A 9-dimensional algebra which is not a block of a finite group

Markus Linckelmann and William Murphy

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Abstract

We rule out a certain 9-dimensional algebra over an algebraically closed field to be the basic algebra of a block of a finite group, thereby completing the classification of basic algebras of dimension at most 12 of blocks of finite group algebras.

1 Introduction

Basic algebras of block algebras of finite groups over an algebraically closed field of dimension at most 12 have been classified in [13], except for one 9-dimensional symmetric algebra over an algebraically closed field k of characteristic 3 with two isomorphism classes of simple modules for which it is not known whether it actually arises as a basic algebra of a block of a finite group algebra. The purpose of this paper is to show that this algebra does not arise in this way. It is shown in [13, Section 2.9] that if A is a 9-dimensional basic algebra over an algebraically closed field k of prime characteristic p with two isomorphism classes of simple modules such that A is isomorphic to a basic algebra of a block B of kG for some finite group G, then the algebra A has the Cartan matrix

$$C = \begin{pmatrix} 5 & 1 \\ 1 & 2 \end{pmatrix},$$

Since the elementary divisors of C are 9 and 1, it follows that $p = 3$ and that a defect group P of B is either cyclic (in which case A is a Brauer tree algebra) or P is elementary abelian of order 9. We will show that the second case does not arise.

Theorem 1.1. *Let k be an algebraically closed field of prime characteristic p. Let G be a finite group and B a block of kG with Cartan matrix C as above. Then $p = 3$, the defect groups of B are cyclic of order 9, and B is Morita equivalent to the Brauer tree algebra of the tree with two edges, exceptional multiplicity 4 and exceptional vertex at the end of the tree.*

The proof of Theorem 1.1 proceeds in the following stages. We first identify in Theorem 2.1 any hypothetical basic algebra A of a block with Cartan matrix C as above and a noncyclic defect group. It turns out that there is only one candidate algebra, up to isomorphism. In Section 3 we give a description of the structure of this candidate A, and we show in Theorem 5.1 that A is not isomorphic to a basic algebra of a block. The proof of Theorem 2.1 amounts essentially to filling in the details in [13, section 2.9]. For the proof of Theorem 5.1 we combine a stable equivalence of Puig [17], a result of Broué in [5] on the invariance of stable centres under stable equivalences of Morita type, results of Kiyota [9] on blocks with an elementary abelian defect group of order 9, and properties of blocks with symmetric stable centres from [8]. A slightly different approach to proving Theorem 5.1 is outlined in the last section, first showing in Proposition 6.1 a more precise result on the stable equivalence class of A, and then using Rouquier's stable equivalences for blocks with elementary abelian defect groups of rank 2 from [18].

Sambale [19] recently extended the classification of blocks with a low-dimensional basic algebra to the dimensions 13 and 14, and in dimension 15 the only open question is whether a certain Brauer tree algebra does arise as a block algebra.

For background material on describing finite-dimensional algebras in terms of their quivers and relations, see [2, Chapter III, Setion 1], and for Brauer tree algebras, as part of the theory of blocks with cyclic defect groups, see [1, Chapter 5, Section 17] and [15, Sections 11.7 and 11.8].

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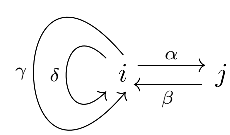
2 The basic algebra A of a noncyclic block with Cartan matrix C

The following result is stated in [6] without proof; for the convenience of the reader we give a detailed proof, following in part the arguments in [13, Section 2.9].

Theorem 2.1. *Let k be an algebraically closed field of prime characteristic p. Let A be a basic algebra with Cartan matrix*

$$C = \begin{pmatrix} 5 & 1 \\ 1 & 2 \end{pmatrix},$$

such that A is Morita equivalent to a block B of kG, for some finite group G, with a noncyclic defect group P. Then $p = 3$, we have $P \cong C_3 \times C_3$, and A is isomorphic to the algebra given by the quiver



with relations $\delta^2 = \gamma^3 = \alpha\beta$, $\delta\gamma = \gamma\delta = 0$, $\delta\alpha = \gamma\alpha = 0$, and $\beta\delta = \beta\gamma = 0$. In particular, we have

$$|\mathrm{Irr}(B)| = \dim_k(Z(A)) = 6 \ ,$$

and the decomposition matrix of B is equal to

$$D = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} \quad .$$

Proof. As mentioned above, since the elementary divisors of the Cartan matrix C are 1 and 9, it follows that $p = 3$ and that B has a defect group P of order 9. Since P is assumed to be noncyclic, it follows that $P \cong C_3 \times C_3$.

