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Citation: Karcanias, N., Vafiadis, D. & Livada, M. (2021). Orientation of Implicit State Space Models and the Partitioning of Kronecker Structure. IFAC-PapersOnLine, 54(9), pp. 108-113. doi: 10.1016/j.ifacol.2021.06.069 ISSN 2405-8963 doi: 10.1016/j.ifacol.2021.06.069

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IFAC PapersOnLine 54-9 (2021) 108–113

Orientation of Implicit State Space Models and the Partitioning of Kronecker Structure

Nicos Karcanias * **Dimitris Vafiadis** * **Maria Livada** *

* *Systems and Control Research Centre*
City University of London, London EC1V 0HB, England

Abstract: Early stages modelling of processes involves issues of classification of variables into inputs, outputs and internal variables, referred to as Model Orientation Problem (MOP) which may be addressed on state space implicit, or matrix pencil descriptions. Defining orientation is equivalent to producing state space models of the regular or singular type. In this paper we consider autonomous differential descriptions defined by matrix pencils and then search for strict equivalence transformations which introduce the partitioning of the implicit vector into states and possible inputs and outputs, referred to as system orientation. The Kronecker invariant structure of the matrix pencil description is shown to be central to the solution of system orientation and this is expressed as a problem of classification and partitioning of the Kronecker invariants. It is shown that the types of Kronecker invariants characterise the nature of the system orientation solutions. Studying the conditions, under which such oriented models may be derived, as well as their structural properties in terms of the Kronecker structure, is the issue considered here.

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Keywords: state–space, singular systems, Kronecker form, matrix pencils, system orientation.

1. INTRODUCTION

The problem considered here is an integral parts of "early design" of processes, Karcanias (1994a), Karcanias (2008), Karcanias and Livada (2018) and it is considered in the context of linear systems using results based on the algebraic structure of systems and in particular the Kronecker structure of matrix pencils Gantmacher (1959). The emphasis here is on the characterisation of the desirable structure and properties of the resulting system. Aspects of modelling where all possible variables are included without their classification into control variables (inputs), command variables (outputs) and other internal variables form part of the early system design Karcanias (2008). Heuristics linked to the specific domain of applications, or methodologies such as graph analysis, Lagrangian methodology etc. may be used in specific cases for handling issues of non-redundancy in representations and classification of variables. This problem has been studied in Karcanias and Vafiadis (2002), where the classification of variables has been introduced not only for state space descriptions, but also based on autoregressive models. A version of the problem related to reduction of potential inputs and outputs has been considered in Karcanias (1994b); Karcanias and Vafiadis (2001).

A natural system description that makes no distinction as far as the role of process variables and their dependence, or independence is for the linear case the matrix pencil model (first order differential descriptions), or the general polynomial Rosenbrock (1970), or autoregressive model. In this paper, we focus on the implicit, or matrix pencil models, Karcanias and Hayton (1981), Aplevich (1991), Lewis (1991) which are defined by a matrix pencil. The classification of the implicit vector into inputs, outputs and internal variables is a problem that has been defined as system, or model orientation problem (MOP) Karcanias and Vafiadis (2002). The solutions to such problems are systems of the standard state space, or extended state space type Lewis (1991). The derivation of such oriented models, the conditions under which MOP is solvable, as well as characterisation of structural properties of solutions, when solutions exist are the main topics considered here.

The autonomous differential system defined by a matrix pencil model provides a natural description for linear Implicit models, when all system variables are included in the implicit vector, Karcanias and Hayton (1981), Eliopoulou and Karcanias (1991). The structure of such models is defined by the Kronecker invariants of the associated matrix pencil. The problem of model orientation is to define a coordinate transformation on the implicit vector that allows the identification of the transformed implicit vector into a form where internal variables and possible inputs and outputs. Such transformation will permit the linking of the transformed model (under strict equivalence, Gantmacher (1959)) to the Kronecker structure of the original pencil. In the paper we show that the Kronecker invariant structure is central to the solution of system orientation and this is expressed as a problem of classification and partitioning of the Kronecker invariants which result to system pencil models allowing the establishment of links between structural invariants and partitioning of the implicit vector. This establishes the link of the different types of Kronecker invariants and the nature of the system ori-

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10.1016/j.ifacol.2021.06.069

entation solutions. Studying the conditions, under which different types of oriented models may be derived, as well as their structural properties in terms of the Kronecker structure, are the issues considered here. Of special interest is the classification of resulting models and the study of properties such as McMillan degree, Singular/Regular, no outputs, output zeroing dynamics etc.

