Topology Induced Coarsening in Language Games

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We investigate how very large populations are able to reach a global consensus, out of local “microscopic” interaction rules, in the framework of a recently introduced class of models of semiotic dynamics, the so-called Naming Game. We compare in particular the convergence mechanism for interacting agents embedded in a low-dimensional lattice with respect to the mean-field case. We highlight that in low-dimensions consensus is reached through a coarsening process which requires less cognitive effort of the agents, with respect to the mean-field case, but takes longer to complete. In 1-d the dynamics of the boundaries is mapped onto a truncated Markov process from which we analytically computed the diffusion coefficient. More generally we show that the convergence process requires a memory per agent scaling as \( N \) and lasts a time \( N^{3+2/d} \) in dimension \( d \leq 4 \) (the upper critical dimension), while in mean-field both memory and time scale as \( N^{3/2} \), for a population of \( N \) agents. We present analytical and numerical evidences supporting this picture.

The past decade has seen an important development of the so-called Semiotic Dynamics, a new field which studies how conventions (or semiotic relations) can originate, spread and evolve over time in populations. This occurred mainly trough the definition of Language interaction games \([1,2]\) in which a population of agents is seen as a complex adaptive system which self-organizes \([3]\) as a result of simple local interactions (games). The interest of physicists for Language Games comes from the fact that they can be easily formulated as non-equilibrium statistical mechanics models of interacting agents: at each time step, an agent updates its state (among a certain set of possible states) through an interaction with its neighbors. An interesting question concerns the possibility of convergence towards a common state for all agents, which emerges without external global coordination and from purely local interaction rules \([1,2,4]\). In this Letter, we focus on the so-called Naming Games, introduced to describe the emergence of conventions and shared lexicons in a population of individuals interacting with each other by negotiations rules, and study how the embedding of the agents on a low-dimensional lattice influences the emergence of consensus, which we show to be reached through a coarsening process. The original model \([5]\) was inspired by a well-known experiment of artificial intelligence called Talking Heads \([6]\), in which embodied software agents develop their vocabulary observing objects through digital cameras, assigning them randomly chosen names and negotiating these names with other agents.

Recently a new minimal version of the Naming Game endowed with simplified interactions rules \([6]\) has been introduced, that reproduces the phenomenology of the experiments and is amenable to analytical treatment. In this model, \( N \) individuals (or agents) observe the same object, trying to communicate its name one to the other. The identical agents have at their disposal an internal inventory, in which they can store an unlimited number of different names or opinions. At the initial time, all individuals have empty inventories, with no innate terms. At each time step, the dynamics consists of a pairwise interaction between randomly chosen individuals. Each agent can take part in the interaction as a “speaker” or as a “hearer”. The speaker transmits to the hearer a possible name for the object at issue; if the speaker does not know a name for the object (its inventory is empty), it invents a name to be passed to the hearer\([14]\), while in the case it already knows more synonyms (stored in the inventory), it chooses one of them randomly. The hearer’s move is deterministic: if it possesses the term pronounced by the speaker, the interaction is a success, and both speaker and hearer retain that name as the right one, canceling all the other terms in their inventories; otherwise, the new name is included in the inventory of the hearer, without any cancellation.

The mean-field (MF) case has been studied in \([4]\): the system initially accumulates a large number of possible names for the object since different agents (speakers) initially invent different names and propagate them. Interestingly however, this profusion of different names leads in the end to an asymptotic absorbing state in which all the agents share the same name. Although this model leads to the convergence of all agents to a common state or “opinion”, it is interesting to notice the important differences with other commonly studied models of opinion formation \([1,2,4]\). For example, each agent can potentially be in an infinite number of possible discrete states (or words, names), contrarily to the Voter model in which each agent has only two possible states \([8]\). Moreover, an agent can here accumulate in its memory different possible names for the object, i.e. wait before reaching a decision. Finally, each dynamical step involves a certain degree of stochasticity, while in the Voter model, an agent deterministically adopts the opinion of one of its neighbors.

In this Letter, we study the Naming Game model on low-dimensional lattices: the agents, placed on a regu-
lar $d$-dimensional lattice, can interact only with their $2d$ nearest neighbors. Numerical and analytical investigations allow us to highlight important differences with the mean-field case, in particular in the time needed to reach consensus, and in the effective size of the inventories or total memory required. We show how the dynamics corresponds to a coarsening of clusters of agents sharing a common name; the interfaces between such clusters are composed by agents who still have more than one possible name.

Relevant quantities in the study of Naming Games are the total number of words in the system $N_w(t)$, which corresponds to the total memory used by the agents, the total number of different words $N_d(t)$, and the average rate of success $S(t)$ of the interactions. Fig. 1 displays the evolution in time of these three quantities for the low-dimentional models, compared to the mean-field case.

In the initial state, all inventories are empty. At short times therefore, each speaker with an empty inventory has to invent a name for the object, and many different words are indeed invented. In this initial phase, the growth of clusters of agents having one single word by the agents accumulate in memory are shown in black. This figure clearly shows the total number of distinct words displays a long plateau. The total memory used scales as $N$, and the time to reach the maximum number of words in the system scales as $N^{\alpha_d}$ with $\alpha_1 = \alpha_2 = 1$ (Fig. 2). No plateau is observed in the total number of distinct words since coarsening of clusters of agents soon start to eliminate words.

