Free Vibration of Timoshenko-Ehrenfest Beams and Frameworks Using Frequency-Dependent Mass and Stiffness Matrices

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Abstract

The frequency-dependent mass and stiffness matrices of a Timoshenko-Ehrenfest beam are developed through extensive application of symbolic computation. Explicit algebraic expressions for the frequency dependent shape functions and each of the independent elements of the frequency-dependent mass and stiffness matrices are presented concisely. The ensuing frequency-dependent mass and stiffness matrices of the Timoshenko-Ehrenfest beam are applied with particular reference to the Wittrick-Williams algorithm to investigate the free vibration characteristics of an individual Timoshenko-Ehrenfest beam and a framework. The results are discussed with significant conclusions drawn. The proposed method retains the exactness of results like the dynamic stiffness method, but importantly, it opens the possibility of including damping in the analysis.

1. Introduction

The original idea of the frequency dependency of mass and stiffness properties of structural elements for free vibration analysis was developed by Przemieniecki [1, 2] who formulated the frequency dependent mass and stiffness matrices of a Timoshenko-Ehrenfest beam and provided series expansions of the matrices by retaining two frequency dependent terms. Przemieniecki's work was further developed by subsequent researchers [3-6] who also relied on power series expansion of the mass and stiffness matrices and truncated the series at some point. By contrast, explicit algebraic expressions for the elements of the frequency-dependent mass and stiffness matrices of a Bernoulli-Euler beam using symbolic computation were published recently [7] which circumvented the limitation of earlier research by including all terms of the infinite series implicitly. This technical brief extends the work of [7] to a Timoshenko-Ehrenfest beam through skilful application of symbolic computation [8-10]. The resulting frequency-dependent mass and stiffness matrices of the Timoshenko-Ehrenfest beam are related to its dynamic stiffness matrix which is finally utilised to compute the natural frequencies of two illustrative examples with particular reference to the Wittrick-Williams algorithm [11].

2. Frequency-Dependent Exact Shape Functions

Figure 1 shows the coordinate system and notations for a Timoshenko-Ehrenfest beam of length *L* with its node 1 located at the origin *O* and node 2 at the other end at a distance *L* from the origin.

In the usual notation, the governing differential equations of motion in free vibration in terms of the flexural displacement *w*(*x*, *t*) and bending rotation θ(*x*, *t*) are [12]

$$-\rho A\frac{\partial^2 w}{\partial t^2}+kAG\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}-\theta\right)=0 \quad (1)$$

$$-\rho I\frac{\partial^2 \theta}{\partial t^2}+EI\frac{\partial^2 \theta}{\partial x^2}+kAG\left(\frac{\partial w}{\partial x}-\theta\right) \quad (2)$$

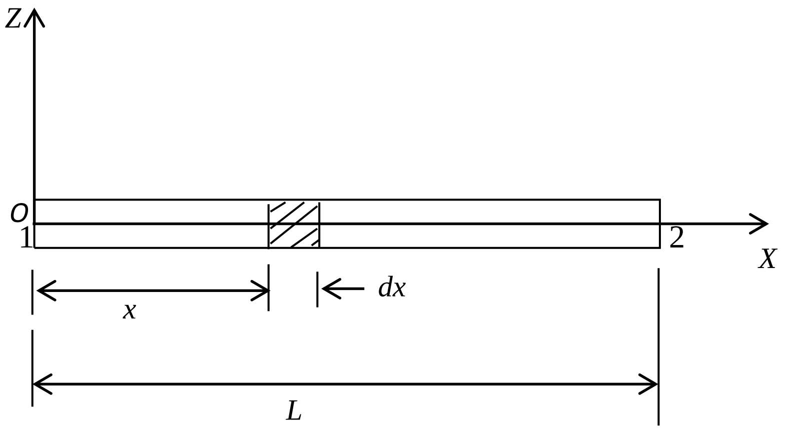

Fig. 1. Coordinate system and notation for a Timoshenko-Ehrenfest beam.

In Eqs. (1) and (2), *A* is the area, and *I* is the second moment of area of the beam cross-section, *E*, *G* and ρ are the Young's modulus, modulus of rigidity (shear modulus) and density of the beam material, respectively, and *k* is the shear correction factor (also known as shape factor) for the beam cross-section which accounts for the non-uniform shear stress distribution through the beam cross-section, approximately.

Introducing the non-dimensional length $\xi = x/L$ and assuming harmonic oscillation with circular or angular frequency ω, Eqs. (1) and (2) can be solved to give the amplitudes of flexural displacement *W* and bending rotation Θ as [12]

$$W(\xi)=A_1\cosh\alpha\xi+A_2\sinh\alpha\xi+A_3\cos\beta\xi+A_4\sin\beta\xi \quad (3)$$

$$\Theta(\xi)=B_1\sinh\alpha\xi+B_2\cosh\alpha\xi+B_3\sin\beta\xi+B_4\cos\beta\xi \quad (4)$$

where α and β are given by

$$\alpha^2 = -\frac{b^2(r^2+s^2)}{2} + \frac{b^2}{2}\sqrt{(r^2+s^2)^2 + \frac{4}{b^2}(1-b^2r^2s^2)} \tag{5}$$

$$\beta^2 = \frac{b^2(r^2+s^2)}{2} + \frac{b^2}{2}\sqrt{(r^2+s^2)^2 + \frac{4}{b^2}(1-b^2r^2s^2)} \tag{6}$$

with

$$b^2 = \frac{\rho A\omega^2L^4}{EI}; \quad r^2 = \frac{I}{AL^2}; \quad s^2 = \frac{EI}{kAGL^2} \tag{7}$$

and A_1-A_4 and B_1-B_4 are two different sets of constants, related as follows [12]

$$B_1 = \frac{k_\alpha}{L}A_1; \quad B_2 = \frac{k_\alpha}{L}A_2; \quad B_3 = -\frac{k_\beta}{L}A_3; \quad B_4 = \frac{k_\beta}{L}A_4 \tag{8}$$

where

$$k_\alpha = \left(\frac{\alpha^2+b^2s^2}{\alpha}\right); \qquad k_\beta = \left(\frac{\beta^2-b^2s^2}{\beta}\right) \tag{9}$$

By eliminating the constants A_1-A_4 and B_1-B_4 from Eqs. (3) and (4) with the help of nodal end conditions at ξ =0 and ξ = 1, respectively, the shape function **N** relating the displacements within the element ($\boldsymbol{\delta}$) to the nodal displacements ($\boldsymbol{\delta}_N$) at node 1 (W_1 and Θ_1) and at node 2 (W_2 and Θ_2) is given by

$$\boldsymbol{\delta} = \mathbf{N}\boldsymbol{\delta}_\mathbf{N} \text{ or } [\boldsymbol{\delta}] = [\mathbf{N_w} \quad \mathbf{N_\theta}]\begin{bmatrix}\boldsymbol{\delta_1}\\ \boldsymbol{\delta_2}\end{bmatrix}, \text{ i.e., } \begin{bmatrix}W\\ \Theta\end{bmatrix} = \begin{bmatrix} N_{w1} & N_{w2} & N_{w3} & N_{w4} \\ N_{\theta1} & N_{\theta2} & N_{\theta3} & N_{\theta4}\end{bmatrix}\begin{bmatrix}W_1\\ \Theta_1\\ W_2\\ \Theta_2\end{bmatrix} \tag{10}$$