2. PROBLEM STATEMENT

Consider the autonomous differential system

$$(pF - G)\underline{\xi} = \underline{\mathbf{0}}, \quad p \triangleq \frac{d}{dt} \tag{1}$$

where the pencil $(pF - G) \in \mathbb{R}^{r\times k}[p]$ and $\underline{\xi} \in \mathbb{R}^k$. We are interested to find a transformation matrix $Q \in \mathbb{R}^{k\times k}$ such that

$$\underline{\xi} = Q\tilde{\underline{\xi}}, \quad \tilde{\underline{\xi}} = \left[\underline{\mathbf{x}}^t, \underline{\mathbf{u}}^t, \underline{\mathbf{y}}^t\right]^t \tag{2}$$

where $\underline{\mathbf{x}} \in \mathbb{R}^n$, $\underline{\mathbf{u}} \in \mathbb{R}^p$, $\underline{\mathbf{y}} \in \mathbb{R}^m$, $n + p + m = k$, represent a state, input, output vector respectively and there exists an $R \in \mathbb{R}^{r\times r}$ such that

$$R(pF - G)Q = \begin{bmatrix} pE - A & -B & 0 \\ -C & -D & -I \end{bmatrix} \tag{3}$$

This is referred to as the general state space orientation problem and it has as subproblems the following:

Subproblem I: Given the square pencil $(pF-G) \in \mathbb{R}^{r\times r}[p]$, find the conditions under which the differential system (1), can represent a regular, or singular system under some appropriate transformations (R, Q). □

Subproblem II: Given the singular pencil $pF - G \in \mathbb{R}^{r\times k}[p]$, find the conditions under which the differential system (1), can represent state space descriptions of the type

$$[pE - A - B] \quad \text{or} \quad \begin{bmatrix} pE - A \\ -C \end{bmatrix} \tag{4}$$

having only states and inputs or states and outputs respectively under appropriate transformations (R, Q).

□

The study of the general state space orientation problem uses the Kronecker invariants of the pencil. A basic lemma that will be discussed subsequently is stated below:

Lemma 2.1. Let $pF-G \in \mathbb{R}^{r\times k}[p]$ and assume that $pF-G$ has μ linear infinite elementary divisors. There always exist a pair of strict equivalence transformations (R, Q) such that

$$R(pF - G)Q = \begin{bmatrix} p\hat{F} - \hat{G} & 0 \\ 0 & I_\mu \end{bmatrix} \tag{5}$$

where $p\hat{F} - \hat{G}$ has all invariants of $pF - G$ apart from linear infinite elementary divisors (i.e.d.).

Proof:

There always exists a pair (R', Q') that reduces $pF - G$ (or its Laplace transform) to the form

$$R'(pF - G)Q' = \begin{bmatrix} pF_w - G_w & 0 \\ 0 & pF' - G' \end{bmatrix} \tag{6}$$

where $pF_w - G_w$ is the Wierstrass form that corresponds to all finite and infinite elementary divisors of $pF - G$, and $pF' - G'$ is a Kronecker form that corresponds to all column minimal indices (c.m.i.) and row minimal indices (r.m.i.) of $pF - G$ (see Gantmacher (1959)). The part of $pF_w - G_w$ that corresponds to infinite elementary divisors is of the form (note that the Laplace variable s is used instead of the differentiation operator p):

$$\begin{bmatrix} sH_\infty & 0 \\ 0 & I_\mu \end{bmatrix}, \quad H_\infty = diag\left\{H_{q_1}, \cdots, H_{q_\nu}\right\} \tag{7a}$$

where $q_i > 1$, $i = 1, \cdots, \nu$ are the orders of nonlinear i.e.d. of $sF - G$ and

$$H_{q_i} = \begin{bmatrix} 1 & -s & 0 & \cdots & 0 & 0 \\ 0 & 1 & -s & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -s \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \tag{7b}$$

