GRASSMANN INEQUALITIES AND EXTREMAL VARIETIES IN $\mathbb{P}\left(\bigwedge^{p}\mathbb{R}^{n}\right)$

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Abstract. In continuation of the work in [9,8], which defines extremal varieties in $\mathbb{P}\left(\bigwedge^{2}\mathbb{R}^{n}\right)$, we define a more general concept of extremal varieties of the real Grassmannian $G_p\left(\mathbb{R}^n\right)$ in $\mathbb{P}\left(\bigwedge^{p}\mathbb{R}^{n}\right)$. This concept is based on the minimization of the sums of squares of the quadratic Plücker relations defining the Grassmannian variety as well as the reverse maximisation problem. Such extremal problems define a set of Grassmannian inequalities on the set of Grassmann matrices, which are essential for the definition of the Grassmann variety and its dual extremal variety. We define and prove these inequalities for a general Grassmannian and we apply the existing results, in the cases $\wedge^2\mathbb{R}^{2n}$ and $\wedge^n\mathbb{R}^{2n}$. The resulting extremal varieties underline the fact which was demonstrated in [10,9], that such varieties are represented by multi-vectors that acquire the property of a unique singular value with total multiplicity. Crucial to these inequalities are the numbers $M_{n,p}$, which are calculated within the cases mentioned above.

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1 Introduction

Grassmann varieties and manifolds play an important role in many areas of mathematics. The main application of Grassmannians in system theory and engineering is the parametrization and usage of fixed structure controllers in feedback systems. Many problems in control theory, linked to frequency assignment, can be written as intersection problems of a Grassmann variety viewed as a projective variety via the Plücker embedding. For this reason the Grassmannian is significant with respect to mathematical control theory, see [1]. The Grassmannian manifolds are also of significant importance in the study of vector or tangent bundles of manifolds or varieties, see [12]. In fact, every tangent bundle of a manifold $\mathscr{M}$ generates a continuous map from $\mathscr{M}$ to a generalized Grassmannian. The study of the vector bundle is reduced to the study of this map. Vector and tangent bundles are frequently encountered in algebraic topology, differential topology, differential geometry, and theoretical physics, see [5]. Another application is multidimensional data reduction or simplification, see [6]. For this reason, the study of Grassmannians is of great significance in the above mentioned areas. If we have tensor data with skew symmetric structure, then one way to simplify these data is to approximate the tensor by a decomposable one. This problem amounts to approximating a point in the Plücker space by an element of the Grassmannian. A study of such a distance problem has applications in multidimensional data reduction. Such data appear in image processing, signal processing, or areas like social sciences where multidimensional data appear. In [10,9], distance problems were examined in the projective space $\mathbb{P}\left(\bigwedge^2\mathbb{R}^n\right)$. Such problems arise in pole and zero assignment in Control Theory where generalized controllers are approximated by realizable controllers belonging to a realizable Grassmannian variety. If $\underline{z} \in P\left(\wedge^2\mathbb{R}^n\right)$, is a multi-vector solution of the linear sub-problems of frequency assignment, then an optimization problem of the form

$$\min_{x_1,x_2\in\mathbb{R}^n} \|\underline{z} - x_1 \wedge x_2\|$$

has to be solved, so that the controller formed by the vectors x_1, x_2 is realizable. The more general problem

$$\min_{x_1,x_2,\ldots,x_p\in\mathbb{R}^n} \left\|z - x_1 \wedge x_2 \wedge \ldots \wedge x_p\right\|$$

has to also be addressed for the general frequency assignment problems of Multi-Input Multi-Output (MIMO) systems. This is basically an approximation problem in an exterior algebra of a multi-vector by a decomposable vector. Similar approximation problems arise in signal processing where a multidimensional vector has to be approximated and simplified by a lower and decomposable vector, gaining in such a way simplicity and storage space. This can also be viewed as a distance problem $d\left(\underline{z}, G_p(\mathbb{R}^n)\right)$ of a multi-vector from the Grassmannian variety consisting of all decomposable vectors. The multi-vectors $\underline{z} \in P\left(\wedge^p\mathbb{R}^n\right)$, which are "badly" approximated are the ones that have the furthest possible distance from $G_p(\mathbb{R}^n)$. Such multi-vectors form a variety called the extremal variety of the Grassmannian and it is a new interesting variety, dual to the Grassmannian variety, worthy of studying. This duality stems from distance problems in the projective space, where distance is measured by the angle between two points in the projective space. This distance can be either maximised or minimised and if the set of vectors with maximal distance to Grassmann variety are considered, then we get the Extremal variety and vice versa. This involution type of mapping is explained in [8]. In [9,8], such varieties were considered when $p = 2$, leaving out the more general case $p > 2$. In the present paper, we extend the definition of these varieties to accommodate the case $p > 2$. The additional advantage is, that these varieties can now be computed, in the sense that their defining equations can be easily calculated. This is done, by maximising the sums of squares of the quadratic Plücker relations defining the Grassmannian. This sum is denoted by $f_{n,p}(\underline{z})$, where $\underline{z} \in \wedge^p\mathbb{R}^n$. By utilizing the results obtained in [4, 6] together with some new results presented in this paper, we derive a set of inequalities relating $f_{n,p}(\underline{z})$ and the Grassmann matrix $\Phi(\underline{z})$. The maximum value of