Through extensive application of symbolic using REDUCE [10], explicit expressions for the shape functions (N_{w1}, N_{w2}, N_{w3}, N_{w4}) and ($N_{\theta1}$, $N_{\theta2}$, $N_{\theta3}$, $N_{\theta4}$) are derived and given below

$$\left.\begin{aligned}
N_{w1} &= (\mu_1 k_\beta \cosh\alpha\xi + \mu_2\, k_\beta \sinh\alpha\xi + \mu_3 k_\alpha \cos\beta\xi - \mu_2\, k_\alpha \sin\beta\xi)/\Delta \\
N_{w2} &= (-\mu_4 \cosh\alpha\xi + \mu_3 \sinh\alpha\xi + \mu_4 \cos\beta\xi + \mu_1 \sin\beta\xi)L/\Delta \\
N_{w3} &= (\mu_6 k_\alpha k_\beta \cosh\alpha\xi - \mu_5 k_\beta \sinh\alpha\xi - \mu_6\, k_\alpha k_\beta \cos\beta\xi + \mu_5 k_\alpha \sin\beta\xi)/\Delta \\
N_{w4} &= (\mu_7 \cosh\alpha\xi + \mu_6 k_\beta \sinh\alpha\xi - \mu_7 \cos\beta\xi - \mu_6 k_\alpha \sin\beta\xi)L/\Delta \\
N_{\theta1} &= (\mu_2 \cosh\alpha\xi + \mu_1 \sinh\alpha\xi - \mu_2 \cos\beta\xi - \mu_3 \sin\beta\xi)k_\alpha\, k_\beta/(\Delta L) \\
N_{\theta2} &= (\mu_3 k_\alpha \cosh\alpha\xi - \mu_4 k_\alpha \sinh\alpha\xi + \mu_1 k_\beta \cos\beta\xi - \mu_4 k_\beta \sin\beta\xi)/\Delta \\
N_{\theta3} &= (-\mu_5 \cosh\alpha\xi + \mu_6\, k_\alpha \sinh\alpha\xi + \mu_5 \cos\beta\xi + \mu_6 k_\beta \sin\beta\xi)k_\alpha\, k_\beta/(\Delta L) \\
N_{\theta4} &= (\mu_6 k_\alpha k_\beta \cosh\alpha\xi + \mu_7\, k_\alpha \sinh\alpha\xi - \mu_6 k_\alpha\, k_\beta \cos\beta\xi + \mu_7 k_\beta \sin\beta\xi)/\Delta
\end{aligned}\right\} \tag{11}$$

with

$$\left.\begin{aligned}
\mu_1 &= k_\alpha\left(1 - C_{h\alpha}C_\beta\right) - k_\beta S_{h\alpha}S_\beta \\
\mu_2 &= k_\alpha S_{h\alpha}C_\beta + k_\beta C_{h\alpha}S_\beta \\
\mu_3 &= k_\alpha S_{h\alpha}S_\beta + k_\beta(1 - C_{h\alpha}C_\beta) \\
\mu_4 &= k_\alpha C_{h\alpha}S_\beta - k_\beta S_{h\alpha}C_\beta \\
\mu_5 &= k_\alpha S_{h\alpha} + k_\beta S_\beta \\
\mu_6 &= C_{h\alpha} - C_\beta \\
\mu_7 &= k_\alpha S_\beta - k_\beta S_{h\alpha} \\
\Delta &= \left(k_\alpha^2 - k_\beta^2\right)S_{h\alpha}S_\beta + 2k_\alpha k_\beta(1 - C_{h\alpha}C_\beta)
\end{aligned}\right\} \tag{12}$$

where

$$C_{h\alpha} = \cosh\alpha\,; \quad S_{h\alpha} = \sinh\alpha\,; \quad C_\beta = \cos\beta\,; \quad S_\beta = \sin\beta \tag{13}$$

3. Frequency-Dependent Mass Matrix

The shape functions derived in Section 2 can now be used to derive the frequency-dependent mass matrix **m** of the Timoshenko-Ehrenfest beam as follows [2, 13]

$$\mathbf{m} = \int_V \rho\mathbf{N}^{\mathrm{T}}\mathbf{N}dv \tag{14}$$

where T denotes a transpose.

Using Eq. (10), the frequency dependent mass matrix of Eq. (14) can be expressed as

$$\mathbf{m}(\omega) = L\int_0^1 \begin{bmatrix} N_{w1} & N_{\theta1} \\ N_{w2} & N_{\theta2} \\ N_{w3} & N_{\theta3} \\ N_{w4} & N_{\theta4} \end{bmatrix} \begin{bmatrix} \rho A & 0 \\ 0 & \rho I \end{bmatrix} \begin{bmatrix} N_{w1} & N_{w2} & N_{w3} & N_{w4} \\ N_{\theta1} & N_{\theta2} & N_{\theta3} & N_{\theta4} \end{bmatrix} d\xi = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{12} & m_{22} & m_{23} & m_{24} \\ m_{13} & m_{23} & m_{33} & m_{34} \\ m_{14} & m_{24} & m_{34} & m_{44} \end{bmatrix} \tag{15}$$

Substituting Eqs. (11) and (12) into Eq. (15) and carrying out the matrix multiplications and integration algebraically with the help of REDUCE [10] yielded explicit expressions for the elements of the frequency-dependent mass matrix $\mathbf{m}(\omega)$ as follows

$$m_{11}(\omega) = m_{33}(\omega) = \rho AL \frac{\Phi_1}{2\alpha\alpha_1\beta\Delta^2} \quad (16)$$

$$m_{12}(\omega) = -m_{34}(\omega) = \rho AL \frac{\Phi_2 L}{2\alpha\alpha_1\beta\Delta^2} \quad (17)$$

$$m_{13}(\omega) = \rho AL \frac{\Phi_3}{2\alpha\alpha_1\beta\Delta^2} \quad (18)$$

$$m_{14}(\omega) = -m_{23}(\omega) = \rho AL \frac{\Phi_4 L}{2\alpha\alpha_1\beta\Delta^2} \quad (19)$$

$$m_{22}(\omega) = m_{44}(\omega) = \rho AL \frac{\Phi_5 L^2}{2\alpha\alpha_1\beta\Delta^2} \quad (20)$$

$$m_{24}(\omega) = \rho AL \frac{\Phi_6 L^2}{2\alpha\alpha_1\beta\Delta^2} \quad (21)$$

where

$$\Phi_1 = 2\alpha\alpha_1\beta_2\mu_2\mu_3 k_\alpha^2 C_\beta^2 + 4\alpha\beta\mu_2 k_\alpha k_\beta(\zeta_1 - \zeta_2 r^2 k_\alpha k_\beta) C_\beta C_{h\alpha} - \alpha\alpha_1\beta_2\delta_1 k_\alpha^2 C_\beta S_\beta + 4\alpha\beta k_\alpha k_\beta(\lambda_1 - \kappa_4 r^2 k_\alpha k_\beta) C_\beta S_{h\alpha} + 2\alpha_1\beta\beta_1\mu_1\mu_2 k_\beta^2 C_{h\alpha}^2 - 4\alpha\beta k_\alpha k_\beta(\lambda_1 r^2 k_\alpha k_\beta + \kappa_4) C_{h\alpha} S_\beta + \alpha_1\beta\beta_1\gamma_1 k_\beta^2 C_{h\alpha} S_{h\alpha} - 4\alpha\beta\mu_2 k_\alpha k_\beta(\zeta_2 + \zeta_1 r^2 k_\alpha k_\beta) S_\beta S_{h\alpha} + \Gamma_1 \quad (22)$$

