# L-fuzzy compactness and related concepts

by

SORAYA ROSANA TORRES KUDRI

A thesis submitted for the degree of

Doctor of Philosophy in Mathematics

Department of Mathematics

City University

London

April 1995

## TABLE OF CONTENTS

**ACKNOWLEDGEMENT**

I should like to express my sincere gratitude to Professor M.W. Warner for her help and guidance throughout my research and my thanks to CAPES for financial support.

**DECLARATION**

I grant powers of discretion to the University Librarian to allow this thesis to be copied in whole or in part without further reference to me. This permission covers only single copies made for study purposes, subject to the normal conditions of acknowledgement.

**ABSTRACT**

The compactness defined by Warner and McLean is extended to arbitrary L-fuzzy sets where L is a fuzzy lattice, i.e., a completely distributive lattice with an order reversing involution. It is shown that with our compactness we can build up a satisfactory theory. The different definitions of compactness in L-fuzzy topological spaces are stated and other characterizations of some of these notions are obtained. We also study their goodness and establish the inter-relations between the compactnesses which are good extensions.

Good definitions of L-fuzzy regularity and normality are proposed.

Following the lines of our compactness we suggest two definitions of L-fuzzy local compactness that are good extensions of the respective ordinary versions. A comparison between them is presented and some of their properties studied. A one point compactification is also obtained.

By introducing a new definition of a locally finite family of L-fuzzy sets and combining it with our definition of compactness, we propose an L-fuzzy paracompactness and study some of its properties.

Good definitions of L-fuzzy countable and sequential compactness and the Lindelöf property are introduced and studied.

We also present, in L-fuzzy topological spaces, good extensions of S-closedness and RS-compactness. Some of their properties are examined.

Good L-fuzzy versions of almost compactness, near compactness and a strong compactness are put forward and studied. A comparison between these compactness related concepts is also presented.

# NOMENCLATURE

The following list contains the most frequently used classical symbols. Some of them will be also used to represent fuzzy concepts. A list of the most frequently used fuzzy notations will be given later on.

| | |
|---|---|
| $\mathbb{N}$ | the set of the natural numbers |
| $\mathbb{Q}$ | the set of the rational numbers |
| $\phi$ | the empty set |
| $\leq, \nleq$ | partial order relation and its negation |
| $<, \nless$ | strict partial order relation and its negation |
| max | maximum |
| $\vee$ | join |
| $\wedge$ | meet |
| $'$ | order reversing involution (def. 1.1.9) |
| $(X,\delta)$ | a topological space |
| $(A,\delta_A)$ | a subspace of a topological space $(X,\delta)$ |
| cl(X) | the closure of the set X |
| int(X) | the interior of the set X |
| $\mathbb{P}(X)$ | the power set of X |
| $\chi_A$ | the characteristic function of A |
| $x \in X$ | x is an element of X |
| $x \notin X$ | x is not an element of X |
| $\{x \in X; P\}$ | the set of all elements x in X satisfying the condition (s) P |
| $\{x\}$ | the singleton set having the element x |
| A-B | the set $\{x;\ x \in A,\ x \notin B\}$ |

$\subset, \not\subset$ the relation "is properly contained in" on a power set and its negation

$\subseteq, \not\subseteq$ the relation "is contained in" on a power set and its negation

$\{A_\alpha\}_{\alpha\in J}$ or $(A_\alpha)_{\alpha\in J}$ indexed family of sets

$\bigcup_{\alpha\in J} A_\alpha$, $\bigcup_{A\in\mathscr{C}} A$ the union of the family $(A_\alpha)_{\alpha\in J}$ (respectively $\mathscr{C}$).

$\bigcap_{\alpha\in J} A_\alpha$ , $\bigcap_{A\in\mathscr{C}} A$ the intersection of the family $(A_\alpha)_{\alpha\in J}$ (respectively $\mathscr{C}$).

$f:X\to Y$ a function from X to Y

$\prod_{\alpha\in J} A_\alpha$ the cartesian product of the family $(A_\alpha)_{\alpha\in J}$

$f(A)$, $f^{-1}(A)$ the image of A and the inverse image of A under f

$f(x)$, $f^{-1}(y)$ the image of x and the inverse image of y under f

$f|_A$ the restriction of the function f to the set A

$(x_m)_{m\in\mathbb{N}}$ or $(x^m)_{m\in\mathbb{N}}$ a sequence of terms $x_m$

$(x_{m_i})_{i\in\mathbb{N}}$ or $(x^{m_i})_{i\in\mathbb{N}}$ a subsequence of $(x_m)_{m\in\mathbb{N}}$ or $(x^m)_{m\in\mathbb{N}}$ respectively

$x^m \to x$ the sequence $(x_m)_{m\in\mathbb{N}}$ converges to x

$x^m \not\to x$ the negation of the statement $x^m \to x$

$C_1$ first countable

$C_2$ second countable

$\forall$ the quantifier "for each"

pr(L) the set of all prime elements (def. 1.1.12) of a lattice L

M(L) the set of all union irreducible elements (def. 1.1.13.) of a lattice L

$\pi_\alpha$ the $\alpha$-th projection map

| | |
|---|---|
| $2^{(\mathcal{A})}$ | the family of all finite subsets of the collection $\mathcal{A}$. |
| 0,1 | the smallest and the largest element of a lattice L |
| $e \ll b$ | e is way below b (def. 1.1.4) |
| $\beta(\alpha)$ | the union of all minimal sets relative to $\alpha$ (def. 1.1.16) |
| $\beta^*(\alpha)$ | the intersection of $\beta(\alpha)$ and M(L) where $\alpha \in L$ |

A list of the most frequently used fuzzy notations will now be given with a reference to where they first appear in the text.

| | |
|---|---|
| $L^X$ | the set of all L-fuzzy sets on X (def. 2.1.1) |
| suppf | the support of an L-fuzzy set f (def. 2.1.4) |
| $\phi$ | the empty fuzzy set in X (def 2.1.1) |
| X | the full fuzzy set in X (def. 2.1.1) |
| $pr(L^X)$ | the set of all prime elements of $L^X$ (remark 2.1.5) |
| $M(L^X)$ | the set of all coprime elements of $L^X$ (remark 2.1.7) |
| $x_p$ | an L-fuzzy point of X (def. 2.1.6) |
| $x_\alpha$ | a coprime element of $L^X$ (def. 2.1.7) |
| $x_p \in f$ | the L-fuzzy point $x_p$ is a member of the L-fuzzy set f (def. 2.1.6) |
| $\bigvee_{i \in J} f_i$ , $\bigvee_{f \in \mathcal{C}} f$ | the join of the family $(f_i)_{i \in J}$ (respectively $\mathcal{C}$) of L-fuzzy sets (remark 2.1.3) |
| $\bigwedge_{i \in J} f_i$ , $\bigwedge_{f \in \mathcal{C}} f$ | the meet of the family $(f_i)_{i \in J}$ (respectively $\mathcal{C}$) of L-fuzzy sets (remark 2.1.3) |
| f(g) | the image of an L-fuzzy set g under a function f (def. 2.2.1) |
| $f^{-1}(g)$ | the inverse image of an L-fuzzy set g under a |

function f (def. 2.2.2)

$(S_m)_{m\in D}$ a net in X of term $S_m \in M(L^X)$ (def. 2.3.1)

supp $S_m$ the support of the term $S_m$ of a net $(S_m)_{m\in D}$ (remark 2.3.2)

$h(S_m)$ the height of the term $S_m$ of a net $(S_m)_{m\in D}$ (remark 2.3.2)

$(x^m_{\alpha_m})_{m\in D}$ an $\alpha$-net of term $x^m_{\alpha_m}$ where $x^m$ is its support and $\alpha_m$ its height (def. 2.3.8)

$(X,\mathcal{T})$ an L-fuzzy topological space (def. 3.1.1)

$(A,\mathcal{T}_A)$ a subspace of an L-fuzzy topological space $(X,\mathcal{T})$ (def. 3.2.3)

cl(f) the closure of an L-fuzzy set f (def. 3.1.5)

int(f) the interior of an L-fuzzy set f (def. 3.1.5)

$S_m \to x_\alpha$ the net $(S_m)_{m\in D}$ converges to $x_\alpha \in M(L^X)$, i.e., $x_\alpha$ is a limit point of $(S_m)_{m\in D}$ (def. 3.1.9(i))

$\omega(\delta)$ the set of all continuous functions from a topological space $(X,\delta)$ to a lattice L with its Scott topology (remark 3.2.5)

$i_p(f)$ the set $\{x\in X;\ f(x)\nleq p\}$ where f belongs to an L-fuzzy topology $\mathcal{T}$ and $p\in pr(L)$ (remark 4.6.9)

$i_L(\mathcal{T})$ the ordinary topology with subbase $\varphi(\mathcal{T}) = \{i_p(f);\ p\in pr(L) \text{ and } f\in\mathcal{T}\} \cup \{X\}$ where $(X,\mathcal{T})$ is an L-fuzzy topological space.

$C_1$ first countable (def. 3.4.1)

$C_2$ second countable (def. 3.4.2)

# INTRODUCTION

The theory of fuzzy sets was introduced by Zadeh [108] in 1965 with the purpose of developing a mathematically precise framework to deal with indefiniteness, with the vagueness that exists in the real world. This has caused great interest among pure and applied mathematicians and experts in other areas. Since then, work has been done by many authors, in several directions, which has resulted in the formation of a new mathematical field called "Fuzzy Mathematics".

Fuzzy set theory can be thought of as a mathematical model for imprecise concepts. A fuzzy set is a sort of generalized "characteristic function", whose "degrees of membership" can be more general than "yes" or "no", that is, a membership function which describes the gradual transition from membership to nonmembership. This notion replaced the rigid membership relation of ordinary set theory by the flexible grade of membership.

In [108], Zadeh defined fuzzy sets in terms of functions from a set to the closed unit interval and introduced basic notions such as fuzzy union, intersection and complement.

In 1967, Goguen [38] extended the concept of fuzzy set by replacing the unit interval by an arbitrary lattice with both a minimal and a maximal element, thus, introducing the notion of an L-fuzzy set. He showed how the language of categories and functors could be used to describe a fuzzification of whole theories in a unified manner.

Interest has been aroused in the application of fuzzy sets to such fields as artificial intelligence, optimization, pattern recognition and decision theory.

General topology was one of the first branches of pure mathematics to which fuzzy sets have been applied systematically. It was in 1968, that Chang [18] made the first attempt to formulate a theory of fuzzy topological spaces. He showed for the first time that basic topological notions can be extended to fuzzy topological spaces. He introduced the notion of fuzzy topological space and also defined fuzzy image and fuzzy inverse image under a function and extended a number of properties of functions, such as continuity, to fuzzy topology. He adopted Zadeh's fuzzy sets.

Since the early eighties, the intensity of research on fuzzy topology, that is a branch of fuzzy mathematics, has sharply increased. As remarked by Lowen [57], while topology classifies objects (spaces, functions, filters etc...) into classes, those which fulfill and those which do not fulfill a certain property (compactness, continuity, convergence, etc...) and the theory is developed mainly on classes of objects which have "good" properties; fuzzy topology, also classifies and studies those objects not having a given property, into subclasses, each of which is characterized by the fact that its objects have an approximate - to a certain degree - form of that property. Concerned about developing a technique with which we can measure a degree to which a space has a given property or not, we have, for instance, the works by Lowen [57], Wuyts and Lowen [104], Lowen and Lowen [52] and Rodabaugh [83].

The notion of point plays an important role in general topology. In spite of the fact that many important results had been obtained in fuzzy topology without this notion, it is impossible not to deal at all with the notion of point. So, the fuzzification of points was necessary. A point is a minimal object in the sense of the relation of belonging, that is, nothing can belong to a point. A peculiarity of fuzzy set mathematics is the absence of such minimal objects.

In Chang's work [18], he did not define fuzzy points, but introduced neighbourhoods and sequences of fuzzy sets.

The fuzzy point problem was avoided by Hutton [44] and others, adopting the so called "pointless approach", where either a fuzzy topological theory or a fuzzy neighbourhood theory is built up without reference to points.

Many mathematicians tried to define fuzzy point and its membership relation. In 1974, Wong [103] based on the notion of fuzzy singleton introduced by Zadeh [109], defined fuzzy points as fuzzy singletons and fuzzy membership with strict inequality. But in 1979, Gottwald [40] showed that Wong's definition of fuzzy membership was not good and some of the results were not correct.

Gottwald [40], Ghanin et al [36] and Kerre [47] used the concept of fuzzy singleton, not mentioning points.

Sarkar [86, 87], Srivastava et al [91, 92], Deng [26, 27], de Mitri and Pascali [69], Bülbül [10, 11] and others worked with a modified version of Wong's fuzzy membership and with his definition of fuzzy point. But Wong's definition as well as this version exclude the crisp points that are the classical points.

Pu and Liu [80] including crisp points in fuzzy points, considered the fuzzy membership relation $\leq$ and introduced quasi-coincidence. These two relations are connected as follows: a fuzzy point belongs to a fuzzy set if and only if it is not quasi-coincident to its complement.

Wang, in [96], introduced the notion of a molecule, which is a kind of fuzzy point.

In [99], Warner, considering a frame L defined L-fuzzy points locale-wise by frame homomorphisms to the two-point set, and so corresponding bijectively to prime elements. Membership emerges in terms of $\not\leq$.

Kerre and Ottoy in [48] gave a detailed survey of the various definitions of fuzzy points and corresponding neighbourhood theories.

In [54], Lowen introduced another definition of fuzzy topology that is a restriction of the point-fuzzy set approach in which L=[0,1] and the fuzzy topology on a set X contains all the constant maps from X to [0,1]. In [54], Lowen gave some reasons to adopt this constant maps approach and in [58] he and Wuyts insist on the advantages of this definition. In [85], Rodabaugh pointed out some reasons to justify non-stratified spaces, that is, spaces without the requirement that all the constant maps are in the topology.

The third definition of fuzzy topological space is due to Hutton [44]. Realising that if L is a fuzzy lattice, a completely distributive lattice with an order-reversing involution, then $L^X$ is also a fuzzy lattice, he defined pointless fuzzy topological spaces. His studies are related to lattice theory.

Different definitions of fuzzy topology and several approaches to fuzzy topology have been pointed out. We have the Chang-Goguen fuzzy topological category which uses the point-fuzzy set approach. We also have Lowen's category called the constant maps approach and Hutton's category which represents the pointless approach to fuzzy topology.

In [84], Rodabaugh introduced a new fuzzy topological category called Fuzz. It is a generalization of ordinary topology, of the pointless approach, the point-fuzzy set approach and the constant maps approach to fuzzy topology. In this work he also summarizes the previous approaches.

In [68], Mingsheng pointed out a new approach for fuzzy topology with fuzzy logic and studied the neighbourhood structure of a point and the convergence of nets and filters. He remarked that a fuzzy topological space was defined as a classical subset of the fuzzy power set of a non-empty classical set which is closed for finite intersection and any union operations, i.e., fuzzy objects were being investigated by crisp methods. Then he used the semantics of fuzzy logic, a fuzzy method, to investigate topology and to propose a topology whose logical foundation is fuzzy, observing that we can consider other topologies based on many-valued logics. In this work he defined bifuzzy topology, fuzzy topology and fuzzifying topology.

In [41] Hazra, Samanta and Chattopadhyay gave a new definition of fuzzy topology by introducing a concept of gradation of openness of fuzzy subsets and then studied fuzzy continuity. In [19] they modified the definition of gradation function and then studied subspaces of fuzzy topological spaces, gradation preserving maps and the category of fuzzy topological spaces. With this modification, their definition of fuzzy topological spaces coincides with the already introduced concept of smooth topological spaces [82], where the lattices L and L′ in Ramadan's smooth topological spaces [82] are taken both equal to [0,1].

In [16], Chakraborty and Ahsanullah introduced another category for fuzzy topological spaces and also a new category for fuzzy sets. Within the categories considered another peculiarity of L-fuzzy spaces, up to the time that [16] was published, was the nonavailability of the subspace topology for any fuzzy subset of any fuzzy space, that is, subspaces made sense only for crisp subsets of L-fuzzy spaces. In the approach of Chakraborty and Ahsanullah [16] to fuzzy topology, subspaces may be defined on any fuzzy set.

In the same year Chakraborty and Banerjee presented in [17] a different category for fuzzy topological spaces, that includes as subcategories, both the categories of Chakraborty and Ahsanullah [16] and Rodabaugh [84] and hence Chang-Goguen [18, 38], Lowen [54] and Hutton's categories [44].

Macho Stadler and Prada Vicente [60], working with Chang's definition of fuzzy topological spaces, defined fuzzy topological subspaces for arbitrary fuzzy sets, which coincides with the usual definition given in [80] for crisp sets of fuzzy spaces.

In the unpublished work "A new approach to fuzzy topological spaces and fuzzy perfect mappings", R.D. Sarma and N. Ajmal [88] proposed yet another approach to defining a fuzzy topological space. This approach is net-theoretic and the fuzzy topological spaces obtained form a category. In their work, they claim that their category, which is a subcategory of the Chang-Goguen category, is free from the drawbacks of Chang-Goguen's category. For instance, in their category, a rich convergence theory can be developed, projections are open and there are many properties of general topology for which it is more suitable.

In [25] Dang et al. worked on fuzzy supratopological spaces.

In [56] Lowen introduced the so called "goodness criterion" and in [98], Warner generalised this criterion to a continuous frame L.

Compactness is one of the most important notions in pure mathematics. Therefore it is natural to pay particular interest to it in fuzzy topology.

The first definition of compactness in a fuzzy topological space was suggested in 1968 by Chang [18]. But very soon the disadvantages of this definition became clear and compactness in fuzzy topology was shown to be far more complex than in general topology.

The next compactness results are due to Goguen [39], who proved a Tychonoff theorem for finite products. Goguen was the first to point out a deficiency in Chang's compactness theory by showing that the Tychonoff theorem is false for infinite products.

After that, Wong [102] treated compactness, defined sequential and countable compactness and in [103] he introduced local compactness.

Weiss [101] dealt with a subfamily of the family of all fuzzy topologies on a fixed set, induced fuzzy topological spaces. Since no member of a Weiss subfamily is compact in the sense of Chang, he introduced a new notion of compactness. Lowen [53] gave a new definition of a compact fuzzy space which, when restricted to Weiss' subfamily, is generalised by Weiss' definition. However, Lowen was able to obtain only a finite Tychonoff theorem.

Lowen, in [54], the work in which he altered the definition of a fuzzy topological space, gave a different definition of compact fuzzy space and obtained a Tychonoff theorem for an arbitrary product. In this paper he also proved that Chang's compactness is not a good extension.

In [35] Gantner, Steinlage and Warren proposed a definition of L-fuzzy compactness, where L is a completely distributive lattice (the so-called $\alpha$-compactness ($\alpha \in L$)). With restrictions on $\alpha$, they obtained a Tychonoff theorem for an arbitrary product and a one-point compactification.

In [56], Lowen studied different kinds of compactness notions that had already been introduced and added two more notions, ultra fuzzy compactness and strong compactness. He worked in fuzzy topological spaces as defined in [54]. He showed which of these compactnesses are good extensions, studied the implications between them and analysed for which notions there is a product theorem.

Hutton, in [44], introduced a strong definition of compactness and proved the Tychonoff theorem in L-fuzzy topology where L is a fuzzy lattice.

Wang [95] defined the notion of $\alpha$-net and introduced a new kind of compactness in fuzzy topological spaces, the so-called N-compactness, by using $\alpha$-nets from the point of view of convergence. N-compactness has almost all the properties that ordinary compactness has in general topology.

Li, in [49], proposed two more kinds of fuzzy compactness, $Q_\alpha$-compactness and strong Q-compactness, based on Q-neighbourhoods and convergence of nets.

Peng [78] and Zhao [110] generalized N-compactness to L-fuzzy topological spaces, where L is a complete completely distributive lattice.

Höhle [42] introduced a different concept of L-fuzzy compactness, called probabilistic compactness, where L is a complete Boolean Algebra. A convergence theory is developed, and the new concept of compactness is introduced by means of 1-ultrafilters. This compactness is useful in probability theory.

Eklund and Gähler [31] defined fuzzy compactness by means of nets and compared this with two modified versions, one of which uses a covering property and the other uses filters. In [32] they defined compactness by means of almost bounded nets.

Ganguly and Saha, in [34], presented a definition of fuzzy compactness by filters.

In [15], Chadwick proposed another fuzzy compactness that is a modification of Wang's N-compactness.

Prada Vicente and Macho Stadler, in [79], introduced the notion of t-prefilter and obtained a characterization of t-compactness [35] by means of maximal t-prefilters.

In [100], Warner and McLean suggested a definition of L-fuzzy compactness, where L is a frame. By considering L a continuous spatial frame they proved the goodness of this definition.

In [66], Meng mentioned that Wang, in a work in Chinese, generalised Lowen's compactness [54] to a general L-fuzzy topological space. In this work Meng obtained new characterizations for Lowen's compactness in L-fuzzy topological spaces, where L is a fuzzy lattice.

Xu, in [105], referred to another L-fuzzy compactness which it would seem exists only in a work in

Chinese, that is weaker than N-compactness [95].

In [67], Meng mentioned that Wang, in a work in Chinese has generalized ultra fuzzy compactness [56] to L-fuzzy topological spaces.

