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Time-dependent $\mathcal{C}$-operators as Lewis-Riesenfeld invariants in non-Hermitian theories

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ABSTRACT

$\mathcal{C}$-operators were introduced as involution operators in non-Hermitian theories that commute with the time-independent Hamiltonians and the parity/time-reversal operator. Here we propose a definition for time-dependent $\mathcal{C}(t)$-operators and demonstrate that for a particular signature they may be expanded in terms of time-dependent biorthonormal left and right eigenvectors of Lewis-Riesenfeld invariants. The vanishing commutation relation between the $\mathcal{C}$-operator and the Hamiltonian in the time-independent case is replaced by the Lewis-Riesenfeld equation in the time-dependent scenario. Thus, $\mathcal{C}(t)$-operators are always Lewis-Riesenfeld invariants, whereas the inverse is only true in certain circumstances. We demonstrate the working of the generalities for a non-Hermitian two-level matrix Hamiltonian. We show that solutions for $\mathcal{C}(t)$ and the time-dependent metric operator may be found that hold in all three $\mathcal{PT}$-regimes, i.e., the $\mathcal{PT}$-regime, the spontaneously broken $\mathcal{PT}$-regime and at the exceptional point.

1. Introduction

By definition, quasi-Hermitian Hamiltonian systems are characterised by the intertwining relation $H^\dagger\mathcal{O} = \mathcal{O}H$ satisfied by their non-Hermitian Hamiltonian H [1–3]. The operator $\mathcal{O}$ could be the indefinite parity operator $\mathcal{P}$ or the positive definite metric operator ρ. The $\mathcal{C}$-operator is then defined [4] as the multiplicative factor that converts the indefinite operator into a definite one, $\mathcal{PC} = \rho$. Time-independent $\mathcal{C}$-operators have been constructed and utilized in obtaining well-defined positive definite metric operators in many non-Hermitian theories, such as for instance theories with complex cubic interaction terms [5], spin chain lattice models [6], quantum field theories [7–9], non-Hermitian versions of quantum electrodynamics [10], in semi-classical approximations [11] and in non-Hermitian theories modelling superconductivity with $\mathcal{PT}$-symmetric Cooper pairing symmetry [12]. In principle, many more models for which the metric operator ρ has been constructed by directly solving the quasi-Hermiticity relation could be listed here as the parity operator $\mathcal{P}$ can usually be identified trivially so that one immediately obtains the $\mathcal{C}$-operators from $\mathcal{C} = \mathcal{P}\rho$. For instance in [13–18] exact solutions for the metric in systems with underlying infinite dimensional Hilbert spaces have been found and the corresponding $\mathcal{P}$-operators were identified. We stress here that the $\mathcal{C}$-operator should not be confused with the charge conjugation operator that maps particles to anti-particles and vice versa in quantum field theoretical systems, even though some of their general properties are shared.

While the scheme of pseudo/quasi-Hermitian $\mathcal{PT}$-symmetric systems [3,19–21] has been generalised to explicitly time-dependent Hamiltonian systems [22–43], time-dependent versions of $\mathcal{C}$-operators have not been discussed so far. We introduce here a definition for time-dependent $\mathcal{C}$-operators and study their properties. We find that the factorisation property $\mathcal{C}(t) = \mathcal{P}\rho(t)$, its involutory nature, $\mathcal{C}^2(t) = \mathbb{I}$, and the vanishing commutation relation with the $\mathcal{PT}$-operator are preserved in the time-dependent setting. However, crucially we will argue here that time-dependent $\mathcal{C}(t)$-operators no longer commute with the Hamiltonian. We will demonstrate that instead they have to satisfy the Lewis-Riesenfeld equation [44]. This means that time-dependent $\mathcal{C}(t)$-operators are in fact Lewis-Riesenfeld invariants. The latter have been devised originally to facilitate the construction of solutions to the time-dependent Schrödinger equation, as they reduce the former to an eigenvalue problem. For time-dependent non-Hermitian systems they were also found to be extremely useful, since they can be utilised to reduce the time-dependent Dyson equation, that is a first order differential equation, to as a much simpler similarity relation [34,36,42,45–47].

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Our manuscript is organised as follows: In section 2 we first recall the key features of time-independent versions of $\mathcal{C}$-operators, propose a definition for their time-dependent versions $\mathcal{C}(t)$ and study their general properties in particular we demonstrate how they are related to Lewis-Riesenfeld invariants. In section 3 we illustrate the working of the general formulae for a two level system, first in a time-independent setting, a scenario with time-independent Hamiltonian, but time-dependent metric and finally in a fully fledged time-dependent scenario. Our conclusions are stated in section 4.

2. $\mathcal{C}$-operators and invariants

2.1. Time-independent $\mathcal{C}$-operators

Before proposing a definition for time-dependent $\mathcal{C}$-operators we recall their definition in the time-independent case and briefly discuss the role they play. When dealing with non-Hermitian Hamiltonians, $H \neq H^\dagger$, with a discrete spectrum standard orthonormal basis have to be replaced with biorthonormal basis comprised of their left and right eigenvectors, $|\Phi\rangle$ and $|\Psi\rangle$, respectively. They are defined by the right and left eigenvalue equations

$$H|\Psi_n\rangle = E_n|\Psi_n\rangle, \qquad H^\dagger|\Phi_n\rangle = E_n|\Phi_n\rangle, \qquad n \in \mathbb{N}, \tag{2.1}$$

respectively. These eigenvectors are biorthonormal to each other and are complete [49,50]

$$\langle\Phi_n|\Psi_m\rangle = \langle\Psi_n|\Phi_m\rangle = \delta_{nm}, \qquad \sum_n |\Phi_n\rangle\langle\Psi_n| = \sum_n |\Psi_n\rangle\langle\Phi_n| = \mathbb{I}. \tag{2.2}$$

The biorthonormal basis can be employed to introduce a new so-called $\mathcal{C}$-operator [4]

$$\mathcal{C} := \sum_n s_n|\Psi_n\rangle\langle\Phi_n|, \qquad s_n = \pm 1, \tag{2.3}$$

with the set $\{s_1, \ldots, s_n\}$ being the signature. In turn, this operator equals the product of the indefinite parity operator $\mathcal{P}$ and the positive-definite Hermitian metric ρ

$$\mathcal{C} = \mathcal{P}\rho, \tag{2.4}$$

with properties

$$H^\dagger\rho = \rho H, \quad \rho^\dagger = \rho \qquad \text{and} \qquad H^\dagger\mathcal{P} = \mathcal{P}H, \quad \mathcal{P}^2 = \mathbb{I}. \tag{2.5}$$