The above transformations may be extended to the parts associated with the finite elementary divisors (f.e.d.) the c.m.i and the r.m.i of $sF - G$ thus there exist strict equivalence transformations (R, Q), Gantmacher (1959), such that

$$R(sF - G)Q = \begin{bmatrix} sH_\infty(s) & & & & \\ & sJ_f(s) & & & \\ & & L_\varepsilon(s) & & \\ & & & L\eta(s) & \\ & & & & I_\mu \end{bmatrix} = \begin{bmatrix} s\hat{F} - \hat{G} & 0 \\ 0 & I_\mu \end{bmatrix} \tag{7c}$$

□

In the statement of the above result we denote

$$\begin{cases} \mu: \text{ The number of linear i.e.d.} \\ \nu: \text{ The number of nonlinear i.e.d.} \\ \varrho: \text{ The number of f.e.d.} \\ \tilde{\pi}: \text{ The number of zero c.m.i.} \\ \tilde{\omega}: \text{ The number of zero r.m.i.} \\ \pi: \text{ The number of nonzero c.m.i.} \\ \omega: \text{ The number of nonzero r.m.i.} \end{cases} \quad (8)$$

This notation will be used subsequently. In the following we will consider the different families of matrix pencils and examine the nature of the associated different systems. We examine first the case of matrix pencils with no linear i.e.d. which will be referred to as normal pencils.

The case of normal regular pencils is considered first.

Case of Square Pencils: If $pF-G$ is normal and square we distinguish two cases ($pF-G \in \mathbb{R}^{r\times r}[p]$)

(i) $pF-G$ is regular pencil and normal
(ii) $pF-G$ is singular and normal

For the normal regular pencil case we have the following obvious result:

Proposition 2.1. If $pF-G$ is a normal regular pencil then

$$(pF-G)\underline{\xi} = \underline{0} \quad (9)$$

is an autonomous system and $\underline{\xi}$ represents a state vector $\underline{\mathbf{x}}$. In particular, if $pF-G$ has no i.e.d., then this represents a regular state space system and, if there are i.e.d., then this is a generalised state space system (singular). □

We consider now the case where $pF-G$ is square, but singular, i.e. $rank_{\mathbb{R}(p)}(pF-G) < r$. This case is part of the general state – space orientation and will be considered there.

Next we consider the case of normal nonsquare pencils $pF-G \in \mathbb{R}^{r\times k}[p]$ which have full rank. Once more, we consider two cases

(i) $pF-G$ is of full rank and $r < k$
(ii) $pF-G$ is of full rank and $r > k$

Case of nonsquare normal and flat full rank pencils: Such pencils are characterised only by c.m.i and possibly f.e.d. and i.e.d. There exist transformations (R,Q) such that

$$R(pF-G)Q = pF_k - G_k \quad (10)$$

where $pF_k - G_k$ is in Kronecker form i.e.

$$pF_k - G_k = \begin{bmatrix} \overset{\tilde{\pi}}{\longleftrightarrow} & & & \\ \vdots & L_{\varepsilon_1}(p) & & \\ 0 \cdots 0 & & \ddots & \\ \vdots & & & L_{\varepsilon_\pi}(p) \\ & & & & pM-N \end{bmatrix} \quad (11)$$

where $pM-N$ is a regular pencil in Weierstrass form. There are $\tilde{\pi}$ zero c.m.i. and π nonzero c.m.i., where L_{ε_i}, $i = 1, \cdots, \pi$ have dimensions $\varepsilon_i \times (\varepsilon_i + 1)$. Clearly by elementary column operations (10) is equivalent to