$$\Phi_2 = -\alpha\alpha_1\beta_2\varepsilon_3 k_\alpha C_\beta^2 + 2\alpha\beta\{k_\beta(\eta_3 r^2 k_\alpha^2 + \kappa_3) + k_\alpha(\eta_1 r^2 k_\beta^2 + \kappa_2)\} C_\beta C_{h\alpha} + \alpha\alpha_1\beta_2\psi_4 k_\alpha C_\beta S_\beta - 2\alpha\beta\nu_1(\zeta_3 r^2 k_\alpha k_\beta + \zeta_4) C_\beta S_{h\alpha} + \alpha_1\beta\beta_1\varepsilon_3 k_\beta C_{h\alpha}^2 + 2\alpha\beta\nu_1(\zeta_4 r^2 k_\alpha k_\beta - \zeta_3) C_{h\alpha} S_\beta - \alpha_1\beta\beta_1\varepsilon_2 k_\beta C_{h\alpha} S_{h\alpha} - 2\alpha\beta\{r^2 k_\alpha k_\beta(\kappa_2 k_\alpha + \kappa_3 k_\beta) - (\eta_3 k_\alpha + \eta_1 k_\beta)\} S_\beta S_{h\alpha} + \Gamma_2 \quad (23)$$

$$\Phi_3 = -\alpha\alpha_1\beta_2\nu_3 k_\alpha^2 C_\beta^2 - 2\alpha\beta\{\mu_2\mu_6 k_\alpha k_\beta(\alpha_1 r^2 k_\alpha k_\beta - \alpha_4) - k_\alpha k_\beta(\beta_3\mu_1\mu_5 - \beta_4\mu_3\mu_5)\} C_\beta C_{h\alpha} - \alpha\alpha_1\beta_2\nu_4 k_\alpha^2 C_\beta S_\beta + 2\alpha\beta k_\alpha k_\beta(\beta_4\xi_1 + 2\beta_3\mu_2\mu_5) C_\beta S_{h\alpha} + \alpha_1\beta\beta_1\xi_2 k_\beta^2 C_{h\alpha}^2 - 2\alpha\beta k_\alpha k_\beta(\beta_3\xi_1 - 2\beta_4\mu_2\mu_5) C_{h\alpha} S_\beta + \alpha_1\beta\beta_1\phi_1 k_\beta^2 C_{h\alpha} S_{h\alpha} - 2\alpha\beta k_\alpha k_\beta\{\mu_2\mu_6(\alpha_4 r^2 k_\alpha k_\beta + \alpha_1) - \mu_5(\beta_3\mu_3 + \beta_4\mu_1)\} S_\beta S_{h\alpha} + \Gamma_3 \quad (24)$$

$$\Phi_4 = \alpha\alpha_1\beta_2\rho_4 k_\alpha C_\beta^2 - 2\alpha\beta\{\mu_6 k_\alpha k_\beta(\beta_3\mu_1 - \beta_4\mu_3) + \mu_2\mu_7(\alpha_1 r^2 k_\alpha k_\beta - \alpha_4)\} C_\beta C_{h\alpha} - \alpha\alpha_1\beta_2\xi_3 k_\alpha C_\beta S_\beta + 2\alpha\beta(\beta_4\xi_4 - 2\beta_3\mu_2\mu_6 k_\alpha k_\beta) C_\beta S_{h\alpha} + \alpha_1\beta\beta_1\phi_2 k_\beta C_{h\alpha}^2 - 2\alpha\beta(2\beta_4\mu_2\mu_6 k_\alpha k_\beta + \beta_3\xi_4) C_{h\alpha} S_\beta + \alpha_1\beta\beta_1\nu_2 k_\beta C_{h\alpha} S_{h\alpha} - 2\alpha\beta\{\mu_6 k_\alpha k_\beta(\beta_3\mu_3 + \beta_4\mu_1) + \mu_2\mu_7(\alpha_4 r^2 k_\alpha k_\beta + \alpha_1)\} S_\beta S_{h\alpha} + \Gamma_4 \quad (25)$$

$$\Phi_5 = -2\alpha\alpha_1\beta_2\mu_1\mu_4 C_\beta^2 - \alpha\alpha_1\beta_2\delta_2 C_\beta S_\beta - 4\alpha\beta\mu_4(\zeta_2 r^2 k_\alpha k_\beta - \zeta_1) C_\beta C_{h\alpha} + 4\alpha\beta(\eta_4 r^2 k_\alpha k_\beta - \kappa_1) C_\beta S_{h\alpha} - 2\alpha_1\beta\beta_1\mu_3\mu_4 C_{h\alpha}^2 + 4\alpha\beta(\kappa_1 r^2 k_\alpha k_\beta + \eta_4) C_{h\alpha} S_\beta + \alpha_1\beta\beta_1(\mu_3^2 + \mu_4^2) C_{h\alpha} S_{h\alpha} - 4\alpha\beta(\lambda_3 r^2 k_\alpha k_\beta + \psi_1) S_\beta S_{h\alpha} + \Gamma_5 \quad (26)$$

$$\Phi_6 = \alpha\alpha_1\beta_2(\mu_4\mu_6 k_\alpha + \mu_1\mu_7) C_\beta^2 + 2\alpha\beta\{\mu_4\mu_6(\alpha_1 r^2 k_\alpha k_\beta - \alpha_4) + \mu_7(\beta_3\mu_1 - \beta_4\mu_3)\} C_\beta C_{h\alpha} + \alpha\alpha_1\beta_2\phi_3 C_\beta S_\beta - 2\alpha\beta(\beta_3\xi_1 - 2\beta_4\mu_4\mu_7) C_\beta S_{h\alpha} - \alpha_1\beta\beta_1\lambda_2 C_{h\alpha}^2 - 2\alpha\beta(\beta_4\xi_1 + 2\beta_3\mu_4\mu_7) C_{h\alpha} S_\beta + \alpha_1\beta\beta_1\phi_4 C_{h\alpha} S_{h\alpha} + 2\alpha\beta\{\mu_4\mu_6(\alpha_4 r^2 k_\alpha k_\beta + \alpha_1) + \mu_7(\beta_3\mu_3 + \beta_4\mu_1)\} S_\beta S_{h\alpha} + \Gamma_6 \quad (27)$$

with

$$\alpha_1 = \alpha^2 + \beta^2;\ \ \alpha_2 = \alpha^2 - \beta^2;\ \ \alpha_3 = -\alpha k_\alpha + \beta k_\beta;\ \ \alpha_4 = \beta k_\alpha - \alpha k_\beta \tag{28}$$

$$\beta_1 = 1 + k_\alpha^2 r^2;\ \ \beta_2 = 1 - k_\beta^2 r^2;\ \ \beta_3 = \alpha k_\alpha k_\beta r^2 - \beta;\ \ \beta_4 = \beta k_\alpha k_\beta r^2 + \alpha \tag{29}$$

$$\theta_1 = 1 - k_\alpha^2 r^2;\ \ \theta_2 = 1 + k_\beta^2 r^2;\ \ \theta_3 = k_\alpha^2 - k_\beta^2;\ \ \theta_4 = \mu_3^2 + \mu_4^2 \tag{30}$$

