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A 9-dimensional algebra which is not a block of a finite group

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Abstract

We rule out a certain 9-dimensional algebra over an algebraically closed field to be the basic algebra of a block of a finite group, thereby completing the classification of basic algebras of dimension at most 12 of blocks of finite group algebras.

1 Introduction

Basic algebras of block algebras of finite groups over an algebraically closed field of dimension at most 12 have been classified in [13], except for one 9-dimensional symmetric algebra over an algebraically closed field k of characteristic 3 with two isomorphism classes of simple modules for which it is not known whether it actually arises as a basic algebra of a block of a finite group algebra. The purpose of this paper is to show that this algebra does not arise in this way. It is shown in [13, Section 2.9] that if A is a 9-dimensional basic algebra over an algebraically closed field k of prime characteristic p with two isomorphism classes of simple modules such that A is isomorphic to a basic algebra of a block B of kG for some finite group G , then the algebra A has the Cartan matrix

$$C = \begin{pmatrix} 5 & 1 \\ 1 & 2 \end{pmatrix},$$

Since the elementary divisors of C are 9 and 1, it follows that $p = 3$ and that a defect group P of B is either cyclic (in which case A is a Brauer tree algebra) or P is elementary abelian of order 9. We will show that the second case does not arise.

Theorem 1.1. *Let k be an algebraically closed field of prime characteristic p . Let G be a finite group and B a block of kG with Cartan matrix C as above. Then $p = 3$, the defect groups of B are cyclic of order 9, and B is Morita equivalent to the Brauer tree algebra of the tree with two edges, exceptional multiplicity 4 and exceptional vertex at the end of the tree.*

The proof of Theorem 1.1 proceeds in the following stages. We first identify in Theorem 2.1 any hypothetical basic algebra A of a block with Cartan matrix C as above and a noncyclic defect group. It turns out that there is only one candidate algebra, up to isomorphism. In Section 3 we give a description of the structure of this candidate A , and we show in Theorem 5.1 that A is not isomorphic to a basic algebra of a block. The proof of Theorem 2.1 amounts essentially to filling in the details in [13, section 2.9]. For the proof of Theorem 5.1 we combine a stable equivalence of Puig [17], a result of Broué in [5] on the invariance of stable centres under stable equivalences

of Morita type, results of Kiyota [9] on blocks with an elementary abelian defect group of order 9, and properties of blocks with symmetric stable centres from [8]. A slightly different approach to proving Theorem 5.1 is outlined in the last section, first showing in Proposition 6.1 a more precise result on the stable equivalence class of A , and then using Rouquier's stable equivalences for blocks with elementary abelian defect groups of rank 2 from [18].

Sambale [19] recently extended the classification of blocks with a low-dimensional basic algebra to the dimensions 13 and 14, and in dimension 15 the only open question is whether a certain Brauer tree algebra does arise as a block algebra.

For background material on describing finite-dimensional algebras in terms of their quivers and relations, see [2, Chapter III, Section 1], and for Brauer tree algebras, as part of the theory of blocks with cyclic defect groups, see [1, Chapter 5, Section 17] and [15, Sections 11.7 and 11.8].

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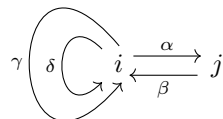
2 The basic algebra A of a noncyclic block with Cartan matrix C

The following result is stated in [6] without proof; for the convenience of the reader we give a detailed proof, following in part the arguments in [13, Section 2.9].

Theorem 2.1. *Let k be an algebraically closed field of prime characteristic p . Let A be a basic algebra with Cartan matrix*

$$C = \begin{pmatrix} 5 & 1 \\ 1 & 2 \end{pmatrix},$$

such that A is Morita equivalent to a block B of kG , for some finite group G , with a noncyclic defect group P . Then $p = 3$, we have $P \cong C_3 \times C_3$, and A is isomorphic to the algebra given by the quiver



with relations $\delta^2 = \gamma^3 = \alpha\beta$, $\delta\gamma = \gamma\delta = 0$, $\delta\alpha = \gamma\alpha = 0$, and $\beta\delta = \beta\gamma = 0$. In particular, we have

$$|\text{Irr}(B)| = \dim_k(Z(A)) = 6,$$

and the decomposition matrix of B is equal to

$$D = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Proof. As mentioned above, since the elementary divisors of the Cartan matrix C are 1 and 9, it follows that $p = 3$ and that B has a defect group P of order 9. Since P is assumed to be noncyclic, it follows that $P \cong C_3 \times C_3$.