$f_{n,p}(\underline{z})$ when $\|\underline{z}\|^2 = 1$, is denoted as $M_{n,p}$ and this way the extremal variety is defined by the equation

$$f_{n.p}(\underline{z}) = M_{n,p}\|\underline{z}\|^4$$

The extremal variety is a real variety defined by a single homogeneous equation of degree 4 in the Plücker space. Hence, it is a real projective variety in the projective Plücker space. To define the extremal variety of $G_p(\mathbb{R}^n)$, denoted by $Extr(n,p)$, the numbers $M_{n,p}$ have to be calculated. We prove that the numbers $M_{n,p}$ satisfy the bound $M_{n,p} \leq \frac{(n-p)p}{n}$. In certain cases $M_{n,p}$ equals this upper bound and in other cases the inequality is strict. In this paper we calculate the exact values $M_{n,p}$ for $\wedge^2\mathbb{R}^{2n}$ and $\wedge^n\mathbb{R}^{2n}$. We also study the cases of $\wedge^3\mathbb{R}^6$ and $\wedge^p\mathbb{R}^{pk}$. Finally, we present representations and spectral properties of the elements $Extr(n,p)$, underlying the fact that these vectors contain multi-vectors of a unique singular value of total multiplicity.
The consideration of reversing an extremisation problem (from min to max or vice versa) also exists in operator theory, i.e. in the definition of operator trigonometric functions [4] and in antieigenvalue theory [3]. In our case, this extremisation reversing reveals an interplay between Grassmann and extremal varieties.
The optimisation problem considered in the present paper, if it is simpified, in a matrix version seems to be related to the orthogonal Procrustes problem [2].
The extremisation approach may be used in various scenarios as demonstrated by ”toy” examples at the end of this paper. In the setup of vector bundles we may project the related line bundle to the extremal variety and examine the new line bundle produced. Secondly, in case that the extremal variety is simpler than the Grassmann variety, we may solve intersection problems on the Grassmannian by considering equivalent problems on the extremal variety and then utilize Poincaré - Miranda type of intermediate value theorems [11].

2 Grassmann Inequalities and Extremal Varieties

In this section we define extremal varieties and their corresponding inequalities, calculations and bounds for the $M_{n,p}$ constants, connections with the Grassmann matrices and some basic representations.

Let $\underline{z} \in \wedge^p\mathbb{R}^n$. Then $\underline{z}$ is decomposable if

$$z = \wedge_{i=1}^{p} x_i,\ x_i \in \mathbb{R}^n$$

In this case, if we consider the line

$$\langle\underline{z}\rangle = \{\lambda\underline{z} :\ \lambda \in \mathbb{R}\}$$

in $\mathbb{P}(\bigwedge^p\mathbb{R}^n)$, the set of all lines of decomposable multi-vectors from a variety called the Grassmann variety . To study decomposability and, as a result, the Grassmann variety we define a linear map

$$T :\ \mathbb{R}^n \to \wedge^{p+1}\mathbb{R}^n$$

defined as

$$T(\underline{u}) = \underline{z} \wedge \underline{u}$$

and the matrix representation of T, denoted as $\Phi_n^p(\underline{z}) \in \mathbb{R}^{\binom{n}{p+1}\times n}$ is called the *Grassmann matrix*. The rank properties of this matrix are related to the decomposability properties of $\underline{z}$. We may define in a similar manner the Grassmann matrix of $\underline{z}^*$, the *Hodge dual* of $\underline{z}$. This matrix is denoted by $\Phi_n^{n-p}(\underline{z}^*)$ and is called the *Hodge-Grassmann* matrix of $\underline{z}$. These two matrices play an important role in the decomposability properties of $\underline{z}$ as well as in the definition of the Grassmann variety. The Grassmann variety $G_p(\mathbb{R}^n)$ in $\mathbb{P}(\bigwedge^p \mathbb{R}^n)$, is defined by the quadratic equations, see [6],

$$\Phi_n^{n-p}(\underline{z}^*)\Phi_n^p(\underline{z})^T = 0$$

called the quadratic Plücker relations (QPRs), or equivalently, the sum of squares of QPRs equals zero.
This sum of squares can be written in trace form as:

$$f_{n,p}(\underline{z}) = tr\left(\Phi_n^p(\underline{z})\Phi_n^{n-p}(\underline{z}^*)^T\Phi_n^{n-p}(\underline{z}^*)\Phi_n^p(\underline{z})^T\right) \tag{1}$$

and $f_{n,p}(\underline{z})$maps the compact and connected sphere $S^{\binom{n}{p}-1}$ to $\mathbb{R}$. Hence, it attains a global minimum, which is equal to zero and defines the set of decomposable vectors. It also attains a global maximum $M_{n,p}$, and so, in view of the fact that $f_{n,p}(\underline{z})$ is a homogeneous function of degree 4, we can rewrite this maximum relation as:

$$M_{n,p}\|\underline{z}\|^4 \geq f_{n,p}(\underline{z}) \geq 0, \text{ for all } \underline{z} \in \wedge^p\mathbb{R}^n. \tag{2}$$

Based on the maximum we define another variety as follows:

Definition 1. *The set* $\{\underline{z} \in \mathbb{P}(\bigwedge^p \mathbb{R}^n)\}$*:* $f_{n,p}(\underline{z}) = M_{n,p}\|\underline{z}\|^4\}$ *defined by the zeros of the 4-th order homogeneous polynomial* $f_{n,p}(\underline{z}) - M_{n,p}\|\underline{z}\|^4$ *is a projective variety in* $\mathbb{P}(\bigwedge^p \mathbb{R}^n)$ *denoted by* $Extr(n,p)$.

The purpose of this paper is to calculate $M_{n,p}$ and explore the structure of the variety $Extr(n,p)$.

Theorem 1. *(See [10]) For any* $\underline{z} \in \wedge^p\mathbb{R}^n$ *the following identity holds:*

$$\Phi_n^{n-p}(\underline{z}^*)^T\Phi_n^{n-p}(\underline{z}^*) + \Phi_n^p(\underline{z})^T\Phi_n^p(\underline{z}) = \|\underline{z}\|^2 I_n \tag{3}$$

where $\Phi_n^p(\underline{z})$ *is the Grassmann matrix and* $\Phi_n^{n-p}(\underline{z}^*)$ *is the Hodge-Grassmann matrix of* $\underline{z}$.

