Optimal Dynamic Allocation of Rental and Sales Inventory for Fashion Apparel Products

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Abstract

There is a growing trend towards renting rather than permanent ownership of various product categories such as designer clothes and accessories. In this paper, we study an emerging retail business model that simultaneously serves rental and sales markets. Specifically, we consider a retailer that primarily focuses on renting while also selectively meeting incidental sales demand. Once a unit is sold, the firm forgoes potentially recurring rental revenues from that unit during the remaining periods. Therefore, it is critical for a retailer to dynamically decide how much of its inventory to allocate for sales and rentals at each period. We first develop a consumer choice model that determines the fraction of the market that chooses renting over purchasing. We characterize the optimal inventory allocation policy and explore how market characteristics and prices impact inventory allocation. We discuss the value of dynamic allocation and observe that the profit improvement can be substantial. In addition, we propose a simple and efficient heuristic policy. Finally, we extend our analysis to study the optimal allocation policies for (i) a retailer that is primarily a seller that selectively meets rental demand, and (ii) a retailer that does not enforce any prioritization between rental and sales demand.

Keywords: Retailing, Operations/Marketing interface, Renting.

1. Introduction

The retailing industry has been witnessing dramatic changes during the last decade. Although we have seen several technological enablers and facilitators of this change, such as the recent developments in analytics-based applications and online retailing in general, an important driver of one of the main new initiatives and innovative business models in the industry has been a shift in consumer behavior resulting in a growing segment of the population preferring not to own. Even though the no-ownership mindset started with high-ticket items such as cars and appliances, over the last few years, it has spread to many low-ticket items as well, collectively fueling what is known as the sharing economy. As a result, new business models based on *renting* began to spring up a few years ago and are becoming an almost disruptive initiative in the retailing industry for the more traditional business models based on sales.

Although we have started to see retailers renting products such as textbooks, smart phones, and even furniture, the fast fashion and apparel retailing industry responded especially well to this new trend and is the main focus of this paper. Whereas leasing items such as cars and copying machines are well-known and accepted practices associated with durable goods, the success of the rental business model in the fast fashion and apparel retailing industry is partly due to the non-durable and "perishable" nature of the product, which aligns well with the highly *fashion-conscious* consumer segment. For example, these consumers prefer to have access to the high-end designer dresses and accessories, but in the age of social media and continuous sharing, they would also generally prefer not to be seen in the same dress among their professional and social circles. According to one survey conducted in 2012, 80% of women bought new outfits out of fear of being tagged in the same clothes more than once on social media (Claire (2012), Greene (2018)). Hence, paying such a high price to own these fashionable items just to wear them once or maybe twice is wasteful on many different levels. Consumers with this mindset have further reduced the useful life of such non-durable products, creating another strong reason for the rental business model to thrive in apparel retailing.

Rent the Runway was founded in 2009 as the "Netflix for dresses," driven by the motto "access is the new ownership," and has become one of the pioneers in this industry in a very short amount of time. The rental apparel industry has since exceeded $1 billion and it grew 24% in 2018 compared to 5% for the wider clothing market. Lending Luxury, Renta-Dress and Tux, Glam Corner, Hirestreetuk, Endless Wardrobe, Rotaro, Rainey's Closet, Style Lend, Zent, National Tuxedo Rentals are just some of the other on-line apparel retailers that are renting dresses, headwear, handbags, and even shoes and accessories and are on their way to revolutionize the entire industry which is still evolving with new business models. For example, recently, the well-known iconic department store Lord & Taylor has been sold to the online apparel rental retailer Le Tote. On the other hand, some of the other well-known traditional retailers such as Bloomingdale's and Banana Republic announced that they will soon start renting clothes while Ann Taylor, American Eagle, Urban Outfitters, Express and Vince have recently launched their rental services (Dowsett and Fares (2019)).

Although the industry started with *pure* renters that are just renting their products at a fraction of the retail price, there are now many apparel retailers that are *simultaneously* renting and selling the same product. In particular, there are these retailers that are primarily known as renters and are now selling on the side (such as Lending Luxury, Poshare) and those that are primarily sellers, but have started renting on the side (Ann Taylor and Vince to name a few). However, unlike some of the other industries in which the duration of the rental can be "long" or extended with various other options, these retailers provide limited options. For example, Rent the Runway offers either a four-day or eight-day rental period at the end of which the products need to be returned to the retailer.

These retailers that offer the same product for *both* sales and rentals need to effectively manage their inventory. Due to the nature of this industry with its long lead times, retailers generally do not have a replenishment opportunity during the season. Hence, they must allocate the right amount of inventory for rentals while leaving enough inventory for sales. Even though the retailer will make more from selling the product than renting it once, the repeat revenue of a rented unit may surpass its sales revenue, depending on how many times it can be rented before the end of the season. Therefore, these types of retailers need to consider the potential repeat rental revenue of a unit before they sell that unit and lose that opportunity for good. As this industry generally faces high demand uncertainty, an upfront fixed allocation of inventory between sales and rental customers at the beginning of the season is likely to lead to a considerable loss of profits. Thus, we believe retailers can increase their profits by doing a dynamic allocation of sales and rental inventory throughout the horizon. In an industry with notoriously slim margins, this approach can make a meaningful difference.

In this paper, we consider such a monopolist retailer that is both renting and selling an apparel product by dynamically managing its inventory throughout the season. We assume the season is divided into multiple periods, with each period corresponding to a rental duration offered by that retailer. We implement a choice model that considers consumer heterogeneity regarding frequency of use for the item, fashion-consciousness (i.e., fraction of valuation retained for repeated uses), strength of outside options, and product valuations that determine the fraction of the market that chooses renting over purchasing. That is, given their valuation for the product, and their type for use frequency, retained valuations for subsequent uses, and strength of outside options, each consumer decides whether to rent or purchase the product or forgo either option. Motivated by the recent developments in the industry, our main focus is on a retailer that is primarily a renter with incidental sales. At each period, the retailer first prioritizes and meets the rental demand as much as possible and then decides how much of its sales demand to meet realizing that once a unit is sold, they will lose its potential repeat rental revenue. These types of retailers can mix their inventory and sell their previously rented units, albeit at lower than regular retail price. We assume any excess inventory including all previously rented units to be liquidated at the end of the season at a particular salvage price.

We first characterize the *optimal dynamic sales admission* policy and discuss how it changes with key market characteristics parameters and sales and rental prices. Considering potential damages to rented units, we then extend our base model to incorporate inventory spoilage and discuss its impact on the optimal policy. Without dynamic allocation, a manager would most likely try to meet all sales demand that arrives in that period as long as they have sufficient inventory after meeting rental demand. Comparing such a myopic allocation strategy with the optimal dynamic allocation strategy, we discuss the value of dynamic allocation and observe that the profit improvement can be substantial (e.g., 40%) for limited inventories in a long selling season. Even with sufficient inventory, we observe that the improvement can be significant. We believe that the narrow margins generally observed in this industry make this outcome even more critical. We also discuss how the value of dynamic allocation strategy changes with rental to sales ratio in the market and demand uncertainty. We then draw a parallel between our problem and the classic two-fare revenue management problem and propose a heuristic policy that is easy to implement and that performs well.

In addition, and as an extension, we also consider a retailer that is primarily a seller with incidental rentals where the retailer prioritizes and first meets sales demand as much as possible and then decides how much of the rental demand it should meet, taking into consideration that once a unit is allocated for rental, it cannot be sold as new. Specifically, these retailers might have an existing focus on selling brand new products and thus need to distinguish between previously rented and new inventory. We characterize the optimal dynamic rental allocation policy for this setting as well and discuss how it is influenced by key market characteristics parameters. We also propose a marginal revenue heuristic policy that performs well. Finally, we also extend our base model to study a retailer that does not prioritize either the sales or rental demand and similarly characterize its optimal dynamic rental allocation policy.

2. Related Literature

Our work is related to several different literature streams. First, we would like to highlight the related work on the leasing and purchasing of durable goods. As durable goods have a long life, initial customers who have higher valuations will be aware of the fact that the firm will eventually reduce the price in order to be able to attract the lower-valuation customers as well and thus they will also demand price reduction at the beginning. This may result in a monopolist preferring to lease its products. This phenomenon was first recognized by Coase (1972) and was then also studied in depth by Bulow (1982). This work has since been extended in various directions. Bucovetsky and Chilton (1986) show that a threat of entry from a competitor would change the preference of a durable-goods monopolist from renting to concurrent renting and selling. Purohit (1995) studies a setting where a firm has to sell or rent its product through an intermediary and show that, in the presence of such an intermediary, the firm prefers to sell rather than rent. Later, based on a two-period model of a duopoly, Desai and Purohit (1999) show that the optimal proportion of leases and sales depends on market competitiveness and product reliability, and that neither firm leases all its units in equilibrium.

Due to the durable nature of the products studied in these papers, the later-stage second-hand market that develops for used products cannibalizes the new ones. As a result, the consumers who return their used products at the end of their leasing agreement will supply a proportion of the used market. Moreover, a consumer can lease the product for an extended period that can represent a significant percentage of the product's life cycle. On the other hand, the perishable and non-durable nature of the fashion-apparel products we study in this paper do not have these characteristics. For example, no significant used market exists within the general population for a given product. Furthermore, the firm rents the product for a shorter period, anticipating that the same unit will be rented several times before the end of the season. Gilbert et al. (2014) and Jalili and Pangburn (2020) study a monopolist that simultaneously sells and offers per-use rentals. Motivated more by *information* goods and the video-entertainment industry in which consumers develop instances of need for these types of products over time, they focus on joint pricing of rentals and sales focusing on different aspects of consumer behavior in this context. We, on the other hand, consider a retailer that is simultaneously selling and renting a *non-durable* good, and focus on the *dynamic inventory allocation* between sales and rental customers for a given fixed amount of inventory, which is a non-issue for information goods.

Inventory management in the context of renting has received somewhat limited attention in the literature. The earliest work by Tainiter (1964) studies a setting where the firm is a pure renter with random rental duration and identifies the optimal initial ordering quantity. Whisler (1967) characterizes the optimal policy for a firm that does not own any equipment but can adjust its inventory by leasing and returning the equipment at every period with the goal of meeting consumers' rental demands for a certain duration. More recently, Slaugh et al. (2016) study a rental inventory system with random rental lifetimes and inventory loss, and characterizes the optimal initial ordering quantity. Jain et al. (2015) studied a rental system in which a firm needs to find an optimal allocation policy for two classes of renters with different return behaviors. Based on optimal control theory, the authors characterize the optimal policy and show that it switches priorities between classes. All this prior work assumed the firm was a pure renter and focused on finding the optimal *initial* quantity. On the other hand, motivated by apparel retailing, we consider a retailer that is simultaneously selling and renting the same product, and study the *dynamic* allocation of its fixed initial inventory between these two types of demand throughout the horizon. To the best of our knowledge, our work is the first to consider the inventory management problem of a retailer that simultaneously sells and rents its product.

Although it is the inventory decision that is being optimized, given that the retailer is trying to maximize its revenue from a *fixed* amount of inventory with no replenishment opportunity, our work can also be seen as a contribution to the general area of revenue management as well. Moreover, the type of heuristic we propose can be associated with the well-known two-fare revenue management problem (Littlewood (1972), Belobaba (1987)). Similarly, Topkis (1968) studies inventory allocation among multi-class demand with and without replenishment; yet unlike their setting, in the context of renting, a portion of the inventory always comes back and because the repeat rental revenue may exceed the sales price, there is no strict ordering among demand classes.

Lastly, we also would like to briefly mention that rental and purchase decisions for consumers has also been recently studied in a peer to peer rental context (e.g., Filippas et al. (2020)). Rather than peer to peer renting, and as described earlier, our focus on this work is the emerging retailer model where firms choose to simultaneously rent and sell their products.

3. Model

3.1 Retailer's Environment

We consider a retailer that simultaneously sells and rents apparel products. At the beginning of the planning horizon, e.g., a fashion season, the retailer first sets an initial stock level, Q, for a product with unit cost c and can serve both the rental and sales market. As is typical in apparel retail industry, we assume that there are no further replenishment opportunities. We also assume that the season is divided into T periods and that each period t represents the rental duration allowed (e.g., one week), in which the retailer faces a random demand D_t that is drawn from an independent and identical distribution $F(\cdot)$ and is split as rental and sales demand as will be described subsequently in Section 3.2. Each rented unit during period t brings the firm a rental price of p_r^t and the unit is then returned back to the firm's inventory and is made available for rentals in period $t+1$. (Later, we also provide an extension for lost/damaged rental units that may reduce the availability of inventory for future rentals.) Each unit sold during period t brings p_s^t and leaves the firm's inventory permanently. As the emphasis of this paper is mainly on apparel rentals, our analysis mainly focuses on a retailer that prioritizes rentals that also serves the sales market with incidental sales and we characterize the retailer's *optimal dynamic sales admission policy.* In Sections 6.1 and 6.2, we also extend our work to study a retailer that is primarily focused on selling with incidental rentals, and a retailer that serves the two types of markets with no priority.

Although the retailer earns more by selling instead of renting a single unit, the repeat revenue the retailer can potentially obtain by renting the same unit multiple times over the season can surpass its sales price. It is important to note that the revenue contributions per unit from rentals and sales do not necessarily have a constant ordering throughout the horizon. That is, while a rented unit may generate a higher expected revenue earlier in the horizon compared to selling that unit, the order of contributions may reverse towards the end of the selling horizon; i.e., the unit contribution from a unit that is sold may be higher than the expected rental contributions from that unit towards the end of the horizon. As the retailer loses potential repeat rental revenue once a unit is sold, it is critical for the retailer to decide on how much of its inventory to allocate for sales and rentals at every period. We present the specifics of the retail strategy and provide insights regarding the corresponding optimal policy as well as how the optimal decision is influenced by market characteristics in the following section. Before doing so, we would first like to introduce the main aspects of the consumer demand model.

3.2 Consumer Demand Model

As discussed in the Introduction, the renting business model we consider is motivated by a combination of different types of customer behavior and the heterogeneity observed in these behaviors.

First, we consider heterogeneity among customers with respect to their usage frequencies for the type of product the retailer offers. A certain segment of the population may be socially and professionally more active compared to others and generally might have more occasions where they need a certain type of an apparel product. For example, a professional consultant who is expecting to have various company-related events and high-profile meetings during a summer season may be more likely to need and use the retailer's products compared to a stay-at-home parent who may not have as many occasions during the same season. To capture the usage frequency heterogeneity, we assume that consumers are either high type (θ_H fraction of the population) or low type ($\theta_L = (1 - \theta_H)$ fraction) with respect to their frequency of use, N_i, which we assume follows a geometric distribution with $N_i \sim$ Geometric(q_i) ($i = L, H$), such that the probability of last use for the high use-frequency consumers, q_H, is less than that for the low use-frequency consumers, q_L (e.g., Feder et al. (2010)).

A second type of heterogeneity among customers we consider is regarding their fashion-consciousness. We know that some consumers are always more fashion-conscious in the sense that they care and enjoy the freshness and diversity of a wardrobe built renting different types of apparel products. We assume that people who are more fashion-conscious retain a lower fraction of their original valuation if they use the same product for a second or third time as they start to experience satiation using the product (e.g., Kahn (1995), Walsh (1995), Farquhar and Rao (1976), McAlister (1982), Meyer and Kahn (1990), McAlister and Pessemier (1982), Seetharaman and H.Che (2009), Kim et al. (2009)). Specifically, we assume ϕ_H fraction of the population is highly fashion-conscious and retain γ_H of their valuations for each subsequent use, and a $\phi_L(= 1 - \phi_H)$ fraction of the population is low fashion-conscious and retain γ_L of their valuations where $\gamma_H < \gamma_L \leq 1$, i.e., those customers who would highly prefer to have another product for their future needs would retain a lower fraction of their initial valuation for a subsequent use.