Let $\{i, j\}$ be a primitive decomposition of 1 in A . Set $S = Ai/J(A)i$ and $T = Aj/J(A)j$. It follows from the entries of the Cartan matrix that we may choose notation such that Ai has composition length 6 and Aj has composition length 3. Since the top and bottom composition factor of Aj are both isomorphic to T , it follows that Aj is uniserial, with composition factors T, S, T (from top to bottom). In what follows, we tend to use the same notation for generators in A corresponding to homomorphisms between projective indecomposables; this reverses the order in relations since $\text{End}_A(A)$ is isomorphic to the opposite algebra A^{op} .

We label the two vertices of the quiver of A by i and j . The quiver of A contains a unique arrow from i to j and no loop at j because $J(A)j/J(A)^2j \cong S$. Thus there is an A -homomorphism

$$\alpha : Ai \rightarrow Aj$$

with image $\text{Im}(\alpha) = V$ uniserial of length 2, with composition factors S, T . Since Aj is uniserial of length 3, it follows that $V = J(A)j$ is the unique submodule of length 2 in Aj .

The symmetry of A implies that the quiver of A contains a path from j to i . This forces that the quiver of A has an arrow from j to i . Since the Cartan matrix of A implies that Ai has exactly one composition factor T , it follows that the quiver of A contains exactly one arrow from j to i . This arrow corresponds to an A -homomorphism

$$\beta : Aj \rightarrow Ai$$

which is not injective as Aj is an injective module. Thus $U = \text{Im}(\beta)$ is a submodule of Ai of length at most 2. The length of U cannot be 1, because the top composition factor of U is T , but the unique simple submodule of Ai is isomorphic to S . Thus U is a uniserial submodule of length 2 of Ai , with composition factors T, S . It follows that $\beta \circ \alpha$ is an endomorphism of Ai with image $\text{soc}(Ai)$.

Since $U = \text{Im}(\beta)$ and β corresponds to an arrow in the quiver of A , it follows that U is not contained in $J(A)^2i$. Thus the simple submodule $U/\text{soc}(Ai)$ of $J(A)i/\text{soc}(Ai)$ is not contained in the radical of $J(A)i/\text{soc}(Ai)$, and therefore must be a direct summand. Let M be a submodule of Ai such that $M/\text{soc}(Ai)$ is a complement of $U/\text{soc}(Ai)$ in $J(A)i/\text{soc}(Ai)$. Then

$$J(A)i = U + M$$

$$\text{soc}(Ai) = U \cap M$$

and, by the Cartan matrix, M has composition length 4, and all composition factors of M are isomorphic to S , and $\text{soc}(M) = \text{soc}(Ai)$. Equivalently, $M/\text{soc}(Ai)$ has length 3, with all composition factors isomorphic to S . We rule out some cases.

(1) $M/\text{soc}(Ai)$ cannot be semisimple. Indeed, if it were semisimple, then $J(A)i/\text{soc}(Ai) = U/\text{soc}(Ai) \oplus M/\text{soc}(Ai)$ would be semisimple. This would imply that $J(A)^3i = \{0\}$. Since also $J(A)^3j = \{0\}$, it would follow that $\ell\ell(A) = 3$. But a result of Okuyama in [16] rules this out. Thus $M/\text{soc}(Ai)$ is not semisimple.

(2) $M/\text{soc}(Ai)$ cannot be uniserial. Indeed, if it were, then the quiver of A would have a unique loop at i , corresponding to an endomorphism γ of Ai mapping Ai onto M (with kernel necessarily

equal to U because M has no composition factor isomorphic to T). Then $\gamma^5 = 0$ and γ^4 has image $\text{soc}(Ai) \cong S$.

