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**ON THE BV STRUCTURE OF THE HOCHSCHILD  
COHOMOLOGY OF FINITE GROUP ALGEBRAS**

DAVE BENSON, RADHA KESSAR AND MARKUS LINCKELMANN



## ON THE BV STRUCTURE OF THE HOCHSCHILD COHOMOLOGY OF FINITE GROUP ALGEBRAS

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**We give a simple algebraic recipe for calculating the components of the BV operator  $\Delta$  on the Hochschild cohomology of a finite group algebra with respect to the centraliser decomposition. We use this to investigate properties of  $\Delta$  and to make some computations for some particular finite groups.**

### 1. Introduction

The Hochschild cohomology of a finite group is a Batalin–Vilkovisky (BV) algebra. This follows, for example, by observations in [Tradler 2008] from the fact that the group algebra over a commutative ring  $k$  is a symmetric algebra. So there is a BV operator  $\Delta: HH^n(kG) \rightarrow HH^{n-1}(kG)$ , which is related to the Gerstenhaber bracket by the formula

$$[x, y] = (-1)^{|x|} \Delta(xy) - (-1)^{|x|} \Delta(x)y - x\Delta(y).$$

Thus, if the cup product and the BV operator are known, so is the Gerstenhaber bracket.

The BV operator on the Hochschild cohomology  $HH^*(kG)$  of a finite group  $G$  over  $k$  coming from the standard symmetrising form on  $kG$  preserves the centraliser decomposition

$$HH^*(kG) \cong \bigoplus_g H^*(C_G(g), k)$$

under which  $\Delta$  is the sum of degree  $-1$  operators  $\Delta_g$  on  $H^*(C_G(g), k)$  (see Section 3). Here  $g$  runs over a set of representatives of the conjugacy classes in  $G$ . An individual component  $\Delta_g$  depends only on  $g$  and  $C_G(g)$  but not on  $G$  itself, and hence in order to describe  $\Delta_g$  we may assume that  $g$  is central in  $G$ . The following theorem gives a description of  $\Delta_g$  in this situation, and some of its properties:<sup>1</sup>

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<sup>1</sup>We have to be careful to distinguish the cup product in Hochschild cohomology from the cup product in ordinary cohomology  $H^*(C_G(g), k)$  in a component of the centraliser decomposition. We use juxtaposition  $xy$  to denote Hochschild cup product and  $x \cdot y$  for cup product inside  $H^*(C_G(g), k)$ .

**Theorem 1.1.** *Let  $k$  be a commutative ring of coefficients. Let  $g$  be a central element in a finite group  $G$ , and let  $z_g: \mathbb{Z} \times G \rightarrow G$  be the map which sends  $(m, h)$  to  $g^m h$ . Then*

$$(z_g)^*: H^n(G, k) \rightarrow H^n(\mathbb{Z} \times G, k) \cong H^n(G, k) \oplus H^{n-1}(G, k)$$

*has the form  $(1, \Delta_g)$ , where  $\Delta_g: H^n(G, k) \rightarrow H^{n-1}(G, k)$  is the component of the BV operator indexed by  $g$ . The following properties hold for the map  $\Delta_g$ :*

- (i)  $\Delta_g$  is a  $k$ -linear derivation with respect to multiplication in  $H^*(G, k)$ : for  $x, y \in H^*(G, k)$  we have

$$\Delta_g(x \cdot y) = \Delta_g(x) \cdot y + (-1)^{|x|} x \cdot \Delta_g(y).$$

*In particular,  $\Delta_g$  is determined by its values on a set of generators for  $H^*(G, k)$ .*

- (ii) *If  $g \in Z(G)$  and  $g' \in Z(G')$ , then the following diagram commutes:*

$$\begin{array}{ccc} H^i(G, k) \otimes_k H^j(G', k) & \longrightarrow & H^{i+j}(G \times G', k) \\ (\Delta_g \otimes 1, 1 \otimes \Delta_{g'}) \downarrow & & \downarrow \Delta_{(g, g')} \\ (H^{i-1}(G, k) \otimes_k H^j(G', k)) \oplus (H^i(G, k) \otimes_k H^{j-1}(G', k)) & \longrightarrow & H^{i+j-1}(G \times G', k), \end{array}$$

*where the horizontal maps are the Künneth maps and*

$$(\Delta_g \otimes 1)(x \otimes y) = \Delta_g(x) \otimes y, \quad (1 \otimes \Delta_{g'})(x \otimes y) = (-1)^{|x|} x \otimes \Delta_{g'}(y).$$

*If  $k$  is a field, then  $\Delta_{(g, g')}$  is determined by  $\Delta_g$  and  $\Delta_{g'}$  by combining this with the Künneth formula.*

- (iii) *If  $\phi: G \rightarrow G'$  is a group homomorphism sending  $g \in Z(G)$  to  $g' \in Z(G')$  then the following diagram commutes:*

$$\begin{array}{ccc} H^n(G', k) & \xrightarrow{\phi^*} & H^n(G, k) \\ \Delta_{g'} \downarrow & & \downarrow \Delta_g \\ H^{n-1}(G', k) & \xrightarrow{\phi^*} & H^{n-1}(G, k). \end{array}$$

- (iv) *If  $\rho: k \rightarrow k'$  is a homomorphism of commutative rings, then the following diagram commutes:*

$$\begin{array}{ccc} H^n(G, k) & \xrightarrow{\rho_*} & H^n(G, k') \\ \Delta_g \downarrow & & \downarrow \Delta_g \\ H^{n-1}(G, k) & \xrightarrow{\rho_*} & H^{n-1}(G, k'). \end{array}$$

- (v) *If  $g$  and  $g'$  are elements of  $Z(G)$ , then we have  $\Delta_{gg'} = \Delta_g + \Delta_{g'}$ .*

(vi) If  $H$  is a subgroup of  $G$  containing  $g \in Z(G)$ , then  $\Delta_g$  commutes with the transfer map: if  $x \in H^n(H, k)$ , then

$$\mathrm{Tr}_{H,G}(\Delta_g(x)) = \Delta_g(\mathrm{Tr}_{H,G}(x)) \in H^n(G, k).$$

(vii) If  $g \in Z(G)$  and  $k$  is a field of prime characteristic  $p$ , then the map  $\Delta_g$  is determined by its action on the cohomology of a Sylow  $p$ -group of  $G$ , as follows. Let  $g = g_p g_{p'} = g_{p'} g_p$  with  $g_p$  a  $p$ -element and  $g_{p'}$  a  $p'$ -element, and let  $P$  be a Sylow  $p$ -subgroup of  $G$ . Then  $g_p \in Z(P)$  and the following diagram commutes:

$$\begin{array}{ccc} H^n(G, k) & \xrightarrow{\mathrm{Res}_{G,P}} & H^n(P, k) \\ \Delta_g \downarrow & & \downarrow \Delta_{g_p} \\ H^{n-1}(G, k) & \xrightarrow{\mathrm{Res}_{G,P}} & H^{n-1}(P, k). \end{array}$$

(viii) In the case  $k = \mathbb{F}_p$ , the map  $\Delta_g$  commutes with the action of Steenrod operations on  $H^*(G, \mathbb{F}_p)$  and with the Bockstein homomorphism.

(ix) If  $x \in H^1(G, k)$ , we can regard  $x$  as an element of  $\mathrm{Hom}(G, k)$ . With this identification, we have  $\Delta_g(x) = x(g)$ .

(x) If  $x \in H^2(G, k)$  is in the image of  $H^2(G, \mathbb{Z}) \rightarrow H^2(G, k)$ , then  $\Delta_g(x) = 0$ .

(xi) If  $x \in H^2(G, k)$  corresponds to a central extension

$$1 \rightarrow k^+ \rightarrow K \rightarrow G \rightarrow 1,$$

then for  $h \in G$  we choose any inverse image  $\hat{h} \in K$ . Then we have

$$\Delta_g(x)(h) = [\hat{g}, \hat{h}] \in k^+.$$

The proof of Theorem 1.1 appears below in the following places. The first assertion is proved in Theorem 4.2. Statements (i), (iii), (iv), (vi), and (viii) are proved in Corollary 4.3, statement (ii) is in Proposition 6.6, statements (v) and (vii) are in Proposition 6.7, statement (ix) is Proposition 4.4, statement (x) is Proposition 4.6, and statement (xi) is Theorem 8.1.

Part (vii) of Theorem 1.1 says that if  $k$  is a field of prime characteristic  $p$ , we may as well suppose that  $G$  is a  $p$ -group for the purpose of computation, and hence we give in Section 9 a number of examples where we compute  $\Delta_g$  in the case where  $g \in Z(G)$  and  $G$  is a finite  $p$ -group. Our first examples are cyclic groups and, more generally, abelian groups. Then we deal with dihedral, quaternion, and semidihedral 2-groups.

Following [Tradler 2008], the operator  $\Delta_g$  is defined as the dual of the Connes operator  $B \circ I$  in Hochschild homology. By work of Burghelea [1985], the Connes exact sequence preserves the centraliser decomposition, and the corresponding components of the Connes exact sequence can be used to describe  $\Delta_g$ ; see 2.4 and 3.4 below for details. For our analysis of  $\Delta_g$ , we first provide a description of

the Connes operator in the context of a general discrete group  $G$ . Again, it only depends on  $g$  and  $C_G(g)$ , and we give the following description for  $g$  central in  $G$ . Consider the map  $z_g: \mathbb{Z} \times G \rightarrow G$ , as in the theorem above. Then

$$(z_g)_*: H_n(\mathbb{Z} \times G, k) \cong H_n(G, k) \oplus H_{n-1}(G, k) \rightarrow H_n(G, k)$$

is equal to  $(1, B \circ I)$ . Dualising this statement to give a statement about Hochschild cohomology requires the use of a symmetrising form on  $kG$ , and we make use of the standard one, whose value on  $(g, h)$  is one if  $gh = 1$  and zero otherwise.

We use Theorem 1.1 to give an explicit description of the BV operator  $\Delta$  in the Hochschild cohomology of finite groups over an arbitrary commutative ring  $k$  in terms of ordinary cohomology, based on a simple general principle using homotopies in order to construct degree  $-1$  operators in cohomology. This general principle, and its relation to  $\Delta_g$ , is expressed in the following theorem, which summarises the main parts of 5.1 and 6.1 below. By a homotopy on a chain complex (respectively, cochain complex), we mean a graded map of degree 1 (respectively,  $-1$ ).

**Theorem 1.2.** *Let  $A$  be an algebra over a commutative ring  $k$ , and let  $U$  and  $V$  be  $A$ -modules. Let  $z \in Z(A)$  such that  $z$  annihilates  $U$  and  $V$ . Let  $(P, \delta)$  be a projective resolution of  $U$ .*

*There is a homotopy  $s$  on  $P$  such that the chain endomorphism  $s \circ \delta + \delta \circ s$  of  $P$  is equal to multiplication by  $z$  on the components of  $P$ . For any such homotopy, the induced homotopy  $s^\vee$  on the cochain complex  $\text{Hom}_A(P, V)$  obtained from applying the functor  $\text{Hom}_A(-, V)$  to  $s$  is a cochain map  $\text{Hom}_A(P, V) \rightarrow \text{Hom}_A(P[1], V)$ . In particular, upon taking cohomology,  $s^\vee$  induces a degree  $-1$  operator, denoted  $D_z$ , on  $\text{Ext}_A^*(U, V)$ , and then  $D_z$  is independent of the choice of  $s$ .*

*If  $G$  is a finite group,  $A = kG$ ,  $U = V = k$ , and  $g \in Z(G)$ , then  $\Delta_g = D_{g-1}$ .*

See Remark 5.2 below for sign conventions for the differentials of the complexes arising in this theorem. While the first part of Theorem 1.2 is a routine verification (see the proof of Theorem 5.1), the identification  $\Delta_g = D_{g-1}$  in the case  $A = kG$  in Theorem 6.1 requires Theorem 4.2, the proof of which is based on the Connes exact sequence relating Hochschild and cyclic cohomology. We give explicit homotopies for the bar resolution in Theorem 7.1. We use this description in Section 8 to give a short proof of the formula in Theorem 1.1 (xi), restated as Theorem 8.1 below, for the components of the BV operator in degree 2, in terms of central extensions corresponding to degree 2 elements in group cohomology.

The theme of deciding when  $HH^1(kG)$  is a soluble Lie algebra has been recently investigated by a number of authors [Rubio y Degraasi et al. 2019; Eisele and Raedschelders 2020; Linckelmann and Rubio y Degraasi 2020]. As an application of Theorem 1.1, we add to these the following results, which are proved as 10.10 and 11.6. We denote by  $\Phi(G)$  the Frattini subgroup of a finite group  $G$ .

**Theorem 1.3.** *If  $G$  is a finite  $p$ -group such that  $|Z(G) : Z(G) \cap \Phi(G)| \geq 3$ , then the Lie algebra  $HH^1(kG)$  is not soluble.*

**Theorem 1.4.** *If  $G$  is an extraspecial  $p$ -group, then the Lie algebra  $HH^1(kG)$  is soluble.*

## 2. Hochschild and cyclic homology of $kG$

The BV operator in string topology on the homology of a free loop space  $LX$  is obtained by applying homology to the rotation map  $S^1 \times LX \rightarrow LX$  and decomposing  $H_*(S^1 \times LX)$  using the Künneth formula together with the fundamental class in  $H_1(S^1)$ . We give a similar description for group (co)homology. Background material for this section may be found in [Benson 1991b, §2.11–2.15]. Other references include [Loday 1992, Chapter 7], [Burghelea 1985], and [Karoubi and Villamayor 1990].

Let  $G$  be a discrete group, and let  $k$  be a commutative ring of coefficients. The Hochschild homology of  $kG$  has a centraliser decomposition

$$HH_*(kG) \cong \bigoplus_g H_*(C_G(g), k),$$

where  $g$  runs over a set of representatives of the conjugacy classes of  $G$ . This decomposition is unique up to unique isomorphism. A similar description of cyclic homology appears in [Burghelea 1985; Karoubi and Villamayor 1990] and was reinterpreted in [Benson 1991b] in terms of extended centralisers, as follows. If  $g \in G$ , we define the extended centraliser  $\hat{C}_G(g)$  to be the quotient  $(\mathbb{R} \times C_G(g))/\mathbb{Z}$ , where  $\mathbb{Z}$  is embedded in  $\mathbb{R} \times C_G(g)$  via the group homomorphism sending 1 to  $(1, g)$ . Recall that for a discrete group  $G$  we have  $H_*(G, k) \cong H_*(BG; k)$ , where  $BG$  is the classifying space of  $G$ . On the other hand, we regard  $\hat{C}_G(g)$  as a one-dimensional Lie group, so we need to use classifying space homology, and we have

$$HC_*(kG) \cong \bigoplus_g H_*(B\hat{C}_G(g); k).$$

The Connes exact sequence (see [Loday and Quillen 1984] and [Connes 1985, §II.4]) connecting Hochschild and cyclic homology

$$(2.1) \quad \cdots \rightarrow HH_{n+2}(kG) \xrightarrow{I} HC_{n+2}(kG) \xrightarrow{S} HC_n(kG) \xrightarrow{B} HH_{n+1}(kG) \rightarrow \cdots$$

respects the centraliser decomposition [Burghelea 1985] and may be described as follows. Applying the classifying space construction to the short exact sequence of Lie groups

$$1 \rightarrow \mathbb{Z} \rightarrow \mathbb{R} \times C_G(g) \rightarrow \hat{C}_G(g) \rightarrow 1,$$

we obtain a fibration sequence

$$(2.2) \quad S^1 \rightarrow BC_G(g) \rightarrow B\hat{C}_G(g),$$

where  $\mathbb{R}/\mathbb{Z} = S^1 = B\mathbb{Z}$ . The (co)homology of  $S^1$  is  $\mathbb{Z}$  in degree 0 and 1 and vanishes in all other degrees. In particular,  $H_1(S^1; \mathbb{Z}) = H_1(B\mathbb{Z}; \mathbb{Z}) \cong \mathbb{Z}$ , and we write  $\nu$  for the element of  $H_1(S^1; \mathbb{Z})$  corresponding to  $1 \in \mathbb{Z}$ . Dually, we write  $\mu$  for the element of  $H^1(S^1; \mathbb{Z}) = H^1(B\mathbb{Z}; \mathbb{Z}) \cong \text{Hom}(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}$  representing the identity element. We use the same letters  $\mu$  and  $\nu$  for their images in  $H^1(\mathbb{Z}, k)$  and  $H_1(\mathbb{Z}, k)$ .

The Künneth formula yields a canonical identification

$$H_n(\mathbb{Z} \times G, k) = H_0(\mathbb{Z}, k) \otimes_k H_n(G, k) \oplus H_1(\mathbb{Z}, k) \otimes_k H_{n-1}(G, k)$$

(see [Weibel 1994, Theorem 3.6.1]), where the summand  $H_0(\mathbb{Z}, k) \otimes_k H_n(G, k)$  is equal to the image of  $H_n(G, k)$  under the map induced by the canonical inclusion  $G \rightarrow \mathbb{Z} \times G$ . Using  $\nu$  yields an identification

$$H_n(\mathbb{Z} \times G, k) = H_n(G, k) \oplus H_{n-1}(G, k).$$

Similarly, using  $\mu$  yields an identification

$$H^n(\mathbb{Z} \times G, k) = H^n(G, k) \oplus H^{n-1}(G, k),$$

where  $H^n(G, k)$  is identified to its image via the map induced by the canonical projection  $\mathbb{Z} \times G \rightarrow G$ . See Proposition 6.4 for more details on this identification.

The Serre spectral sequence of the fibration (2.2) has two nonvanishing rows, and therefore induces a long exact sequence

$$(2.3) \quad \cdots \rightarrow H_{n+2}(BC_G(g); k) \xrightarrow{I} H_{n+2}(B\hat{C}_G(g); k) \\ \xrightarrow{S} H_n(B\hat{C}_G(g); k) \xrightarrow{B} H_{n+1}(BC_G(g); k) \rightarrow \cdots,$$

where we have used  $\nu$  to identify  $E_{*,1}^2$  with  $H_*(B\hat{C}_G(g); k)$ . Note that the map  $I$  is induced by the inclusion  $BC_G(g) \rightarrow B\hat{C}_G(g)$ .

It was observed by Burghelea [1985] that the Connes sequence is the direct sum of these sequences. Note that the maps in these sequences depend only on  $g$  and  $C_G(g)$ , but not on  $G$  itself. So in the following theorem, we assume that  $g$  is central in  $G$ , and we identify  $H_n(\mathbb{Z} \times G, k)$  with  $H_n(G, k) \oplus H_{n-1}(G, k)$  as described above, using the element  $\nu \in H_1(\mathbb{Z}, k)$ :

**Theorem 2.4.** *Let  $G$  be a discrete group with a central element  $g \in Z(G)$ . Consider the group homomorphism  $z_g: \mathbb{Z} \times G \rightarrow G$  sending  $(m, h)$  to  $g^m h$ . The induced map*

$$(z_g)_*: H_n(\mathbb{Z} \times G, k) = H_n(G, k) \oplus H_{n-1}(G, k) \rightarrow H_n(G, k)$$

*has the form  $\begin{pmatrix} 1 \\ \psi \end{pmatrix}$ , where  $\psi$  is equal to the composite*

$$H_{n-1}(G, k) \xrightarrow{I} H_{n-1}(\hat{G}, k) \xrightarrow{B} H_n(G, k).$$

*Proof.* The map  $G \rightarrow \mathbb{Z} \times G$  sending  $h$  to  $(0, h)$  is a section both of  $z_g$  and of the canonical projection  $\mathbb{Z} \times G \rightarrow G$ , so  $(z_g)_*$  has the form  $\begin{pmatrix} 1 \\ \psi \end{pmatrix}$ .