A third heterogeneity is due to consumer valuation. We assume customers know their per-use valuation v_t for the product but are heterogeneous in their valuations with $v_t \sim U(0, \bar{v}_t)$. If a customer with valuation v_t, of use-frequency type i, and of fashion-consciousness type j, where $i, j = \{L, H\}$, purchases and uses the product only once (i.e., with last use probability q_i), her total valuation from the product will be her single period valuation, v_t. If she uses the product a second time, she only retains γ_j fraction of her original valuation. Similarly, each subsequent use is associated with retaining a fraction γ_j of her earlier valuation. On the other hand, if this same customer rents this product for one period, she will use it for her first need and will have the flexibility to go with other products for her future $(N_i - 1)$ needs. We assume these rental customers have outside options for their future needs (which could simply be renting a different designer dress) and we assume ω_H percent is high-type with an average outside utility of u_H^o and $1 - \omega_H$ percent is low-type with an average outside utility of u_L^o such that $u_L^o < u_H^o$.

Hence, the expected utility of purchasing the product for a customer with valuation v_t, use-frequency type i, fashion-conscious type j and outside option type k, where $i, j, k = \{L, H\}$, is given by $v_t\Big(1 + \gamma_j(1 - q_i) + \gamma_j^2(1 - q_i)^2 + \cdots\Big) - p_s^t$, or equivalently by

$$\frac{v_t}{1 - \gamma_j(1 - q_i)} - p_s^t \tag{1}$$

whereas the expected utility for renting is given by $v_t - p_r^t + E[N_i - 1]u_k^o$, or equivalently by

$$v_t - p_r^t + \frac{1 - q_i}{q_i} u_k^o \tag{2}$$

As discussed earlier, we observe that the retailers in this business generally offer a limited number of rental-duration options such as one or two weeks, partly to be able to increase the predictability of their inventory levels as they make it available for future rentals. Therefore, we assume that consumers can rent only for that single period and that all rented units are brought back to the retailer at the end of the same period to be put back on the shelf for re-renting.

Consumers compare their utilities in equations (1) and (2) and purchase the product if $\Big(\frac{v_t}{1-\gamma_j(1-q_i)} - p_s^t > v_t - p_r^t + \frac{1-q_i}{q_i} u_k^o\Big)$ and $v_t \geq p_s^t(1 - \gamma_j(1 - q_i))$. Based on the incentive compatibility and individual rationality constraints for all eight segments, arrivals in period t, D_t, are split as *sales* demand, $D_t^s = \alpha_t D_t$, as *rental* demand, $D_t^r = \beta_t D_t$, or as no-transaction, $(1 - \alpha_t - \beta_t)D_t$, where

$$\alpha_t = \sum_{i=L,H} \sum_{j=L,H} \sum_{k=L,H} \alpha_{ijk}^t \theta_i \phi_j \omega_k, \tag{3}$$

$$\beta_t = \sum_{i=L,H} \sum_{j=L,H} \sum_{k=L,H} \beta_{ijk}^t \theta_i \phi_j \omega_k, \tag{4}$$

and where $\alpha_{ijk}^t = (1 - \frac{v_{ijk}^{'}}{\bar{v}_t})$ and $\beta_{ijk}^t = \frac{(v_{ijk}^{'} - v_{ijk}^{''})^+}{\bar{v}_t}$ with $v_{ijk}^{'} = \min\Big(\max\Big(\frac{p_s^t - p_r^t}{\gamma_j(1-q_i)}(1 - \gamma_j(1 - q_i)) + \frac{u_k^o}{\gamma_j q_i}(1 - \gamma_j(1 - q_i)), p_s^t(1 - \gamma_j(1 - q_i))\Big), \bar{v}_t\Big)$ and $v_{ijk}^{''} = (p_r^t - \frac{1-q_i}{q_i} u_k^o)^+$.

When $v_{ijk}^{'} < \bar{v}_t$ for $\forall i, j, k = \{H, L\}$, customers will be purchasing the product from all eight segments, but it is possible to have no customers purchasing from a given segment if the condition is reversed. Note that when the rental price p_r^t is less than the threshold $(p_s^t(1-\gamma_j(1-q_i)) + u_k^o(\frac{1-q_i}{q_i} + \frac{1-\gamma_j(1-q_i)}{\gamma_j q_i})\gamma_j(1-q_i))$, there will be renting customers $(v_{ijk}^{'} > v_{ijk}^{''})$ and when it is less than the threshold $(u_k^o(\frac{1-q_i}{q_i}))$, all market will be covered and everyone will either purchase or rent. We also would like to remark that the preceding model also allows fashion-consciousness and usage-frequencies to be correlated, which can be captured through a joint distribution for ϕ_i and θ_j. Note that equations (3) and (4) aggregates potential purchasing and renting customers from all eight segments giving us the total sales and rental split for a given period.

Based on the above segmentation, we can make some observations. While the segment that is highly fashion-conscious (γ_H) with a higher outside utility option (u_H^o) would be more likely to rent, the segment that is low fashion-conscious (γ_L) with a lower outside utility option (u_L^o) would be more likely to purchase. We find that the impact of usage frequency (q_i) and its interaction with other parameters are less intuitive. On the one hand, an increase in usage frequency may justify purchasing over renting. On the other hand, it may also lead to a decrease in consumer valuation especially for the highly fashion-conscious group, for which the flexibility of renting and the outside utility option can start to outweigh the diminishing valuations from repeated usage. We will discuss the impact of this segmentation on retailer's decisions further in the following sections.

We also want to briefly note that it is possible to incorporate *additional* utilities that might come with renting. For example, some renting customers may enjoy the fact that their choice is more environmentally-friendly since they are sharing the same unit with other customers. Such an addition will impact the parameters regarding the rental and sales demand as follows. If we let u^a denote the added utility into a renting customer's utility function, the renting customer will compare her total utility of renting and purchasing and will purchase if $\frac{v}{1-\gamma_j(1-q_i)} - p_s > \frac{v-p_r+u^a}{q_i}$. This update leads to $v_{ijk}^{'} = \min\Big(\max\Big(\frac{p_s-p_r+u_a}{\gamma_j(1-q_i)}(1-\gamma_j(1-q_i)) + \frac{u_k^o}{\gamma_j q_i}(1-\gamma_j(1-q_i)), p_s(1-\gamma_j(1-q_i))\Big), \bar{v}\Big)$ with $v_{ijk}^{''} = (p_r - u_a - \frac{1-q_i}{q_i}u_k^o)^+$ which decreases the fraction of purchasing customers (and increases the fraction of rental customers).

4. Analysis

As discussed in the Introduction and motivated by the recent emerging business models in the apparel industry, we consider a retailer that *primarily* serves a rental market with accompanying incidental (supplementary) sales. Besides their main operations focusing on rental units, these types of retailers also *sell* their units (rented and returned) at a discounted price compared to that of the brand new ones but do not directly serve the new product market.

We assume that at the beginning of the planning horizon, the retailer first decides on an initial order quantity that will be used to serve the overall uncertain demand throughout the planning horizon. Subsequently, at each period t, the firm observes its current inventory level, x_t, and the total arrivals within the period, D_t, a fraction α_t and β_t of which opt for purchasing and renting, respectively, as indicated in

(3) and (4). As the firm is *primarily* concerned with serving the rental market, it prioritizes the rental demand and first satisfies as much rental demand as its inventory permits. During that period, the firm also has the option to decide how much of the sales demand to satisfy based on its intermediate inventory level that excludes the units that are currently away for rentals. We refer to this decision as the retailer's "sales admission" decision. While each unit sold brings in p_s^t and is removed from the retailer's inventory permanently, the rented units bring a per-period rental revenue of p_r^t and are assumed to be returned to the retailer's inventory at the beginning of the next period to be made available for subsequent rentals.

We assume the retailer incurs a one-time transaction cost of c_a when it allocates a unit for sales; this represents the cost of moving units in the system from rental to sales as well as the cost of potential updates required for these units such as changing tags or physically relocating items. As rented units are returned to the retailer at the end of the period, the retailer incurs another type of cost, c_r, to recondition each rented unit so that it can be re-rented to a new customer.

4.1 Problem Formulation

We first present the retailer's dynamic sales admission decision for a given inventory position which determines how much of the sales demand the retailer satisfies each period. We then portray the retailer's initial order quantity decision at the beginning of the planning horizon.

We formulate the retailer's dynamic sales admission decision through a multi-period stochastic dynamic programming model assuming a planning horizon of T periods. We let $V_t(x_t)$ denote the expected optimal profit-to-go function at the beginning of period t when the retailer's inventory level is x_t, where x_t is a non-negative real number. As described earlier, the firm first prioritizes the rental demand and meets as much of this demand type as its inventory permits. Having observed the overall demand D_t and its remaining available inventory, it then decides how much of the sales portion of the demand it should meet. Once a unit is allocated to meet sales demand in a particular period, the firm essentially forgoes potentially recurring rental revenues from that unit during the remaining periods in the horizon. As apparel rental business models generally appeal to consumers that have upcoming special occasions with preset dates regarding when they would need the item, we assume that the items in our model are not backlogged. We allow excess units to be salvaged at a unit salvage value of $h < c$ at the end of the planning horizon. Below, we provide the dynamic programming recursions starting at state x_t for each period $t = 1, 2, ..., T$.

$$V_t(x_t) = (p_r^t - c_r)E_{D_t}\Big[\min(x_t, \beta_t D_t)\Big] + E_{D_t}\Bigg[\max_{\substack{s_t \\ s_t \le \alpha_t D_t \\ 0 \le s_t \le (x_t - \beta_t D_t)^+}} (p_s^t - c_a)\, s_t + V_{t+1}(x_t - s_t)\Bigg] \tag{5}$$

where $V_{T+1}(x) := h\,x$. In equation (5), the first term corresponds to the expected profit from rentals during the period, where $(p_r^t - c_r)$ is the rental contribution margin per unit and per period, and $E_{D_t}\big[\min(x_t, \beta_t D_t)\big]$ is the expected number of units rented, with priority given to the rental segment of demand. The second term is the expected profit based on the sales admission decision, where the retailer determines the amount of sales demand to satisfy, s_t, taking into account only the available inventory net of currently rented units. That is, sales allocation cannot exceed the intermediate inventory level $x_t - \beta_t d_t$ for a particular realization d_t of the current period demand D_t. As described earlier, any units sold, s_t, is permanently removed from the retailer's inventory. Thus, with the rental units returned at the beginning of the next period and having sold s_t units, the retailer's inventory position at the beginning of the next period becomes $x_t - s_t$. In the following, we characterize the optimal dynamic sales admission decision and provide insights on how key problem parameters influence this decision.

4.2 Structure of the Optimal Policy

4.2.1 Optimal Sales Admission Policy

The retailer's decision on whether to satisfy sales demand essentially hinges on the comparison between the one-off revenue it would receive from selling the product now versus the prospective repeat rental revenues it would obtain by keeping the item for subsequent period(s). The main difficulty in resolving this trade-off is due to the state- and time-dependency of the expected revenues generated from an item over the remaining periods resulting in neither the rental nor the sales channel necessarily dominating the other across the entire horizon. For example, keeping a particular unit for rentals may be more valuable for the firm earlier in the horizon as the firm can accrue higher rental revenues over a longer duration. In addition, each additional unit in inventory makes it less likely for the firm to be able to obtain rental revenues for that particular unit. Hence, one might expect that if the firm has a higher inventory level or the revenue potential from a rental unit decreases with time, it might be inclined to satisfy more of the sales demand in the current period. Our first result presented below formally characterizes the structure of the optimal sales admission policy.

Proposition 1 *The optimal sales admission policy can be described by a rental rationing threshold $\bar{r}_t$ such that (i) for $x_t \leq (\alpha_t + \beta_t)D_t$, it is optimal for the firm to satisfy $s_t^* = \min\big((x_t - \bar{r}_t)^+, x_t - \beta_t D_t\big)$ units of sales demand in period t, and (ii) for $x_t > (\alpha_t + \beta_t)D_t$, it is optimal for the firm to satisfy $s_t^* = \min\big((x_t - \bar{r}_t)^+, \alpha_t D_t\big)$ units of sales demand in period t. Further, the optimal sales admission quantity s_t^* increases with the current period inventory level x_t, increases with the fraction of customers that prefer to purchase, α_t, and decreases with the fraction of customers that prefer to rent, β_t.*

Proposition 1 indicates that the firm's optimal decision regarding how much of sales demand it should meet in a given period is governed by a (state-independent) rental rationing threshold. Specifically, the retailer aims to begin the next period by at least $\bar{r}_t$ units available for rentals and hence satisfies the current period sales demand at most to the extent its inventory level does not fall below this rationing level. Along with the rationing level, either the retailer's intermediate inventory level after satisfying the current period rentals or the amount of sales demand might also limit the actual units sold by the retailer. When the retailer's inventory level is relatively low compared to the realized demand, specifically, when $x_t \leq (\alpha_t+\beta_t)D_t$, the intermediate inventory level is the potentially limiting factor. On the other hand, when the retailer's inventory level is relatively high compared to the realized demand, i.e., when $x_t > (\alpha_t + \beta_t)D_t$, the retailer will satisfy as much of the sales demand as its rationing threshold permits. Consequently, we find that the amount of sales demand the retailer meets is non-monotonic with respect to the demand realization. In addition, the retailer will meet more of the sales demand if it begins the period with a higher inventory level. Further, we find that the retailer satisfies a larger amount of sales demand if the fraction of the customers who prefer to purchase is higher; similarly, the retailer satisfies fewer sales demand if the fraction of the customers who prefer to rent is higher.

We also would like to briefly comment on how the rental rationing threshold changes with the remaining time as well as the end of horizon salvage value. Our numerical tests indicate that the rental rationing threshold decreases as we approach the end of the horizon. As the expected cumulative revenue from a rental unit decreases towards the end of the horizon, the firm is more inclined to sell these units rather than save them for future rentals. Furthermore, we also find that the rental rationing threshold increases with the salvage value h. This is also expected as a higher salvage value increases the expected cumulative revenue contributions from a rental unit compared to selling the unit.

4.2.2 Initial Order Quantity Decision

Having characterized the structure of the optimal dynamic sales admission decision across the planning horizon, next we comment on the firm's initial quantity decision. The retailer selects an initial order quantity, Q, at the beginning of the planning horizon in order to serve its rental demand along with the portion of sales demand it will satisfy throughout the horizon. Assuming that each unit is procured at a cost of c, the retailer's initial order quantity problem can be expressed as follows:

$$\max_{Q} \; V_1(Q) - c\,Q \tag{6}$$

where $V_1(Q)$ denotes the optimal value function as defined earlier in (5) when starting the first period with Q units of inventory. An underlying structural property underpinning Proposition 1 (as shown in its corresponding proof) is that the value function $V_t(Q)$ is strictly concave. Therefore, letting $V_1^1(Q)$ denote the first derivative of $V_1(Q)$ with respect to its argument, the following result immediately follows.

Proposition 2 *There exists a unique optimal initial order quantity Q^*, which satisfies $V_1^1(Q^*) = c$.*

In words, the firms sets an initial order quantity Q^* that balances the marginal benefit from an additional unit of inventory with its marginal cost. Ordering Q^* units and dynamically setting the sales admission decision s_t^* as characterized in Proposition 1 throughout the horizon will maximize the retailer's total profit.