By construction, α maps U to $\text{soc}(Aj)$ and β maps V to $\text{soc}(Ai)$. Thus $\beta \circ \alpha$ sends Ai onto $\text{soc}(Ai)$. Thus γ^4 and $\beta \circ \alpha$ differ at most by nonzero scalar. We may choose α such that $\gamma^4 = \beta \circ \alpha$.

The homomorphism α sends M to zero, because Aj contains no simple submodule isomorphic to S . Thus $\alpha \circ \gamma = 0$. Also, since U is the kernel of γ , we have $\gamma \circ \beta = 0$. Using the same letters α, β, γ for the elements in iAj, jAi, iAi , respectively, it follows that A is generated by $\{i, j, \alpha, \beta, \gamma\}$ with the (now opposite) relations $\gamma^4 = \alpha\beta, \gamma\alpha = 0 = \beta\gamma$, and all the obvious relations using that i, j are orthogonal idempotents whose sum is 1.

We will show next that these relations that A is a Brauer tree algebra, of a tree with two edges, exceptional multiplicity 4, and exceptional vertex at an end of the Brauer tree. By [15, Theorem 11.8.1] and its proof, such a Brauer tree algebra is generated by two orthogonal idempotents i, j whose sum is 1, and two elements r, s satisfying $ir = ri, jr = rj, is = sj, js = si, ir^4 + is^2 = 0$ and $jr + js^2 = 0$. Since $p = 3$ and k is algebraically closed, we may multiply s by a fourth root of unity, so that the latter two relations become $ir^4 = is^2$ and $jr = js^2$. One verifies that the assignment $r \mapsto \gamma + \beta\alpha$ and $s \mapsto \alpha + \beta$, together with the obvious assignments on the primitive idempotents, induces a surjective algebra homomorphism from this Brauer tree algebra to A . To see this, one first needs to verify that the above images of r and s satisfy the relations in A corresponding to those involving r and s in the Brauer tree algebra. This follows easily from the given relations for the generating set of A . For the surjectivity one needs to observe that α, β, γ are in the image of this map. This follows from multiplying r, s and their images by the primitive idempotents in the two algebras. Since both the Brauer tree algebra and A have dimension 9, it follows that they are isomorphic.

This, however, would force P to be cyclic, contradicting the current assumption that $P \cong C_3 \times C_3$.

(3) $M/\text{soc}(Ai)$ cannot be indecomposable. Indeed, if it were, then it would have Loewy length 2 because it has composition length 3, but is neither of length 1 (because it is not semisimple) nor of length 3 (because it is not uniserial). But then either its socle or its top is simple, and therefore it would have to be either a quotient of Ai , or a submodule of Ai . We rule out both cases.

Suppose first that $M/\text{soc}(Ai)$ is a quotient of Ai . Note that then M itself has a simple top, isomorphic to S , hence is a quotient of Ai because Ai is projective. Comparing composition lengths yields $M \cong Ai/U$. But also $U + M = J(A)i$, so the image of M in Ai/U is the unique maximal submodule $J(A)i/U$ of $Ai/U \cong M$. Thus $J(A)M$ is the unique maximal submodule of M , and that maximal submodule is isomorphic to a quotient of M , hence has itself a unique maximal submodule. This however would imply that $M/\text{soc}(Ai)$ is uniserial of length 3, which was ruled out earlier.

Suppose finally that $M/\text{soc}(Ai)$ is a submodule of Ai . Then it must be a submodule of M , because it does not have a composition factor T . Moreover, M and the image of $M/\text{soc}(Ai)$ in M both have the same simple socle $\text{soc}(Ai)$. Thus $M/\text{soc}(Ai)$ divided by its socle (which is simple) is a submodule of $M/\text{soc}(Ai)$, which has a simple socle. Thus the first and second socle series quotients are both simple, again forcing $M/\text{soc}(Ai)$ to be uniserial, which is not possible.

(4) Combining the above, it follows that $M/\text{soc}(Ai)$ is a direct sum of S and a uniserial module of length 2 with both composition factors S . That is, we have

$$M = M_1 + M_2$$

for some submodules M_i of M with

$$M_1 \cap M_2 = \text{soc}(Ai) = \text{soc}(M)$$

$$M_1/\text{soc}(Ai) \cong S$$

and $M_2/\text{soc}(Ai)$ uniserial of length 2. It follows that M_1 and M_2 are uniserial, of lengths 2 and 3, respectively.