$$\gamma_1 = \mu_1^2 + \mu_2^2;\ \ \gamma_2 = \mu_2^2 + \mu_3^2;\ \ \gamma_3 = \mu_1^2 - \mu_2^2;\ \ \gamma_4 = \mu_1^2 - \mu_3^2 \tag{31}$$

$$\delta_1 = \mu_2^2 - \mu_3^2;\ \ \delta_2 = \mu_1^2 - \mu_4^2;\ \ \delta_3 = \mu_3^2 - \mu_4^2;\ \ \delta_4 = \mu_1^2 + \mu_4^2 \tag{32}$$

$$\varepsilon_1 = \mu_1\mu_4 + \mu_2\mu_3;\ \ \varepsilon_2 = \mu_1\mu_4 - \mu_2\mu_3;\ \ \varepsilon_3 = \mu_1\mu_3 - \mu_2\mu_4;\ \ \varepsilon_4 = \mu_1\mu_2 - \mu_3\mu_4 \tag{33}$$

$$\zeta_1 = \alpha\mu_3 + \beta\mu_1;\ \ \zeta_2 = \alpha\mu_1 - \beta\mu_3;\ \ \zeta_3 = \alpha\mu_2 + \beta\mu_4;\ \ \zeta_4 = \alpha\mu_4 - \beta\mu_2 \tag{34}$$

$$\eta_1 = \alpha\mu_1^2 + \beta\mu_2\mu_4;\ \ \eta_2 = \alpha\mu_2^2 - \beta\mu_2\mu_3;\ \ \eta_3 = \beta\mu_3^2 + \alpha\mu_2\mu_4;\ \ \eta_4 = -\beta\mu_4^2 + \alpha\mu_1\mu_3 \tag{35}$$

$$\kappa_1 = \alpha\mu_4^2 + \beta\mu_1\mu_3;\ \ \kappa_2 = \alpha\mu_3^2 - \beta\mu_2\mu_4;\ \ \kappa_3 = -\beta\mu_1^2 + \alpha\mu_2\mu_4;\ \ \kappa_4 = \alpha\mu_2^2 - \beta\mu_1\mu_3 \tag{36}$$

$$\lambda_1 = \alpha\mu_1\mu_3 + \beta\mu_2^2;\ \ \lambda_2 = k_\beta\mu_4\mu_6 - \mu_3\mu_7;\ \ \lambda_3 = \alpha\mu_3\mu_4 + \beta\mu_1\mu_4;\ \ \lambda_4 = \alpha\mu_1\mu_7 - \beta\mu_3\mu_7 \tag{37}$$

$$\psi_1 = \alpha\mu_1\mu_4 - \beta\mu_3\mu_4;\ \ \psi_2 = \alpha\mu_3\mu_5 + \beta\mu_1\mu_5;\ \ \psi_3 = \alpha\mu_3\mu_6 + \beta\mu_1\mu_6;\ \ \psi_4 = \mu_1\mu_2 + \mu_3\mu_4 \tag{38}$$

$$\nu_1 = k_\alpha\mu_3 - k_\beta\mu_1;\ \ \nu_2 = k_\beta\mu_2\mu_6 + \mu_1\mu_7;\ \ \nu_3 = k_\beta\mu_2\mu_6 + \mu_3\mu_5;\ \ \nu_4 = k_\beta\mu_3\mu_6 - \mu_2\mu_5 \tag{39}$$

$$\xi_1 = k_\alpha\mu_3\mu_6 - k_\beta\mu_1\mu_6;\ \xi_2 = k_\alpha\mu_2\mu_6 - \mu_1\mu_5;\ \xi_3 = k_\alpha\mu_2\mu_6 + \mu_3\mu_7;\ \xi_4 = k_\alpha\mu_3\mu_7 - k_\beta\mu_1\mu_7 \tag{40}$$

$$\rho_1 = \alpha k_\alpha\mu_3^2 - \beta k_\beta\mu_1^2;\ \ \rho_2 = \beta k_\alpha\mu_3^2 + \alpha k_\beta\mu_1^2;\ \ \rho_3 = \beta\mu_2^2 r^2 k_\beta^2 - \mu_2\mu_3;\ \ \rho_4 = k_\alpha\mu_3\mu_6 - \mu_2\mu_7 \tag{41}$$

$$\phi_1 = k_\alpha\mu_1\mu_6 - \mu_2\mu_5;\ \ \phi_2 = k_\beta\mu_1\mu_6 + \mu_2\mu_7;\ \ \phi_3 = k_\alpha\mu_1\mu_6 - \mu_4\mu_7;\ \ \phi_4 = k_\beta\mu_3\mu_6 - \mu_4\mu_7 \tag{42}$$

and

$$\Gamma_1 = -\alpha\alpha_1\beta\gamma_4 r^2 k_\alpha^2 k_\beta^2 + 2\alpha\alpha_1\rho_3 k_\alpha^2 + 2\alpha_2\mu_2\zeta_1 r^2 k_\alpha^2 k_\beta^2 + \alpha\alpha_1\beta\gamma_2 k_\alpha^2 - 4\alpha\beta\mu_2\zeta_1 k_\alpha k_\beta + \alpha\alpha_1\beta\gamma_3 k_\beta^2 - 2\alpha_1\beta\mu_1\mu_2 k_\beta^2 \tag{43}$$

$$\Gamma_2 = -\alpha\alpha_1\beta\left(\varepsilon_1\theta_1 k_\beta + \varepsilon_4\theta_2 k_\alpha\right) + \alpha_1\mu_1\mu_3\left(\alpha\beta_2 k_\alpha - \beta\beta_1 k_\beta\right) - \alpha_2\mu_2\mu_4\left(\alpha\beta_2 k_\alpha + \beta\beta_1 k_\beta\right) - 2\alpha\beta(\rho_2 r^2 k_\alpha k_\beta + \rho_1) \tag{44}$$

$$\Gamma_3 = \alpha\alpha_1\beta\mu_6 k_\alpha k_\beta\left(\mu_1\theta_1 k_\beta - \mu_3\theta_2 k_\alpha\right) + \alpha_2\mu_2\mu_6 k_\alpha k_\beta\left(\beta\beta_1 k_\beta + \alpha\beta_2 k_\alpha\right) - \alpha\mu_3\mu_5 k_\alpha^2\left(\alpha_2 r^2 k_\beta^2 - \alpha_1\right) - \beta\mu_1\mu_5 k_\beta^2(\alpha_2 r^2 k_\alpha^2 - \alpha_1) - \alpha\alpha_1\beta\mu_2\mu_5\theta_3 - 2\alpha\beta k_\alpha k_\beta(\alpha_1\mu_2\mu_5 r^2 k_\alpha k_\beta - \psi_2) \tag{45}$$

$$\Gamma_4 = \alpha\mu_3\mu_6 k_\alpha^2\left(\alpha_2 r^2 k_\beta^2 - \alpha_1\right) + \beta\mu_1\mu_6 k_\beta^2(\alpha_2 r^2 k_\alpha^2 - \alpha_1) + \alpha\alpha_1\beta\left(\mu_1\mu_7\theta_1 k_\beta + \mu_2\mu_6\theta_3\right) + \alpha_2\mu_2\mu_7\left(\beta\beta_1 k_\beta + \alpha\beta_2 k_\alpha\right) - \alpha\alpha_1\beta\mu_3\mu_7\theta_2 k_\alpha + 2\alpha\beta k_\alpha k_\beta(\alpha_1\mu_2\mu_6 r^2 k_\alpha k_\beta - \psi_3) \tag{46}$$