Corollary 1. *Let* $(\sigma_i)_{i=1}^n$ *and* $(\sigma_i^{'})_{i=1}^n$ *the singular values of* $\Phi_n^p(\underline{z})$ *and* $\Phi_n^{n-p}(\underline{z}^*)$*, respectively. Then, these singular values can be paired so that:*

$$\sigma_i^2 + \left(\sigma_i^{'}\right)^2 = \|\underline{z}\|^2 \text{ and } 0 \leq \sigma_i, \sigma_i^{'} \leq \|\underline{z}\| \tag{4}$$

Proof. *The matrices* $\Phi_n^{n-p}(\underline{z}^*)^T \cdot \Phi_n^{n-p}(\underline{z}^*)$ *and* $\Phi_n^p(\underline{z})^T \cdot \Phi_n^p(\underline{z})$ *are simultaneously diagonalizable, and so, in view of* (3) *relation* (4) *follows.*

Lemma 1. *For any $\underline{z} \in \wedge^p \mathbb{R}^n$ the following identity holds:*

$$tr\left(\Phi_n^p(\underline{z})^T \Phi_n^p(\underline{z})\right) = (n-p)\|\underline{z}\|^2 \tag{5}$$

where $\Phi_n^p(\underline{z})$ is the Grassmann matrix of $\underline{z}$.

Proof. *Indeed, let e_i be the canonical bases of $\mathbb{R}^n$ and $q \in Q_p^n$ the selections of p indices from $\{1,2,...,n\}$, then*

$$tr\left(\Phi_n^p(\underline{z})^T \Phi_n^p(\underline{z})\right) = \sum_i \langle \underline{z} \wedge e_i, \underline{z} \wedge e_i \rangle = \sum_{1 \le i \le n} \sum_{q \in Q_p^n, i \notin q} \underline{z}_q^2 = (n-p)\|\underline{z}\|^2$$

Lemma 2. *It holds that*

$$tr\left(\Phi_n^p(\underline{z})\Phi_n^{n-p}(\underline{z}^*)^T \Phi_n^{n-p}(\underline{z}^*)\Phi_n^p(\underline{z})^T\right) = (n-p)\|\underline{z}\|^4 - tr\left[\left(\Phi_n^p(\underline{z})^T \Phi_n^p(\underline{z})\right)^2\right]. \tag{6}$$

Proof. *In view of* (3)*, we have*

$$tr\left(\Phi_n^p(\underline{z})\Phi_n^{n-p}(\underline{z}^*)^T \Phi_n^{n-p}(\underline{z}^*)\Phi_n^p(\underline{z})^T\right) = tr\left[\Phi_n^p(\underline{z})\left(\|\underline{z}\|^2 I_n - \Phi_n^p(\underline{z})^T \Phi_n^p(\underline{z})\right)\Phi_n^p(\underline{z})^T\right].$$

From this and in view of (5)*, one may easily see that*

$$tr\left(\Phi_n^p(\underline{z})\Phi_n^{n-p}(\underline{z}^*)^T \Phi_n^{n-p}(\underline{z}^*)\Phi_n^p(\underline{z})^T\right) = (n-p)\|\underline{z}\|^4 - tr\left[\left(\Phi_n^p(\underline{z})^T \Phi_n^p(\underline{z})\right)^2\right].$$

Using identities (5) and (6), the extremal problem to be considered is:

$$\left\{\begin{array}{c} \max \text{ (or min) } tr\left[\left(\Phi_n^p(\underline{z})^T \Phi_n^p(\underline{z})\right)^2\right], \\ tr\left(\Phi_n^p(\underline{z})^T \Phi_n^p(\underline{z})\right) = (n-p)\|\underline{z}\|^2 \text{ and } \|\underline{z}\|^2 = 1. \end{array}\right\} \tag{7}$$

If σ_i are the singular values of $\frac{1}{\|\underline{z}\|}\Phi_n^p(\underline{z})$, by taking into consideration (4), the extremal problem described by (7) is equivalent with the following extremal problem:

$$\left\{\begin{array}{c} \max \text{ (or min) } \sum_i \sigma_i^4, \\ \sum_i \sigma_i^2 = n-p \text{ and } \sigma_i^2 \le 1. \end{array}\right\} \tag{8}$$

Remark 1. By setting $\sigma_i^2 = w_i$, the extremal problem described by (8) is equivalent with the following extremal problem:

$$\left\{\begin{array}{c} \max \text{ (or min) } \sum_i w_i^2 \\ \sum_i w_i = n-p \text{ and } 0 \le w_i \le 1 \end{array}\right\}$$

(9)

The ratio

$$\frac{\sum_i w_i^2}{\sum_i w_i}$$

is the measure of dispersion of w_i and is called **Herfindahl-Hirschman Index**. The maximum value of the ratio is attained when $\{w_i\}$ have maximum dispersion, i.e when the $n-p$ highest w_i are equal to 1 and the rest equal to zero. The minimum value of the ratio is taken when all w_i are equal to $\frac{n-p}{n}$.

In view of remark 1, the following theorem is now clear.

Theorem 2. *The following statements are true:*

(i) *The maximum value of* $tr\left[\left(\Phi_n^p(\underline{z})^T\Phi_n^p(\underline{z})\right)^2\right]$ *is attained when*

$$\frac{1}{\|\underline{z}\|^2}\Phi_n^p(\underline{z})^T\Phi_n^p(\underline{z}) = \left[\begin{array}{c|c} I_{n-p} & 0_{n-p\times p} \\ \hline 0_{p\times n-p} & 0_{p\times p}\end{array}\right] = G \tag{10}$$

(ii) *The minimum value of* $tr\left[\left(\Phi_n^p(\underline{z})^T\Phi_n^p(\underline{z})\right)^2\right]$ *is attained when*

$$\frac{1}{\|\underline{z}\|^2}\Phi_n^p(\underline{z})^T\Phi_n^p(\underline{z}) = \left(\frac{n-p}{n}\right)I_n \tag{11}$$

Remark 2. Theorem 1 gives rise to the following Grassmann inequality:

$$(n-p)\,\|\underline{z}\|^4 \geq tr\left[\left(\Phi_n^p(\underline{z})^T\Phi_n^p(\underline{z})\right)^2\right] \geq \frac{(n-p)^2}{n}\|\underline{z}\|^4 \tag{12}$$