Thus, many papers dealing with compactness have been written and various kinds of fuzzy compactnesses have been introduced and studied. However, each of them has its own limitation, some more and others less. As Eklund [30] remarked, in fuzzy topology the notion of compactness is almost a nuisance.

In this work we propose an extension to arbitrary L-fuzzy sets, of the compactness defined in [100]. Working with a fuzzy lattice L, we study some properties of this new definition. We also propose some good new definitions of countable, sequential, local, almost, nearly, strong and RS-compactness, as well as, new good ones of paracompactness, Lindelöffness and S-closedness and study some of their properties. Good definitions of regularity and normality are also introduced. We compare our definition of an L-fuzzy compact set with the other good definitions already introduced in L-fuzzy topological spaces, as well as present a comparison between our concepts of compactness; S-closedness; almost, nearly, strong and RS-compactness.

The thesis is divided into twelve chapters as follows:

| | |
|---|---|
| I | Lattice theory |
| II | L-fuzzy set theory |
| III | Fuzzy topological spaces |
| IV | Compactness in L-fuzzy topological spaces |
| V | Countable compactness, Sequential compactness and Lindelöfness |
| VI | Local compactness in L-fuzzy topological spaces |
| VII | Paracompactness in L-fuzzy topological spaces |
| VIII | Some weaker forms of compactness |
| IX | S-closedness in L-fuzzy topological spaces |
| X | RS-compactness in L-fuzzy topological spaces |
| XI | S-compactness in L-fuzzy topological spaces |
| XII | A comparison between the concepts introduced in chapters VIII, IX, X and XI and some related properties |

From chapter V on, L will be always a fuzzy lattice.

# CHAPTER I

## Lattice Theory

This chapter consists essentially of some definitions and results on lattice theory upon which this work is based. Our purposed is to make this work reasonably self-contained. For more details we refer to Johnstone [45], Gierz et al. [37] and Birkhoff [8].

We divide this chapter in two sections.

We devote the first section to some basic definitions.

The second section is reserved for some related properties.

## 1. Basic definitions

Definition 1.1.1. Birkhoff [8]

A directed set D is a set with a partial order ≥ such that for each pair m,j of elements of D, there exists an element k of D having the property that k≥m and k≥j.

Definition 1.1.2. Birkhoff [8]

A lattice L=L (≤,$^\wedge$,$^\vee$) is a set L equipped with a partial order ≤, in which every finite subset has a join and a meet, where meet and join are denoted by $^\wedge$ and $^\vee$ respectively.

Definition 1.1.3. Birkhoff [8]

A complete lattice is a lattice in which every set has a join and a meet. We denote L the largest element of L, $^\vee$L, by 1 and its smallest element, $^\wedge$L, by 0. We consider 0 as the join of the empty set and 1 as the meet of the empty set.

Definition 1.1.4. Gierz et al. [37]

Let L be a complete lattice. We say that e is way below b, in symbols e«b, if and only if for any directed subset D of L the relation b≤$^\vee$D always implies the existence of d∈D with e≤d.

Definition 1.1.5. Gierz et al. [37]

A continuous lattice L is a complete lattice in which for all $e \in L$, $e = \vee\{x \in L;\ x \ll e\}$.

<u>Definition</u> <u>1.1.6.</u> Johnstone [45]

A <u>locale</u> or a <u>frame</u> L is a complete lattice satisfying the infinite distributive law

$$e \wedge (\vee S) = \vee\{e \wedge x;\ x \in S\}$$

for all $e \in L$ and all $S \subseteq L$.

<u>Definition</u> <u>1.1.7.</u> Gierz et al. [37]

A lattice L is called <u>completely</u> <u>distributive</u> if and only if it is complete and the following condition holds:

$$\bigwedge_{i \in I} \left( \bigvee_{j \in J_i} e_{i,j} \right) = \bigvee_{f \in K} \left( \bigwedge_{i \in I} e_{i,f(i)} \right),$$

where for each $i \in I$ and for each $j \in J_i$, $e_{i,j} \in L$, and K is the set of all maps $f: I \to \cup J_i$ such that for every $i \in I$, $f(i) \in J_i$.

<u>Definition</u> <u>1.1.8.</u> Birkhoff [8]

Let A be a set that is equipped with a partial order. Then $B \subset A$ is called a <u>chain</u> in A if and only if each two elements in B are related.

<u>Definition</u> <u>1.1.9.</u> Birkhoff [8]

An <u>order</u> <u>reversing</u> <u>involution</u> on a lattice L is a map $x \to x'$ from L to L satisfying:

(i) if $e \le b$ then $b' \le e'$

(ii) $(e')' = e$

Definition 1.1.10. Hutton [44]

A fuzzy lattice is a completely distributive lattice with an order reversing involution.

Definition 1.1.11. Johnstone [45]

A frame morphism is a map between frames which preserves finite meets and arbitrary joins.

Definition 1.1.12. Gierz et al. [37]

An element p of a lattice L is prime if and only if $p \neq 1$ and whenever $e, b \in L$ with $e \wedge b \le p$ then $e \le p$ or $b \le p$. The set of all prime elements of a lattice L will be denoted by pr(L).

Definition 1.1.13. Gierz et al. [37]

An element $\alpha$ of a lattice L is coprime or union irreducible if and only if $\alpha \neq 0$ and whenever $e, b \in L$ with $\alpha \le e \vee b$ then $\alpha \le e$ or $\alpha \le b$. The set of all coprime elements of a lattice L will be denoted by M(L).

Definition 1.1.14. Johnstone [45]

A frame L is called spatial if and only if for all $e, b \in L$ with $e \not\le b$ there is a prime p with $e \not\le p \ge b$.

Definition 1.1.15. Gierz et al. [37]

A subset U of a complete lattice L is called Scott open if and only if it is an upper set and is inaccessible by directed joins, i.e.:

(i) if $e \in U$ and $e \leq b$ then $b \in U$

(ii) if D is a directed subset of L with $\vee D \in U$ then there is a $d \in D$ with $d \in U$.

The set of all Scott open subsets of L is a topology and is called the Scott topology of L.

Definition 1.1.16. [51, 97, 50, 110]

Let L be a complete lattice, $\alpha \in L$ and $\phi \neq B \subset L$. B is called a minimal set relative to $\alpha$ if and only if $\vee B = \alpha$ and for each set $A \subset L$ satisfying $\vee A \geq \alpha$ and for each $b \in B$, there exists $e \in A$ such that $b \leq e$.

Remark 1.1.17. Wang [97]

The union of minimal sets relative to $\alpha$ is a minimal set relative to $\alpha$. We shall denote the union of all minimal sets relative to $\alpha \in L$ by $\beta(\alpha)$ and by $\beta^*(\alpha)$ the set $\beta(\alpha) \cap M(L)$.

Definition 1.1.18. Birkhoff [8]

A maximal element of a subset A of a partially ordered set L is an element x such that for no $e \in A$ it is true that $x < e$.

Definition 1.1.19. Johnstone [45]

A point of a frame L is a frame morphism from L to the frame $\{0,1\}$.

Remark 1.1.20. Johnstone [45]

When L is a frame, there is a one-to-one correspondence between the points of L and the prime elements of the lattice L. Therefore we can regard points of L as prime elements of L.

## 2. Some properties

Proposition 1.2.1. Blyth and Janowitz [9]

Let $(L,')$ be a complete lattice with an order reversing involution. Then for any family $(e_i)_{i\in J}$ of members of L we have:

(i) $$\left(\bigvee_{i\in J} e_i\right)' = \bigwedge_{i\in J} e_i'$$

(ii) $$\left(\bigvee_{i\in J} e_i\right)' = \bigwedge_{i\in J} e_i'$$

Proof

See theorem 17.1 of [9].

Proposition 1.2.2. Johnstone [45]

A locale L is spatial if and only if every element is a meet of primes, that is, every element is a join of coprimes.

Proof

See in Johnstone [45] pp43 and proposition 2.17 in Wang [97].

Proposition 1.2.3. Gierz et al. [37]

Every completely distributive lattice is a continuous frame and is therefore spatial.

Proof

See in Gierz et al. [37] pp72 and section I.3.7 and I.3.12.

Proposition 1.2.4. Johnstone [45]

Let L be a frame. Then the power set $\mathbb{P}(pr(L))$ is a frame and the map $\phi: L \to \mathbb{P}(pr(L))$ defined by $\phi(x) = \{p \in pr(L);\ p \ngeq x\}$ is a frame morphism. Its image is therefore a topology on pr(L). The map $\phi$ is injective if and only if L is spatial.

Proof

See Johnstone [45] pp41 and 42.

Lemma 1.2.5. (Zorn's lemma) Dugundji [29]

If each chain in a nonempty partially ordered set has an upper bound, then the set has a maximal element.

Proposition 1.2.6. Wang [97]

Let L be a complete lattice. Then L is a completely distributive lattice if and only if for every $\alpha \in L$, $\alpha$ has a minimal set, and hence $\beta(\alpha)$ exists.

Proof

See theorem 2.11 in [97].

Proposition 1.2.7. Zhao [110]

Let L be a completely distributive lattice. If $\alpha \in L \backslash \{0\}$, then $\beta^*(\alpha)$ is a minimal set relative to $\alpha$. Furthermore if $\alpha \in M(L)$ then $\beta^*(\alpha)$ is a directed set.

Proof

See lemma 4.1 in [110].

Example 1.2.8. Wang [97]

Let L = [0,1].

Thus pr(L) = [0,1), M(L) = (0,1], for every $\alpha \in (0,1]$ $\beta(\alpha) = [0,\alpha)$ and $\beta(0) = \{0\}$.

Proposition 1.2.9. Gierz et al. [37]

Let L be a continuous lattice. Then the sets of the form $\{q \in L;\ e_0 \ll q\}$ form a basis for the Scott topology on L.

Proof

See remark 3.2 pp68 and proposition 1.10 (ii) pp104 in [37].

Proposition 1.2.10. Gierz et al. [37]

Let L be a completely distributive lattice. Then the sets of the form $\{x \in L;\ x \nleq e\}$ generate the Scott topology.

Proof

See [37] pp166 and pp205.

Proposition 1.2.11. Warner and McLean [100]

The Scott topology of a completely distributive lattice L is generated by the sets of the form $\{x \in L;\ x \nleq p\}$ where $p \in pr(L)$.

Proof

See proposition 2.1 in [100].

# CHAPTER II

## L-fuzzy set theory

This chapter is concerned with the definitions and some properties of L-fuzzy sets, L-fuzzy points, image and inverse image of L-fuzzy sets that we shall use later on. We also present the definitions of nets, $\alpha$-nets, sequences and $\alpha$-sequences in the L-fuzzy context.

We divide this chapter in three sections.

The first section contains the definitions and some properties related to L-fuzzy sets and L-fuzzy points.

Section two is devoted to image and inverse image of L-fuzzy sets.

The third section is reserved for the definitions of nets, $\alpha$-nets, sequences and $\alpha$-sequences.

Although, some of the definitions, results and proofs we refer to are given only for L-fuzzy sets where $L=[0,1]$, they are totally similar for a fuzzy lattice L.

## 1. L-fuzzy sets and L-fuzzy points

In the following, let X be a nonempty set and let $L=L(\leq,\vee,\wedge,')$ be a fuzzy lattice with a smallest element 0 and a largest element 1 ($0\neq 1$). We consider 0 as the join of the empty set and 1 as the meet of the empty set.

Definition 2.1.1. Goguen [38]

An L-fuzzy set f on X is a function $f:X\to L$. The set of all L-fuzzy sets on X will be denoted by $L^X$.

The L-fuzzy sets on X defined by $f(x) = 0$ for every $x\in X$ and $g(x) = 1$ for every $x\in X$ will be denoted by $\phi$ and X respectively. We call them the empty L-fuzzy set and the full L-fuzzy set on X respectively.

Definition 2.1.2. Weiss [101]

A crisp set on X is an ordinary subset of X. In particular, its characteristic function from X to L is an L-fuzzy set. We shall denote the characteristic function of a set $A\subseteq X$ by $\chi_A$.

Remark 2.1.3. Goguen [38]

To the set $L^X$, of all L-fuzzy sets, can be given whatever operations L has and these operations in $L^X$ will obey any law valid in L which extends point by point. Thus, since L is a fuzzy lattice, $L^X$ is also a fuzzy lattice, with the partial ordering $f\leq g$ if and only if $f(x)\leq g(x)$ for all $x\in X$, for $f,g\in L^X$, and the operations of meet and join as:

(i) $(f\wedge g)(x) = f(x)\wedge g(x)$ for each $x\in X$ and for $f,g\in L^X$

(ii) $(f\vee g)(x) = f(x)\vee g(x)$ for each $x\in X$ and for $f,g\in L^X$

(iii) $\left(\bigvee_{i\in J} f_i\right)(x) = \bigvee_{i\in J} f_i(x) = \vee\{f_i(x);\ i\in J\}$ for each $x\in X$ and for $f_i\in L^X$ for each $i\in J$.

(iv) $\left(\bigwedge_{i\in J} f_i\right)(x) = \bigwedge_{i\in J} f_i(x) = \wedge\{f_i(x);\ i\in J\}$ for each $x\in X$ and for $f_i\in L^X$ for each $i\in J$.

We shall call $f\vee g$ the union of f and g, $f\wedge g$ the intersection of f and g and read $f\le g$ as "f is contained in g".

Definition 2.1.4. Weiss [101]

Let f be an L-fuzzy set on X. The support of f is defined by $\text{supp}f = \{x\in X;\ f(x)>0\}$.

Remark 2.1.5.

Warner [99] determined the prime elements of the frame $L^X$ of all L-fuzzy sets on X. We have $pr(L^X) = \{x_p;\ x\in X \text{ and } p\in pr(L)\}$ where for each $x\in X$ and each $p\in pr(L)$ $x_p: X\to L$ is the L-fuzzy set defined by $x_p(y) = \begin{cases} p & \text{if } y=x \\ 1 & \text{otherwise} \end{cases}$

By remark 1.1.20., the points of the frame $L^X$ are in one-to-one correspondence with the prime elements of $L^X$. Therefore we have the following:

Definition 2.1.6. Warner [99]

These $x_p$ , in remark 2.1.5., are called the L-fuzzy points of X and we say that $x_p$ is a member of an L-fuzzy set f on X and write $x_p \in f$ if and only if $f(x) \nleq p$.

Remark 2.1.7.

Since the prime elements of $L^X$ are the functions $x_p : X \to L$ defined by $x_p(y) = \begin{cases} p & \text{if } y=x \\ 1 & \text{otherwise} \end{cases}$ where $x \in X$ and $p \in pr(L)$, the coprime elements of $L^X$ are the functions $x_\alpha : X \to L$ defined by $x_\alpha(y) = \begin{cases} \alpha & \text{if } y=x \\ 0 & \text{otherwise} \end{cases}$ where $x \in X$ and $\alpha \in M(L)$. As these $x_\alpha$ may be identified with the L-fuzzy points $x_p$ of X, we shall refer to them as the L-fuzzy points $x_\alpha \in M(L^X)$ where $M(L^X)$ is the set of all coprime elements of $L^X$. In this case, that is, when $x_\alpha \in M(L^X)$, we shall call x and $\alpha$ the support of $x_\alpha$ ($x = \operatorname{supp} x_\alpha$) and the height of $x_\alpha$ ($\alpha = h(x_\alpha)$), respectively.

Remark 2.1.8. Warner [99]

Since L is spatial, by proposition 1.2.2. we have that every L-fuzzy set on X is a meet of L-fuzzy points in $pr(L^X)$ and so, every L-fuzzy set on X is a join of L-fuzzy points in $M(L^X)$.

## 2. Image and inverse image of L-fuzzy sets

In the following, let X and Y be nonempty ordinary sets, let $f:X\to Y$ be a function and let L be a fuzzy lattice.

Definition 2.2.1. Chang [18]

For an L-fuzzy set g on X, the image of g under f is the L-fuzzy set on Y defined by
$f(g)(y) = \vee\{g(x);\ x\in f^{-1}(y)\}$ for each $y\in Y$.

Definition 2.2.2. Chang [18]

For an L-fuzzy set g on Y, the inverse image of g under f is the L-fuzzy set on X defined by $f^{-1}(g)(x) = g(f(x))$ for each $x\in X$.

The following is well-known.

Proposition 2.2.3. [18, 62, 81]

For a family $(g_i)_{i\in J}$ of L-fuzzy sets on X and a family $(h_i)_{i\in K}$ of L-fuzzy sets on Y we have:

(i) $f^{-1}(h_i') = (f^{-1}(h_i))'$

(ii) if $h_{i_1} \le h_{i_2}$ then $f^{-1}(h_{i_1}) \le f^{-1}(h_{i_2})$

(iii) if $g_{i_1} \le g_{i_2}$ then $f(g_{i_1}) \le f(g_{i_2})$

(iv) $f(f^{-1}(h_i)) \le h_i$. If f is onto then $f(f^{-1}(h_i)) = h_i$

(v) $(f(g_i))' \leq f(g_i')$ if f is onto

(vi) $g_i \leq f^{-1}(f(g_i))$. If f is injective then

$f^{-1}(f(g_i)) = g_i$

(vii) $f^{-1}\left(\bigvee_{i\in K} h_i\right) = \bigvee_{i\in K} f^{-1}(h_i)$

(viii) $f^{-1}\left(\bigwedge_{i\in K} h_i\right) = \bigwedge_{i\in K} f^{-1}(h_i)$

(ix) $f\left(\bigvee_{i\in J} g_i\right) = \bigvee_{i\in J} f(g_i)$

(x) $f\left(\bigwedge_{i\in J} g_i\right) \leq \bigwedge_{i\in J} f(g_i)$

(xi) $f(g_i') \leq (f(g_i))'$ if f is injective

Proposition 2.2.4. Malghan and Benchalli [62]

For a family $(g_i)_{i\in J}$ of L-fuzzy sets on X and an L-fuzzy set h on Y we have:

(i) if $g_{i_1} \leq g_{i_2}$ then $\text{supp}\, g_{i_1} \subseteq \text{supp}\, g_{i_2}$

(ii) $\text{supp}\left(\bigvee_{i\in J} g_i\right) = \bigcup_{i\in J} \text{supp}\, g_i$

(iii) $\text{supp}\left(\bigwedge_{j=1}^{m} g_{i_j}\right) = \bigcap_{j=1}^{m} \text{supp}\, g_{i_j}$

(iv) $f(\text{supp}\, g_i) = \text{supp}\, f(g_i)$

(v) $f^{-1}(\text{supp}\, h) = \text{supp}\, f^{-1}(h)$

## 3. Nets and Sequences

In the following, let X be a nonempty ordinary set and let L be a fuzzy lattice.

Definition 2.3.1. Zhao [110]

Let D be a directed set, X a nonempty ordinary set and let $M(L^X)$ be the set of all coprimes of $L^X$. A net in X is a function $S:D \to M(L^X)$. For $m \in D$, we shall denote $S(m)$ by $S_m$ and the net S by $(S_m)_{m \in D}$.

Remark 2.3.2.

If $(S_m)_{m \in D}$ is a net in X, then $S_m$ is an L-fuzzy point in $M(L^X)$. Thus we shall denote by supp $S_m$ and $h(S_m)$ the support and height of $S_m$, respectively.

Definition 2.3.3.

Let $f \in L^X$ and let $(S_m)_{m \in D}$ be a net in X. The net $(S_m)_{m \in D}$ is called a net contained in f if and only if $S_m \le f$ for each $m \in D$, i.e., $f(\text{supp } S_m) \ge h(S_m)$ where $h(S_m)$ is the height of the L-fuzzy point $S_m$ in $M(L^X)$.

Definition 2.3.4.

A sequence in X is a function $S:\mathbb{N} \to M(L^X)$ where $\mathbb{N}$ is the set of all natural numbers. We shall denote $S(m)$ by $S_m$ and S by $(S_m)_{m \in \mathbb{N}}$.

Remark 2.3.5.

We say that the sequence $(S_m)_{m \in \mathbb{N}}$ in X is contained in $f \in L^X$ if and only if $f(\text{supp } S_m) \geq h(S_m)$ for each $m \in \mathbb{N}$, where $h(S_m)$ is the height of $S_m$.

<u>Definition</u> <u>2.3.6.</u>

A net $T = (T_m)_{m \in E}$ in X is called a <u>subnet</u> of a net $S = (S_k)_{k \in D}$ in X if and only if there is a function $J: E \to D$ such that $T_i = S_{J(i)}$ for each $i \in E$ and for each $k \in D$ there is $m \in E$ such that $J(p) \geq k$ whenever $E \ni p \geq m$.

<u>Definition</u> <u>2.3.7.</u>

A sequence $T = (T_m)_{m \in \mathbb{N}}$ in X is called a <u>subsequence</u> of a sequence $S = (S_m)_{m \in \mathbb{N}}$ in X if and only if there is a sequence J of natural numbers such that $T_i = S_{J(i)}$ for each $i \in \mathbb{N}$ and for each $m \in \mathbb{N}$ there is $r \in \mathbb{N}$ such that $J(i) \geq m$ whenever $i \geq r$.

<u>Definition</u> <u>2.3.8.</u> Zhao [110]

A net $(S_m)_{m \in D}$ is called an <u>$\alpha$-net</u> ($\alpha \in M(L)$) if and only if for each $\gamma \in \beta^*(\alpha)$, the net $h(S) = (h(S_m))_{m \in D}$ is eventually greater than $\gamma$, i.e., there is $m_0 \in D$ such that $h(S_m) \geq \gamma$ whenever $m \geq m_0$, where $h(S_m)$ is the height of the L-fuzzy point $S_m \in M(L^X)$. If $h(S_m) = \alpha$ for all $m \in D$, then we shall say that $(S_m)_{m \in D}$ is a <u>constant</u> <u>$\alpha$-net</u> .