We choose now M_2 as follows. By construction, we have a direct sum

$$J(A)i/\text{soc}(Ai) = U/\text{soc}(Ai) \oplus M_1/\text{soc}(Ai) \oplus M_2/\text{soc}(Ai)$$

Thus we have

$$J(A)i/(U + M_1) \cong (J(A)i/\text{soc}(Ai))/(U/\text{soc}(Ai) \oplus M_1/\text{soc}(Ai)) \cong M_2/\text{soc}(Ai) .$$

This is a uniserial module with two composition factors isomorphic to S . Thus $Ai/(U + M_1)$ is uniserial with three composition factors isomorphic to S , because $Ai/J(A)i \cong S$. Since in particular its socle is simple, isomorphic to S , this module is isomorphic to a submodule of Ai . Choose an embedding $A/(U + M_1) \rightarrow Ai$ and replace M_2 by the image of this embedding. Then the composition of canonical maps

$$\gamma : Ai \rightarrow Ai/(U + M_1) \rightarrow Ai$$

is an A -endomorphism of Ai with kernel $U + M_1$ and uniserial image M_2 of length three. Note that M_1 is uniserial of length two, so both a quotient and a submodule of Ai . Thus there is an endomorphism

$$\delta : Ai \rightarrow Ai$$

with image M_1 . Since $M_1 \subseteq \ker(\gamma)$, we have

$$\gamma \circ \delta = 0 .$$

We show next that we also have

$$\delta \circ \gamma = 0 .$$

One way to see this is to observe that this is a calculation in the split local 5-dimensional symmetric algebra $\text{End}_A(Ai) \cong (iAi)^{\text{op}}$, which as a consequence of [10, B. Theorem], is commutative.

There is a (slightly more general) argument that works in this case. Since the A -module Ai , and hence also the image of γ , is generated by i , it suffices to show that $\delta(\gamma(i)) = 0$. Now since $\gamma \circ \delta = 0$, we have

$$0 = \gamma(\delta(i)) = \gamma(\delta(i)i) = \delta(i)\gamma(i)$$

Note that $\delta(i) = \delta(i^2) = i\delta(i) \in iAi$, and similarly, $\gamma(i) \in iAi$. Since $\text{Im}(\delta) = M_2$ has length 2, we have $\text{Im}(\delta) \subseteq \text{soc}^2(A)$. Thus $\delta(i) \in \text{soc}^2(A) \cap iAi \subseteq \text{soc}^2(iAi)$, and since iAi is symmetric, we have $\text{soc}^2(iAi) \subseteq Z(iAi)$. It follows that

$$\delta(i)\gamma(i) = \gamma(i)\delta(i) = \delta(\gamma(i)i) = \delta(\gamma(i))$$

whence $\delta(\gamma(i)) = 0$, and so $\delta \circ \gamma = 0$ by the previous remarks. Thus $M_2 \subseteq \ker(\delta)$. Since $\text{Im}(\delta) = M_1$ has no composition factor T , it follows that $U \subseteq \ker(\delta)$. Together we get that $U + M_2 \subseteq \ker(\delta)$. Comparing composition lengths yields

$$\ker(\delta) = U + M_2 .$$

This implies that

$$\ker(\delta) \cap \text{Im}(\delta) = \text{soc}(Ai)$$

$$\ker(\gamma) \cap \text{Im}(\gamma) = \text{soc}(Ai)$$

and hence the endomorphisms δ^2 and γ^3 both map Ai onto $\text{soc}(Ai)$. Thus they differ by a nonzero scalar. Up to adjusting δ, β , we may therefore assume that

$$\delta^2 = \gamma^3 = \beta \circ \alpha$$

Since $\ker(\alpha)$ contains $M_1 + M_2$, it follows that

$$\alpha \circ \delta = \alpha \circ \gamma = 0 .$$

By taking these relations into account, it follows that $\text{End}_A(A)$ is spanned k -linearly by the set

$$\{i, j, \alpha, \beta, \gamma, \gamma^2, \delta, \delta^2, \alpha \circ \beta\}$$