Let $\{i, j\}$ be a primitive decomposition of 1 in A. Set $S = Ai/J(A)i$ and $T = Aj/J(A)j$. It follows from the entries of the Cartan matrix that we may choose notation such that Ai has composition length 6 and Aj has composition length 3. Since the top and bottom composition factor of Aj are both isomorphic to T, it follows that Aj is uniserial, with composition factors T, S, T (from top to bottom). In what follows, we tend to use the same notation for generators in A corresponding to homomorphisms between projective indecomposables; this reverses the order in relations since $\mathrm{End}_A(A)$ is isomorphic to the opposite algebra A^{op}.

We label the two vertices of the quiver of A by i and j. The quiver of A contains a unique arrow from i to j and no loop at j because $J(A)j/J(A)^2 j \cong S$. Thus there is an A-homomorphism

$$\alpha : Ai \to Aj$$

with image $\mathrm{Im}(\alpha) = V$ uniserial of length 2, with composition factors S, T. Since Aj is uniserial of length 3, it follows that $V = J(A)j$ is the unique submodule of length 2 in Aj.

The symmetry of A implies that the quiver of A contains a path from j to i. This forces that the quiver of A has an arrow from j to i. Since the Cartan matrix of A implies that Ai has exactly one composition factor T, it follows that the quiver of A contains exactly one arrow from j to i. This arrow corresponds to an A-homomorphism

$$\beta : Aj \to Ai$$

which is not injective as Aj is an injective module. Thus $U = \mathrm{Im}(\beta)$ is a submodule of Ai of length at most 2. The length of U cannot be 1, because the top composition factor of U is T, but the unique simple submodule of Ai is isomorphic to S. Thus U is a uniserial submodule of length 2 of Ai, with composition factors T, S. It follows that $\beta \circ \alpha$ is an endomorphism of Ai with image $\mathrm{soc}(Ai)$.

Since $U = \mathrm{Im}(\beta)$ and β corresponds to an arrow in the quiver of A, it follows that U is not contained in $J(A)^2 i$. Thus the simple submodule $U/\mathrm{soc}(Ai)$ of $J(A)i/\mathrm{soc}(Ai)$ is not contained in the radical of $J(A)i/\mathrm{soc}(Ai)$, and therefore must be a direct summand. Let M be a submodule of Ai such that $M/\mathrm{soc}(Ai)$ is a complement of $U/\mathrm{soc}(Ai)$ in $J(A)i/\mathrm{soc}(Ai)$. Then

$$J(A)i = U + M$$

$$\mathrm{soc}(Ai) = U \cap M$$

and, by the Cartan matrix, M has composition length 4, and all composition factors of M are isomorphic to S, and $\mathrm{soc}(M) = \mathrm{soc}(Ai)$. Equivalently, $M/\mathrm{soc}(Ai)$ has length 3, with all composition factors isomorphic to S. We rule out some cases.

(1) $M/\mathrm{soc}(Ai)$ cannot be semisimple. Indeed, if it were semisimple, then $J(A)i/\mathrm{soc}(Ai) = U/\mathrm{soc}(Ai) \oplus M/\mathrm{soc}(Ai)$ would be semisimple. This would imply that $J(A)^3 i = \{0\}$. Since also $J(A)^3 j = \{0\}$, it would follow that $\ell\ell(A) = 3$. But a result of Okuyama in [16] rules this out. Thus $M/\mathrm{soc}(Ai)$ is not semisimple.

(2) $M/\mathrm{soc}(Ai)$ cannot be uniserial. Indeed, if it were, then the quiver of A would have a unique loop at i, corresponding to an endomorphism γ of Ai mapping Ai onto M (with kernel necessarily equal to U because M has no composition factor isomorphic to T). Then $\gamma^5 = 0$ and γ^4 has image $\mathrm{soc}(Ai) \cong S$.