4.3 Sensitivity of the Optimal Sales Admission Policy

In this section, we provide additional insights on how the firm's optimal admission policy changes with respect to several key problem parameters. We first focus on the influence of rental and sales prices, followed by a discussion on the impact of parameters that define the market characteristics and composition. The following sensitivity results are based on changes that take place only in one parameter at a time while the values of all other parameters are kept constant. Furthermore, we use the terms 'increases' and 'decreases' in a weak sense to denote 'nondecreasing' and 'nonincreasing', respectively.

4.3.1 Impact of Rental and Sales Prices

Proposition 3 *The optimal rental rationing threshold $\bar{r}_t$ decreases with an increase in the current period sales price p_s^t and does not change with the current period rental price p_r^t. Consequently, when the optimal sales admission decision is defined by $s_t^* = (x_t - \bar{r}_t)^+$, the optimal sales quantity s_t^* increases with the current period sales price p_s^t and does not change with the current period rental price p_r^t. Otherwise, for $s_t^* = x_t - \beta_t D_t$ or $s_t^* = \alpha_t D_t$ (i.e., for inventory levels for which the optimal decision lies on a boundary), the optimal sales quantity s_t^* decreases with the current period sales price p_s^t and increases with the current period rental price p_r^t.*

Proposition 3 indicates that the optimal rental rationing threshold $\bar{r}_t$ decreases with the current period sales price and does not change with the current period rental price. Regarding the resulting optimal sales quantity, we find that the firm's optimal sales quantity may either increase or decrease with the sales price and increases with the rental price. Specifically, having a higher sales price for the product makes selling part of the available inventory more attractive than saving it for future rentals, thus the firm reduces its rationing threshold. We also find that the rental price does not impact the rationing threshold (due to rentals being prioritized). Consequently, for sales decisions directly defined by the rationing threshold and away from the boundaries of either selling all remaining units or meeting all sales demand, we find that an increase in the current period sales price results in a higher number of units sold during the period (due to the lower rationing threshold). Similarly, for these instances, we find that the optimal sales quantity does not change with the current period rental price. If the optimal sales quantity equals selling all remaining units (i.e., $s_t^* = x_t - \beta_t D_t$), then an increase in the sales price, which in turn implies a higher rental demand, results in fewer units available to be sold during the period. Thus the optimal sales quantity decreases. In a similar fashion, an increase in the rental price results in a lower rental demand, implying more units available to be sold during the period. If the optimal sales quantity is defined such that all sales demand is met during the period (i.e., $s_t^* = \alpha_t D_t$), then an increase in the sales price, which implies a lower sales demand, also results in fewer units to be sold during the period. Similarly, in such instances, an increase in the rental price implies a higher sales demand and thus results in more units to be sold during the period.

4.3.2 Impact of Market Characteristics and Composition

In this section, we discuss how the optimal sales admission decision changes with respect to some of the market characteristics. In the following proposition, we first show the impact of consumers' frequency of use, fashion-consciousness, and outside utilities.

Proposition 4 *The optimal sales quantity s_t^* increases with consumers' retained valuation γ_j, decreases with consumers' outside utilities, u_k^o, and may increase or decrease with consumers' last use probability q_i where $i, j, k = \{L, H\}$.*

As consumers become less fashion-conscious and retain a higher fraction of their future valuation, their utility from owning the product increases, which creates further incentive for them to buy the product instead of renting it for short-term use. Therefore, we find that the firm satisfies more sales demand in a given period as consumers' retained valuations increases. On the other hand, if the outside utility of consumers increases, the fraction of customers who prefer to purchase decreases while the fraction of customers who prefer to rent increases, which leads to a lower sales quantity. However, the change in the sales quantity with last use probability is quite unintuitive. Note that as q_i decreases (and frequency of use increases), consumers can make more use of the product they purchase and that leads to an increase in their purchase utility; in response to that, the retailer will increase its sales quantity. However, as q_i keeps decreasing (and frequency of use is increasing further), consumers' fashion-consciousness results in retaining only a smaller fraction of their valuations for the now more likely additional repeated usages. On the other hand, the rental option frees the customer after one period and hence the customer can benefit from a potentially higher valued outside option for their remaining needs. Therefore, when q_i decreases beyond a certain level, the aforementioned negative effect from purchasing combined with the positive effect from renting start to dominate consumer behavior making them more likely to rent and that makes the retailer decrease its sales quantity.

Next, we would like to further comment on how the market compositions across these heterogeneities affect retailer's allocation decision.

Proposition 5 *For given market characteristics parameters q_i,γ_j and u_k^o, where $i, j, k = \{L, H\}$, the optimal sales quantity in period t, s_t^*, (i) decreases with the current period fraction of high-type customers with respect to fashion-consciousness (ϕ_H) (ii) decreases with the current period fraction of high-type customers with respect to outside utility (ω_H) and (iii) may increase or decrease with the current period fraction of high-type customers with respect to frequency of use (θ_H).*

When consumers are more fashion-conscious, they would retain a lower fraction of their original valuation which will reduce their overall utility from purchasing. Consequently when the fraction of customers that are more fashion-conscious increases with respect to those that are less fashion-conscious, there will be more customers renting the product. Therefore, the retailer decreases its sales quantity during that period. Similarly, more customers will be renting when the utility of that outside option is higher. Hence, the retailer will again decrease its sales quantity during that period when the fraction of high-type customers with respect to outside utility increases. On the other hand, we find that the retailer's response to an increase in the fraction of high-type customers with respect to frequency of usage is not necessarily monotonic. Though a higher fraction of consumers likely having further usages of the product may initially increase the sales quantity, after a certain extent, and as discussed in the related result regarding the impact of q_i in Proposition 4, it may also start decreasing the sales quantity, and thus the optimal sales allocation.

4.4 An Extension for Lost/Damaged Rented Units

In apparel renting business, the same product is generally rented several times before it is sold at a clearance price. However, it is possible that a rented unit can be lost or damaged while being used by a customer. To gain insights into the impact of lost/damaged units on the optimal policy, we consider an extension to our base model where we let ℓ_t denote the fraction of currently rented items that can no longer be rented due to excessive wear or some damage. That is, when we rent $\min(x_t, \beta_t D_t)$ units, only $(1 - \ell_t)$ fraction of the rented units can be used for rentals in the subsequent period. We let the retailer be compensated for any lost items by a fee of p_l^t. The problem with inventory loss can be then stated as follows:

$$V_t(x_t) = (p_r^t - c_r)E_{D_t}\Big[\min(x_t, \beta_t D_t)\Big] + E_{D_t}\Bigg[\max_{\substack{s_t \\ s_t \le \alpha_t D_t \\ 0 \le s_t \le (x_t - \beta_t D_t)^+}} (p_s^t - c_a)\; s_t + p_l^t \ell_t \min(x_t, \beta_t D_t) + V_{t+1}(x_t - s_t - \ell_t \min(x_t, \beta_t D_t))\Bigg] \qquad (7)$$

where $V_{T+1}(x) := h\,x$. Below, we characterize the optimal sales admission policy for a retailer that experiences inventory loss for rented units.

Proposition 6 *The optimal sales admission policy can be described by a rental rationing threshold $\bar{r}_t(D_t)$ such that (i) for $x_t \le (\alpha_t + \beta_t)D_t$, it is optimal for the firm to satisfy $s_t^* = \min\big((x_t - \bar{r}_t(D_t))^+, \; x_t - \beta_t D_t\big)$ units of sales demand in period t, and (ii) for $x_t > (\alpha_t + \beta_t)D_t$, it is optimal for the firm to satisfy $s_t^* = \min\big((x_t - \bar{r}_t(D_t))^+, \; \alpha_t D_t\big)$ units of sales demand in period t. The rental rationing threshold $\bar{r}_t(D_t)$ increases with D_t. Further, the optimal rental rationing threshold increases with an increase in the current period loss fraction ℓ_t and does not change with an increase in the current period lost fee p_l^t.*

Note that with such inventory loss, the structure of the optimal policy is similar except that the rental rationing threshold now depends on the demand realization of that period. A higher demand realization will lead to more rentals which in turn will lead to less units returned, to which the retailer responds by increasing the rationing threshold. Furthermore, we also find that the rationing threshold increases with the fraction of items damaged/lost. That is, if the firm expects fewer units to be available for rentals due to damaged/lost items, it prefer to retain more of its inventory and increases its rationing threshold level. Lastly, we find that the firm's rationing threshold does not depend on the current period loss fee as the rentals are met as a priority and the rationing decision is not impacted by the current period loss fee.

5. Numerical Studies

5.1 Value of Optimal Dynamic Allocation

In this section, we evaluate the *value* of optimal dynamic allocation and discuss under which circumstances it may be more important for retailers to implement the dynamic allocation strategy. Note that the dynamic allocation strategy intentionally protects units from sales that can otherwise be rented in future periods. In other words, the dynamic allocation strategy may choose to ration inventory for rentals and consequently turn down a customer who is interested in buying a unit in that period if the potential repeat rental revenue of that unit outweighs the purchase price. We let Π_{OPT} denote the total profit of the retailer when it applies the optimal dynamic allocation strategy.

Without such a strategy, a manager would most likely try to meet *all* sales demand that arrives in that period as long as they have sufficient inventory after meeting rental demand. (Note that both strategies prioritizes rental demand before considering sales demand, which is motivated by the rental business model that we are considering.) Let Π_o be the total profit of the retailer when it tries to meet all sales demand, i.e., there is no allocation decision made by the retailer at every period to protect units from sales. We are interested in understanding the profit improvement the optimal dynamic allocation strategy would achieve compared to the "no allocation" (i.e. "meet all") strategy. We will compare the two strategies by finding the percent improvement in profit ($\frac{\Pi_{OPT}-\Pi_o}{\Pi_o}$ %) for a range of parameters. Figure 1 shows profit improvement percentage ($\frac{\Pi_{OPT}-\Pi_o}{\Pi_o}$) for various initial inventory values, x_o and across three different lengths of time horizons of $T = \{20, 30, 40\}$. When the inventory is very limited, although the dynamic allocation strategy will start to perform better by actively protecting units for rentals, a considerable amount of rental demand cannot be met and that limits this strategy's effectiveness. As inventory increases, while the optimal dynamic allocation policy protects some units for future rentals, the no-allocation strategy will instead give away any available units to meet sales demand. As a result, the profit gap between the two strategies increases. Our numerical studies demonstrate that this profit difference can be substantial, e.g., approximately 40% for a

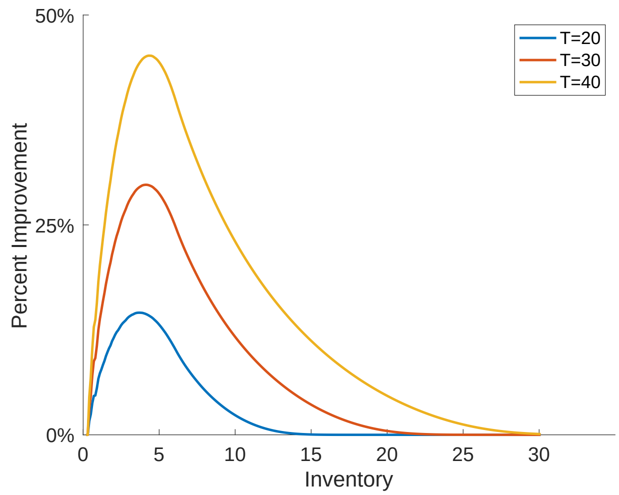
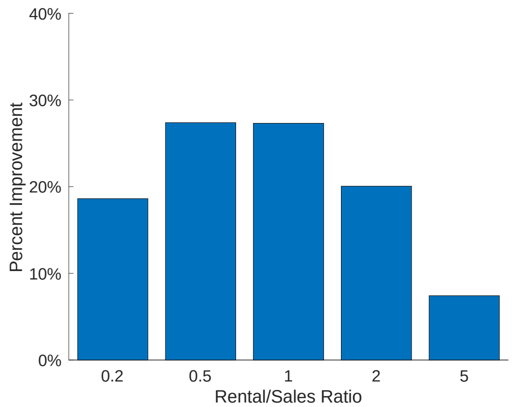


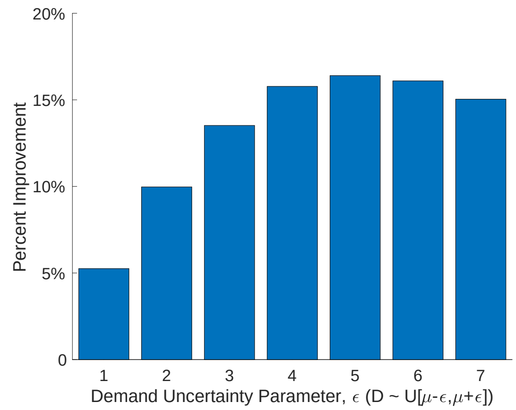
Figure 1: Percent profit improvement across different initial inventory levels and horizon length

horizon of $T = 40$ rental periods. (Note: The remaining parameters are set at the base Case 0 values as indicated and described further in the subsequent Section 5.2.) However, as inventory further increases, the retailer will have enough inventory to meet every rental demand at every period and the dynamic allocation strategy that actively protects units for rental demand will be less critical. Therefore, we see that profit improvement between two strategies starts to drop. In addition, as the selling season T (i.e., potential number of rental periods) increases, a unit can be rented more, leading to higher repeat rental revenue. From again Figure 1, we see that the profit gap between the two strategies widens for a given initial inventory level as the renting/selling season increases. Finally, we also want to note that if the firm is able to start the season with optimal initial quantities corresponding to the two strategies of dynamic allocation and no allocation the value of dynamic allocation over the no allocation may be diminished. For example, considering the T=30 period problem, while the maximum value improvement across all initial inventory levels between 0 and 30 units is 29.8%, and the average value improvement across these inventory levels is 8.3%, the value improvement for the optimal initial inventory level is 0.5%. As a side note, we believe having insights into the value gained from various possible initial inventory levels are important as in practice a retailer may not necessarily obtain the optimal initial quantity for all its products due to potential supply restrictions or longer lead times which is common in apparel industry.

We next comment on the value of the optimal policy for various proportions of rental and sales demand. When either rental or sales demand is very low, the allocation decision is not as critical. However, when rental and sales demands are more comparable, the dynamic allocation strategy provides higher benefits in terms of profitability as seen in Figure 2(a). In particular, when rental to sales demand ratio in the market is low, the dynamic allocation strategy will protect units from sales, but because of the low rental demand,



(a) Impact of rental-to-sales ratio in the market (i.e., β/α)



(b) Impact of demand uncertainty

Figure 2: Impact of (a) rental-to-sales ratio in the market, (b) demand uncertainty

the profit improvement is low. At the other end, when the rental to sales demand ratio is very high, the dynamic allocation strategy will help meet all that rental demand by protecting units from sales early in the horizon; however, because the sales demand is low, the no-allocation strategy's disadvantage of premature inventory selling will be less pronounced which will reduce the benefit of dynamic allocation. On the other hand, we observe that the biggest profit improvement occurs when the rental to sales demand ratio is in the middle range. In that range, the dynamic allocation strategy needs to protect units to meet its rental demand while the no-allocation strategy tries to actively meet the relatively high sales demand; in other words, the competition for that single unit becomes more intense and makes a difference. Note that various parameters (q, γ, u_o, p_s^t, p_r^t and market composition parameters) affect the aforementioned rental-sales ratio in the market. Therefore, depending on the combinations of these parameters and the rental-sales ratio range that these combinations lead to, we may see that the profit improvement between the two strategies increases or decreases.