The Hermiticity of $\rho(t)$ implies that $\mathcal{C}$ is pseudo-Hermitian with regard to the adjoint action of the parity operator $\mathcal{P}$. This follows by conjugating (2.4)

$$\mathcal{C}^\dagger = \rho^\dagger\mathcal{P} = \mathcal{P}\mathcal{P}\rho\mathcal{P} = \mathcal{P}\mathcal{C}\mathcal{P}. \tag{2.6}$$

As a consequence of the previous relations one obtains the three constraints

$$\mathcal{C}^2 = \mathbb{I}, \qquad [\mathcal{P}\mathcal{T}, \mathcal{C}] = 0, \qquad [H, \mathcal{C}] = 0. \tag{2.7}$$

The first property simply follows from the defining relation (2.3) together with the orthonormality relation for the left and right eigenvectors (2.2). The second equation in (2.7) follows by multiplying (2.6) from the right and left by the time-reversal operator $\mathcal{T}$. The anti-linear time-reversal operator with general property $\mathcal{T}^2 = \mathbb{I}$ is here simply taken as the complex conjugation. Using the fact that $\mathcal{C}^\dagger\mathcal{T} = \mathcal{T}\mathcal{C}$, which follows right away from the defining relation (2.3), we deduce $[\mathcal{P}\mathcal{T}, \mathcal{C}] = 0$ when $\mathcal{C}$ is symmetric. In case $\mathcal{C}$ is not symmetric this relation must be read as $\mathcal{P}\mathcal{T}\mathcal{C} = \mathcal{C}^\intercal\mathcal{P}\mathcal{T}$, which is similar to how the commutation relation for $\mathcal{P}\mathcal{T}$-symmetric Hamiltonians is changed from $[\mathcal{P}\mathcal{T}, H]$ to $\mathcal{P}\mathcal{T}H = H^\intercal\mathcal{P}\mathcal{T}$ when H is not symmetric. The third relation follows by multiplying the first equation in (2.5) from the right by $\mathcal{P}$ together with the last two relations in (2.5) and (2.4).

$$\mathcal{P}H^\dagger\rho = \mathcal{P}\rho H \quad \Rightarrow \quad \mathcal{P}H^\dagger\mathcal{P}\mathcal{C} = \mathcal{P}\mathcal{P}\mathcal{C}H \quad \Rightarrow \quad H\mathcal{C} = \mathcal{C}H. \tag{2.8}$$

Instead of employing the left and right eigenvectors to obtain an explicit expression for the $\mathcal{C}$-operator, it was suggested in [5] that one may also solve the equation (2.7) to construct it algebraically. Especially for systems with an infinite dimensional Hilbert space this is advantageous, as in the definition (2.3) one is even in the rare cases of exactly solvable models left with an infinite sum. See [5–12] for examples where this algebraic approach has been carried out in one form or another.

While ρ and $\mathcal{P}$ satisfy the same equations, the metric operator is positive definite, whereas the parity operator is in general not positive definite. Thus we may also read equation (2.4) as $\mathcal{P}\mathcal{C} = \rho$, so that the $\mathcal{C}$-operator can be interpreted as the operator that by multiplication converts the indefinite operator $\mathcal{P}$ into a positive definite operator ρ, which in turn serves to define the ρ-inner product $\langle\Psi|\tilde{\Psi}\rangle_\rho := \langle\Psi|\rho\tilde{\Psi}\rangle$. Moreover, the metric can be decomposed as $\rho = \eta^\dagger\eta$ where η is the Dyson map that adjointly maps the non-Hermitian Hamiltonian H to a Hermitian one $h = h^\dagger = \eta H\eta^{-1}$, by relating the eigenstates $\eta|\Psi\rangle = |\phi\rangle$ of the corresponding eigenvalue equation $h|\phi\rangle = E|\phi\rangle$.

2.2. Time-dependent $\mathcal{C}$-operators versus Lewis-Riesenfeld invariants

Let us now see how the above is generalised and the $\mathcal{C}$-operator is naturally introduced for the time-dependent setting. For explicitly time-dependent systems we propose the definition

$$\mathcal{C}(t) := \sum_n s_n |\Psi_n(t)\rangle\langle\Phi_n(t)|, \qquad s_n = \pm 1, \tag{2.9}$$

and assume that the states $|\Psi_n(t)\rangle$ and $|\Phi_n(t)\rangle$ satisfy the time-dependent Schrödinger equations

$$i\hbar\partial_t|\Psi_n(t)\rangle = H(t)|\Psi_n(t)\rangle, \qquad \text{and} \qquad i\hbar\partial_t|\Phi_n(t)\rangle = H^\dagger(t)|\Phi_n(t)\rangle. \tag{2.10}$$

It should be stressed that in the time-dependent case the Hamiltonian is distinct from the energy operator, see e.g. [27,31]. The set of states $|\Psi_n(t)\rangle$ and $|\Phi_n(t)\rangle$ comprise a biorthonormal basis whose existence is guaranteed by means of their close relation to the invariants as explained after equation (2.15).

We will now argue that the first two properties in (2.7) also hold in the time-dependent case, whereas the third constraint needs modification as is seen by differentiating (2.9) with respect to time

$$i\hbar\partial_t\mathcal{C}(t) = i\hbar\sum_n s_n\left\{\partial_t|\Psi_n(t)\rangle\langle\Phi_n(t)| + |\Psi_n(t)\rangle\partial_t\langle\Phi_n(t)|\right\} = [H, \mathcal{C}(t)], \tag{2.11}$$

where we simply used the time-dependent Schrödinger equations (2.10).