<u>Remark</u> <u>2.3.9.</u>

When D is the set of all natural numbers in definition 2.3.8., $(S_m)_{m \in \mathbb{N}}$ is called an <u>$\alpha$-sequence</u>.

# CHAPTER III

## Fuzzy topological spaces

Different definitions of fuzzy topology have been stated in the literature [e.g. 18, 54, 44, 88, 41, 39]. The first basic notion of fuzzy topology is due to Chang [18]. In [54], Lowen required that a fuzzy topology had one more axiom, which included the constant fuzzy sets. Here we adopt Chang's definition of fuzzy topology and consider that of Lowen as a special case (definition 3.2.1.).

This chapter is divided in four sections.

Our main goal in section 1 is to establish the basic notions and results of L-fuzzy topology that here we shall deal with. We also present some results, obtained by us, that will be necessary later on, as well as, some definitions that we propose.

Section 2 is reserved for some special L-fuzzy topological spaces and some related properties.

The third section is devoted to some special functions with some of their properties.

In the fourth section we concentrate on countability and separation axioms. Here we propose new good definitions of regularity and normality.

Although, some of the definitions, results and proofs we refer to are given only for [0,1]-fuzzy topological spaces, they are totally similar for an L-fuzzy topological space where L is a fuzzy lattice.

## 1. Some basic definitions and assertions

In the following, let X be a nonempty set and let $L=L(\le,\vee,\wedge,')$ be a fuzzy lattice with a smallest element 0 and a largest element $1 (0\neq 1)$.

Definition 3.1.1. Chang [18]

An L-fuzzy topology on X is a subset $\mathcal{T}$ of $L^X$, the set of all L-fuzzy sets, having the following properties:

(i) the L-fuzzy sets $\phi$ and X belong to $\mathcal{T}$.

(ii) if f, g are in $\mathcal{T}$ then $f\wedge g$ is in $\mathcal{T}$.

(iii) if $(f_j)_{j\in J}$ is a family in $\mathcal{T}$ then $\bigvee_{j\in J} f_j$ is in $\mathcal{T}$.

The pair $(X,\mathcal{T})$, where $\mathcal{T}$ is an L-fuzzy topology on X, is called an L-fuzzy topological space (for short L-fts).

If $(X,\mathcal{T})$ is an L-fuzzy topological space, we say that an L-fuzzy set f is open or $\mathcal{T}$-open in the L-fuzzy topological space $(X,\mathcal{T})$ if and only if $f\in\mathcal{T}$. We say that $f\in L^X$ is closed or $\mathcal{T}$-closed in the L-fts $(X,\mathcal{T})$ if and only if $f'\in\mathcal{T}$.

Definition 3.1.2. Wong [103]

Let $(X,\mathcal{T})$ be an L-fts. A collection $\mathcal{B}\subset\mathcal{T}$ is said to be a base for $\mathcal{T}$ if and only if for each $f\in\mathcal{T}$, there is a collection $\mathcal{C}\subset\mathcal{B}$ such that $f = \bigvee_{g\in\mathcal{C}} g$.

Definition 3.1.3. Wong [103]

Let $(X,\mathcal{T})$ be an L-fts. A collection $\mathcal{S}\subset\mathcal{T}$ is said to be a <u>subbase</u> for $\mathcal{T}$ if and only if the family of all finite intersection of members of $\mathcal{S}$ forms a base for $\mathcal{T}$.

<u>Proposition 3.1.4.</u>

Let $(X,\mathcal{T})$ be an L-fts. Then $f\in\mathcal{T}$ if and only if for all $p\in pr(L)$ and for every $x\in X$ such that $f(x)\nleq p$, there is $g\in\mathcal{T}$ with $g\leq f$ and $g(x)\nleq p$.

<u>Proof</u>

<u>Necessity:</u>

Take $g=f$.

<u>Sufficiency:</u>

Suppose that $f\notin\mathcal{T}$. Let $\mathcal{T}^* = \{g\in\mathcal{T};\ g<f\}$.

Since $\bigvee_{g\in\mathcal{T}^*} g \in \mathcal{T}$, $f\neq \bigvee_{g\in\mathcal{T}^*} g$. But for all $g\in\mathcal{T}^*$ we have $g\leq \bigvee_{g\in\mathcal{T}^*} g<f$. Thus, there is $x\in X$ with $g(x)\leq \left(\bigvee_{g\in\mathcal{T}^*} g\right)(x)<f(x)$ for all $g\in\mathcal{T}^*$. So, $f(x)\nleq\left(\bigvee_{g\in\mathcal{T}^*} g\right)(x)$.

By the spatiality of L (proposition 1.2.3.), there exists $q\in pr(L)$ such that $f(x)\nleq q$ and $\left(\bigvee_{g\in\mathcal{T}^*} g\right)(x)\leq q$. Then $f(x)\nleq q$ and $g(x)\leq q$ for all $g\in\mathcal{T}^*$, yielding a contradiction.

<u>Definition</u> <u>3.1.5.</u> Pu and Liu [80]

Let $(X,\mathcal{T})$ be an L-fts and let $f\in L^X$. The <u>interior</u> <u>of</u> <u>f</u>, int(f), and the <u>closure</u> <u>of</u> <u>f</u>, cl(f), are defined as follows:

$$\mathrm{int}(f) = \bigvee\{g\in\mathcal{T};\ g\leq f\}$$

$$\mathrm{cl}(f) = \bigwedge\{g\in L^X;\ g\geq f \text{ and } g'\in\mathcal{T}\}$$

Remark 3.1.6.

(i) Evidently int(f) is the largest open L-fuzzy set contained in f and int(int(f)) = int(f). Similarly cl(f) is the smallest closed L-fuzzy set containing f and cl(cl(f)) = cl(f). Pu and Liu [80]

(ii) (cl(f))′ = int(f′) and (int(f))′ = cl(f′) Pu and Liu [80]

(iii) for a family $(f_j)_{j\in J}$ of L-fuzzy sets we have: $\bigvee_{j\in J} cl(f_j) \leq cl\left(\bigvee_{j\in J} f_j\right)$, $\bigvee_{j\in J} int(f_j) \leq int\left(\bigvee_{j\in J} f_j\right)$ and if J is finite, $\bigvee_{j\in J} cl(f_j) = cl\left(\bigvee_{j\in J} f_j\right)$ Azad [7].

Definition 3.1.7.

Let $(X,\mathcal{T})$ be an L-fts and let $f\in L^X$. The L-fuzzy set f is called:

(i) Regularly open [7] if and only if f=int(cl(f)).

(ii) Regularly closed [7] if and only if f=cl(int(f)).

(iii) Semiopen [7] if and only if there exists $g\in\mathcal{T}$ such that $g\leq f\leq cl(g)$.

(iv) Semiclosed [7] if and only if there exists a closed L-fuzzy set g such that $int(g)\leq f\leq g$.

(v) Regularly semiopen [23] if and only if there exists a regularly open L-fuzzy set g such that $g\leq f\leq cl(g)$.

(vi) Regularly semiclosed [23] if and only if there exists a regularly closed L-fuzzy set g such that $int(g)\leq f\leq g$.

(vii) pre-open [75] if and only if $f\leq int(cl(f))$.

(viii) pre-closed [75] if and only if $cl(int(f)) \leq f$.

Remark 3.1.8. Azad [7]

Let $(X,\mathcal{T})$ be an L-fts and let $f \in L^X$.

(i) f is semiclosed if and only if $int(cl(f)) \leq f$.

(ii) f is semiopen if and only if $cl(int(f)) \geq f$.

(iii) the closure of an open L-fuzzy set is a regularly closed L-fuzzy set.

(iv) the interior of a closed L-fuzzy set is a regularly open L-fuzzy set.

Definition 3.1.9.

Let $(X,\mathcal{T})$ be an L-fts, let $x_\alpha$ be an L-fuzzy point in $M(L^X)$ and let $(S_m)_{m \in D}$ be a net. The L-fuzzy point $x_\alpha \in M(L^X)$ is called a:

(i) limit point [110] of $(S_m)_{m \in D}$ (or $(S_m)_{m \in D}$ converges to $x_\alpha$) if and only if for each closed L-fuzzy set f with $f(x) \ngeq \alpha$ there exists $m_0 \in D$ such that $m \geq m_0$ implies that $S_m \nleq f$, i.e., $f(\text{supp } S_m) \ngeq h(S_m)$ where supp $S_m$ and $h(S_m)$ are the support and the height of $(S_m)_{m \in D}$ respectively.

Notation $S_m \to x_\alpha$ .

(ii) cluster point [110] of $(S_m)_{m \in D}$ if and only if for each closed L-fuzzy set f with $f(x) \ngeq \alpha$ we have that for all $j \in D$ there is $m \in D$ such that $m \geq j$ and $S_m \nleq f$, i.e., $f(\text{supp } S_m) \ngeq h(S_m)$.

(iii) $\theta$ - cluster point of $(S_m)_{m \in D}$ if and only if for each closed L-fuzzy set f with $f(x) \ngeq \alpha$ we have that for all $j \in D$ there is $m \in D$ such that $m \geq j$ and $S_m \nleq int(f)$, i.e.,

$(\mathrm{int}(f))(\mathrm{supp}\ S_m) \nleq h(S_m)$.

(iv) <u>$\delta$ - cluster point</u> of $(S_m)_{m\in D}$ if and only if for each closed L-fuzzy set f with $f(x) \nleq \alpha$ we have that for all $j\in D$ there is $m\in D$ such that $m\geq j$ and $S_m \nleq \mathrm{cl}(\mathrm{int}(f))$, i.e., $(\mathrm{cl}(\mathrm{int}(f)))(\mathrm{supp}\ S_m) \nleq h(S_m)$.

(v) <u>Semi - $\delta$ - cluster point</u> of $(S_m)_{m\in D}$ if and only if for each semiclosed L-fuzzy set f with $f(x) \nleq \alpha$ we have that for all $j\in D$ there is $m\in D$ such that $m\geq j$ and $S_m \nleq \mathrm{cl}(\mathrm{int}(f))$, i.e., $(\mathrm{cl}(\mathrm{int}(f)))(\mathrm{supp}\ S_m) \nleq h(S_m)$.

(vi) <u>Semi - $\theta$ - cluster point</u> of $(S_m)_{m\in D}$ if and only if for each semiclosed L-fuzzy set f with $f(x) \nleq \alpha$ we have that for all $j\in D$ there is $m\in D$ such that $m\geq j$ and $S_m \nleq \mathrm{int}(f)$, i.e., $(\mathrm{int}(f))(\mathrm{supp}\ S_m) \nleq h(S_m)$.

(vii) <u>pre-cluster point</u> of $(S_m)_{m\in D}$ if and only if for each pre-closed L-fuzzy set f with $f(x) \nleq \alpha$ we have that for all $j\in D$ there is $m\in D$ such that $m\geq j$ and $S_m \nleq f$, i.e., $f(\mathrm{supp}\ S_m) \nleq h(S_m)$.

<u>Proposition 3.1.10.</u>

Let $(X,\mathcal{T})$ be an L-fts. An L-fuzzy point $x_\alpha \in M(L^X)$ is a cluster point of a net in $(X,\mathcal{T})$ if and only if this net has a subnet converging to $x_\alpha$.

<u>Proof</u>

The same as that given by Pu and Liu [80] in theorem 13.2.

## 2. Some special L-fuzzy topological spaces

Definition 3.2.1. Pu and Liu [81]

An L-fuzzy topological space $(X,\mathcal{T})$ is fully stratified if and only if each L-fuzzy set on X taking a constant value on X is open.

Definition 3.2.2. Mashhour et al. [64]

An L-fts $(X,\mathcal{T})$ is said to be extremally disconnected if and only if $cl(f) \in \mathcal{T}$ for every $f \in \mathcal{T}$.

Definition 3.2.3. Gantner et al. [35]

Let $(X,\mathcal{T})$ be an L-fts and $A \subset X$. Let $\mathcal{T}_A$ be the set of restrictions $\{f|_A;\ f \in \mathcal{T}\}$. Then $\mathcal{T}_A$ is an L-fuzzy topology on A and we say that $(A, \mathcal{T}_A)$ is an L-fuzzy subspace of $(X,\mathcal{T})$.

Definition 3.2.4. Wong [103]

Let $\{(X_\lambda, \mathcal{T}_{X_\lambda})\}_{\lambda \in J}$ be a family of L-fts's, let X be the cartesian product $\prod_{\lambda \in J} X_\lambda$ and let $\pi_\lambda : X \to X_\lambda$ be the $\lambda$th-projection map. Let $\mathcal{S} = \{\pi_\lambda^{-1}(\mu_\lambda);\ \lambda \in J \text{ and } \mu_\lambda \in \mathcal{T}_{X_\lambda}\}$ and let $\mathcal{B}$ be the family of all finite intersections of members of $\mathcal{S}$. The L-fuzzy topology $\mathcal{T}$ on X, having $\mathcal{S}$ as a subbase and $\mathcal{B}$ as a base, is called the product topology. The pair $(X,\mathcal{T})$ is called the L-fuzzy product space of the L-fts's $(X_\lambda, \mathcal{T}_{X_\lambda})$, $\lambda \in J$.

Remark 3.2.5. Warner [98]

Let $(X,\delta)$ be a topological space. The set F of all continuous functions from $(X,\delta)$ to L with its Scott topology forms an L-fuzzy topology, which will be denoted by $\omega(\delta)$.

When L=[0,1], then F is the set $\omega(\delta)$ of lower semicontinuous functions from $(X,\delta)$ to [0,1] (Lowen [54]).

Remark 3.2.6.

Let $(X,\delta)$ be a topological space. Lowen [54] has called a [0,1]-fuzzy topological property $P_f$ a good extension of a topological property P if and only if: $(X,\delta)$ has P if and only if $(X,\omega(\delta))$ has $P_f$, where $\omega(\delta)$ is the [0,1]-fuzzy topology of lower semicontinuous functions. Warner [98] has extended the definition of goodness to an L-fuzzy property where L is a continuous frame. Then we have the following:

Definition 3.2.7.

Let $(X,\delta)$ be a topological space. An L-fuzzy topological property $P_f$ is a good extension of a topological property P if and only if:

the topological space $(X,\delta)$ has P if and only if the L-fuzzy topological space $(X,\omega(\delta))$ has $P_f$, where $\omega(\delta)$ is the L-fuzzy topology of continuous functions from $(X,\delta)$ to L with its Scott topology.

Definition 3.2.8. Lowen [54]

Let $(X,\mathcal{T})$ be an L-fts. The L-fuzzy topology $\mathcal{T}$ on X is called topological if and only if there is a topology $\delta$ on X with $\mathcal{T} = \omega(\delta)$.

Proposition 3.2.9.

Let $(X,\delta)$ be a topological space. Then we have the following:

(i) An L-fuzzy set $f \in L^X$ is open in $(X,\omega(\delta))$ if and only if $f^{-1}(\{t \in L;\ t \nleq e\}) \in \delta$ for every $e \in L$.

(ii) An L-fuzzy set $f \in L^X$ is open in $(X,\omega(\delta))$ if and only if $f^{-1}(\{t \in L;\ t \nleq p\}) \in \delta$ for every $p \in pr(L)$.

(iii) An L-fuzzy set $f \in L^X$ is closed in $(X,\omega(\delta))$ if and only if $f^{-1}(\{t \in L;\ t \geq b\})$ is closed in $(X,\delta)$ for every $b \in L$.

Proof

(i) Necessity:

If $f \in \omega(\delta)$ then f is a continuous function from $(X,\delta)$ to L with its Scott topology. Thus, since for any $e \in L$ the set $\{t \in L;\ t \nleq e\}$ is Scott open (proposition 1.2.10.), we have that $f^{-1}(\{t \in L;\ t \nleq e\}) \in \delta$.

Sufficiency:

Since L is completely distributive, the sets of the form $\{t \in L;\ t \nleq e\}$ where $e \in L$ generate the Scott topology (proposition 1.2.10.). Therefore if $f^{-1}(\{t \in L;\ t \nleq e\}) \in \delta$ for every $e \in L$ then $f \in \omega(\delta)$.

(ii) This follows as in (i) from the fact that the Scott topology of a completely distributive lattice L is generated by the sets of the form $\{t\in L;\ t\nleq p\}$ where $p\in pr(L)$ (proposition 1.2.11.).

(iii) $f\in L^X$ is closed in $(X,\omega(\delta))$ if and only if $f'\in\omega(\delta)$ if and only if $f'^{-1}(\{t\in L;\ t\nleq e\})\in\delta$ for every $e\in L$ if and only if $f^{-1}(\{t\in L;\ t\ngeq e'=b\})\in\delta$ for every $b\in L$ if and only if $(f^{-1}(\{t\in L;\ t\ngeq b\}))'$ is closed in $(X,\delta)$ for every $b\in L$ if and only if $f^{-1}(\{t\in L;\ t\geq b\})$ is closed in $(X,\delta)$ for every $b\in L$.

<u>Proposition</u> <u>3.2.10.</u>

Let $(X,\delta)$ be a topological space. Then $A\in\delta$ if and only if $\chi_A$ is open in $(X,\omega(\delta))$.

<u>Proof</u>

This immediately follows from proposition 3.2.9..

<u>Proposition</u> <u>3.2.11.</u> Warner [99]

Let $(X,\delta)$ be a topological space. The family $(f_i^{e_i U_i})_{i\in J}$, where $f_i^{e_i U_i}(x)=\begin{cases} e_i & \text{if } x\in U_i \\ 0 & \text{if } x\notin U_i \end{cases}$, is a base for $\omega(\delta)$.

<u>Proof</u>

See lemma 6 in Warner [99].

<u>Lemma</u> <u>3.2.12.</u>

Let $(X,\delta)$ be a topological space, let f be an L-fuzzy set in the L-fts $(X,\omega(\delta))$ and $p\in pr(L)$. Then

$$(cl(f))^{-1}(\{t\in L;\ t\nleq p\})\subseteq cl(f^{-1}(\{t\in L;\ t\nleq p\})).$$

Proof

Firstly we are going to prove that any closed set C in $(X,\delta)$ with $C \supseteq f^{-1}(\{t \in L;\ t \nleq p\})$ satisfies $C \supseteq (cl(f))^{-1}(\{t \in L;\ t \nleq p\})$.

Let C be a closed set in $(X,\delta)$ with $C \supseteq f^{-1}(\{t \in L;\ t \nleq p\})$ and let $g: X \to L$ be defined by $g(x) = 1$ if $x \in C$ and $g(x) = p$ otherwise.

Since for every $e \in L$ $g^{-1}(\{t \in L;\ t \geq e\}) = \begin{cases} X & \text{if } e \leq p \\ C & \text{if } e \nleq p \end{cases}$, we have $g^{-1}(\{t \in L;\ t \geq e\})$ closed in $(X,\delta)$ for all $e \in L$. Thus, by proposition 3.2.9., g is closed in $(X,\omega(\delta))$.

We also have $g \geq f$. Thus, $g \geq cl(f)$. Then, $C = g^{-1}(\{t \in L;\ t \nleq p\}) \supseteq (cl(f))^{-1}(\{t \in L;\ t \nleq p\})$.

Therefore, since $f^{-1}(\{t \in L;\ t \nleq p\}) \subseteq cl(f^{-1}(\{t \in L;\ t \nleq p\}))$ and $cl(f^{-1}(\{t \in L;\ t \nleq p\}))$ is closed in $(X,\delta)$, we have $cl(f^{-1}(\{t \in L;\ t \nleq p\}) \supseteq (cl(f))^{-1}(\{t \in L;\ t \nleq p\})$.

Proposition 3.2.13.

Let $(X,\delta)$ be a topological space and $A \subseteq X$. Considering the L-fts $(X,\omega(\delta))$, $e \in L$ and $f(x) = \begin{cases} e & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$, we have $cl(f)(x) = \begin{cases} e & \text{if } x \in cl(A) \\ 0 & \text{otherwise} \end{cases}$ and $int(f)(x) = \begin{cases} e & \text{if } x \in int(A) \\ 0 & \text{otherwise} \end{cases}$.

Proof

Let $g(x) = \begin{cases} e & \text{if } x \in cl(A) \\ 0 & \text{otherwise} \end{cases}$ and $h(x) = \begin{cases} e & \text{if } x \in int(A) \\ 0 & \text{otherwise} \end{cases}$.

We shall prove that $cl(f) = g$ and $int(f) = h$.

Since for every $b \in L$ $g^{-1}(\{t \in L;\ t \geq b\}) = \begin{cases} X & \text{if } b = 0 \\ cl(A) & \text{if } e \geq b \text{ and } b \neq 0 \\ \phi & \text{if } e \ngeq b \text{ and } b \neq 0 \end{cases}$ is closed in $(X,\delta)$, by proposition

3.2.9., $g$ is closed in $(X,w(\delta))$. We also have $g \geq f$.

Thus, $f \leq cl(f) \leq cl(g) = g$. Therefore, we have $cl(f)(x) = 0$ for all $x \notin cl(A)$ and $cl(f) = e$ for all $x \in A$.