We also explore the impact of demand uncertainty on the value of the dynamic allocation policy. Figure 2(b) shows an example setting where we assume that demand is uniformly distributed within the interval $[\mu - \epsilon, \mu + \epsilon]$ where $\mu = 7$ is the mean demand. (Other parameters are again set at the base values corresponding to the previous figures.). We vary ϵ to capture lower and higher demand uncertainty. As seen in Figure 2(b), as demand uncertainty increases, the dynamic allocation strategy first leads to higher levels of profit improvement as it ensures that the retailer has enough units even if the realized rental demand is high. However, we also observe that the improvement may start to decrease if uncertainty is too high. A reason for the diminishing value is that as the realized rental demand may now also be too low, the retailer might be better off selling units early on, as the no-allocation strategy does. Hence, the benefit of the the dynamic allocation strategy may start to decrease.

Lastly, we comment on the value of the optimal dynamic allocation policy for instances with inventory spoilage. In the base model, we had assumed that all rented units were returned. However, as argued in the earlier section, some of these units may not be re-rented because of a damage. When we incorporate such a loss in inventory, we observe that the benefit of the dynamic allocation strategy seems to decline. As the units actively protected by this strategy for rental may now be lost, the retailer may be better off by selling them early.

5.2 A Heuristic Sales Admission Policy

Before concluding this section, we also would like to utilize the structure of the optimal policy to construct and test an efficient and simple heuristic policy that considers the main trade-off of over-and-under rationing of inventory for rentals by taking into account the decreasing expected returns from a rental unit when approaching the end of the horizon.

Specifically, consider a retailer that currently has x_t units of inventory at the beginning of period t and is assessing the marginal revenue from an additional unit to be rationed for rentals from period t onwards, i.e., the revenue implications of increasing the rental rationing from r to $(r+1)$ units for the remaining $(T-t)$ periods. Let $F_R(\cdot)$ be defined as the single-period cumulative distribution of the rental demand per period. An additional unit rationed for rentals from period t onwards is expected to generate a total rental profit of $(p_r^t - c_r)\,(T-t)\,(1-F_R(r))$ across the remaining $(T-t)$ periods where $(1-F_R(r))$ corresponds to the probability that the unit is rented in any given period. Then, the retailer should increase its rationing level as long as $g(r) := (p_r^t - c_r)\,(T-t)\,(1-F_R(r)) - (p_s^t - c_a) > 0$. Consequently, setting $g(r)$ to zero and re-arranging the terms results in the heuristic rental rationing level, $\tilde{r}_t$: $F_R(\tilde{r}_t) = 1 - (p_s^t - c_a)/\big((p_r^t - c_r)(T-t)\big)$. The heuristic rationing level therefore has a close resemblance to Littlewood's rule for protection levels with two fare classes, where the low and high fares are reflected by $(p_s^t - c_a)$ and $(p_r^t - c_r)(T-t)$. It is important to note however that the ordering of the two fares can flip in our setting as the firm approaches the end of the horizon and thus the rationing (protection) level may be zero during part of the time horizon.

Next, we conduct a numerical study to evaluate the performance of the heuristic policy. To do so, we first define a base case assuming a market composition in which $\phi_H = 0.5$ fraction of customers are of the high fashion-conscious type with a retained valuation of $\gamma_H = 0.4$ for each subsequent use, i.e., they value a subsequent use of the item at 40% of their current valuation. Similarly, we assume that the remaining customers are of low fashion-conscious type and retain $\gamma_L = 0.8$ of their valuation for a subsequent use. In addition, we assume $\theta_H = 0.5$ fraction of consumers are high use-frequency type with a last-use probability of $q_H = 0.1$. The remaining consumers are deemed as low use-frequency type with a last use probability of $q_L = 0.8$. Moreover, we assume a fraction $\omega_H = 0.5$ of consumers to be high-type in terms of utilities from outside options with an outside utility of $u_H^o = 10$ per use and the remaining fraction to be low-type with an outside utility of zero. We set the purchase and rental prices at $p_s^t = 200$ and $p_r^t = 50$, respectively. We assume

Table 1: Performance of the heuristic policy across various parameter instances

Case	Fashion Consciousness			Use Frequency			Outside Flexibility		Prices and Valuation			Optimal Profit	Heuristic Profit	% diff.
#	ϕ_H	γ_H	γ_L	θ_H	q_H	q_L	ω_H	u_H	p_s^t	p_r^t	$\bar{v}$			
0	0.5	0.4	0.8	0.5	0.1	0.8	0.5	10	200	50	200	2035.7	2032.7	0.15
1	0.25	0.4	0.8	0.5	0.1	0.8	0.5	10	200	50	200	2239.4	2231.7	0.34
2	1.0	0.4	0.8	0.5	0.1	0.8	0.5	10	200	50	200	1319.4	1319.4	0.00
3	0.5	0.2	0.8	0.5	0.1	0.8	0.5	10	200	50	200	2035.7	2032.7	0.15
4	0.5	0.6	0.8	0.5	0.1	0.8	0.5	10	200	50	200	2172.3	2166.8	0.25
5	0.5	0.4	0.5	0.5	0.1	0.8	0.5	10	200	50	200	1378.9	1378.9	0.00
6	0.5	0.4	1.0	0.5	0.1	0.8	0.5	10	200	50	200	2252.1	2245.2	0.31
7	0.5	0.4	0.8	0.25	0.1	0.8	0.5	10	200	50	200	1721.7	1721.1	0.03
8	0.5	0.4	0.8	1.0	0.1	0.8	0.5	10	200	50	200	2332.0	2317.7	0.61
9	0.5	0.4	0.8	0.5	0.05	0.8	0.5	10	200	50	200	2028.8	2026.0	0.14
10	0.5	0.4	0.8	0.5	0.2	0.8	0.5	10	200	50	200	1943.4	1941.5	0.10
11	0.5	0.4	0.8	0.5	0.1	0.5	0.5	10	200	50	200	2035.7	2032.7	0.15
12	0.5	0.4	0.8	0.5	0.1	1.0	0.5	10	200	50	200	2035.7	2032.7	0.15
13	0.5	0.4	0.8	0.5	0.1	0.8	0.25	10	200	50	200	2070.6	2067.2	0.16
14	0.5	0.4	0.8	0.5	0.1	0.8	1.0	10	200	50	200	1947.9	1945.9	0.10
15	0.5	0.4	0.8	0.5	0.1	0.8	0.5	5	200	50	200	2070.4	2067.0	0.16
16	0.5	0.4	0.8	0.5	0.1	0.8	0.5	20	200	50	200	1947.9	1945.9	0.10
17	0.5	0.4	0.8	0.5	0.1	0.8	0.5	10	100	50	200	1587.8	1558.2	1.86
18	0.5	0.4	0.8	0.5	0.1	0.8	0.5	10	400	50	200	2035.8	2035.8	0.00
19	0.5	0.4	0.8	0.5	0.1	0.8	0.5	10	200	25	200	1349.5	1349.5	0.00
20	0.5	0.4	0.8	0.5	0.1	0.8	0.5	10	200	100	200	2728.2	2692.8	1.30
21	0.5	0.4	0.8	0.5	0.1	0.8	0.5	10	200	50	100	1283.4	1283.0	0.03
22	0.5	0.4	0.8	0.5	0.1	0.8	0.5	10	200	50	400	2366.3	2358.0	0.35

that the overall demand in each period of a $T = 10$ period problem is uniformly distributed on the interval $[0, 10]$, and that the consumers' valuations per use for their initial use of the item is uniformly distributed on $[0, \bar{v}]$ with $\bar{v} = 200$. For the remaining parameters, we set the rental reconditioning cost as $c_r = 10$ per rental and set the cost to shift a unit from sales to rental as zero. We then systematically increase and decrease key problem parameters regarding the market characteristics $(\phi_H, \theta_H, \omega_H)$ and $(\gamma_H, \gamma_L, q_H, q_L, u_H^o)$, product prices ($p_s^t, p_s^t$, which we keep stationary for numerical purposes), and consumer valuations $(\bar{v})$.

For each parameter set, we first solve optimally the dynamic program given in (5) using discretizations of 0.1 unit increments for the inventory and demand values. (We solve the optimal profit through the value iteration algorithm and search for the optimal sales admission quantity directly without imposing the structural results outlined in Proposition 1.) We then compute the heuristic profit using the heuristic rationing levels obtained by the procedure described above. We compute both the optimal and the heuristic profits starting from a range of inventory values from 0.1 up to 20 units. The profit values reported in Table 1 for each parameter set correspond to the average profit across the range of the starting inventory positions. Overall, we find that the heuristic policy performed well across all parameter instances tested with an average difference from the optimal policy of 0.28%.

As a side note, we also tested how a simpler heuristic policy that does not rely on demand parameter estimation but instead decides on the allocation purely by comparing the sales price with the unit rental revenue (rental price net of reconditioning costs) across the remaining time horizon. Specifically, when there are t periods remaining, the heuristic policy allocates all units to rentals as long as $(p_r - c_r)t \geq p_s$ and allocates all units to sales otherwise. Thus, the heuristic captures the main dynamic of preferring rentals more earlier in the horizon when there are more opportunities for repeat rental revenue rather than towards the end of the horizon, when sales are more preferred. However, as it ignores demand estimates it is not capable of a finer inventory allocation based on the inventory levels with respect to the two types of demands. We tested this simpler heuristic across all the instances outlined in Table 1 (in Appendix B), and observed that the average difference from the optimal profit was 17.0%. For instances with lower starting inventory levels (within a range of initial inventory levels from 0.1 to 5 units), the average performance gap was 7.7%, where better performance at lower inventory levels are reasonable due to the allocate all or none nature of this simpler heuristic. The performance gap not only emphasizes the good performance of our original heuristic but also indicates that it is important to be able to estimate the demand parameters and thus implement a finer allocation policy as outlined in our original heuristic policy.

6. Extensions

6.1 Primarily Sales with Incidental Rentals

In this section, motivated by the very recent developments in the industry, we study retail business models that primarily focus on sales and view rentals as a supplementary channel, such as Ann Taylor Infinite Style and Express Style Trial as previously referred to in the Introduction section. Specifically, we consider a retailer that first prioritizes its sales demand but may subsequently decide in each period to reallocate some of its inventory to serve the rental market. Since these retailers are primarily known as sellers with an established brand recognition in the industry, a key differentiating characteristic of their model, compared to the one studied in the earlier section, is that they need to be able to distinguish between new and rental inventory as units allocated for rentals previously cannot be sold as new for the sales channel. Similar to the previous framework, the retailer now first meets the sales demand as much as its inventory permits. In addition, if the retailer has any previous allocations for the rental market, it also satisfies as much of the rental portion of demand as its rental inventory permits. Subsequently, the retailer also has the option to reallocate further units from new inventory to rental inventory to meet any excess rental demand observed in the current period. Units reallocated and rented can no longer be sold as new and therefore are used to satisfy only the rental demand until the end of the horizon.

6.1.1 Problem Formulation

We again present the retailer's optimal rental allocation problem first and comment on its initial order quantity decision subsequently. We let (x_t, y_t) denote the retailer's inventory position at the beginning of period t of a horizon consisting of T time periods, where x_t refers to *new inventory* and y_t refers to *rental inventory*. A sales demand can only be satisfied through new inventory while both new and rental inventories can be used to satisfy the rental demand in a given period, in which case any further allocation from new inventory to satisfy rental demand can only be used for rentals subsequently. Within the extension, we consider stationary rental and sales prices. We again allow excess units to be salvaged at the end of the planning horizon at unit salvage values of h_s and h_r for new and rental inventories, respectively, where $h_r < h_s < p_s$. For consistency, we retain much of the previous notation and use the same utility model derived in Section 3. Letting $V_t(x_t, y_t)$ denote the expected discounted profit-to-go function under the optimal policy when the retailer begins period t in state (x_t, y_t), we can write the dynamic programming recursions as follows:

$$V_t(x_t, y_t) = E_{D_t}\left[p_s \min(x_t, \alpha_t D_t) + (p_r - c_r)\min(y_t, \beta_t D_t)\right] + E_{D_t}\left[\max_{\substack{0 \le r_t \le (x_t - \alpha_t D_t)^+ \\ r_t \le (\beta_t D_t - y_t)^+}} (p_r - c_r - c_a)\, r_t + V_{t+1}\Big((x_t - \alpha_t D_t)^+ - r_t, y_t + r_t\Big)\right] \tag{8}$$

where $V_T(x, y) := h_s x + h_r y$. In the above, the term $E_{D_t}[p_s \min(x_t, \alpha_t D_t) + (p_r - c_r)\min(y_t, \beta_t D_t)]$ represents the expected profit from satisfying the sales and rental demand based on the inventory available at the beginning of the period. Specifically, the retailer is able to satisfy up to x_t units of the sales demand $\alpha_t D_t$ and receives p_s for each unit sold. Similarly, the retailer uses its previously allocated rental inventory y_t to satisfy as much of the rental demand realization $\beta_t D_t$ as possible and receives a net contribution of $(p_r - c_r)$ per unit. The remaining part of the expression consists of the expected immediate profit along with the profit-to-go based on selecting the optimal allocation policy after observing the current period demand, where the additional allocation is denoted by r_t. For any particular realization d_t of demand D_t in the current period, the amount that the retailer can allocate to satisfy excess rental demand is limited by the intermediate inventory level for the new inventory, $(x_t - \alpha_t d_t)^+$, and by the level of excess rental demand $(\beta_t d_t - y_t)^+$. For each additional unit the retailer allocates to satisfy any excess rental demand, it receives a net contribution of $(p_r - c_r - c_a)$ where $p_r - c_r$ denotes the net contribution per period as described earlier and c_a represents any administrative and allocation related costs the retailer may incur. With r_t additional units rented in period t (and thus further allocated for rentals for the subsequent periods), the retailer begins the next period with $(x_t - \alpha_t D_t)^+ - r_t$ units of new inventory and $y_t + r_t$ units of rental inventory.

6.1.2 Optimal Rental Allocation Policy

As described in the problem formulation, a retailer deciding whether to satisfy an additional excess rental demand during the current period from new inventory weighs the benefits of potential rentals for this particularly allocated unit over the remaining horizon with the opportunity cost to potentially sell the unit as new at a higher price. The expected rental revenues from each additional unit allocated for rentals decreases with the level of existing rental inventory as well as the remaining time in the horizon. In a similar fashion, the expected revenue from reserving the unit for a potential new sale in the future decreases with the amount of existing new inventory as well as the remaining time in the horizon. Below, we describe the retailer's optimal rental allocation decision that balances these two potential revenue opportunities.