We notice that (2.11) is identical to the non-Hermitian extension for the Lewis-Riesenfeld invariants $I_H(t)$ [34,36,45,46,48], i.e. the Lewis-Riesenfeld equation [44]

$$i\hbar\partial_t I_H(t) = [H, I_H(t)], \tag{2.12}$$

suggesting therefore a possible relation between the two. In general, the main advantage of employing these invariants is that they reduce the time-dependent Schrödinger equation to an eigenvalue problem in which time simply plays the role of a standard parameter. The invariants satisfy the eigenvalue equations

$$I_H(t)\left|\Psi^I(t)\right\rangle = \Lambda\left|\Psi^I(t)\right\rangle, \quad |\Psi(t)\rangle = e^{i\alpha_I(t)}\left|\Psi^I(t)\right\rangle, \quad \partial_t\Lambda = 0, \tag{2.13}$$

where the real phase $\alpha_I(t)$ relating the solution of the time-dependent Schrodinger equation to eigenstates of the Lewis-Riesenfeld invariants is determined from

$$\dot{\alpha}_I = \left\langle\psi^I(t)\right| i\partial_t - h(t)/\hbar \left|\psi^I(t)\right\rangle = \left\langle\Psi^I(t)\right| i\partial_t - H(t)/\hbar \left|\Psi^I(t)\right\rangle_\rho. \tag{2.14}$$

Here $\left|\psi^I(t)\right\rangle$ denote the eigenstates of the Hermitian invariant $I_h(t)$ and the inner product has to be changed for the calculation in the non-Hermitian framework with the new metric ρ as explained after equation (2.8). For more details on (2.14) see equation (3.3) in [36] and also [34,45,46]. Since the Lewis-Riesenfeld phase for a Hermitian Hamiltonian is real and identical to the one of the non-Hermitian Hamiltonian, also the latter has to be real.

Since for the non-Hermitian Hamiltonian H the invariant $I_H(t)$ must also be non-Hermitian, it possesses in addition to the set of right eigenvectors in (2.13) a set of left eigenvectors $\left|\Phi^I(t)\right\rangle$ with

$$I_H^\dagger(t)\left|\Phi^I(t)\right\rangle = \Lambda\left|\Phi^I(t)\right\rangle, \quad |\Phi(t)\rangle = e^{i\alpha_I(t)}\left|\Phi^I(t)\right\rangle. \tag{2.15}$$

We have indicated here that the Lewis-Riesenfeld-phases for the left and right eigenequations are identical. As this is an important property to be used in what follows, we briefly establish this. Differentiating the second equation in (2.15) with respect to time and using the TDSE (2.10) we derive

$$\dot{\alpha}_I^L\left|\Phi^I(t)\right\rangle = \left(i\partial_t - H^\dagger/\hbar\right)\left|\Phi^I(t)\right\rangle, \tag{2.16}$$

which upon using $\left|\Phi^I(t)\right\rangle = \rho\left|\Psi^I(t)\right\rangle$ becomes

$$\dot{\alpha}_I^L\rho\left|\Psi^I(t)\right\rangle = \left(i\rho\partial_t + i\partial_t\rho - H^\dagger\rho/\hbar\right)\left|\Psi^I(t)\right\rangle. \tag{2.17}$$

Multiplying (2.17) now from the left by $\langle\Psi^I(t)|$ and using time-dependent quasi-Hermiticity relation $H^\dagger\rho - \rho H = i\hbar\partial_t\rho(t)$, see [22–43], to eliminate $H^\dagger$, we precisely obtain (2.14) with $\dot{\alpha}_I^L = \dot{\alpha}_I$.

We assume now that the invariant $I_H(t)$ possesses a discrete spectrum with eigenvalues Λ_n and eigenfunctions $\left|\Psi_n^I(t)\right\rangle$. This discretisation is then inherited by the solutions of the time-dependent Schrödinger equations in (2.10) via (2.13) and an immediate consequence of these relations is that we can also expand the time-dependent $\mathcal{C}$-operator in terms of the left and right eigenstates of the Lewis-Riesenfeld invariant

$$\mathcal{C}(t) := \sum_n s_n|\Psi_n^I(t)\rangle\langle\Phi_n^I(t)|, \qquad s_n = \pm 1. \tag{2.18}$$

The phases $\alpha_I(t)$ from (2.13) and (2.14) cancel out in (2.3) so that the solutions of the time-dependent Schrödinger equation are replaced by the eigenstates of the Lewis-Riesenfeld invariants. This means all time-dependent $\mathcal{C}$-operators as defined in (2.18) are also Lewis-Riesenfeld invariants. The inverse does not always hold and one may easily construct invariants that are not $\mathcal{C}$-operators. However, noting that we may expand the invariants as

$$I_H(t) := \sum_n \Lambda_n |\Psi_n^I(t)\rangle\langle\Phi_n^I(t)|, \tag{2.19}$$

we can obviously achieve the equality $\mathcal{C}(t) = I_H(t)$, if we can tune the eigenvalues for the invariant such that $\{\Lambda_1, \ldots, \Lambda_n\} = \{s_1, \ldots, s_n\}$. This may be achieved by enforcing further constraints on the invariants so that they become $\mathcal{C}$-operators. Let us now see how this is achieved. First we compute

$$I_H^2(t) = \sum_{n,m} \Lambda_n \Lambda_m |\Psi_n^I(t)\rangle\langle\Phi_n^I(t)|\Psi_m^I(t)\rangle\langle\Phi_m^I(t)| = \sum_n \Lambda_n^2 |\Psi_n^I(t)\rangle\langle\Phi_n^I(t)|, \tag{2.20}$$

which equals $\mathbb{I}$ when $\Lambda_n^2 = 1$, that is when the eigenvalues of the Lewis-Riesenfeld invariants become identical to the signatures of the $\mathcal{C}$-operator.

We keep here $\mathcal{T}$ and $\mathcal{P}$ as time-independent also in the time-dependent case, being defined in the same manner as stated above. We may then establish how the time-dependent $\mathcal{C}$-operator relates to the time-dependent metric and demonstrate that the relation (2.4) directly generalises to the time-dependent scenario as

$$\mathcal{C}(t) = \mathcal{P}\rho(t). \tag{2.21}$$

This follows by solving (2.21) for the metric and differentiating the resulting equation with respect to t. In this way we compute

$$i\hbar\partial_t\rho(t) = \mathcal{P}i\hbar\partial_t\mathcal{C}(t) \tag{2.22}$$

$$= \mathcal{P}H\mathcal{C}(t) - \mathcal{P}\mathcal{C}(t)H \tag{2.23}$$

$$= H^\dagger\mathcal{P}\mathcal{C}(t) - \mathcal{P}\mathcal{C}(t)H \tag{2.24}$$

$$= H^\dagger\rho(t) - \rho(t)H, \tag{2.25}$$

which is precisely the time-dependent quasi-Hermiticity relation that generalises (2.5), see [22–43]. Thus (2.21) holds consistently for the definition (2.9) together with the properties of the parity operator as stated in (2.5). Using now (2.21) instead of (2.4) we can use the same argument as after equation (2.7) to establish that $\mathcal{C}$ commutes with the $\mathcal{PT}$ operator.