Theorem 3. *A vector* $\underline{z}\in P(\wedge^p\mathbb{R}^n)$ *satisfies the following equation*

$$tr\left[\left(\Phi_n^p(\underline{z})^T\Phi_n^p(\underline{z})\right)^2\right] = (n-p)\,\|\underline{z}\|^4 \tag{13}$$

if and only if, $\underline{z}$ *is decomposable.*

Proof. In this case

$$\Phi_n^p(\underline{z})^T\Phi_n^p(\underline{z}) = G\|\underline{z}\|^4 \tag{14}$$

Here, $\Phi_n^p(\underline{z})$ *has a p-dimensional right kernel. If*

$$\underline{v}\in Rker(\Phi_n^p(\underline{z})),$$

then

$$\underline{z}\wedge\underline{v}=0,$$

which means that $\underline{v}$ *is a factor of* $\underline{z}$*. If*

$$\underline{v}_1,\underline{v}_2,\ldots,\underline{v}_p$$

is an orthonormal basis of Rker$(\Phi_n^p(\underline{z}))$, *then*

$$\underline{z} = \|\underline{z}\|\, \underline{v}_1 \wedge \underline{v}_2 \wedge \ldots \wedge \underline{v}_p, \tag{15}$$

implying that $\underline{z}$ *is decomposable.*
Conversely, if (15) *holds, then one can easily see that* (14) *holds, and so,* (13) *also holds. The proof is complete.*

Theorem 4. *In the special case where* $n = 2n$ *and* $p = 2$ *the extremal variety* $Extr(n, p)$ *can be also defined by the following trace equation:*

$$tr\left[\left(\Phi_n^p(\underline{z})^T \Phi_n^p(\underline{z})\right)^2\right] = \frac{(n-p)^2}{n}\|\underline{z}\|^4 \tag{16}$$

where $\underline{z} \in \mathbb{P}(\bigwedge^p \mathbb{R}^n)$.

Proof. *Indeed, if* $\underline{z} \in \wedge^2\mathbb{R}^{2n}$, *then consider* $\underline{z}$ *to be*

$$\underline{z} = \sigma \underline{x}_1 \wedge \underline{y}_1 + \ldots \sigma \underline{x}_n \wedge \underline{y}_n,$$

where $\left\{\underline{x}_1, \ldots, \underline{x}_n, \underline{y}_1, \ldots, \underline{y}_n\right\}$ *is an orthonormal basis. Then,*

$$n\sigma^2 = \|\underline{z}\|^2,\ \Phi_{2n}^{2n-2}(\underline{z}^*)^T \Phi_{2n}^{2n-2}(\underline{z}^*) = \sigma^2 I_{2n},$$

and in view of (3),

$$\Phi_{2n}^2(\underline{z})^T \Phi_{2n}^2(\underline{z}) = \|\underline{z}\|^2 I_{2n} - \sigma^2 I_{2n} = \frac{n-1}{n}\|\underline{z}\|^2 I_{2n},$$

implies that

$$tr\left[\left(\Phi_{2n}^2(\underline{z})^T \Phi_{2n}^2(\underline{z})\right)^2\right] = \left(\frac{n-1}{n}\right)^2 2n\|\underline{z}\|^4 = \frac{2(n-1)^2}{n}\|\underline{z}\|^4 = \frac{(2n-2)^2}{2n}\|\underline{z}\|^4 \tag{17}$$

Hence, $Extr(2n, 2)$ *defined by* (16) *is nonempty.*
Conversely, if (17) *holds, then in view of* (3),

$$\Phi_{2n}^{2n-2}(\underline{z}^*)^T \Phi_{2n}^{2n-2}(\underline{z}^*) = \frac{1}{n}\|\underline{z}\|^2 I_{2n},$$

and hence,

$$\underline{z} = \frac{\|\underline{z}\|}{\sqrt{n}}\left(\underline{x}_1 \wedge \underline{y}_1 + \ldots + \underline{x}_n \wedge \underline{y}_n\right), \tag{18}$$

for some orthonormal basis $\left\{\underline{x}_1, \ldots, \underline{x}_n, \underline{y}_1, \ldots, \underline{y}_n\right\}$. *So, for the case of* $Extr(2n, 2)$ *the variety defined by* (17), *contains exactly all vectors of the form* (18).
In the case of $\wedge^n\mathbb{R}^{2n}$, *if we consider the vector*

$$\underline{z} = \frac{\|\underline{z}\|}{\sqrt{2}}\left(\underline{x}_1 \wedge \ldots \wedge \underline{x}_n + \underline{y}_1 \wedge \ldots \wedge \underline{y}_n\right) \tag{19}$$

where $\left\{\underline{x}_1,...,\underline{x}_n,\underline{y}_1,...,\underline{y}_n\right\}$ *is an orthonormal basis, then*

$$\Phi_{2n}^{n}(\underline{z})^T \Phi_{2n}^{n}(\underline{z}) = \frac{\|\underline{z}\|^2}{2} I_{2n}$$

and

$$tr\left[\left(\Phi_{2n}^{n}(\underline{z})^T \Phi_{2n}^{n}(\underline{z})\right)^2\right] = \frac{\|\underline{z}\|^4}{4}2n = \frac{n}{2}\|\underline{z}\|^4 = \frac{(2n-n)^2}{2n}\|\underline{z}\|^4$$

Hence, $Extr(2n,n)$ *is non-void, as it contains all vectors of the form* (19)*. The proof of the theorem is complete.*

The definition of the extremal variety given in (1) suggests that it is a real projective variety in the Plücker space as it is defined by the zero set of a homogeneous polynomial of degree 4 on the Plücker coordinates. This definition is unique and well defined and the variety is non empty. In the case of theorem 4, this variety can also be written by a trace formula of an operator, i.e. $tr\left[\left(\Phi_{n}^{p}(\underline{z})^T \Phi_{n}^{p}(\underline{z})\right)^2\right]$ which is obviously invariant under orthogonal coordinate transformations of the starting space $\mathbb{R}^n$ (in these specific two cases $p \to 2$, $n \to 2n$ and $p \to n$, $n \to 2n$). There are many other instances that this trace formula can give an alternative definition for the extremal variety.