By construction, α maps U to $\mathrm{soc}(Aj)$ and β maps V to $\mathrm{soc}(Ai)$. Thus $\beta \circ \alpha$ sends Ai onto $\mathrm{soc}(Ai)$. Thus γ^4 and $\beta \circ \alpha$ differ at most by nonzero scalar. We may choose α such that $\gamma^4 = \beta \circ \alpha$.

The homomorphism α sends M to zero, because Aj contains no simple submodule isomorphic to S. Thus $\alpha \circ \gamma = 0$. Also, since U is the kernel of γ, we have $\gamma \circ \beta = 0$. Using the same letters α, β, γ for the elements in iAj, jAi, iAi, respectively, it follows that A is generated by $\{i, j, \alpha, \beta, \gamma\}$ with the (now opposite) relations $\gamma^4 = \alpha\beta$, $\gamma\alpha = 0 = \beta\gamma$, and all the obvious relations using that i, j are orthogonal idempotents whose sum is 1.

We will show next that these relations that A is a Brauer tree algebra, of a tree with two edges, exceptional multiplicity 4, and exceptional vertex at an end of the Brauer tree. By [15, Theorem 11.8.1] and its proof, such a Brauer tree algebra is generated by two orthogonal idempotents i, j whose sum is 1, and two elements r, s satisfying $ir = ri$, $jr = rj$, $is = sj$, $js = si$, $ir^4 + is^2 = 0$ and $jr + js^2 = 0$. Since $p = 3$ and k is algebraically closed, we may multiply s by a fourth root of unity, so that the latter two relations become $ir^4 = is^2$ and $jr = js^2$. One verifies that the assignment $r \mapsto \gamma + \beta\alpha$ and $s \mapsto \alpha + \beta$, together with the obvious assignments on the primitive idempotents, induces a surjective algebra homomorphism from this Brauer tree algebra to A. To see this, one first needs to verify that the above images of r and s satisfy the relations in A corresponding to those involving r and s in the Brauer tree algebra. This follows easily from the given relations for the generating set of A. For the surjectivity one needs to observe that α, β, γ are in the image of this map. This follows from multiplying r, s and their images by the primitive idempotents in the two algebras. Since both the Brauer tree algebra and A have dimension 9, it follows that they are isomorphic.

This, however, would force P to be cyclic, contradicting the current assumption that $P \cong C_3 \times C_3$.

(3) $M/\mathrm{soc}(Ai)$ cannot be indecomposable. Indeed, if it were, then it would have Loewy length 2 because it has composition length 3, but is neither of length 1 (because it is not semisimple) nor of length 3 (because it is not uniserial). But then either its socle or its top is simple, and therefore it would have to be either a quotient of Ai, or a submodule of Ai. We rule out both cases.

Suppose first that $M/\mathrm{soc}(Ai)$ is a quotient of Ai. Note that then M itself has a simple top, isomorphic to S, hence is a quotient of Ai because Ai is projective. Comparing composition lengths yields $M \cong Ai/U$. But also $U + M = J(A)i$, so the image of M in Ai/U is the unique maximal submodule $J(A)i/U$ of $Ai/U \cong M$. Thus $J(A)M$ is the unique maximal submodule of M, and that maximal submodule is isomorphic to a quotient of M, hence has itself a unique maximal submodule. This however would imply that $M/\mathrm{soc}(Ai)$ is uniserial of length 3, which was ruled out earlier.

Suppose finally that $M/\mathrm{soc}(Ai)$ is a submodule of Ai. Then it must be a submodule of M, because it does not have a composition factor T. Moreover, M and the image of $M/\mathrm{soc}(Ai)$ in M both have the same simple socle $\mathrm{soc}(Ai)$. Thus $M/\mathrm{soc}(Ai)$ divided by its socle (which is simple) is a submodule of $M/\mathrm{soc}(Ai)$, which has a simple socle. Thus the first and second socle series quotients are both simple, again forcing $M/\mathrm{soc}(Ai)$ to be uniserial, which is not possible.

(4) Combining the above, it follows that $M/\mathrm{soc}(Ai)$ is a direct sum of S and a uniserial module of length 2 with both composition factors S. That is, we have

$$M = M_1 + M_2$$

for some submodules M_i of M with

$$M_1 \cap M_2 = \mathrm{soc}(Ai) = \mathrm{soc}(M)$$

$$M_1/\mathrm{soc}(Ai) \cong S$$

and $M_2/\mathrm{soc}(A_i)$ uniserial of length 2. It follows that M_1 and M_2 are uniserial, of lengths 2 and 3, respectively.