$\Gamma_5 = \alpha\alpha_1\beta\delta_3 r^2 k_\alpha^2 + 2\beta\mu_3\mu_4(\alpha_1 r^2 k_\alpha^2 - \alpha_2) + \alpha\alpha_1\beta(\delta_4 r^2 k_\beta^2 + \gamma_4) - 2\alpha\mu_1\mu_4(\alpha_1 r^2 k_\beta^2 - \alpha_2) + 2\alpha\beta\mu_4(2\zeta_2 r^2 k_\alpha k_\beta +$

$\alpha_1\mu_4)$ (47)

$\Gamma_6 = -\alpha\alpha_1\beta\mu_6(\mu_3\theta_1 k_\beta + \mu_1\theta_2 k_\alpha) - \alpha_2\mu_4\mu_6(\beta\beta_1 k_\beta + \alpha\beta_2 k_\alpha) + \alpha\alpha_1\beta\mu_4\mu_7\theta_3 r^2 - \beta\mu_3\mu_7(\alpha_1 r^2 k_\alpha^2 - \alpha_2) +$

$\alpha\mu_1\mu_7(\alpha_1 r^2 k_\beta^2 - \alpha_2) - 2\alpha\beta(\lambda_4 r^2 k_\alpha k_\beta + \alpha_1\mu_4\mu_7)$ (48)

4. Frequency-Dependent Stiffness Matrix

The shape functions derived in Section 2 can likewise be used to derive the frequency-dependent stiffness matrix of the Timoshenko-Ehrenfest beam by using the following relationship [2, 13]

$$\mathbf{k}(\omega) = L\int_0^1 [(\mathbf{N}'_\theta)^{\mathrm{T}} \quad (\mathbf{N}'_\mathbf{w} - \mathbf{N}_\theta)^{\mathrm{T}}]\begin{bmatrix} EI & 0 \\ 0 & kAG \end{bmatrix}\begin{bmatrix} \mathbf{N}'_\theta \\ \mathbf{N}'_\mathbf{w} - \mathbf{N}_\theta \end{bmatrix} d\xi \tag{49}$$

Substituting $\mathbf{N}_\mathrm{w}$ and $\mathbf{N}_\theta$ from Eq. (10) into Eq. (49) gives

$$\mathbf{k}(\omega) = L\int_0^1 \begin{bmatrix} N'_{\theta_1} & N'_{w_1} - N_{\theta_1} \\ N'_{\theta_2} & N'_{w_2} - N_{\theta_2} \\ N'_{\theta_3} & N'_{w_3} - N_{\theta_3} \\ N'_{\theta_4} & N'_{w_4} - N_{\theta_4} \end{bmatrix}\begin{bmatrix} EI & 0 \\ 0 & kAG \end{bmatrix}\begin{bmatrix} N'_{\theta_1} & N'_{\theta_2} & N'_{\theta_3} & N'_{\theta_4} \\ N'_{w_1} - N_{\theta_1} & N'_{w_2} - N_{\theta_2} & N'_{w_3} - N_{\theta_3} & N'_{w_4} - N_{\theta_4} \end{bmatrix} d\xi$$

$$= \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{12} & k_{22} & k_{23} & k_{24} \\ k_{13} & k_{23} & k_{33} & k_{34} \\ k_{14} & k_{24} & k_{34} & k_{44} \end{bmatrix} \tag{50}$$

Performing all matrix operations and subsequent integration using REDUCE [10], explicit algebraic expressions of the elements of the frequency-dependent stiffness matrix $\mathbf{k}(\omega)$ are generated as follows

$$k_{11}(\omega) = k_{33}(\omega) = \left(\frac{EI}{L^3}\right)\frac{\Psi_1}{2\alpha\alpha_1\beta s^2\Delta^2} \tag{51}$$

$$k_{12}(\omega) = -k_{34}(\omega) = \left(\frac{EI}{L^3}\right)\frac{\Psi_2 L}{2\alpha\alpha_1\beta s^2\Delta^2} \tag{52}$$

$$k_{13}(\omega) = \left(\frac{EI}{L^3}\right)\frac{\Psi_3}{2\alpha\alpha_1\beta s^2\Delta^2} \tag{53}$$

$$k_{14}(\omega) = -k_{23}(\omega) = \left(\frac{EI}{L^3}\right)\frac{\Psi_4 L}{2\alpha\alpha_1\beta s^2\Delta^2} \tag{54}$$

$$k_{22}(\omega) = k_{44}(\omega) = \left(\frac{EI}{L^3}\right)\frac{\Psi_5 L^2}{2\alpha\alpha_1\beta s^2\Delta^2} \tag{55}$$

$$k_{24}(\omega) = \left(\frac{EI}{L^3}\right)\frac{\Psi_6 L^2}{2\alpha\alpha_1\beta s^2\Delta^2} \tag{56}$$

where

$$\Psi_1 = 2\alpha\alpha_1\bar{\alpha}_4\mu_2\mu_3k_\alpha^2C_\beta^2 - 4\alpha\beta\mu_2k_\alpha k_\beta(\alpha\beta\zeta_1s^2k_\alpha k_\beta + \zeta_2g_\alpha g_\beta)C_\beta C_{h\alpha} - \alpha\alpha_1\bar{\alpha}_4\delta_1k_\alpha^2C_\beta S_\beta - 4\alpha\beta k_\alpha k_\beta(\alpha\beta\lambda_1s^2k_\alpha k_\beta + \kappa_4g_\alpha g_\beta)C_\beta S_{h\alpha} + 2\alpha_1\bar{\alpha}_1\beta\mu_1\mu_2k_\beta^2C_{h\alpha}^2 + 4\alpha\beta k_\alpha k_\beta(\alpha\beta\kappa_4s^2k_\alpha k_\beta - \lambda_1g_\alpha g_\beta)C_{h\alpha}S_\beta + \alpha_1\bar{\alpha}_1\beta\gamma_1k_\beta^2C_{h\alpha}S_{h\alpha} + 4\alpha\beta\mu_2k_\alpha k_\beta(\alpha\beta\zeta_2s^2k_\alpha k_\beta - \zeta_1g_\alpha g_\beta)S_\beta S_{h\alpha} + \bar{\Gamma}_1 \quad (57)$$

$$\Psi_2 = -\alpha\alpha_1\bar{\alpha}_4\varepsilon_3k_\alpha C_\beta^2 + 2\alpha\beta(-\alpha\beta\kappa_2s^2k_\alpha^2k_\beta - \alpha\beta\kappa_3s^2k_\alpha k_\beta^2 + \eta_3g_\alpha g_\beta k_\alpha + \eta_1g_\alpha g_\beta k_\beta)C_\beta C_{h\alpha} + \alpha\alpha_1\bar{\alpha}_4\psi_4k_\alpha C_\beta S_\beta + 2\alpha\beta\{\alpha\beta\zeta_4s^2k_\alpha k_\beta(\mu_3k_\alpha - \mu_1k_\beta) - \zeta_3g_\alpha g_\beta(\mu_3k_\alpha - \mu_1k_\beta)\}C_\beta S_{h\alpha} + \alpha_1\bar{\alpha}_1\beta\varepsilon_3k_\beta C_{h\alpha}^2 + 2\alpha\beta\nu_1(\alpha\beta\zeta_3s^2k_\alpha k_\beta + \zeta_4g_\alpha g_\beta)C_{h\alpha}S_\beta - \alpha_1\bar{\alpha}_1\beta\varepsilon_2k_\beta C_{h\alpha}S_{h\alpha} - 2\alpha\beta\{\alpha\beta s^2k_\alpha k_\beta(\eta_3k_\alpha + \eta_1k_\beta) + g_\alpha g_\beta(\kappa_2k_\alpha + \kappa_3k_\beta)\}S_\beta S_{h\alpha} + \bar{\Gamma}_2 \quad (58)$$