Theorem 5. *It holds that*

$$\frac{(n-p)\,p}{n}\|\underline{z}\|^4 \geq f_{n,p}(\underline{z}) \geq 0 \text{ and } M_{n,p} \leq \frac{(n-p)\,p}{n}. \tag{20}$$

Proof. *The proof follows directly from* (12) *in view of* (6)*. The proof is complete.*

Corollary 2. *In the case of theorem 4 for the specific choices of* n,p *we have that* $M_{n,p} = \frac{(n-p)p}{n}$*. Therefore, we can calculate the following characteristic numbers for these cases:*

$$M_{2n,2} = \frac{(2n-2)\,2}{2n} = \frac{2\,(n-1)}{n} \text{ and } M_{2n,n} = \frac{(2n-n)\,n}{2n} = \frac{n}{2}.$$

Remark 3. Extremal varieties are defined by the equation $f_{n,p} = M_{n,p}\|\underline{z}\|^4$. Theorem 5 states that $M_{n,p} \leq \frac{(n-p)p}{n}$. The following theorem 6 demonstrates a case where the above inequality is strict.

Theorem 6. *If* $n = 5$ *and* $p = 2$*, then*

$$M_{n,p} = 1 \neq \frac{(5-2)\,2}{5} = \frac{6}{5}.$$

Proof. *Consider* $z \in \wedge^2\mathbb{R}^5$ *and* $\underline{z}_1 = \sigma_1\underline{x}_1 \wedge \underline{y}_1 + \sigma_2\underline{x}_2 \wedge \underline{y}_2$*.*
Then,

$$\Phi_5^3(\underline{z}^*)\,\Phi_5^3(\underline{z}^*) \sim \begin{bmatrix} \sigma_1^2 & & & & 0 \\ & \sigma_1^2 & & & \\ & & \sigma_2^2 & & \\ & & & \sigma_2^2 & \\ 0 & & & & 0 \end{bmatrix}.$$

Hence,

$$\Phi_5^2(\underline{z}^*)\,\Phi_5^2(\underline{z}^*) \sim \begin{bmatrix} \sigma_2^2 & & & & 0 \\ & \sigma_2^2 & & & \\ & & \sigma_1^2 & & \\ & & & \sigma_1^2 & \\ 0 & & & & \|\underline{z}\|^2 \end{bmatrix}$$

and

$$tr\left[\left(\Phi_5^2(\underline{z})^T \Phi_5^2(\underline{z})\right)^2\right] = 2\sigma_1^4 + 2\sigma_2^4 + \|\underline{z}\|^4.$$

The extremal problem is

$$\min\ tr\left[\left(\Phi_5^2(z)^T \Phi_5^2(z)\right)^2\right]\ s.\text{t. } \sigma_2^2 + \sigma_1^2 = \|\underline{z}\|^2.$$

In view of (6)*,*

$$\Phi_5^2(\underline{z})^T \Phi_5^2(\underline{z}) + \Phi_5^3(\underline{z}^*)^T \Phi_5^3(\underline{z}^*) = \|\underline{z}\|^2 I_5,$$

we have $2\sigma_1^2 = 2\sigma_2^2 = \|z\|^2$*, and so,*

$$tr\left[\left(\Phi_5^2(z)^T \Phi_5^2(z)\right)^2\right] = 2\frac{\|z\|^4}{4} + 2\frac{\|z\|^4}{4} + \|z\|^4 = 2\|z\|^4.$$

Hence, (6) *implies that*

$$\max_{\|\underline{z}\|^2=1} f_{5,2}(z) = (5-2)\cdot 1^4 - 2\cdot 1^4 = 1 \neq \frac{(5-2)\,2}{5} = \frac{6}{5}.$$

The proof of the theorem is complete.

For the case of $\wedge^{2n}\mathbb{R}^{4n}$, the *Hodge star operator* $(*)$ is an involution, that is $(\underline{z}^*)^*=\underline{z}$, and it has two eigenvalues ± 1. The eigenspaces V_{+1} and V_{-1} of this involution are related to the extremal variety $Extr(4n,2n)$.

Theorem 7. *$Extr(4n,2n)$ contains the projectivizations of V_{+1} and V_{-1}. In the case of $Extr(4,2)$ these two projectivizations are exactly $Extr(4,2)$.*

Proof. For

$$\underline{x} \in V_{+1} \cup V_{-1}$$

we have

$$\underline{x}^* = \pm x \text{ and } \Phi_{2n}^n(\underline{x}^*)^T \Phi_{2n}^n(\underline{x}^*) = \Phi_{2n}^n(\underline{x})^T \Phi_{2n}^n(\underline{x})$$

In view of (6)*, we find*

$$\Phi_{2n}^n(\underline{x}^*)^T \Phi_{2n}^n(\underline{x}^*) + \Phi_{2n}^n(\underline{x})^T \Phi_{2n}^n(\underline{x}) = 2\Phi_{2n}^n(\underline{x})^T \Phi_{2n}^n(\underline{x}) = \|\underline{x}\|^2 I_{2n}.$$

Therefore,

$$\Phi_{2n}^n(\underline{x})^T \Phi_{2n}^n(\underline{x}) = \frac{\|\underline{x}\|^2}{2} I_{2n},$$

and so, $\underline{x} \in Extr(4n,2n)$.
In the case of $Extr(4,2)$ *the reverse is also true, as the maximum value of* $|QPR(\underline{x})|$ *is equal to* $\frac{\|\underline{x}\|^2}{2}$*, that is, if* $\underline{x}_0 = argmax|QPR(\underline{x})|$*, then*

$$\frac{\|\underline{x}_0\|^2}{2} = \max|QPR(\underline{x}_0)|,$$

which, if written in coordinates we have:

$$x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+x_6^2 = \pm 2(x_1x_6 - x_2x_5 + x_3x_4)$$
$$\Leftrightarrow (x_1 \pm x_6)^2 + (x_2 \mp x_5)^2 + (x_3 \pm x_4)^2 = 0.$$

Hence,

$$x_1 = \pm x_6,\ x_2 = \mp x_5 \text{ and } x_3 = \pm x_4$$

proving the reverse, that is

$$\underline{x}^* = \pm x \text{ and } \underline{x} \in V_{+1} \cup V_{-1}.$$

The proof is complete.