Consider the diagram of groups

$$\begin{array}{ccccccccc}
 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} \times G & \longrightarrow & G & \longrightarrow & 1 \\
 & & \parallel & & \downarrow & & \downarrow & & \\
 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{R} \times G & \longrightarrow & \hat{G} & \longrightarrow & 1.
 \end{array}$$

Here, the upper sequence is just the direct product sequence, and the lower sequence is the defining sequence for  $\hat{G}$ . The middle vertical map sends  $(n, h)$  to  $(n, z_g(n, h)) = (n, g^n h)$ , while the right-hand vertical map is the inclusion. It is easy to check that the two squares commute. Taking classifying spaces, we have a map of fibrations

$$\begin{array}{ccccc}
 S^1 & \longrightarrow & S^1 \times BG & \longrightarrow & BG \\
 \parallel & & \downarrow (z_g)_* & & \downarrow \\
 S^1 & \longrightarrow & BG & \longrightarrow & B\hat{G},
 \end{array}$$

and hence a map of long exact sequences

$$\begin{array}{ccccccc}
 \cdots & \xrightarrow{0} & H_{n-1}(BG; k) & \xrightarrow{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} & H_n(BG; k) \oplus H_{n-1}(BG; k) & \xrightarrow{(1,0)} & H_n(BG; k) & \xrightarrow{0} & \cdots \\
 & & \downarrow I & & \downarrow (1, \psi) & & \downarrow I & & \\
 \cdots & \longrightarrow & H_{n-1}(B\hat{G}; k) & \xrightarrow{B} & H_n(BG; k) & \xrightarrow{I} & H_n(B\hat{G}; k) & \longrightarrow & \cdots
 \end{array}$$

The commutativity of the left-hand square proves the theorem.  $\square$

### 3. The BV operator $\Delta$ on $HH^*(kG)$

For any algebra  $\Lambda$  which is projective as a module over the coefficient ring  $k$ , to set up a duality between Hochschild homology and cohomology, we need to use coefficients  $\Lambda$  in homology and  $\Lambda^\vee = \text{Hom}_k(\Lambda, k)$  in cohomology. This is because if  $P_*$  is a projective resolution of  $\Lambda$  as a  $\Lambda$ - $\Lambda$ -bimodule, then

$$\text{Hom}_{\Lambda^e}(P_*, \Lambda^\vee) \cong \text{Hom}_k(\Lambda \otimes_{\Lambda^e} P_*, k).$$

Thus, if  $k$  is a field, then  $HH^n(\Lambda, \Lambda^\vee) \cong (HH_n(\Lambda, \Lambda))^\vee$ ; but, for example, for  $k = \mathbb{Z}$ , we have a universal coefficient sequence

$$0 \rightarrow \text{Ext}_{\mathbb{Z}}^1(HH_{n-1}(\Lambda, \Lambda), \mathbb{Z}) \rightarrow HH^n(\Lambda, \Lambda^\vee) \rightarrow \text{Hom}_{\mathbb{Z}}(HH_n(\Lambda, \Lambda), \mathbb{Z}) \rightarrow 0.$$

Similarly, cyclic cochains on  $\Lambda$  are described in terms of the Hochschild cochain complex for  $\Lambda$  with coefficients in  $\Lambda^\vee$  and are dual to cyclic chains described in terms of the Hochschild chain complex for  $\Lambda$  with coefficients in  $\Lambda$ . The Connes sequence for cyclic cohomology takes the form

$$(3.1) \quad \cdots \rightarrow HH^{n+1}(\Lambda, \Lambda^\vee) \xrightarrow{B} HC^n(\Lambda) \xrightarrow{S} HC^{n+2}(\Lambda) \xrightarrow{I} HH^n(\Lambda, \Lambda^\vee) \rightarrow \cdots.$$

For  $g$ , an element in a discrete group  $G$ , the Serre spectral sequence in cohomology of the fibration (2.2) induces a long exact sequence

$$\cdots \rightarrow H^{n+1}(BC_G(g); k) \xrightarrow{B} H^n(B\hat{C}_G(g); k) \xrightarrow{S} H^{n+2}(B\hat{C}_G(g); k) \xrightarrow{I} H^{n+2}(BC_G(g); k) \rightarrow \cdots,$$

where we used the class  $\mu \in H^1(\mathbb{Z}, k)$  introduced in the previous section to identify  $E_2^{*,1}$  with  $H^*(B\hat{C}_G(g); k)$ . Note that the map  $I$  is induced by  $BC_G(g) \rightarrow B\hat{C}_G(g)$ .

**Theorem 3.2.** *Let  $G$  be a discrete group, and let  $k$  be a commutative ring of coefficients. Then we have a centraliser decomposition*

$$HH^*(kG, kG^\vee) \cong \prod_g H^*(C_G(g), k),$$

where  $g$  runs over a set of representatives of the conjugacy classes in  $G$ . The Connes sequence is a direct product of the sequences

$$\cdots \rightarrow H^{n+1}(BC_G(g); k) \xrightarrow{B} H^n(B\hat{C}_G(g); k) \xrightarrow{S} H^{n+2}(B\hat{C}_G(g); k) \xrightarrow{I} H^{n+2}(BC_G(g); k) \rightarrow \cdots$$

*Proof.* For the centraliser decomposition, see [Benson 1991b, Theorem 2.11.2]. The statement on the Connes sequence is dual to [Burghlea 1985], with essentially the same proof. See also [Benson 1991b, §2.11–2.15].  $\square$

If  $\Lambda$  is a symmetric algebra over  $k$ , finitely generated and projective as a  $k$ -module, then  $\Lambda^\vee$  is isomorphic to  $\Lambda$  as a  $\Lambda$ - $\Lambda$ -bimodule, but the isomorphism depends on the choice of symmetrising form.

From now on, we assume that  $G$  is a finite group. Then for  $kG$ , there is a canonical symmetrising form. This is the bilinear pairing  $kG \otimes_k kG \rightarrow k$  which sends  $g \otimes g'$  to  $1 \in k$  if  $gg' = 1$  and zero otherwise. Using this bilinear pairing, we obtain an isomorphism between  $kG^\vee$  and  $kG$ , and hence between  $HH^*(kG, kG^\vee)$  and  $HH^*(kG, kG)$ . From now on, we write  $HH^*(kG)$  for  $HH^*(kG, kG)$ . This is a graded commutative ring, whose product structure was elucidated by Siegel and Witherspoon [1999].

**Definition 3.3** (cf. [Burghlea 1985; Tradler 2008]). For  $G$  finite, we define the operator  $\Delta: HH^n(kG) \rightarrow HH^{n-1}(kG)$  to be the map induced by  $I \circ B$  under the isomorphism between  $HH^*(kG)$  and  $HH^*(kG, kG^\vee)$  given by the symmetrising form on  $kG$  described above.

**Theorem 3.4.** *For  $G$  a finite group, we have a centraliser decomposition*

$$HH^*(kG) \cong \bigoplus_g H^*(C_G(g), k).$$

The map  $\Delta$  preserves the centraliser decomposition of  $HH^*(kG)$  and is the sum of the maps  $\Delta_g = I \circ B: H^*(C_G(g), k) \rightarrow H^{*-1}(C_G(g), k)$ .

*Proof.* The centraliser decomposition in the finite case comes from Theorem 3.2, see also [Siegel and Witherspoon 1999]. Also by Theorem 3.2, the Connes sequence is a direct sum of the Connes sequences for each centraliser, and hence the components  $\Delta_g$  of  $\Delta$  are as stated.  $\square$

**Definition 3.5.** A *Gerstenhaber algebra* is a graded  $k$ -algebra  $R$  which is associative and graded commutative:

$$yx = (-1)^{|x||y|}xy$$

and has a Lie bracket  $[x, y]$  of degree  $-1$  such that:

(i) Anticommutativity holds:

$$[y, x] = -(-1)^{(|x|-1)(|y|-1)}[x, y].$$

(ii) The graded Jacobi identity holds:

$$[[x, y], z] = [x, [y, z]] - (-1)^{(|x|-1)(|y|-1)}[y, [x, z]].$$

(iii) The bracket is a derivation with respect to the product (Leibniz identity):

$$[x, yz] = [x, y]z + (-1)^{(|x|-1)|y|}y[x, z].$$

**Definition 3.6.** A *Batalin–Vilkovisky algebra* (or BV algebra) is a Gerstenhaber algebra together with an operator  $\Delta$  of degree  $-1$  satisfying  $\Delta \circ \Delta = 0$  and

$$[x, y] = (-1)^{|x|}\Delta(xy) - (-1)^{|x|}(\Delta x)y - x(\Delta y).$$

Thus, the multiplication and the BV operator  $\Delta$  determine the Lie bracket.

For background on BV algebra structures in Hochschild cohomology and their relationship with loop space topology, see [Abbaspour 2015]. It is pointed out in [Rubio y Degraasi 2017, Remark 5.1] that the restricted Lie algebra structure cannot, in general, be read off from the BV algebra structure.

**Proposition 3.7** [Menichi 2004; Tradler 2008]. *The operator  $\Delta$  of Definition 3.3 defines a BV operator on  $HH^*(kG)$  making it a BV algebra in which the Lie bracket  $[-, -]$  is the Gerstenhaber bracket in Hochschild cohomology.*

**Remark 3.8.** Tradler gives the following formula for  $\Delta$  at the level of Hochschild cochains, on an algebra  $\Lambda$  with a symmetric, invariant, nondegenerate bilinear form  $\langle -, - \rangle: \Lambda \times \Lambda \rightarrow k$ . For  $f \in C^n(\Lambda, \Lambda)$ , define  $\Delta f \in C^{n-1}(\Lambda, \Lambda)$  by

$$\langle \Delta f(a_1, \dots, a_{n-1}), a_n \rangle = \sum_{i=1}^n (-1)^{i(n-1)} \langle f(a_i, \dots, a_{n-1}, a_n, a_1, \dots, a_{i-1}), 1 \rangle.$$

Note that this formula depends on the choice of the above symmetrising form.

Explicit calculations of the BV structure on Hochschild cohomology of finite groups have been made in a number of different cases, see the references in Section 9. Also relevant are the papers of Le and Zhou [2014] and Volkov [2016]. For  $k$  a field, Liu and Zhou [2016] have given an explicit description of the BV operator on

$HH^*(kG)$  in terms of Hochschild cochains, combining the centraliser decomposition and Tradler's description of the BV operator in [Tradler 2008].

#### 4. The map $\Delta_g$

In this section, we dualise the construction in Theorem 2.4 and use it to describe the components

$$\Delta_g: H^n(G, k) \rightarrow H^{n-1}(G, k)$$

of the BV operator  $\Delta$ , where  $G$  is a finite group. As mentioned previously, for  $g \in Z(G)$ , the long exact sequence in cohomology for the fibration

$$S^1 \rightarrow BG \rightarrow B\hat{G}$$

has the form

$$(4.1) \quad \dots \rightarrow H^{n+1}(BG; k) \xrightarrow{B} H^n(B\hat{G}; k) \xrightarrow{S} H^{n+2}(B\hat{G}; k) \xrightarrow{I} H^{n+2}(BG; k) \rightarrow \dots,$$

where we have used the class  $\mu \in H^1(\mathbb{Z}, k)$  introduced in Section 2 to identify  $E_2^{*,1}$  with  $H^*(B\hat{G}; k)$ . Again, the map  $I$  is induced by  $BG \rightarrow B\hat{G}$ . Next, we describe the map  $\Delta_g: H^n(G, k) \rightarrow H^{n-1}(G, k)$  coming from the element  $g \in Z(G)$ .

**Theorem 4.2.** *Let  $G$  be a finite group, and let  $g \in Z(G)$ . Consider the group homomorphism  $z_g: \mathbb{Z} \times G \rightarrow G$  which sends  $(m, h)$  to  $g^m h$ . The induced map*

$$(z_g)^*: H^n(G, k) \rightarrow H^n(\mathbb{Z} \times G, k) = H^n(G, k) \oplus H^{n-1}(G, k)$$

has the form  $(1, \Delta_g)$ , where  $\Delta_g = I \circ B$  is equal to the composite

$$H^n(G, k) \xrightarrow{B} H^{n-1}(\hat{G}, k) \xrightarrow{I} H^{n-1}(G, k).$$

*Proof.* The proof is dual to the proof of Theorem 2.4. □

The next corollary proves properties (i), (iii), (iv), (vi), and (viii) in Theorem 1.1:

**Corollary 4.3.** *Let  $G$  be a finite group, and let  $g \in Z(G)$ . Then:*

(i) *The map*

$$\Delta_g: H^*(G, k) \rightarrow H^{*-1}(G, k)$$

*is a derivation with respect to the ordinary cohomology cup product. In particular,  $\Delta_g$  is determined by its values on a generating set of  $H^*(G, k)$ .*

- (ii) *The map  $\Delta_g$  is natural with respect to group homomorphisms and with respect to homomorphisms of coefficient rings.*
- (iii) *If  $H$  is a subgroup of  $G$  containing  $g$ , then  $\Delta_g$  commutes with the transfer map and the restriction map.*
- (iv) *In the case  $k = \mathbb{F}_p$ , the map  $\Delta_g$  commutes with the Steenrod operations and with the Bockstein homomorphism.*

*Proof.* The map

$$(z_g)^*: H^*(BG, k) \rightarrow H^*(S^1 \times BG, k) = H^*(S^1, k) \otimes_k H^*(BG, k)$$

is a  $k$ -algebra homomorphism. Let  $\mu$  be the generator for  $H^1(S^1, k)$  chosen so that for  $x \in H^*(BG, k)$  we have

$$(z_g)^*(x) = 1 \otimes x + \mu \otimes \Delta_g(x).$$

Then,

$$\begin{aligned} (z_g)^*(x \cdot y) &= (1 \otimes x + \mu \otimes \Delta_g(x)) \cdot (1 \otimes y + \mu \otimes \Delta_g(y)) \\ &= 1 \otimes x \cdot y + \mu \otimes (\Delta_g(x) \cdot y + (-1)^{|x|} x \cdot \Delta_g(y)). \end{aligned}$$

Examining the coefficient of  $\mu$ , we see that

$$\Delta_g(x \cdot y) = \Delta_g(x) \cdot y + (-1)^{|x|} x \cdot \Delta_g(y).$$

This proves (i). The naturality statements (ii)–(iv) follow directly from the fact that  $(z_g)^* = (1, \Delta_g)$ .  $\square$

For degree one elements, it is easy to describe  $\Delta_g$ ; this is statement (ix) in Theorem 1.1.

**Proposition 4.4.** *Let  $G$  be a finite group, and let  $g \in Z(G)$ . Then identifying  $H^1(G, k)$  with  $\text{Hom}(G, k)$ , we have*

$$\Delta_g(x) = x(g)$$

for any  $x \in H^1(G, k)$ .

*Proof.* The composite  $\mathbb{Z} \times G \xrightarrow{z_g} G \xrightarrow{x} k$  is equal to  $1 \otimes x + \mu \otimes x(g)$ .  $\square$

**Remark 4.5.** It follows from Proposition 4.4 that if  $g$  is in the subgroup generated by commutators and  $p$ -th powers of elements in  $G$ , then  $\Delta_g$  is identically zero on  $H^1(G, k)$ .

The next result is statement (x) in Theorem 1.1.

**Proposition 4.6.** *Let  $G$  be a finite group, and let  $g \in Z(G)$ . Then the map  $\Delta_g$  vanishes on the image of  $H^2(G, \mathbb{Z}) \rightarrow H^2(G, k)$ .*

*Proof.* Consider the commutative diagram

$$\begin{array}{ccc} H^2(G, \mathbb{Z}) & \xrightarrow{\Delta_g} & H^1(G, \mathbb{Z}) \\ \downarrow & & \downarrow \\ H^2(G, k) & \xrightarrow{\Delta_g} & H^1(G, k). \end{array}$$

The proposition now follows from the fact that since  $G$  is finite, we have  $H^1(G, \mathbb{Z}) = \text{Hom}(G, \mathbb{Z}) = 0$ .  $\square$

### 5. Degree $-1$ operators on $\text{Ext}_A^*(U, V)$

We describe an elementary construction principle for degree  $-1$  operators on  $\text{Ext}_A^*(U, V)$  determined by a central element in an algebra  $A$  which annihilates both modules  $U$  and  $V$ . We use this in the next section to interpret the BV operator in terms of this construction principle. Let  $k$  be a commutative ring.

**Theorem 5.1.** *Let  $A$  be a  $k$ -algebra, let  $z \in Z(A)$ , and let  $U$  and  $V$  be  $A$ -modules. Suppose that  $z$  annihilates both  $U$  and  $V$ . Let  $P = (P_n)_{n \geq 0}$  together with a surjective  $A$ -homomorphism  $\pi : P_0 \rightarrow U$  be a projective resolution of  $U$ , with differential  $\delta = (\delta_n : P_n \rightarrow P_{n-1})_{n \geq 1}$ . For notational convenience, set  $P_i = 0$  for  $i < 0$  and  $\delta_i = 0$  for  $i \leq 0$ . Then the following hold:*

- (i) *There is a graded  $A$ -homomorphism  $s : P \rightarrow P$  of degree 1 such that the chain endomorphism  $\delta \circ s + s \circ \delta$  of  $P$  is equal to multiplication by  $z$  on  $P$ .*
- (ii) *The graded  $k$ -linear map*

$$s^\vee = \text{Hom}_A(s, V) : \text{Hom}_A(P, V) \rightarrow \text{Hom}_A(P[1], V)$$

*sending  $f \in \text{Hom}_A(P_n, V)$  to  $f \circ s \in \text{Hom}_A(P_{n-1}, V)$  for all  $n \geq 0$  is a homomorphism of cochain complexes. In particular,  $s^\vee$  induces a graded  $k$ -linear map of degree  $-1$*

$$D_z^A = H^*(s^\vee) : \text{Ext}_A^*(U, V) \rightarrow \text{Ext}_A^{*-1}(U, V).$$

- (iii) *The graded map  $D_z^A$  is independent of the choice of the projective resolution  $P$  and the choice of the homotopy  $s$  satisfying (i). In particular, we have  $D_0^A = 0$ .*

**Remark 5.2.** If  $A$  is obvious from the context, we write  $D_z$  instead of  $D_z^A$ . Note that  $D_z$  is a graded map which depends on  $U$  and  $V$ . With the notation of the theorem, we use the following sign conventions. For an integer  $i$ , the shifted complex  $P[i]$  is equal, in degree  $n$ , to  $P_{n-i}$ , with differential  $(-1)^i \delta$ . The cochain complex  $\text{Hom}_A(P, V)$  has differential in degree  $n$  given by precomposing with  $(-1)^{n+1} \delta_{n+1}$ . (This is consistent with the standard sign conventions, as described in [Benson 1991a, §2.7], for total complexes of double complexes of the form  $\text{Hom}_A(P, Q)$ , where  $Q$  is another chain complex, modulo regarding  $\text{Hom}_A(P, V)$  as a chain complex.) Combining the above sign conventions for shifts and total complexes, we get that the chain complex  $P[1]$  has differential  $-\delta$ , and the cochain complex  $\text{Hom}_A(P[1], V)$  has, in degree  $n$ , the differential sending  $f \in \text{Hom}_A(P_{n-1}, V)$  to  $-(-1)^{n+1} f \circ \delta_n = (-1)^n f \circ \delta_n \in \text{Hom}_A(P_n, V)$ . The sign convention for cochain complexes of the form  $\text{Hom}_A(P, V)$  has no impact on the definition of the operators  $D_z$ , but it does matter for the signs of Bockstein homomorphisms. Had we chosen the differential of  $\text{Hom}_A(P, V)$  simply being given by precomposing with  $\delta$ , then the Bockstein homomorphisms in Proposition 5.10 below would anticommute with the operators  $D_z$ .