In summary, the three constraining relations (2.7) underlying the algebraic approach in the time-independent case and the factorisation of the $\mathcal{C}$-operator into the parity operator and the metric (2.4) have to be replaced by

$$\mathcal{C}^2(t) = \mathbb{I}, \qquad [\mathcal{PT}, \mathcal{C}(t)] = 0, \qquad [H, \mathcal{C}(t)] = i\hbar\partial_t\mathcal{C}(t), \qquad \mathcal{C}(t) = \mathcal{P}\rho(t) \tag{2.26}$$

in the time-dependent scenario. Let us now demonstrate that one may indeed construct a time-dependent metric operator $\rho(t)$ by starting with the construction of a Lewis-Riesenfeld invariant $I_H(t)$, which is then constraint to become a time-dependent $\mathcal{C}(t)$-operator that contains $\rho(t)$ as a factor.

3. A two level system

In order to illustrate the consistent working of the proposed formulae above we will present a simple worked out example i) in the time-independent case, ii) for time-independent Hamiltonian and time-dependent metric and iii) for the fully time-dependent scenario.

As many techniques have been devised to construct Lewis-Riesenfeld invariants [34,36,42,45–47], their construction will be our starting point in finding time-dependent $\mathcal{C}(t)$-operators from which we subsequently compute the metric operators by means of (2.4). We briefly explain one general method by considering the most general two level matrix Hamiltonian and invariant expanded in terms of Pauli matrices $\sigma_{x,y,z}$ as

$$H(t) = h_0(t)\mathbb{I} + h_1(t)\sigma_x + h_2(t)\sigma_y + h_3(t)\sigma_z, \quad I_H(t) = \iota_0(t)\mathbb{I} + \iota_1(t)\sigma_x + \iota_2(t)\sigma_y + \iota_3(t)\sigma_z, \tag{3.1}$$

with time-dependent coefficient functions $h_i(t), \iota_i(t) \in \mathbb{C}$, $i = 0, 1, 2, 3$. When substituting these expressions into the Lewis-Riesenfeld equation (2.12), it reduces to

$$\partial_t\iota_0 = 0, \qquad \partial_t\vec{\iota} = \frac{2}{\hbar}M\vec{\iota}, \quad \text{with} \quad M_{ij} = -\epsilon_{ijk}h_k = \begin{pmatrix} 0 & -h_3 & h_2 \\ h_3 & 0 & -h_1 \\ -h_2 & h_1 & 0 \end{pmatrix}_{ij}. \tag{3.2}$$

The general solution of (3.2) is then

$$\vec{\iota}(t) = T\exp\left[\frac{2}{\hbar}\int_0^t M(s)ds\right]\vec{\iota}(0) \tag{3.3}$$

$$= \sum_{n=0}^{\infty}\left(\frac{2}{\hbar}\right)^n T\left[\int_0^t M(t_1)dt_1\int_0^{t_1} M(t_2)dt_2\ldots\int_0^{t_{n-1}} M(t_{n-1})dt_n\right]\vec{\iota}(0), \tag{3.4}$$

with $t > t_1 > \ldots t_n > 0$ and T denoting the time-ordering operator. In what follows we will consider some special cases of this solution for which the time-ordered exponential can be computed explicitly.

3.1. A time-independent system

For reference purposes we start with a well-known two level example for which all of the above quantities can be calculated easily

$$H = -\frac{1}{2}\left(\omega\mathbb{I} + \lambda\sigma_z + i\kappa\sigma_x\right) = -\frac{1}{2}\begin{pmatrix} \omega+\lambda & i\kappa \\ i\kappa & \omega-\lambda \end{pmatrix}, \qquad \omega, \lambda, \kappa \in \mathbb{R}. \tag{3.5}$$

The eigenenergies together with the left and right eigenvectors are obtained as

$$E_\pm = -\frac{1}{2}\omega \pm \frac{1}{2}\sqrt{\lambda^2-\kappa^2}, \quad |\Psi_\pm\rangle = \frac{1}{\sqrt{N_\pm}}\begin{pmatrix} i\left(-\lambda \pm \sqrt{\lambda^2-\kappa^2}\right) \\ \kappa \end{pmatrix}, \quad |\Phi_\pm\rangle = \mp\mathcal{P}|\Psi_\pm\rangle. \tag{3.6}$$

The parity operator is identified from (2.5) as $\mathcal{P} = \sigma_z$ and the normalisation constant $N_\pm = 2(\lambda\sqrt{\lambda^2-\kappa^2} \pm \kappa^2 \mp \lambda^2)$ is determined by (2.2). For $|\lambda| > |\kappa|$ these states are $\mathcal{PT}$ symmetric, $\mathcal{PT}|\Psi_\pm\rangle = -|\Psi_\pm\rangle$ when we identify $\mathcal{PT} := \sigma_z\tau$ with τ being a complex conjugation. The Hamiltonian respects the same symmetry, i.e., $[H, \mathcal{PT}] = 0$.

With signature $\{+,-\}$, the $\mathcal{C}$-operator is directly computed from the defining relation (2.3)

$$\mathcal{C} = \frac{1}{\sqrt{\lambda^2-\kappa^2}}\begin{pmatrix} \lambda & i\kappa \\ i\kappa & -\lambda \end{pmatrix}. \tag{3.7}$$

Thus the metric is $\rho = \mathcal{PC}$, from which one obtains the Dyson map η by solving $\rho = \eta^\dagger\eta$. We can also obtain η directly by defining it in term eigenvectors of H as column vectors $\eta = \{\psi_+, \psi_-\}^\intercal$, since the adjoint action of this operator will always diagonalise the Hamiltonian with $h = \mathrm{diag}\{E_+, E_-\}$.