We choose now M_2 as follows. By construction, we have a direct sum

$$J(A)i/\mathrm{soc}(Ai) = U/\mathrm{soc}(Ai) \oplus M_1/\mathrm{soc}(Ai) \oplus M_2/\mathrm{soc}(Ai)$$

Thus we have

$$J(A)i/(U + M_1) \cong (J(A)i/\mathrm{soc}(Ai))/(U/\mathrm{soc}(Ai) \oplus M_1/\mathrm{soc}(Ai)) \cong M_2/\mathrm{soc}(Ai) \ .$$

This is a uniserial module with two composition factors isomorphic to S. Thus $Ai/(U + M_1)$ is uniserial with three composition factors isomorphic to S, because $Ai/J(A)i \cong S$. Since in particular its socle is simple, isomorphic to S, this module is isomorphic to a submodule of Ai. Choose an embedding $A/(U + M_1) \to Ai$ and replace M_2 by the image of this embedding. Then the composition of canonical maps

$$\gamma : Ai \to Ai/(U + M_1) \to Ai$$

is an A-endomorphism of Ai with kernel $U + M_1$ and uniserial image M_2 of length three. Note that M_1 is uniserial of length two, so both a quotient and a submodule of Ai. Thus there is an endomorphism

$$\delta : Ai \to Ai$$

with image M_1. Since $M_1 \subseteq \ker(\gamma)$, we have

$$\gamma \circ \delta = 0 \ .$$

We show next that we also have

$$\delta \circ \gamma = 0 \ .$$

One way to see this is to observe that this is a calculation in the split local 5-dimensional symmetric algebra $\mathrm{End}_A(Ai) \cong (iAi)^{\mathrm{op}}$, which as a consequence of [10, B. Theorem], is commutative.

There is a (slightly more general) argument that works in this case. Since the A-module Ai, and hence also the image of γ, is generated by i, it suffices to show that $\delta(\gamma(i)) = 0$. Now since $\gamma \circ \delta = 0$, we have

$$0 = \gamma(\delta(i)) = \gamma(\delta(i)i) = \delta(i)\gamma(i)$$

Note that $\delta(i) = \delta(i^2) = i\delta(i) \in iAi$, and similarly, $\gamma(i) \in iAi$. Since $\mathrm{Im}(\delta) = M_2$ has length 2, we have $\mathrm{Im}(\delta) \subseteq \mathrm{soc}^2(A)$. Thus $\delta(i) \in \mathrm{soc}^2(A) \cap iAi \subseteq \mathrm{soc}^2(iAi)$, and since iAi is symmetric, we have $\mathrm{soc}^2(iAi) \subseteq Z(iAi)$. It follows that

$$\delta(i)\gamma(i) = \gamma(i)\delta(i) = \delta(\gamma(i)i) = \delta(\gamma(i))$$

whence $\delta(\gamma(i)) = 0$, and so $\delta \circ \gamma = 0$ by the previous remarks. Thus $M_2 \subseteq \ker(\delta)$. Since $\mathrm{Im}(\delta) = M_1$ has no composition factor T, it follows that $U \subseteq \ker(\delta)$. Together we get that $U + M_2 \subseteq \ker(\delta)$. Comparing composition lengths yields

$$\ker(\delta) = U + M_2 \ .$$

This implies that

$$\ker(\delta) \cap \mathrm{Im}(\delta) = \mathrm{soc}(Ai)$$
$$\ker(\gamma) \cap \mathrm{Im}(\gamma) = \mathrm{soc}(Ai)$$

and hence the endomorphisms δ^2 and γ^3 both map Ai onto $\mathrm{soc}(Ai)$. Thus they differ by a nonzero scalar. Up to adjusting δ, β, we may therefore assume that

$$\delta^2 = \gamma^3 = \beta \circ \alpha$$

Since $\ker(\alpha)$ contains $M_1 + M_2$, it follows that

$$\alpha \circ \delta = \alpha \circ \gamma = 0 \ .$$

By taking these relations into account, it follows that $\mathrm{End}_A(A)$ is spanned k-linearly by the set