*Proof of Theorem 5.1.* Multiplication by  $z$  annihilates  $U$ . Thus the chain endomorphism of  $P$  induced by multiplication with  $z$  is homotopic to zero. This proves the existence of a homotopy  $s$  satisfying (i). Let  $n$  be a nonnegative integer. In order to show that  $s^\vee$  is a cochain map, we need to show that the following diagram of  $k$ -linear maps:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \mathrm{Hom}_A(P_n, V) & \xrightarrow{(-1)^{n+1}\delta_{n+1}^\vee} & \mathrm{Hom}_A(P_{n+1}, V) & \longrightarrow & \cdots \\ & & \downarrow s_{n-1}^\vee & & \downarrow s_n^\vee & & \\ \cdots & \longrightarrow & \mathrm{Hom}_A(P_{n-1}, V) & \xrightarrow{(-1)^n\delta_n^\vee} & \mathrm{Hom}_A(P_n, V) & \longrightarrow & \cdots \end{array}$$

is commutative. The commutativity of this diagram is equivalent to

$$(-1)^{n+1} f \circ \delta_{n+1} \circ s_n = (-1)^n f \circ s_{n-1} \circ \delta_n,$$

hence to

$$f \circ (\delta_{n+1} \circ s_n + s_{n-1} \circ \delta_n) = 0,$$

for all  $f \in \mathrm{Hom}_A(P_n, V)$ . By the choice of the homotopy  $s$ , the left side is equal to  $f \circ \zeta$ , where  $\zeta : P_n \rightarrow P_n$  is equal to multiplication by  $z$ . Since  $f(zP_n) = zf(P_n) \subseteq zV = \{0\}$ , it follows that  $f \circ \zeta = 0$ . This shows that  $s^\vee$  is a cochain map. Taking cohomology,  $s^\vee$  induces a degree  $-1$  map  $D_z$  as stated, whence (ii).

Let  $P'$  be a projective resolution of  $U$ , with differential  $\delta'$  and quasi-isomorphism  $P' \rightarrow U$  given by a map  $\pi' : P'_0 \rightarrow U$ . Let  $s'$  be a homotopy on  $P'$  with the property that the chain endomorphism  $\delta' \circ s' + s' \circ \delta'$  of  $P'$  is equal to multiplication by  $z$  on  $P'$ . Let  $a : P \rightarrow P'$  be a chain homotopy equivalence lifting the identity on  $U$ , via the maps  $\pi$  and  $\pi'$ . We need to show that the homotopies  $a \circ s$  and  $s' \circ a$  from  $P[1]$  to  $P'$  induce the same map upon applying  $\mathrm{Hom}_A(-, V)$ . Set  $t = a \circ s - s' \circ a$ . We will use the same letter  $\zeta$  for the chain endomorphisms of  $P$  and  $P'$  given by multiplication with  $z$ . Using that  $a$  commutes with the differentials of  $P$  and  $P'$ ,

$$\delta' \circ t + t \circ \delta = a \circ \delta \circ s - \delta' \circ s' \circ a + a \circ s \circ \delta - s' \circ \delta' \circ a = a \circ \zeta - \zeta \circ a = 0.$$

Taking into account that the differential of  $P[1]$  is  $-\delta$ , this implies that  $t$  is in fact a chain map from  $P[1]$  to  $P'$ , or equivalently, from  $P$  to  $P'[-1]$ . The homotopy class of such a chain map represents an element in  $\mathrm{Ext}_A^{-1}(U, U) = \{0\}$ , and hence the chain map  $t$  is homotopic to zero. That is, there is a graded map  $u : P[1] \rightarrow P'$  of degree 1 such that  $t = \delta' \circ u - u \circ \delta$ , where, as before, the sign comes from the fact that the differential of  $P[1]$  is  $-\delta$ . Since  $t$  is a chain map, it follows that  $t^\vee = \mathrm{Hom}_A(t, V) : \mathrm{Hom}_A(P', V) \rightarrow \mathrm{Hom}_A(P[1], V)$  is a cochain map. The functor  $\mathrm{Hom}_A(-, V)$  sends the homotopy  $u$  to a homotopy  $u^\vee$  satisfying  $t^\vee = u^\vee \circ \delta^\vee - \delta^\vee \circ u^\vee$ . We need to adjust  $u^\vee$  with the signs needed to compensate for the signs in the differentials of  $\mathrm{Hom}_A(P', V)$  and  $\mathrm{Hom}_A(P[1], V)$  according to the sign convention in Remark 5.2. More precisely, one checks that  $((-1)^{n+1}u_n^\vee)$  is

the homotopy which is needed to show that the cochain map  $t^\vee$  is homotopic to zero. Thus  $t^\vee$  induces the zero map on cohomology. This shows that the maps  $s^\vee$  and  $(s')^\vee$  induce the same map  $D_z$  upon taking cohomology, hence, in particular,  $D_0 = 0$ . This proves (iii).  $\square$

Let  $A$  be an algebra over a commutative ring  $k$ . Any element  $z \in Z(A)$  induces a chain endomorphism on a projective resolution  $P$  of an  $A$ -module  $U$ , and hence a graded linear endomorphism on  $\text{Ext}_A^*(U, V)$ , for any two  $A$ -modules  $U$  and  $V$ . In this way, the space of graded  $k$ -linear endomorphisms of  $\text{Ext}_A^*(U, V)$  of any fixed degree becomes a module over  $Z(A)$ . Since multiplication by  $z$  induces an element in the centre of the module category of  $A$ , it follows easily that this module structure does not depend on the choice of the projective resolution  $P$ , and moreover, for the same reason, it makes no difference whether we compose the endomorphism on  $U$  given by  $z$  with an element in  $\text{Ext}_A^*(U, V)$  or compose this element with the endomorphism on  $V$  given by  $z$ .

**Theorem 5.3.** *Let  $A$  be a  $k$ -algebra, let  $U$  and  $V$  be  $A$ -modules, and let  $y, z \in Z(A)$ .*

- (i) *Suppose that  $y$  and  $z$  annihilate  $U$  and  $V$ . Then  $y + z$  annihilates  $U$  and  $V$ , and we have  $D_{y+z} = D_y + D_z$ .*
- (ii) *Suppose that  $z$  annihilates  $U$  and  $V$ . Then  $yz$  annihilates  $U$  and  $V$ , and we have  $D_{yz} = yD_z$ .*
- (iii) *Suppose that  $y$  annihilates  $V$  and that  $z$  annihilates  $U$  and  $V$ . Then  $D_{yz} = 0$ .*
- (iv) *Let  $e$  be an idempotent in  $Z(A)$  such that  $e$  annihilates  $U$  and  $V$ . Then  $D_e = 0$ .*

*Proof.* For (i), suppose that  $y$  and  $z$  annihilate  $U$  and  $V$ . Then clearly  $y + z$  annihilates  $U$  and  $V$ . Let  $P$  be a projective resolution of  $U$ , with differential  $\delta$ . Let  $s$  and  $t$  be homotopies on  $P$  such that  $\delta \circ s + s \circ \delta$  is the chain endomorphism of  $P$  given by multiplication with  $y$  and such that  $\delta \circ t + t \circ \delta$  is the chain endomorphism of  $P$  given by multiplication with  $z$ . Then  $\delta \circ (s+t) + (s+t) \circ \delta$  is the endomorphism of  $P$  given by multiplication with  $y + z$ . Statement (i) follows. For (ii), suppose that  $z$  annihilates  $U$  and  $V$ . Then clearly  $yz$  annihilates  $U$  and  $V$ . As before, let  $t$  be a homotopy on  $P$  such that  $\delta \circ t + t \circ \delta$  is the chain endomorphism of  $P$  given by multiplication with  $z$ . Denote by  $y \cdot t$  the homotopy obtained from composing  $t$  with the endomorphism given by multiplication with  $y$ . Then  $\delta \circ (y \cdot t) + (y \cdot t) \circ \delta$  is equal to multiplication on  $P$  by  $yz$ , which shows that  $D_{yz} = yD_z$ . Statement (iii) follows from (ii) and the fact that multiplication by  $y$  annihilates  $V$ , hence it annihilates the space  $\text{Ext}_A^*(U, V)$ . Since  $e = e^2$ , statement (iv) is a special case of (iii).  $\square$

The operators  $D_z^A$  are compatible with the restriction to subalgebras  $B$  containing  $z$  such that  $A$  is projective as a  $B$ -module.

**Proposition 5.4.** *Let  $A$  be a  $k$ -algebra, and let  $B$  be a subalgebra of  $A$  such that  $A$  is projective as a left  $B$ -module. Let  $z \in Z(A) \cap B$ , and let  $U$  and  $V$  be  $A$ -modules. Suppose that  $z$  annihilates  $U$  and  $V$ . Then we have a commutative diagram of graded maps*

$$\begin{array}{ccc} \mathrm{Ext}_A^*(U, V) & \xrightarrow{D_z^A} & \mathrm{Ext}_A^{*-1}(U, V) \\ \downarrow & & \downarrow \\ \mathrm{Ext}_B^*(\mathrm{Res}_B^A(U), \mathrm{Res}_B^A(V)) & \xrightarrow{D_z^B} & \mathrm{Ext}_B^{*-1}(\mathrm{Res}_B^A(U), \mathrm{Res}_B^A(V)), \end{array}$$

where the vertical maps are induced by the restriction to  $B$ .

*Proof.* Let  $P$  be a projective resolution of  $U$ . By the assumptions on  $B$ , the restriction to  $B$  of  $P$  is a projective resolution of  $\mathrm{Res}_B^A(U)$ . Thus if  $s$  is a homotopy on  $P$  which defines  $D_z^A$  as in Theorem 5.1, then the restriction to  $B$  of  $s$  is the corresponding homotopy for  $D_z^B$ . The commutativity of the diagram follows immediately from the construction of the maps  $D_z^A$  and  $D_z^B$ .  $\square$

**Remark 5.5.** As pointed out by the referee, Proposition 5.4 holds even if  $A$  is not projective as a  $B$ -module. In that case,  $\mathrm{Res}_B^A(P)$  is still a resolution of  $\mathrm{Res}_B^A(U)$ , albeit not necessarily a projective one. But since the homotopy  $s$  on  $P$  restricts to a homotopy on  $\mathrm{Res}_B^A(P)$ , the proof of Theorem 5.1 (ii) still yields an operator of degree  $-1$  upon taking cohomology in  $\mathrm{Hom}_B(\mathrm{Res}_B^A(P), \mathrm{Res}_B^A(V))$ . To show that this yields a commutative diagram as stated, one observes in the proof of Theorem 5.1 (iii) that it suffices to assume that  $P'$  is a resolution which need not be projective but admits a homotopy  $s'$  as in (i).

**Proposition 5.6.** *Let  $G$  a finite group,  $H$  a subgroup of  $G$ , and  $z \in Z(kG) \cap kH$ . Let  $U$  and  $V$  be  $kG$ -modules, and suppose that  $z$  annihilates  $U$  and  $V$ . We have a commutative diagram of graded maps*

$$\begin{array}{ccc} \mathrm{Ext}_{kH}^*(\mathrm{Res}_H^G(U), \mathrm{Res}_H^G(V)) & \xrightarrow{D_z^{kH}} & \mathrm{Ext}_{kH}^{*-1}(\mathrm{Res}_H^G(U), \mathrm{Res}_H^G(V)) \\ \mathrm{Tr}_H^G \downarrow & & \downarrow \mathrm{Tr}_H^G \\ \mathrm{Ext}_{kG}^*(U, V) & \xrightarrow{D_z^{kG}} & \mathrm{Ext}_B^{*-1}(U, V). \end{array}$$

*Proof.* Let  $P$  be a projective resolution of the  $kG$ -module  $U$ , with differential denoted  $\delta$ . By the assumptions on  $z$  and by Theorem 5.1, there is a homotopy  $s$  on  $P$  such that  $\delta \circ s + s \circ \delta$  is equal to the chain endomorphism of  $P$  given by multiplication with  $z$ . The operator  $D_z^{kG}$  is induced by the map sending  $f \in \mathrm{Hom}_{kG}(P_n, V)$  to  $f \circ s_{n-1}$ . Since  $\mathrm{Res}_H^G(P)$  is a projective resolution of  $\mathrm{Res}_H^G(U)$ , it follows that  $D_z^{kH}$  is induced by the map sending  $f' \in \mathrm{Hom}_{kH}(\mathrm{Res}_H^G(P_n), \mathrm{Res}_H^G(V))$  to  $f' \circ s_{n-1}$ . Since  $s_{n-1}$  is a  $kG$ -homomorphism, we have  $\mathrm{Tr}_H^G(f') \circ s_{n-1} = \mathrm{Tr}_H^G(f' \circ s_{n-1})$ .  $\square$

The operators  $D_z^A$  satisfy a Künneth formula. We suppress the superscripts in what follows, since the central element subscripts determine which algebra we are working in:

**Proposition 5.7.** *Let  $A$  and  $B$  be  $k$ -algebras that are finitely generated projective as  $k$ -modules, and let  $U$  and  $U'$  be  $A$ -modules and  $V$  and  $V'$  be  $B$ -modules, all finitely generated projective as  $k$ -modules. Set  $W = U \otimes_k V$  and  $W' = U' \otimes_k V'$ . Let  $z \in Z(A)$  and  $w \in Z(B)$  such that  $z$  annihilates  $U$  and  $U'$  and  $w$  annihilates  $V$  and  $V'$ .*

*Then  $z \otimes 1$  and  $1 \otimes w$  annihilate the  $A \otimes_k B$ -modules  $U \otimes_k V$  and  $U' \otimes_k V'$ , and we have a commutative diagram*

$$\begin{array}{ccc} \text{Ext}_A^i(U, U') \otimes_k \text{Ext}_B^j(V, V') & \longrightarrow & \text{Ext}_{A \otimes_k B}^{i+j}(W, W') \\ (D_z \otimes 1, 1 \otimes D_w) \downarrow & & \downarrow D_{z \otimes 1 + 1 \otimes w} \\ (\text{Ext}_A^{i-1}(U, U') \otimes_k \text{Ext}_B^j(V, V')) \oplus (\text{Ext}_A^i(U, U') \otimes_k \text{Ext}_B^{j-1}(V, V')) & \longrightarrow & \text{Ext}_{A \otimes_k B}^{i+j-1}(W, W'), \end{array}$$

where  $i, j$  are nonnegative integers. Moreover, we have  $D_{z \otimes w} = 0$ . In particular, if  $k$  is a field, then  $D_{z \otimes 1 + 1 \otimes w}$  is determined by  $D_z, D_w$ , combined with the Künneth formula.

*Proof.* Note the following sign convention (as in statement (ii) of Theorem 1.1): the second component  $1 \otimes D_w$  of the left vertical map sends  $\eta \otimes \theta$  to  $(-1)^i \eta \otimes D_w(\theta)$ , where  $\eta \in \text{Ext}_A^i(U, U')$  and  $\theta \in \text{Ext}_B^j(V, V')$ .

The assumptions on  $z$  and  $w$  imply that  $z \otimes 1$  and  $1 \otimes w$  annihilate the  $A \otimes_k B$ -modules  $U \otimes_k V$  and  $U' \otimes_k V'$ . Let  $(P, \delta)$  be a projective resolution of  $U$  and  $(Q, \epsilon)$  a projective resolution of  $V$ . Since  $A$  and  $B$  are projective as  $k$ -modules, it follows that  $P \otimes_k Q$  is a projective resolution of the  $A \otimes_k B$ -module  $U \otimes_k V$ . Note the signs in the differential  $\delta \otimes \epsilon$  of  $P \otimes_k Q$ ; more precisely, the differential  $\delta \otimes \epsilon$  sends  $u \otimes v \in P_i \otimes_k Q_j$  to

$$(\delta_i(u) \otimes v, (-1)^i u \otimes \epsilon_j(v))$$

in  $(P_{i-1} \otimes_k Q_j) \oplus (P_i \otimes_k Q_{j-1})$ .

We have a canonical isomorphism of cochain complexes

$$\text{Hom}_A(P, U') \otimes_k \text{Hom}_B(Q, V') \cong \text{Hom}_{A \otimes_k B}(P \otimes_k Q, U' \otimes_k V')$$

thanks to the assumptions that the involved algebras and modules are finitely generated projective as  $k$ -modules. Upon taking cohomology, this induces the horizontal maps

$$\text{Ext}_A^i(U, U') \otimes_k \text{Ext}_B^j(V, V') \rightarrow \text{Ext}_{A \otimes_k B}^{i+j}(U \otimes_k V, U' \otimes_k V')$$

in the statement.

Let  $s$  be a homotopy on  $P$  such that  $s \circ \delta + \delta \circ s$  is equal to multiplication by  $z$  on  $P$ . Similarly, let  $t$  be a homotopy on  $Q$  such that  $t \circ \epsilon + \epsilon \circ t$  is equal to multiplication by  $w$  on  $Q$ . Define the homotopy  $\sigma$  on  $P \otimes_k Q$  by sending  $u \otimes v \in P_i \otimes_k Q_j$  to

$$\sigma(u \otimes v) = (s_i(u) \otimes v, (-1)^i u \otimes t_j(v))$$

in  $(P_{i+1} \otimes_k Q_i) \oplus (P_i \otimes_k Q_{j+1})$ . The sign  $(-1)^i$  is needed because of the above mentioned sign convention for  $1 \otimes D_w$ . We need to show that this homotopy has the property that the chain endomorphism of  $P \otimes_k Q$  given by

$$\sigma \circ (\delta \otimes \epsilon) + (\delta \otimes \epsilon) \circ \sigma$$

is equal to multiplication by  $z \otimes 1 + 1 \otimes w$  on  $P \otimes_k Q$ .

Let  $u \otimes v \in P_i \otimes_k Q_j$ . We calculate first the image of  $u \otimes v$  under  $\sigma \circ (\delta \otimes \epsilon)$ . The differential  $\delta \otimes \epsilon$  sends  $u \otimes v$  to the element

$$(\delta_i(u) \otimes v, (-1)^i u \otimes \epsilon_j(v))$$

in  $(P_{i-1} \otimes_k Q_j) \oplus (P_i \otimes_k Q_{j-1})$ . The homotopy  $\sigma$  sends this to the element

$$((-1)^{i-1} \delta_i(u) \otimes t_j(v), s_{i-1}(\delta_i(u)) \otimes v + u \otimes t_{j-1}(\epsilon_j(v)), (-1)^i s_i(u) \otimes \epsilon_j(v))$$

in  $(P_{i-1} \otimes_k Q_{j+1}) \oplus (P_i \otimes_k Q_j) \oplus (P_{i+1} \otimes_k Q_{j-1})$ .

We calculate next the image of  $u \otimes v$  under  $(\delta \otimes \epsilon) \circ \sigma$ . The homotopy  $\sigma$  sends  $u \otimes v$  to the element

$$(s_i(u) \otimes v, (-1)^i u \otimes t_j(v))$$

in  $(P_{i+1} \otimes_k Q_j) \oplus (P_i \otimes_k Q_{j+1})$ . Applying the differential  $\delta \otimes \epsilon$  to this element yields

$$((-1)^i \delta_i(u) \otimes t_j(v), \delta_{i+1}(s_i(u)) \otimes v + u \otimes \epsilon_{j+1}(t_j(v)), (-1)^{i+1} s_i(u) \otimes \epsilon_j(v))$$

in  $(P_{i-1} \otimes_k Q_{j+1}) \oplus (P_i \otimes_k Q_j) \oplus (P_{i+1} \otimes_k Q_{j-1})$ . The sum of the images of  $u \otimes v$  under the two maps  $\sigma \circ (\delta \otimes \epsilon)$  and  $(\delta \otimes \epsilon) \circ \sigma$  is therefore equal to

$$(0, ((z \otimes 1) + (1 \otimes w))(u \otimes v), 0),$$

as claimed. The first statement follows. Since  $z \otimes w = (z \otimes 1)(1 \otimes w)$ , the second statement follows from Theorem 5.3 (iii).  $\square$

**Proposition 5.8.** *Let  $A$  be a  $k$ -algebra, let  $z, w \in Z(A)$ , and let  $U$  and  $V$  be  $A$ -modules. Suppose that  $z, w$  annihilate  $U$  and  $V$ . Then the following hold:*

- (i)  $D_z \circ D_z = 0$ .
- (ii)  $D_w \circ D_z = -D_z \circ D_w$ .