3.2. The metric picture

When introducing a time-dependence, an interesting new option emerges that does not exist in the time-independent case, which allows to include the time-dependence into the metric $\rho(t)$, while keeping the Hamiltonian still time-independent. We now demonstrate that the $\mathcal{C}$-operator resulting in this case is in fact identical to the Lewis-Riesenfeld invariant. For this purpose we present a simple solution to the Lewis-Riesenfeld equation (2.11) for the time-independent Hamiltonian (3.5) resulting from the general scheme (3.1)-(3.4). Setting in equation (3.1) the time-dependent coefficient functions to constants $h_0(t) = -1/2\omega$, $h_1(t) = -i/2\kappa$, $h_2(t) = 0$, $h_3(t) = -1/2\lambda$, we obtain the time-independent Hamiltonian in (3.5). Next we construct from (3.3) invariants that are all of the form

$$I_H(t) = \frac{1}{\xi}\begin{pmatrix} -\delta & \gamma_+ \\ \gamma_- & \delta \end{pmatrix}, \tag{3.8}$$

with $\det[I_H(t)] = -1$ by making different choices for the initial conditions. Choosing $\iota_0(t) = 0$ and $\vec{\iota}(0) = \{i\sqrt{2}\kappa/\xi, i, \sqrt{2}\lambda/\xi\}$ the evaluation of (3.3) leads to an invariant (3.8) in the $\mathcal{PT}$-symmetric regime with

$$\mathcal{PT}: \xi = \sqrt{\lambda^2-\kappa^2},\ \delta = -\sqrt{2}\lambda - \kappa\sin(\xi t),\ \gamma_\pm = \pm\xi\cos(\xi t) + i\left[\sqrt{2}\kappa + \lambda\sin(\xi t)\right]. \tag{3.9}$$

Taking instead $\iota_0(t) = 0$ and $\vec{\iota}(0) = \{i(\sqrt{2}\lambda - \kappa)/\xi, 0, (\sqrt{2}\kappa - \lambda)/\xi\}$ we obtain a solution for the spontaneously broken $\mathcal{PT}$-regime with

$$b\mathcal{PT}: \xi = \sqrt{\kappa^2-\lambda^2},\ \delta = \lambda - \sqrt{2}\kappa\cosh(\xi t),\ \gamma_\pm = \pm\sqrt{2}\xi\sinh(\xi t) + i\left[\sqrt{2}\lambda\cosh(\xi t) - \kappa\right]. \tag{3.10}$$

Setting $\lambda = \kappa$ with initial conditions $\iota_0(t) = 0$ and $\vec{\iota}(0) = \{0, i, \sqrt{2}\}$ the solution (3.3) becomes valid at the exceptional point with

$$\mathrm{EP}: \xi = 1,\ \delta = -\frac{\kappa^2 t^2}{\sqrt{2}} - \kappa t - \sqrt{2},\ \gamma_\pm = \pm\left(1 + \sqrt{2}\kappa t\right) + i\left[\frac{\kappa^2 t^2}{\sqrt{2}} + \kappa t\right]. \tag{3.11}$$

The most general solution involves four integration constants corresponding to the initial conditions, which have been chosen here conveniently to keep our expressions simple.

Since the invariant $I_H(t)$ is associated to a non-Hermitian Hamiltonian, it is by (2.12) itself also non-Hermitian, and therefore possesses a set of left and right eigenvectors that form a biorthonormal basis

$$|\Psi_\pm\rangle = \frac{1}{\sqrt{N_\pm}}\begin{pmatrix} \delta \mp \xi \\ -\gamma_- \end{pmatrix}, \quad |\Phi_\pm\rangle = \mathcal{P}|\Psi_\pm\rangle, \quad I_H(t)|\Psi_\pm\rangle = \pm|\Psi_\pm\rangle, \quad I_H^\dagger(t)|\Phi_\pm\rangle = \pm|\Phi_\pm\rangle, \tag{3.12}$$

satisfying the biorthonormality relations (2.2), with normalisation constants $N_\pm = 2(\xi^2 \mp \delta\xi)$. Using the definition (2.3) of the time-dependent $\mathcal{C}$-operator, we obtain

$$\mathcal{C}(t) = |\Psi_+\rangle\langle\Phi_+| - |\Psi_-\rangle\langle\Phi_-| = I_H(t). \tag{3.13}$$

We verify that the pseudo-Hermitian relation (2.6) holds for this operator. For the signatures $\{\pm, \pm\}$ we simply obtain the trivial solutions $\mathcal{C}(t) = I_H(t) = \pm\mathbb{I}$ and evidently for $\{-, +\}$ equations (3.13) is just multiplied by -1. Thus, we have found $\mathcal{C}$-operators identical to the Lewis-Riesenfeld invariant in all three $\mathcal{PT}$-regimes.

From relation (2.21) we easily obtain a metric in each of the three regimes

$$\rho_{b\mathcal{PT}}(t) = \frac{1}{\xi}\begin{pmatrix} \sqrt{2}\kappa\cosh(\xi t) - \lambda & \sqrt{2}\xi\sinh(\xi t) - i\left[\kappa + \sqrt{2}\lambda\cosh(\xi t)\right] \\ \sqrt{2}\xi\sinh(\xi t) + i\left[\kappa + \sqrt{2}\lambda\cosh(\xi t)\right] & \sqrt{2}\kappa\cosh(\xi t) - \lambda \end{pmatrix},$$
$$\rho_{\mathcal{PT}}(t) = \frac{1}{\xi}\begin{pmatrix} \sqrt{2}\lambda + \kappa\sin(\xi t) & \xi\cos(\xi t) + i\left[\sqrt{2}\kappa + \lambda\sin(\xi t)\right] \\ \xi\cos(\xi t) - i\left[\sqrt{2}\kappa + \lambda\sin(\xi t)\right] & \sqrt{2}\lambda + \kappa\sin(\xi t) \end{pmatrix}, \tag{3.14}$$
$$\rho_{\mathrm{EP}}(t) = \begin{pmatrix} \frac{\kappa^2 t^2}{\sqrt{2}} + \kappa t + \sqrt{2} & 1 + \sqrt{2}\kappa t + \frac{1}{2}i\kappa t\left(\sqrt{2}\kappa t + 2\right) \\ 1 + \sqrt{2}\kappa t - \frac{1}{2}i\kappa t\left(\sqrt{2}\kappa t + 2\right) & \frac{\kappa^2 t^2}{\sqrt{2}} + \kappa t + \sqrt{2} \end{pmatrix},$$

which are Hermitian as long as $\xi \in \mathbb{R}$, which is guaranteed for the appropriate choices of ξ in the different regimes. Moreover, ρ is positive definite in these regimes with $\det \rho(t) = 1$. Thus in any of the three $\mathcal{PT}$-regimes one may also find a well-defined metric from a given Lewis-Riesenfeld invariant/$\mathcal{C}$-operator and parity operator. However, we notice that we can not cross over smoothly from one $\mathcal{PT}$-regime to the other, so that the three solutions in (3.14) correspond to different theories. As we will see in the next section this is not always the case and one can also construct solutions for *one* theory that is defined in all three regimes.