$$\{i, j, \alpha, \beta, \gamma, \gamma^2, \delta, \delta^2, \alpha \circ \beta\}$$

so this is a basis of $\mathrm{End}_A(A)$. We have identified here i, j with the canonical projections of A onto Ai and Aj. Note that $\mathrm{End}_A(A)$ is the algebra opposite to A. This accounts for the reverse order in the relations of the generators in A (denoted abusively by the same letters). This shows that the quiver with relations of A is as stated. The equation $C = (D^t)D$ implies that the second column of D has exactly two nonzero entries and that these are equal to 1. The first row has either five entries equal to 1, which yields $|\mathrm{Irr}(B)| = 6$ and the decomposition matrix D as stated. Or the first row has one entry 2 and one entry 1. This would lead to a decomposition matrix of the form

$$D = \begin{pmatrix} 2 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} \ .$$

In particular, this would yield $|\mathrm{Irr}(B)| = 3$. But this is not possible, since $\dim_k(Z(A))$ is clearly greater than 3; indeed, $Z(A)$ contains the linearly independent elements 1, δ, γ, γ^2. This concludes the proof. □

3 The structure of the algebra A

Let k be an algebraically closed field. Throughout this section we denote by A the k-algebra given in Theorem 2.1. We keep the notation of this theorem and identify the generators i, j, α, β, γ, δ with their images in A.

Lemma 3.1.

(i) *The set* $\{i, j, \alpha, \beta, \beta\alpha, \gamma, \gamma^2, \delta, \delta^2\}$ *is a* k*-basis of* A.

(ii) *The set* $\{\alpha, \beta, \alpha\beta - \beta\alpha\}$ *is a* k*-basis of* $[A, A]$.

(iii) *The set* $\{1, \gamma, \gamma^2, \delta, \delta^2, \beta\alpha\}$ *is a* k*-basis of* $Z(A)$.

(iv) *The set* $\{\alpha\beta, \beta\alpha\}$ *is a* k*-basis of* $\mathrm{soc}(A)$.

Proof. This follows immediately from the relations of the quiver of A. □

Lemma 3.2. *There is a unique symmetrising form* $s : A \to k$ *such that*

$$s(\alpha\beta) = s(\beta\alpha) = 1$$

and such that

$$s(i) = s(j) = s(\alpha) = s(\beta) = s(\gamma) = s(\gamma^2) = s(\delta) = 0$$

The dual basis with respect to the form s *of the basis*

$$\{i, j, \alpha, \beta, \beta\alpha, \gamma, \gamma^2, \delta, \delta^2\}$$

is, in this order, the basis

$$\{\alpha\beta, \beta\alpha, \beta, \alpha, j, \gamma^2, \gamma, \delta, i\}$$

Proof. Straightforward verification. □

See [5, §5.B] or [14, Definition 2.16.10] for details regarding the definitions and some properties of the *projective ideal* $Z^{\mathrm{pr}}(A)$ in $Z(A)$ and the *stable centre* $\underline{Z}(A) = Z(A)/Z^{\mathrm{pr}}(A)$.

Lemma 3.3. *Let* $\mathrm{char}(k) = 3$*. The projective ideal* $Z^{\mathrm{pr}}(A)$ *is one-dimensional, with basis* $\{\alpha\beta - \beta\alpha\}$*, we have an isomorphism of* k*-algebras*

$$\underline{Z}(A) \cong k[x, y]/(x^3 - y^2, xy, y^3)$$

induced by the map sending x *to* γ *and* y *to* δ*, and after identifying* x *and* y *with their images in the quotient, the following statements hold:*

(i) *The set* $\{1, x, x^2, y, y^2\}$ *is a* k*-basis of* $\underline{Z}(A)$*, and in particular* $\dim_k(\underline{Z}(A)) = 5$.

(ii) *The set* $\{x, x^2, y, y^2\}$ *is a* k*-basis of* $J(\underline{Z}(A))$.

(iii) *The set* $\{x^2, y^2\}$ *is a* k*-basis of* $J(\underline{Z}(A))^2$.

(iv) *The set* $\{y^2\}$ *is a* k*-basis of* $\mathrm{soc}(\underline{Z}(A))$*, and* $J(\underline{Z}(A))^3 = \mathrm{soc}(\underline{Z}(A))$.