From $cl(f) \leq g$ we obtain $(cl(f))^{-1}(\{t \in L;\ t \neq e\}) \subseteq g^{-1}(\{t \in L;\ t \neq e\}) = (cl(A))'$. Hence $cl(f)(x) = e$ for all $x \in cl(A)$ and $cl(f)(x) = 0$ for all $x \notin cl(A)$ and $cl(f) = g$.

Similarly, for every $b \in L$, $h^{-1}(\{t \in L;\ t \nleq b\}) = \begin{cases} \phi & \text{if } e \leq b \\ int(A) & \text{if } e \nleq b \end{cases}$ is in $\delta$, so, by proposition 3.2.9., $h \in \omega(\delta)$. We also have $h \leq f$. Then, $h \leq int(f) \leq f$. Thus, we have $int(f)(x) = 0$ for all $x \notin A$ and $int(f)(x) = e$ for all $x \in int(A)$.

Since $int(f) \in \omega(\delta)$ and $int(f) \leq f$ we obtain $(int(f))^{-1}(\{t \in L;\ t \neq 0\}) \subseteq f^{-1}(\{t \in L;\ t \neq 0\}) = A$ and $(int(f))^{-1}(\{t \in L;\ t \neq 0\}) \subseteq int(A)$.

Hence $int(f)(x) = \begin{cases} e & \text{if } x \in int(A), \\ 0 & \text{otherwise} \end{cases}$ that is, $int(f) = h$

<u>Corollary</u> <u>3.2.14.</u>

Let $(X,\delta)$ be a topological space and $A \subseteq X$. Considering the L-fts $(X,\omega(\delta))$, $e \in L$ and $f(x) = \begin{cases} e & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$, we have $int(cl(f))(x) = \begin{cases} e & \text{if } x \in int(cl(A)) \\ 0 & \text{otherwise} \end{cases}$

<u>Proof</u>

This immediately follows from proposition 3.2.13.

<u>Proposition</u> <u>3.2.15.</u>

Let $(X,\delta)$ be a topological space. Then we have the following:

(i) If $A$ is a pre-open set in $(X,\delta)$ then $\chi_A$ is a pre-open L-fuzzy set in $(X,\omega(\delta))$.

(ii) If A is a semiopen set in $(X,\delta)$ then $\chi_A$ is a semiopen L-fuzzy set in $(X,\omega(\delta))$.

Proof

(i) Since A is a pre-open set in $(X,\delta)$ we have $A\subseteq \text{int}(\text{cl}(A))$. Thus $\chi_A \leq \chi_{\text{int}(\text{cl}(A))}$ and, by corollary 3.2.14., $\chi_{\text{int}(\text{cl}(A))} = \text{int}(\text{cl}(\chi_A))$. Therefore $\chi_A \leq \text{int}(\text{cl}(\chi_A))$.

Hence $\chi_A$ is pre-open in $(X,\omega(\delta))$.

(ii) Since A is a semiopen set in $(X,\delta)$, there exists $U\in\delta$ such that $U\subseteq A\subseteq \text{cl}(U)$. Thus, $\chi_U\leq\chi_A\leq\chi_{\text{cl}(U)}$ and by proposition 3.2.13 $\chi_{\text{cl}(U)}=\text{cl}(\chi_U)$. Since $U\in\delta$, by proposition 3:2.10., $\chi_U\in\omega(\delta)$.

Hence $\chi_A$ is semiopen in $(X,\omega(\delta))$.

Lemma 3.2.16.

Let $(X,\mathcal{T})$ be an L-fts and let $\mathcal{B}$ be a nonempty family of L-fuzzy sets. Then an L-fuzzy set f is a union of elements of $\mathcal{B}$ if and only if for all $p\in pr(L)$ and for all $x\in X$ with $f(x)\nleq p$ there is $g\in\mathcal{B}$ such that $g\leq f$ and $g(x)\nleq p$.

Proof

Necessity:

Let f be an L-fuzzy set, $p\in pr(L)$, let $x\in X$ with $f(x)\nleq p$ and let $f = \bigvee_{g\in\mathcal{B}} g$.

Thus $\left(\bigvee_{g\in\mathcal{B}} g\right)(x)\nleq p$, which implies that there is $g\in\mathcal{B}$ such that $g(x)\nleq p$.

Sufficiency:

Let f be an L-fuzzy set such that for all $p\in pr(L)$ and for all $x\in X$ with $f(x)\nleq p$ there is $g\in\mathcal{B}$ with $g\le f$ and $g(x)\nleq p$. Let $\mathcal{C} = \{h\in\mathcal{B};\ h\le f\}$ and suppose that $\bigvee_{h\in\mathcal{C}} h < f$.

Therefore there exists $x\in X$ with $\left(\bigvee_{h\in\mathcal{C}} h\right)(x)<f(x)$, so $f(x)\nleq\left(\bigvee_{h\in\mathcal{C}} h\right)(x)$. By the spatiality of L (proposition 1.2.3.), there is $q\in pr(L)$ such that $f(x)\nleq q$ and $\left(\bigvee_{h\in\mathcal{C}} h\right)(x)\le q$, yielding a contradiction.

Lemma 3.2.17.

Let $(X,\delta)$ be a topological space. If f is a semiopen L-fuzzy set in $(X,\omega(\delta))$ then f is semicontinuous as a function from $(X,\delta)$ to L with its Scott topology.

Proof

Let f be a semiopen L-fuzzy set in $(X,\omega(\delta))$. We want to prove that f is semicontinuous, i.e., $f^{-1}(V)$ is a semiopen set in $(X,\delta)$ for each Scott open set V in L. Since $(\{t\in L;\ t\nleq p\})_{p\in pr(L)}$ is a base for the Scott topology (proposition 1.2.11.), $f^{-1}(\bigcup_i V_i) = \bigcup_i f^{-1}(V_i)$ and any union of semiopen sets is a semiopen set, it will suffice to prove that $f^{-1}(\{t\in L;\ t\nleq p\})$ is semiopen in $(X,\delta)$ for every $p\in pr(L)$.

Because f is a semiopen L-fuzzy set in $(X,\omega(\delta))$, there is $g\in\omega(\delta)$ with $g\le f\le cl(g)$. Thus $g^{-1}(\{t\in L;\ t\nleq p\})\subseteq f^{-1}(\{t\in L;\ t\nleq p\})\subseteq(cl(g))^{-1}(\{t\in L;\ t\nleq p\})\subseteq cl(g^{-1}(\{t\in L;\ t\nleq p\})$ for all $p\in pr(L)$ where the last inclusion is due to lemma 3.2.12.

Since $g\in\omega(\delta)$, by proposition 3.2.9., $g^{-1}(\{t\in L;\ t\nleq p\})$

$\in\delta$ for all $p\in pr(L)$. Hence $f^{-1}(\{t\in L;\ t\nleq p\})$ is semiopen in $(X,\delta)$ for all $p\in pr(L)$.

<u>Proposition 3.2.18.</u>

Let $(X,\delta)$ be a topological space. Then every semiopen L-fuzzy set in $(X,\omega(\delta))$ is a union of elements of the collection $\mathcal{B} = (f_i)_{i\in J}$ where $f_i(x) = \begin{cases} e_i & \text{if } x\in U_i \text{ semiopen in } (X,\delta) \\ 0 & \text{otherwise} \end{cases}$.

<u>Proof</u>

Let $p\in pr(L)$, $x\in X$ and let g be a semiopen L-fuzzy set in $(X,\omega(\delta))$ with $g(x)\nleq p$. From lemma 3.2.16. it will suffice to prove that there is $i\in J$ such that $f_i\le g$ and $f_i(x)\nleq p$.

Since $g(x)\nleq p$, by the continuity of L (proposition 1.2.3.), there is $b\in L$ such that $b\ll g(x)$, $b\nleq p$.

Take $e_0\in L$ such that $b\ll e_0\ll g(x)$.

Therefore $g(x)\in H = \{q\in L;\ e_0\ll q\}$ which is Scott open in L (proposition 1.2.9.). By lemma 3.2.17., since g is semiopen in $(X,\omega(\delta))$, g is semicontinuous. By the semicontinuity of g, there is a semiopen set $U_0$ in $(X,\delta)$ with $x\in U_0$ such that $g(U_0)\subseteq H$.

Thus $g(x)\ge e_0$ for all $x\in U_0$ and $e_0\nleq p$ since $e_0\gg b\nleq p$. Therefore, $f_0\le g$ and $f_0(x)\nleq p$ where $f_0(x) = \begin{cases} e_0 & \text{if } x\in U_0 \\ 0 & \text{otherwise} \end{cases}$.

<u>Proposition 3.2.19.</u> McLean and Warner [65]

If L is a frame and $\mathcal{T}$ is an L-fuzzy topology on a set X, then the map $\phi:\mathcal{T}\to\mathbb{P}(X\times pr(L))$ defined by $\phi(f) =$

$\{(x,p);\ p \nleq f(x)\}$ for every $f \in \mathcal{T}$ is a frame morphism and $\phi(\mathcal{T})$ is a topology on $X \times pr(L)$. If L is spatial then the L-fuzzy topology $\mathcal{T}$ is isomorphic as a frame to the topology $\phi(\mathcal{T})$.

Proof

Firstly we would like to remark that since L is a frame, $\mathcal{T}$ is itself a frame [99]. For the proof of our proposition we refer to theorem 6 in [65].

Proposition 3.2.20. McLean and Warner [65]

Let L be a continuous frame and let $(X,\delta)$ be a topological space. Then the topology $\phi(\omega(\delta))$ on $X \times pr(L)$ is the product topology $\delta \times \phi(L)$.

Proof

See theorem 6 in McLean and Warner [65].

## 3. **Some special functions**

Definition 3.3.1.

Let $(X,\mathcal{T}_X)$ and $(Y,\mathcal{T}_Y)$ be L-fts's. A function $f:(X,\mathcal{T}_X)\to(Y,\mathcal{T}_Y)$ is called:

(i) continuous [18] if and only if $f^{-1}(g)\in \mathcal{T}_X$ for every $g\in\mathcal{T}_Y$.

(ii) semicontinuous [7] if and only if $f^{-1}(g)$ is semiopen in $(X,\mathcal{T}_X)$ for every $g\in\mathcal{T}_Y$, i.e., $f^{-1}(g)$ is semiclosed in $(X,\mathcal{T}_X)$ for every closed L-fuzzy set g in $(Y,\mathcal{T}_Y)$.

(iii) almost continuous [7] if and only if $f^{-1}(g)\in \mathcal{T}_X$ for all regularly open L-fuzzy set g in $(Y,\mathcal{T}_Y)$.

(iv) weakly continuous [7] if and only if $f^{-1}(g)\le \mathrm{int}(f^{-1}(\mathrm{cl}(g)))$ for all $g\in\mathcal{T}_Y$.

(v) open [103] if and only if $f(g)\in \mathcal{T}_Y$ for every $g\in\mathcal{T}_X$.

(vi) almost open [74] if and only if $f(g)\in \mathcal{T}_Y$ for every regularly open L-fuzzy set g in $(X,\mathcal{T}_X)$.

(vii) strongly continuous [33] if and only if $f(\mathrm{cl}(g))\le f(g)$ for every $g\in L^X$.

(viii) irresolute [23] if and only if $f^{-1}(g)$ is semiopen in $(X,\mathcal{T}_X)$ for every semiopen L-fuzzy set g in $(Y,\mathcal{T}_Y)$, i.e., $f^{-1}(g)$ is semiclosed in $(X,\mathcal{T}_X)$ for every semiclosed L-fuzzy set g in $(Y,\mathcal{T}_Y)$.

(ix) pre-continuous [75] if and only if $f^{-1}(g)$ is pre-open in $(X,\mathcal{T}_X)$ for every $g\in\mathcal{T}_Y$.

(x) M-pre-continuous [75] if and only if $f^{-1}(g)$ is pre-open in $(X,\mathcal{T}_X)$ for every pre-open L-fuzzy set g in $(Y,\mathcal{T}_Y)$.

Definition 3.3.2. Nanda [75]

Let $(X,\mathcal{T})$ and $(Y,\mathcal{T}^*)$ be L-fts's and let $\mathcal{T}_\phi$ be the L-fuzzy topology on X which has the set of all pre-open L-fuzzy sets of $(X,\mathcal{T})$ as a subbase. A mapping $f:(X,\mathcal{T})\to(Y,\mathcal{T}^*)$ is called $\phi$-continuous if and only if $f:(X,\mathcal{T}_\phi)\to(Y,\mathcal{T}^*)$ is continuous and $f:(X,\mathcal{T})\to(Y,\mathcal{T}^*)$ is said to be $\phi'$-continuous if and only if $f:(X,\mathcal{T}_\phi)\to(Y,\mathcal{T}^*_\phi)$ is continuous.

Proposition 3.3.3. Pu and Liu [81]

Let $\{(X_\lambda, \mathcal{T}_{X_\lambda})\}_{\lambda\in J}$ be a family of L-fts's and let $(X,\mathcal{T})$ be the L-fuzzy product space of the L-fts's $(X_\lambda, \mathcal{T}_{X_\lambda})$, $\lambda\in J$. Thus:

(i) For every $\lambda\in J$ the $\lambda$th-projection map $\pi_\lambda:X\to X_\lambda$ is continuous.

(ii) If $(X_\lambda, \mathcal{T}_{X_\lambda})$ is fully stratified, then the projection $\pi_\lambda:X\to X_\lambda$ is an open map.

Proof

See theorem 2.2. in Pu and Liu [81].

Proposition 3.3.4. Mukherjee and Sinha [71]

Let $(X,\mathcal{T}_X)$ and $(Y,\mathcal{T}_Y)$ be L-fts's and let $f:(X,\mathcal{T}_X)\to(Y,\mathcal{T}_Y)$ be an almost continuous and almost open map. Then the inverse image of any regularly open L-fuzzy set in $(Y,\mathcal{T}_Y)$ is a regularly open L-fuzzy set in $(X,\mathcal{T}_X)$, i.e., the inverse image of any regularly closed L-fuzzy set in $(Y,\mathcal{T}_Y)$ is a regularly closed L-fuzzy set in $(X,\mathcal{T}_X)$.

<u>Proof</u>

See theorem 3.5. in [71].

<u>Proposition</u> <u>3.3.5.</u> Mukherjee and Sinha [71]

Let $(X,\mathcal{T}_X)$ and $(Y,\mathcal{T}_Y)$ be L-fts's and let $f:(X,\mathcal{T}_X)\rightarrow(Y,\mathcal{T}_Y)$ be an almost continuous map with $f^{-1}(cl(h))\leq cl(f^{-1}(h))$ for all $h\in\mathcal{T}_Y$. Then the inverse image of any regularly open L-fuzzy set in $(Y,\mathcal{T}_Y)$ is a regularly open L-fuzzy set in $(X,\mathcal{T}_X)$.

<u>Proof</u>

See theorem 3.6. in [71].

<u>Proposition</u> <u>3.3.6.</u> Allam and Zahran [3]

Let $(X,\mathcal{T}_X)$ and $(Y,\mathcal{T}_Y)$ be L-fts's and let $f:(X,\mathcal{T}_X)\rightarrow(Y,\mathcal{T}_Y)$ be a weakly continuous map with $f^{-1}(cl(h)\leq cl(f^{-1}(h))$ for every regularly open L-fuzzy set h in $(Y,\mathcal{T}_Y)$. Then the inverse image of any regularly open L-fuzzy set in $(Y,\mathcal{T}_Y)$ is regularly open in $(X,\mathcal{T}_X)$.

<u>Proof</u>

See theorem 3.10. in [3].

Proposition 3.3.7. Nanda [75]

Let $(X,\mathcal{T})$ and $(Y,\mathcal{T}^*)$ be L-fts's. If $f:(X,\mathcal{T})\to(Y,\mathcal{T}^*)$ is M-pre-continuous, then f is $\phi'$-continuous.

Proof

See theorem 3.3. in [75].

Proposition 3.3.8. Allam and Zahran [3,26]

Let $(X,\mathcal{T}_X)$ and $(Y,\mathcal{T}_Y)$ be L-fts's and let $f:(X,\mathcal{T}_X)\to(Y,\mathcal{T}_Y)$ be a weakly continuous map with $int(f^{-1}(h))\le f^{-1}(int(h))$ for every regularly semiopen L-fuzzy set h in $(Y,\mathcal{T}_Y)$. Then the inverse image of any regularly closed L-fuzzy set in $(Y,\mathcal{T}_Y)$ is regularly closed in $(X,\mathcal{T}_X)$, i.e., the inverse image of any regularly open L-fuzzy set in $(Y,\mathcal{T}_Y)$ is regularly open in $(X,\mathcal{T}_X)$.

Proof

See theorem 3.10. in [3] and corollary 4.12. in [26].

Proposition 3.3.9. Singal and Rajvanshi [90]

Let $(X,\mathcal{T}_X)$ and $(Y,\mathcal{T}_Y)$ be L-fts's and let $f:(X,\mathcal{T}_X)\to(Y,\mathcal{T}_Y)$ be a weakly continuous open map. Then f is almost continuous.

Proof

See theorem 3.4. in [90].

Proposition 3.3.10. Mukherjee and Sinha [70].

Let $(X,\mathcal{T}_X)$ and $(Y,\mathcal{T}_Y)$ be L-fts's and let $f:(X,\mathcal{T}_X)\rightarrow(Y,\mathcal{T}_Y)$ be an irresolute map. Then the inverse image of any semiclosed L-fuzzy set in $(Y,\mathcal{T}_Y)$ is semiclosed in $(X,\mathcal{T}_X)$.

Proof

See theorem 2.5 in [70].

Proposition 3.3.11. Mukherjee and Sinha [71]

Let $(X,\mathcal{T}_X)$ and $(Y,\mathcal{T}_Y)$ be L-fts's. Then a map $f:(X,\mathcal{T}_X)\rightarrow(Y,\mathcal{T}_Y)$ is almost continuous if and only if $cl(f^{-1}(g))\leq f^{-1}(cl(g))$ for all semiopen L-fuzzy sets g in $(Y,\mathcal{T}_Y)$.

Proof

See theorem 3.4. in [71].

## 4. Countability and Separation axioms

Definition 3.4.1. McLean and Warner [65]

An L-fts $(X,\mathcal{T})$ is first countable, $C_1$, if and only if for each $x \in X$ and each $\alpha \in M(L)$ there exists a countable family of closed L-fuzzy sets $(f_i)_{i \in J}$ with $f_i(x) \not\geq \alpha$ such that for every closed L-fuzzy set g with $g(x) \not\geq \alpha$ there is $i \in J$ with $g \leq f_i$.

Definition 3.4.2. McLean and Warner [65]

An L-fts is said to be $C_2$ or satisfies the second axiom of countability if and only if it has a countable base.

Remark 3.4.3.

If $(X,\delta)$ is a topological space then the L-fts $(X,\omega(\delta))$ is first countable (second countable) if and only if the topological spaces $(pr(L), \phi(L))$ and $(X,\delta)$ are both first countable (second countable) [65].

Proposition 3.4.4.

Let $(X,\mathcal{T})$ be a $C_1$ L-fts. If an L-fuzzy point $x_\alpha \in M(L^X)$ is a cluster point of a sequence $(S_m)_{m \in \mathbb{N}}$ in $(X,\mathcal{T})$, then $(S_m)_{m \in \mathbb{N}}$ has a subsequence converging to $x_\alpha$.

Proof

This is totally similar to the proof of theorem 13.4. (4) in [80].

Definition 3.4.5. Warner and McLean [100]

An L-fts $(X,\mathcal{T})$ is Hausdorff if and only if for every $p,q \in pr(L)$ and every pair $x,y$ of distinct elements of X, there exist $f,g \in\mathcal{T}$ with $f(x)\nleq p$, $g(y)\nleq q$ and $(\forall z\in X)$ $f(z)=0$ or $g(z)=0$.

Proposition 3.4.6. (The goodness of Hausdorfness) Warner and McLean [100]

Let $(X,\delta)$ be a topological space. Then $(X,\delta)$ is Hausdorff if and only if $(X,\omega(\delta))$ is Hausdorff.

Proof

See proposition 3.1. and definition (H3) in [100].

Proposition 3.4.7.

Let $(X,\mathcal{T})$ be a Hausdorff L-fts. Then no net in $(X,\mathcal{T})$ converges to two L-fuzzy points in $M(L^X)$ with different supports.

Proof

Let $(S_m)_{m\in D}$ be a net in $(X,\mathcal{T})$ converging to two fuzzy points $x_\alpha$, $y_\beta \in M(L^X)$ with $x\neq y$.

Thus, by definition 3.1.9.(i), for all closed L-fuzzy sets f,g with $f(x)\nleq\alpha$ and $g(y)\nleq\beta$ there are $m_0$, $m_1$ $\in D$ such that $m\geq m_0$ implies that $f(\text{supp } S_m)\nleq h(S_m)$ and $m\geq m_1$ implies that $g(\text{supp } S_m)\nleq h(S_m)$ where $h(S_m)$ is the height of $S_m$. Therefore there is $m_2\in D$ such that $m\geq m_2$ implies $f(\text{supp } S_m) \nleq h(S_m)$ and $g(\text{supp } S_m) \nleq h(S_m)$.

Let $d\in D$ with $d\geq m_2$ and $S_d=z_\gamma$ .

Thus $f(z)\nleq\gamma$ and $g(z)\nleq\gamma$. Hence there exists $z\in X$ with $f(z)\neq 1$ and $g(z)\neq 1$, contradicting the Hausdorfness of $(X,\mathcal{T})$.