*Proof.* Let  $P = (P_n)_{n \geq 0}$  together with a surjective  $A$ -homomorphism  $\pi : P_0 \rightarrow U$  be a projective resolution of  $U$ , with differential

$$\delta = (\delta_n : P_n \rightarrow P_{n-1})_{n \geq 1}.$$

As above, for notational convenience, we set  $P_i = 0$  for  $i < 0$  and  $\delta_i = 0$  for  $i \leq 0$ .

Let  $s$  be a homotopy on  $P$  such that  $s \circ \delta + \delta \circ s$  is equal to multiplication by  $z$  on  $P$ . By the construction from Theorem 5.1, the map  $D_z$  is induced by the map sending  $f \in \text{Hom}_A(P_n, V)$  to  $f \circ s$ . Thus  $D_z \circ D_z$  is induced by the map sending  $f \in \text{Hom}_A(P_n, V)$  to  $f \circ s \circ s$ . In order to show  $D_z \circ D_z = 0$ , we need to show that if  $f$  is a cocycle (that is,  $f \circ \delta_{n+1} = 0$ ), then  $f \circ s \circ s = 0$ . For this, it suffices to show that the graded degree 2 map  $s \circ s$  is a chain map from  $P[2]$  to  $P$ . Indeed, any such chain map is homotopic to zero (as it represents an element in  $\text{Ext}_A^{-2}(U, U) = 0$ ), hence, upon applying the contravariant functor  $\text{Hom}_A(-, V)$ , it induces a cochain map  $\text{Hom}_A(P, V) \rightarrow \text{Hom}_A(P[2], V)$  which is still homotopic to zero and which therefore induces the zero map in cohomology.

Composing the chain map  $s \circ \delta + \delta \circ s$  with  $s$  in either order yields the equations (of graded endomorphisms of  $P$  of degree 1)

$$\begin{aligned} s \circ s \circ \delta + s \circ \delta \circ s &= s \cdot z, \\ s \circ \delta \circ s + \delta \circ s \circ s &= z \cdot s. \end{aligned}$$

The right sides of the two equations are equal, since  $s$  is a (graded)  $A$ -homomorphism, so commutes with the action of  $z$ . Taking the difference of these two equations therefore yields

$$s \circ s \circ \delta - \delta \circ s \circ s = 0.$$

This shows that  $s \circ s$  is indeed a chain map  $P[2] \rightarrow P$ , which by the previous paragraph completes the proof of (i). Statement (ii) follows from applying (i) to  $z + w$  and using Theorem 5.3 (i).  $\square$

Let  $A$  be a  $k$ -algebra,  $z \in Z(A)$ , and let  $U$  and  $V$  be  $A$ -modules which are annihilated by  $z$ . The operator  $D_z$  on  $\text{Ext}_A^*(U, V)$  can also be described using an injective resolution  $(I, \epsilon)$  of  $V$  instead of a projective resolution  $(P, \delta)$  of  $U$ . Let  $p : P \rightarrow U$  and  $i : V \rightarrow I$  be quasi-isomorphisms, where  $U$  and  $V$  are regarded as complexes concentrated in degree 0. Denote by  $K(A)$  the homotopy category of chain complexes of  $A$ -modules. By standard facts (see, e.g., [Benson 1991a, §2.7] or [Weibel 1994, §2.7]), the space  $\text{Ext}_A^*(U, V)$  can be identified with any of

$$\text{Hom}_{K(A)}(P, V[n]) \xrightarrow{\cong} \text{Hom}_{K(A)}(P, I[n]) \xleftarrow{\cong} \text{Hom}_{K(A)}(U, I[n]),$$

where the isomorphisms are induced by composing with  $i$  and precomposing with  $p$ . We reindex complexes as chain complexes, if necessary (so in particular, an injective resolution of  $V$  is of the form  $I_0 \rightarrow I_{-1} \rightarrow I_{-2} \rightarrow \cdots$ ).

**Theorem 5.9.** *Let  $A$  be a  $k$ -algebra, let  $z \in Z(A)$ , and let  $U$  and  $V$  be  $A$ -modules which are both annihilated by  $z$ . Let  $(I, \epsilon)$  be an injective resolution of  $V$  with quasi-isomorphism  $i : V \rightarrow I$ . Let  $t$  be a homotopy on  $I$  such that  $\epsilon \circ t + t \circ \epsilon$  is equal to multiplication by  $z$  on the terms of  $I$ . Then the graded  $k$ -linear map  $t_\vee : \text{Hom}_A(U, I) \rightarrow \text{Hom}_A(U[1], I)$  sending  $g \in \text{Hom}_A(U, I_{-n})$  to  $(-1)^n t_{-n} \circ g$  is a chain map, and the induced map in cohomology is equal to  $D_z$ , where we identify the cohomology of  $\text{Hom}_A(U, I)$  and  $\text{Ext}_A^*(U, V)$  using the preceding isomorphisms.*

*Proof.* Since  $z$  annihilates  $V$ , the existence of a homotopy  $t$  on  $I$  such that  $\epsilon \circ t + t \circ \epsilon$  is equal to multiplication by  $z$  is obvious. The verification that the assignment  $g \mapsto (-1)^n t \circ g$  is a chain map is analogous to the first part of the proof of Theorem 5.1. (The sign  $(-1)^n$  comes from the fact that  $g$  is regarded as a chain map  $U \rightarrow I[n]$ , and since the differential of  $I[n]$  is  $(-1)^n \epsilon$ , one needs to use the homotopy  $(-1)^n t$  in order to obtain multiplication by  $z$  as the chain map determined by this homotopy on  $I[n]$ .)

Let  $n$  be a nonnegative integer. Let  $g \in \text{Hom}_A(U, I_{-n})$  be a cocycle; that is,  $\epsilon \circ g = 0$ . Note that this is equivalent to stating that  $g : U \rightarrow I[n]$  is a chain map, and hence  $\tilde{g} = g \circ p : P \rightarrow I[n]$  is a chain map, where, as before,  $P$  is a projective resolution of  $U$  with differential  $\delta$  and quasi-isomorphism  $p : P \rightarrow U$ . Similarly, let  $f \in \text{Hom}_A(P_n, V)$  be a cocycle; that is,  $f \circ \delta = 0$ . As before, this means that  $f : P \rightarrow V[n]$  is a chain map, and hence  $\tilde{f} = i[n] \circ f : P \rightarrow I[n]$  is a chain map.

Assume now that  $f$  and  $g$  represent the same class in  $\text{Ext}_A^n(U, V)$ . This is equivalent to requiring that the chain maps  $\tilde{f}, \tilde{g}$  from  $P$  to  $I[n]$  are homotopic. Thus, there is a homotopy  $u$  from  $P$  to  $I[n]$  such that

$$\tilde{f} - \tilde{g} = u \circ \delta + (-1)^n \epsilon \circ u,$$

where the sign  $(-1)^n$  comes from the fact that the differential of  $I[n]$  is  $(-1)^n \epsilon$ .

Let  $s$  be a homotopy on  $P$  such that  $\delta \circ s + s \circ \delta$  is equal to multiplication by  $z$  on  $P$ . By the construction of  $D_z$ , the image of the class of  $f$  under  $D_z$  is represented by  $f \circ s$ . Note that  $f \circ s : P \rightarrow V[n]$  is a chain map (this was noted already in the proof of  $D_z$  being well-defined: since  $z$  annihilates  $V$ , we have

$$0 = f \cdot z = f \circ \delta \circ s + f \circ s \circ \delta = f \circ s \circ \delta,$$

where we use the assumption  $f \circ \delta = 0$ , and hence  $f \circ s \circ \delta = 0$ ). We need to show that  $f \circ s$  and  $(-1)^n t \circ g$  represent the same class in  $\text{Ext}_A^{n-1}(U, V)$ . That is, we need to show that the chain maps  $\tilde{f} \circ s$  and  $(-1)^n t \circ \tilde{g}$  from  $P$  to  $I[n-1]$  are homotopic, or equivalently, we need to show that their difference  $\tilde{f} \circ s - (-1)^n t \circ \tilde{g}$  is homotopic to zero.

Note that  $f : P \rightarrow V[n]$ , and hence

$$\tilde{f} = i[n] \circ f : P \rightarrow I[n]$$

is also a chain map which is zero in all degrees other than  $n$ . Since  $I[n]$  is zero in all degrees bigger than  $n$ , it follows that  $t \circ \tilde{f} = 0$ . Similarly, we have  $\tilde{g} \circ s = 0$ . It follows that

$$\begin{aligned} \tilde{f} \circ s - (-1)^{n-1} t \circ \tilde{g} &= (\tilde{f} - \tilde{g}) \circ s + (-1)^{n-1} t \circ (\tilde{f} - \tilde{g}) \\ &= u \circ \delta \circ s + (-1)^n \epsilon \circ u \circ s + (-1)^{n-1} t \circ u \circ \delta + (-1)^{2n-1} t \circ \epsilon \circ u \\ &= u \circ \delta \circ s + (-1)^n \epsilon \circ u \circ s + (-1)^{n-1} t \circ u \circ \delta - t \circ \epsilon \circ u. \end{aligned}$$

Since  $u \cdot z = u \circ (\delta \circ s + s \circ \delta)$ , we have  $u \circ \delta \circ s = u \cdot z - u \circ s \circ \delta$ . Similarly, we have  $t \circ \epsilon \circ u = z \cdot u - \epsilon \circ t \circ u$ . Inserting these two equations into the displayed equality and cancelling  $u \cdot z = z \cdot u$  yields the expression

$$-(u \circ s \circ \delta + (-1)^{n-1} \epsilon \circ u \circ s) + (-1)^{n-1} (t \circ u \circ \delta + (-1)^{n-1} \epsilon \circ t \circ u).$$

The two summands in this equation are contractible chain maps from  $P$  to  $I[n-1]$ , via the homotopies  $u \circ s$  and  $t \circ u$ , respectively. This shows the result.  $\square$

**Proposition 5.10.** *Let  $A$  be a  $k$ -algebra, let  $z \in Z(A)$ , and let  $n$  be a nonnegative integer.*

- (i) *The map  $D_z : \text{Ext}_A^n(U, V) \rightarrow \text{Ext}_A^{n-1}(U, V)$  is functorial in  $A$ -modules  $U$  and in  $V$  which are annihilated by  $z$ .*
- (ii) *For any  $A$ -module  $U$  and any short exact sequence of  $A$ -modules*

$$0 \longrightarrow V \longrightarrow W \longrightarrow X \longrightarrow 0$$

*such that  $z$  annihilates  $U, V, W, X$ , the map  $D_z$  commutes with the connecting homomorphisms  $\gamma^n : \text{Ext}_A^n(U, X) \rightarrow \text{Ext}_A^{n+1}(U, V)$ ; that is, we have*

$$\gamma^{n-1} \circ D_z = D_z \circ \gamma^n : \text{Ext}_A^n(U, X) \rightarrow \text{Ext}_A^n(U, V).$$

- (iii) *For any short exact sequence of  $A$ -modules*

$$0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0$$

*and any  $A$ -module  $X$  such that  $z$  annihilates  $U, V, W, X$ , the map  $D_z$  commutes with the connecting homomorphisms  $\gamma^n : \text{Ext}_A^n(U, X) \rightarrow \text{Ext}_A^{n+1}(W, X)$ ; that is,*

$$\gamma^{n-1} \circ D_z = D_z \circ \gamma^n : \text{Ext}_A^n(U, X) \rightarrow \text{Ext}_A^n(W, X).$$

*Proof.* Let  $P$  be a projective resolution of  $U$ . Since the operator  $D_z$  on  $\text{Ext}_A^n(U, V)$  is induced by precomposing  $f \in \text{Hom}_A(P_n, V)$  with a homotopy  $s$  on  $P$ , the functoriality in  $V$  is obvious. Using an injective resolution  $I$  of  $V$  and Theorem 5.9 yields the functoriality in  $U$ . This proves (i).

Let  $U$  be an  $A$ -module which is annihilated by  $z$ , and let as before  $P$  be a projective resolution of  $U$ . Let

$$0 \longrightarrow V \xrightarrow{i} W \xrightarrow{p} X \longrightarrow 0$$

be a short exact sequence of  $A$ -modules  $V, W, X$  which are annihilated by  $z$ . Since  $P$  consists of projective  $A$ -modules, applying  $\text{Hom}_A(P, -)$  yields a short exact sequence of cochain complexes

$$0 \longrightarrow \text{Hom}_A(P, V) \longrightarrow \text{Hom}_A(P, W) \longrightarrow \text{Hom}_A(P, X) \longrightarrow 0.$$

The connecting homomorphism  $\gamma$  associated to this sequence is constructed as follows. Let  $f \in \text{Hom}_A(P_n, X)$  be a cocycle; that is,  $f \circ \delta_{n+1} = 0$ . This represents a class  $\underline{f}$  in  $\text{Ext}_A^n(U, X)$ . Let  $g \in \text{Hom}_A(P_n, W)$  such that  $p \circ g = f$ . Then  $g \circ \delta_{n+1}$  satisfies  $p \circ g \circ \delta_{n+1} = f \circ \delta_{n+1} = 0$ , so  $g \circ \delta_{n+1}$  factors through  $i$ . Let  $h \in \text{Hom}_A(P_{n+1}, V)$  such that

$$i \circ h = (-1)^{n+1} g \circ \delta_{n+1}.$$

Then  $h$  is a cycle, and by our sign conventions from Remark 5.2 regarding the differential of  $\text{Hom}_A(P, W)$ , the class of  $h$  in  $\text{Ext}_A^{n+1}(U, V)$  is  $\gamma^n(\underline{f})$ . As before, we suppress subscripts and superscripts to  $\delta, s, \gamma$ , and write abusively  $h = \gamma(f)$ . Now  $D_z(\underline{f})$  is represented by  $f \circ s$ , and clearly  $g \circ s$  lifts  $f \circ s$  through  $p$ . Therefore, by the same construction as before, applied to  $f \circ s$ , the class of  $\gamma(f \circ s)$  is represented by the map  $m$  satisfying  $i \circ m = (-1)^n g \circ s \circ \delta$ . Since  $s \circ \delta + \delta \circ s$  is equal to multiplication by  $z$  on  $P$  and since  $z$  annihilates  $W$  and commutes with  $g$ , it follows that  $g \circ (s \circ \delta + \delta \circ s) = 0$ , and hence

$$i \circ m = (-1)^n g \circ s \circ \delta = -(-1)^n g \circ \delta \circ s = i \circ h \circ s.$$

Since  $i$  is a monomorphism,  $m = h \circ s$ . The map  $m$  represents the class of  $\gamma(D_z(f))$ , and  $h \circ s$  represents the class of  $D_z(\gamma(f))$ . Statement (ii) follows. A similar argument, using an injective resolution of  $X$  and Theorem 5.9, yields (iii).  $\square$

**Remark 5.11.** With the notation of Theorem 5.1, if  $A$  is a finite-dimensional self-injective algebra over a field  $k$ , then the construction principle of degree  $-1$  operators in Theorem 5.1 extends to Tate-Ext, by replacing a projective resolution of  $U$  with a complete resolution of  $U$ . More precisely, let  $(P, \delta)$  be a complete resolution of  $U$ ; that is,  $P$  is an acyclic chain complex of projective  $A$ -modules together with an isomorphism  $\text{Im}(\delta_0) \cong U$ . Then  $\widehat{\text{Ext}}_A^n(U, V) \cong H^n(\text{Hom}_A(P, V))$  for all integers  $n$ ; for  $n$  positive, this coincides with  $\text{Ext}_A^n(U, V)$ . If  $z \in Z(A)$  annihilates  $U$ , then multiplication by  $z$  on  $P$  is a chain endomorphism which is homotopic to zero, and thus there is a homotopy  $s$  on  $P$  such that  $s \circ \delta + \delta \circ s$  is equal to multiplication by  $z$ . (In fact, for this part of the construction, it suffices to assume that the endomorphism of  $U$  given by multiplication with  $z$  factors through a projective module). If  $z$  also annihilates  $V$ , then just as in the proof of Theorem 5.1 the correspondence sending  $f \in \text{Hom}_A(P_n, V)$  to  $f \circ s$  induces for any integer  $n$  an operator  $\hat{D}_z : \widehat{\text{Ext}}_A^n(U, V) \rightarrow \widehat{\text{Ext}}_A^{n-1}(U, V)$ , which for  $n \geq 2$  coincides with the operator  $D_z$ . For  $G$  a finite group, the Tate-Hochschild cohomology  $\widehat{HH}^*(kG)$  of  $kG$  admits a centraliser decomposition analogous to that of  $HH^*(kG)$  in terms

of Tate-Ext of centralisers of group elements, and hence the above construction yields a degree  $-1$  operator on  $\widehat{HH}^*(kG)$ . It would be interesting to check that this coincides with the extension of the BV operator to  $\widehat{HH}^*(kG)$  in [Liu et al. 2021].

**Remark 5.12.** A construction analogous to that in Theorem 5.1 exists for Tor. Let  $A$  be a  $k$ -algebra, let  $U$  be an  $A$ -module with a projective resolution  $(P, \delta)$ , let  $W$  be a right  $A$ -module, and let  $z \in Z(A)$  such that  $z$  annihilates  $U$  and  $W$ . Then there is a homotopy  $s : P \rightarrow P[-1]$  such that  $s \circ \delta + \delta \circ s$  is the graded chain endomorphism of  $P$  given by multiplication with  $z$ . Since  $z$  also annihilates  $W$ , it follows that  $\text{Id}_W \otimes s : W \otimes_A P \rightarrow W \otimes_A P[-1]$  is a chain map. Taking homology yields a degree 1 operator  $\text{Tor}_n^A(W, U) \rightarrow \text{Tor}_{n+1}^A(W, U)$ , for all  $n \geq 0$ , which satisfies the formal properties analogous to those developed for the operators  $D_z$  in this section. A similar construction, in which  $z$  is an integer, has been used for calculating torsion in loop space homology in [Levi 1996].

## 6. The BV operator in terms of homotopies on projective resolutions

Let  $k$  be a commutative ring. For a finite group  $G$  and an element  $g \in Z(G)$ , we denote as before for any positive integer  $n$  by  $\Delta_g = I \circ B : H^n(G, k) \rightarrow H^{n-1}(G, k)$  the map obtained from the long exact sequences (3.1). The following result shows that  $\Delta_g$  can be obtained as a special case of the construction described in Theorem 5.1, implying in particular that the component  $\Delta_1$  of the BV operator  $\Delta$  on the summand  $H^*(G, k)$  in the centraliser decomposition of  $\widehat{HH}^*(kG)$  corresponding to the unit element 1 of  $G$  is zero.

**Theorem 6.1.** *Let  $G$  be a finite group and  $g \in Z(G)$ . With the notation from Theorem 5.1, we have  $\Delta_g = D_{g-1}$ . In particular, we have  $\Delta_1 = D_0 = 0$ .*

In order to show this, we will use Theorem 4.2. As before, we denote by

$$z_g : \mathbb{Z} \times G \rightarrow G$$

the group homomorphism sending  $(n, h)$  to  $g^n h$ , where  $n \in \mathbb{Z}$  and  $h \in G$ . Note that  $z_1$  is the canonical projection onto the second component of  $\mathbb{Z} \times G$ . In order to describe the map induced by  $z_g$  on cohomology, we choose a projective resolution  $P_G$  of  $k$  as a  $kG$ -module and a projective resolution  $P_{\mathbb{Z}}$  of  $k$  as a  $k\mathbb{Z}$ -module. Here,  $k\mathbb{Z}$  is the group algebra over  $k$  of the infinite cyclic group  $(\mathbb{Z}, +)$ . We will need to describe quasi-isomorphisms  $P_{\mathbb{Z}} \otimes_k P_G \rightarrow z_g^*(P_G)$  as complexes of  $k(\mathbb{Z} \times G)$ -modules.