so this is a basis of $\text{End}_A(A)$. We have identified here i, j with the canonical projections of A onto Ai and Aj . Note that $\text{End}_A(A)$ is the algebra opposite to A . This accounts for the reverse order in the relations of the generators in A (denoted abusively by the same letters). This shows that the quiver with relations of A is as stated. The equation $C = (D^t)D$ implies that the second column of D has exactly two nonzero entries and that these are equal to 1. The first row has either five entries equal to 1, which yields $|\text{Irr}(B)| = 6$ and the decomposition matrix D as stated. Or the first row has one entry 2 and one entry 1. This would lead to a decomposition matrix of the form

$$D = \begin{pmatrix} 2 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} .$$

In particular, this would yield $|\text{Irr}(B)| = 3$. But this is not possible, since $\dim_k(Z(A))$ is clearly greater than 3; indeed, $Z(A)$ contains the linearly independent elements $1, \delta, \gamma, \gamma^2$. This concludes the proof. \square

3 The structure of the algebra A

Let k be an algebraically closed field. Throughout this section we denote by A the k -algebra given in Theorem 2.1. We keep the notation of this theorem and identify the generators $i, j, \alpha, \beta, \gamma, \delta$ with their images in A .

Lemma 3.1.

- (i) *The set $\{i, j, \alpha, \beta, \beta\alpha, \gamma, \gamma^2, \delta, \delta^2\}$ is a k -basis of A .*

- (ii) The set $\{\alpha, \beta, \alpha\beta - \beta\alpha\}$ is a k -basis of $[A, A]$.
- (iii) The set $\{1, \gamma, \gamma^2, \delta, \delta^2, \beta\alpha\}$ is a k -basis of $Z(A)$.
- (iv) The set $\{\alpha\beta, \beta\alpha\}$ is a k -basis of $\text{soc}(A)$.

Proof. This follows immediately from the relations of the quiver of A . □

Lemma 3.2. *There is a unique symmetrising form $s : A \rightarrow k$ such that*

$$s(\alpha\beta) = s(\beta\alpha) = 1$$

and such that

$$s(i) = s(j) = s(\alpha) = s(\beta) = s(\gamma) = s(\gamma^2) = s(\delta) = 0$$

The dual basis with respect to the form s of the basis

$$\{i, j, \alpha, \beta, \beta\alpha, \gamma, \gamma^2, \delta, \delta^2\}$$

is, in this order, the basis

$$\{\alpha\beta, \beta\alpha, \beta, \alpha, j, \gamma^2, \gamma, \delta, i\}$$

Proof. Straightforward verification. □

See [5, §5.B] or [14, Definition 2.16.10] for details regarding the definitions and some properties of the *projective ideal* $Z^{\text{pr}}(A)$ in $Z(A)$ and the *stable centre* $\underline{Z}(A) = Z(A)/Z^{\text{pr}}(A)$.

Lemma 3.3. *Let $\text{char}(k) = 3$. The projective ideal $Z^{\text{pr}}(A)$ is one-dimensional, with basis $\{\alpha\beta - \beta\alpha\}$, we have an isomorphism of k -algebras*

$$\underline{Z}(A) \cong k[x, y]/(x^3 - y^2, xy, y^3)$$

induced by the map sending x to γ and y to δ , and after identifying x and y with their images in the quotient, the following statements hold:

- (i) The set $\{1, x, x^2, y, y^2\}$ is a k -basis of $\underline{Z}(A)$, and in particular $\dim_k(\underline{Z}(A)) = 5$.
- (ii) The set $\{x, x^2, y, y^2\}$ is a k -basis of $J(\underline{Z}(A))$.
- (iii) The set $\{x^2, y^2\}$ is a k -basis of $J(\underline{Z}(A))^2$.
- (iv) The set $\{y^2\}$ is a k -basis of $\text{soc}(\underline{Z}(A))$, and $J(\underline{Z}(A))^3 = \text{soc}(\underline{Z}(A))$.
- (v) The k -algebra $\underline{Z}(A)$ is a symmetric algebra.