Definition 3.4.8.

An L-fts $(X,\mathcal{T})$ is <u>regular</u> if and only if for every $p\in pr(L)$, for each $x\in X$ and each closed L-fuzzy set f such that there is $y\in X$ with $y_p\notin f'$ (i.e. $f(y)\geq p'$) and $f(x) = 0$, there are $u,v\in\mathcal{T}$ with $x_p\in u$ (i.e., $u(x)\nleq p$), $y_p\in v$ for every $y_p\notin f'$ and $(\forall z\in X)$ $u(z)=0$ or $v(z)=0$.

Theorem 3.4.9. (The goodness of regularity)

Let $(X,\delta)$ be a topological space. Then $(X,\delta)$ is regular if and only if $(X,\omega(\delta))$ is a regular L-fts.

Proof

Necessity:

Let $p\in pr(L)$, $x\in X$ and let f be a closed L-fuzzy set in $(X,\omega(\delta))$ such that there is $y\in X$ with $f(y)\geq p'$ and $f(x) = 0$.

Therefore, $F=\{t\in X;\ f(t)\geq p'\}\neq\phi$ is closed in $(X,\delta)$ (proposition 3.2.9. (iii)) and $x\notin F$. From the regularity of $(X,\delta)$, there are $U_x$, $U_F\in\delta$ with $x\in U_x$, $F\subset U_F$ and $U_x\cap U_F = \phi$.

Let $u=\chi_{U_x}$ and $v=\chi_{U_F}$.

Thus $u,v\in\omega(\delta)$ (proposition 3.2.10.), $u(x) = 1\nleq p$ and for every $y_p\notin f'$, $y_p\in v$ because $y_p\notin f'$ implies that $y\in F$, then $y\in U_F$ and $v(y) = 1\nleq p$.

We also have $(\forall z\in X)$ $u(z)=0$ or $v(z)=0$ because if

$z \in X$ and $u(z) \neq 0$ then $z \in U_x$ which implies $z \in U'_F$ and $v(z)=0$.

Hence $(X, \omega(\delta))$ is regular.

Sufficiency:

Let $x \in X$, $p \in pr(L)$ and let F be a closed set in $(X, \delta)$ with $x \notin F \neq \phi$.

Consider $f : X \to L$ defined by $f(y) = \begin{cases} p' \text{ if } y \in F \\ 0 \text{ if } y \notin F \end{cases}$ for each $y \in X$.

We have f closed in $(X, \omega(\delta))$, there is $y \in X$ with $y_p \notin f'$ and $f(x)=0$. In fact, since $F \neq \phi$ there is $y \in F$ and then $f(y)=p'$ which implies $y_p \notin f'$ and we also have $f(x)=0$ since $x \notin F$. By proposition 3.2.9. (iii), f is closed in $(X, \omega(\delta))$, since $f^{-1}(\{t \in L;\ t \geq q\}) =$
$\begin{cases} X \text{ if } q=0 \\ F \text{ if } q \neq 0 \text{ and } p' \geq q \\ \phi \text{ if } q \neq 0 \text{ and } p' \not\geq q \end{cases}$ is closed in $(X, \delta)$.

Thus, from the regularity of $(X, \omega(\delta))$, there are open L-fuzzy sets u,v with $u(x) \not\leq p$; for every $y_p \notin f'$, $y_p \in v$ and $(\forall z \in X)$ $u(z)=0$ or $v(z)=0$.

Let $U_x = \{t \in X;\ u(t) \not\leq p\}$ and $U_F = \{t \in X;\ v(t) \not\leq p\}$.

Therefore, $U_x$, $U_F \in \delta$ (proposition 3.2.9. (i)), $x \in U_x$ and $F \subset U_F$ because if $y \in F$ then $y_p \notin f'$ which implies $y_p \in v$ and then $y \in U_F$. We also have $U_x \cap U_F = \phi$ because if $t \in U_x$ then $u(t) \not\leq p$ which implies $v(t)=0$ and $t \notin U_F$.

Hence $(X, \delta)$ is regular.

Definition 3.4.10.

An L-fts $(X, \mathcal{T})$ is normal if and only if for all $p \in pr(L)$ and for every pair f,g of closed L-fuzzy sets such that there are $x, y \in X$ with $x_p \notin f'$ $(f(x) \geq p')$ and

$y_p \notin g'$ $(g(y) \geq p')$ and $(\forall z \in X)$ $f(z)=0$ or $g(z)=0$, there are $u, v \in \mathcal{T}$ with for every $z_p \notin f'$, $z_p \in v$ $(v(z) \not\leq p)$; for every $z_p \notin g'$, $z_p \in u$ and $(\forall z \in X)$ $u(z)=0$ or $v(z)=0$.

<u>Proposition</u> <u>3.4.11.</u> (The goodness of normality)

Let $(X, \delta)$ be a topological space. Then $(X, \delta)$ is normal if and only if $(X, \omega(\delta))$ is normal.

<u>Proof</u>

<u>Necessity</u>:

Let $p \in pr(L)$ and let $f, g$ be closed L-fuzzy sets in $(X, \omega(\delta))$ such that there are $x, y \in X$ with $f(x) \geq p'$ and $g(y) \geq p'$ and $(\forall z \in X)$ $f(z)=0$ or $g(z)=0$.

Therefore, $F = \{t \in X;\ f(t) \geq p'\}$ and $G = \{t \in X;\ g(t) \geq p'\}$ are closed sets in $(X, \delta)$ (proposition 3.2.9. (iii)) and $F \cap G = \phi$ because if $t \in F$ then $f(t) \geq p'$ which implies $g(t) = 0$ and then $t \notin G$.

From the normality of $(X, \delta)$, there are $U_F$, $U_G \in \delta$ with $F \subset U_F$, $G \subset U_G$ and $U_F \cap U_G = \phi$.

Let $u = \chi_{U_G}$ and $v = \chi_{U_F}$.

Thus, $u, v \in \omega(\delta)$ (proposition 3.2.10.), for every $z_p \notin f'$, $z_p \in v$; for every $z_p \notin g'$, $z_p \in u$ and $(\forall z \in X)$ $u(z)=0$ or $v(z)=0$. In fact, if $z_p \notin f'$ then $f(z) \geq p'$ which implies $z \in F$ and then $z \in U_F$ and $v(z)=1 \not\leq p$, i.e., $z_p \in v$. In the same way we obtain $z_p \in u$ for every $z_p \notin g'$. We also have $(\forall z \in X) u(z)=0$ or $v(z)=0$ because if $z \in X$ and $u(z) \neq 0$ then $z \in U_G$ which implies $z \notin U_F$ and then $v(z)=0$.

Hence $(X, \omega(\delta))$ is normal.

Sufficiency:

Let $F \neq \phi$ and $G \neq \phi$ be closed sets in $(X,\delta)$ with $F \cap G = \phi$ and let $p \in pr(L)$.

Consider $f: X \to L$ and $g: X \to L$ defined by

$$f(y) = \begin{cases} p' & \text{if } y \in F \\ 0 & \text{if } y \notin F \end{cases} \quad \text{and} \quad g(y) = \begin{cases} p' & \text{if } y \in G \\ 0 & \text{if } y \notin G \end{cases} \text{ for each } y \in X,$$

respectively.

We have that $f, g$ are closed L-fuzzy sets in $(X,\omega(\delta))$ (as in sufficiency of theorem 3.4.9.). Since $F \neq \phi$ and $G \neq \phi$, there are $x, y \in X$ with $x_p \notin f'$ and $y_p \notin g'$. We also have $(\forall z \in X)$ $f(z)=0$ or $g(z)=0$ because if $z \in X$ and $f(z) \neq 0$ then $z \in F$ which implies $z \notin G$ and then $g(z)=0$.

Thus, from the normality of $(X,\omega(\delta))$, there are $u, v \in \omega(\delta)$ with $z_p \in v$ for every $z_p \notin f'$; $z_p \in u$ for all $z_p \notin g'$ and $(\forall z \in X)$ $u(z)=0$ or $v(z)=0$.

Let $U_F = \{t \in X;\ v(t) \nleq p\}$ and $U_G = \{t \in X;\ u(t) \nleq p\}$.

Therefore, $F \subset U_F$, $U_G \supset G$ because $y \in F$ implies $f(y)=p'$ and then $y_p \notin f'$ thus $y_p \in v$, i.e., $y \in U_F$ and in the same way we obtain $G \subset U_G$. We also have $U_F$ and $U_G \in \delta$ (proposition 3.2.9. (i)) and $U_F \cap U_G = \phi$ because if $t \in U_A$ then $v(t) \nleq p$ which implies $u(t)=0$ and then $t \in U_B$ .

Hence $(X,\delta)$ is normal.

## Chapter IV

## Compactness in L-fuzzy topological spaces

This chapter is concerned with compactness in L-fuzzy topological spaces where L is a completely distributive lattice. We present our definition of compactness for arbitrary L-fuzzy sets and study some of its properties. We then concentrate on the other versions of L-fuzzy compactness introduced by various authors, set up their goodness and establish their interrelations.

Lowen [54] introduced in 1976 a good fuzzy compactness for [0,1]-fuzzy topological spaces, which we shall call here Lowen fuzzy compactness. Meng pointed out in [66] that in 1988, in a work in Chinese, Wang generalised it to L-fuzzy topological spaces by means of $\alpha$-nets, for L a fuzzy lattice. In [66], Meng obtained some other characterizations for Lowen's compactness in L-fuzzy topological spaces (which we shall call here Lowen L-fuzzy compactness) by means of remote neighbourhood families, R-covers, $\alpha$-filters and families of closed L-fuzzy subsets which have the finite intersection property. Lowen [55], in 1977, proved the Tychonoff product theorem for Lowen fuzzy compactness.

In 1978, Gantner, Steinlage and Warren [35] proposed, for a fuzzy lattice L, the so-called $\alpha$-compactness and $\alpha^*$-compactness in L-fuzzy topological spaces, observing that it is possible to have degrees of compactness. In this work they obtained the Tychonoff product theorem but with restrictions on $\alpha$ or on the lattice L.

In the same year, 1978, Lowen [56] suggested two more good definitions of compactness in [0,1]-fuzzy topological spaces, namely the well known strong fuzzy compactness and ultra-fuzzy compactness. For both, a Tychonoff product theorem was obtained in the same work. In this paper, he also showed that, in [0,1]-fuzzy spaces, $\alpha$-compactness [35] is a good extension but $\alpha^*$-compactness [35] is not.

In 1993, Warner and McLean [100] generalized strong compactness [56] to an L-fuzzy topological space, for a completely distributive lattice L. It was proved that it is a good extension and also that compact Hausdorff L-fuzzy spaces are topological. In this work we shall call it just <u>compactness</u>.

Also in 1993, Meng [67] mentioned that, Wang, in a work in Chinese, generalized ultra-fuzzy compactness to L-fuzzy topological spaces, for L a fuzzy lattice. In [67], Meng also presented another characterization of this generalization that here will be called <u>ultra-L-fuzzy compactness</u>.

Hutton [44], in 1980, obtained the Tychonoff product theorem using a definition of compactness called here <u>H-compactness</u>. For doing so, in his pointless framework he gave a "pointless" definition of the product of fuzzy topological spaces. He worked in a fuzzy lattice $L^X$.

In 1983, Wang [95] introduced N-compactness, by means of $\alpha$-nets, in [0,1]-fuzzy topological spaces. This compactness is defined for arbitrary fuzzy sets and has some desirable properties. But, as remarked by Chadwick [15], because an N-compact fuzzy set has to attain a maximum value, it is possible to have fuzzy sets which are never N-compact, even if the fuzzy topology has only a finite number of open fuzzy sets. In 1987 Zhao [110] generalized N-compactness to L-fuzzy topological spaces where L is a fuzzy lattice. He also proved that this generalization, which will be called here N-L-compactness, has the same properties as N-compactness. In this work, Zhao, besides the $\alpha$-nets characterization of N-L-compactness, presented a geometrical characterization by means of R-neighbourhoods.

In 1993, Xu [55] mentioned that in a paper by him, in Chinese, a new L-fuzzy compactness was introduced. This compactness we shall call here X-compactness.

Thus, many papers on L-fuzzy compactness have been written and different kinds of compactness have been introduced. However, each of them has its own disadvantage. Some of them are not good extensions, some do not satisfy results related to separation axioms, for some the Tychonoff product theorem does not hold and so on. In spite of N-L-compactness having good properties, it also has its disadvantage as mentioned above.

Our aim is to suggest for arbitrary fuzzy sets a good definition of compactness with the satisfactory properties of N-L-compactness whilst avoiding Chadwick's drawback. This might well have applications not accessible to the stronger N-L-compactness.

We divide this chapter into seven sections.

In the first section we introduce our definition of compactness for arbitrary fuzzy sets and study some of its properties.

The second section is devoted to the Tychonoff Product Theorem for compactness.

In the third section we obtain results related to separation axioms.

The fourth section contains other characterizations of this compactness.

The fifth section is reserved for the goodness of N-L-compactness and a comparison of compactness with N-L-compactness.

In the sixth section we state the other definitions of compactness in L-fuzzy topological spaces where L is a completely distributive lattice and study them with respect to their goodness.

Finally in the seventh section we establish the interrelations between the compactness which are good extensions.

## 1. Compactness in L-fuzzy topological spaces

Definition 4.1.1. Gantner, Steinlage and Warren [35]

Let $(X,\mathcal{T})$ be an L-fuzzy topological space, where L is a completely distributive lattice with an order reversing involution and let $\alpha\in L$. A collection $\mathcal{B}\subset\mathcal{T}$ is called an <u>$\alpha$-shading</u> (resp. $\alpha^*$-shading) of X if, for each $x\in X$, there exists $f\in\mathcal{B}$ with $f(x)>\alpha$ (resp. $f(x)\geq\alpha$). A subcollection $\mathcal{C}$ of an $\alpha$-shading (resp. $\alpha^*$-shading) $\mathcal{B}$ of X that is also an $\alpha$-shading (resp. $\alpha^*$-shading) is called an <u>$\alpha$-subshading</u> (resp. $\alpha^*$-subshading) of $\mathcal{B}$. The L-fuzzy topological space $(X,\mathcal{T})$ is called <u>$\alpha$-compact</u> (resp. <u>$\alpha^*$-compact</u>) if and only if each $\alpha$-shading (resp. $\alpha^*$-shading) of X has finite $\alpha$-subshading (resp. $\alpha^*$-subshading).

Definition 4.1.2. Lowen [56]

Let $(X,\mathcal{T})$ be a $[0,1]$-fuzzy topological space. $(X,\mathcal{T})$ is called <u>strong fuzzy compact</u> if and only if it is $\alpha$-compact for each $\alpha\in[0,1)$.

Definition 4.1.3. Warner and McLean [100]

Let $(X,\mathcal{T})$ be an L-fts where L is a completely distributive lattice. $(X,\mathcal{T})$ is called <u>compact</u> if and only if for every prime p of L and every collection $(f_i)_{i\in J}$ of open L-fuzzy sets with $\left(\bigvee_{i\in J} f_i\right)(x)\not\leq p$ for all $x\in X$, there is a finite subset F of J with $\left(\bigvee_{i\in F} f_i\right)(x)\not\leq p$ for all $x\in X$.

Now we give the definition proposed by us.

Definition 4.1.4.

Let $(X,\mathcal{T})$ be an L-fts where L is a fuzzy lattice and let $g \in L^X$. The L-fuzzy set g is said to be compact if and only if for very prime $p \in L$ and every collection $(f_i)_{i\in J}$ of open L-fuzzy sets with $\left(\bigvee_{i\in J} f_i\right)(x) \nleq p$ for all $x \in X$ with $g(x) \geq p'$, there exists a finite subset F of J with $\left(\bigvee_{i\in F} f_i\right)(x) \nleq p$ for all $x \in X$ with $g(x) \geq p'$.

If the L-fuzzy set is the whole space X, then we say that the L-fts $(X,\mathcal{T})$ is compact. In this case, definition 4.1.4. reduces to definition 4.1.3.

Remark 4.1.5..

We would like to draw attention to the fact that, when we say, for all $x \in X$ with $g(x) \geq p'$, it means for all L-fuzzy points $x_p \in pr(L^X)$ such that $x_p \notin g'$.

This can be restated as follows:

The L-fuzzy set g is compact if and only if for every prime $p \in L$, every collection $(f_i)_{i\in J}$ of open L-fuzzy sets such that for all L-fuzzy points $x_p \in pr(L^X)$ with $x_p \notin g'$ there exists $i \in J$ with $x_p \in f_i$, has a finite subcollection with this property.

Theorem 4.1.6. (The goodness of compactness)

Let $(X,\delta)$ be a topological space. Then $(X,\delta)$ is compact if and only if $(X,\omega(\delta))$ is a compact L-fts.

Proof

In [100], Warner and McLean proved the goodness of compactness for L a continuous lattice (proposition 4.4. in [100]). Since we are working with L a fuzzy lattice (definition 1.1.10.), we have L a completely distributive lattice. By proposition 1.2.3. we know that every completely distributive lattice is a continuous frame. Hence we have our result.

Proposition 4.1.7.

Let $(X,\mathcal{T})$ be an L-fts where $\mathcal{T}$ is a finite subset of $L^X$. Then $(X,\mathcal{T})$ is compact.

Proof

This immediately follows from definition 4.1.4.

Proposition 4.1.8. Warner and McLean [100]

Let $(X,\mathcal{T})$ be an L-fts where X is a finite set. Then $(X,\mathcal{T})$ is compact.

Proof

See proposition 4.6. in Warner and McLean [100].

In spite of the fact that the proof of our next theorem has already been given by Warner and McLean [100], we included it here because we shall need it later in proposition 4.7.8..

Theorem 4.1.9. Warner and McLean [100]

Let $(X,\mathcal{T})$ be a fully stratified compact Hausdorff L-fts. Then $(X,\mathcal{T})$ is topological.

<u>Proof</u>

By definition 3.2.8. we need to exhibit a topology $\delta$ on X such that $\mathcal{T} = \omega(\delta)$.

Let $\delta = \{U\in\mathbb{P}(X);\ \chi_U \in \mathcal{T}\}$.

We shall show that $\mathcal{T} = \omega(\delta)$.

Since by proposition 3.2.11. $\omega(\delta)$ is generated by $\delta$ and the constant functions from X to L, we have that $\omega(\delta)\subseteq\mathcal{T}$.

Now we are going to show that $\mathcal{T}\subseteq\omega(\delta)$, i.e., every $f\in\mathcal{T}$ is a continuous map from $(X,\delta)$ to L with its Scott topology. By proposition 3.2.9. it is sufficient to prove that for every $f\in\mathcal{T}$ and for all $p\in pr(L)$ we have $\{x\in X;\ f(x)\not\leq p\} \in \delta$, i.e., $\chi_{\{x\in X;\ f(x)\not\leq p\}} \in \mathcal{T}$.

Let $f\in\mathcal{T}$ , $p\in pr(L)$ and let $e\in X$ such that $f(e)\not\leq p$ and $q\in pr(L)$.

Then, by the Hausdorffness of $(X,\mathcal{T})$ (definition 3.4.5.), for each $x\in X$ such that $f(x)\leq p$, there are $g_x$, $h_x \in \mathcal{T}$ with $g_x(e)\not\leq q$, $h_x(x)\not\leq p$ and $(\forall z\in X)\ g_x(z)=0$ or $h_x(z)=0$.

Therefore, $\left(\bigvee_{\substack{x\in X \\ f(x)\leq p}} h_x \vee f\right)(x)\not\leq p$ for all $x\in X$. So, by compactness there are $x_1,\ldots,x_m \in X$ with $f(x_i)\leq p$ $\forall i\in\{1,\ldots,m\}$ such that $\left(f\vee h_{x_1}\vee\ldots\vee h_{x_m}\right)(x) \not\leq p$ for all $x\in X$.

Thus, for all $y\in X$ with $f(y)\leq p$, there is $j\in\{1,\ldots,m\}$ with $h_{x_j}(y)\not\leq p$ which implies that $g_{x_j}(y) = 0$.

Let $k_q = \bigwedge_{i=1}^{m} g_{x_i}$ .

We have that $k_q \in \mathcal{T}$, $k_q(y)=0$ for all $y \in X$ with $f(y) \leq p$ and since $g_{x_i}(e) \nleq q$ for every i and $q \in pr(L)$, follows that $k_q(e) \nleq q$.

So, for each $e \in X$ with $f(e) \nleq p$ and for each $q \in pr(L)$ there is $k_q \in \mathcal{T}$ with $k_q(e) \nleq q$ and $k_q(y)=0$ for all $y \in X$ with $f(y) \leq p$.

Let $s_e = \bigvee_{q \in pr(L)} k_q$.

We have that $s_e \in \mathcal{T}$, $s_e(y)=0$ for all $y \in X$ with $f(y) \leq p$ and $s_e(e) \nleq q$ for all $q \in pr(L)$.

Since L is spatial by proposition 1.2.3., each of its elements, except 1, is a meet of primes by proposition 1.2.2. and hence $s_e(e)=1$.

Let $g = \bigvee_{\substack{e \in X \\ f(e) \leq p}} s_e$ .