3.3. Fully time-dependent scenario

By slightly modifying (3.5), let us now consider the explicitly time-dependent Hamiltonian

$$H = -\frac{1}{2}\left(\omega\mathbb{I} + \lambda\tau(t)\sigma_z + i\kappa\tau(t)\sigma_x\right) = -\frac{1}{2}\begin{pmatrix} \omega + \lambda\tau(t) & i\kappa\tau(t) \\ i\kappa\tau(t) & \omega - \lambda\tau(t) \end{pmatrix}, \qquad \omega, \lambda, \kappa, \tau(t) \in \mathbb{R}. \tag{3.15}$$

Setting in equation (3.1) the time-dependent coefficient functions to $h_0(t) = -1/2\omega$, $h_1(t) = -i/2\kappa\tau(t)$, $h_2(t) = 0$, $h_3(t) = -1/2\lambda\tau(t)$, $\iota_0(t) = 0$ we obtain the time-dependent Hamiltonian in (3.15). Furthermore when choosing the initial condition in (3.3) to $\vec{\iota}(0) = \{0, 0, 1\}$, we find a simple invariant $I_H(t)$ of the same general form as in (3.8) with re-defined entries

$$\xi = \kappa^2 - \lambda^2, \;\; \delta = \lambda^2 - \kappa^2\cosh[\mu(t)], \;\; \gamma_\pm = \pm\kappa\sqrt{\xi}\sinh[\mu(t)] + i\kappa\lambda\{\cosh[\mu(t)] - 1\}, \tag{3.16}$$

where $\mu(t) = \sqrt{\xi}\int^t \tau(s)\,ds$. Unlike the invariants (3.9), (3.10) and (3.11), this invariant is meaningful in all three $\mathcal{PT}$-regimes. Moreover, the expressions for the left and right eigenstates (3.12) still hold with (3.16). Hence we have also in this case the relation (3.13) and the $\mathcal{C}$-operator is identical to the Lewis-Riesenfeld invariant.

A Hermitian positive definite metric operator valid in all three different $\mathcal{PT}$-regimes is then obtained from (2.21) as

$$\rho(t) = \frac{1}{\xi}\begin{pmatrix} \kappa^2\cosh[\mu(t)] - \lambda^2 & \kappa\sqrt{\xi}\sinh[\mu(t))] + i\kappa\lambda\{\cosh[\mu(t)] - 1\} \\ \kappa\sqrt{\xi}\sinh[\mu(t)] - i\kappa\lambda\{\cosh[\mu(t)] - 1\} & \kappa^2\cosh[\mu(t)] - \lambda^2 \end{pmatrix}. \tag{3.17}$$

The left and right limit to the exceptional point, that is approaching either from the $\mathcal{PT}$-symmetric or the spontaneously broken regime, can be carried out in a smooth manner

$$\lim_{\lambda\uparrow\kappa}\rho(t) = \lim_{\lambda\downarrow\kappa}\rho(t) = \begin{pmatrix} 1 + \frac{\kappa^2\tilde{\mu}^2}{2} & \kappa\tilde{\mu} + i\frac{\kappa^2\tilde{\mu}^2}{2} \\ \kappa\tilde{\mu} - i\frac{\kappa^2\tilde{\mu}^2}{2} & 1 + \frac{\kappa^2\tilde{\mu}^2}{2} \end{pmatrix}, \tag{3.18}$$

where $\tilde{\mu}(t) = \int^t \tau(s)ds$.

Notice that these solutions also hold for $\tau(t) = 1$, in which case we obtain a solution for $\rho(t)$ in the metric picture with time-independent Hamiltonian valid in all three $\mathcal{PT}$-regimes.

The metric ρ is indeed positive definite as its two eigenvalues are positive for all times and all values of κ, λ in all $\mathcal{PT}$-regimes. Fig. 1 displays the eigenvalues as functions of time for a specific choice of the time-dependent function $\tau(t)$ in the Hamiltonian. We observe some standard degeneracy when the metric reduces to the identity matrix at specific values of time $t = \pi/2 + n\pi$ in all regime, and in the $\mathcal{PT}$-symmetric regimes also at $t = \arccos(2n\pi/\sqrt{\lambda^2 - \kappa^2})$ with $n \in \mathbb{Z}$.

4. Conclusions

We have shown that time-dependent $\mathcal{C}$-operators can be defined in terms of solutions of the left and right Schrödinger equation (2.9). However, as these solutions are related to the left and right eigenstates of the non-Hermitian Lewis-Riesenfeld invariants simply by phase factors, they can also be expanded in terms of the latter as the phase factors of the left and right solutions cancel each other out. The three key properties of time-dependent $\mathcal{C}$-operators (2.7) that serve as the set up for an algebraic approach to find $\mathcal{C}$, have to be replaced by (2.26) for their time-dependent versions. Since the last equation in (2.26) is the Lewis-Riesenfeld equation it implies that time-dependent $\mathcal{C}$-operators are always identical to Lewis-Riesenfeld invariants. The inverse only holds when the eigenvalues of the Lewis-Riesenfeld invariants are identical to the signature of the $\mathcal{C}$-operators. Thus, there are plenty of Lewis-Riesenfeld invariants $I_H(t)$ that are