(v) *The* k*-algebra* $\underline{Z}(A)$ *is a symmetric algebra.*

Proof. It follows from lemma 3.2 that the relative trace map Tr_1^A from A to $Z(A)$ is given by

$$\mathrm{Tr}_1^A(u) = iu\alpha\beta + ju\beta\alpha + \alpha u\beta + \beta u\alpha + \beta\alpha uj + \gamma u\gamma^2 + \gamma^2 u\gamma + \delta u\delta + \delta^2 ui$$

for all $u \in A$. One checks, using $\mathrm{char}(k) = 3$, that

$$\mathrm{Tr}_1^A(i) = -\mathrm{Tr}_1^A(j) = \beta\alpha - \alpha\beta$$

and that Tr_1^A vanishes on all basis elements different from i, j. Statement (i) then follows from the relations in the quiver of A and Lemma 3.1. The algebra $\underline{Z}(A)$ is split local, proving statement (ii), whilst a straightforward computation shows both statement (iii) and (iv). Finally, a simple verification proves that the map $s : \underline{Z}(A) \to k$ such that

$$s(y^2) = 1$$

and such that

$$s(1) = s(x) = s(x^2) = s(y) = 0$$

is a symmetrising form on $\underline{Z}(A)$. One verifies also that the dual basis with respect to the form s of the basis

$$\{1, x, y, x^2, y^2\}$$

is, in this order, the basis

$$\{y^2, x^2, y, x, 1\}.$$

This completes the proof. □

Remark 3.4. Note that by a result of Erdmann [7, I.10.8(i)], A is of wild representation type.

4 The stable centre of the group algebra $k(P \rtimes C_2)$.

Let k be a field of characteristic 3. Set $P = C_3 \times C_3$ and E the subgroup of $\mathrm{Aut}(P)$ of order 2 such that the nontrivial element t of E acts as inversion on P. Denote by $H = P \rtimes E$ the corresponding semidirect product; this is a Frobenius group. Denote by r and s generators of the two factors C_3 of P. The following Lemma holds in greater generality (see Remark 4.1 in [8]); we state only what we need in this paper.

Lemma 4.1. *The projective ideal $Z^{\mathrm{pr}}(kH)$ is one-dimensional, with k-basis $\{\sum_{x \in P} xt\}$, and we have an isomorphism of k-algebras*

$$\underline{Z}(kH) \cong (kP)^E$$

induced by the map sending $x + x^{-1}$ in $(kP)^E$ to its image in $\underline{Z}(kH)$. In particular, we have $\dim_k(\underline{Z}(kH)) = 5$, and the image of the set $\{1, r + r^2, s + s^2, r^2 s + rs^2, rs + r^2 s^2\}$ is a k-basis of $\underline{Z}(kH)$.

Proof. The relative trace map Tr_1^H from kH to $Z(kH)$ satisfies $\mathrm{Tr}_1^H = \mathrm{Tr}_P^H \circ \mathrm{Tr}_1^P$. We calculate for all $a \in P$

$$\mathrm{Tr}_1^P(a) = \sum_{g \in P} gag^{-1} = \sum_{|P|} a = 9 \cdot a = 0$$

Thus for every $c \in kP$ we have $\mathrm{Tr}_1^H(c) = \mathrm{Tr}_P^H(\mathrm{Tr}_1^P(c)) = 0$. On the other hand, for every element of the form at in H, where $a \in P$, we have

$$\begin{aligned}
\mathrm{Tr}_1^H(at) &= \sum_{g\in P} g(at)g^{-1} + \sum_{g\in P}(gt)(at)(gt)^{-1} \\
&= (a+a^{-1})\sum_{x\in P} xt \\
&= 2\cdot\big(\sum_{x\in P} xt\big)
\end{aligned}$$

The conjugacy classes of G are given by $\{1\}$, $\{r, r^2\}$, $\{s, s^2\}$, $\{r^2s, rs^2\}$, $\{rs, r^2s^2\}$ and $\{xt \mid x \in P\}$. The last statement follows. □