$$\Psi_3 = -\alpha\alpha_1\bar{\alpha}_4\nu_3k_\alpha^2C_\beta^2 - 2\alpha\beta k_\alpha k_\beta\{\alpha\beta s^2k_\alpha k_\beta(\alpha_4\mu_2\mu_6 - \mu_5\zeta_1) + g_\alpha g_\beta(\alpha_1\mu_2\mu_6 - \zeta_2\mu_5)\}C_\beta C_{h\alpha} - \alpha\alpha_1\bar{\alpha}_4\nu_4k_\alpha^2C_\beta S_\beta - 2\alpha\beta k_\alpha k_\beta\{\beta\mu_6\nu_1(\alpha^2s^2k_\alpha k_\beta - g_\alpha g_\beta) - 2\alpha\bar{\beta}_3\mu_2\mu_5\}C_\beta S_{h\alpha} + \alpha_1\bar{\alpha}_1\beta\xi_2k_\beta^2C_{ha}^2 - 2\alpha\beta k_\alpha k_\beta\{\alpha\mu_6\nu_1(\beta^2s^2k_\alpha k_\beta + g_\alpha g_\beta) + 2\beta\bar{\beta}_2\mu_2\mu_5\}C_{h\alpha}S_\beta + \alpha_1\bar{\alpha}_1\beta\phi_1k_\beta^2C_{h\alpha}S_{h\alpha} + 2\alpha\beta k_\alpha k_\beta\{\alpha\beta s^2k_\alpha k_\beta(\alpha_1\mu_2\mu_6 - \mu_5\zeta_2) - g_\alpha g_\beta(\alpha_4\mu_2\mu_6 - \mu_5\zeta_1)\}S_\beta S_{h\alpha} + \bar{\Gamma}_3 \quad (59)$$

$$\Psi_4 = \alpha\alpha_1\bar{\alpha}_4\rho_4k_\alpha C_\beta^2 - 2\alpha\beta\{\mu_6k_\alpha k_\beta(\beta\bar{\beta}_2\mu_3 + \alpha\bar{\beta}_3\mu_1) + \alpha\alpha_4\beta\mu_2\mu_7s^2k_\alpha k_\beta + \alpha_1\mu_2\mu_7g_\alpha g_\beta\}C_\beta C_{h\alpha} - \alpha\alpha_1\bar{\alpha}_4\xi_3k_\alpha C_\beta S_\beta - 2\alpha\beta\{2\alpha\bar{\beta}_3\mu_2\mu_6k_\alpha k_\beta + \beta\bar{\beta}_2\mu_7(\mu_3k_\alpha - \mu_1k_\beta)\}C_\beta S_{h\alpha} + \alpha_1\bar{\alpha}_1\beta\phi_2k_\beta C_{h\alpha}^2 + 2\alpha\beta\{2\beta\bar{\beta}_2\mu_2\mu_6k_\alpha k_\beta - \alpha\bar{\beta}_3\mu_7(\mu_3k_\alpha - \mu_1k_\beta)\}C_{h\alpha}S_\beta + \alpha_1\bar{\alpha}_1\beta\nu_2k_\beta C_{h\alpha}S_{h\alpha} + 2\alpha\beta\{\mu_6k_\alpha k_\beta(\beta\bar{\beta}_2\mu_1 - \alpha\bar{\beta}_3\mu_3) + \mu_2\mu_7(\alpha\alpha_1\beta s^2k_\alpha k_\beta - \alpha_4g_\alpha g_\beta)\}S_\beta S_{h\alpha} + \bar{\Gamma}_4 \quad (60)$$

$$\Psi_5 = -2\alpha\alpha_1\bar{\alpha}_4\mu_1\mu_4C_\beta^2 - 4\alpha\beta\mu_4(\beta\bar{\beta}_2\mu_3 + \alpha\bar{\beta}_3\mu_1)C_\beta C_{h\alpha} - \alpha\alpha_1\bar{\alpha}_4\delta_2C_\beta S_\beta + 4\alpha\beta(\beta\bar{\beta}_2\mu_4^2 + \alpha\bar{\beta}_3\mu_1\mu_3)C_\beta S_{h\alpha} - 2\alpha_1\bar{\alpha}_1\beta\mu_3\mu_4C_{h\alpha}^2 + 4\alpha\beta(\alpha\bar{\beta}_3\mu_4^2 - \beta\bar{\beta}_2\mu_1\mu_3)C_{h\alpha}S_\beta + \alpha_1\bar{\alpha}_1\beta\theta_4C_{h\alpha}S_{h\alpha} + 4\alpha\beta\mu_4(\beta\bar{\beta}_2\mu_1 - \alpha\bar{\beta}_3\mu_3)S_\beta S_{h\alpha} + \bar{\Gamma}_5 \quad (61)$$

$$\Psi_6 = \alpha\alpha_1\bar{\alpha}_4(\mu_4\mu_6k_\alpha + \mu_1\mu_7)C_\beta^2 + 2\alpha\beta\{\mu_4\mu_6(\alpha\bar{\beta}_3k_\alpha - \beta\bar{\beta}_2k_\beta) + \mu_7(\alpha\beta\zeta_1s^2k_\alpha k_\beta + \zeta_2g_\alpha g_\beta)\}C_\beta C_{h\alpha} + \alpha\alpha_1\bar{\alpha}_4\phi_3C_\beta S_\beta - 2\alpha\beta\{\alpha\bar{\beta}_3\mu_6(\mu_3k_\alpha - \mu_1k_\beta) + 2\beta\bar{\beta}_2\mu_4\mu_7\}C_\beta S_{h\alpha} - \alpha_1\bar{\alpha}_1\beta\lambda_2C_{h\alpha}^2 + 2\alpha\beta\{\beta\bar{\beta}_2\mu_6(\mu_3k_\alpha - \mu_1k_\beta) - 2\alpha\bar{\beta}_3\mu_4\mu_7\}C_{h\alpha}S_\beta + \alpha_1\bar{\alpha}_1\beta\phi_4C_{h\alpha}S_{h\alpha} - 2\alpha\beta\{\mu_4\mu_6(\beta\bar{\beta}_2k_\alpha + \alpha\bar{\beta}_3k_\beta) + \mu_7(\beta\bar{\beta}_2\mu_1 - \alpha\bar{\beta}_3\mu_3)\}S_\beta S_{h\alpha} + \bar{\Gamma}_6 \quad (62)$$