Proposition 7 *(a) If the current period inventory position* (x_t, y_t) *satisfies* $x_t\beta_t \leq y_t\alpha_t$*, then, for any demand realization* D_t*, the current period rental allocation is zero. Otherwise, i.e., if* $x_t\beta_t > y_t\alpha_t$*, the optimal rental allocation policy in period* t *is defined by a sales rationing threshold* $\bar{s}_t(w_t)$ *where* w_t *denotes the total intermediate inventory position, i.e.,* $w_t = x_t - \alpha_t D_t + y_t$*, such that:*

(i) for $(x_t + y_t) \leq (\alpha_t + \beta_t)D_t$*, the optimal rental allocation is* $r_t^* = (x_t - \alpha_t D_t - \bar{s}_t(w_t))^+$

(ii) for $(x_t + y_t) > (\alpha_t + \beta_t)D_t$*, the optimal rental allocation is* $r_t^* = \min\left((x_t - \alpha_t D_t - \bar{s}_t(w_t))^+, \beta_t D_t - y_t\right)$

(b) The sales rationing threshold $\bar{s}_t(w_t)$ *increases with* w_t*.*

(c) The optimal rental allocation quantity r_t^* *increases with* x_t*, decreases with* y_t*, and may increase or decrease with* D_t*.*

(d) The optimal rental allocation quantity r_t^* *decreases with* α_t *and increases with* β_t*.*

The first part of Proposition 7 part (a) simply indicates that when $x_t\beta_t \leq y_t\alpha_t$, any particular demand realization will lead to either the case in which no new inventory is available for further allocation to rentals or the case where all rental demand can be met with existing rental inventories and hence there is no excess rental demand to be met within the period. Therefore the firm's allocation decision only arises if $x_t\beta_t > y_t\alpha_t$. For this case, we find that the optimal rental allocation is described by a sales rationing threshold which is a function of the *total intermediate inventory* level w_t, i.e., the remaining new product inventory after the current period sales demand is met $(x_t - \alpha_t D_t)$, plus the current rental inventory level y_t. The sales rationing threshold increases with the intermediate inventory level. It is optimal for the firm to reallocate new inventory to rentals aiming to preserve at least $\bar{s}_t(w_t)$ units of new inventory for the subsequent period. Moreover, the firm's additional rental allocation is also constrained by the excess rental demand $\beta_t D_t - y_t$ when $(x_t + y_t) > (\alpha_t + \beta_t)D_t$. That is, overall, the firm will reallocate as many units as either its new inventory permits or is required to meet the rental demand as long as its inventory level for the new product does not fall below $\bar{s}_t(w_t)$ at the beginning of the next period. Part (b) of Proposition 7 states that the firm rations more units if its intermediate inventory level is higher. Consequently, we also find that the optimal reallocation amount increases with the firm's new inventory level, decreases with its existing rental inventory level and may increase or decrease with the realized demand. Lastly, we find that the rental allocation decreases with the fraction of customers who prefer to purchase and increases with the fraction of customers who prefer to rent.

Next, we highlight the sensitivity of the optimal rental allocation with respect to key characteristics of consumer behavior.

Proposition 8 *The optimal rental allocation r_t^* decreases with consumers' retained valuation γ_j, increases with consumers' outside utility, u_k^o, and may increase or decrease with consumers' last use probability, q_i, where $i, j, k = \{L, H\}$.*

As argued earlier, fashion-conscious consumers will retain a smaller fraction of their future valuations creating an incentive for them to rent more; hence, the retailer will decrease its rental allocation as consumers' retained valuation increases. We will observe the opposite outcome with the outside utility as higher outside utility would incentivize more customers to rent making the retailer increase its rental allocation. On the other hand, based on the aforementioned positive and negative effects of use-frequency on purchasing and renting utility of a consumer, the rental allocation may increase or decrease.

Note that, similar to the earlier setting, the retailer may select an initial order quantity, Q, at the beginning of the planning horizon by maximizing $\max_Q V_1(Q, 0) - c\,Q$ from which we can find that there exists a unique initial optimal order quantity Q^*. Ordering Q^* units and then dynamically setting the rental allocation decision r_t^* as characterized in Proposition 7 throughout the horizon will optimize retailer's total profit.

So far, we have characterized the retailer's optimal rental allocation decision for a given market. As mentioned earlier, the market consists of consumers that are heterogeneous in three dimensions that were assumed to have fixed proportions in the market and we would like to also understand how these market characteristics affect retailer's decision.

Proposition 9 *For given market characteristics parameters q_i,γ_j and u_k^o, where $i, j, k = \{L, H\}$, the optimal rental allocation in period t, r_t^* (i) decreases with the current period fraction of low-type customers with respect to fashion-consciousness (ϕ_L), (ii) decreases with the current period fraction of low-type customers with respect to outside utility (ω_L), and (iii) increases or decreases with the current period fraction of low-type customers with respect to frequency of use (θ_L).*

When consumers are less fashion-conscious, they would retain a higher fraction of their original valuation which will increase their overall utility from purchasing. Hence, when the fraction of customers that are less fashion-conscious increases, there will be more customers purchasing the product. In response, the retailer will decrease its rental allocation during that period. In a similar fashion, more customers will be purchasing to be able to use the product longer when the utility of the outside option is lower. Therefore, the retailer will decrease its rental allocation during that period if the fraction of low-type customers with respect to outside utility increases. Lastly, the retailer's response to an increase in the fraction of low-type customers with respect to frequency of use is non-monotonic due to the combination of similar factors explained in section 4.3.

As we did for the main model of Section 4, we also construct and test a heuristic policy for this extension setting where the retailer primarily sells and rents on the side. We describe the heuristic and discuss the results in Appendix B.

6.2 A No-Priority Rental Allocation Model

Motivated by the business models in the industry, both the base and the extension models prioritized either the rental or the sales demand. In this section, we study a retailer that does enforce any prioritization between rental and sales demand. Moreover, considering the ease of implementation in practice, we consider a retailer that dynamically reallocates part of its new inventory to the rental market at the beginning of each period and then meets the sales and rental demand based on the resulting inventory levels.

6.2.1 Problem Formulation

We use the much of the same notation as introduced in the main body of the paper and the previous subsection. Consider a retailer that starts period t with an inventory position of (x_t, y_t) where x_t and y_t denote, respectively, the inventory level that can satisfy the sales and rental markets. At the beginning of each period, the retailer decides how many more units (r_t) to allocate for rentals. This decision reduces current sales inventory by r_t and increases available rental inventory by r_t. Uncertain demand D_t is realized and split as sales and rental customers, as described earlier. Both kinds of customer demand are satisfied from their respective inventory positions. At the end of the period, rented units are returned to the retailer and sales inventory is updated based on the realized demand. As in the previous extension, we again consider stationary rental and sales prices. We allow excess units to be salvaged at the end of the planning horizon at unit salvage values of h_s and h_r for new and rental inventories, respectively, where $h_r < h_s < p_s$. We can rewrite the dynamic programming recursion for this problem as follows:

$$V_t(x_t, y_t) = \max_{0 \le r_t \le x_t} \Bigg(p_s E_{D_t}[\min(x_t - r_t, \alpha_t D_t)] + (p_r - c_r) E_{D_t}[\min(y_t + r_t, \beta_t D_t)] - c_a r_t \\ + E_{D_t} V_{t+1}\Big((x_t - r_t - \alpha D_t)^+, y_t + r_t\Big)\Bigg) \tag{9}$$

where $V_T(x, y) := h_s x + h_r y$ with $h_r < h_s < p_s$. In equation (9), the first term is the expected revenue from sales, the second term is the expected revenue from rentals, the third term is the cost of allocating further inventory for rentals, and the last term is the expected profit-to-go function. The retailer decides on r_t and cannot allocate more than the available on hand new product inventory x_t. After the rental allocation r_t and upon a particular realization d_t of demand D_t, the retailer starts the next period with a new product inventory position of $x_t - r_t - \alpha D_t)^+$ for sales and with an inventory position of $y_t + r_t$ for rentals. Lastly, as before, we assume that all rented units and any remaining excess inventory are sold at salvage prices as indicated in the previous subsection.

6.2.2 Characterization of the Optimal Allocation

Similar to the previous dynamics, the retailer again has to balance the opportunity cost of lost revenue from sales with the opportunity cost of repeat rental revenue from that one unit. Depending on how much inventory the retailer holds at any given period, it will have to balance this trade-off and find the optimal allocation. We formally characterize the optimal policy in the next result.

Proposition 10 *The optimal rental-allocation policy is defined by a state-dependent rental target threshold* $\bar{r}_t(x_t, y_t)$. *It is optimal for the firm to allocate an additional* $r_t^* = \min\Big(x_t,\ \big(\bar{r}_t(x_t, y_t) - y_t\big)^+\Big)$ *units from sales to rental in period* t. *Furthermore,* $\bar{r}_t(x_t, y_t)$ *is increasing in* x_t *and* y_t *and* r_t^* *increases with* x_t *and decreases with* y_t.

Proposition 10 indicates that the optimal rental allocation in each period is defined by a dynamically set rental target, which is a function of the current period sales and rental inventory. Specifically, the firm will reallocate as much units to the rental inventory as needed to bring the rental inventory level up to this rental target inventory threshold. We find that the rental target threshold increases with either type of inventory. Further, we also find that the optimal number of units to be allocated from the sales inventory to rental inventory increases with the sales inventory and decreases with the rental inventory observed at the beginning of the period.

Lastly, we provide additional results on how the actual quantity of newly allocated rental units change with respect to other problem parameters.

Proposition 11 *The optimal rental allocation r_t^* decreases with consumers' retained valuation γ_j, increases with consumers' outside utility, u_k^o, and may increase or decrease with consumers' last use probability, q_i, where $i, j, k = \{L, H\}$.*

Proposition 11 indicates that the sensitivity of the optimal allocation policy with respect to consumer behavior remains the same as in earlier models and can be explained as discussed in the earlier sections.

7. Conclusion

We have started to see many innovative business models in the retail industry. One such model that has recently become quite successful is renting and we see that its popularity has been consistently rising among consumers in fashion-apparel retailing. As a growing segment of consumers prefer renting over buying, many on-line apparel retailers are now either renting or simultaneously selling and renting fashion-apparel products. Although a retailer may earn more by selling instead of renting a single unit, the repeat revenue the retailer can potentially make by renting the same unit multiple times over the season can well exceed its sales price. Therefore it is critical for a retailer to dynamically decide its inventory allocation decision for sales and rental at every period. Motivated by these recent trends in the industry and the changes in consumer behavior, we study a retailer that starts a season with a fixed amount of inventory and is planning to both sell and rent during the season. Because retailers in this business may have different objectives and market reputation among consumers, they generally adopt different types of renting strategies. In our base model, we study a retailer that is primarily focused on renting with incidental sales.

Towards that objective, we consider a multi-period setting where each period corresponds to the minimum duration for which the product can be rented. We develop a consumer demand model based on consumers' inherent fashion-consciousness, frequency of use and the value of outside option. We assume demand in each period is uncertain and splits as renters and buyers based on this consumer behavior model. We first characterize the optimal dynamic sales admission policy and show that the firm's optimal decision regarding how much of sales demand it should meet in a given period is governed by a rental rationing threshold. We find that the firm satisfies more sales demand in a given period as consumers' retained valuations increases (i.e. become less fashion-consciousness) and satisfies less sales demand as their outside utility increases. On the other hand, we show that the sales allocation first increases as frequency of use increases and then decreases beyond a certain level. We also discuss how the retailer's policy changes with sales and rental price.

In apparel renting business, the same product is generally rented several times throughout the horizon. It is possible that a rented unit can be lost or may be damaged during this period. We extend our model to understand the impact of such inventory spoilage and show that the structure of the optimal policy is similar except that the rental rationing threshold now depends on the demand realization of that period.

Since a higher demand realization leads to more rentals which in turn will lead to less units returned, the retailer responds to this by increasing the rationing threshold, which also increases with the fraction of items damaged/lost.

We then discuss the value of optimal dynamic allocation. Without such a proactive allocation strategy, we assume that the retailer would try to meet all sales demand at every period. We show that the profit improvement can be significant and be as high as 40% with relatively limited inventory and longer horizons and is quite substantial even with large inventory. We also discuss how the value of dynamic allocation changes with sales-to-rental demand ratio and demand uncertainty. We then propose a marginal revenue heuristic that balances the aforementioned economic trade-off between selling and renting. After we establish an interesting link between this heuristic and the classic two-fare revenue management and the newsvendor model, we show it performs very well with respect to the optimal policy for both models.

We then extend our model to study a retailer that is primarily focused on selling with incidental rentals and characterize its optimal dynamic rental allocation policy and then discuss how it changes with market characteristics. Finally, we study a retailer that does not prioritize either of the sales or rental demand and similarly characterize its optimal dynamic rental allocation policy.

We believe this work can be extended in various directions. In apparel retailing, pricing is an important decision. Given the focus of this paper on dynamic inventory allocation, we assumed that the prices were fixed, but could argue how inventory allocations would change with rental and sales prices. An extension that completely focuses on the pricing issues of an apparel retailer that is in the simultaneous sales and rental business would be a very interesting and valuable contribution. Assortment optimization for retailers in this business can be further complicated due to consumers' different preferences. Managing returns, especially in on-line retailing, is a challenge and we believe that future work that studies the impact of the rental-business model on retailers' return policies and assortment decisions would be a fruitful area of research. Moreover, in fashion apparel retailing, there may be interesting network effects that could influence future demand, hence we also believe that an extension that studies correlated demand across periods could bring further interesting insights.

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Supplemental Online Materials for "Optimal Dynamic Allocation of Rental and Sales Inventory for Fashion Apparel Products" by M. Altug and O. Ceryan

Appendix A

Proof of Proposition 1

To facilitate the proof of Proposition 1, we first rewrite (5) as follows:

$$V_t(x_t) = (p_r - c_r)\int_0^{x_t/\beta_t} \beta_t z f(z)dz + (p_r - c_r)\int_{x_t/\beta_t}^{\infty} x_t f(z)dz + \int_0^{x_t/\beta_t} G_t(x_t, z) f(z)dz + \int_{x_t/\beta_t}^{\infty} V_{t+1}(x_t) f(z)dz$$

where $G_t(x_t, z) = \max_{s_t} \ (p_s - c_a)\ s_t + V_{t+1}(x_t - s_t)$ subject to $0 \le s_t \le (x_t - \beta_t z)$ and $s_t \le \alpha_t z$. We make an inductional assumption that the value function in period $t+1$, $V_{t+1}(x_t)$ is strictly concave in x_t. For expositional clarity, we let $V^1_{t+1}()$ and $V^{11}_{t+1}()$ denote, repsectively, the first and second derivatives of $V_{t+1}()$ with respect to its argument. After characterizing the optimal policy structure in period t, we will then show that the assumption also holds for $V_t(x_t)$. We note that as the analysis for showing that $V_{T-1}(x_t)$ (one period to the last) is strictly concave uses similar arguments for concavity preservation for an arbitrary period t, for brevity, we relegate its proof to the end. The Lagrangian for the subproblem is given by $(p_s - c_a)\ s_t + V_{t+1}(x_t - s_t) + \lambda_{1t} s_t - \lambda_{2t}(s_t - x_t + \beta_t z) - \lambda_{3t}(s_t - \alpha_t z)$ with $\lambda_{it} \ge 0$ for $i = \{1, 2, 3\}$. We distinguish four possible cases. (a): $\lambda_{1t} = 0, \lambda_{2t} = 0, \lambda_{3t} = 0$. For this case, the first order condition yields: $(p_s - c_a) - V^1_{t+1}(x_t - s_t)|_{s^*_t} = 0$. Differentiating this with respect to x_t, we get $-V^{11}_{t+1}(x_t - s_t)|_{s^*_t}\ (1 - \frac{\partial s^*_t}{\partial x_t}) = 0$, which results in $\frac{\partial s^*_t}{\partial x_t} = 1$. Similarly, we also find $\frac{\partial s^*_t}{\partial z} = 0$. Further, let $\bar{r}^*_t$ denote the optimal rationing threshold defined as $\bar{r}^*_t = x_t - s^*_t$. We have $\frac{\partial \bar{r}^*_t}{\partial x_t} = 0$ and $\frac{\partial \bar{r}^*_t}{\partial z} = 0$. Case (b): $\lambda_{1t} > 0, \lambda_{2t} = 0, \lambda_{3t} = 0$. This case immediately implies $s^*_t = 0$, or equivalently, the rationing threshold can be defined as $\bar{r}^*_t = \infty$. Case (c): $\lambda_{1t} = 0, \lambda_{2t} > 0, \lambda_{3t} = 0$ implies $s^*_t = x_t - \beta_t z$ for which $\frac{\partial s^*_t}{\partial x_t} = 1$, $\frac{\partial s^*_t}{\partial z} = -\beta_t < 0$. Further, we also have $\frac{\partial s^*_t}{\partial \beta_t} = -z < 0$. We define the rationing threshold for this case as zero. Case (d): $\lambda_{1t} = 0, \lambda_{2t} = 0, \lambda_{3t} > 0$ implies $s^*_t = \alpha_t z$ for which $\frac{\partial s^*_t}{\partial x_t} = 0$, $\frac{\partial s^*_t}{\partial z} = \alpha_t > 0$. Further, we also have $\frac{\partial s^*_t}{\partial \alpha_t} = z > 0$. We define the rationing threshold for cases (c) and (d) as $\bar{r}^*_t = 0$. Together, cases (a)-(c) correspond to the result in part (i) of the Proposition, while cases (a), (b), and (d) correspond to part (ii) of the Proposition. Hence, overall the optimal selling admission quantity s^*_t can be described through a state-independent rationing threshold $\bar{r}^*_t$ where it is optimal for the firm to meet as much of the sales demand $\alpha_t z$ with its inventory availability after demand realization $x_t - \beta_t z$ as long as its inventory level (for the next period) does not fall below $\bar{r}^*_t$. When $\bar{r}^*_t = 0$, the amount the firm will sell is limited by either the sales quantity $\alpha_t z$ or the available inventory $x_t - \beta_t z$. Hence when $x_t \le (\alpha_t + \beta_t)z_t$, the sales amount will be limited by $x_t - \beta_t z$ and when

$x_t > (\alpha_t + \beta_t) z_t$, the sales amount will be limited by $\alpha_t z$.