Identify  $k\mathbb{Z} = k[u, u^{-1}]$  for some indeterminate  $u$  via the unique algebra isomorphism sending  $1_{\mathbb{Z}}$  to  $u$ . We choose for  $P_{\mathbb{Z}}$  the two-term complex (in degrees 1 and 0) of the form

$$k[u, u^{-1}] \xrightarrow{u-1} k[u, u^{-1}],$$

where the superscript  $u-1$  is the map given multiplication with  $u-1$ . This is a projective resolution of  $k$  as a  $k[u, u^{-1}]$ -module, together with the augmentation

map  $k[u, u^{-1}] \rightarrow k$  sending  $u$  to 1. Note that pairs consisting of an infinite cyclic group and a generator are unique up to unique isomorphism. Choosing a generator is equivalent to choosing a projective resolution of the form above.

We will make use of the following special case of the Tensor-Hom-adjunction. We adopt the following shorthand: for any  $kG$ -module  $M$ , we write

$$M[u, u^{-1}] = k[u, u^{-1}] \otimes_k M.$$

**Lemma 6.2.** *With the notation above, let  $M$  and  $N$  be  $kG$ -modules. We have a natural isomorphism of  $k$ -modules*

$$\mathrm{Hom}_{k(\mathbb{Z} \times G)}(M[u, u^{-1}], z_g^*(N)) \cong \mathrm{Hom}_{kG}(M, N),$$

sending a  $k(\mathbb{Z} \times G)$ -homomorphism  $f: M[u, u^{-1}] \rightarrow z_g^*(N)$  to the  $kG$ -homomorphism  $M \rightarrow N$  given by  $m \mapsto f(1 \otimes m)$  for all  $m \in M$ .

Let  $P_G = (P_n)_{n \geq 0}$  be a projective resolution of the trivial  $kG$ -module, with differential  $(\delta_n)_{n \geq 1}$ . We adopt the convention  $P_{-1} = \{0\}$  and  $\delta_0 = 0$ . Note that the  $k(\mathbb{Z} \times G)$ -module structure of  $z_g^*(P_n)$  is given via  $u^i \otimes h$  acting as left multiplication by  $g^i h$  on  $P_n$ , where  $n \geq 0$ . The degree  $n$  term of  $P_{\mathbb{Z}} \otimes_k P_G$  is equal to

$$P_n[u, u^{-1}] \oplus P_{n-1}[u, u^{-1}]$$

for any  $n \geq 0$ . Denote by  $\eta = (\eta_n)_{n \geq 1}$  the differential of  $P_{\mathbb{Z}} \otimes_k P_G$ . Use the same letter  $\delta_n$  for the obvious extension  $\mathrm{Id} \otimes \delta_n$  of  $\delta_n$  to  $P_n[u, u^{-1}]$ . The differential  $\eta$  is given in degree  $n \geq 1$  by

$$(6.3) \quad \eta_n : P_n[u, u^{-1}] \oplus P_{n-1}[u, u^{-1}] \xrightarrow{\begin{pmatrix} \delta_n & u^{-1} \\ 0 & -\delta_{n-1} \end{pmatrix}} P_{n-1}[u, u^{-1}] \oplus P_{n-2}[u, u^{-1}].$$

We describe the identification  $H^n(\mathbb{Z} \times G, k) = H^n(G, k) \oplus H^{n-1}(G, k)$  from Theorem 4.2 as follows:

**Proposition 6.4.** *With the notation above, the canonical split exact sequences*

$$0 \longrightarrow P_n[u, u^{-1}] \longrightarrow P_n[u, u^{-1}] \oplus P_{n-1}[u, u^{-1}] \longrightarrow P_{n-1}[u, u^{-1}] \longrightarrow 0$$

define a degreewise split short exact sequence of chain complexes of  $k(\mathbb{Z} \times G)$ -modules

$$0 \longrightarrow k[u, u^{-1}] \otimes_k P_G \longrightarrow P_{\mathbb{Z}} \otimes_k P_G \longrightarrow k[u, u^{-1}] \otimes_k P_G[1] \longrightarrow 0.$$

Applying  $\mathrm{Hom}_{k(\mathbb{Z} \times G)}(-, k)$ , with the appropriate signs for the differentials, yields a short exact sequence of cochain complexes of  $k$ -modules

$$0 \longrightarrow \mathrm{Hom}_{kG}(P_G[1], k) \longrightarrow \mathrm{Hom}_{k(\mathbb{Z} \times G)}(P_{\mathbb{Z}} \otimes_k P_G, k) \longrightarrow \mathrm{Hom}_{kG}(P_G, k) \longrightarrow 0,$$

which splits canonically, and hence yields a canonical identification

$$H^n(\mathbb{Z} \times G, k) = H^n(G, k) \oplus H^{n-1}(G, k).$$

This is the identification in Theorem 4.2.

*Proof.* The exact sequence of chain complexes of  $k(\mathbb{Z} \times G)$ -modules is a special case of a tensor product of the chain complex  $P_G$  with a two-term chain complex and easily verified. The exact sequence of cochain complexes of  $k$ -modules is obtained by applying the contravariant functor  $\text{Hom}_{k(\mathbb{Z} \times G)}(-, k)$  to the previous sequence and then using the canonical adjunctions

$$\text{Hom}_{k(\mathbb{Z} \times G)}(P_n[u, u^{-1}], k) \cong \text{Hom}_{kG}(P_n, k)$$

from Lemma 6.2. The fact that this sequence splits canonically follows from the observation that multiplication by  $u - 1$  has image in the kernel of any  $k(\mathbb{Z} \times G)$ -homomorphism  $P_n[u, u^{-1}] \rightarrow k$ , and so the nondiagonal entry  $u - 1$  in the differential (6.3) of  $P_{\mathbb{Z}} \otimes_k P_G$  becomes zero upon applying the functor  $\text{Hom}_{k(\mathbb{Z} \times G)}(-, k)$ . To see that this is the identification in Theorem 4.2, consider the class  $\mu$  in  $H^1(\mathbb{Z}, k)$  corresponding to the group homomorphism  $\mathbb{Z} \rightarrow k$  sending  $1_{\mathbb{Z}}$  to  $1_k$ . This is the class of the 1-cocycle (abusively still denoted by the same letter)  $\mu : k[u, u^{-1}] \rightarrow k$  sending  $u$  to 1. The explicit description of the maps in the statement implies that upon taking cohomology in degree  $n$ , the map  $\text{Hom}_{kG}(P_G[1], k) \rightarrow \text{Hom}_{k(\mathbb{Z} \times G)}(P_{\mathbb{Z}} \otimes_k P_G, k)$  induces a map which sends  $x \in H^{n-1}(G, k)$  to the image of  $\mu \otimes x$  in  $H^n(\mathbb{Z} \times G, k)$ , as required.  $\square$

For  $n \geq 0$ , denote by

$$e_n : P_n[u, u^{-1}] \rightarrow z_g^*(P_n)$$

the  $k(\mathbb{Z} \times G)$ -homomorphism defined by  $e_n(u^i \otimes v) = g^i v$  for all  $i \in \mathbb{Z}$  and  $v \in P_n$ . Equivalently,  $e_n$  corresponds to the identity on  $P_n$  under the adjunction from Lemma 6.2. The following theorem parametrises homotopies in Theorem 5.1 in terms of certain quasi-isomorphisms  $P_{\mathbb{Z}} \otimes_k P_G \rightarrow z_g^*(P_G)$  lifting the identity on  $k$ :

**Theorem 6.5.** *With the notation above, for any  $n \geq -1$  let  $f_n : P_n[u, u^{-1}] \rightarrow z_g^*(P_{n+1})$  be a  $k(\mathbb{Z} \times G)$ -homomorphism and let  $s_n : P_n \rightarrow P_{n+1}$  be the corresponding  $kG$ -homomorphism sending  $a \in P_n$  to  $f_n(1 \otimes a)$ . The following are equivalent:*

(i) *The graded  $k(\mathbb{Z} \times G)$ -homomorphism*

$$(e_n, f_{n-1})_{n \geq 0} : P_{\mathbb{Z}} \otimes_k P_G \rightarrow z_g^*(P_G)$$

*is a quasi-isomorphism of chain complexes which lifts the identity on  $k$ .*

(ii) *The graded  $kG$ -homomorphism*

$$(s_{n-1})_{n \geq 0} : P_G[1] \rightarrow P_G$$

*is a homotopy with the property that the chain map  $\delta \circ s + s \circ \delta$  is equal to the endomorphism of  $P_G$  given by left multiplication with  $g - 1$ .*

*Proof.* In degree 0, the map  $e_0$  clearly lifts the identity on  $k$ , so we need to show that  $(e_n, f_{n-1})_{n \geq 0}$  is a chain map if and only if  $s \circ \delta + \delta \circ s$  is equal to multiplication

by  $g-1$ . By the definition of the differential of  $P_{\mathbb{Z}} \otimes_k P_Q$ , we have that  $(e_n, f_{n-1})_{n \geq 0}$  is a chain map if and only if for any  $n \geq 1$  we have

$$(e_{n-1}, f_{n-2}) \circ \begin{pmatrix} \delta_n & u-1 \\ 0 & -\delta_{n-1} \end{pmatrix} = \delta_n \circ (e_n, f_{n-1}),$$

where we use as before the same letter  $\delta_n$  for the extension  $\text{Id} \otimes \delta_n$  of  $\delta_n$  to  $P_n[u, u^{-1}]$ . This is equivalent to

$$(e_{n-1} \circ \delta_n, e_{n-1} \circ (u-1) - f_{n-2} \circ \delta_{n-1}) = (\delta_n \circ e_n, \delta_n \circ f_{n-1}),$$

where we have used the notation  $(u-1)$  for the map given by multiplication with  $u-1$ . In the first component, this holds automatically since  $(e_n)_{n \geq 0}$  is a chain map. Thus the previous condition is equivalent to

$$e_{n-1} \circ (u-1) = \delta_n \circ f_{n-1} + f_{n-2} \circ \delta_{n-1}.$$

Note that the left side is equal to  $(g-1) \circ e_{n-1}$ , where  $(g-1)$  denotes the map given by multiplication with  $g-1$ . Through the obvious versions of the adjunction from Lemma 6.2, this is equivalent to the statement that  $\delta \circ s + s \circ \delta$  is equal to left multiplication by  $g-1$  on  $P_G$  as stated.  $\square$

*Proof of Theorem 6.1.* The fact that  $s^\vee$  is a cochain map is the special case of Theorem 5.1, applied to  $A = kG$ ,  $U = V = k$ , and  $z = g-1$ . Thus,  $H^n(s^\vee) = D_{g-1}$  for  $n \geq 0$ . By Theorem 6.5 (and with the notation of that theorem) the graded map  $(s_{n-1})_{n \geq 0}: P_G[1] \rightarrow P_G$  induces a quasi-isomorphism

$$(e_n, f_{n-1})_{n \geq 0}: P_{\mathbb{Z}} \otimes_k P_G \rightarrow z_g^*(P_G),$$

which lifts the identity on  $k$ . Applying the functor  $\text{Hom}_{k(\mathbb{Z} \times G)}(-, k)$ , with the appropriate signs for the differentials, and making use of the adjunction Lemma 6.2, yields a cochain map

$$\text{Hom}_{kG}(P_G, k) \rightarrow \text{Hom}_{k(\mathbb{Z} \times G)}(P_{\mathbb{Z}} \otimes_k P_G, k).$$

Taking cohomology, and using the identification in Proposition 6.4, yields for any  $n \geq 0$  a map

$$H^n(G, k) \rightarrow H^n(\mathbb{Z} \times G, k) = H^n(G, k) \oplus H^{n-1}(G, k).$$

By construction, the second component of this map is induced by  $s^\vee$ , hence equal to  $D_{g-1}$ . By Theorem 3.4, the second component is also equal to  $\Delta_g$  as stated.  $\square$

The following result is Theorem 1.1 (ii):

**Proposition 6.6.** *Let  $G$  and  $H$  be finite groups, and let  $g \in Z(G)$  and  $h \in Z(H)$ . Let  $i$  and  $j$  be nonnegative integers, and let  $x \in H^i(G, k)$  and  $y \in H^j(H, k)$ .*

Identify  $k(G \times H) \cong kG \otimes_k kH$  and identify  $x \otimes y$  with its canonical image in  $H^{i+j}(G \times H, k)$ . Then  $\Delta_{(g,h)}(x \otimes y)$  is equal to the canonical image of

$$\Delta_g(x) \otimes y + (-1)^i x \otimes \Delta_h(y)$$

in  $H^{i+j-1}(G \times H, k)$ .

*Proof.* We apply Proposition 5.7 to the case  $A = kG$ ,  $U = U' = k$ ,  $B = kH$ ,  $V = V' = k$ ,  $z = g - 1$ , and  $w = h - 1$ . We have

$$\Delta_g(x) \otimes y + (-1)^i x \otimes \Delta_h(y) = D_{g-1}(x) \otimes y + (-1)^i x \otimes D_{h-1}(y).$$

By Proposition 5.7, the image in  $H^{i+j}(G \times H, k)$  of this element is equal to

$$D_{(g-1) \otimes 1 + 1 \otimes (h-1)}(x \otimes y).$$

It follows from the last statement in Proposition 5.7 that  $D_{(g-1) \otimes (h-1)} = 0$ . Since

$$(g-1) \otimes (h-1) = (g \otimes h) - (1 \otimes 1) - ((g-1) \otimes 1) - (1 \otimes (h-1)),$$

this implies that

$$0 = D_{(g-1) \otimes (h-1)} = D_{(g \otimes h) - (1 \otimes 1)} - D_{(g-1) \otimes 1} - D_{1 \otimes (h-1)},$$

or equivalently,

$$D_{(g \otimes h) - (1 \otimes 1)} = D_{(g-1) \otimes 1} + D_{1 \otimes (h-1)} = D_{(g-1) \otimes 1 + 1 \otimes (h-1)}.$$

The left side in the last equation is  $\Delta_{(g,h)}$ , whence the result.  $\square$

The following proposition implies the statements (v) and (iv) in Theorem 1.1:

**Proposition 6.7.** *Let  $G$  be a finite group and  $g, h \in Z(G)$ .*

- (i) *We have  $\Delta_{gh} = \Delta_g + \Delta_h$ .*
- (ii) *We have  $\Delta_{g^m} = m\Delta_g$  for any positive integer  $m$ .*
- (iii) *If the order of  $g$  is invertible in  $k$ , then  $\Delta_g = 0$ . In particular, if  $k$  is a field of prime characteristic  $p$  and  $g$  is a  $p'$ -element in  $Z(G)$ , then  $\Delta_g = 0$ .*
- (iv) *If  $k$  is a field of prime characteristic  $p$ , then  $\Delta_g = \Delta_{g_p}$ , where  $g_p$  is the  $p$ -part of  $g$ .*
- (v) *If  $k$  is a field of prime characteristic  $p$ , denoting by  $P$  a Sylow  $p$ -subgroup of  $G$ , we have  $g_p \in Z(P)$ , and for any positive integer  $n$  the diagram*

$$\begin{array}{ccc} H^n(G, k) & \xrightarrow{\text{Res}_{G,P}} & H^n(P, k) \\ \Delta_g \downarrow & & \downarrow \Delta_{g_p} \\ H^{n-1}(G, k) & \xrightarrow{\text{Res}_{G,P}} & H^{n-1}(P, k) \end{array}$$

*is commutative with injective horizontal maps.*

*Proof.* The identity  $(g-1)(h-1) = (gh-1) - (g-1) - (h-1)$  implies that  $D_{(g-1)(h-1)} = D_{gh-1} - D_{g-1} - D_{h-1}$ . By Theorem 5.3 (iii), we have  $D_{(g-1)(h-1)} = 0$ , and hence  $D_{gh-1} = D_{g-1} + D_{h-1}$ . By Theorem 6.1, this yields the equality stated in (i), and (ii) is an immediate consequence of (i). By Theorem 6.1 we have  $\Delta_1 = 0$ . Thus (iii) follows from (ii). Assume that  $k$  is a field of prime characteristic  $p$ . Note that since  $g$  is central, so are the  $p$ -part  $g_p$  and the  $p'$ -part  $g_{p'}$  of  $g$ . By (i) we have  $\Delta_g = \Delta_{g_p} + \Delta_{g_{p'}}$ , and by (iii) this is equal to  $\Delta_{g_p}$ , whence (iv). Since  $g$ , and hence also  $g_p$ , is central, it follows that  $g_p$  is contained in any Sylow  $p$ -subgroup of  $G$ . Statement (v) follows from (iv), the compatibility of  $\Delta_{g_p}$  with restriction from  $G$  to  $P$  by Corollary 4.3 (iii), together with the standard fact that the restriction map  $H^*(G, k) \rightarrow H^*(P, k)$  is injective.  $\square$

## 7. The BV operator on the bar resolution

Let  $k$  be a commutative ring,  $G$  a finite group, and  $g \in Z(G)$ . The purpose of this section is to calculate explicitly a homotopy  $s$  on the bar resolution of the trivial  $kG$ -module as in Theorem 6.1. This will be used in the proof of Theorem 8.1. As before, we denote by  $z_g: \mathbb{Z} \times G \rightarrow G$  the group homomorphism sending  $(m, h)$  to  $g^m h$ .

We choose for  $P_G$  the projective resolution of the trivial  $kG$ -module which in degree  $n$  term is equal to  $kG^{n+1}$ , where  $n \geq 0$  and where  $G^{n+1}$  is the direct product of  $n+1$  copies of  $G$ , with differential  $\delta_n$  given for  $n \geq 1$  by

$$\delta_n(a_0, a_1, \dots, a_n) = \sum_{i=1}^{n-1} (-1)^i (a_0, \dots, a_i a_{i+1}, \dots, a_n) + (-1)^n (a_0, a_1, \dots, a_{n-1}),$$

where the  $a_i$  are elements in  $G$ . The left  $kG$ -module structure on the terms  $kG^{n+1}$  is given by left multiplication with  $G$  on the first copy of  $G$ . (This is the resolution obtained from tensoring the Hochschild resolution of  $kG$  by  $-\otimes_{kG} k$ .) In particular, in degree 1, we have

$$\delta_1(a_0, a_1) = a_0 a_1 - a_0.$$

As earlier, we identify  $k\mathbb{Z} = k[u, u^{-1}]$  and  $k(\mathbb{Z} \times G) = k[u, u^{-1}] \otimes_k kG$ . The  $k(\mathbb{Z} \times G)$ -module structure of  $z_g^*(kG^{n+1})$  is given by the action of  $u^i \otimes h$  acting as left multiplication by  $g^i h$  on the first copy of  $G$  in  $G^{n+1}$ .

**Theorem 7.1.** *With the notation above, for any  $n \geq 1$  denote by  $s_{n-1}: kG^n \rightarrow kG^{n+1}$  the  $kG$ -homomorphism defined by*

$$s_{n-1}(a_0, a_1, \dots, a_{n-1}) = \sum_{i=0}^{n-1} (-1)^i (a_0, \dots, a_i, g, a_{i+1}, \dots, a_{n-1}),$$

where  $a_i \in G$  for  $0 \leq i \leq n-1$ . Applying the functor  $\text{Hom}_{kG}(-, k)$  to the graded map

$$s = (s_{n-1})_{n \geq 1}: P_G[1] \rightarrow P_G$$

yields a map of cochain complexes of  $k$ -modules

$$s^\vee : \text{Hom}_{kG}(P_G, k) \rightarrow \text{Hom}_{kG}(P_G[1], k)$$

such that, for any  $n \geq 1$ , we have

$$\Delta_g = H^n(s^\vee) : H^n(G, k) \rightarrow H^{n-1}(G, k).$$

*Proof.* By Theorem 6.5, it suffices to show that  $s$  is a homotopy such that  $\delta \circ s + s \circ \delta$  is the chain map given by multiplication with  $g - 1$ . This is the content of the next theorem, whence the result.  $\square$

As in the previous section, for  $n \geq 0$ , denote by

$$e_n : kG^{n+1}[u, u^{-1}] \rightarrow z_g^*(kG^{n+1})$$

the  $k(\mathbb{Z} \times G)$ -homomorphism defined by  $e_n(u^m \otimes h) = g^m h$  for all  $m \in \mathbb{Z}$  and  $h \in G$ . For  $n \geq 1$  denote by

$$f_{n-1} : kG^n[u, u^{-1}] \rightarrow z_g^*(kG^{n+1})$$

the  $k(\mathbb{Z} \times G)$ -homomorphism given by

$$f_{n-1}(u^m \otimes (a_0, a_1, \dots, a_{n-1})) = \sum_{j=0}^{n-1} (-1)^j (g^m a_0, \dots, a_j, g, a_{j+1}, \dots, a_{n-1}),$$

with  $m \in \mathbb{Z}$  and  $a_j \in G$ . Through an adjunction as in Lemma 6.2, the map  $f_{n-1}$  corresponds to the  $kG$ -homomorphism  $s_{n-1} : kG^n \rightarrow kG^{n+1}$  defined in the previous theorem.