$$p\tilde{F}_k - \tilde{G}_k = \begin{bmatrix} \hat{L}_{\varepsilon_1}(p) & & & \overset{\tilde{\pi}}{\longleftrightarrow} & -\underline{\mathbf{e}}_1^{\varepsilon_1} & & \\ & \ddots & & \vdots & & \ddots & \\ & & \hat{L}_{\varepsilon_\pi}(p) & 0 \cdots 0 & & & -\underline{\mathbf{e}}_1^{\varepsilon_\pi} \\ & & & pM-N & & & 0 \end{bmatrix} \quad (12a)$$

where

$$\hat{L}_{\varepsilon_i}(p) = \begin{bmatrix} p & 1 & 0 & \cdots & 0 \\ 0 & p & 1 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & p \end{bmatrix} \in \mathbb{R}^{\varepsilon_i \times \varepsilon_i}[p], \underline{\mathbf{e}}_1^{\varepsilon_i} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^{\varepsilon_i} \quad (12b)$$

and thus we may write

$$p\tilde{F}_k - \tilde{G}_k = \left[p\tilde{E} - \tilde{D} | -\tilde{B}\right], \tilde{B} \in \mathbb{R}^{r\times\sigma}, \sigma = \tilde{\pi} + \pi \quad (12c)$$

This analysis leads to the following result:

Theorem 2.1. Given the normal pencil $pF-G \in \mathbb{R}^{r\times k}[p]$, $r < k$, $rank_{\mathbb{R}(p)}(pF-G) = r$, there always exists a pair of strict equivalence transformations (R,Q) such that

$$R(pF-G)Q = \left[p\tilde{E} - \tilde{D} | -\tilde{B}\right] = p\tilde{F}_k - \tilde{G}_k \quad (13)$$

where $p\tilde{F}_k - \tilde{G}_k$ is defined by condition (8). With the partitioned form of $p\tilde{F}_k - \tilde{G}_k$ we can always associate the vector $\underline{\tilde{\xi}}$, where

$$\left[p\tilde{E} - \tilde{D} | -\tilde{B}\right] \begin{bmatrix} \underline{\tilde{\mathbf{x}}} \\ \underline{\tilde{\mathbf{u}}} \end{bmatrix} = \underline{\mathbf{0}} \text{ and } \underline{\xi} = \begin{bmatrix} \underline{\mathbf{x}} \\ \underline{\mathbf{u}} \end{bmatrix} = Q = \begin{bmatrix} \underline{\tilde{\mathbf{x}}} \\ \underline{\tilde{\mathbf{u}}} \end{bmatrix} \quad (14)$$

and $\underline{\tilde{\mathbf{x}}} \in \mathbb{R}^{\varphi}$, $\varphi = \sum_{i=1}^{\pi} \varepsilon_i + \nu$, $\nu \triangleq$ total number of finite and infinite elementary divisors and $\underline{\tilde{\mathbf{u}}} \in \mathbb{R}^{\sigma}$, which indicates a partitioning of $\underline{\xi}$ into a state and input as indicated by condition (14). □

The proof of the above result follows from the previous analysis. Some interesting Corollaries are stated next:

Corollary 2.1. For the pencil $pF-G \in \mathbb{R}^{r\times k}[p]$, $r < k$, $rank_{\mathbb{R}(p)}(pF-G) = r$, there always exists a linear system associated with a partitioning of $\underline{\xi}$ of the type $[pE-A, -B]$ such that:

(i) B is of full rank if and only if $pF-G$ has no zero c.m.i.

(ii) The matrix E is singular if and only if $pF-G$ has i.e.d. □

Similar results to Theorem 2.1 and Corollary 2.1 may be stated by using duality for pencils $pF-G \in \mathbb{R}^{r\times k}[p]$, $r > k$, $rank_{\mathbb{R}(p)}(pF-G) = k$. In the following we consider the case of a general pencil, not necessarily normal, that is characterised by: The notation introduced in (8), where $rank_{\mathbb{R}(p)}(pF-G) \leq min(r,k)$.