3 The Extremal Variety $Extr(6,3)$

In the following section we define representations for the extremal variety $Extr(6,3)$ and calculations for $M_{(6,3)}$.

Theorem 8. *The extremal variety* $Extr(6,3)$ *is given by the set* $\mathscr{A}$*, which is the solution set of the equations*

$$tr\left[\left(\Phi_6^3(\underline{z})^T \Phi_6^3(\underline{z})\right)^2\right] = \frac{3}{2} \text{ and } \|\underline{z}\|^2 = 1 \tag{21}$$

or equivalently the equations,

$$tr\left[\left(\Phi_6^3(\underline{z})^T \Phi_6^3(\underline{z})\right)^2\right] = \frac{3}{2}\|\underline{z}\|^4. \tag{22}$$

Proof. *In view of theorem 2, the minimum value of* $tr\left[\left(\Phi_6^3(\underline{z})^T \Phi_6^3(\underline{z})\right)^2\right]$ *is attained when all eigenvalues are equal, or equivalently when condition* (11) *holds, in which case the set of homogeneous equations*

$$tr\left[\left(\Phi_6^3(\underline{z})^T \Phi_6^3(\underline{z})\right)^2\right] = \frac{3}{2}\|\underline{z}\|^4$$

defines a projective variety in $\mathbb{P}^{\binom{6}{3}-1}(\mathbb{R}) = \mathbb{P}^{19}(\mathbb{R})$*, which is non-void, since the vectors*

$$\frac{1}{\sqrt{2}}\underline{e}_1 \wedge \underline{e}_2 \wedge \underline{e}_3 \pm \frac{1}{\sqrt{2}}\underline{e}_4 \wedge \underline{e}_5 \wedge \underline{e}_6 \tag{23}$$

belong to this variety.

(23) is a representation of a specific element of the extremal veriety $Extr(3,6)$ in the projective space $\mathbb{P}(\bigwedge^3 \mathbb{R}^6)$. For this specific projective space there is a minimal representation of its elements in terms of an orthonormal basis and 5 decomposable multivectors (see [7]) (26). The representation in (23) can be generalised to describe all elements of the extremal variety $Extr(3,6)$ utilising the representation in [7]. This is shown in theorem 9.

Remark 4. The following inequalities hold true:

$$3\|z\|^4 \geq tr\left[\left(\Phi_6^3(z)^T \Phi_6^3(z)\right)^2\right] \geq \frac{3}{2}\|z\|^4. \tag{24}$$

When the left part is an equality, then we have the Grassmann variety, whereas when the right part is equality we have the extremal variety.

Next, we define a basis representation of the elements of $Extr(6,3)$.

Theorem 9. *If $\underline{z} \in Extr(6,3)$, then there exists an orthonormal basis $\{x_1,x_2,x_3,y_1,y_2,y_3\}$ of $\mathbb{R}^6$, such that*

$$z = \frac{1}{\sqrt{2}} x_1 \wedge x_2 \wedge x_3 + \frac{1}{\sqrt{2}} y_1 \wedge y_2 \wedge y_3. \tag{25}$$

Proof. As in [5], every element $\underline{z} \in \wedge^3\mathbb{R}^6$, can be written as

$$\underline{z} = \lambda_1 x_1 \wedge x_2 \wedge x_3 + \lambda_2 y_1 \wedge y_2 \wedge x_1 + \lambda_3 y_1 \wedge y_3 \wedge x_2 + \lambda_4 y_2 \wedge y_3 \wedge x_3 + \lambda_5 y_1 \wedge y_2 \wedge y_3 \tag{26}$$

where $\{x_1,x_2,x_3,y_1,y_2,y_3\}$ is an orthonormal set and

$$\lambda_1 \geq |\lambda_2| \geq |\lambda_3| \geq |\lambda_4| \text{ and } \lambda_1^2 \geq \lambda_2^2 + \lambda_5^2.$$

In this case, the matrix $\Phi_6^3(\underline{z})^T \Phi_6^3(\underline{z})$ is similar to: (by an orthonormal change of basis)

$$G_6 = \left[\begin{array}{cc|cc|cc} \lambda_3^2+\lambda_4^2+\lambda_5^2 & -\lambda_2\lambda_5 & 0 & 0 & 0 & 0 \\ -\lambda_2\lambda_5 & \lambda_1^2+\lambda_2^2 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \lambda_2^2+\lambda_4^2+\lambda_5^2 & \lambda_3\lambda_5 & 0 & 0 \\ 0 & 0 & \lambda_3\lambda_5 & \lambda_1^2+\lambda_3^2 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \lambda_2^2+\lambda_3^2+\lambda_5^2 & -\lambda_4\lambda_5 \\ 0 & 0 & 0 & 0 & -\lambda_4\lambda_5 & \lambda_1^2+\lambda_4^2 \end{array}\right]$$

In view of Theorem 2, in this case $\underline{z} \in Extr(6,3)$,

$$G_6 = \frac{\|\underline{z}\|^2}{2} I_6.$$

Hence,

$$\lambda_2\lambda_5 = \lambda_3\lambda_5 = \lambda_4\lambda_5 = 0.$$

This can happen, if and only if, either *(a)* $\lambda_5 = 0$ *and* $\lambda_i^2 = \frac{\|z\|^2}{4}$ $i = 1,2,4$

or

(b) $\lambda_2 = \lambda_3 = \lambda_4 = 0$ *and* $\lambda_1^2 = \lambda_5^2 = \frac{\|z\|^2}{2}$