Furthermore, the time needed to reach consensus, $t_{\text{conv}}$, grows as $N^{\beta_d}$ with $\beta_1 \simeq 3$ in $d = 1$ and $\beta_2 \simeq 2$ in $d = 2$, while $\beta_{\text{MF}} \simeq 1.5$ (Fig. 2). We will now see how such behaviors emerge from a more detailed numerical and analytical analysis of the dynamical evolution.

Fig. 2 reports a typical evolution of agents on a one-dimensional lattice, by displaying one below the other a certain number of (linear) configurations corresponding to successive equally separated temporal steps. Each agent having one single word in memory is presented by a colored point while agents having more than one word in memory are shown in black. This figure clearly shows the growth of clusters of agents having one single word by diffusion of interfaces made of agents having more than one word in memory. The fact that the interfaces remain
thin is however not obvious a priori: an agent having e.g. two words in memory can propagate them to its neighbors, leading to possible clusters of agents having more than one word.

In order to rationalize and quantify such evolution, we consider a single interface between two linear clusters of agents: in each cluster, all the agents share the same unique word, say $A$ in the left-hand cluster and $B$ in the other. The interface is a string of length $m$ composed of sites in which both states $A$ and $B$ are present. We call $C_m$ this state $(A+B)^m$. A $C_0$ corresponds to two directly neighboring clusters ($\cdots AAABBB\cdots$), while $C_m$ means that the interface is composed by $m$ sites in the state $C = A+B$ ($\cdots AAAC\cdots CBBC\cdots$). Note that, in the actual dynamics, two clusters of states $A$ and $B$ can be separated by a more complex interface. For instance a $C_m$ interface can break down into two or more smaller sets of $C$-states spaced out by $A$ or $B$ clusters, causing the number of interfaces to grow. Numerical investigation shows that such configurations are however eliminated in the early times of the dynamics.

Bearing in mind these hypotheses, an approximate expression for the stationary probability that two neighboring clusters are separated by a $C_m$ interface can be computed in the following way. In a one-dimensional line composed of $N$ sites and initially divided into two clusters of $A$ and $B$, the probability to select the unique $C_0$ interface is $1/N$, and the interacting rules say that the only possible product is a $C_1$ interface. Thus, there is a probability $p_{0,1} = 1/N$ that a $C_0$ interface becomes a $C_1$ interface in a single time step, otherwise it stays in $C_0$. From $C_1$ the interface can evolve into a $C_0$ or a $C_2$ interface with probabilities $p_{1,0} = 2/N$ and $p_{1,2} = 2/N$ respectively. This procedure is easily extended to higher values of $m$. The numerics suggest that we can safely truncate this study at $m \leq 3$. In this approximation, the problem corresponds to determine the stationary probabilities of the Markov chain reported in Fig. 4 and defined by transition matrix

$$
\mathcal{M} = \begin{pmatrix}
\frac{N-1}{N} & \frac{N}{N} & 0 & 0 \\
\frac{N}{N} & \frac{N-2}{N} & \frac{1}{N} & 0 \\
\frac{1}{2N} & \frac{1}{2N} & \frac{1}{2N} & \frac{3}{2N} \\
\frac{3}{2N} & \frac{1}{2N} & \frac{1}{2N} & \frac{1}{2N} \\
\end{pmatrix},
$$

in which the basis is $\{C_0, C_1, C_2, C_3\}$ and the contribution $\frac{1}{2N}$ from $C_3$ to $C_1$ has been neglected (see Fig. 4). The stationary probability vector $\mathbf{P} = \{P_0, P_1, P_2, P_3\}$ is computed by imposing $\mathbf{P}(t + 1) - \mathbf{P}(t) = 0$, i.e. $(\mathcal{M}^T - I)\mathbf{P} = 0$, that gives $P_0 = 133/227 \approx 0.586$, $P_1 = 78/227 \approx 0.344$, $P_2 = 14/227 \approx 0.062$, $P_3 = 2/227 \approx 0.0088$. Direct numerical simulations of the evolution of a line $\cdots AAABBB\cdots$ yields $P_0 \approx 0.581$, $P_1 = 0.344$, $P_2 = 0.063$, $P_3 = 0.01$, thus clearly confirming the correctness of our approximation.