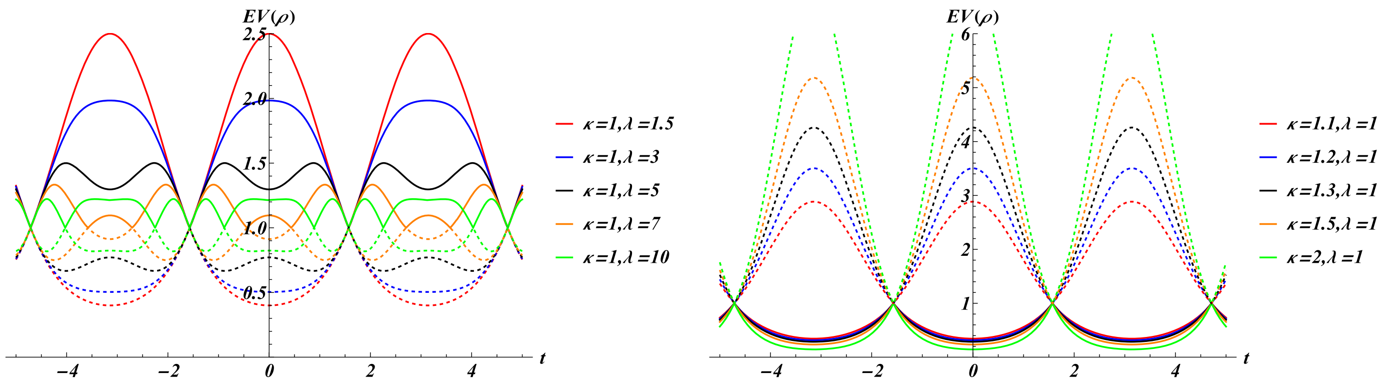

Fig. 1. Positive-definiteness of the metric $\rho(t)$: Eigenvalues of $\rho(t)$ in (3.17) as function of time for different values of κ and λ for $\tau(t) = \sin(t)$ in the $\mathcal{PT}$-symmetric regime, panel (a), and spontaneously broken $\mathcal{PT}$-regime, panel (b). Two eigenvalues corresponding to the same values of κ and λ are of the same colour drawn as dashed and solid lines. (For interpretation of the colours in the figure, the reader is referred to the web version of this article.)

not $\mathcal{C}(t)$-operators and only when we also impose the first two equations in (2.26) does the equality $I_H(t) = \mathcal{C}(t)$ hold. Subsequently the $\mathcal{C}(t)$-operator can be utilised to construct time-dependent metric operators $\rho(t)$. The well-known non-uniqueness of the metric is reflected here in the different choices of the signature in the defining relation of the $\mathcal{C}$-operator (2.3). We have not rigorously proven that the latter is the only way to define this operator.

$$I_H \;\rightarrow\; I_H^2 = \mathbb{I}, [\mathcal{PT}, I_H] = 0 \;\Rightarrow\; \mathcal{C}(t) \;\Rightarrow \rho(t) = \mathcal{PC}(t). \tag{4.1}$$

Thus our proposed scheme is yet another option to construct $\rho(t)$. Previously, [34–43], it was shown how to solve directly the coupled system of coupled equations arising from the time-dependent quasi-Hermiticity or time-dependent Dyson equation. As this is rather involved, alternative schemes where proposed. For instance, one may utilize Lewis-Riesenfeld invariants I_H for the non-Hermitian Hamiltonian and subsequently simply solve $I_H = \eta(t) I_h \eta^{-1}$ for $\eta(t)$, similar to the time-independent case, see e.g. [45,46,38–43]. This approach can even be pursued if the eigenvalue equation for the invariant can not be solved exactly [41]. Here we propose yet another approach as summarized in (4.1).

Once more we have seen that in a time-dependent setting the spontaneously broken regime is mended [33], as one can find positive definite metric operators in all three $\mathcal{PT}$-regimes. While some of the solutions only hold in a particular regime and break down at their boundaries (3.14), there exist also solutions (3.17) that can be continued smoothly across all three $\mathcal{PT}$-regimes. In turn this allows for very interesting applications and new types of behaviour as for instance seen for the entropy in the three different regimes, see [37,39,47].

CRediT authorship contribution statement

Andreas Fring: Writing – review & editing, Writing – original draft, Supervision, Methodology, Investigation, Formal analysis, Conceptualization. **Takanobu Taira:** Writing – review & editing, Writing – original draft, Methodology, Funding acquisition, Formal analysis, Conceptualization. **Rebecca Tenney:** Writing – review & editing, Writing – original draft, Methodology, Investigation, Formal analysis, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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References

[1] J. Dieudonné, Quasi-Hermitian operators, in: Proceedings of the International Symposium on Linear Spaces, Jerusalem, 1960, Pergamon, Oxford, 1961, pp. 115–122.
[2] F.G. Scholtz, H.B. Geyer, F. Hahne, Quasi-Hermitian operators in quantum mechanics and the variational principle, Ann. Phys. 213 (1992) 74–101.
[3] A. Mostafazadeh, Pseudo-Hermitian representation of quantum mechanics, Int. J. Geom. Methods Mod. Phys. 7 (2010) 1191–1306.
[4] C.M. Bender, D.C. Brody, H.F. Jones, Complex extension of quantum mechanics, Phys. Rev. Lett. 89 (2002) 270401.
[5] C.M. Bender, D.C. Brody, H.F. Jones, Extension of PT-symmetric quantum mechanics to quantum field theory with cubic interaction, Phys. Rev. D 70 (2004) 025001.
[6] O.A. Castro-Alvaredo, A. Fring, A spin chain model with non-Hermitian interaction: the Ising quantum spin chain in an imaginary field, J. Phys. A 42 (2009) 465211.
[7] C.M. Bender, D.C. Brody, H.F. Jones, Scalar quantum field theory with a complex cubic interaction, Phys. Rev. Lett. 93 (2004) 251601.
[8] C.M. Bender, H.F. Jones, R.J. Rivers, Dual PT-symmetric quantum field theories, Phys. Lett. B 625 (2005) 333–340.
[9] A. Fring, T. Taira, Goldstone bosons in different PT-regimes of non-Hermitian scalar quantum field theories, Nucl. Phys. B 950 (2020) 114834.
[10] C.M. Bender, I. Cavero-Pelaez, K.A. Milton, K.V. Shajesh, PT-symmetric quantum electrodynamics, Phys. Lett. B 613 (2005) 97–104.

