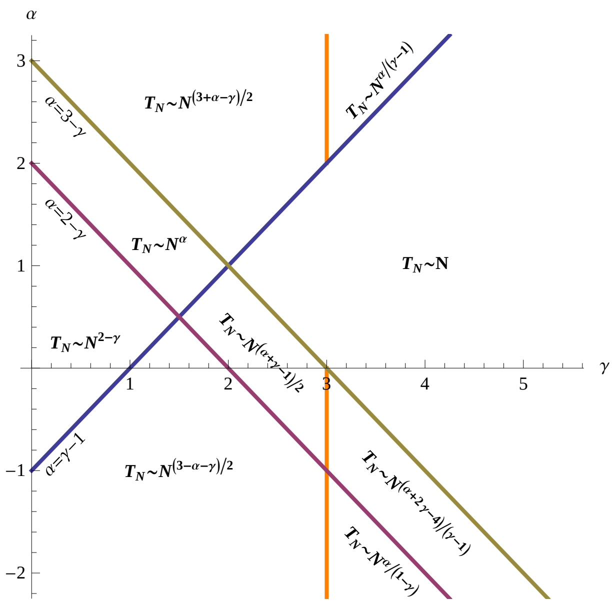

Fig. 1. Phase diagram of the heterogeneous voter model with variable opinion strengths, in which the strength of the opinion of an agent (its willingness to change opinion) is related to its degree by the relation $f(k) = k^{\alpha}$.

processes on complex networks, we obseve that for $3-\gamma < \alpha < \gamma-1$ the simple scaling relation $T_N \sim N$ is recovered. In analogy with a similar phenomenon observed in the case of weighted networks in Section 4.2, the value $\gamma = 3$ does not act as frontier and the same scaling law is observed, regardless of the value of γ. Larger values of α break the balance that ensues the $T_N \sim N$ behavior and lead to non-trivial size dependence of the consensus time, with exponents that depend on the degree distribution exponent γ. If we restrict our analysis to the case of scale-free networks, corresponding to the $2 < \gamma < 3$ strip in the phase diagram, we can easily see from the results in Figure 1 that values of α in the range $|\alpha| < \gamma - 1$ lead to either linear or sub-linear size scaling of T_N, whereas for $|\alpha| > \gamma-1$ super-linear scaling is encountered. This translates into the simple observation that in scale-free networks, *large* degree selectivity makes consensus harder to reach in larger systems, regardless of the sign of α, i.e., of whether the degree selectivity makes low-degree individuals or high-degree individuals more vulnerable to opinion change.

In order to corroborate our predictions, we have performed numerical simulations of the voter dynamics at hand in complex networks with power-law distributed degrees, generated with the Uncorrelated Model (UCM) [32]. We check the scaling behavior of the consensus time, $T_N \sim N^{\beta}$, for three pairs of (γ, α) values of the phase diagram, located in the regions where super-linear, linear and sub-linear scaling of T_N are observed respectively.

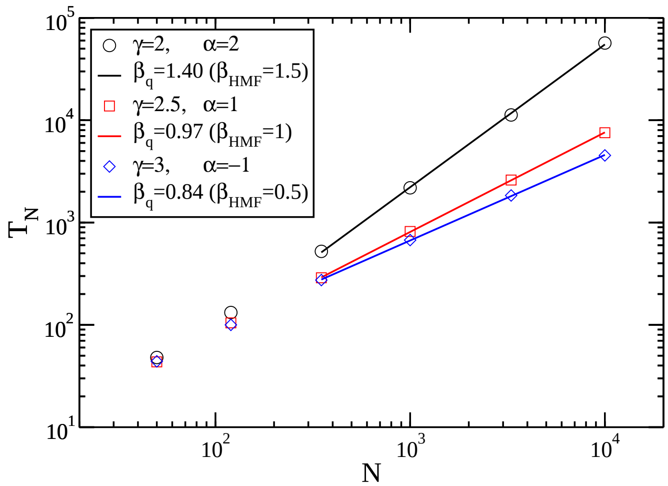

Fig. 2. Scaling behavior of the consensus time T_N as a function of the network size N for different values of γ and α.

In Fig. 2 we compare the scaling exponent predicted by Eq. (41), β_{HMF}, with the exponent β_q obtained by fitting the numerical simulations run on quenched networks. The results are summarized in the same Fig. 2. We note that the HMF theory predicts *qualitatively* well the behavior of the voter dynamics, in the sense that the scaling of T_N with N is super-linear for $\gamma = 2$ and $\alpha = 2$, linear for $\gamma = 2.5$ and $\alpha = 1$, and sub-linear for $\gamma = 3$ and $\alpha = -1$, as expected. As for the exact values of the β exponents, the HMF prediction is *quantitatively* accurate only is certain regions of the phase diagram, where the annealed-network approximation appears to hold. Elsewhere, quenched-network effects take over and sensible deviations with respect to the HMF value of β are encountered in simulations, in analogy with results for the weighted voter model, as discussed in Section 4.2.

6 Conclusions

In this Chapter, we have presented a generalized model of consensus formation, which is able to encompass all previous formulations of copy/invasion processes inspired by variations on the voter model and the Moran process. We considered the implementation of such generalized dynamics on a heterogeneous contact pattern, represented by a complex network, and derived the theoretical predictions for the relevant dynamical quantities, within the assumptions of the heterogeneous mean-field theory. We provided a brief review of previous results that can be recovered by our generalized formalism, and finally we considered a novel application to the case of opinion formation in a social network. In particular, we addressed the case in which the opinion strength of an individual is related to his/her degree centrality in the network. We found that in scale-free networks strong selectivity rules (which make less connected individuals much proner to change their opinions than more-connected ones or vice versa) lead to a steeper growth of consensus time with the system size, making the ordering process slower in general. Numerical simulations on quenched networks show that the HMF theory is able to predict such behavior with reasonable accuracy. Slight deviations from the theoretical predictions are encountered in certain regions of the phase diagram, but they are due to quenched-network effects that the HMF theory is not be able to capture.

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