To complete the proof of Proposition 1, we next show that strict concavity of $V_t(x_t)$ is preserved after the dynamic programming recursion. To do so, we first find that $G_t(x_t, z)$ is weakly concave. Through envelope theorem, case (a) results in $G_t^{ij} = 0$ where G_t^{ij} denotes the second partials of $G_t(x_t, z)$. Similarly, case (b) results in $G_t^{11} = V_{t+1}^{11}$, $G_t^{12} = 0$, $G_t^{21} = 0$, and $G_t^{22} = 0$, case (c) results in $G_t^{11} = 0$, $G_t^{12} = 0$, $G_t^{21} = 0$, and $G_t^{22} = V_{t+1}^{11}$, and case (d) results in $G_t^{11} = V_{t+1}^{11}$, $G_t^{12} = \alpha V_{t+1}^{11}$, $G_t^{21} = -\alpha V_{t+1}^{11}$, and $G_t^{22} = \alpha^2 V_{t+1}^{11}$. Thus $G_t(x_t, z)$ is weakly concave. We also write , $V_t^{11}(x_t) = (p_r - c_r)(-\frac{1}{\beta_t}) f(\frac{x_t}{\beta_t}) + \int_0^{x_t/\beta_t} G_t^{11}(x_t, z_t) f(z) dz + \int_{x_t/\beta_t}^{\infty} V_{t+1}^{11}(x_t) f(z) dz$. In this expression, the first term is negative, the second term is non-positive due to $G_t^{11} \leq 0$ as shown previously, and the third terms is negative due to the inductional assumption. Therefore, $V_t(x_t)$ is strictly concave in x_t. Lastly, we show that $V_{T-1}(x_t)$ (one period to the last) is also strictly concave. We have $V_{T-1}^{11}(x_{T-1}) = (p_r - c_r)(-\frac{1}{\beta_{T-1}}) f(\frac{x_{T-1}}{\beta_{T-1}}) + \int_0^{x_{T-1}/\beta_{T-1}} G_{T-1}^{11}(x_{T-1}, z_{T-1}) f(z) dz + \int_{x_{T-1}/\beta_{T-1}}^{\infty} V_T^{11}(x_T) f(z) dz$, which is negative as the first term is negative, the second term is non-positive following similar arguments as above, and the last term is also non-positive due to $V_T(x_T) = h x_T$ being linear in x_T.

Proof of Proposition 2

The proof of Proposition 2 immediately follows from the strict concavity of $V_t(x_t)$ (at $t = 1$), which is shown in the proof of Proposition 1.

Proof of Proposition 3

We recall from the Proof of Proposition 1 the Lagrangian for the subproblem as given by $(p_s - c_a)\ s_t + V_{t+1}(x_t - s_t) + \lambda_{1t} s_t - \lambda_{2t}(s_t - x_t + \beta_t z) - \lambda_{3t}(s_t - \alpha_t z)$ with $\lambda_{it} \geq 0$ for $i = \{1, 2, 3\}$ and again distinguish four cases: (a) $\lambda_{1t} = 0, \lambda_{2t} = 0, \lambda_{3t} = 0$. For this case, the first order condition yields: $(p_s - c_a) - V_{t+1}^{1}(x_t - s_t)|_{s_t^*} = 0$. Differentiating this with respect to p_s^t, we get $1 + V_{t+1}^{11}\ (\frac{\partial s_t^*}{\partial p_s^t}) = 0$, which results in $\frac{\partial s_t^*}{\partial p_s^t} = -1/V_{t+1}^{11} > 0$ since $V_{t+1}^{11} < 0$. Further, for this case we have $s_t^* = x_t - \bar{r}_t$, hence we also have $\frac{\partial \bar{r}_t}{\partial p_s^t} = -\frac{\partial s_t^*}{\partial p_s^t} > 0$.. Hence, the rationing threshold $\bar{r}_t$ is increasing in p_s^t and the optimal sales quantity s_t^* is decreasing in p_s^t. Differentiating the first order condition with respect to p_r^t results in $\frac{\partial s_t^*}{\partial p_r^t} = 0$ and thus also $\frac{\partial \bar{r}_t}{\partial p_s^t} = 0$. (b) $\lambda_{1t} > 0, \lambda_{2t} = 0, \lambda_{3t} = 0$. This case immediately implies $s_t^* = 0$, and hence $\frac{\partial s_t^*}{\partial p_s^t} = 0$ and $\frac{\partial \bar{r}_t}{\partial p_s^t} = 0$. Similarly, we also have $\frac{\partial s_t^*}{\partial p_r^t} = 0$ and $\frac{\partial \bar{r}_t}{\partial p_r^t} = 0$. (c) $\lambda_{1t} = 0, \lambda_{2t} > 0, \lambda_{3t} = 0$ implies $s_t^* = x_t - \beta_t z$ for which $\frac{\partial s_t^*}{\partial p_s^t} = -\frac{\partial \beta_t^*}{\partial p_s^t} z_t < 0$ and $\frac{\partial s_t^*}{\partial p_r^t} = -\frac{\partial \beta_t^*}{\partial p_r^t} z_t > 0$. We recall that the rationing threshold is defined as zero for this case and thus $\frac{\partial \bar{r}_t}{\partial p_s^t} = 0$ $\frac{\partial \bar{r}_t}{\partial p_r^t} = 0$. (d) $\lambda_{1t} = 0, \lambda_{2t} = 0, \lambda_{3t} > 0$ implies $s_t^* = \alpha_t z$ and a similar reasoning as in case (c) also leads to the same result of $\frac{\partial s_t^*}{\partial p_s^t} < 0$ and $\frac{\partial s_t^*}{\partial p_r^t} > 0$ as well as $\frac{\partial \bar{r}_t}{\partial p_s^t} = 0$ $\frac{\partial \bar{r}_t}{\partial p_r^t} = 0$.

Proof of Proposition 4

The proof follows from the monotonicity of the optimal rental allocation s_t^* with respect to α_t and β_t along with the monotonicities of α_t and β_t with respect to the parameters of interest as shown below: We find $\frac{\partial \alpha_t}{\partial \gamma_j} = \sum_i \sum_j \sum_k ((1-q_i)\frac{(p_s - p_r)}{\bar{v}_t \gamma_j (1-q_i)} + \frac{(p_s-p_r)(1-\gamma_j(1-q_i))}{\gamma_j^2(1-q_i)\bar{v}_t} + \frac{u_k^o}{\gamma_j q_i}(1-\gamma_j(1-q_i))(1-q_i)$
$+ (\frac{u_k^o}{\gamma_j^2 q_i}(1-\gamma_j(1-q_i))))\phi_i\theta_j\omega_k > 0$; $\frac{\partial \alpha_t}{\partial u_k^o} = -\sum_i \sum_j (\frac{1-\gamma_j(1-q_i)}{\gamma_j q_i \bar{v}_t})\phi_j\theta_i < 0$; and
$\frac{\partial \alpha_t}{\partial q_i} = -\sum_i \sum_j \sum_k \frac{u_k^o}{\bar{v}_t q_i}(\frac{1-\gamma_j}{\gamma_j q_i} - \frac{(p_s-p_r)}{\gamma_j(1-q_i)^2\bar{v}_t})\phi_i\theta_j\omega_k$ and $\frac{\partial^2 \alpha_t}{\partial q_i^2} < 0$, so α_t is concave in q_i. Through a similar analysis we also have $\frac{\partial \beta_t}{\partial u_k^o} > 0$; $\frac{\partial \beta_t}{\partial \gamma_j} < 0$ and $\frac{\partial \beta_t}{\partial q_i^2} \geq 0$. Hence, with respect to q_i, there is a threshold such that s_t^* increases up to that threshold and decreases beyond the threshold. The other monotonicity results directly follow.

Proof of Proposition 5

(i) $\frac{d\, v'_{ijk}}{d\, \gamma_j} < 0$ and $\frac{d\, \alpha_{ijk}^t}{d\, \gamma_j} > 0$ which would make $\alpha_{iLk}^t > \alpha_{iHk}^t$ for $i,k = L,H$ since $\gamma_L > \gamma_H$. We can rewrite α_t as:

$$\alpha_t = \phi_H \sum_{i=L,H} \sum_{k=L,H} \alpha_{iHk}^t \theta_i \omega_k + (1-\phi_H) \sum_{i=L,H} \sum_{k=L,H} \alpha_{iLk}^t \theta_i \omega_k$$

Hence, as ϕ_H increases α_t decreases.
$\frac{d\, \beta_{ijk}^t}{d\, \gamma_j} > 0$, so $\beta_{iHk}^t > \beta_{iLk}^t$ for $i,k = L,H$ We can rewrite β_t as:

$$\beta_t = \phi_H \sum_{i=L,H} \sum_{k=L,H} \beta_{iHk}^t \theta_i \omega_k + (1-\phi_H) \sum_{i=L,H} \sum_{k=L,H} \beta_{iLk}^t \theta_i \omega_k$$

Hence, as ϕ_H increases, β_t increases as well; combining these two observations with the result that s_t^* increases with α_t and decreases with β_t proves the result.

(ii.) $\frac{d\, v'_{ijk}}{d\, u_k^o} > 0$ and $\frac{d\, v''_{ijk}}{d\, u_k^o} < 0$; $\frac{d\, \alpha_{ijk}^t}{d\, u_k^o} < 0$; $\frac{d\, \beta_{ijk}^t}{d\, u_k^o} > 0$. Rewriting α_t as:

$$\alpha_t = \omega_H \sum_{i=L,H} \sum_{j=L,H} \alpha_{ijH}^t \theta_i \phi_j + (1-\omega_H) \sum_{i=L,H} \sum_{j=L,H} \alpha_{ijL}^t \theta_i \phi_j$$

Hence α_t decreases with ω_H. Similarly, β_t can be rewritten as:

$$\beta_t = \omega_H \sum_{i=L,H} \sum_{j=L,H} \beta_{ijH}^t \theta_i \phi_j + (1-\omega_H) \sum_{i=L,H} \sum_{j=L,H} \beta_{ijL}^t \theta_i \phi_j$$

Hence, as ω_H increases, β_t increases as well; combining these two observations with the result that s_t^* increases with α_t and decreases with β_t proves the result.

On the other hand, α_t and β_t may increase or decrease with q_i as argued in the earlier result, so s_t^* increases when α_t increases and β_t decreases with q_i and s_t^* decreases when α_t decreases and β_t increases with q_i.

Proof of Proposition 6

To aid our analysis, we first rewrite (7) explicitly as follows:

$$V_t(x_t) = (p_r - c_r)\int_0^{x_t/\beta_t} \beta_t z f(z)dz + (p_r - c_r)\int_{x_t/\beta_t}^{\infty} x_t f(z)dz$$

$$+\int_0^{x_t/\beta_t} G_t(x_t, z) f(z)dz + \int_{x_t/\beta_t}^{\infty} \Big(p_l^t \ell_t x_t + V_{t+1}((1-\ell_t)x_t)\Big) f(z)dz$$

where $G_t(x_t, z) = \max_{s_t}\ (p_s - c_a)\ s_t + p_l^t \ell_t \beta_t z + V_{t+1}(x_t - \ell_t \beta_t z - s_t)$ subject to $0 \leq s_t \leq (x_t - \beta_t z)$ and $s_t \leq \alpha_t z$. The proof follows similar similar steps as in the proof of Proposition 1. We again make an inductional assumption that the value function in period $t+1$, $V_{t+1}(x_t)$ is strictly concave in x_t. The Lagrangian is given by $(p_s - c_a)\ s_t + p_l^t \ell_t \beta_t z + V_{t+1}(x_t - \ell_t \beta_t z - s_t) + \lambda_{1t} s_t - \lambda_{2t}(s_t - x_t + \beta_t z) - \lambda_{3t}(s_t - \alpha_t z)$ with $\lambda_{it} \geq 0$ for $i = \{1, 2, 3\}$. We again distinguish four possible cases. Case (a): $\lambda_{1t} = 0, \lambda_{2t} = 0, \lambda_{3t} = 0$. For this case, the first order condition yields: $(p_s - c_a) - V_{t+1}^1(x_t - \ell_t \beta_t z - s_t)|_{s_t^*} = 0$. Differentiating this with respect to x_t, we get $-V_{t+1}^{11}(x_t - \ell_t \beta_t z - s_t)|_{s_t^*}\ (1 - \frac{\partial s_t^*}{\partial x_t}) = 0$, which results in $\frac{\partial s_t^*}{\partial x_t} = 1$. We also find $\frac{\partial s_t^*}{\partial z} = -\ell_t \beta_t < 0$, $\frac{\partial s_t^*}{\partial \ell_t} = -\beta_t z < 0$, $\frac{\partial s_t^*}{\partial \beta_t} = -\ell_t z < 0$ and $\frac{\partial s_t^*}{\partial p_l^t} = 0$. Therefore, s_t^* is increasing in x_t, is decreasing in ℓ_t, z, and β_t, and does not change with p_l^t. Letting $\bar{r}_t^*$ denote the optimal rationing threshold defined as $\bar{r}_t^* = x_t - s_t^*$, we have $\frac{\partial \bar{r}_t^*}{\partial x_t} = 0$, $\frac{\partial \bar{r}_t^*}{\partial z} = \ell_t \beta_t > 0$. Therefore, the rationing threshold $\bar{r}_t^*$ is increasing with the observed demand z. Further, the rationing threshold also increases with ℓ_t, β_t, and does not change with p_l^t. Case (b) in which $\lambda_{1t} > 0, \lambda_{2t} = 0, \lambda_{3t} = 0$ yields $s_t^* = 0$, for which the rationing threshold can be defined as $\bar{r}_t^* = \infty$. In Case (c), with $\lambda_{1t} = 0, \lambda_{2t} > 0, \lambda_{3t} = 0$, we have $s_t^* = x_t - \beta_t z$, which leads to $\frac{\partial s_t^*}{\partial x_t} = 1 > 0$, $\frac{\partial s_t^*}{\partial z} = -\beta_t < 0$, $\frac{\partial s_t^*}{\partial \beta_t} = -z < 0$, $\frac{\partial s_t^*}{\partial \ell_t} = 0$ and $\frac{\partial s_t^*}{\partial p_l^t} = 0$. Further, for this case, we define $\bar{r}_t^* = 0$. Collectively, Cases (a)-(c) correspond to part (i) of the Proposition. In a similar fashion, Cases (a), (b), and (d) (where case (d) is defined by $\lambda_{1t} = 0, \lambda_{2t} = 0, \lambda_{3t} > 0$) collectively correspond to part (ii) of the Proposition, for which we omit the details for brevity. To complete the proof, we also note that a similar analysis as the one in the proof of Proposition 1 verifies the preservation of concavity for $V_t(x_t)$, thus completing the induction.