The second case (b) means (given that $\|z\| = 1$*) that*

$$\underline{z} = \frac{1}{\sqrt{2}} x_1 \wedge x_2 \wedge x_3 \pm \frac{1}{\sqrt{2}} y_1 \wedge y_2 \wedge y_3$$

and so, the result follows in this case.
The first case (a) is equivalent to

$$\underline{z} = \frac{1}{2} x_1 \wedge x_2 \wedge x_3 \pm \frac{1}{2} y_1 \wedge y_2 \wedge x_1 \pm \frac{1}{2} y_1 \wedge y_3 \wedge x_2 \pm \frac{1}{2} y_2 \wedge y_3 \wedge x_3,$$

which can be written in the form (25) *as:*

$$\begin{aligned} &\tfrac{1}{\sqrt{2}}\left(\tfrac{1}{\sqrt{2}}(-x_1+y_3) \wedge \tfrac{1}{\sqrt{2}}(-x_2-y_2) \wedge \tfrac{1}{\sqrt{2}}(x_3+y_1)\right) \\ &+\tfrac{1}{\sqrt{2}}\left(\tfrac{1}{\sqrt{2}}(x_1+y_3) \wedge \tfrac{1}{\sqrt{2}}(x_2-y_2) \wedge \tfrac{1}{\sqrt{2}}(x_3-y_1)\right). \end{aligned}$$

This proves that all elements of $Extr(6,3)$ *are of the form* (25)*. The proof of the theorem is complete.*

4 The Extremal Variety $Extr(n,p)$ when $n = pk$

In this section we define representations for the extremal variety $Extr(pk,p)$ and calculations for $M_{(pk,p)}$.

Theorem 10. *When* $n = pk$*, then*

$$M_{n,p} = \frac{(n-p)\,p}{n} = \frac{k-1}{k} \cdot p \tag{27}$$

Proof. *Consider an orthonormal basis of* $\mathbb{R}^n$ *as follows:*

$$\underline{e}_1^1, \underline{e}_2^1, \ldots, \underline{e}_p^1, \underline{e}_1^2, \underline{e}_2^2, \ldots, \underline{e}_p^2, \ldots, \underline{e}_1^k, \underline{e}_2^k, \ldots, \underline{e}_p^k$$

and consider also the multi-vector

$$\underline{z} = \frac{1}{\sqrt{k}} \underline{e}_1^1 \wedge \underline{e}_2^1 \wedge \ldots \wedge \underline{e}_p^1 + \frac{1}{\sqrt{k}} \underline{e}_1^2 \wedge \underline{e}_2^2 \wedge \ldots \wedge \underline{e}_p^2 + \ldots + \frac{1}{\sqrt{k}} \underline{e}_1^k \wedge \underline{e}_2^k \wedge \ldots \wedge \underline{e}_p^k$$

Then one may easily check that:

(a)

$$\left\langle \underline{z} \wedge \underline{e}_j^i, \underline{z} \wedge \underline{e}_j^i \right\rangle = \underbrace{\left(\frac{1}{\sqrt{k}}\right)^2 + \ldots + \left(\frac{1}{\sqrt{k}}\right)^2}_{k-1 \text{ times}} = \frac{k-1}{k} \tag{28}$$

(b)

$$\left\langle \underline{z} \wedge \underline{e}_j^i, \underline{z} \wedge \underline{e}_\lambda^\rho \right\rangle = 0, \text{ for all } (\rho, \lambda) \neq (i, j) \tag{29}$$

Hence,

$$\left(\Phi_n^p(\underline{z})^T \Phi_n^p(\underline{z}) \right)^2 = diag\left(\frac{(k-1)^2}{k^2}, \frac{(k-1)^2}{k^2}, \ldots, \frac{(k-1)^2}{k^2} \right)$$

and

$$f_{n,p}(\underline{z}) = (n-p)\,\|\underline{z}\|^n - n \cdot \frac{(k-1)^2}{k^2} = p\,(k-1) - pk \cdot \frac{(k-1)^2}{k^2} = p\,(k-1)\left(\frac{k-(k-1)}{k} \right) = p \cdot \frac{(k-1)}{k}$$

as the eigenvalues of $\Phi_n^p(\underline{z})^T \Phi_n^p(\underline{z})$ have the minimum dispersion, that is zero dispersion. Therefore,

$$\max_{\|\underline{z}\|=1} f_{n,p}(\underline{z}) = \frac{p\,(k-1)}{k}.$$

The proof is complete.

5 Applications to Control Theory

In the following section we present how Grassmann inequalities can be used as an alternative tool in the Pole placement problem, which is one of the main problems in Algebraic Control Theory.

5.1 Pole Placement Assessment

An application of Grassmann inequalities, that is discussed in this paper, is the assessment of pole placement for a specific MIMO system S with n states, p inputs, m outputs and a known pole polynomial $\underline{p}$ of degree n with static output feedback. This problem is one of the basic problems in Algebraic Control Theory. This problem may be described as a central projection problem from a projective space parametrised by the Plücker coordinates to another projective space parametrised by the pole polynomials. The desirable solutions are elements of the Grassmannian into this projective space which are depicted in the pole polynomial through central projection. The aforementioned problem can be mathematicaly formulated as follows:

$$\underline{z}^T P_S = \underline{p} \tag{30}$$

$$\underline{z}^T \in G_p(\mathbb{R}^{m+p}) \subset P(\wedge^p \mathbb{R}^{m+p}) \tag{31}$$

The set of solutions of equation (30) for given P_S and $\underline{p}$ is denoted by $\mathscr{L}_{P_S,\underline{p}}$. When $\underline{z}^T$ does not satisfy the requirement in (30) and belongs in general in the projective space, then the following Grassmann inequalities hold true:

$$m\|\underline{z}\|^4 \geq tr\left[\left(\Phi_{m+p}^p(\underline{z})^T \Phi_{m+p}^p(\underline{z}) \right)^2 \right] \geq \frac{m^2}{m+p}\|\underline{z}\|^4 \tag{32}$$

If normalised, so that $\|\underline{z}\| = 1$

$$m \geq tr\left[\left(\Phi^p_{m+p}(\underline{z})^T \Phi^p_{m+p}(\underline{z})\right)^2\right] \geq \frac{m^2}{m+p} \tag{33}$$

A solution $\underline{z} \in \mathscr{L}_{P_S,\underline{p}}$ satisfies also equation (31) if

$$tr\left[\left(\Phi^p_{m+p}(\underline{z})^T \Phi^p_{m+p}(\underline{z})\right)^2\right] = m \tag{34}$$