Therefore $g \in \mathcal{T}$, $g(y) = 0$ for all $y \in X$ with $f(y) \leq p$ and $g(y) = 1$ for all $y \in X$ with $f(y) \nleq p$. Hence $\chi_{\{x \in X;\ f(x) \nleq p\}} \in \mathcal{T}$.

**Proposition 4.1.10.**

Let $(X,\mathcal{T})$ be an L-fts. If h and g are compact L-fuzzy sets, then $h \vee g$ is compact as well.

**Proof**

Let h and g be compact L-fuzzy sets. Let $p \in pr(L)$ and let $(f_i)_{i \in J}$ be a family of open L-fuzzy sets with $\left(\bigvee_{i \in J} f_i\right)(x) \nleq p$ for all $x \in X$ such that $(h \vee g)(x) \geq p'$. But if $(h \vee g)(x) \geq p'$ then $h(x) \geq p'$ or $g(x) \geq p'$ because $p \in pr(L)$ and we always have that if $h(x) \geq p'$ or $g(x) \geq p'$ then $(h \vee g)(x) \geq p'$. So, from the compactness of h and g, there are finite subsets $F_1$, $F_2$ of J with $\left(\bigvee_{i \in F_1} f_i\right)(x) \not\geq p$ for all $x \in X$ such that $h(x) \geq p'$ and $\left(\bigvee_{i \in F_2} f_i\right)(x) \not\geq p$ for all $x \in X$ such that $g(x) \geq p'$. Then, $\left(\bigvee_{i \in F_1 \cup F_2} f_i\right)(x) \not\geq p$ for all $x \in X$ such that $h(x) \geq p'$ or $g(x) \geq p'$.

Thus, $\left(\bigvee_{i \in F_1 \cup F_2} f_i\right)(x) \not\geq p$ for all $x \in X$ with $(h \vee g)(x) \geq p'$. Hence $h \vee g$ is compact.

<u>Corollary</u> <u>4.1.11.</u>

Let $(X, \mathcal{T})$ be an L-fts. Every L-fuzzy set g with finite support is compact.

<u>Proof</u>

Let g be an L-fuzzy set with finite support. By remark 2.1.8. we have that each L-fuzzy set is a join of functions of the form

$$g^x_{p'} : X \to L, \quad y \to \begin{cases} p' & \text{if } y = x \\ 0 & \text{otherwise} \end{cases}$$

where $p \in pr(L)$. So, g is a finite join of functions $g^x_{p'}$. Thus, from proposition 4.1.10. it will suffice to prove that any $g^x_{p'}$ is compact.

Let $p \in pr(L)$, $p_0 \in pr(L)$, $y \in X$ and let

$$g^y_{p_0'} : X \to L, \quad x \to \begin{cases} p_0' & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases}.$$

We have $H = \{x \in X;\ g^y_{p_0'}(x) \geq p'\} = \begin{cases} \{y\} & \text{if } p_0' \geq p' \\ \phi & \text{if } p_0' \not\geq p' \end{cases}$.

For every family $(f_i)_{i\in J}$ of open L-fuzzy sets with $\left(\bigvee_{i\in J} f_i\right)(x) \not\geq p$ for all $x\in H$, we must show that there is a finite subset F of J with $\left(\bigvee_{i\in F} f_i\right)(x) \not\geq p$ for all $x\in H$.

If $p'_0 \not\geq p'$ we have $H = \phi$.

If $p'_0 \geq p$ we have $H = \{y\}$, then $\left(\bigvee_{i\in J} f_i\right)(y) \not\geq p$ implies that there is $i_0\in J$ such that $f_{i_0}(y) \not\geq p$. Thus there exists a finite subset $F = \{i_0\}$ of J with $\left(\bigvee_{i\in F} f_i\right)(y) \not\geq p$. Hence $g^y_{p'_0}$ is compact.

Proposition 4.1.12.

Let $(X,\mathcal{T})$ be an L-fts. If g is a compact L-fuzzy set, then for each closed L-fuzzy set h, $h\wedge g$ is compact.

Proof

Let g be a compact L-fuzzy set and let h be a closed L-fuzzy set.

Let $p\in pr(L)$ and let $(f_i)_{i\in J}$ be a family of open L-fuzzy sets with $\left(\bigvee_{i\in J} f_i\right)(x) \not\geq p$ for all $x\in X$ with $(h\wedge g)(x) \geq p'$.

Thus, $\mathcal{B} = (f_i)_{i\in J} \cup \{h'\}$ is a family of open L-fuzzy sets in $(X,\mathcal{T})$ with $\left(\bigvee_{k\in\mathcal{B}} k\right)(x) \not\geq p$ for all $x\in X$ with $g(x) \geq p'$. In fact, for each $x\in X$ with $g(x) \geq p'$, if $h(x) \geq p'$ then $(h\wedge g)(x) \geq p'$ which implies that $\left(\bigvee_{i\in J} f_i\right)(x) \not\geq p$, thus $\left(\bigvee_{k\in\mathcal{B}} k\right)(x) \not\geq p$. If $h(x) \not\geq p'$ then $h'(x) \not\geq p$ which implies that $\left(\bigvee_{k\in\mathcal{B}} k\right)(x) \not\geq p$. From the compactness of g, there is a finite subfamily $\mathcal{C}$ of $\mathcal{B}$, say $\mathcal{C} = \{f_1, \ldots, f_m, h'\}$ with $\left(\bigvee_{k\in\mathcal{C}} k\right)(x) \not\geq p$ for all $x\in X$ with $g(x) \geq p'$. Then, $\left(\bigvee_{i\in\{1,\ldots,m\}} f_i\right)(x) \not\geq p$ for all $x\in X$ with $(h\wedge g)(x) \geq p'$. In fact, if $(h\wedge g)(x)\geq p'$ then $g(x)\geq p'$, hence $\left(\bigvee_{k\in\mathcal{C}} k\right)(x)\not\leq p$. Therefore, there is $k\in\mathcal{C}$ such that $k(x)\not\leq p$. However $h(x)\geq p'$, that is, $h'(x)\leq p$, so $\left(\bigvee_{i\in\{1,\dots,m\}} f_i\right)(x)\not\leq p$ for all $x\in X$ with $(h\wedge g)(x)\geq p'$. Hence $h\wedge g$ is compact.

Corollary 4.1.13.

Let $(X,\mathcal{T})$ be an L-fts. If g is a compact L-fuzzy set, then each closed L-fuzzy set contained in g is compact as well.

Proof

This immediately follows from proposition 4.1.12.

Proposition 4.1.14.

Let $(X,\mathcal{T}_X)$ and $(Y,\mathcal{T}_Y)$ be L-fts's and let $f:(X,\mathcal{T}_X)\to(Y,\mathcal{T}_Y)$ be a continuous mapping such that $f^{-1}(y)$ is finite for every $y\in Y$. If g is a compact L-fuzzy set in $(X,\mathcal{T}_X)$, then $f(g)$ is a compact L-fuzzy set in $(Y,\mathcal{T}_Y)$.

Proof

Let g be a compact L-fuzzy set in $(X,\mathcal{T}_X)$. Let $p\in pr(L)$ and let $(f_i)_{i\in J}$ be a family of open L-fuzzy sets in $(Y,\mathcal{T}_Y)$ with $\left(\bigvee_{i\in J} f_i\right)(y)\not\leq p$ for all $y\in Y$ with $f(g)(y)\geq p'$. Then $(f^{-1}(f_i))_{i\in J}$ is a family of open L-fuzzy sets in $(X,\mathcal{T}_X)$ with $\left(\bigvee_{i\in J} f^{-1}(f_i)\right)(x)\not\leq p$ for all $x\in X$ with $g(x)\geq p'$. In fact, since each $f_i\in\mathcal{T}_Y$ and f is continuous, $f^{-1}(f_i)\in\mathcal{T}_X$ for every $i\in J$. We also have $\left(\bigvee_{i\in J} f^{-1}(f_i)\right)(x)\not\leq p$ for all $x\in X$ with $g(x)\geq p'$ because if $g(x)\geq p'$ then $f(g)(f(x))\geq p'$. So, $\left(\bigvee_{i\in J} f^{-1}(f_i)\right)(x) = \left(\bigvee_{i\in J} f_i\right)(f(x))\not\leq p$.

From the compactness of g in $(X,\mathcal{T}_X)$ there exists a finite subset F of J with $\left(\bigvee_{i\in F} f^{-1}(f_i)\right)(x)\not\geq p$ for all $x\in X$ with $g(x)\geq p'$. Then $\left(\bigvee_{i\in F} f_i\right)(y)\not\geq p$ for all $y\in Y$ with $f(g)(y)\geq p'$. In fact, if $f(g)(y)\geq p'$ then, by definition 2.2.1., $\bigvee_{x\in f^{-1}(y)} \{g(x)\}\geq p'$, which implies that there is $x\in X$ with $g(x)\geq p'$ and $f(x) = y$. So, $\left(\bigvee_{i\in F} f_i\right)(y) = \left(\bigvee_{i\in F} f_i\right)(f(x)) = \left(\bigvee_{i\in F} f^{-1}(f_i)\right)(x)\not\geq p$. Hence $f(g)$ is compact in $(Y,\mathcal{T}_Y)$.

When $g = \chi_A$ for some $A\subseteq X$, $f^{-1}(y)$ does not need to be finite.

<u>Remark 4.1.15.</u>

If $(X,\mathcal{T})$ is a Hausdorff L-fts and f is a compact L-fuzzy set, we do not necessarily have f a closed L-fuzzy set. For example:

Let $X = [0,1]$ and let $\mathcal{T}$ be the $[0,1]$-fuzzy topology $\{\chi_U$ ; U usual open crisp set in X$\}$.

$(X,\mathcal{T})$ is Hausdorff because for every $q,p\in [0,1) = pr([0,1])$ and for every pair x,y of X with $x\neq y$ there exist $f,g\in\mathcal{T}$ with $f(x)\not\geq p$, $g(y)\not\geq q$ and $(\forall z\in X)$ $f(z)=0$ or $g(z)=0$. In fact, given $x\neq y$ in $[0,1]$, there are open crisp sets $U_1$, $U_2$ such that $x\in U_1$, $y\in U_2$ and $U_1\cap U_2 = \phi$. Then take $f = \chi_{U_1}$ and $g=\chi_{U_2}$. So, f and g are in $\mathcal{T}$, $f(x)=1\not\geq p$ for every $p\in pr(L)$, $g(y)=1\not\geq q$ for every $q\in pr(L)$. We also have that if $z\in X$ and $f(z)=\chi_{U_1}(z)\neq 0$ then $z\in U_1$, so $z\notin U_2$ because $U_1\cap U_2=\phi$. Thus $g(z)=\chi_{U_2}(z)=0$.

Let $f: X \to L = [0,1]$

$x \to \frac{1}{3}$

Clearly f is not closed. But f is compact. In fact, if $p \in [0,2/3)$ then $\{x \in X;\ f(x) \geq p'\} = \{x \in [0,1];\ p' \leq 1/3\} = \phi$ and given $p \in [2/3, 1]$ and a family $(f_i)_{i\in J}$ of open [0,1]-fuzzy sets with $\left(\bigvee_{i\in J} f_i\right)(x) > p$ for all $x \in X$ with $f(x) \geq p'$, i.e., for all $x \in X$, we have that $\left(\bigvee_{i\in J} f_i\right)(x) = 1$ for all $x \in X$ because $f_i = \chi_{U_i}$ where $U_i$ is an open set in the topological space $X = [0,1]$. So, $\bigcup_{i\in J} U_i = X$ and by compactness there is a finite subset F of J with $\bigcup_{i\in F} U_i = X$. Thus, $\left(\bigvee_{i\in F} f_i\right)(x) = 1 > p$ for all $x \in X$. Hence f is compact.

Proposition 4.1.16.

Let $(X,\mathcal{T})$ be a Hausdorff L-fts and let $F \subset X$ such that $\chi_F$ is a compact L-fuzzy set in $(X,\mathcal{T})$. Then $\chi_F$ is a closed L-fuzzy set in $(X,\mathcal{T})$.

Proof

Let $p \in pr(L)$ and $x \in F'$. We shall show that there exists $f \in \mathcal{T}$ with $f(x) \nleq p$ and $f \leq \chi_{F'}$. Therefore by proposition 3.1.4. we have $\chi_{F'} \in \mathcal{T}$ and so $\chi_F$ is a closed L-fuzzy set.

For all $y \in F$, by the Hausdorffness of $(X,\mathcal{T})$, there are $g_y$, $h_y \in \mathcal{T}$ with $g_y(x) \nleq p$, $h_y(y) \nleq p$ and $(\forall z \in X)$ $h_y(z) = 0$ or $g_y(z) = 0$. Thus, $\mathcal{B} = (h_y)_{y\in F}$ is a family of open L-fuzzy sets with $\left(\bigvee_{y\in F} h_y\right)(z) \nleq p$ for all $z \in F$. Then by the compactness of $\chi_F$, there is a finite subfamily $\mathcal{B}_1$ of $\mathcal{B}$, say $\mathcal{B}_1 = \left\{h_{y_1}, \ldots, h_{y_m}\right\}$ with $\left(\bigvee_{i\in\{1,\ldots,m\}} h_{y_i}\right)(z) \not\geq p$ for all $z\in F$.

Let $f = \bigwedge_{i\in\{1,\ldots,m\}} g_{y_i}$.

Then $f\in\mathcal{T}$, $f(x)\not\leq p$ and $f\leq\chi_{F'}$. In fact, since each $g_{y_i}\in\mathcal{T}$, $f\in\mathcal{T}$ and from $g_{y_i}(x)\not\leq p$ for all $i\in\{1,\ldots,m\}$ and $p\in pr(L)$, $f(x)\not\leq p$. We also have $f\leq\chi_{F'}$ because if $z\in F'$, $\chi_{F'}(z)=1$ and if $z\in F$, there is $k\in\{1,\ldots,m\}$ with $h_{y_k}(z)\not\geq p$, so $g_{y_k}(z)=0$ and $f(z)=0$ .

<u>Lemma</u> <u>4.1.17.</u>

Let $(X,\mathcal{T})$ be an L-fts, $f\in L^X$, $p\in pr(L)$ and let f be a compact L-fuzzy set in $(X,\mathcal{T})$. Then $\{x\in X;\ f(x)\geq p'\}$ is compact in the ordinary topological space $(X,\delta)$ where $\delta = \{U\subset X;\ \chi_U\in\mathcal{T}\}$ .

<u>Proof</u>

Let $(F_i)_{i\in J}$ be a family of open sets in the subspace $H = \{x\in X;\ f(x)\geq p'\}$ of $(X,\delta)$ with $H = \bigcup_{i\in J} F_i$. Therefore, for each $i\in J$ there is $0_i\in\delta$ such that $F_i = 0_i\cap H$. So, $(\chi_{0_i})_{i\in J}$ is a family of open L-fuzzy sets in $(X,\mathcal{T})$ with $\left(\bigvee_{i\in J}\chi_{0_i}\right)(x)=1\not\leq p$ for all $x\in H$. From the compactness of f there is a finite subset K of J with $\left(\bigvee_{i\in K}\chi_{0_i}\right)(x)\not\leq p$, i.e., $\left(\bigvee_{i\in K}\chi_{0_i}\right)(x)=1$ for all $x\in H$. Then $\bigcup_{i\in K} F_i = H$ and H is compact in $(X,\delta)$.

<u>Lemma</u> <u>4.1.18.</u> Liu and Luo [50]

Let $(X,\mathcal{T})$ be an L-fts, $x_0 \in X$, $f \in L^X$ such that for every $e \in L$, $\chi_{\{x\in X;\ f(x)\geq e\}}$ is closed in $(X,\mathcal{T})$. Then $f(x_0) = \bigwedge_{U\in\mathcal{B}} \bigvee_{y\in U} f(y)$, where $\mathcal{B}$ is a neighbourhood base at the point $x_0$ in the topological space $(X,\delta)$ where $\delta = \{U\subset X;\ \chi_U \in\mathcal{T}\}$.

<u>Proof</u>

See lemma 3 in Liu and Luo [50].

<u>Lemma</u> <u>4.1.19.</u> Liu and Luo [50]

Let $(X,\mathcal{T})$ be a fully stratified L-fts and let $f \in L^X$ such that for every $b \in L$ $\chi_{\{x\in X;\ f(x)\geq b\}}$ is closed. Then f is closed as well.

<u>Proof</u>

See proposition 1 in Liu and Luo [50].

<u>Proposition</u> <u>4.1.20.</u>

Let $(X,\mathcal{T})$ be a fully stratified L-fts and $(X,\delta)$ be a Hausdorff topological space, where $\delta = \{U\subset X;\ \chi_U\in\mathcal{T}\}$. Then each compact L-fuzzy set is closed.

<u>Proof</u>

Let f be a compact L-fuzzy set in $(X,\mathcal{T})$. Then from lemma 4.1.17. the set $H = \{x\in X;\ f(x) \geq p'\}$ is compact in $(X,\delta)$ for every $p\in pr(L)$. Because $(X,\delta)$ is Hausdorff by hypothesis, H is closed in $(X,\delta)$, which implies that $\chi_H$ is closed in $(X,\mathcal{T})$ for every $p\in pr(L)$.

Since, by proposition 1.2.2., for all $b\in L$, $b = \bigvee \{p';\ p\in pr(L) \text{ and } p'\leq b\}$, we have that $\chi_{\{x\in X;\ f(x)\geq b\}} =$

$\bigwedge_{\substack{p\in pr(L)\\ p'\leq b}} \chi_H$ is closed in $(X,\mathcal{T})$. Then by lemma 4.1.19 f is closed in $(X,\mathcal{T})$.

<u>Lemma</u> <u>4.1.21.</u> Warner and McLean [100]

Let $(X,\mathcal{T})$ be an L-fts and let $\phi(\mathcal{T})$ be the topology on $X \times pr(L)$ given by the image of $\phi : L^X \to \mathbb{P}(X\times pr(L))$ where $\phi(f) = \{(x,p\}; p\not\geq f(x)\}$ (proposition 3.2.19). Then the following are equivalent:

(i) $(X,\mathcal{T})$ is compact

(ii) for every $p\in pr(L)$, $X\times\{p\}$ is a compact subspace of $(X\times pr(L), \phi(\mathcal{T}))$.

(iii) for every $p\in pr(L)$, $X\times\{q\in pr(L); q\leq p\}$ is a compact subspace of $(X\times pr(L), \phi(\mathcal{T}))$.

<u>Proof</u>

See lemma 4.3 in Warner and McLean [100].

## 2. The Tychonoff Product Theorem

Theorem 4.2.1. Alexander's subbase theorem

Let $(X,\mathcal{T})$ be an L-fts, $g\in L^X$ and let $\mathcal{S}$ be a subbase for $\mathcal{T}$. If for every prime p of L and every collection $(f_i)_{i\in J}$ of subbasic open L-fuzzy sets with $\left(\bigvee_{i\in J} f_i\right)(x) \not\leq p$ for all $x\in X$ with $g(x) \geq p'$, there is a finite subset F of J with $\left(\bigvee_{i\in F} f_i\right)(x)\not\leq p$ for all $x\in X$ with $g(x)\geq p'$, then g is compact in $(X,\mathcal{T})$.

Proof

Let $p\in pr(L)$. Let us say $\mathcal{C}\subset\mathcal{T}$ has the finite union property (for short FUP) in $H = \{x\in X;\ g(x)\geq p'\}$ if and only if for any $c_1,\ldots,c_m\in\mathcal{C}$, there exists $x\in H$ with $c_1(x)\vee\ldots\vee c_m(x)\leq p$. Then g is compact if and only if no $\mathcal{C}\subset\mathcal{T}$ with the FUP in H satisfies $\left(\bigvee_{f\in\mathcal{C}} f\right)(x)\not\leq p$ for all $x\in H$.

Let $\mathcal{C}\subset\mathcal{T}$ have the FUP in H and let $\mathcal{F} = \{\mathcal{B};\ \mathcal{C}\subset\mathcal{B}\subset\mathcal{T}$ and $\mathcal{B}$ has the FUP in H$\}$.

Then $\mathcal{F}$ is nonempty since it contains $\mathcal{C}$ and $\mathcal{F}$ is partially ordered by inclusion. We now prove that each chain in $\mathcal{F}$ has an upper bound.

Let $\{\mathcal{B}_i \in\mathcal{F}$ for $i\in I\}$ be a chain. Then clearly $\mathcal{C}\subset \bigcup_{i\in I} \mathcal{B}_i$ and to conclude that $\bigcup_{i\in I} \mathcal{B}_i \in\mathcal{F}$ it remains to show that $\bigcup_{i\in I} \mathcal{B}_i$ has the FUP in H. Let $\mathcal{B}_0 \subset \bigcup_{i\in I} \mathcal{B}_i$ be finite; then each element of $\mathcal{B}_0$ appears first in some $\mathcal{B}_i$, therefore all of $\mathcal{B}_0$ appears in the largest, say $\mathcal{B}_j$, of this finite set of $\{\mathcal{B}_i\}$. Since $\mathcal{B}_j$ has the FUP in H, for any $b_1, \ldots b_r \in \mathcal{B}_j$, there exists $x\in H$ with $b_1(x)\vee\ldots\vee b_r(x) \leq p$. Therefore, $\bigcup_{i\in I} \mathcal{B}_i$ is an upper bound for the chain $\{\mathcal{B}_i\}$.

Hence by lemma 1.2.5., there exists a maximal member $\mathcal{D}$.