with

$$\bar{\alpha}_1 = \alpha^2s^2k_\alpha^2 + g_\alpha^2;\ \ \bar{\alpha}_2 = \alpha^2s^2k_\alpha^2 - g_\alpha^2;\ \ \bar{\alpha}_3 = \beta^2s^2k_\beta^2 + g_\beta^2;\ \ \bar{\alpha}_4 = \beta^2s^2k_\beta^2 - g_\beta^2;\ g_\alpha = \alpha - k_\alpha;\ g_\beta = \beta - k_\beta \quad (63)$$

$$\bar{\beta}_1 = \alpha k_\alpha g_\beta^2 + \beta k_\beta g_\alpha^2;\ \bar{\beta}_2 = \alpha^2s^2k_\alpha k_\beta - g_\alpha g_\beta;\ \bar{\beta}_3 = \beta^2s^2k_\alpha k_\beta + g_\alpha g_\beta;\ \bar{\beta}_4 = k_\alpha^2g_\beta^2 + k_\beta^2g_\alpha^2 \quad (64)$$

and

$$\bar{\Gamma}_1 = \alpha\alpha_1\bar{\alpha}_2\beta\mu_1^2k_\beta^2 - \alpha^3\beta s^2k_\alpha^2k_\beta^2(\alpha^2\mu_2^2 - \beta^2\mu_3^2) - 2\alpha\alpha_2\beta s^2k_\alpha^2k_\beta^2(\alpha\mu_1\mu_2 - \beta\mu_2\mu_3) + \alpha\bar{\alpha}_3\beta^3\gamma_2k_\alpha^2 + \alpha^3\beta\gamma_2g_\beta^2k_\alpha^2 + 2\alpha\alpha_1\mu_2\mu_3g_\beta^2k_\alpha^2 + 4\alpha\beta g_\alpha g_\beta k_\alpha k_\beta(\alpha\mu_1\mu_2 - \beta\mu_2\mu_3) + \alpha\alpha_1\beta\mu_2^2g_\alpha^2k_\beta^2 - 2\alpha_1\beta\mu_1\mu_2g_\alpha^2k_\beta^2 \quad (65)$$

$$\bar{\Gamma}_2 = -\alpha\alpha_1\bar{\alpha}_2\beta\varepsilon_1k_\beta + \alpha\alpha_1\alpha_3\beta\mu_1\mu_3s^2k_\alpha k_\beta + \alpha\alpha_1\alpha_2\beta\mu_2\mu_4s^2k_\alpha k_\beta + 2\alpha^2\beta^2\rho_1s^2k_\alpha k_\beta - \alpha\alpha_1\bar{\alpha}_3\beta\varepsilon_4k_\alpha - \alpha_1\bar{\beta}_1\varepsilon_3 - 2\alpha\beta\eta_3g_\alpha g_\beta k_\alpha - 2\alpha\beta\eta_1g_\alpha g_\beta k_\beta \quad (66)$$

$$\bar{\Gamma}_3 = \alpha\alpha_1\bar{\alpha}_2\beta\mu_1\mu_6k_\alpha k_\beta^2 - \alpha\alpha_1\alpha_2\beta\mu_2\mu_6s^2k_\alpha^2k_\beta^2 - \alpha\alpha_1\beta^3\mu_3\mu_6s^2k_\alpha^2k_\beta^3 + \alpha\alpha_1\alpha_2\beta\mu_2\mu_5s^2k_\alpha^2k_\beta^2 + \alpha^2\alpha_2\beta\mu_1\mu_5s^2k_\alpha^2k_\beta^2 - \alpha\alpha_2\beta^2\mu_3\mu_5s^2k_\alpha^2k_\beta^2 - \alpha\alpha_1\mu_6g_\beta^2k_\alpha^2k_\beta(\mu_2 + \beta\mu_3) + 2\alpha\beta g_\alpha g_\beta k_\alpha k_\beta(\alpha_1\mu_2\mu_6 - \zeta_2\mu_5) - \alpha\alpha_1\beta\bar{\beta}_4\mu_2\mu_5 - \alpha\alpha_1\mu_3\mu_5g_\beta^2k_\alpha^2 - \alpha_1\beta\xi_2g_\alpha^2k_\beta^2 \quad (67)$$

$$\bar{\Gamma}_4 = -\alpha\alpha_1\alpha_2\beta\mu_2\mu_6s^2k_\alpha^2k_\beta^2 - \alpha\alpha_2\beta\mu_6\zeta_2s^2k_\alpha^2k_\beta^2 + \alpha\alpha_1\bar{\alpha}_2\beta\mu_1\mu_7k_\beta - \alpha\alpha_1\alpha_2\beta\mu_2\mu_7s^2k_\alpha k_\beta + \alpha\alpha_1\beta\bar{\beta}_4\mu_2\mu_6 + \alpha\alpha_1\mu_3\mu_6g_\beta^2k_\alpha^2 - \alpha\alpha_1\beta^3\mu_3\mu_7s^2k_\alpha k_\beta^2 + 2\alpha\beta g_\alpha g_\beta(\mu_6\zeta_2k_\alpha k_\beta + \alpha_1\mu_2\mu_7) - \alpha\alpha_1\mu_7g_\beta^2k_\alpha(\mu_2 + \beta\mu_3) - \alpha_1\beta\phi_2g_\alpha^2k_\beta \quad (68)$$

$$\bar{\Gamma}_5 = -\alpha^3\alpha_1\beta\delta_3s^2k_\alpha^2 + 2\alpha_1\bar{\alpha}_1\beta\mu_3\mu_4 + 4\alpha\beta\mu_4(\alpha\beta\zeta_1s^2k_\alpha k_\beta + \zeta_2g_\alpha g_\beta) + \alpha\alpha_1\beta^3\delta_4s^2k_\beta^2 + 2\alpha\alpha_1\bar{\alpha}_4\mu_1\mu_4 + \alpha\alpha_1\beta\delta_4g_\beta^2 + \alpha\alpha_1\beta\delta_3g_\alpha^2 \quad (69)$$

$$\bar{\Gamma}_6 = -\alpha\alpha_1\bar{\alpha}_2\beta\mu_3\mu_6k_\beta + \alpha\alpha_1\alpha_2\beta\mu_4\mu_6s^2k_\alpha k_\beta - \alpha\alpha_1\bar{\alpha}_2\beta\mu_4\mu_7 - \alpha_1\bar{\alpha}_1\beta\mu_3\mu_7 - \alpha\alpha_1\bar{\alpha}_3\beta\mu_1\mu_6k_\alpha - 2\alpha\beta\mu_7(\alpha\beta\zeta_1s^2k_\alpha k_\beta + \zeta_2g_\alpha g_\beta) + \alpha_1\bar{\beta}_1\mu_4\mu_6 - 2\alpha\alpha_1\beta\mu_4\mu_6g_\alpha g_\beta - \alpha\alpha_1\bar{\alpha}_4\mu_1\mu_7 - \alpha\alpha_1\bar{\alpha}_3\beta\mu_4\mu_7 \quad (70)$$

and all the rest of the terms have already been defined before.

5. Application of the Frequency-Dependent Mass and Stiffness Matrices

The frequency-dependent mass and stiffness matrices in axial motion already exist in the literature [2, 10] which can be combined with the above theory for free vibration analysis of frame structures consisting of Timoshenko-Ehrenfest beam elements. In the expressions for the mass and stiffness elements given above, the frequency ω must not be set to exactly zero, but a small value e.g., $\omega = 10^{-4}$ or 10^{-5} rad/s can be used for most of the practical problems so that any numerical overflow or ill-conditioning can be avoided and yet the degenerate case giving the frequency-independent mass and stiffness matrices that are generally used in the FEM can be obtained.