[11] C.M. Bender, H.F. Jones, Semiclassical calculation of the C operator in PT-symmetric quantum mechanics, Phys. Lett. A 328 (2004) 102–109.
[12] A. Ghatak, T. Das, Theory of superconductivity with non-Hermitian and parity-time reversal symmetric Cooper pairing symmetry, Phys. Rev. B 97 (2018) 014512.
[13] H.F. Jones, J. Mateo, An equivalent Hermitian Hamiltonian for the non-Hermitian $-x^4$ potential, Phys. Rev. D 73 (2006) 085002.
[14] C. Figueira de Morisson Faria, A. Fring, Isospectral Hamiltonians from Moyal products, Czechoslov. J. Phys. 56 (2006) 899–908.
[15] A. Mostafazadeh, Metric operators for quasi-Hermitian Hamiltonians and symmetries of equivalent Hermitian Hamiltonians, J. Phys. A 41 (2008) 055304.
[16] P.E.G. Assis, A. Fring, Metrics and isospectral partners for the most generic cubic $\mathcal{PT}$-symmetric non-Hermitian Hamiltonian, J. Phys. A 41 (2008) 244001.
[17] D.P. Musumbu, H.B. Geyer, W.D. Heiss, Choice of a metric for the non-Hermitian oscillator, J. Phys. A 40 (2007) F75–F80.
[18] P.E.G. Assis, A. Fring, Non-Hermitian Hamiltonians of Lie algebraic type, J. Phys. A 42 (2009) 015203.
[19] C.M. Bender, S. Boettcher, Real spectra in non-Hermitian Hamiltonians having PT symmetry, Phys. Rev. Lett. 80 (1998) 5243–5246.
[20] F. Bagarello, J.-P. Gazeau, F.H. Szafraniec, M. Znojil, Non-selfadjoint Operators in Quantum Physics: Mathematical Aspects, John Wiley & Sons, 2015.
[21] C.M. Bender, P.E. Dorey, C. Dunning, A. Fring, D.W. Hook, H.F. Jones, S. Kuzhel, G. Levai, R. Tateo, PT symmetry, in: Quantum and Classical Physics, World Scientific, Singapore, 2019.
[22] C. Figueira de Morisson Faria, A. Fring, Time evolution of non-Hermitian Hamiltonian systems, J. Phys. A 39 (2006) 9269–9289.
[23] C. Figueira de Morisson Faria, A. Fring, Non-Hermitian Hamiltonians with real eigenvalues coupled to electric fields: from the time-independent to the time dependent quantum mechanical formulation, Laser Phys. 17 (2007) 424–437.
[24] A. Mostafazadeh, Time-dependent pseudo-Hermitian Hamiltonians defining a unitary quantum system and uniqueness of the metric operator, Phys. Lett. B 650 (2007) 208–212.
[25] M. Znojil, Time-dependent version of crypto-Hermitian quantum theory, Phys. Rev. D 78 (2008) 085003.
[26] H. Bíla, Adiabatic time-dependent metrics in PT-symmetric quantum theories, arXiv preprint, arXiv:0902.0474, 2009.
[27] A. Fring, M.H.Y. Moussa, Unitary quantum evolution for time-dependent quasi-Hermitian systems with nonobservable Hamiltonians, Phys. Rev. A 93 (2016) 042114.
[28] A. Fring, M.H.Y. Moussa, Non-Hermitian Swanson model with a time-dependent metric, Phys. Rev. A 94 (2016) 042128.
[29] J. Gong, Q.-H. Wang, Time-dependent PT-symmetric quantum mechanics, J. Phys. A, Math. Theor. 46 (2013) 485302.
[30] A. Mostafazadeh, Time-dependent pseudo-Hermitian Hamiltonians and a hidden geometric aspect of quantum mechanics, Entropy 22 (2020) 471.
[31] A. Fring, T. Frith, Exact analytical solutions for time-dependent Hermitian Hamiltonian systems from static unobservable non-Hermitian Hamiltonians, Phys. Rev. A 95 (2017) 010102(R).
[32] A. Fring, T. Frith, Metric versus observable operator representation, higher spin models, Eur. Phys. J. Plus 133 (2018) 1–9.
[33] A. Fring, T. Frith, Mending the broken PT-regime via an explicit time-dependent Dyson map, Phys. Lett. A 381 (2017) 2318–2323.
[34] A. Fring, T. Frith, Solvable two-dimensional time-dependent non-Hermitian quantum systems with infinite dimensional Hilbert space in the broken PT-regime, J. Phys. A, Math. Theor. 51 (2018) 265301.
[35] A. Fring, T. Frith, Quasi-exactly solvable quantum systems with explicitly time-dependent Hamiltonians, Phys. Lett. A 383 (2019) 158–163.
[36] J. Cen, A. Fring, T. Frith, Time-dependent Darboux (supersymmetric) transformations for non-Hermitian quantum systems, J. Phys. A, Math. Theor. 52 (2019) 115302.
[37] A. Fring, T. Frith, Eternal life of entropy in non-Hermitian quantum systems, Phys. Rev. A 100 (2019) 010102.
[38] A. Fring, R. Tenney, Time-independent approximations for time-dependent optical potentials, Eur. Phys. J. Plus 135 (2020) 163.
[39] T. Frith, Exotic entanglement for non-Hermitian Jaynes–Cummings Hamiltonians, J. Phys. A, Math. Theor. 53 (2020) 485303.
[40] A. Fring, R. Tenney, Spectrally equivalent time-dependent double wells and unstable anharmonic oscillators, Phys. Lett. A 384 (2020) 126530.
[41] A. Fring, R. Tenney, Perturbative approach for strong and weakly coupled time-dependent for non-Hermitian quantum systems, Phys. Scr. 96 (2021) 045211.
[42] A. Fring, R. Tenney, Exactly solvable time-dependent non-Hermitian quantum systems from point transformations, Phys. Lett. B 410 (2021) 127548.
[43] A. Fring, R. Tenney, Infinite series of time-dependent Dyson maps, J. Phys. A, Math. Theor. 54 (2021) 485201.
[44] H. Lewis, W. Riesenfeld, An exact quantum theory of the time dependent harmonic oscillator and of a charged particle time dependent electromagnetic field, J. Math. Phys. 10 (1969) 1458–1473.
[45] M. Maamache, O.K. Djeghiour, N. Mana, W. Koussa, Pseudo-invariants theory and real phases for systems with non-Hermitian time-dependent Hamiltonians, Eur. Phys. J. Plus 132 (2017) 383.
[46] B. Khantoul, A. Bounames, M. Maamache, On the invariant method for the time-dependent non-Hermitian Hamiltonians, Eur. Phys. J. Plus 132 (2017) 258.
[47] T. Frith, Time-dependence in non-Hermitian quantum systems, arXiv:2002.01977, PhD thesis, City, University of London, 2019.
[48] X.-C. Gao, J.-B. Xu, T.-Z. Qian, Invariants and geometric phase for systems with non-Hermitian time-dependent Hamiltonians, Phys. Rev. A 46 (1992) 3626.
[49] J. Dieudonné, On biorthogonal systems, Mich. Math. J. 2 (1953) 7–20.
[50] S. Weigert, Completeness and orthonormality in $\mathcal{PT}$-symmetric quantum systems, Phys. Rev. A 68 (2003) 062111.