Proof. It follows from lemma 3.2 that the relative trace map Tr_1^A from A to $Z(A)$ is given by

$$\text{Tr}_1^A(u) = iu\alpha\beta + ju\beta\alpha + \alpha u\beta + \beta u\alpha + \beta\alpha u j + \gamma u\gamma^2 + \gamma^2 u\gamma + \delta u\delta + \delta^2 u i$$

for all $u \in A$. One checks, using $\text{char}(k) = 3$, that

$$\text{Tr}_1^A(i) = -\text{Tr}_1^A(j) = \beta\alpha - \alpha\beta$$

and that Tr_1^A vanishes on all basis elements different from i, j . Statement (i) then follows from the relations in the quiver of A and Lemma 3.1. The algebra $\underline{Z}(A)$ is split local, proving statement (ii), whilst a straightforward computation shows both statement (iii) and (iv). Finally, a simple verification proves that the map $s : \underline{Z}(A) \rightarrow k$ such that

$$s(y^2) = 1$$

and such that

$$s(1) = s(x) = s(x^2) = s(y) = 0$$

is a symmetrising form on $\underline{Z}(A)$. One verifies also that the dual basis with respect to the form s of the basis

$$\{1, x, y, x^2, y^2\}$$

is, in this order, the basis

$$\{y^2, x^2, y, x, 1\}.$$

This completes the proof. \square

Remark 3.4. Note that by a result of Erdmann [7, I.10.8(i)], A is of wild representation type.

4 The stable centre of the group algebra $k(P \rtimes C_2)$.

Let k be a field of characteristic 3. Set $P = C_3 \times C_3$ and E the subgroup of $\text{Aut}(P)$ of order 2 such that the nontrivial element t of E acts as inversion on P . Denote by $H = P \rtimes E$ the corresponding semidirect product; this is a Frobenius group. Denote by r and s generators of the two factors C_3 of P . The following Lemma holds in greater generality (see Remark 4.1 in [8]); we state only what we need in this paper.

Lemma 4.1. *The projective ideal $Z^{\text{pr}}(kH)$ is one-dimensional, with k -basis $\{\sum_{x \in P} xt\}$, and we have an isomorphism of k -algebras*

$$\underline{Z}(kH) \cong (kP)^E$$

induced by the map sending $x + x^{-1}$ in $(kP)^E$ to its image in $\underline{Z}(kH)$. In particular, we have $\dim_k(\underline{Z}(kH)) = 5$, and the image of the set $\{1, r + r^2, s + s^2, r^2s + rs^2, rs + r^2s^2\}$ is a k -basis of $\underline{Z}(kH)$.

Proof. The relative trace map Tr_1^H from kH to $\underline{Z}(kH)$ satisfies $\text{Tr}_1^H = \text{Tr}_P^H \circ \text{Tr}_1^P$. We calculate for all $a \in P$

$$\text{Tr}_1^P(a) = \sum_{g \in P} gag^{-1} = \sum_{|P|} a = 9 \cdot a = 0$$

Thus for every $c \in kP$ we have $\text{Tr}_1^H(c) = \text{Tr}_P^H(\text{Tr}_1^P(c)) = 0$. On the other hand, for every element of the form at in H , where $a \in P$, we have

$$\begin{aligned}
\mathrm{Tr}_1^H(at) &= \sum_{g \in P} g(at)g^{-1} + \sum_{g \in P} (gt)(at)(gt)^{-1} \\
&= (a + a^{-1}) \sum_{x \in P} xt \\
&= 2 \cdot \left(\sum_{x \in P} xt \right)
\end{aligned}$$

The conjugacy classes of G are given by $\{1\}$, $\{r, r^2\}$, $\{s, s^2\}$, $\{r^2s, rs^2\}$, $\{rs, r^2s^2\}$ and $\{xt \mid x \in P\}$. The last statement follows. \square

Lemma 4.2. *There is an isomorphism of k -algebras*

$$\underline{Z}(kH) \cong (k[x, y]/(x^3, y^3))^E$$

with inverse induced by the map sending x to $r - 1$ and y to $s - 1$, where the nontrivial element t of E acts by $x^t = x^2 + 2x$ and $y^t = y^2 + 2y$. After identifying x and y with their images in $k[x, y]/(x^3, y^3)$, the following statements hold:

- (i) *The image of the set $\{1, x^2, y^2, xy + x^2y + xy^2, x^2y^2\}$ is a k -basis of $\underline{Z}(kH)$.*
- (ii) *The set $\{x^2, y^2, xy + x^2y + xy^2, x^2y^2\}$ is a k -basis of $J(\underline{Z}(kH))$.*
- (iii) *The set $\{x^2y^2\}$ is a k -basis of $\mathrm{soc}(\underline{Z}(kH))$, and $J(\underline{Z}(kH))^2 = \mathrm{soc}(\underline{Z}(kH))$. In particular, $\dim_k(J(\underline{Z}(kH))^2) = 1$.*
- (iv) *The k -algebra $\underline{Z}(kH)$ is symmetric.*

Proof. By Lemma 4.1 we have $\underline{Z}(kH) \cong (kP)^E$. Since k has characteristic 3, we have an isomorphism $kP \cong k[x, y]/(x^3, y^3)$ induced by the map given in the statement of the lemma. Under this isomorphism, the action of t on x and y is given by $x^t = x^2 + 2x$ and $y^t = y^2 + 2y$ as stated. It is straightforward to then verify that this isomorphism gives

$$\begin{aligned}
r + r^t &\mapsto x^2 + 2, \\
s + s^t &\mapsto y^2 + 2, \\
rs + (rs)^t &\mapsto 2 + x^2 + y^2 + 2xy + 2x^2y + 2xy^2 + x^2y^2, \\
r^2s + (r^2s)^t &\mapsto 2 + x^2 + y^2 + xy + x^2y + xy^2.
\end{aligned}$$

This proves the statement (i) and (ii). A straightforward computation proves statement (iii). The final statement is given in general in [8, Corollary 1.3], with an explicit symmetrising form $s : \underline{Z}(kH) \rightarrow k$ given by $s(x^2y^2) = 1$ and sending all other basis elements to 0. \square

5 Proof of Theorem 1.1

Theorem 1.1 will be an immediate consequence of Theorem 2.1 and the following result.

Theorem 5.1. *Let k be an algebraically closed field of prime characteristic p , and let A be the algebra given in Theorem 2.1. Then A is not isomorphic to a basic algebra of a block of a finite group algebra over k .*

Proof. Arguing by contradiction, suppose that A is isomorphic to a basic algebra of a block B of kG , for some finite group G . Denote by P a defect group of B . By Theorem 2.1 we have $p = 3$ and $P \cong C_3 \times C_3$. By Lemma 3.3, the stable centre $\underline{Z}(A)$ is symmetric, hence so is $\underline{Z}(B)$, as A and B are Morita equivalent. It follows from [8, Proposition 3.8] that we have an algebra isomorphism

$$\underline{Z}(A) \cong (kP)^E$$

where E is the inertial quotient of the block B . Again by Lemma 3.3, we have $\dim_k((kP)^E) = 5$, or equivalently, E has five orbits in P . The list of possible inertial quotients in Kiyota's paper [9] shows that E is isomorphic to one of $1, C_2, C_2 \times C_2, C_4, C_8, D_8, Q_8, SD_{16}$. In all cases except for $E \cong C_2$ is the action of E on P determined, up to equivalence, by the isomorphism class of E . Thus if E contains a cyclic subgroup of order 4, then E has at most 3 orbits, and if E is the Klein four group, then E has 4 orbits. Therefore we have $E \cong C_2$. If the nontrivial element t of E has a nontrivial fixed point in P (or equivalently, if t centralises one of the factors C_3 of P and acts as inversion on the other), then E has 6 orbits. Thus t has no nontrivial fixed point in P , and the group $H = P \rtimes E$ is the Frobenius group considered in the previous section. By a result of Puig [17, 6.8] (also described in [15, Theorem 10.5.1]), there is a stable equivalence of Morita type between B and kH , hence between A and kH . By a result of Broué [5, 5.4] (see also [14, Corollary 2.17.14]), there is an algebra isomorphism $\underline{Z}(A) \cong \underline{Z}(kH)$. This, however, contradicts the calculations in the Lemmas 3.3 and 4.2, which show that the dimension of $J(\underline{Z}(A))^2$ and of $J(\underline{Z}(kH))^2$ are different. This contradiction completes the proof. \square