The frequency-dependent mass and stiffness matrices for the Timoshenko-Ehrenfest beam $\mathbf{m}(\omega)$ and $\mathbf{k}(\omega)$ derived in Sections 3 and 4 can be related to its dynamic stiffness matrix $\mathbf{k_D}(\omega)$ by the following relationship [10].

$$\mathbf{k_D}(\omega) = \mathbf{k}(\omega) - \omega^2\mathbf{m}(\omega) \quad (71)$$

Equation (71) can now be applied either to an individual Timoshenko-Ehrenfest beam or to a plane or space frame when investigating the free vibration characteristics. For frameworks, the frequency-dependent mass and stiffness matrices in axial motion [2, 10] must be incorporated into the corresponding flexural mass and stiffness matrices given by Eqs. (16)-(21) and Eqs. (51)-(56), and then for all individual elements in the frame, the $\mathbf{k}(\omega)$ and $\mathbf{m}(\omega)$ matrices should be assembled using conventional transformation based on the orientation of the elements, as commonly employed in FEM. Once the overall frequency-dependent mass and stiffness matrices

$\mathbf{K}(\omega)$ and $\mathbf{M}(\omega)$ and hence the overall dynamic stiffness matrix $\mathbf{K_D}(\omega)$ of the final structure is constructed the natural frequencies and mode shapes follow from the application of the Wittrick-Williams algorithm [11].

6. Numerical Results and Discussions

The theory developed above was applied to a wide range of problems and it was ascertained that the frequency-dependent mass and stiffness matrices give the same results that can be obtained by the conventional dynamic stiffness method which uses a single matrix containing both the mass and stiffness properties, rather than using separate mass and stiffness matrices. Two illustrative examples are presented here. The first example focuses on the free vibration analysis of a Timoshenko-Ehrenfest beam with simple support-simple support (S-S), clamped-free (C-F) and clamped-simple support (C-S) boundary conditions for which Chen et al. [13] quoted numerical results up to eight significant figures for the natural frequencies although Huang [12] gave exact expressions for the frequency equation of each of the above boundary conditions from which results to any desired accuracy can be computed. The data used were taken from Chen et al. [13] which are: Young's modulus E = 210 GPa, density ρ = 7850 kg/m^3, length L = 0.4m, width b = 0.02m, depth or height h = 0.08m, shear correction factor k = 2/3 and Poisson's ratio ν =1/3. The shear modulus G was calculated by relating it to E through the Poisson's ratio ν to give $G = 3E/8$ as reported in [13]. Using these data, the beam parameters were worked out as bending rigidity EI = 1.792×10^5 Nm^2, shear rigidity kAG= 8.4×10^7 N, and mass per unit length ρA=12.56 kg/m. The first nine natural frequencies of the beam with S-S, C-F and C-S boundary conditions computed using the present theory are shown in Table 1. (To be consistent with [13], the axial natural frequencies are not included in the analysis.)

The natural frequencies shown in Table 1 agreed with the exact natural frequencies quoted in [13]. These results were further checked using the exact frequency equations reported by Huang [12] and again complete agreement was found. Since symbolic algebra has been used, such a high degree of accuracy in results is certainly important so that interested readers can check their own theory or computer programs in future research.

The second example is a plane frame shown in Fig. 2 which is that of [14]. Each element of the frame has the same uniform geometrical, cross sectional and material properties and the data used in the analysis for each element are: EI = 4.0×10^6 Nm^2, EA = 8.0×10^8 N, kAG = 2.0×10^8 N, ρA = 30 kg/m, ν = 1/3, k = 2/3. Using the present theory, nine natural frequencies were computed within the low, medium, and high frequency ranges which are 1st, 3rd, 4th, 50th, 70th, 90th, 150th, 200th and 250th natural frequencies. The results are shown in Table 2. For comparison purposes the corresponding natural frequencies using the Bernoulli-Euler theory-based dynamic stiffness theory [14] were also computed and shown in Table 2. As expected, the Bernoulli-Euler theory gives poor results for higher order natural frequencies.

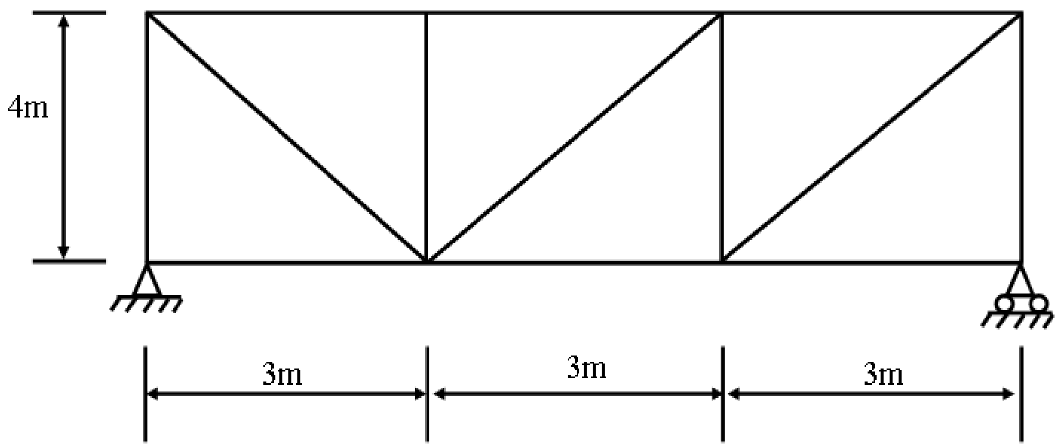

Fig. 2. A plane frame comprising Timoshenko-Ehrenfest beams for free vibration analysis using frequency-dependent mass and stiffness matrices.

Table 1. Natural frequencies of a Timoshenko-Ehrenfest beam for S-S, C-F and C-S boundary conditions.

Natural frequency no (i)	ω_i (rad/s)		
	Boundary conditions		
	S-S	C-F	C-S
1	6838.8336	2529.4927	9741.9469
2	23190.827	13279.905	26150.251
3	43443.493	31044.791	45545.510
4	64939.186	50825.834	66211.994
5	86710.899	71565.047	87376.643
6	108431.34	91994.824	108601.14
7	111981.29	110975.98	114295.44
8	120647.24	119244.57	128739.40
9	130003.61	131606.52	131610.63

Table 2: Natural frequencies of a plane frame

Natural Frequency Number (i)	Natural Frequency ω_i (rad/s)	
	Timoshenko-Ehrenfest theory	Bernoulli-Euler theory
1	222.29030	224.75854
3	263.74414	267.38209
4	318.72268	322.91808
50	3553.5494	3776.0281
70	5209.2737	5870.9164
90	6967.8612	8175.5854
150	13518.984	17167.288
200	19153.400	25695.288
250	24771.930	35091.591

7. Conclusions

Explicit algebraic expressions for the elements of the frequency-dependent mass and stiffness matrices of a Timoshenko-Ehrenfest beam have been derived by extensive application of symbolic computation. The investigation allows exact free vibration analysis of Timoshenko-Ehrenfest beams and frameworks, but importantly it paves the way for the inclusion of damping in exact free vibration analysis of such structures. Numerical examples by applying the Wittrick-Williams algorithm as solution technique are given to demonstrate the capability of the theory.

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