There always exists a pair of strict equivalence transformations (R,Q) such that $pF-G$ is reduced to its Kronecker form of the type:

$$pF-G = \left[\begin{array}{c:c:c:c:c:c} 0 & L_{\varepsilon_1(p)} & & & & \\ & & \ddots & & & \\ & & & L_{\varepsilon_\pi}(p) & & \\ \hdashline 0_{\bar{\omega}\times\bar{\pi}} & 0 & & & & \\ \hdashline & & & L_{\eta_1}(p) & & \\ & & & & \ddots & \\ & & & & & L_{\eta_\omega}(p) \\ & & & & & & pM-N \end{array}\right] \tag{15a}$$

where $pM-N$ is in Wierstrass form. Note that by elementary transformations each of the above blocks $L_{\varepsilon_i}(p)$ may be reduced to

$$L_{\varepsilon_i}(p) \sim \left[\hat{L}_{\varepsilon_i}(p), \underline{\mathbf{e}}_1^{\varepsilon_i}\right] \tag{15b}$$

where $\hat{L}_{\varepsilon_i}(p), \underline{\mathbf{e}}_1^{\varepsilon_i}$ are defined as in (12b). Similarly for $L_{\eta_j}(p)$ we have

$$\hat{L}_{\eta_j}(p) = \left[\begin{array}{cccc} p & 0 & \cdots & 0 \\ -1 & p & \cdots & 0 \\ 0 & -1 & \ddots & 0 \\ \vdots & \vdots & \ddots & p \\ 0 & 0 & \cdots & -1 \end{array}\right] \sim \left[\begin{array}{cccc} p & 0 & \cdots & 0 \\ -1 & p & \cdots & 0 \\ 0 & -1 & \ddots & 0 \\ \vdots & \vdots & \ddots & p \\ \hdashline 0 & 0 & \cdots & 1 \end{array}\right] = \left[\begin{array}{c} \hat{L}_{\eta_j}(p) \\ \hdashline \underline{\mathbf{e}}_{\eta_j}^{\eta_j} \end{array}\right] \tag{15c}$$

where $\hat{L}_{\eta_j}(p) \in \mathbb{R}^{\eta_j\times\eta_j}[p]$ and $\underline{\mathbf{e}}_{\eta_j}^{\eta_j} \in \mathbb{R}^{\eta_j}$. Similarly for the regular part of $pM-N$ we have that each block associated with a nonlinear i.e.d. may be reduced by strict equivalence transformations to the form indicated below

$$\left[\begin{array}{c:cccc} -1 & p & 0 & \cdots & 0 & 0 \\ 0 & -1 & p & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & p \\ \hdashline 0 & 0 & 0 & \cdots & 0 & -1 \end{array}\right] \sim \left[\begin{array}{c:c} \underline{\mathbf{e}}_{q_i-1}^1 & \hat{H}_{q_i-1} \\ \hdashline 0 & \underline{\mathbf{e}}_{q_i-1}^{q_i-1\,t} \end{array}\right] \tag{15d}$$

where H_{q_i} is the standard idempotent matrix. From the (15) expressions we get the following result:

Theorem 2.2. Given the general pencil Given the normal pencil $pF-G \in \mathbb{R}^{r\times k}[p]$, $rank_{\mathbb{R}(p)}(pF-G) \leq min(r,k)$, there always exists a pair of strict equivalence transformations (R,Q) such that

$$R(pF-G)Q = p\tilde{F}_k - \tilde{G}_k \tag{16a}$$

$$p\tilde{F}_k - \tilde{G}_k = \left[\begin{array}{c:c:c:c:c:c:c} \hat{L}_\varepsilon(p) & & & & & E_1^\varepsilon & \\ & \hat{L}_\eta(p) & & & & 0 & \\ & & \hat{H}_q(p) & & & E_q^1 & \\ & & & pI-J_f & & 0 & \\ \hdashline & & & & 0_{\bar{\omega}\times\bar{\pi}} & & \\ \hdashline & E_\eta^\eta & & & & & \\ & & E_q^{q\,t} & & & & \\ \hdashline & & & & & & I_\mu \end{array}\right] \tag{16b}$$

where

$\hat{L}_\varepsilon(p) = bl-diag\{\hat{L}_{\varepsilon_i}(p), i = 1,\cdots,\pi\}$