Proof of Theorem 1.1. Arguing by contradiction, if a defect P of B is not cyclic, then $P \cong C_3 \times C_3$ because the Cartan matrix of B has elementary divisors 9 and 1. But then B has a basic algebra isomorphic to the algebra A in Theorem 2.1. This, however, is ruled out by Theorem 5.1. \square

6 Further remarks

Using the arguments of the proof of Theorem 5.1 it is possible to prove some slightly stronger statements about the stable equivalence class of the algebra A from Theorem 2.1.

Proposition 6.1. *Let k be an algebraically closed field of prime characteristic p and let A be the algebra in Theorem 2.1. Let P be a finite p -group, E a p' -subgroup of $\text{Aut}(P)$, and $\tau \in H^2(E; k^\times)$. There does not exist a stable equivalence of Morita type between A and the twisted group algebra $k_\tau(P \rtimes E)$.*

Proof. Arguing by contradiction, suppose that there is a stable equivalence of Morita type between A and $k_\tau(P \rtimes E)$. Note that $k_\tau(P \rtimes E)$ is a block of a central p' -extension of $P \rtimes E$ with defect group P , so its Cartan matrix has a determinant divisible by $|P|$. By [14, Proposition 4.14.13], the

Cartan matrices of the algebras A and $k_\tau(P \rtimes E)$ have the same determinant, which is 9. Since A is clearly not of finite representation type (cf. Remark 3.4), it follows that P is not cyclic, hence $P \cong C_3 \times C_3$. Using as before Broué's result [5, 5.4], we have an isomorphism $\underline{Z}(A) \cong \underline{Z}(k_\tau(P \rtimes E))$. Since $\underline{Z}(A)$ is symmetric, so is $\underline{Z}(k_\tau(P \rtimes E))$. Since $k_\tau(P \rtimes E)$ is a block of a central p' -extension of $P \rtimes E$ with defect group P and inertial quotient E , it follows again from [8, Proposition 3.8] that $\underline{Z}(A) \cong (kP)^E$. From this point onward, the rest of the proof follows the proof of Theorem 5.1, whence the result. \square

Remark 6.2. By results of Rouquier [18, 6.3] (see also [12, Theorem A2]), for any block B with an elementary abelian defect group of rank 2 there is a stable equivalence of Morita type between B and its Brauer correspondent, which by a result of Külshammer [11], is Morita equivalent to a twisted semidirect product group algebra as in Proposition 6.1. Thus Theorem 5.1 can be obtained as a consequence of Proposition 6.1 and Rouquier's stable equivalence.

Remark 6.3. A slightly different proof of Theorem 5.1 makes use of Broué's surjective algebra homomorphism $Z(B) \rightarrow (kZ(P))^E$ from [4, Proposition III (1.1)], induced by the Brauer homomorphism Br_P , where here P is a (not necessarily abelian) defect group of a block B of a finite group algebra kG , with k an algebraically closed field of prime characteristic p . If P is normal in G , then it is easy to see that Broué's homomorphism is split surjective, but this is not known in general. If B is a block with P nontrivial such that there exists a stable equivalence of Morita type between B and its Brauer correspondent, then this implies the existence of at least *some* split surjective algebra homomorphism $\underline{Z}(B) \rightarrow kZ(P)^E$.

Kiyota's list in [9] shows that if A were isomorphic to a basic algebra of a block with defect group $P \cong C_3 \times C_3$, then E would be isomorphic to one of C_2 or D_8 (subcase (b) in Kiyota's list). The case C_2 can be ruled out as above, and the case D_8 can be ruled out by using Rouquier's stable equivalence, and by showing that if $E \cong D_8$, then $(kP)^E$ is uniserial of dimension 3, but $\underline{Z}(A)$ admits no split surjective algebra homomorphism onto a uniserial algebra of dimension 3. Note that $\underline{Z}(A)$ does though admit a surjective algebra homomorphism onto a uniserial algebra of dimension 3, so the splitting is an essential point in this argument, and may warrant further investigation.

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