Proof of Proposition 7

We provide the proofs of parts (a)-(d) together. Similar to our notation in the proof of Proposition 1, and to aid expositional clarity, we let $V_t^1(x_t, y_t) := \frac{\partial V_t(x_t, y_t)}{\partial x_t}$, $V_t^2(x_t, y_t) := \frac{\partial V_t(x_t, y_t)}{\partial y_t}$. In a similar fashion, we also let $V_t^{ij}(x_t, y_t)$ denote the second partials, e.g., $V_t^{12}(x_t, y_t) = \frac{\partial^2 V_t(x_t, y_t)}{\partial x_t \partial y_t}$. In order to proceed with the proof of Proposition 7, we first introduce an inductional assumption on the properties of the value function $V_{t+1}(x_t, y_t)$.

Inductional Assumption: The value function $V_{t+1}(x_{t+1}, y_{t+1})$ possesses the following properties.

(i) $V_{t+1}(x_{t+1}, y_{t+1})$ is submodular and its second partials satisfy the following diagonal dominance conditions: $V_{t+1}^{11}(x_{t+1}, y_{t+1}) < V_{t+1}^{21}(x_{t+1}, y_{t+1}) \leq 0$ and $V_{t+1}^{22}(x_{t+1}, y_{t+1}) < V_{t+1}^{12}(x_{t+1}, y_{t+1}) \leq 0$

(ii) $V^1_{t+1}(x_{t+1}, y_{t+1}) \leq p_s$

Note that the above submodularity and strict diagonal dominance condition also immediately implies strict concavity. After characterizing the optimal policy, we will show that the assumption also holds for $V_t(x_t, y_t)$ through the dynamic programming recursions.

First, we observe that if $x_t\beta_t \leq y_t\alpha_t$, any particular demand realization will result in either no unmet rental demand or no available intermediate new product inventory to allocate for rentals. Hence, the firm's rental allocation decision only arises when $x_t\beta_t > y_t\alpha_t$. To facilitate our analysis, we rewrite (8) as follows:

$$V_t(x_t, y_t) = p_s \int_0^{x_t/\alpha_t} \alpha_t z f(z)dz + ps \int_{x_t/\alpha_t}^{\infty} x_t f(z)dz + (p_r - c_r)\int_0^{y_t/\beta_t} \beta_t z f(z)dz + (p_r - c_r)\int_{y_t/\beta_t}^{\infty} y_t f(z)dz$$

$$+ \begin{cases} \int_0^{x_t/\alpha_t} V_{t+1}(x_t - \alpha_t z, y_t) f(z)dz + \int_{x_t/\alpha_t}^{\infty} V_{t+1}(0, y_t) f(z)dz & \text{if } x_t\beta_t \leq y_t\alpha_t \\ \int_0^{y_t/\beta_t} V_{t+1}(x_t - \alpha_t z, y_t) f(z)dz + \int_{y_t/\beta_t}^{x_t/\alpha_t} G_t(x_t, y_t, z) f(z)dz + \int_{x_t/\alpha_t}^{\infty} V_{t+1}(0, y_t) f(z)dz & \text{if } x_t\beta_t > y_t\alpha_t \end{cases}$$

where $G_t(x_t, y_t, z) = \max_{s_t} \ (p_r - c_r - c_a)\ r_t + V_{t+1}(x_t - \alpha_t z - r_t, y_t + r_t)$ subject to $0 \leq r_t \leq (x_t - \alpha_t z)$ and $r_t \leq \beta_t z - y_t$.

The Lagrangian for the subproblem is given by $(p_r - c_r - c_a)\ r_t + V_{t+1}(x_t - \alpha_t z - r_t, y_t) + \lambda_{1t} r_t - \lambda_{2t}(r_t - x_t + \alpha_t z) - \lambda_{3t}(r_t - \beta_t z + y_t)$ with $\lambda_{it} \geq 0$ for $i = \{1, 2, 3\}$. There are four possible cases. (a): $\lambda_{1t} = 0, \lambda_{2t} = 0, \lambda_{3t} = 0$.

In this case, through the first order condition, we have $(p_r - c_r - c_a) - V^1_{t+1}(x_t - \alpha_t z - r_t, y_t + r_t)|_{r^*_t} + V^2_{t+1}(x_t - \alpha_t z - r_t, y_t + r_t)|_{r^*_t} = 0$. Differentiating with respect to x_t, we get $\frac{\partial r^*_t}{\partial x_t} = \frac{V^{11}_{t+1} - V^{21}_{t+1}}{V^{11}_{t+1} - V^{12}_{t+1} - V^{21}_{t+1} + V^{22}_{t+1}} > 0$ due to the induction assumption. Similarly, we find $\frac{\partial r^*_t}{\partial y_t} = \frac{V^{12}_{t+1} - V^{22}_{t+1}}{V^{11}_{t+1} - V^{12}_{t+1} - V^{21}_{t+1} + V^{22}_{t+1}} < 0$, and $\frac{\partial r^*_t}{\partial z_t} = -\alpha_t \frac{V^{12}_{t+1} - V^{22}_{t+1}}{V^{11}_{t+1} - V^{12}_{t+1} - V^{21}_{t+1} + V^{22}_{t+1}} < 0$. Let $\bar{s}^* := x_t - \alpha_t z - r^*_t$, that is, the firm allocates $r^*_t = x_t - \alpha_t z - \bar{s}^*$ so that its remaining inventory, i.e., rationing level for sales is $\bar{s}^*$. We have $\frac{\partial \bar{s}^*}{\partial x_t} = \frac{V^{22}_{t+1} - V^{12}_{t+1}}{V^{11}_{t+1} - V^{12}_{t+1} - V^{21}_{t+1} + V^{22}_{t+1}}$, $\frac{\partial \bar{s}^*}{\partial y_t} = -\frac{V^{22}_{t+1} - V^{12}_{t+1}}{V^{11}_{t+1} - V^{12}_{t+1} - V^{21}_{t+1} + V^{22}_{t+1}}$, and $\frac{\partial \bar{s}^*}{\partial z} = -\alpha_t \frac{V^{22}_{t+1} - V^{12}_{t+1}}{V^{11}_{t+1} - V^{12}_{t+1} - V^{21}_{t+1} + V^{22}_{t+1}}$. Hence the rationing level remains constant with a unit increase in x_t and a unit decrease in y_t. Similarly, the rationing level remains the same with a α_t unit increase in x_t and a unit decrease in z. Consequently, the rationing level is a function of the total intermediate inventory position $w_t = x_t - \alpha_t z + y_t$, such that $\frac{\partial \bar{s}^*}{\partial w_t} > 0$. Further, in this case we also have $\frac{\partial r^*_t}{\partial \alpha_t} = -z \frac{V^{11}_{t+1} - V^{21}_{t+1}}{V^{11}_{t+1} - V^{12}_{t+1} - V^{21}_{t+1} + V^{22}_{t+1}} < 0$ and $\frac{\partial r^*_t}{\partial \beta_t} = 0$. Case (b) $\lambda_{1t} > 0, \lambda_{2t} = 0, \lambda_{3t} = 0$. This case immediately implies $r^*_t = 0$, or equivalently, the sales rationing threshold can be defined as $\bar{s}^*_t = \infty$. Case (c): $\lambda_{1t} = 0, \lambda_{2t} > 0, \lambda_{3t} = 0$ implies $r^*_t = x_t - \beta_t z$ for which $\frac{\partial r^*_t}{\partial x_t} = 1 > 0$, $\frac{\partial r^*_t}{\partial y_t} = 0$, $\frac{\partial s^*_t}{\partial z} = -\alpha_t < 0$. We also have $\frac{\partial s^*_t}{\partial \alpha_t} = -z < 0$. We define the rationing threshold for this case as zero. For future reference we also find $\frac{\lambda^*_{2t}}{\partial x_t} = -V^{12}_{t+1} + V^{22}_{t+1}$, $\frac{\lambda^*_{2t}}{\partial y_t} = -V^{12}_{t+1} + V^{22}_{t+1}$, and $\frac{\lambda^*_{2t}}{\partial z} = \alpha_t(V^{12}_{t+1} - V^{22}_{t+1})$. Case (d): $\lambda_{1t} = 0, \lambda_{2t} = 0, \lambda_{3t} > 0$ implies $r^*_t = \beta_t z - y_t$ and similarly yields $\frac{\partial r^*_t}{\partial x_t} = 0$, $\frac{\partial r^*_t}{\partial y_t} = -1 < 0$, $\frac{\partial s^*_t}{\partial z} = \beta_t > 0$. We also have $\frac{\partial r^*_t}{\partial \beta_t} = z > 0$. We again define the rationing threshold for this case as zero. For future reference we also find $\frac{\lambda^*_{3t}}{\partial x_t} = V^{21}_{t+1} - V^{11}_{t+1}$, $\frac{\lambda^*_{3t}}{\partial y_t} = V^{21}_{t+1} - V^{11}_{t+1}$, and $\frac{\lambda^*_{3t}}{\partial z} = \alpha_t(V^{11}_{t+1} - V^{21}_{t+1}) + \beta_t(V^{11}_{t+1} - V^{12}_{t+1} - V^{21}_{t+1} + V^{22}_{t+1})$. Hence the optimal rental allocation quantity r^*_t can be described through a rationing threshold $\bar{s}^*_t(w_t)$ where w_t is the total intermediate inventory level as described previously. It is optimal for the firm to meet as much of the excess rental demand $y_t - \beta_t z$ with its new inventory availability after demand realization $x_t - \beta_t z$ as long as its total inventory level (for the next period) does not fall below $\bar{s}^*_t(w_t)$. When $\bar{s}^*_t = 0$, the amount the firm will sell is limited by either the excess rental demand quantity $y_t - \beta_t z$ or the available inventory $x_t - \alpha_t z$. Hence when $(x_t + y_t) \leq (\alpha_t + \beta_t)z_t$, the sales amount will be limited by $x_t - \beta_t z$ and when $(x_t + y_t) > (\alpha_t + \beta_t)z_t$, the sales amount will be limited by $y_t - \beta_t z$.

Lastly, we need to show that the inductional assumptions are preserved after the dynamic programming recursion. To show part (i) of the induction assumption, it is sufficient to show that $G^{11}_t \leq G^{21}_t \leq 0$ and that $G^{22}_t \leq G^{12}_t \leq 0$. For expositional clarity, let $\hat{V} := V^{11}_{t+1}V^{22}_{t+1} - V^{12}_{t+1}V^{21}_{t+1}$ and let $\hat{V} := V^{11}_{t+1} - V^{12}_{t+1} - V^{21}_{t+1} + V^{22}_{t+1}$. Through the envelope theorem, for case (a), we have $G^{11}_t = \hat{V}/\hat{V}$, $G^{12}_t = \hat{V}/\hat{V}$, and $G^{13}_t = -\alpha_t\hat{V}/\hat{V}$. Similarly, we find $G^{21}_t = \hat{V}/\hat{V}$, $G^{22}_t = \hat{V}/\hat{V}$, and $G^{23}_t = -\alpha_t\hat{V}/\hat{V}$, and $G^{31}_t = -\alpha_t\hat{V}/\hat{V}$, $G^{32}_t = -\alpha_t\hat{V}/\hat{V}$, and $G^{33}_t = \alpha^2_t\hat{V}/\hat{V}$. For case (b) we find $G^{11}_t = V^{11}_{t+1}$, $G^{12}_t = V^{12}_{t+1}$, and $G^{13}_t = -\alpha_t V^{11}_{t+1}$, $G^{21}_t = V^{21}_{t+1}$, $G^{22}_t = V^{22}_{t+1}$, and $G^{23}_t = -\alpha_t V^{21}_{t+1}$, $G^{31}_t = -\alpha_t V^{11}_{t+1}$, $G^{12}_t = -\alpha_t V^{12}_{t+1}$, and $G^{13}_t = \alpha^2_t V^{11}_{t+1}$. For brevity we omit the expressions for G^{ij}_t for the remaining two cases, which can be obtained similarly. For all cases, we have $G^{11}_t \leq G^{21}_t \leq 0$ and $G^{22}_t \leq G^{12}_t \leq 0$. Thus, part (i) also holds for period t. For part (ii), it is sufficient to show that $G^1_t \leq 0$ which holds through the envelope theorem. Finally, it can be verified that the inductional assumptions also hold for the trivial case of $x_t\beta_t \leq y_t\alpha_t$ and is preserved across the boundary.

Proof of Proposition 8

The proof of Proposition 8 is similar to the proof of Proposition 4 and follows from the corresponding results of Proposition 7.

Proof of Proposition 9

Similar to the proof of Proposition 5, we can show that as ϕ_L increases α_t increases and β_t decreases. Combining these two observations with the result that r^*_t decreases with α_t and increases with β_t proves the result. Similarly, we can show that as ω_L increases, α_t increases and β_t decreases and combining that with the aforementioned r^*_t result also proves the second result. On the other hand, α_t and β_t may increase or decrease with q_i as argued earlier, so r^*_t increases when α_t decreases and β_t increases with q_i and r^*_t decreases when α_t increases and β_t decreases with q_i.

Proof of Proposition 10

To facilitate the analysis, we let $J_t(x_t, y_t, r_t) = p_s E_{D_t}[\min(x_t - r_t, \alpha_t D_t)] + (p_r - c_r)E_{D_t}[\min(y_t + r_t, \beta_t D_t)] -$

$c_a r_t + E_{D_t} V_{t+1}\Big((x_t - r_t - \alpha D_t)^+, y_t + r_t\Big)$. In the remainder, we assume that $V_t(x_t, y_t)$ is twice-continuously differentiable and, for expositional clarity, we let $V_t^1(x_t, y_t)$ represent its first partial with respect to its first argument, i.e., $V_t^1(x_t, y_t) := \frac{\partial V_t(x_t,y_t)}{\partial x_t}$, and similarly let $V_t^{12}(x_t, y_t)$ denote the second partials, i.e., $V_t^{12}(x_t, y_t) = \frac{\partial^2 V_t(x_t,y_t)}{\partial x_t \partial y_t}$. We also define $V_t^2(x_t, y_t)$, $V_t^{11}(x_t, y_t)$, and $V_t^{22}(x_t, y_t)$ analogously. We first introduce an inductional assumption on the properties of the value function $V_{t+1}(x_{t+1}, y_{t+1})$. After characterizing the optimal policy, we will then show that the assumption hold through the dynamic programming recursions.

Inductional Assumption: The value function $V_t(x_t, y_t)$ possesses the following properties for $t = t+1, ..., T$.