**Theorem 7.2.** *With the notation above, the following hold:*

- (i) *The graded  $k(\mathbb{Z} \times G)$ -homomorphism  $(e_n, f_{n-1})_{n \geq 0} : P_{\mathbb{Z}} \otimes_k P_G \rightarrow z_g^*(P_G)$  is a quasi-isomorphism which lifts the identity on  $k$ .*
- (ii) *The graded  $kG$ -homomorphism  $(s_{n-1})_{n \geq 1} : P_G[1] \rightarrow P_G$  is a homotopy with the property that  $\delta \circ s + s \circ \delta$  is equal to the endomorphism of  $P_G$  given by left multiplication with  $g - 1$ .*

*Proof.* By Theorem 6.5, the two statements are equivalent. We prove the second statement. That is, for  $n \geq 1$  and  $x = (a_0, a_1, \dots, a_{n-1}) \in P_{n-1}$ , we need to prove the equality

$$\delta_n(s_{n-1}(x)) = (g - 1)x - s_{n-2}(\delta_{n-1}(x)).$$

We start with the left side. We have

$$\delta_n(s_{n-1}(x)) = \sum_{j=0}^{n-1} (-1)^j \delta_n(a_0, \dots, a_j, g, a_{j+1}, \dots, a_{n-1}).$$

We need to calculate the summands

$$\delta_n(a_0, \dots, a_j, g, a_{j+1}, \dots, a_{n-1}).$$

The definition of  $\delta_n$  yields an alternating sum over an index  $i$  running from 0 to  $n - 1$ , which we will need to break up according to whether  $0 \leq i < j$ ,  $i = j$ ,  $i = j + 1$ , or  $j + 1 < i \leq n - 1$ . We have

$$\begin{aligned} & \delta_n(a_0, \dots, a_j, g, a_{j+1}, \dots, a_{n-1}) \\ &= \sum_{i=0}^{j-1} (-1)^i (a_0, \dots, a_i a_{i+1}, \dots, a_j, g, a_{j+1}, \dots, a_{n-1}) \\ & \quad + (-1)^j (a_0, \dots, a_j g, a_{j+1}, \dots, a_{n-1}) \\ & \quad + (-1)^{j+1} (a_0, \dots, a_j, g a_{j+1}, \dots, a_{n-1}) \\ & \quad + \sum_{i=j+2}^{n-1} (-1)^j (a_0, \dots, a_j, g, a_{j+1}, \dots, a_{i-1} a_i, \dots, a_{n-2}) \\ & \quad + (-1)^n (a_0, \dots, a_j, g, a_{j+1}, \dots, a_{n-2}). \end{aligned}$$

For  $j = 0$  or  $j = n - 1$  or  $j = n - 2$ , this formula takes a slightly different form. If  $j = 0$ , then the first sum is empty (hence zero by convention). The fourth term (that is, the sum indexed  $\sum_{i=j+2}^{n-1}$ ) is empty if  $j$  is one of  $n - 1, n - 2$ , so zero. In addition, if  $j = n - 1$ , then the third term does not appear (because the component  $g a_{j+1}$  is not defined in that case) and the last summand takes the form  $(-1)^n (a_0, \dots, a_{n-1})$ , and if  $j = n - 2$ , then the last summand takes the form  $(-1)^n (a_0, \dots, a_{n-2}, g)$ . Then

$$\begin{aligned} & \delta_n(s_{n-1}(x)) \\ &= \sum_{j=1}^{n-1} \sum_{i=0}^{j-1} (-1)^{i+j} (a_0, \dots, a_i a_{i+1}, \dots, a_j, g, a_{j+1}, \dots, a_{n-1}) \\ & \quad + \sum_{j=0}^{n-1} (a_0, \dots, a_j g, a_{j+1}, \dots, a_{n-1}) - \sum_{j=0}^{n-2} (a_0, \dots, a_j, g a_{j+1}, \dots, a_{n-1}) \\ & \quad + \sum_{j=0}^{n-3} \sum_{i=j+2}^{n-1} (-1)^{i+j} (a_0, \dots, a_j, g, a_{j+1}, \dots, a_{i-1} a_i, \dots, a_{n-1}) \\ & \quad + \sum_{j=0}^{n-1} (-1)^{n+j} (a_0, \dots, a_j, g, a_{j+1}, \dots, a_{n-2}). \end{aligned}$$

Note that in the last term, the summand when  $j = n - 1$  takes the form  $-(a_0, a_1, \dots, a_{n-1})$ . Since  $g$  is central, the third sum cancels against the second sum for  $j \geq 1$ . Also, we may reindex the second double sum and replace  $i$  by  $i - 1$ . This yields

$$\begin{aligned} & \delta_n(f_n(x)) \\ &= \sum_{j=1}^{n-1} \sum_{i=0}^{j-1} (-1)^{i+j} (a_0, \dots, a_i a_{i+1}, \dots, a_j, g, a_{j+1}, \dots, a_{n-1}) + (a_0 g, a_1, \dots, a_{n-1}) \\ & \quad + \sum_{j=0}^{n-3} \sum_{i=j+1}^{n-2} (-1)^{i+j+1} (a_0, \dots, a_j, g, a_{j+1}, \dots, a_i a_{i+1}, \dots, a_{n-1}) \\ & \quad + \sum_{j=0}^{n-1} (-1)^{n+j} (a_0, \dots, a_j, g, a_{j+1}, \dots, a_{n-2}), \end{aligned}$$

where as before the summand for  $j = n - 1$  in the last sum is  $-(a_0, a_1, \dots, a_{n-1})$ . We need to show that this is equal to the expression

$$(g-1)x - s_{n-2}(\delta_{n-1}(x)) = (ga_0, a_1, \dots, a_{n-1}) - (a_0, \dots, a_{n-1}) \\ - \sum_{i=0}^{n-2} (-1)^i s_{n-2}(a_0, \dots, a_i a_{i+1}, \dots, a_{n-1}) \\ - (-1)^{n-1} s_{n-2}(a_0, \dots, a_{n-2}).$$

Expanding  $s_{n-2}$  and adjusting indexing yields that this is equal to

$$(ga_0, a_1, \dots, a_{n-1}) - (a_0, \dots, a_{n-1}) \\ + \sum_{i=0}^{n-2} \sum_{j=0}^{i-1} (-1)^{i+j+1} (a_0, \dots, a_j, g, a_{j+1}, \dots, a_i a_{i+1}, \dots, a_{n-1}) \\ + \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} (-1)^{i+j} (a_0, \dots, a_i a_{i+1}, \dots, a_j, g, a_{j+1}, \dots, a_{n-1}) \\ + \sum_{j=0}^{n-2} (-1)^{n+j} (a_0, \dots, a_j, g, a_{j+1}, \dots, a_{n-2}).$$

We need to match all summands to those for the expression of  $\delta_n(s_{n-1}(x))$ . The first summand cancels against the second summand in  $\delta_n(s_{n-1}(x))$ . The second summand cancels against the summand for  $j = n - 1$  in the fourth (and last) sum of  $\delta_n(s_{n-1}(x))$ . The last sum cancels against the remaining summands of the fourth sum in  $\delta_n(s_{n-1}(x))$ . The first double sum cancels against the second double sum in  $\delta_n(s_{n-1}(x))$  because both can be written as sum indexed by pairs  $(i, j)$  with  $0 \leq i < j \leq n - 2$ . Similarly, the remaining double sum cancels against the first double sum in  $\delta_n(s_{n-1}(x))$ , because both can be written as a sum indexed by pairs  $(i, j)$  with  $0 \leq i < j \leq n - 1$ .  $\square$

## 8. The BV operator in degree 2

For convenience, we restate Theorem 1.1 (xi). For elements  $x$  and  $y$  in a group  $G$  we write  $[x, y]$  for  $xyx^{-1}y^{-1}$ , and we recall that  $[x, yz] = [x, y]y[x, z]y^{-1}$ .

**Theorem 8.1.** *Let  $G$  be a finite group,  $g \in Z(G)$ , and  $x \in H^2(G, k)$ . Suppose that  $x$  corresponds to a central extension*

$$1 \rightarrow k^+ \rightarrow K \rightarrow G \rightarrow 1.$$

*For any  $h \in G$ , choose an inverse image  $\hat{h} \in K$ . Then identifying  $H^1(G, k)$  with  $\text{Hom}(G, k)$ , we have*

$$[\hat{g}, \hat{h}] = \Delta_g(x)(h) \in k^+.$$

We combine Theorem 7.1 and the following Lemma to prove Theorem 8.1:

**Lemma 8.2.** *Let  $1 \rightarrow Z \rightarrow \hat{G} \rightarrow G \rightarrow 1$  be a central extension of an abelian group  $G$ . For  $g \in G$  choose an inverse image  $\hat{g}$  of  $g$  in  $\hat{G}$  such that  $\hat{1}_G = \hat{1}_{\hat{G}}$ . Let  $\alpha: G \times G \rightarrow Z$  be the 2-cocycle defined by*

$$\alpha(g, h) = \hat{g}\hat{h}\widehat{gh}^{-1}$$

for  $g, h \in G$ . Let  $\beta, \gamma: G \times G \rightarrow Z$  be the maps defined by

$$\begin{aligned}\beta(g, h) &= \alpha(h, g), \\ \gamma(g, h) &= [\hat{g}, \hat{h}] = \hat{g}\hat{h}\hat{g}^{-1}\hat{h}^{-1}\end{aligned}$$

for  $g, h \in G$ . Then  $\beta, \gamma$  are 2-cocycles, and we have  $\alpha\beta^{-1} = \gamma$ .

*Proof.* All parts are trivial verifications. □

*Proof of Theorem 8.1.* By Theorem 1.1 (iii),  $\Delta_g$  commutes with restriction to subgroups of  $G$  containing  $g$ . Hence, it suffices to consider the case where  $G = \langle g, h \rangle$ . Therefore, in order to prove the formula for  $\Delta_g(x)(h)$  we may assume that  $G$  is abelian. Note that with the notation of Theorem 7.1, we have

$$s_1(a_0, a_1) = (a_0, g, a_1) - (a_0, a_1, g),$$

and hence its dual  $(s_1)^\vee$  sends a  $kG$ -homomorphism  $\zeta: kG^3 \rightarrow k$  to the  $kG$ -homomorphism  $\zeta \circ s_1: kG^2 \rightarrow k$  sending  $(a_0, a_1)$  to  $\zeta(a_0, g, a_1) - \zeta(a_0, a_1, g)$ . The identification of  $H^2(G; k)$  in terms of classes of cocycles is given via the adjunction map

$$\mathrm{Hom}_{kG}(kG^3, k) \cong \mathrm{Hom}_k(kG^2, k)$$

sending a  $kG$ -homomorphism  $\zeta: kG^3 \rightarrow k$  to the  $k$ -linear map  $kG^2 \rightarrow k$  determined by the assignment  $(a_1, a_2) \mapsto \zeta(1, a_1, a_2)$ . Similarly, the identification  $H^1(G, k) = \mathrm{Hom}(G, k^+)$  is given via the adjunction  $\mathrm{Hom}_{kG}(kG^2, k) \cong \mathrm{Hom}_k(kG, k)$ . Thus, if  $\alpha$  is a 2-cocycle representing the class  $x$ , then the 1-class determined by  $\alpha \circ s_1$  is given by the assignment

$$a_1 \mapsto \alpha(g, a_1) - \alpha(a_1, g).$$

By Lemma 8.2 applied with  $Z = k^+$  (written additively) this is equal to the map

$$a_1 \mapsto [\hat{a}_1, \hat{g}].$$

Writing  $h$  instead of  $a_1$ , we get that  $\Delta_g(x)(h) = [\hat{h}, \hat{g}]$  for all  $h \in G$  as claimed. □

## 9. Examples

Throughout this section  $p$  is a prime and  $k$  is a field of characteristic  $p$ . As a consequence of Theorem 1.1 (vii), in order to calculate the maps  $\Delta_g$ , where  $g$  is a

central element in a finite group  $G$ , it suffices to calculate in the case where  $G$  is a finite  $p$ -group. See [Adem and Milgram 1994] for background material.

**Example 9.1.** Let  $G$  be a finite cyclic  $p$ -group, and let  $g \in G$ . Then  $H^*(G, k)$  contains a polynomial subalgebra  $k[x]$  with  $x$  in degree 2 such that  $H^*(G, k)$  is generated as a module over  $k[x]$  by 1 and a degree 1 element  $y$ . Then  $x$  is in the image of the map  $H^2(G, \mathbb{Z}) \rightarrow H^*(G, k)$ , and hence  $\Delta_g(x) = 0$  by Theorem 1.1 (x). Moreover, by Theorem 1.1 (ix), we have  $\Delta_g(y) = y(g)$ . Thus, using that  $\Delta_g$  is a derivation by Theorem 1.1 (i), for any nonnegative integer  $n$ , we have  $\Delta_g(x^n) = 0$  and  $\Delta_g(x^n y) = x^n y(g)$ .

Using the canonical identification  $HH^*(kG) = kG \otimes_k H^*(G, k)$  from [Holm 1996; Cibils and Solotar 1997], this determines the BV operator  $\Delta$  on  $HH^*(kG)$  as follows: we have

$$\begin{aligned}\Delta(g \otimes x^n) &= 0, \\ \Delta(g \otimes x^n y) &= y(g) \cdot g \otimes x^n.\end{aligned}$$

Other papers dealing with BV and Gerstenhaber structures on Hochschild cohomology of cyclic groups include [Sánchez-Flores 2012], [Yang 2013], and [Angel and Duarte 2017].

**Example 9.2.** Let  $G$  be a finite abelian  $p$ -group, and let  $g \in G$ . Combining Example 9.1 with the Künneth formula Proposition 6.6 determines  $\Delta_g$ . Using again the canonical identification of algebras  $HH^*(kG) = kG \otimes_k H^*(G, k)$  from [Holm 1996; Cibils and Solotar 1997], the BV operator  $\Delta$  on  $HH^*(kG)$  is determined by  $\Delta(g \otimes \zeta) = g \otimes \Delta_g(\zeta)$ .

**Example 9.3.** Let  $G$  be the generalised quaternion group  $Q_{2^n}$  of order  $2^n$  ( $n \geq 3$ ), and let  $\gamma$  be the central element of order two in  $G$ . Then  $H^*(G, \mathbb{F}_2)$  is generated by two elements  $x$  and  $y$  in degree one and an element  $z$  of degree four. The elements  $x$  and  $y$  are nilpotent and generate a finite Poincaré duality algebra with top degree three. Since  $z$  is not a zero divisor, we have

$$\sum_{i=0}^{\infty} t^i \dim_{\mathbb{F}_2} H^i(Q_{2^n}, \mathbb{F}_2) = \frac{1 + 2t + 2t^2 + t^3}{1 - t^4}.$$

The exact relations between  $x$  and  $y$  depend on whether  $n = 3$  or  $n > 3$ . If  $n = 3$ , then  $x^2 + xy + y^2 = 0$  and  $x^2y + xy^2 = 0$ ; whereas if  $n > 3$ , then the relations are  $xy = 0$  and  $x^3 + y^3 = 0$ . Again using Theorem 1.1 (ix), and using the fact that  $\gamma$  belongs to the derived subgroup of  $G$ , we have  $\Delta_\gamma(x) = \Delta_\gamma(y) = 0$ , and it remains to compute  $\Delta_\gamma(z)$ .

In both cases,  $H^3(G, \mathbb{F}_2)$  is one dimensional, and we have  $H^3(G, \mathbb{Z}) = 0$ . Moreover, there is an element  $z' \in H^4(G, \mathbb{Z})$  with  $2^n z' = 0$  and such that  $z$  is the

image of  $z'$  under the reduction mod two map  $H^4(G, \mathbb{Z}) \rightarrow H^4(G, \mathbb{F}_2)$ . Therefore  $\Delta_\gamma(z') = 0$ , and the commutativity of the diagram

$$\begin{array}{ccc} H^4(G, \mathbb{Z}) & \xrightarrow{\Delta_\gamma} & H^3(G, \mathbb{Z}) \\ \downarrow & & \downarrow \\ H^4(G, \mathbb{F}_2) & \xrightarrow{\Delta_\gamma} & H^3(G, \mathbb{F}_2) \end{array}$$

shows that  $\Delta_\gamma(z) = 0$ . Since  $\Delta_\gamma$  is a derivation, it follows that it is zero on all elements of  $H^*(G, \mathbb{F}_2)$ .

Other papers considering the BV structure on Hochschild cohomology of quaternion groups include [Ivanov 2014] and [Ivanov et al. 2015].

**Example 9.4.** Let  $G$  be the dihedral group of order  $2^n$  with  $n \geq 3$ ,

$$G = \langle g, h \mid g^2 = h^2 = 1, (gh)^{2^{n-1}} = 1 \rangle,$$

and let  $\gamma$  be the central involution  $(gh)^{2^{n-2}}$  in  $G$ . Then  $H^*(G, \mathbb{F}_2) = \mathbb{F}_2[x, y, z]/(xy)$ , where  $|x| = |y| = 1$  and  $|z| = 2$ . Using Theorem 1.1 (ix) and the fact that  $\gamma$  belongs to the derived subgroup of  $G$ , we have  $\Delta_\gamma(x) = \Delta_\gamma(y) = 0$ , so it remains to evaluate  $\Delta_\gamma(z)$ . Now  $z$  corresponds to the central extension

$$1 \rightarrow \mathbb{F}_2^+ \rightarrow Q \rightarrow G \rightarrow 1,$$

where  $Q$  is the generalised quaternion group of order  $2^{n+1}$

$$Q = \langle \hat{g}, \hat{h} \mid \hat{g}^2 = \hat{h}^2 = (\hat{g}\hat{h})^{2^{n-1}}, (\hat{g}\hat{h})^{2^n} = 1 \rangle.$$

The inverse image of  $\gamma$  in  $Q$  is not central, so it follows from Theorem 1.1 (xi) that  $\Delta_\gamma(z) \neq 0$ . Now  $G$  has an automorphism of order two swapping  $g$  and  $h$ , and swapping  $x$  and  $y$ . It lifts to an automorphism of  $Q$  swapping  $\hat{g}$  and  $\hat{h}$ , and therefore fixes  $z$ . So it also fixes  $\Delta_\gamma(z)$ . The only nonzero fixed element of degree one is  $x + y$ , and therefore we have

$$\Delta_\gamma(z) = x + y.$$

Since  $\Delta_\gamma$  is a derivation, this determines its value on all elements of  $H^*(G, \mathbb{F}_2)$ . We have

$$\Delta_\gamma(x^i z^j) = j x^{i+1} z^{j-1}, \quad \Delta_\gamma(y^i z^j) = j y^{i+1} z^{j-1}.$$

**Remark 9.5.** As well as the proof given in Section 8, Theorem 1.1 (xi) can be proved by a direct computation as follows. Note that  $[\hat{g}, \hat{h}] \in k^+$  is central, hence equal to  $[\hat{h}^{-1}, \hat{g}]$ , and the formula for  $\Delta_g(x)$  as stated does indeed define a group homomorphism from  $G$  to  $k^+$ . Next, since  $\Delta_g$  commutes with restriction to subgroups of  $G$  containing  $g$ , it suffices to consider the case where  $G = \langle g, h \rangle$ . This is an abelian group, and so by Example 9.2, in principle we are done. If  $H^1(G, k)$  is one

dimensional, both sides are zero. Otherwise, it is two dimensional, and  $H^2(G, k)$  modulo the image of  $H^2(G, \mathbb{Z}) \rightarrow H^2(G, k)$  is one dimensional, spanned by the product of two degree one elements. This reduces the proof to an explicit and slightly tedious computation.