Lemma 4.2. *There is an isomorphism of k-algebras*

$$\underline{Z}(kH) \cong \big(k[x,y]/(x^3,y^3)\big)^E$$

with inverse induced by the map sending x to $r-1$ and y to $s-1$, where the nontrivial element t of E acts by $x^t = x^2+2x$ and $y^t = y^2+2y$. After identifying x and y with their images in $k[x,y]/(x^3,y^3)$, the following statements hold:

(i) *The image of the set $\{1, x^2, y^2, xy + x^2y + xy^2, x^2y^2\}$ is a k-basis of $\underline{Z}(kH)$.*

(ii) *The set $\{x^2, y^2, xy + x^2y + xy^2, x^2y^2\}$ is a k-basis of $J(\underline{Z}(kH))$.*

(iii) *The set $\{x^2y^2\}$ is a k-basis of $\mathrm{soc}(\underline{Z}(kH))$, and $J(\underline{Z}(kH))^2 = \mathrm{soc}(\underline{Z}(kH))$. In particular, $\dim_k(J(\underline{Z}(kH))^2) = 1$.*

(iv) *The k-algebra $\underline{Z}(kH)$ is symmetric.*

Proof. By Lemma 4.1 we have $\underline{Z}(kH) \cong (kP)^E$. Since k has characteristic 3, we have an isomorphism $kP \cong k[x,y]/(x^3,y^3)$ induced by the map given in the statement of the lemma. Under this isomorphism, the action of t on x and y is given by $x^t = x^2+2x$ and $y^t = y^2+2y$ as stated. It is straightforward to then verify that this isomorphism gives

$$r + r^t \mapsto x^2 + 2,$$

$$s + s^t \mapsto y^2 + 2,$$

$$rs + (rs)^t \mapsto 2 + x^2 + y^2 + 2xy + 2x^2y + 2xy^2 + x^2y^2,$$

$$r^2s + (r^2s)^t \mapsto 2 + x^2 + y^2 + xy + x^2y + xy^2.$$

This proves the statement (i) and (ii). A straightforward computation proves statement (iii). The final statement is given in general in [8, Corollary 1.3], with an explicit symmetrising form $s : \underline{Z}(kH) \to k$ given by $s(x^2y^2) = 1$ and sending all other basis elements to 0. □

5 Proof of Theorem 1.1

Theorem 1.1 will be an immediate consequence of Theorem 2.1 and the following result.

Theorem 5.1. *Let k be an algebraically closed field of prime characteristic p, and let A be the algebra given in Theorem 2.1. Then A is not isomorphic to a basic algebra of a block of a finite group algebra over k.*

Proof. Arguing by contradiction, suppose that A is isomorphic to a basic algebra of a block B of kG, for some finite group G. Denote by P a defect group of B. By Theorem 2.1 we have $p = 3$ and $P \cong C_3 \times C_3$. By Lemma 3.3, the stable centre $\underline{Z}(A)$ is symmetric, hence so is $\underline{Z}(B)$, as A and B are Morita equivalent. It follows from [8, Proposition 3.8] that we have an algebra isomorphism

$$\underline{Z}(A) \cong (kP)^E$$

where E is the inertial quotient of the block B. Again by Lemma 3.3, we have $\dim_k((kP)^E) = 5$, or equivalently, E has five orbits in P. The list of possible inertial quotients in Kiyota's paper [9] shows that E is isomorphic to one of 1, C_2, $C_2 \times C_2$, C_4, C_8, D_8, Q_8, SD_{16}. In all cases except for $E \cong C_2$ is the action of E on P determined, up to equivalence, by the isomorphism class of E. Thus if E contains a cyclic subgroup of order 4, then E has at most 3 orbits, and if E is the Klein four group, then E has 4 orbits. Therefore we have $E \cong C_2$. If the nontrivial element t of E has a nontrivial fixed point in P (or equivalently, if t centralises one of the factors C_3 of P and acts as inversion on the other), then E has 6 orbits. Thus t has no nontrivial fixed point in P, and the group $H = P \rtimes E$ is the Frobenius group considered in the previous section. By a result of Puig [17, 6.8] (also described in [15, Theorem 10.5.1]), there is a stable equivalence of Morita type between B and kH, hence between A and kH. By a result of Broué [5, 5.4] (see also [14, Corollary 2.17.14]), there is an algebra isomorphism $\underline{Z}(A) \cong \underline{Z}(kH)$. This, however, contradicts the calculations in the Lemmas 3.3 and 4.2, which show that the dimension of $J(\underline{Z}(A))^2$ and of $J(\underline{Z}(kH))^2$ are different. This contradiction completes the proof. □