(i) $V_t(x_t, y_t)$ is jointly concave in (x_t, y_t)

(ii) $V_t(x_t, y_t)$ is submodular and its second partials satisfy the following diagonal dominance conditions: $\frac{\partial^2 V_t(x_t,y_t)}{\partial x_t^2} \leq \frac{\partial^2 V_t(x_t,y_t)}{\partial x_t \partial y_t} \leq 0$ and $\frac{\partial^2 V_t(x_t,y_t)}{\partial y_t^2} \leq \frac{\partial^2 V_t(x_t,y_t)}{\partial x_t \partial y_t} \leq 0$

(iii) $\frac{\partial V_t(0,y_t)}{\partial x_t} \leq p_s$

We note that these properties hold for the last period's value function $V_T(x_T, y_T) = h_s x_T + h_r y_T$ as it is linear in x_T and y_T, and $\frac{\partial V_T(0,y_T)}{\partial x_t} < p_s$ (since $h_s < p_s$). We start by showing that $J_t(x_t, y_t, r_t)$ is strictly concave in r_t for any given (x_t, y_t). Differentiating $J_t(x_t, y_t, r_t)$ twice with respect to r_t results in $\frac{d^2 J_t(\cdot)}{dr_t^2} = -\frac{p_s}{\alpha_t} f(\frac{x_t - r_t}{\alpha_t}) - \frac{p_r - c_r}{\beta_t} f(\frac{y_t + r_t}{\beta_t}) + \frac{1}{\alpha_t} V_{t+1}^1(0, y_t + r_t) f(\frac{x_t - r_t}{\alpha_t}) + \int_0^{\frac{x_t - r_t}{\alpha_t}} (V_t^{11} - V_t^{12} - V_t^{21} + V_t^{22}) f(\xi_t)\, d\xi_t + \int_{\frac{x_t - r_t}{\alpha_t}}^{\infty} V_t^{22}(0, y_t + r_t) f(\xi_t)\, d\xi_t < 0$ where the inequality follows immediately from the inductional assumptions.

Next, let $\lambda_t^1 \geq 0$ and $\lambda_t^2 \geq 0$ denote Lagrangian variables associated with the constraints $r_t \geq 0$ and $r_t \leq x_t$, respectively. Then, the Lagrangian is $J_t(x_t, y_t, r_t) + \lambda_t^1(r_t) - \lambda_t^2(r_t - x_t)$. We distinguish four cases. (i) When $\lambda_t^1 = \lambda_t^2 = 0$, the optimal r_t^* solves the first order condition $\frac{\partial J_t(x_t,y_t,r_t^*)}{\partial r_t} = 0$. Next, we identify how r_t^* changes with respect to x_t and y_t. Differentiating the first order condition with respect to x_t and solving for $\frac{\partial r_t^*}{\partial x_t}$, we get $\frac{\partial r_t^*}{\partial x_t} =$

$$\frac{-\frac{p_s}{\alpha_t} f(\frac{x_t - r_t}{\alpha_t}) + \frac{1}{\alpha_t} V_{t+1}^1(0, y_t + r_t) f(\frac{x_t - r_t}{\alpha_t}) + \int_0^{\frac{x_t - r_t}{\alpha_t}} (V_t^{11} - V_t^{21}) f(\xi_t)\, d\xi_t}{-\frac{p_s}{\alpha_t} f(\frac{x_t - r_t}{\alpha_t}) - \frac{p_r}{\beta_t} f(\frac{y_t + r_t}{\beta_t}) + \frac{1}{\alpha_t} V_{t+1}^1(0, y_t + r_t) f(\frac{x_t - r_t}{\alpha_t}) + \int_0^{\frac{x_t - r_t}{\alpha_t}} (V_t^{11} - V_t^{12} - V_t^{21} + V_t^{22}) f(\xi_t)\, d\xi_t + \int_{\frac{x_t - r_t}{\alpha_t}}^{\infty} V_t^{22}(0, y_t + r_t) f(\xi_t)\, d\xi_t}$$

for which both the numerator and the denominator can be shown to be negative due to the inductional assumption. Thus, $\frac{\partial r_t^*}{\partial x_t} > 0$. A similar analysis verifies that $\frac{\partial r_t^*}{\partial y_t} < 0$. Further, we also find $1 - \frac{\partial r_t^*}{\partial x_t} > 0$ and $1 + \frac{\partial r_t^*}{\partial y_t} > 0$. That is, we have $0 < \frac{\partial r_t^*}{\partial x_t} < 1$, and $-1 < \frac{\partial r_t^*}{\partial y_t} < 0$. Now, define $\bar{r}_t(x_t, y_t) = y_t + r_t^*$, that is $\bar{r}_t(x_t, y_t)$ denotes a rental target level $y_t + r_t^*$ to be achieved after r_t^* amount of uprades. $\bar{r}_t(x_t, y_t)$ is increasing in x_t as r_t^* is increasing in x_t. Further, $\bar{r}_t(x_t, y_t)$ is increasing in y_t as $1 + \frac{\partial r_t^*}{\partial y_t} > 0$. Therefore, whenever additional allocation is preferable, i.e., $r_t^* > 0$, and is unconstrained by the available sales units $r_t^* < x_t$, $\bar{r}_t(x_t, y_t)$ denotes the rental target level. The remaining cases consider when there is no longer available sales inventory that could be used for further allocation and/or no more allocation is profitable. (ii) Consider now the case in which $\lambda_t^1 > 0$ and $\lambda_t^2 = 0$. then, due to the active constraint, we have $r_t^* = 0$. For future, reference, through the differentiation of the first order condition with respect to the Lagrangian variable λ_t^1, we also find $\frac{\partial \lambda_t^{1*}}{\partial x_t} < 0$ and $\frac{\partial \lambda_t^{1*}}{\partial y_t} > 0$. For this case, $\bar{r}_t(x_t, y_t) = y_t$. (iii) The case with $\lambda_t^1 = 0$ and $\lambda_t^2 > 0$ similarly leads to $r_t^* = x_t$, with $\frac{\partial \lambda_t^{2*}}{\partial x_t} < 0$ and $\frac{\partial \lambda_t^{2*}}{\partial y_t} < 0$. For this case, $\bar{r}_t(x_t, y_t) = y_t + x_t$. (iv) The case for which $\lambda_t^1 > 0$ and $\lambda_t^2 > 0$ implies $x_t = 0$, and subsequently, $r_t^* = 0$. This also yields $\frac{\partial \lambda_t^{1*}}{\partial y_t} - \frac{\partial \lambda_t^{2*}}{\partial y_t} > 0$. For this case, $\bar{r}_t(x_t, y_t) = y_t$.

To complete the proof of Theorem 1, we next show that the Inductional Assumption propagates to period t. Through the Envelope Theorem, we have $\frac{\partial V_t}{\partial x_t} = \frac{\partial J_t(x_t, y_t, r_t^*)}{\partial x_t} + \lambda_t^{2*}$, $\frac{\partial V_t}{\partial y_t} = \frac{\partial J_t(x_t, y_t, r_t^*)}{\partial x_t}$. Further, $\frac{\partial^2 V_t}{\partial x_t^2} = \frac{\partial^2 J_t(x_t, y_t, r_t^*)}{\partial x_t^2} + \frac{\partial \lambda_t^{2*}}{\partial x_t}$, $\frac{\partial^2 V_t}{\partial x_t \partial y_t} = \frac{\partial^2 J_t(x_t, y_t, r_t^*)}{\partial x_t \partial y_t} + \frac{\partial \lambda_t^{2*}}{\partial y_t}$, $\frac{\partial^2 V_t}{\partial y_t \partial x_t} = \frac{\partial^2 J_t(x_t, y_t, r_t^*)}{\partial x_t^2}$, and $\frac{\partial^2 V_t}{\partial y_t^2} = \frac{\partial^2 J_t(x_t, y_t + r_t^*)}{\partial y_t^2}$. Consider first part (iii), i.e.,verifying that $\frac{\partial V_t(0, y_t)}{\partial x_t} \leq p_s$. When $\lambda_t^{2*} = 0$, we obtain $\frac{\partial V_t(0, y_t)}{\partial x_t} = p_s + \int_0^{\frac{x_t - r_t}{\alpha_t}} (V_{t+1}^1 - p_s) f(\xi_t)\, d\xi_t \leq 0$ where the inequality follows from the inductional assumption. When $\lambda_t^{2*} > 0$, we have $r_t^* = x_t$, leading to $\frac{\partial V_t(0, y_t)}{\partial x_t} = 0$. Now consider part (ii) of the inductional assumption outlining the diagonal dominance and submodularity condition. When $\lambda_t^{1*} = \lambda_t^{2*} = 0$, we have $\frac{\partial^2 V_t}{\partial x_t \partial y_t} = (\frac{p_s}{\alpha_t})(\frac{\partial r_t^*}{\partial y_t}) f(\frac{x_t - r_t^*}{\alpha_t}) - \frac{\partial r_t^*}{\partial y_t} V_{t+1}^1(0, y_t + r_t^*) f(\frac{x_t - r_t^*}{\alpha_t}) + \int_0^{\frac{x_t - r_t^*}{\alpha_t}} \big[- \frac{\partial r_t^*}{\partial y_t} V_{t+1}^{11} + (1 + \frac{\partial r_t^*}{\partial y_t}) V_{t+1}^{12} \big] f(\xi_t)\, d\xi_t \leq 0$. Thus, $\frac{\partial^2 V_t}{\partial x_t \partial y_t}$ is submodular. The inequality follows from $-1 \leq \frac{\partial r_t^*}{\partial y_t} < 0$ (as established in Case (i) above), and the inductional assumptions. For this case, through a similar analysis, we also find $\frac{\partial^2 V_t}{\partial x_t^2} = \frac{\partial^2 V_t}{\partial x_t \partial y_t}$. A similar analysis also leads to $\frac{\partial^2 V_t}{\partial x_t^2} = \frac{\partial^2 V_t}{\partial x_t \partial y_t} \leq 0$. The proofs for the other conditions are similar and omitted for brevity. Part (i) of the inductional assumptions follow immediately from the diagonal dominance and submodularity conditions.

Proof of Proposition 11

The proof follows from the monotonicity of the optimal rental allocation r_t^* with α_t and β_t. Specifically, through differentiation of the first order conditions with respect to α_t for each of the four cases on the signs of λ_t^{1*}, and λ_t^{2*} as defined earlier in the proof of Proposition 10, we find that r_t^* decreases with α_t and increases with β_t. We omit the details for brevity.

Appendix B

A Heuristic Policy for Primarily Sales with Incidental Rentals Model

Here, we introduce and discuss the heuristic for the *Primarily Sales with Incidental Rentals* model analyzed in section 6.

Case	Fashion Consciousness			Use Frequency			Outside Flexibility		Prices and Valuation			Optimal Profit	Heuristic Profit	% diff.
#	ϕ_H	γ_H	γ_L	θ_H	q_H	q_L	ω_H	u_H	p_s	p_r	$\bar{v}$			
0	0.25	0.6	0.9	0.75	0.1	0.8	0.5	5	200	50	200	1895.6	1882.1	0.7
1	0.1	0.6	0.9	0.75	0.1	0.8	0.5	5	200	50	200	1966.2	1953.9	0.6
2	0.5	0.6	0.9	0.75	0.1	0.8	0.5	5	200	50	200	1763.2	1748.6	0.8
3	0.25	0.4	0.9	0.75	0.1	0.8	0.5	5	200	50	200	1843.3	1829.5	0.7
4	0.25	0.8	0.9	0.75	0.1	0.8	0.5	5	200	50	200	1976.6	1965.2	0.6
5	0.25	0.6	0.8	0.75	0.1	0.8	0.5	5	200	50	200	1797.2	1784.1	0.7
6	0.25	0.6	1.0	0.75	0.1	0.8	0.5	5	200	50	200	1989.4	1974.0	0.8
7	0.25	0.6	0.9	0.5	0.1	0.8	0.5	5	200	50	200	1655.6	1641.2	0.9
8	0.25	0.6	0.9	1.0	0.1	0.8	0.5	5	200	50	200	2030.4	2022.3	0.4
9	0.25	0.6	0.9	0.75	0.05	0.8	0.5	5	200	50	200	1930.9	1916.2	0.8
10	0.25	0.6	0.9	0.75	0.2	0.8	0.5	5	200	50	200	1776.6	1763.0	0.8
11	0.25	0.6	0.9	0.75	0.1	0.5	0.5	5	200	50	200	1923.2	1909.9	0.7
12	0.25	0.6	0.9	0.75	0.1	1.0	0.5	5	200	50	200	1895.6	1882.1	0.7
13	0.25	0.6	0.9	0.75	0.1	0.8	0.25	5	200	50	200	1928.3	1915.1	0.7
14	0.25	0.6	0.9	0.75	0.1	0.8	1.0	5	200	50	200	1856.4	1842.6	0.7
15	0.25	0.6	0.9	0.75	0.1	0.8	0.5	0	200	50	200	1946.0	1932.6	0.7
16	0.25	0.6	0.9	0.75	0.1	0.8	0.5	10	200	50	200	1861.8	1847.9	0.7
17	0.25	0.6	0.9	0.75	0.1	0.8	0.5	5	100	50	200	1103.8	1102.8	0.1
18	0.25	0.6	0.9	0.75	0.1	0.8	0.5	5	300	50	200	2454.8	2411.3	1.8
19	0.25	0.6	0.9	0.75	0.1	0.8	0.5	5	200	25	200	1746.4	1659.1	5.0
20	0.25	0.6	0.9	0.75	0.1	0.8	0.5	5	200	100	200	2066.6	2065.6	0.0
21	0.25	0.6	0.9	0.75	0.1	0.8	0.5	5	200	50	100	1518.5	1511.2	0.5
22	0.25	0.6	0.9	0.75	0.1	0.8	0.5	5	200	50	400	2050.9	2034.8	0.8

Table 2: Performance of the heuristic rental allocation policy across various parameter instances

Suppose the retailer has an intermediate inventory level of w_t units and is considering how much of this inventory, r, should ideally be allocated for rentals, thus leaving $(w_t - r)$ units protected for sales for the subsequent period. We define $F_S(\cdot)$ as the distribution of the total demand for sales over the remaining horizon and $F_R(\cdot)$ as the single-period distribution of the rental demand. An additional unit allocated for rentals is expected to generate a total rental profit of $(p_r - c_r)(T - t)(1 - F_R(r))$ across the remaining $(T - t)$ periods where $(1 - F_R(r))$ corresponds to the probability that the unit is rented in any given period. However, this unit instead could be sold as a new product and bring in p_s with probability $1 - F_R(w_t - r)$.

Thus, the firm's ideal rental allocation $\tilde{r}$ can be found by $F_S(w_t - \tilde{r}) = \frac{p_s-(p_r-c_r)+(p_r-c_r)(T-t)\bar{F}_R(\tilde{r})}{p_s}$. Hence, if the firm begins the current period with a rental inventory of y_t units, the heuristic policy allocates as much as the excess rental demand requires and up to $(y_t - \tilde{r})^+$ additional units for rentals.

The average difference between the heuristic profit and the optimal profit across all problem instances is 0.88%. We note that the profit values reported in the table correspond to the average profit obtained across a range of initial inventory levels (between 1 and 20 units.) We find that the performance of the heuristic further improves when the profits are compared based on a desirable initial order quantity. For example, for the parameter instances reported in Case 19, the best initial quantities for the optimal and the heuristic policies are 14.3 units and 14.9 units, respectively. With these initial order quantities, the optimal profit is 1665.0 whereas the heuristic profit is 1648.5, indicating a performance difference of 1.0%.