**Example 9.6.** Let  $G$  be the semidihedral group of order  $2^n$  ( $n \geq 4$ ); that is,

$$G = \langle g, h \mid g^2 = 1, h^{2^{n-1}} = 1, ghg = h^{2^{n-2}-1} \rangle.$$

The cohomology ring was computed by Munkholm (unpublished) and [Evens and Priddy 1985]. We have

$$H^*(G, \mathbb{F}_2) = \mathbb{F}_2[x, y, z, w]/(x^3, xy, xz, z^2 + y^2w),$$

where  $|x| = |y| = 1$ ,  $|z| = 3$ , and  $|w| = 4$ . Evens and Priddy also observed that the cohomology is detected on the dihedral subgroup of order eight,  $D = \langle g, h^{2^{n-3}} \rangle$  and the quaternion subgroup of order eight,  $Q = \langle gh, h^{2^{n-3}} \rangle$ . Let  $\gamma$  be the central element  $h^{2^{n-2}}$  of order two. Using Theorem 1.1 (ix) and the fact that  $\gamma$  belongs to the derived subgroup of  $G$ , we have  $\Delta_\gamma(x) = \Delta_\gamma(y) = 0$ .

Applying  $\Delta_\gamma$  to the relation  $z^2 = y^2w$  and using the fact that  $\Delta_\gamma$  is a derivation, we have  $y^2\Delta_\gamma(w) = 0$ . Now multiplication by  $y^2$  from  $H^2(G, \mathbb{F}_2)$  to  $H^4(G, \mathbb{F}_2)$  is injective, so we deduce that  $\Delta_\gamma(w) = 0$ .

It remains to compute  $\Delta_\gamma(z)$ . This has degree two, so it is a linear combination of  $x^2$  and  $y^2$ . To determine which, we use the information at the bottom of page 71 in [Evens and Priddy 1985] on restriction to  $D$  and  $Q$ . First note that our  $x$  and  $y$  are their  $x$  and  $x + y$ ; this is determined by which nonzero element of  $H^1(G, \mathbb{F}_2)$  is nilpotent and what it annihilates. So  $y$  restricts to zero on  $Q$ , while  $x$  and  $x^2$  have nonzero restriction. Since  $\Delta_\gamma$  is zero on  $H^*(Q, \mathbb{F}_2)$  by Example 9.3, it follows that  $\Delta_\gamma(z)$  is a multiple of  $y^2$ . The restriction of  $z$  to  $H^*(D, \mathbb{F}_2)$  is a degree three element which is not in the subring generated by  $H^1(D, \mathbb{F}_2)$ . So using Example 9.4, it follows that  $\Delta_\gamma$  is nonzero on the restriction of  $z$ , and hence  $\Delta_\gamma(z)$  cannot be zero. Hence, we have  $\Delta_\gamma(z) = y^2$ .

Using the fact that  $\Delta_\gamma$  is a derivation, it is determined by the information that  $\Delta_\gamma(x) = 0$ ,  $\Delta_\gamma(y) = 0$ ,  $\Delta_\gamma(z) = y^2$ , and  $\Delta_\gamma(w) = 0$ .

## 10. The Gerstenhaber bracket

Throughout this section, let  $p$  be a prime and let  $k$  be a field of characteristic  $p$ . In this section, we combine the formula

$$(10.1) \quad [x, y] = (-1)^{|x|} \Delta(xy) - (-1)^{|x|} \Delta(x)y - x\Delta(y)$$

relating the BV operator  $\Delta$  to the Gerstenhaber bracket and products in  $HH^*(kG)$ , with the Siegel–Witherspoon formula

$$(10.2) \quad xy = \sum_u \mathrm{Tr}_{C_G(g) \cap C_G(\bar{u})}^{C_G(g\bar{u})} (\mathrm{Res}_{C_G(g) \cap C_G(\bar{u})}^{C_G(g)}(x) \cdot \mathrm{Res}_{C_G(g) \cap C_G(\bar{u})}^{C_G(\bar{u})}(u^*(y))),$$

where  $\bar{u} = uhu^{-1}$ , for products in  $HH^*(kG)$  in terms of the centraliser decomposition from [Siegel and Witherspoon 1999]. Here,  $G$  is a finite group,  $g, h \in G$ ,  $x \in H^*(C_G(g), k)$  and  $y \in H^*(C_G(h), k)$ , and  $u$  runs over a set of double coset representatives of  $C_G(g)$  and  $C_G(h)$  in  $G$ . The notation  $u^*(y)$  denotes the image of  $y$  in  $H^*(C_G(uhu^{-1}), k)$  under conjugation by  $u$ . The left side is the product of  $x$  and  $y$  regarded as elements in  $HH^*(kG)$  via the centraliser decomposition. The summand on the right-hand side indexed by  $u$  is in the summand in the centraliser decomposition corresponding to  $guhu^{-1}$ , and the multiplication of the restrictions on the right is the usual cup product in  $H^*(C_G(g) \cap C_G(guhu^{-1}), k)$ .

Combining (10.1) and (10.2), we have

$$(10.3) \quad [x, y] = \sum_u \left( (-1)^{|x|} \Delta_{guhu^{-1}} \operatorname{Tr}_{C_G(g) \cap C_G(uhu^{-1})}^{C_G(guhu^{-1})} \left( \operatorname{Res}_{C_G(g) \cap C_G(uhu^{-1})}^{C_G(g)}(x) \right. \right. \\ \left. \left. \cdot \operatorname{Res}_{C_G(g) \cap C_G(uhu^{-1})}^{C_G(uhu^{-1})}(u^*(y)) \right) \right. \\ \left. + \operatorname{Tr}_{C_G(g) \cap C_G(uhu^{-1})}^{C_G(guhu^{-1})} \left( -(-1)^{|x|} \operatorname{Res}_{C_G(g) \cap C_G(uhu^{-1})}^{C_G(g)}(\Delta_g(x)) \right. \right. \\ \left. \left. \cdot \operatorname{Res}_{C_G(g) \cap C_G(uhu^{-1})}^{C_G(uhu^{-1})}(u^*(y)) \right. \right. \\ \left. \left. - \operatorname{Res}_{C_G(g) \cap C_G(uhu^{-1})}^{C_G(g)}(x) \cdot \operatorname{Res}_{C_G(g) \cap C_G(uhu^{-1})}^{C_G(uhu^{-1})}(u^*(\Delta_h(y))) \right) \right).$$

Note that if  $g \in Z(G)$  then this formula simplifies considerably. Namely,  $C_G(g) = G$ ,  $C_G(g) \cap C_G(h) = C_G(gh)$ , and the transfers do not do anything. There is only one double coset, and we may take  $u = 1$ . The formula then becomes

$$[x, y] = (-1)^{|x|} \Delta_{gh} \left( \operatorname{Res}_{C_G(gh)}^G(x) \cdot y \right) - (-1)^{|x|} \operatorname{Res}_{C_G(gh)}^G(\Delta_g(x)) \cdot y \\ - \operatorname{Res}_{C_G(gh)}^G(x) \cdot \Delta_h(y),$$

as an element of  $H^*(C_G(gh), k)$  in the centraliser decomposition. Then, using Theorem 1.1 (i) and (v), we expand the first term on the right side into four terms. Using Theorem 1.1 (iii), two of these cancel with the remaining two terms to leave

$$(10.4) \quad [x, y] = (-1)^{|x|} \Delta_h \left( \operatorname{Res}_{C_G(gh)}^G(x) \right) \cdot y + \operatorname{Res}_{C_G(gh)}^G(x) \cdot \Delta_g(y).$$

If  $x$  and  $y$  have degree one, then using Theorem 1.1 (ix) this formula becomes

$$(10.5) \quad [x, y] = -x(h)y + y(g) \operatorname{Res}_{C_G(gh)}^G(x).$$

Now assume that  $G$  is a finite  $p$ -group. If  $g \in Z(G) \cap \Phi(G)$ , where  $\Phi(G)$  is the Frattini subgroup of  $G$ , then we have  $y(g) = 0$ , and this simplifies further to

$$[x, y] = -x(h)y.$$

Again, these formulas are as elements of the  $H^*(C_G(gh), k)$  component in the centraliser decomposition.

If both  $g$  and  $h$  are in  $Z(G)$ , then notationally, it helps to keep track of  $g$  and  $h$  by writing  $g \otimes x$  and  $h \otimes y$ , since the subring of  $HH^*(kG)$  corresponding to elements in the centre in the centraliser decomposition is isomorphic to  $kZ(G) \otimes H^*(G, k)$ . With this notation, (10.4) becomes

$$[g \otimes x, h \otimes y] = gh \otimes ((-1)^{|x|} \Delta_h(x) \cdot y + x \cdot \Delta_g(y)).$$

In particular,  $kZ(G) \otimes H^*(G, k) \subseteq HH^*(kG)$  is a Lie subalgebra. Finally, if  $x$  and  $y$  have degree one, this becomes

$$(10.6) \quad [g \otimes x, h \otimes y] = gh \otimes (-x(h)y + y(g)x).$$

If  $g, h \in Z(G) \cap \Phi(G)$  then the terms  $x(h)$  and  $y(g)$  vanish, and the Lie bracket is equal to zero.

We record these observations in the following proposition:

**Proposition 10.7.** *Let  $G$  be a finite  $p$ -group. Suppose that  $g \in Z(G) \cap \Phi(G)$  and  $h \in G$ . If  $x \in H^1(C_G(g), k)$  and  $y \in H^1(C_G(h), k)$  in the centraliser decomposition of  $HH^1(kG)$ , then*

$$[x, y] = -x(h)y$$

*as an element of  $H^1(C_G(gh), k)$  in the centraliser decomposition.*

*In particular, the Lie bracket is identically zero on*

$$k(Z(G) \cap \Phi(G)) \otimes H^1(G, k) \leq HH^1(kG).$$

**Remark 10.8.** If  $g \in Z(G) \cap \Phi(G)$  and  $h$  is not in  $\Phi(G)$ , then we can choose  $x \in H^1(C_G(g), k)$  such that  $x(h) = -1$ , and then for all  $y \in H^1(C_G(h), k)$  the element  $[x, y]$  is  $y$ , but as an element of  $H^1(C_G(gh), k)$  in the centraliser decomposition.

Taking  $g = 1$ , we see that given  $h \notin \Phi(G)$ , there exists  $x \in H^1(C_G(1), k)$  such that for every  $y \in H^1(C_G(h), k)$  we have  $[x, y] = y$ . This proves in particular the following result:

**Proposition 10.9.** *Let  $G$  be a nontrivial finite  $p$ -group. Then the Lie algebra  $HH^1(kG)$  is not nilpotent.*

On the other hand, we have the following theorem, whose proof is modelled on the method in [Jacobson 1943]:

**Theorem 10.10.** *Let  $G$  be a finite  $p$ -group such that  $|Z(G) : Z(G) \cap \Phi(G)| \geq 3$ . Then the Lie subalgebra  $kZ(G) \otimes H^1(G, k) \subseteq HH^1(kG)$  is not soluble, and therefore nor is  $HH^1(kG)$ .*

*Proof.* We compute inside  $kZ(G) \otimes H^*(G, k) \subseteq HH^*(kG)$ , as described above. Since by assumption we have  $|Z(G) : Z(G) \cap \Phi(G)| \geq 3$ , either  $p$  is odd or  $Z(G)/(Z(G) \cap \Phi(G))$  is noncyclic. We treat these two cases separately.

Case (i). Suppose that  $p$  is odd. Choose an element  $g \in Z(G) \setminus \Phi(G)$ , and choose  $x \in H^1(G, k)$  with  $x(g) = 1$ . Set  $e = g \otimes x$ ,  $f = -g^{-1} \otimes x$ , and  $h = -1 \otimes 2x$ . Since  $g \neq g^{-1}$ , these elements are linearly independent, and using (10.6) we have

$$\begin{aligned} [e, f] &= -1 \otimes (x(g)x - x(g^{-1})x) = h, \\ [h, e] &= -g \otimes (x(1)2x - 2x(g)x) = 2e, \\ [h, f] &= g^{-1} \otimes (x(1)2x - 2x(g^{-1})x) = -2f. \end{aligned}$$

Thus  $e$ ,  $f$ , and  $h$  span a copy of the Lie algebra  $\mathfrak{sl}(2)$  inside  $HH^1(kG)$ . Therefore, it is not soluble.

Case (ii). Suppose that  $Z(G)/(Z(G) \cap \Phi(G))$  is noncyclic. Choose elements  $g$  and  $h$  in  $Z(G)$  so that their images in  $Z(G)/(Z(G) \cap \Phi(G))$  generate distinct cyclic subgroups. Choose elements  $x$  and  $y$  in  $H^1(G, k)$  with  $x(g) = 1$ ,  $y(g) = 0$ ,  $x(h) = 0$ , and  $y(h) = 1$ . Then we compute

$$\begin{aligned} [g^{-1} \otimes x, g \otimes y] &= 1 \otimes y & [h^{-1} \otimes y, h \otimes x] &= 1 \otimes x \\ [g \otimes x, 1 \otimes x] &= g \otimes x & [h \otimes y, 1 \otimes y] &= h \otimes y \\ [g \otimes y, 1 \otimes x] &= g \otimes y & [h \otimes x, 1 \otimes y] &= h \otimes x \\ [g^{-1} \otimes x, 1 \otimes x] &= -g^{-1} \otimes x & [h^{-1} \otimes y, 1 \otimes y] &= -h^{-1} \otimes y \\ [g^{-1} \otimes y, 1 \otimes x] &= -g^{-1} \otimes y & [h^{-1} \otimes x, 1 \otimes y] &= -h^{-1} \otimes x. \end{aligned}$$

Letting  $U$  be the linear span in  $HH^1(kG)$  of the elements appearing in these computations, we see that  $[U, U] \supseteq U$ . It follows that any Lie subalgebra containing  $U$  is not soluble, and hence  $kZ(G) \otimes H^1(G, k)$  and  $HH^1(kG)$  are not soluble.  $\square$

**Remark 10.11.** If  $G$  is the cyclic group of order two and  $k$  has characteristic two, then  $HH^1(kG)$  is a Lie algebra of dimension two, and is therefore soluble. This shows that the condition  $|Z(G) : Z(G) \cap \Phi(G)| \geq 3$  cannot be weakened to  $|Z(G) : Z(G) \cap \Phi(G)| \geq 2$ .

## 11. Extraspecial $p$ -groups

As an illustration of the methods developed in this paper, in this section we examine the Lie structure of  $HH^1(kG)$  when  $G$  is an extraspecial  $p$ -group. The methods also cover some other  $p$ -groups of class two, so we formulate them in more generality. We begin with a lemma.

**Lemma 11.1.** *Suppose that  $G$  is a finite  $p$ -group, and that  $g, h \in G$  satisfy*

$$\Phi(C_G(g)) = \Phi(C_G(h)) = \Phi(G),$$

*and  $C_G(g) \cap C_G(h)$  is a proper subgroup of  $C_G(gh)$ . If  $x \in H^*(C_G(g), k)$  and  $y \in H^*(C_G(h), k)$  are elements of degree zero or one, then*

$$\mathrm{Tr}_{C_G(g)} \cap C_G(h)^{C_G(gh)} (\mathrm{Res}_{C_G(g) \cap C_G(h)}^{C_G(g)}(x) \cdot \mathrm{Res}_{C_G(g) \cap C_G(h)}^{C_G(h)}(y)) = 0.$$

*Proof.* Since  $\Phi(C_G(g)) = \Phi(G)$ , we may write  $x = \text{Res}_{C_G(g)}^G(\hat{x})$  with  $\hat{x} \in H^*(G, k)$ . Similarly,  $y = \text{Res}_{C_G(h)}^G(\hat{y})$  with  $\hat{y} \in H^*(G, k)$ . Then,

$$\begin{aligned} \text{Tr}_{C_G(g) \cap C_G(h)}^{C_G(gh)} (\text{Res}_{C_G(g) \cap C_G(h)}^{C_G(g)}(x) \cdot \text{Res}_{C_G(g) \cap C_G(h)}^{C_G(h)}(y)) \\ = \text{Tr}_{C_G(g) \cap C_G(h)}^{C_G(gh)} \text{Res}_{C_G(g) \cap C_G(h)}^G(\hat{x} \cdot \hat{y}) \\ = |C_G(gh) : C_G(g) \cap C_G(h)| \text{Res}_{C_G(gh)}^G(\hat{x} \cdot \hat{y}) = 0, \end{aligned}$$

since  $|C_G(gh) : C_G(g) \cap C_G(h)|$  is zero in  $k$ .  $\square$

**Hypothesis 11.2.** *We suppose that  $G$  is a finite  $p$ -group satisfying  $\Phi(G) = Z(G)$ , and  $g, h$  are elements of  $G$  such that for all  $u \in G$ , and for all  $x \in H^*(C_G(g), k)$ ,  $y \in H^*(C_G(h), k)$  of degree zero or one, we have either:*

- (i)  $C_G(g) \cap C_G(uhu^{-1}) = C_G(guhu^{-1})$ , or
- (ii)  $\text{Tr}_{C_G(g) \cap C_G(uhu^{-1})}^{C_G(guhu^{-1})} (\text{Res}_{C_G(g) \cap C_G(uhu^{-1})}^{C_G(g)}(x) \cdot \text{Res}_{C_G(g) \cap C_G(uhu^{-1})}^{C_G(uhu^{-1})}(u^*(y))) = 0$ .

We remark that if  $\Phi(C_G(g)) = \Phi(C_G(h)) = \Phi(G) = Z(G)$  then Hypothesis 11.2 holds by Lemma 11.1.

Suppose that this hypothesis holds, let  $x \in H^1(C_G(g), k)$  and  $y \in H^1(C_G(h), k)$ , and let  $u$  be a double coset representative of  $C_G(g)$  and  $C_G(h)$  in  $G$ . Suppose first that the containment  $C_G(g) \cap C_G(uhu^{-1}) \leq C_G(guhu^{-1})$  is proper. Then by Hypothesis 11.2, the term corresponding to  $u$  in the Siegel–Witherspoon formula (10.2) for  $xy$  is zero. The same argument holds for  $\Delta_g(x)y$  and  $x\Delta_h(y)$ , and so the term corresponding to  $u$  in (10.3) vanishes.