We must show that $\mathcal{C}$ does not satisfy $\left(\bigvee_{f\in\mathcal{C}} f\right)(x)\nleq p$ for all $x\in H$.

Since $\mathcal{C}\subset\mathcal{D}\subset\mathcal{T}$ it suffices to show that $\mathcal{D}$ does not satisfy $\left(\bigvee_{f\in\mathcal{D}} f\right)(x)\nleq p$ for all $x\in H$.

By our assumptions on the subbase $\mathcal{S}$, no subcollection $\mathcal{S}^*$ of $\mathcal{S}$ that has the FUP in H satisfies $\left(\bigvee_{f\in\mathcal{S}^*} f\right)(x)\nleq p$ for all $x\in H$. Since $\mathcal{D}$ has the FUP in H, $\mathcal{D}\cap\mathcal{S}$ is a subcollection of $\mathcal{S}$ that has the FUP in H, so $\mathcal{D}\cap\mathcal{S}$ does not satisfy $(\bigvee_{f\in\mathcal{D}\cap\mathcal{S}} f)(x)\nleq p$ for all $x\in H$. Hence, it will suffice to prove that $\bigvee_{f\in\mathcal{D}} f \le \bigvee_{f\in\mathcal{D}\cap\mathcal{S}} f$ .

Since $\mathcal{S}$ is a subbase (definition 3.1.3.), each $d\in\mathcal{D}$ is of the form $\bigvee_{i\in J} (s_{i1}\wedge\ldots\wedge s_{im_i})$, for $m_i \in\mathbb{N}$, $s_{ij} \in\mathcal{S}$ for each $i\in J$ and $j\in\{1, \ldots, m_i\}$. Then $s_{i1}\wedge\ldots\wedge s_{im_i}\le d$ for all $i\in J$. We are going to prove that, for each $i\in J$ we must have some $j_i\in\{1, \ldots, m_i\}$ with $s_{ij_i} \in \mathcal{S}\cap\mathcal{D}$. It follows that $d = \bigvee_{i\in J}(s_{i1} \wedge\ldots\wedge s_{im_i})\le\bigvee_{i\in J} s_{ij_i}$ , for $s_{ij_i} \in\mathcal{S}\cap\mathcal{D}$, that is, $\bigvee_{f\in\mathcal{D}}f \le \bigvee_{f\in\mathcal{D}\cap\mathcal{S}}f$ and the proof will be complete.

Given $d\in\mathcal{D}$ and $c_1, \ldots, c_m \in\mathcal{T}$ such that $c_1\wedge\ldots\wedge c_m\le d$, we are going to show that $c_i\in\mathcal{D}$, for some i.

If $c\notin\mathcal{D}$ but $c\in\mathcal{T}$, then no open set containing c belongs to $\mathcal{D}$. In fact, suppose that $c\in\mathcal{T}$, $c\in\mathcal{D}$, $b\in\mathcal{T}$ and that $c\le b$. Then $\mathcal{D}\cup\{c\}$ does not have FUP in H by maximality of $\mathcal{D}$, so there are $d_1, \ldots, d_m \in\mathcal{D}$ such that, for every $x\in H$, $(d_1\vee\ldots\vee d_m)(x)\nleq p$. But then $(d_1\vee\ldots\vee d_m\vee b)(x)\nleq p$ for all $x\in H$, so we must have $b\notin\mathcal{D}$.

If $c$, $b \notin \mathcal{D}$ but $c, b \in \mathcal{T}$, then $c \wedge b \notin \mathcal{D}$. In fact, $\mathcal{D} \cup \{c\}$ and $\mathcal{D} \cup \{b\}$ do not have FUP in H, so there exist $d_1, \ldots, d_{m+j} \in \mathcal{D}$ such that, for all $x \in H$ $(d_1 \vee \ldots \vee d_j \vee c)(x) \not\geq p$ and $(d_{j+1} \vee \ldots \vee d_{j+m} \vee b)(x) \not\geq p$. By letting $d = d_1 \vee \ldots \vee d_{j+m}$, we then have that for all $x \in H$ $(d \vee c)(x) \not\geq p$ and $(d \vee b)(x) \not\geq p$, whence $(d \vee (c \wedge b))(x) \not\geq p$ since p is prime. Hence $c \wedge b \notin \mathcal{D}$.

These results extended to $c_1, \ldots, c_m \notin \mathcal{D}$ and $c_1, \ldots, c_m \in \mathcal{T}$ imply $c_1 \wedge \ldots \wedge c_m \notin \mathcal{D}$.

It now follows that, if $c_1, \ldots, c_m \notin \mathcal{D}$ but are open, and $d \geq c_1 \wedge \ldots \wedge c_m$ and $d \in \mathcal{T}$, then $d \notin \mathcal{D}$. The contrapositive of this says that if $d \in \mathcal{D}$ and $c_1 \wedge \ldots \wedge c_m \leq d$ for $c_i \in \mathcal{T}$, then $c_i \in \mathcal{D}$ for some i.

<u>Theorem</u> <u>4.2.2.</u>

Let $\left((X_\lambda, \mathcal{T}_{X_\lambda})\right)_{\lambda \in J}$ be a family of L-fts's and let $g_\lambda$ be a compact L-fuzzy set in $(X_\lambda, \mathcal{T}_{X_\lambda})$ for each $\lambda \in J$. Then the product set $g = \bigwedge_{\lambda \in J} \pi_\lambda^{-1}(g_\lambda)$ is compact in the L-fuzzy product space $(X, \mathcal{T})$.

<u>Proof</u>

To prove this result we apply theorem 4.2.1. to the subbase $\mathcal{S} = \left\{\pi_\lambda^{-1}(u_\lambda);\ \lambda \in J \text{ and } u_\lambda \in \mathcal{T}_{X_\lambda}\right\}$ of the L-fuzzy product topology $\mathcal{T}$ on X (definition 3.2.4.).

We must show that given $p \in pr(L)$, no $\mathcal{C} \subset \mathcal{S}$ having the FUP in $H = \{x \in X;\ g(x) \geq p'\}$, satisfies $\left(\bigvee_{f \in \mathcal{C}} f\right)(x) \not\geq p$ for all $x \in H$.

Let $p \in pr(L)$, $\mathcal{C} \subset \mathcal{S}$ having the FUP in H and for each

$\lambda \in J$, let $\mathscr{C}_\lambda = \{h \in \mathscr{T}_{X_\lambda} ; \pi_\lambda^{-1}(h) \in \mathscr{C}\}$.

Then each $\mathscr{C}_\lambda$ has the FUP in H. Indeed, if $h_1, \ldots, h_m \in \mathscr{T}_{X_\lambda}$, then since $\mathscr{C}$ has the FUP in H, there exists $x \in H$ such that $\pi_\lambda^{-1}(h_1)(x) \vee \ldots \vee \pi_\lambda^{-1}(h_m)(x) \leq p$. In other words, $\pi_\lambda(x) = x_\lambda \in H_\lambda = \{x_\lambda \in X_\lambda ; g_\lambda(x_\lambda) \geq p'\}$ and $h_1(x_\lambda) \vee \ldots \vee h_m(x_\lambda) \leq p$. Therefore, for each $\lambda \in J$, $\mathscr{C}_\lambda$ does not satisfy $\left(\bigvee_{f \in \mathscr{C}_\lambda} f\right)(x_\lambda) \nleq p$ for all $x_\lambda \in H_\lambda$, because $g_\lambda$ is compact. It follows that, for each $\lambda \in J$, there exists $y_\lambda \in H_\lambda$ with $\left(\bigvee_{f \in \mathscr{C}_\lambda} f\right)(y_\lambda) \leq p$.

Let $y = (y_\lambda)_{\lambda \in J} \in H$ and for each $\lambda \in J$ define $\mathscr{C}_\lambda^* = \{\pi_\lambda^{-1}(h) ; h \in \mathscr{C}_\lambda\}$.

Then $\mathscr{C} \subset \mathscr{S}$ implies $\mathscr{C} = \bigcup_{\lambda \in J} \mathscr{C}_\lambda^*$ ; and $\left(\bigvee_{f \in \mathscr{C}_\lambda^*} f\right)(y) = \bigvee\left\{(\pi_\lambda^{-1}(h))(y) ; h \in \mathscr{C}_\lambda\right\} = \bigvee\left\{h(\pi_\lambda(y)) ; h \in \mathscr{C}_\lambda\right\} = \bigvee_{h \in \mathscr{C}_\lambda} \{h(y_\lambda)\} = \left(\bigvee_{f \in \mathscr{C}_\lambda} f\right)(y_\lambda) \leq p$. Therefore, $\left(\bigvee_{f \in \mathscr{C} = \cup \mathscr{C}_\lambda^*} f\right)(y) \leq p$ and then $\left(\bigvee_{f \in \mathscr{C}} f\right)(x) \nleq p$ is not satisfied for all $x \in H$.

<u>Theorem</u> <u>4.2.3.</u>

The L-fuzzy product space $(X, \mathscr{T})$ of the indexed family $\left\{(X_\lambda, \mathscr{T}_{X_\lambda})\right\}_{\lambda \in J}$ of L-fuzzy spaces is compact if and only if for each $\lambda \in J$ $(X_\lambda, \mathscr{T}_{X_\lambda})$ is compact.

<u>Proof</u>

<u>Necessity</u>:

This follows from proposition 4.1.14. and the fact that the projection maps $\pi_\lambda : X \to X_\lambda$ are continuous, onto and $(X, \mathscr{T})$ by hypothesis is compact.

<u>Sufficiency</u>:

This immediately follows from theorem 4.2.2.

## 3. Separation axioms

Theorem 4.3.1.

Let $(X,\mathcal{T})$ be a compact Hausdorff L-fts. Then $(X,\mathcal{T})$ is regular.

Proof

Let $p \in pr(L)$, $x \in X$ and let f be a closed L-fuzzy set in $(X,\mathcal{T})$ such that there is $y \in X$ with $y_p \notin f'$ and $f(x)=0$. By definition 3.4.8. we need to prove that there are u, $v \in \mathcal{T}$ with $x_p \in u$; for every $y_p \notin f'$, $y_p \in v$ and $(\forall z \in X)$ $u(z)=0$ or $v(z)=0$.

Let $F = \{t \in X;\ f(t) \geq p'\}$. We have that $x \notin F$ because $f(x)=0$ and $p' \neq 0$ since $p \in pr(L)$.

Since $(X,\mathcal{T})$ is Hausdorff for each $y \in F$ there exist $f_y$, $g_y \in \mathcal{T}$ with $f_y(x) \nleq p$, $g_y(y) \nleq p$ and $(\forall z \in X)$ $f_y(z)=0$ or $g_y(z)=0$.

Let $\mathcal{A} = (g_y)_{y \in F}$.

We have $\left(\bigvee_{h \in \mathcal{A}} h\right)(z) \nleq p$ for all $z \in F$. In fact, if $z \in F$, $g_z(z) \nleq p$.

Since f is closed and $(X,\mathcal{T})$ is compact, by corollary 4.1.13. we have f compact.

Therefore, there is a finite subfamily $\mathcal{B}$ of $\mathcal{A}$, say $\mathcal{B} = \{g_{y_1}, \ldots, g_{y_k}\}$ with $\left(\bigvee_{h \in \mathcal{B}} h\right)(z) \nleq p$ for all $z \in F$.

Let $u = \bigwedge_{i=1}^{k} f_{y_i}$ and $v = \bigvee_{i=1}^{k} g_{y_i}$.

We have u, $v \in \mathcal{T}$, $u(x) \nleq p$; for every $y_p \notin f'$, $y_p \in v$ and $(\forall z \in X)$ $u(z)=0$ or $v(z)=0$. In fact, since each $f_{y_i}$ and $g_{y_i}$ is open we have $u, v \in \mathcal{T}$. For each $y_i$, $f_{y_i}(x) \nleq p$, so

$u(x) \nleq p$ since p is prime and $v(y) = \left(\bigvee_{i=1}^{k} g_{y_i}\right)(y) = \left(\bigvee_{h \in \mathcal{B}} h\right)(y) \nleq p$ for every $y_p \notin f'$, i.e., for every $y \in F$. We also have that $(\forall z \in X)$ $u(z)=0$ or $v(z)=0$ because if $z \in X$ and $u(z) \neq 0$ then for all $i \in \{1, \ldots, k\}$ $f_{y_i}(z) \neq 0$ which implies that for all $i \in \{1, \ldots, k\}$ $g_{y_i}(z)=0$. So $v(z)=0$.

Hence $(X, \mathcal{T})$ is regular.

<u>Theorem</u> <u>4.3.2.</u>

Let $(X, \mathcal{T})$ be a compact Hausdorff L-fts. Then $(X, \mathcal{T})$ is normal.

<u>Proof</u>

Let $p \in pr(L)$ and let f,g be closed L-fuzzy sets such that there are $x, y \in X$ with $x_p \notin f'$ and $y_p \notin g'$ and $(\forall z \in X)$ $f(z)=0$ or $g(z)=0$. By definition 3.4.10. we need to prove that there are $u, v \in \mathcal{T}$ with for every $z_p \notin f'$, $z_p \in u$; for every $z_p \notin g'$, $z_p \in v$ and $(\forall z \in X)$ $u(z)=0$ or $v(z)=0$.

For every $x \in X$ with $x_p \notin f'$, i.e., $f(x) \geq p'$, we have $g(x)=0$ because $p \in pr(L)$, so $p' \neq 0$ and by hypothesis we must have $g(x)=0$. So, since $(X, \mathcal{T})$ is regular by theorem 4.3.1., by definition 3.4.8. for each $x \in X$ with $f(x) \geq p'$ there are $u_x, v_x \in \mathcal{T}$ with $x_p \in u_x$; for every $z_p \notin g'$, $z_p \in v_x$ and $(\forall z \in X)$ $u_x(z)=0$ or $v_x(z)=0$.

Let $\mathcal{A} = (u_x)_{x \in F}$ where $F = \{t \in X;\ f(t) \geq p'\}$.

We have $\left(\bigvee_{h \in \mathcal{A}} h\right)(z) \nleq p$ for all $z \in F$ because for every $z \in F$ $u_z(z) \nleq p$.

Since f is closed and $(X, \mathcal{T})$ is compact, by corollary 4.1.13. we have f compact.

Therefore, there is a finite subfamily $\mathcal{B}$ of $\mathcal{A}$, say $\mathcal{B} = \left\{u_{x_1}, \ldots, u_{x_k}\right\}$ with $\left(\bigvee_{h\in\mathcal{B}} h\right)(z) \not\geq p$ for all $z \in F$.

Let $u = \bigvee_{j=1}^{k} u_{x_j}$ and $v = \bigwedge_{j=1}^{k} v_{x_j}$.

We have $u, v \in \mathcal{T}$; for every $z_p \notin f'$, $z_p \in u$; for every $z_p \notin g'$, $z_p \in v$ and $(\forall z \in X)$ $u(z)=0$ or $v(z)=0$. In fact, evidently $u, v \in \mathcal{T}$ and $u(z) = \left(\bigvee_{j=1}^{k} u_{x_j}\right)(z) = \left(\bigvee_{h\in\mathcal{B}} h\right)(z) \not\geq p$ for all $z_p \notin f'$, i.e., for all $z \in F$. Since for each $j \in \{1, \ldots, k\}$ $v_{x_j}(z) \not\geq p$ for all $z_p \notin g'$ and $p \in pr(L)$ we have $v(z) = \bigwedge_{i=1}^{k} v_{x_j}(z) \not\geq p$ for all $z_p \notin g'$. We also have $(\forall z \in X)$ $u(z)=0$ or $v(z)=0$ because if $z \in X$ and $u(z) \neq 0$ then there is $j \in \{1, \ldots, k\}$ such that $u_{x_j}(z) \neq 0$ which implies $v_{x_j}(z)=0$ and then $v(z)=0$.

Hence $(X, \mathcal{T})$ is normal.

## 4. Other characterizations of compactness

Proposition 4.4.1.

Let $(X,\mathcal{T})$ be an L-fts. Then $g \in L^X$ is compact if and only if for every $\alpha \in M(L)$ and every family $(f_i)_{i\in J}$ of closed L-fuzzy sets with $\left(\bigwedge_{i\in J} f_i\right)(x) \not\geq \alpha$ for all $x \in X$ with $g(x) \geq \alpha$, there exists a finite subset F of J with $\left(\bigwedge_{i\in F} f_i\right)(x) \not\geq \alpha$ for all $x \in X$ with $g(x) \geq \alpha$.

Proof

This immediately follows from definition 4.1.4.

Theorem 4.4.2.

Let $(X,\mathcal{T})$ be an L-fts. Then $g \in L^X$ is compact if and only if every constant $\alpha$-net $(S_m)_{m\in D}$ contained in g, i.e., $S_m \leq g$ for every $m \in D$, has a cluster point $x_\alpha \in M(L^X)$, with height $\alpha$, contained in g, i.e., $x_\alpha \leq g$, that is, $g(x) \geq \alpha$, for each $\alpha \in M(L)$.

Proof

Necessity:

Let $\alpha \in M(L)$ and let $(S_m)_{m\in D}$ be a constant $\alpha$-net (definition 2.3.8.) contained in g without any cluster point (definition 3.1.9.) with height $\alpha$ contained in g.

Then, for each $x \in X$ with $g(x) \geq \alpha$, $x_\alpha \in M(L^X)$ (remark 2.1.6.) is not a cluster point of $(S_m)_{m\in D}$, i.e., there are $N_x \in D$ and a closed L-fuzzy set $f_x$ with $f_x(x) \not\geq \alpha$ and $S_m \leq f_x$ for each $m \geq N_x$.

Let $x^1, \ldots, x^k$ be elements of X with $g(x^i) \geq \alpha$ for each $i \in \{1, \ldots, k\}$. Then there are $N_{x^1}, \ldots, N_{x^k}$ in D and closed L-fuzzy sets $f_{x^i}$ with $f_{x^i}(x^i) \not\geq \alpha$ and $S_m \leq f_{x^i}$ for each $m \geq N_{x^i}$ and for each $i \in \{1, \ldots, k\}$. Because D is a directed set, there is $N \in D$, $N \geq N_{x^i}$ for every $i \in \{1, \ldots, k\}$, such that $S_m \leq f_{x^i}$ for each $i \in \{1, \ldots, k\}$ and for each $m \geq N$.

Let $\mathcal{A} = (f_x)_{\substack{x \in X \text{ with} \\ g(x) \geq \alpha}}$.

Then $\bigwedge_{f_x \in \mathcal{A}} f_x \; (y) \not\geq \alpha$ for all $y \in X$ with $g(y) \geq \alpha$ because $f_y(y) \not\geq \alpha$. We also have that for any finite subfamily $\mathcal{B} = \{f_{x^1}, \ldots, f_{x^k}\} \subset \mathcal{A}$ there is $y \in X$ with $g(y) \geq \alpha$ and $\left(\bigwedge_{i=1}^{k} f_{x^i}\right)(y) \geq \alpha$ since $S_m \leq \bigwedge_{i=1}^{k} f_{x^i}$ for each $m \geq N$ because $S_m \leq f_{x^i}$ for each $i \in \{1, \ldots, k\}$ and for each $m \geq N$.

Hence g is not compact by proposition 4.4.1.

<u>Sufficiency</u>:

Suppose that g is not compact.

Then by proposition 4.4.1. there exists $\alpha \in M(L)$ and a collection $\mathcal{A} = (f_i)_{i \in J}$ of closed L-fuzzy sets with $\left(\bigwedge_{i \in J} f_i\right)(x) \not\geq \alpha$ for all $x \in X$ with $g(x) \geq \alpha$ but for any finite subfamily $\mathcal{B}$ of $\mathcal{A}$ there is $x \in X$ with $g(x) \geq \alpha$ and $\left(\bigwedge_{f_i \in \mathcal{B}} f_i\right)(x) \geq \alpha$.

Consider the family of all finite subsets of $\mathcal{A}$, $2^{(\mathcal{A})}$, with the order $\mathcal{B}_1 \leq \mathcal{B}_2$ if and only if $\mathcal{B}_1 \subset \mathcal{B}_2$. Then $2^{(\mathcal{A})}$ is a directed set.

So, writing $x_\alpha$ as $S_{\mathcal{B}}$ for every $\mathcal{B} \in 2^{(\mathcal{A})}$, $(S_{\mathcal{B}})_{\mathcal{B} \in 2^{\mathcal{A}}}$ is a constant $\alpha$-net contained in g because the height of $S_{\mathcal{B}}$ for all $\mathcal{B} \in 2^{(\mathcal{A})}$ is $\alpha$ and $S_{\mathcal{B}} \leq g$ for all $\mathcal{B} \in 2^{\mathcal{A}}$, i.e., $g(x) \geq \alpha$.

$(S_{\mathcal{B}})_{\mathcal{B} \in 2^{(\mathcal{A})}}$ also satisfies the condition that for each closed L-fuzzy set $f_i \in \mathcal{B}$ we have $x_\alpha = S_\mathcal{B} \leq f_i$.

Let $y \in X$ with $g(y) \geq \alpha$.

Therefore $\left(\bigwedge_{i\in J} f_i\right)(y) \not\geq \alpha$, i.e., there exists $j \in J$ with $f_j(y) \not\geq \alpha$.

Let $\mathcal{B}_0 = \{f_j\}$.

So, for any $\mathcal{B} \geq \mathcal{B}_0$, $S_\mathcal{B} \leq \bigwedge_{f_i \in \mathcal{B}} f_i \leq \bigwedge_{f_i \in \mathcal{B}_0} f_i = f_j$ .