Proof of Theorem 1.1. Arguing by contradiction, if a defect P of B is not cyclic, then $P \cong C_3 \times C_3$ because the Cartan matrix of B has elementary divisors 9 and 1. But then B has a basic algebra isomorphic to the algebra A in Theorem 2.1. This, however, is ruled out by Theorem 5.1. □

6 Further remarks

Using the arguments of the proof of Theorem 5.1 it is possible to prove some slightly stronger statements about the stable equivalence class of the algebra A from Theorem 2.1.

Proposition 6.1. *Let k be an algebraically closed field of prime characteristic p and let A be the algebra in Theorem 2.1. Let P be a finite p-group, E a p'-subgroup of* $\mathrm{Aut}(P)$, *and $\tau \in H^2(E; k^\times)$. There does not exist a stable equivalence of Morita type between A and the twisted group algebra $k_\tau(P \rtimes E)$.*

Proof. Arguing by contradiction, suppose that there is a stable equivalence of Morita type between A and $k_\tau(P \rtimes E)$. Note that $k_\tau(P \rtimes E)$ is a block of a central p'-extension of $P \rtimes E$ with defect group P, so its Cartan matrix has a determinant divisible by $|P|$. By [14, Proposition 4.14.13], the

Cartan matrices of the algebras A and $k_\tau(P \rtimes E)$ have the same determinant, which is 9. Since A is clearly not of finite representation type (cf. Remark 3.4), it follows that P is not cyclic, hence $P \cong C_3 \times C_3$. Using as before Broué's result [5, 5.4], we have an isomorphism $\underline{Z}(A) \cong \underline{Z}(k_\tau(P \rtimes E))$. Since $\underline{Z}(A)$ is symmetric, so is $\underline{Z}(k_\tau(P \rtimes E))$. Since $k_\tau(P \rtimes E)$ is a block of a central p'-extension of $P \rtimes E$ with defect group P and inertial quotient E, it follows again from [8, Proposition 3.8] that $\underline{Z}(A) \cong (kP)^E$. From this point onward, the rest of the proof follows the proof of Theorem 5.1, whence the result. □

Remark 6.2. By results of Rouquier [18, 6.3] (see also [12, Theorem A2]), for any block B with an elementary abelian defect group of rank 2 there is a stable equivalence of Morita type between B and its Brauer correspondent, which by a result of Külshammer [11], is Morita equivalent to a twisted semidirect product group algebra as in Proposition 6.1. Thus Theorem 5.1 can be obtained as a consequence of Proposition 6.1 and Rouquier's stable equivalence.

Remark 6.3. A slightly different proof of Theorem 5.1 makes use of Broué's surjective algebra homomorphism $Z(B) \to (kZ(P))^E$ from [4, Proposition III (1.1)], induced by the Brauer homomorphism Br_P, where here P is a (not necessarily abelian) defect group of a block B of a finite group algebra kG, with k an algebraically closed field of prime characteristic p. If P is normal in G, then it is easy to see that Broué's homomorphism is split surjective, but this is not known in general. If B is a block with P nontrivial such that there exists a stable equivalence of Morita type between B and its Brauer correspondent, then this implies the existence of at least *some* split surjective algebra homomorphism $\underline{Z}(B) \to kZ(P)^E$.

Kiyota's list in [9] shows that if A were isomorphic to a basic algebra of a block with defect group $P \cong C_3 \times C_3$, then E would be isomorphic to one of C_2 or D_8 (subcase (b) in Kiyota's list). The case C_2 can be ruled out as above, and the case D_8 can be ruled out by using Rouquier's stable equivalence, and by showing that if $E \cong D_8$, then $(kP)^E$ is uniserial of dimension 3, but $\underline{Z}(A)$ admits no split surjective algebra homomorphism onto a uniserial algebra of dimension 3. Note that $\underline{Z}(A)$ does though admit a surjective algebra homomorphism onto a uniserial algebra of dimension 3, so the splitting is an essential point in this argument, and may warrant further investigation.

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