Since our analysis shows that the width of the interfaces remains small, we assume that they are punctual objects localized around their central position $x$: in the previously analyzed case, denoting by $x_1$ the position of the right-most site of cluster $A$ and by $x_r$ the position of the left-most site of cluster $B$, it is given by

$$
x = \frac{x_1 + x_r}{2}.
$$

An interaction involving sites of an interface, i.e. an interface transition $C_{m} \rightarrow C_{m'}$, corresponds to a set of possible movements for the central position $x$. The set of transition rates are obtained by enumeration of all possible cases: denoting by $W(x \rightarrow x \pm \delta)$ the transition probability that an interface centered in $x$ moves to the position $x \pm \delta$, in our approximation only three symmetric contributions are present. We obtain

$$
W(x \rightarrow x \pm \frac{1}{2}) = \frac{1}{2N}P_0 + \frac{1}{2N}P_1 + \frac{1}{2N}P_2 + \frac{1}{2N}P_3, \quad W(x \rightarrow x \pm 1) = \frac{1}{2N}P_2 + \frac{1}{2N}P_3, \quad W(x \rightarrow x \pm \frac{3}{2}) = \frac{1}{2N}P_3.
$$

Using the expressions for the stationary probability $P_0, \ldots, P_3$, we finally get $W(x \rightarrow x \pm \frac{1}{2}) = \frac{319}{354N}$, $W(x \rightarrow x \pm 1) = \frac{8}{227N}$, and $W(x \rightarrow x \pm \frac{3}{2}) = \frac{1}{227N}$.

The knowledge of these transition probabilities allows

FIG. 3: Typical evolution of a one-dimensional system ($N = 1000$). Black color corresponds to interfaces (sites with more than one word). The other colors identify different single state clusters. The vertical axis represents the time ($1000 \times N$ sequential steps), the one-dimensional snapshots are reported on the horizontal axis.

FIG. 4: Truncated Markov process associated with interface width dynamics - schematic evolution of a $C_0$ interface $\cdots AAABBB \cdots$, cut at the maximal width $m = 3$. 

faces disappears, and the two interfaces coalesce. Such a number of different words in the lattice, that is about estimate the number of interfaces at this time with the thin interfaces (see the first lines in Fig. 3). We can pairwise interactions create the lattice can then be described as follows: at short times, the distribution of a time at which unsuccessful interactions can take place, decays points to the existence of an effective surface tension. The density of interfaces, \( \langle x^2 \rangle = 2D_{exp}t/N \) with \( D_{exp} \approx 0.224 \) in agreement with the theoretical prediction.

These results are confirmed by numerical simulations as illustrated in Fig. 5 where the numerical probability \( P(x,t) \) is shown to be a Gaussian around the initial position, while the mean-square distance reached by the interface at the time \( t \) follows the diffusion law \( \langle x^2 \rangle = 2D_{exp}t/N \) with \( D_{exp} \approx 0.224 \approx D \). The dynamical evolution of the Naming Game on a one-dimensional lattice can then be described as follows: at short times, pairwise interactions create \( O(N) \) small clusters, divided by thin interfaces (see the first lines in Fig. 5). We can estimate the number of interfaces at this time with the number of different words in the lattice, that is about \( N/2 \). The interfaces then start diffusing. When two interfaces meet, the cluster situated in between the interfaces disappears, and the two interfaces coalesce. Such a coarsening leads to the well-known growth of the typical size \( \xi \) of the clusters as \( t^{1/2} \). The density of interfaces, at which unsuccessful interactions can take place, decays as \( 1/\sqrt{t} \), so that \( 1 - S(t) \) also decays as \( 1/\sqrt{t} \). Moreover, starting from a lattice in which all agents have no words, a time \( N \) is needed to reach a size of order 1, so that in fact \( \xi \) grows as \( \sqrt{t/N} \) (as also shown by the fact that the diffusion coefficient is \( D/N \), which explains the time \( t_{\text{conv}} \approx N^3 \) needed to reach consensus, i.e. \( \xi = N \). This framework can be extended to the case of higher dimensions. The interfaces, although quite rough, are well defined and their width does not grow in time, which points to the existence of an effective surface tension. The numerical computation of equal-time pair correlation function in dimension \( d = 2 \) (not shown) indicates that the characteristic length scale \( \xi \) grows as \( \sqrt{t/N} \) (a time \( O(N) \) is needed to initialize the agents to at least one word and therefore to reach a cluster size of order 1), in agreement with coarsening dynamics for non-conserved fields [10]. Since \( t_{\text{conv}} \) corresponds to the time needed to reach \( \xi = N^{1/d} \), we can argue \( t_{\text{conv}} \sim N^{1+2/d} \), that has been verified by numerical simulations in \( d = 2 \) and \( d = 3 \). This scaling and the observed coarsening behavior suggest that the upper critical dimension for this system is \( d = 4 \) [10].

In conclusion, the study of the low-dimensional Naming Game using statistical physics methods provides a deeper understanding of the macroscopical collective dynamics of the model. We have shown how it presents a very different behavior in low-dimensional lattices than in mean-field, indicating the existence of a finite upper-critical dimension. Low-dimensional dynamics is initially more effective, less memory per node is required, preventing agents from learning a large part of the many different words created. The dynamics then proceeds by the growth of clusters by coarsening, yielding a slow convergence to consensus. In contrast with other models of opinion dynamics (e.g. the Voter model [11, 12]), the Naming Game presents an effective surface tension that is reminiscent of the non-equilibrium zero-temperature Ising model [11]. In this respect, it seems interesting to investigate the dynamics of the Naming Game in other topologies, such as complex networks in which each node have a finite number of neighbors combined with "long-range" links [13].

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[14] Each word is associated with a numerical label. Thus, the invention of a new word simply corresponds to the extraction of a random number.