$\hat{L}_{\varepsilon_i}(p) = pI_{\varepsilon_i} - H_{\varepsilon_i} \in \mathbb{R}^{\varepsilon_i\times\varepsilon_i}[p]$

$\hat{L}_\eta(p) = bl-diag\{\hat{L}\eta_i(p), i = 1,\cdots,\omega\}$

$\hat{L}_{\eta_i}(p) = pI_{\eta_i} - H_{\varepsilon_i}^t \in \mathbb{R}^{\eta_i\times\eta_i}[p]$

$\hat{H}_q(p) = bl-diag\{\hat{H}q_i(p), i = 1,\cdots,\nu\}$

$\hat{H}_{q_i}(p) = pH_{q_i-1}^t - I_{q_i-1} \in \mathbb{R}^{(q_i-1)\times(q_i-1)}[p]$

$E_1^\varepsilon = bl-diag\{\underline{\mathbf{e}}_i^{\varepsilon_i} : \underline{\mathbf{e}}_i^{\varepsilon_i} \in \mathbb{R}^{\varepsilon_i}, i = 1,\cdots,\pi\}$

$E_\eta^\eta = bl-diag\{\underline{\mathbf{e}}_i^{\eta_i} : \underline{\mathbf{e}}_i^{\eta_i} \in \mathbb{R}^{\eta_i}, i = 1,\cdots,\omega\}$

$E_q^1 = bl-diag\{\underline{\mathbf{e}}_1^{q_i-1} : \underline{\mathbf{e}}_1^{q_i-1} \in \mathbb{R}^{q_i-1}, i = 1,\cdots,\nu\}$

$E_q^q = bl-diag\{\underline{\mathbf{e}}_{q_i-1}^{q_i-1\,t} : \underline{\mathbf{e}}_{q_i-1}^{1\times(q_i-1)\,t} \in \mathbb{R}^{q_i-1}, i = 1,\cdots,\nu\}$

□

A short way of expressing (16b) is

$$p\tilde{F}_k - \tilde{G}_k = \left[\begin{array}{c|c|c} \hat{A} & \hat{B} & 0 \\ \hline \hat{C} & 0 & 0 \\ \hline 0 & 0 & I_\mu \end{array}\right] \tag{17}$$

Based on the above description we have the following result.

Corollary 2.2. For the pencil $pF - G$ with a description $p\tilde{F}_k - \tilde{G}_k$ we have the following properties:

(i) The square matrix $\hat{A}(p)$ has dimensions:

$$\sum_{i=1}^{\pi} \varepsilon_i + \sum_{i=1}^{\omega} \eta_i + \sum_{i=1}^{\nu} (q_i - 1) = \varphi$$

(ii) The matrix $\hat{B}$ has dimensions:

$$\varphi \times \left(\sum_{i=1}^{\pi} \varepsilon_i + \sum_{i=1}^{\nu} (q_i - 1) + \tilde{\pi} \right) = \varphi \times \vartheta$$

(iii) The matrix $\hat{C}$ has dimensions:

$$\left(\sum_{i=1}^{\omega} \eta_i + \sum_{i=1}^{\nu} (q_i - 1) + \tilde{\omega} \right) \times \varphi = \tau \times \varphi$$

□

Corollary 2.3. For the pencil $pF - G$ with a description $p\tilde{F}_k - \tilde{G}_k$ we have the following properties:

(i) If $pF - G$ has no linear i.e.d., then the last block in (17) does not exist

(ii) If $pF - G$ has no zero c.m.i., then $\hat{B}$ has full rank

$$rank(\hat{B}) = \sum_{i=1}^{\pi} \varepsilon_i + \sum_{i=1}^{\nu} (q_i - 1) \tag{18a}$$

(iii) If $pF - G$ has no zero r.m.i., then $\hat{C}$ has full rank

$$rank(\hat{C}) = \sum_{j=1}^{\omega} \eta_j + \sum_{j=1}^{\nu} (q_j - 1) \tag{18b}$$

□

The description (17) is completely defined by the Kronecker invariants of $pF - G$ and it is also a canonical form and can be referred to as the Systems Matrix Kronecker Form (SMFK).

We now address the Kronecker structure of a pencil that is in the Systems Matrix Form i.e.

$$pF - G = \left[\begin{array}{ccc} pE - A & -B & 0 \\ -C & -D & -I \end{array}\right] \tag{19}$$

that characterises the system

$$(pF - G)\underline{\xi} = \left[\begin{array}{ccc} pE - A & -B & 0 \\ -C & -D & -I \end{array}\right] \left[\begin{array}{c} \underline{\mathbf{x}} \\ \underline{\mathbf{u}} \\ \underline{\mathbf{y}} \end{array}\right] = \underline{\mathbf{0}} \tag{20}$$

It is clear that $pF - G$ may be brought by elementary column transformations (strict equivalence) to the form

$$p\tilde{F} - \tilde{G} = \left[\begin{array}{cc|c} pE - A & -B & 0 \\ \hline 0 & 0 & -I \end{array}\right] \tag{21}$$

From the above we have the result

Proposition 2.2. the Kronecker structure of the system matrix pencil (19) is characterised by linear i.e.d., where their number is equal to the number of outputs and the Kronecker invariants of

$$C(p) = [\, pE - A \,|\, -B \,] \tag{22}$$

□

Note that with $C(p)$ we may associate the system

$$[\, pE - A \;\; -B \,|\, 0 \,] \left[\begin{array}{c} \underline{\mathbf{x}} \\ \underline{\mathbf{u}} \end{array}\right] = \underline{\mathbf{0}} \tag{23a}$$

and by assuming that $\underline{\mathbf{y}} = \underline{\mathbf{0}}$ we can have

$$pF - G = \left[\begin{array}{cc|c} pE - A & -B & 0 \\ \hline 0 & 0 & -I \end{array}\right] \left[\begin{array}{c} \underline{\mathbf{x}} \\ \underline{\mathbf{u}} \\ \underline{\mathbf{y}} \end{array}\right] = \underline{\mathbf{0}} \tag{23b}$$

Given that (23b) expresses an output zeroing problem on the system defined by (22), and using elementary column operations we are led to the following result:

Proposition 2.3. Given the system defined by the pencil

$$C(p) = [\, pE - A \,|\, -B \,] \tag{22}$$

then any pencil defined by

$$S(p) = \left[\begin{array}{ccc} pE - A & -B & 0 \\ -C & -D & -I \end{array}\right] \tag{24}$$

where C, D, I are of appropriate dimensions, but otherwise arbitraty, defines an output zeroing problem for the system defined by $C(p)$ and some appropriate vector $\underline{\xi} = \left[\underline{\mathbf{x}}^t, \underline{\mathbf{u}}^t, \underline{\mathbf{y}}^t\right]^t$.

□

3. CONCLUSIONS

The problem of model orientation problem (MOP) has been considered using strict equivalence transformations. This has established the link between Kronecker invariants, partitioning of the implicit vector and the nature of the resulted state space model. It is worth noting that the strict equivalence transformation defining the solution to model orientation is not necessarily unique. Parameterising these families of solutions, investigating whether there exist physical variable solutions and characterisation of non–structural properties of the resulting solutions are

problems under current investigation. Extending the result to the classification of variables of the implicit vector of autoregressive system descriptions (polynomial matrix models) is a topic for further research and it will be treated within the framework of algebraic system theory Rosenbrock (1970), Forney (1975).

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