So the formula (10.3) becomes

$$(11.3) \quad [x, y] = \sum_u \left( -\Delta_{guhu^{-1}} (\text{Res}_{C_G(guhu^{-1})}^{C_G(g)}(x) \cdot \text{Res}_{C_G(guhu^{-1})}^{C_G(uhu^{-1})}(u^*(y))) \right. \\ \left. + \text{Res}_{C_G(guhu^{-1})}^{C_G(g)}(\Delta_g(x)) \cdot \text{Res}_{C_G(guhu^{-1})}^{C_G(uhu^{-1})}(u^*(y)) \right. \\ \left. - \text{Res}_{C_G(guhu^{-1})}^{C_G(g)}(x) \cdot \text{Res}_{C_G(guhu^{-1})}^{C_G(uhu^{-1})}(u^*(\Delta_h(y))) \right),$$

where  $u$  runs over those double coset representatives with

$$C_G(guhu^{-1}) = C_G(g) \cap C_G(uhu^{-1}).$$

For such a representative  $u$ ,  $g$  commutes with  $uhu^{-1}$ . Since  $uhu^{-1}h^{-1} \in [G, G] \leq Z(G)$ , this implies that  $g$  commutes with all conjugates of  $h$ . Now applying Theorem 1.1 (iii), we have

$$(11.4) \quad [x, y] = \sum_u \left( -\Delta_{guhu^{-1}} (\text{Res}_{C_G(guhu^{-1})}^{C_G(g)}(x) \cdot \text{Res}_{C_G(guhu^{-1})}^{C_G(uhu^{-1})}(u^*(y))) \right. \\ \left. + \Delta_g \text{Res}_{C_G(guhu^{-1})}^{C_G(g)}(x) \cdot \text{Res}_{C_G(guhu^{-1})}^{C_G(uhu^{-1})}(u^*(y)) \right. \\ \left. - \text{Res}_{C_G(guhu^{-1})}^{C_G(g)}(x) \cdot \Delta_{uhu^{-1}} \text{Res}_{C_G(guhu^{-1})}^{C_G(uhu^{-1})}(u^*(y)) \right).$$

Set

$$x' = \text{Res}_{C_G(guhu^{-1})}^{C_G(g)}(x) \quad \text{and} \quad y' = \text{Res}_{C_G(guhu^{-1})}^{C_G(uhu^{-1})}(u^*(y)).$$

By Theorem 1.1 (v), we have  $\Delta_{guhu^{-1}} = \Delta_g + \Delta_{uhu^{-1}}$  on  $H^*(C_G(guhu^{-1}), k)$ . Hence, using Theorem 1.1 (i) and (ix), the component of  $[x, y]$  coming from  $u$  is

$$\begin{aligned} (11.5) \quad & -\Delta_{guhu^{-1}}(x'y') + \Delta_g(x')y' - x'\Delta_{uhu^{-1}}(y') \\ & = -(\Delta_g(x')y' + \Delta_{uhu^{-1}}(x')y' - x'\Delta_g(y') - x'\Delta_{uhu^{-1}}(y')) \\ & \quad + \Delta_g(x')y' - x'\Delta_{uhu^{-1}}(y') \\ & = x'\Delta_g(y') - \Delta_{uhu^{-1}}(x')y' \\ & = y(u^{-1}gu) \text{Res}_{C_G(guhu^{-1})}^{C_G(g)}(x) - x(uhu^{-1}) \text{Res}_{C_G(guhu^{-1})}^{C_G(uhu^{-1})}(u^*(y)), \end{aligned}$$

where the multiplications are ordinary cohomology cup products performed inside the summand  $H^*(C_G(guhu^{-1}), k)$  corresponding to  $guhu^{-1}$  in the centraliser decomposition.

We partition a set of conjugacy class representatives in  $G$  into subsets  $C_i$ , where  $g$  is in  $C_i$  if  $|G : C_G(g)| = p^i$ , and for  $i \geq 0$  we set

$$X_i = \bigoplus_{g \in C_i} H^*(C_G(g), k) \quad \text{and} \quad Y_i = \bigoplus_{j \geq i} X_j.$$

Here, as usual, the direct sum is over  $G$ -conjugacy classes of  $g \in C_i$ . Then,

$$HH^*(kG) = \bigoplus_{i=0}^n X_i.$$

It follows from (11.5) that if  $i \leq j$  then  $[X_i, X_j] \subseteq Y_j$ , and hence  $[Y_i, Y_j] \subseteq Y_j$ . So the  $Y_i$  are Lie ideals, and  $HH^1(kG)$  is soluble if and only if each of the  $Y_i/Y_{i+1}$  is soluble. Note that by Proposition 10.7, since  $Z(G) = \Phi(G)$  we have  $[X_0, X_0] = 0$ , and  $[X_0, X_i] = X_i$  for  $i > 0$ .

As an example, we apply these methods to extraspecial  $p$ -groups.

**Theorem 11.6.** *Let  $G$  be an extraspecial  $p$ -group. Then  $HH^1(kG)$  is a soluble Lie algebra. The derived length is two, except in the case where  $G$  has order  $p^3$  and is isomorphic to  $\mathbb{Z}/p^2 \rtimes \mathbb{Z}/p$ , in which case it has derived length three.*

*Proof.* Let  $G$  be extraspecial, and let  $Z = Z(G) = \Phi(G)$ , a group of order  $p$ . Every centraliser in  $G$  is either equal to  $G$  or has index  $p$  in  $G$ . We divide into two cases. The second case deals with extraspecial groups which are semidirect products  $\mathbb{Z}/p^2 \rtimes \mathbb{Z}/p$ , and the first case covers all other extraspecial groups. So in the first case, if  $|G| = p^3$  then  $G$  is either an extraspecial group of exponent  $p$  ( $p$  odd) or the quaternion group of order eight ( $p = 2$ ).

*Case (i).*  $G \not\cong \mathbb{Z}/p^2 \rtimes \mathbb{Z}/p$ . In this case, we claim that Hypothesis 11.2 holds for every  $g$  and  $h$  in  $G$ . If  $|G| \geq p^5$  then for every  $g$  and  $h$  in  $G$  we have

$\Phi(C_G(g)) = \Phi(C_G(h)) = \Phi(G) = Z(G)$ , so by Lemma 11.1, the hypothesis holds. On the other hand, if  $|G| = p^3$  with  $p$  odd, and part (i) of the hypothesis does not hold, then neither  $g$  nor  $h$  is central, and we have

$$|C_G(g) \cap C_G(uhu^{-1})| = p \quad \text{and} \quad |C_G(guhu^{-1})| = p^2$$

. So  $C_G(guhu^{-1})$  is elementary abelian, and then the transfer map from any proper subgroup is zero. Finally, if  $G$  is a quaternion group  $Q_8$ , then the restriction maps from subgroups of order four to  $Z$  is zero. This completes the proof that Hypothesis 11.2 holds for every  $g$  and  $h$  in  $G$ .

So we may apply the theory derived in this section. Using the notation above, we have  $HH^1(kG) = X_0 \oplus X_1$ , and since  $Z(G) = \Phi(G)$  we have  $[X_0, X_0] = 0$  and  $[X_0, X_1] \leq X_1$ . It remains to examine  $[X_1, X_1]$ . Let  $g$  and  $h$  be noncentral elements of  $G$ , and let  $x \in H^1(C_G(g), k)$  and  $y \in H^1(C_G(h), k)$ .

If  $C_G(g) \neq C_G(h)$ , then there are no double coset representatives  $u$  satisfying  $C_G(g) \cap C_G(uhu^{-1}) = C_G(guhu^{-1})$  in the formula (11.3), because the intersection has index  $p^2$  in  $G$ , and so  $[x, y] = 0$ .

On the other hand, if  $C_G(g) = C_G(h)$  then writing  $C$  for their common value, the double cosets are just cosets of  $C$ . Choosing  $v \in G \setminus C$ , we may take the double coset representatives to be  $1, v, \dots, v^{p-1}$ . Now  $v$  does not commute with  $h$ , so defining  $z = v h v^{-1} h^{-1}$ , we have  $Z = \langle z \rangle$ . For some  $2 \leq m \leq p-1$ , we have  $z^{m-1} = v g v^{-1} g^{-1}$ ,  $z^m = v g h v^{-1} (gh)^{-1}$ . So for  $0 \leq i \leq p-1$ , we have

$$v^i h v^{-i} = h z^i, \quad v^i g v^{-i} = g z^{(m-1)i}, \quad \text{and} \quad v^i g h v^{-i} = g h z^{mi}.$$

Choose  $n$  with  $mn$  congruent to one modulo  $p$ . Then  $v^{ni} g h v^{-ni} = g h z^i$ .

All contributions to  $[x, y]$  are landing in the same summand in the centraliser decomposition, but need conjugating to match the elements being centralised. The contribution coming from  $v^i$  is

$$y(v^{-i} g v^i) x - x(v^i h v^{-i}) (v^i)^*(y)$$

in the  $C_G(g v^i h v^{-i})$  component, namely the  $C_G(g h z^i) = C_G(v^{ni} g h v^{-ni})$  component. So we must conjugate to get

$$y(v^{-i} g v^i) (v^{-ni})^*(x) - x(v^i h v^{-i}) (v^{-(n-1)i})^*(y)$$

in the  $C_G(gh)$  component. If  $|G| > p^3$ , we have  $Z \leq \Phi(C)$ , and so  $x$  and  $y$  vanish on  $Z$ . So conjugating by  $v^i$  has no effect. It follows that the above term is independent of  $i$ , and when we sum from  $i = 0$  to  $p-1$  we get zero.

On the other hand, if  $|G| = p^3$ , then  $x$  and  $y$  need not vanish on the central element  $z$ . In this case, we obtain

$$\begin{aligned} y(v^{-i} g v^i) &= y(g z^{-(m-1)i}) = y(g) - (m-1)iy(z), \\ x(v^i h v^{-i}) &= x(h z^i) = x(h) + ix(z). \end{aligned}$$

So for  $\gamma \in C$  we have

$$[x, y](\gamma) = \sum_{i=0}^{p-1} ((y(g) - (m-1)iy(z))(x(\gamma) + nirx(z)) - (x(h) + ix(z))(y(\gamma) + (n-1)iry(z))).$$

Since  $p \geq 3$ , the expressions  $\sum_{i=0}^{p-1} 1$  and  $\sum_{i=0}^{p-1} i$  are zero in  $k$ , and so all but the quadratic term vanish. If  $p \geq 5$  the quadratic term vanishes too since  $\sum_{i=0}^{p-1} i^2 = 0$  in  $k$ , but when  $p = 3$  it equals  $-1$ , and we have

$$\begin{aligned} [x, y](\gamma) &= (-1)(-(m-1)y(z)nrx(z) - x(z)(n-1)ry(z)) \\ &= r((m-1)n + (n-1))x(z)y(z) \\ &= r(mn-1)x(z)y(z). \end{aligned}$$

This is equal to zero since  $m$  and  $n$  are inverses modulo  $p$ . This completes the proof in the case  $|G| = p^3$ , and we are done.

*Case 2.*  $G \cong \mathbb{Z}/p^2 \rtimes \mathbb{Z}/p$ . In this case, the failure of Hypothesis 11.2 comes from the fact that transfer from  $Z$  to a subgroup  $\mathbb{Z}/p^2$  is nonzero in degree one (but zero in degree two). Restriction in degree one from  $\mathbb{Z}/p^2$  to  $Z$  is zero, however, and transfer from  $\mathbb{Z}/p$  to  $(\mathbb{Z}/p)^2$  is zero in all degrees, so for the analysis above to fail, we must restrict a degree one element from  $(\mathbb{Z}/p)^2$  to  $Z$  and then transfer to  $\mathbb{Z}/p^2$ . So we write  $X_1 = X'_1 \oplus X''_1$ , where  $X'_1$  is the sum of the terms in the centraliser decomposition of  $HH^1(kG)$  with  $C_G(g)$  elementary abelian of order  $p^2$  and  $X''_1$  the sum of the terms with  $C_G(g)$  cyclic of order  $p^2$ .

As usual we have  $[X_0, X_0] = 0$ ,  $[X_0, X'_1] = X'_1$ , and  $[X_0, X''_1] = X''_1$ . Furthermore, if  $g, h$ , and  $gh$  have the same centraliser, then the same argument as in Case (i) shows that  $[x, y] = 0$ . So we may assume that  $C_G(g) \cap C_G(h) = Z$  and  $|C_G(gh)| = p^2$ .

If  $p$  is odd, then there is a unique subgroup isomorphic to  $(\mathbb{Z}/p)^2$  in  $G$ , and so if  $C_G(g) \cong (\mathbb{Z}/p)^2$  and  $C_G(h) \cong \mathbb{Z}/p^2$  then  $C_G(gh) \cong \mathbb{Z}/p^2$ . Hence  $[X'_1, X'_1] = 0$ ,  $[X'_1, X''_1] \leq X''_1$ , and  $[X''_1, X''_1] = 0$ . So  $HH^1(kG)$  is soluble of derived length three.

On the other hand, if  $p = 2$  then there is a unique subgroup isomorphic to  $\mathbb{Z}/4$  in  $G$ , and so if  $C_G(g) \cong (\mathbb{Z}/2)^2$  and  $C_G(h) \cong \mathbb{Z}/4$  then  $C_G(gh) \cong (\mathbb{Z}/2)^2$ . This time we have  $[X'_1, X'_1] \leq X''_1$  and  $[X'_1, X''_1] = 0$ , and  $[X''_1, X''_1] = 0$ . So  $HH^1(kG)$  is again soluble of derived length three.  $\square$

**Remark 11.7.** The cases of the extraspecial 2-groups  $D_8$  and  $Q_8$  of order eight were also considered in [Rubio y Degraasi et al. 2019; Eisele and Raedschelders 2020].

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## References

- [Abbaspour 2015] H. Abbaspour, “On algebraic structures of the Hochschild complex”, pp. 165–222 in *Free loop spaces in geometry and topology*, edited by J. Latschev and A. Oancea, IRMA Lect. Math. Theor. Phys. **24**, Eur. Math. Soc., Zürich, 2015. MR Zbl
- [Adem and Milgram 1994] A. Adem and R. J. Milgram, *Cohomology of finite groups*, Grundlehren der Math. Wissenschaften **309**, Springer, 1994. MR Zbl
- [Angel and Duarte 2017] A. Angel and D. Duarte, “The BV-algebra structure of the Hochschild cohomology of the group ring of cyclic groups of prime order”, pp. 353–372 in *Geometric, algebraic and topological methods for quantum field theory* (Villa de Leyva, Colombia, 2013), edited by L. Cano et al., World Sci., Hackensack, NJ, 2017. MR Zbl
- [Benson 1991a] D. J. Benson, *Representations and cohomology, I: Basic representation theory of finite groups and associative algebras*, Cambridge Stud. Adv. Math. **30**, Cambridge Univ. Press, 1991. MR Zbl
- [Benson 1991b] D. J. Benson, *Representations and cohomology, II: Cohomology of groups and modules*, Cambridge Stud. Adv. Math. **31**, Cambridge Univ. Press, 1991. MR Zbl
- [Burghilea 1985] D. Burghilea, “The cyclic homology of the group rings”, *Comment. Math. Helv.* **60**:3 (1985), 354–365. MR Zbl
- [Cibils and Solotar 1997] C. Cibils and A. Solotar, “Hochschild cohomology algebra of abelian groups”, *Arch. Math. (Basel)* **68**:1 (1997), 17–21. MR Zbl
- [Connes 1985] A. Connes, “Noncommutative differential geometry”, *Inst. Hautes Études Sci. Publ. Math.* **62** (1985), 257–360. MR Zbl
- [Eisele and Raedschelders 2020] F. Eisele and T. Raedschelders, “On solvability of the first Hochschild cohomology of a finite-dimensional algebra”, *Trans. Amer. Math. Soc.* **373**:11 (2020), 7607–7638. MR Zbl
- [Evens and Priddy 1985] L. Evens and S. Priddy, “The cohomology of the semidihedral group”, pp. 61–72 in *Conference on algebraic topology in honor of Peter Hilton* (Saint John’s, NL, 1983), edited by R. Piccinini and D. Sjerve, Contemp. Math. **37**, Amer. Math. Soc., Providence, RI, 1985. MR Zbl
- [Holm 1996] T. Holm, “The Hochschild cohomology ring of a modular group algebra: the commutative case”, *Comm. Algebra* **24**:6 (1996), 1957–1969. MR Zbl
- [Ivanov 2014] A. A. Ivanov, “The BV-algebra structure on the Hochschild cohomology of local algebras of quaternion type in characteristic 2”, *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov.* **430**:27 (2014), 136–185. In Russian; translated in *J. Math. Sci. (N.Y)* **219**:3 (2016), 427–461. MR Zbl

- [Ivanov et al. 2015] A. Ivanov, S. O. Ivanov, Y. Volkov, and G. Zhou, “BV structure on Hochschild cohomology of the group ring of quaternion group of order eight in characteristic two”, *J. Algebra* **435** (2015), 174–203. MR Zbl
- [Jacobson 1943] N. Jacobson, “Classes of restricted Lie algebras of characteristic  $p$ , II”, *Duke Math. J.* **10** (1943), 107–121. MR Zbl
- [Karoubi and Villamayor 1990] M. Karoubi and O. E. Villamayor, “Homologie cyclique d’algèbres de groupes”, *C. R. Acad. Sci. Paris Sér. I Math.* **311**:1 (1990), 1–3. MR Zbl
- [Le and Zhou 2014] J. Le and G. Zhou, “On the Hochschild cohomology ring of tensor products of algebras”, *J. Pure Appl. Algebra* **218**:8 (2014), 1463–1477. MR Zbl
- [Levi 1996] R. Levi, “Torsion in loop space homology of rationally contractible spaces”, *J. Pure Appl. Algebra* **114**:1 (1996), 69–87. MR Zbl
- [Linckelmann and Rubio y Degrassi 2020] M. Linckelmann and L. Rubio y Degrassi, “On the Lie algebra structure of  $HH^1(A)$  of a finite-dimensional algebra  $A$ ”, *Proc. Amer. Math. Soc.* **148**:5 (2020), 1879–1890. MR Zbl
- [Liu and Zhou 2016] Y. Liu and G. Zhou, “The Batalin–Vilkovisky structure over the Hochschild cohomology ring of a group algebra”, *J. Noncommut. Geom.* **10**:3 (2016), 811–858. MR Zbl
- [Liu et al. 2021] Y. Liu, Z. Wang, and G. Zhou, “The Batalin–Vilkovisky structure on the Tate–Hochschild cohomology ring of a group algebra”, *Int. Math. Res. Not.* **2021**:6 (2021), 4079–4139. MR Zbl
- [Loday 1992] J.-L. Loday, *Cyclic homology*, Grundlehren der Math. Wissenschaften **301**, Springer, 1992. MR Zbl
- [Loday and Quillen 1984] J.-L. Loday and D. Quillen, “Cyclic homology and the Lie algebra homology of matrices”, *Comment. Math. Helv.* **59**:4 (1984), 569–591. MR Zbl
- [Menichi 2004] L. Menichi, “Batalin–Vilkovisky algebras and cyclic cohomology of Hopf algebras”, *K-Theory* **32**:3 (2004), 231–251. MR Zbl
- [Rubio y Degrassi 2017] L. Rubio y Degrassi, “Invariance of the restricted  $p$ -power map on integrable derivations under stable equivalences”, *J. Algebra* **469** (2017), 288–301. MR Zbl
- [Rubio y Degrassi et al. 2019] L. Rubio y Degrassi, S. Schroll, and A. Solotar, “The first Hochschild cohomology as a Lie algebra”, preprint, 2019. arXiv
- [Sánchez-Flores 2012] S. Sánchez-Flores, “The Lie structure on the Hochschild cohomology of a modular group algebra”, *J. Pure Appl. Algebra* **216**:3 (2012), 718–733. MR Zbl
- [Siegel and Witherspoon 1999] S. F. Siegel and S. J. Witherspoon, “The Hochschild cohomology ring of a group algebra”, *Proc. Lond. Math. Soc.* (3) **79**:1 (1999), 131–157. MR Zbl
- [Tradler 2008] T. Tradler, “The Batalin–Vilkovisky algebra on Hochschild cohomology induced by infinity inner products”, *Ann. Inst. Fourier (Grenoble)* **58**:7 (2008), 2351–2379. MR Zbl
- [Volkov 2016] Y. V. Volkov, “BV differential on Hochschild cohomology of Frobenius algebras”, *J. Pure Appl. Algebra* **220**:10 (2016), 3384–3402. MR Zbl
- [Weibel 1994] C. A. Weibel, *An introduction to homological algebra*, Cambridge Stud. Adv. Math. **38**, Cambridge Univ. Press, 1994. MR Zbl
- [Yang 2013] T. Yang, “A Batalin–Vilkovisky algebra structure on the Hochschild cohomology of truncated polynomials”, *Topology Appl.* **160**:13 (2013), 1633–1651. MR Zbl

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
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