This extra condition (34) does not always hold. In this case, we have to assess how close is the set $\mathscr{L}_{P_S,\underline{p}}$ to have a decomposable solution. If this question is answered, then the problem of approximate pole placement can be solved. Grassmann inequalities offer a tool to assess the effectiveness of approximate pole assignability. This can be done as follows:

We consider the set:

$$\mathrm{SL}_{P_S,\underline{p}} = \left\{ \frac{z}{\|\underline{z}\|} : \underline{z} \in \mathscr{L}_{P_S,\underline{p}} \right\} \tag{35}$$

and its closure: $\overline{\mathrm{SL}}_{P_S,\underline{p}}$, which contains also the point:

$$\left\{ \frac{z}{\|\underline{z}\|} : \underline{z}P_S = \underline{0} \right\} \tag{36}$$

Consider the map:

$$\overline{\mathrm{SL}}_{P_S,\underline{p}} \xrightarrow{F} \left[\frac{m^2}{m+p}, m\right]$$

where, $F(\underline{z}) = tr\left[\left(\Phi^p_{m+p}(\underline{z})^T \Phi^p_{m+p}(\underline{z})\right)^2\right]$ and $F(\overline{\mathrm{SL}}_{P_S,\underline{p}}) = \left[a_{S,\underline{p}}, b_{S,\underline{p}}\right] \subseteq \left[\frac{m^2}{m+p}, m\right]$.

Since $\overline{\mathrm{SL}}_{P_S,\underline{p}}$ is a compact and connected spherical segment, then the index:

$$0 \leq k_{(S,\underline{p})} = \frac{m - b_{S,\underline{p}}}{m - \frac{m^2}{m+p}} \leq 1 \tag{37}$$

measures how close is this problem, for a given pair $(S, \underline{p})$, to have a decomposable solution, i.e. to find a realizable controller to assign the poles closed to the required ones. Equivalently, the index:

$$0 \leq l_{(S,\underline{p})} = \frac{b_{S,\underline{p}} - a_{S,\underline{p}}}{m - \frac{m^2}{m+p}} \leq 1 \tag{38}$$

indicates how sensitive any numerical method to solve this approximate problem is.

5.2 Pole Placement Solvability Conditions

Consider a linear system S of 2 inputs, 2 outputs and n states. We require to solve the arbitrary pole placement via static output feedback. As in [6,7,10] we have to solve simultaneously a system of nonlinear and 1 bi-linear equations:

$$\underline{x} \cdot P_S = \underline{0},\ P_S \in \mathbb{R}^{6\times n} \tag{39}$$

$$\underline{x} \wedge \underline{x} = \underline{0}, \ \underline{x} \in P(\wedge^2 \mathbb{R}^4) \tag{40}$$

in the 5-dimensional projective space $\mathbb{P}(\bigwedge^2 \mathbb{R}^4)$. We call the solution space in $\mathbb{P}(\bigwedge^2 \mathbb{R}^4)$ as Σ_0. We have to state conditions on "n" such that $\Sigma_0 \neq \varnothing$. We call Σ_+, Σ_- the solution space of the system as above with $\underline{x} \wedge \underline{x} = 0$ replaced by $\underline{x} \wedge \underline{x} = \|x\|^2$ and $\underline{x} \wedge \underline{x} = -\|x\|^2$ respectively.
The three solution spaces Σ_0, Σ_+, Σ_- correspond to the intersection of the linear space, L_S, defined by (39) with the Grassmanian $G_2(\mathbb{R}^4)$ and V_+, V_- the sub-spaces of $Extr(4,2)$ respectively. If we prove that Σ_+, $\Sigma_- \neq \varnothing$, then by the intermediate value theorem we will have $\Sigma_0 \neq \varnothing$ as well. The introduction of Σ_+, Σ_- facilitates the problem as we replace the the nonlinear equations $\underline{x} \wedge \underline{x} = 0$ with linear equations defining V_+, V_-. In the last case Σ_+, Σ_- are non-empty if

$$dim(\Sigma_+) + dim(L_S) \geq dim(P(\wedge^2 \mathbb{R}^4)) \tag{41}$$

i.e. $5-3+5-n \geq 5$ and thus, if $2 \geq n$. So, we calculate that if $n \leq 2$, then the arbitrary pole placement problem is generically solvable.

5.3 Example 2

We consider the torus $S_1 \times S_1$ embedded in $\mathbb{R}^4$. The tangent bundle of $S_1 \times S_1$ defines a map:

$$\mathscr{M} : \ S_1 \times S_1 \to G_2(\mathbb{R}^4) \tag{42}$$

and as a result a line bundle map:

$$\wedge^2 \mathscr{M} : \ S_1 \times S_1 \to P(\wedge^2 \mathbb{R}^4) \tag{43}$$

which for $(z_1, z_2) \in S_1 \times S_1$ is given by the tensor product $z_1 \otimes_{\mathbb{R}} z_2$. If we project $\wedge^2 \mathscr{M}(z_1, z_2)$ to the extremal variety $Extr(4,2) \subseteq P(\wedge^2 \mathbb{R}^4)$ we get another line bundle map which we call $Extr(\wedge^2 \mathscr{M})$, which in this case is equivalently defined by $z_1.z_2$ (the complex multiplication of z_1, z_2). So we have that

$$Extr(z_1 \otimes_{\mathbb{R}} z_2) = z_1.z_2 \tag{44}$$

6 Conclusion

We considered special types of varieties related to the Grassmann varieties called extremal varieties. These are defined via new types of inequalities called Grassmann inequalities. The Grassmannian and extremal varieties are obtained as the two extremes of the inequalities. The numbers $M_{n,p}$ arising in these inequalities were calculated for specific values of n and p. Additionally, various representations of the extremal varieties for specific values of n and p were presented demonstrating the total multiplicity property for the singular values of the Grassmann matrix. Further work has to be done towards calculating $M_{n,p}$ for all values of n and p and also towards exploring the structure of $Extr(n,p)$ for each and everyone of these cases. Finally, another step would be to relate the properties of $Extr(n,p)$ to those of the Grassmann variety $G_p(\mathbb{R}^n)$.

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