Thus, we got a closed L-fuzzy set $f_j$, with $f_j(y) \not\geq \alpha$ and $\mathcal{B}_0 \in 2^{(\mathcal{A})}$ such that for any $\mathcal{B} \geq \mathcal{B}_0$ $S_\mathcal{B} \leq f_j$, that means that $y_\alpha \in M(L^X)$ is not a cluster point of $(S_\mathcal{B})_{\mathcal{B}\in 2^{(\mathcal{A})}}$ for all $y \in X$ with $g(y) \geq \alpha$.

Hence the constant $\alpha$-net $(S_\mathcal{B})_{\mathcal{B}\in 2^{(\mathcal{A})}}$ has no cluster point with height $\alpha$, contained in g.

## 5. The goodness of N-L-compactness and a comparison of compactness with N-L-compactness

Definition 4.5.1. Wang [95]

Let $(X,\mathcal{T})$ be a [0,1] - fuzzy topological space and let $g\in[0,1]^X$. The fuzzy set g is said to be N-compact if and only if every $\alpha$-net $(S_m)_{m\in D}$ contained in f, i.e., $S_m\leq f$ for each $m\in D$; has a cluster point with height $\alpha$, $x_\alpha\in M([0,1]^X)$, contained in f $(x_\alpha\leq f)$, for each $\alpha\in(0,1]$.

Remark 4.5.2.

In definition 4.5.1. the notion of $\alpha$-net is different from our definition 2.3.9.. To Wang a net $(S_m)_{m\in D}$ is called an $\alpha$-net $(\alpha\in(0,1])$ if and only if the net $(h(S_m))_{m\in D}$ converges to $\alpha$ in (0,1].

Definition 4.5.3. Zhao [110]

Let $(X,\mathcal{T})$ be an L-fts where L is a fuzzy lattice. Let $\alpha\in M(L)$ and let $\mathcal{A} = (f_i)_{i\in J}$ be a family of closed L-fuzzy sets. The family $\mathcal{A}$ is called a family of $\alpha$-R-neighbourhoods of an L-fuzzy set f if and only if for every $x\in X$ with $f(x)\geq\alpha$ there is $f_i\in\mathcal{A}$ such that $f_i(x)\ngeq\alpha$. The family $\mathcal{A}$ is called a family of $\alpha^-$-R-Neighbourhoods of f if and only if there exists $\gamma\in\beta^*(\alpha)$ such that $\mathcal{A}$ is a family of $\gamma$-R-Neighbourhoods of f. The L-fuzzy set f is said to be N-L-compact if and only if for each $\alpha\in M(L)$, each family $\mathcal{A}$ of $\alpha$-R-Neighbourhoods of f has a finite subfamily $(f_i)_{i\in F} = \mathcal{B}$ such that $\mathcal{B}$ is a family of $\alpha^-$-R-Neighbourhoods of f.

Remark 4.5.4.

Let $(X,\mathcal{T})$ be an L-fts. Even if the L-fuzzy topology $\mathcal{T}$ has only a finite number of open L-fuzzy sets it is possible to have L-fuzzy sets which are not N-L-compact. For example:

Consider $X = (0,1)$, $L = [0,1]$, $f: \underset{x \to x}{X \to L}$ and $\mathcal{T} = \{x, \phi, f'\}$.

The L-fts $(X,\mathcal{T})$ is not N-L-compact. In fact, considering $\alpha=1 \in M(L) = (0,1]$ and the closed L-fuzzy set f we have that $f(x)<1$ for all $x \in X$ but there is not $\gamma \in \beta^*(1) = (0,1)$ such that $f(x)<\gamma$ for all $x \in X$.

Proposition 4.5.5. Zhao [110]

Let $(X,\mathcal{T})$ be an L-fts and $f \in L^X$. Then f is N-L-compact if and only if every $\alpha$-net $(S_m)_{m \in D}$ contained in f has a cluster point with height $\alpha$, $x_\alpha \in M(L^X)$, contained in f, for each $\alpha \in M(L)$.

Proof

See theorem 6.2 in Zhao [110].

Remark 4.5.6.

We have $\alpha$-net in proposition 4.5.5. meaning the same as in definition 2.3.9.

Proposition 4.5.7.

Let $(X,\mathcal{T})$ be an L-fts. If $g \in L^X$ is N-L-compact then it is compact.

Proof

This immediately follows from proposition 4.5.5. and theorem 4.4.2. or by definition 4.5.3. and proposition 4.4.1. as well since $\vee\beta^*(\alpha) = \alpha$ for every $\alpha \in M(L)$.

Theorem 4.5.8. (The goodness of N-L-compactness)

Let $(X,\delta)$ be a topological space. Then $(X,\delta)$ is compact if and only if $(X,\omega(\delta))$ is an N-L-compact L-fts.

Proof

Necessity:

Let $\alpha \in M(L)$, let $(S_m)_{m \in D}$ be an $\alpha$-net in the L-fts $(X,\omega(\delta))$ with $x^m = \text{supp } S_m$ for every $m \in D$. By proposition 4.5.5., to prove the N-L-compactness of $(X,\omega(\delta))$, it will suffice to show that $(S_m)_{m \in D}$ has a cluster point in $(X,\omega(\delta))$.

Since $(S_m)_{m \in D}$ is an $\alpha$-net in $(X,\omega(\delta))$, $(x^m)_{m \in D}$ is a net in $(X,\delta)$. From the compactness of $(X,\delta)$, the net $(x^m)_{m \in D}$ has a cluster point $x$ in $(X,\delta)$.

We shall prove that $x_\alpha \in M(L^X)$ is a cluster point of $(S_m)_{m \in D}$, that is, for each closed L-fuzzy set $f$ with $f(x) \not\geq \alpha$ we have that for all $j \in D$ there is $m \in D$ such that $m \geq j$ and $f(x^m) \not\geq h(S_m)$.

For each closed L-fuzzy set $f$ with $f(x) \not\geq \alpha$, since by proposition 1.2.7. $\beta^*(\alpha)$ is a minimal set relative to $\alpha$, there is $\lambda \in \beta^*(\alpha)$ with $f(x) \not\geq \lambda \leq \alpha$. Since $(S_m)_{m \in D}$ is an $\alpha$-net and $\lambda \in \beta^*(\alpha)$, by definition 2.3.9., there is $m_0 \in D$ such that $h(S_m) = \lambda_m \geq \lambda$ for all $m \geq m_0$.

Let $H = \{t \in X;\ f'(t) \not\geq \lambda'\}$.

Since $f' \in \omega(\delta)$, by proposition 3.2.9. $H \in \delta$. We also have that $x \in H$, so by the fact that x is a cluster point of $(x^m)_{m \in D}$ in $(X,\delta)$, for each $j \in D$ there is $m \in D$ such that $m \geq j$, $m \geq m_0$ and $x^m \in H$. We have then $f(x^m) \not\geq \lambda$. Thus, $f(x^m) \not\geq \lambda \leq \lambda_m = h(S_m)$.

Hence $x_\alpha \in M(L^X)$ is a cluster point of $(S_m)_{m \in D}$ and by proposition 4.5.5. $(X,\omega(\delta))$ is N-L-compact.

Sufficiency:

If $(X,\omega(\delta))$ is N-L-compact, by proposition 4.5.7. it is compact. Then by theorem 4.1.6. $(X,\delta)$ is compact.

Definition 4.5.9.

Let $(X,\mathcal{T})$ be an L-fts and $f \in L^X$. The L-fuzzy set f is said to be M(L) accessible if and only if for every directed subset H of $J = \{\alpha \in M(L);\ f(x) \geq \alpha$ for some $x \in X\}$ with $\vee H \in M(L)$ we have $\vee H \in J$.

Proposition 4.5.10.

Let $(X,\mathcal{T})$ be an L-fts and let f be an N-L-compact set in $(X,\mathcal{T})$. Then f is M(L) accessible.

Proof

Let H be a directed subset of $J = \{\alpha \in M(L);\ f(x) \geq \alpha$ for some $x \in X\}$ with $\vee H \in M(L)$. We need to prove that f being N-L-compact implies $\vee H \in J$, i.e., there is $x \in X$ with $f(x) \geq \vee H$. Since by proposition 4.5.5., f N-L-compact implies that every $\alpha$-net contained in f has a cluster point with height $\alpha$, contained in f, which implies that there is $x \in X$ with $f(x) \geq \alpha$, it will be sufficient to prove that there exists an $\alpha$-net contained in f where $\alpha = \vee H$.

So, we are going to exhibit an $\vee H$-net contained in f.

Since $H \subset J$, for each $m \in H$ there is $x^m \in X$ such that $f(x^m) \geq m$. We also have, by proposition 1.2.7., that $\beta^*(\vee H)$ is a minimal set relative to $\vee H \in M(L)$ and is a directed set as well. From the fact that $\beta^*(\vee H)$ is a minimal set relative to $\vee H$ we have for each $\gamma \in \beta^*(\vee H)$, there is $m_\gamma \in H$ such that $m_\gamma \geq \gamma$.

So, for each $\gamma \in \beta^*(\vee H)$ there exists $m_\gamma \in H$ and $x^{m_\gamma} \in X$ with $f(x^{m_\gamma}) \geq m_\gamma \geq \gamma$.

Consider the net $(S_\gamma)_{\gamma \in \beta^*(\vee H)}$ where supp $S_\gamma = x^{m_\gamma}$ and $h(S_\gamma) = \gamma$.

Actually $(S_\gamma)_{\gamma \in \beta^*(\vee H)}$ is an $\alpha$-net where $\alpha = \vee H$ because for each $b \in \beta^*(\alpha)$, there exists $\gamma_0 \in \beta^*(\alpha)$ such that $h(S_\gamma) = \gamma \geq b$ whenever $\gamma \geq \gamma_0$ (take $\gamma_0 = b$). We also have $(S_\gamma)_{\gamma \in \beta^*(\vee H)}$ contained in f, i.e., $f(\text{supp } S_\gamma) \geq h(S_\gamma)$ because $f(x^{m_\gamma}) \geq \gamma$ for each $\gamma \in \beta^*(\vee H)$.

Now we are going to state a result obtained by Zhao in [110] which we shall use to prove our next theorem.

Proposition 4.5.11. Zhao [110]

Let $(X, \mathcal{T})$ be an L-fts. If g is an N-L-compact L-fuzzy set, then for each closed L-fuzzy set h, $g \wedge h$ is N-L-compact.

Proof

See theorem 4.9. in Zhao [110].

Theorem 4.5.12.

Let $(X,\mathcal{T})$ be an L-fts and $g\in L^X$. The L-fuzzy set g is N-L-compact if and only if g is compact and for each closed L-fuzzy set h in $(X,\mathcal{T})$, $g\wedge h$ is M(L) accessible.

Proof

Necessity:

From proposition 4.5.7., g N-L-compact implies g compact. If h is a closed L-fuzzy set then by proposition 4.5.11. $g\wedge h$ is also N-L-compact. Then by proposition 4.5.10. $g\wedge h$ is M(L) accessible.

Sufficiency:

Suppose that g is not N-L-compact. Then there exist $\alpha_0\in M(L)$ and a family $\mathcal{A} = (f_i)_{i\in K}$ of closed L-fuzzy sets with $\left(\bigwedge_{i\in K} f_i\right)(x)\not\geq\alpha_0$ for all $x\in X$ with $g(x)\geq\alpha_0$ but for any finite subfamily $\mathcal{B}$ of $\mathcal{A}$ and any $\gamma\in\beta^*(\alpha_0)$ there is $x\in X$ with $g(x)\geq\gamma$ and $\left(\bigwedge_{f_i\in\mathcal{B}} f_i\right)(x)\geq\gamma$.

From the compactness of g, by proposition 4.4.1., there is a finite subset F of K with $\left(\bigwedge_{i\in F} f_i\right)(x)\not\geq\alpha_0$ for all $x\in X$ with $g(x)\geq\alpha_0$.

Let $h = \bigwedge_{i\in F} f_i$.

So, h is a closed L-fuzzy set. Now we are going to prove that $g\wedge h$ is not M(L) accessible.

We need to exhibit a directed set H, $H\subset J = \{\alpha\in M(L);\ (g\wedge h)(x)\geq\alpha$ for some $x\in X\}$ with $\vee H \in M(L)$ but $\vee H \notin J$.

Take $H = \beta^*(\alpha_0)$.

By proposition 1.2.7., $\vee\beta^*(\alpha_0) = \alpha_0$ and since $\alpha_0 \in M(L)$, $\beta^*(\alpha_0)$ is a directed set. Then $\vee H = \alpha_0 \in M(L)$ and $\vee H=\alpha_0 \notin J$ because $(g\wedge h)(x)\not\geq\alpha_0$ for every $x\in X$ because if

$g(x) \geq \alpha_0$ then $\left(\bigwedge_{i\in F} f_i\right)(x) \ngeq \alpha_0$ which implies $(g\wedge h)(x) \ngeq \alpha_0$ and if $g(x) \ngeq \alpha_0$ then immediately $(g\wedge h)(x) \ngeq \alpha_0$. We also have $H\subset J$ because for any $\gamma\in H = \beta^*(\alpha_0)$ there is $x\in X$ with $g(x) \geq \gamma$ and $h(x) = \left(\bigwedge_{i\in F} f_i\right)(x) \geq \gamma$, which implies $(g\wedge h)(x) \geq \gamma$.

<u>Remark</u> <u>4.5.13.</u>

Let $(X,\mathcal{T})$ be a $[0,1]$-fuzzy topological space and let $f\in[0,1]^X$. The $[0,1]$-fuzzy set $f$ is $(0,1]$ accessible if and only if there is $x_0\in X$ such that $f(x_0) = \vee\{f(t);\ t\in X\}$.

<u>Necessity</u>:

In fact, if $f$ is $(0,1]$ accessible then for every $Y\subseteq J=\{\alpha\in(0,1];\ f(x)\geq\alpha \text{ for some } x\in X\}$ with $\vee Y \in (0,1]$ we have $\vee Y \in J$. Let $Z = (0,1]\cap\{f(t);\ t\in X\}$. Thus, we have $Z\subseteq J$ and $\vee Z \in (0,1]$ then $\vee Z \in J$. Therefore there is $x_0\in X$ such that $f(x_0) \geq \vee Z$ and then we have $f(x_0) = \vee Z$. Hence $f(x_0) = \vee\{f(t);\ t\in X\}$.

<u>Sufficiency</u>:

Let $Y\subseteq J = \{\alpha\in(0,1];\ f(x)\geq\alpha \text{ for some } x\in X\}$ with $\vee Y \in(0,1]$. If there is $x_0\in X$ with $f(x_0) = \vee\{f(t);\ t\in X\}$, then $Y\subseteq J = (0, f(x_0)]$. Thus $\vee Y \leq \vee J = f(x_0)$ and since $\vee Y\neq 0$, we have $\vee Y \in J$. Hence $f$ is $(0,1]$ accessible.

<u>Remark</u> <u>4.5.14.</u>

By remark 4.5.13., proposition 4.5.10. is a generalization of the following result obtained by Wang in [95]

"If $(X,\mathcal{T})$ is a $[0,1]$-fts and $f$ is N-compact in $(X,\mathcal{T})$ then there is $x_0\in X$ such that $f(x_0)=\vee\{f(t);\ t\in X\}$".

## 6. Other compactnesses in L-fuzzy topological spaces

Definition 4.6.1. Hutton [44]

Let $(L,\mathcal{T})$ be a fuzzy topological space where L is a fuzzy lattice. An open cover $\mathcal{A}$ of a fuzzy set f is a collection of open fuzzy sets such that $f \le \bigvee_{g\in\mathcal{A}} g$. A subcollection $\mathcal{B}$ of a cover $\mathcal{A}$ of f that is also a cover of f, i.e., $f \le \bigvee_{g\in\mathcal{B}} g$, is called a subcover of f. The fts $(L,\mathcal{T})$ is said to be H-compact if and only if every open cover $\mathcal{A}$ of a closed fuzzy set f has a finite subcover.

Remark 4.6.2.

In 1968, Chang [18], defined compactness for a [0,1]-fts $(X,\mathcal{T})$ as follows: "$(X,\mathcal{T})$ is Chang compact if and only if every open cover of X has a finite subcover".

Lowen [54] showed that Chang compactness is not a good extension by exhibiting a compact topological space $(X,\delta)$ such that the [0,1]-fts $(x,\omega(\delta))$ is not Chang compact.

Since H-compactness implies Chang compactness, we can conclude that H-compactness is not good as well.

Definition 4.6.3. Lowen [54]

Let $(X,\mathcal{T})$ be a [0,1]-fts. $(X,\mathcal{T})$ is called fuzzy compact if and only if for each $\alpha\in[0,1]$ and each collection $(f_i)_{i\in J}$ of open fuzzy sets with $\left(\bigvee_{i\in J} f_i\right)(x) \ge\alpha$ for every $x\in X$ and for each $\varepsilon\in(0,\alpha]$, there is a finite subcollection $(f_i)_{i\in F}$ with $\left(\bigvee_{i\in F} f_i\right)(x)\ge\alpha-\varepsilon$ for every $x\in X$.

Definition 4.6.4. Wang [66]

Let $(X,\mathcal{T})$ be an L-fts where L is a fuzzy lattice.

$(X,\mathcal{T})$ is said to be <u>Lowen L-fuzzy compact</u> if and only if for each $\alpha \in M(L)$ and every $\alpha$-net $(S_m)_{m\in D}$ in $(X,\mathcal{T})$ and each $r\in\beta^*(\alpha)$, $(S_m)_{m\in D}$ has a cluster point $x_r \in M(L^X)$, with height r.

<u>Proposition 4.6.5.</u> Meng [66]

Let $(X,\mathcal{T})$ be an L-fts. Then $(X,\mathcal{T})$ is Lowen L-fuzzy compact if and only if for each $\alpha\in M(L)$ and each $r\in\beta^*(\alpha)$, each family $\mathcal{A}$ of r-R-neighbourhoods of X has a finite subfamily $\mathcal{B}$ such that $\mathcal{B}$ is a family of $\alpha$-R-Neighbourhoods of X.

<u>Proof</u>

See theorem 2.5 in Meng [66].

<u>Proposition 4.6.6.</u>

Let $(X,\mathcal{T})$ be an L-fts. Then $(X,\mathcal{T})$ is Lowen L-fuzzy compact if and only if for every $p\in pr(L)$, every $\gamma\in L$ such that $\gamma'\in\beta^*(p')$ and every family $(f_i)_{i\in J}$ of open L-fuzzy sets such that $\left(\bigvee_{i\in J} f_i\right)(x)\not\geq\gamma$ for all $x\in X$, there exists a finite subset F of J with $\left(\bigvee_{i\in F} f_i\right)(x)\not\geq p$ for all $x\in X$.

<u>Proof</u>

By proposition 4.6.5. we have that $(X,\mathcal{T})$ is Lowen L-fuzzy compact if and only if for every $\alpha\in M(L)$, every $\delta\in\beta^*(\alpha)$ and every family $(g_i)_{i\in J}$ of closed L-fuzzy sets such that $\left(\bigwedge_{i\in J} g_i\right)(x)\not\leq\delta$ for all $x\in X$, there exists a finite subset F of J with $\left(\bigwedge_{i\in F} g_i\right)(x)\not\leq\alpha$ for all $x\in X$.

So, $(X,\mathcal{T})$ is Lowen L-fuzzy compact if and only if for every $\alpha'=p \in pr(L)$, every $\delta'=\gamma\in L$ such that $\gamma'=\delta\in$

$\beta^*(p'=\alpha)$ and every family $(f_i)_{i\in J}$ of open L-fuzzy sets $(f_i = g_i')$ such that $\left(\bigvee_{i\in J} f_i\right)(x) \not\geq \gamma$ for all $x\in X$, there exists a finite subset F of J with $\left(\bigvee_{i\in F} f_i\right)(x) \not\geq p$ for all $x\in X$.

Proposition 4.6.7.

Let $(X,\mathcal{T})$ be a compact L-fts. Then $(X,\mathcal{T})$ is Lowen L-fuzzy compact.

Proof

Let $p\in pr(L)$, $\gamma\in L$ such that $\gamma' \in\beta^*(p')$ and let $(f_i)_{i\in J}$ be a family of open L-fuzzy sets with $\left(\bigvee_{i\in J} f_i\right)(x)\not\geq\gamma$ for all $x\in X$.

Since $\gamma'\in\beta^*(p')$ implies that $\gamma'\in M(L)$, i.e., $\gamma\in pr(L)$; by the compactness of $(X,\mathcal{T})$ there is a finite subset F of J with $\left(\bigvee_{i\in F} f_i\right)(x)\not\geq\gamma$ for all $x\in X$. But, because $\gamma'\in\beta^*(p')$ and by proposition 1.2.7. $\bigvee\beta^*(p') = p'$, we have $\gamma'\leq p'$; so $\gamma\geq p$. Then $\left(\bigvee_{i\in F} f_i\right)(x)\not\geq p$ for all $x\in X$. Hence, by proposition 4.6.6., $(X,\mathcal{T})$ is Lowen L-fuzzy compact.

Theorem 4.6.8. (The goodness of Lowen L-fuzzy compactness)

Let $(X,\delta)$ be a topological space. Then $(X,\delta)$ is compact if and only if the L-fts $(X,\omega(\delta))$ is Lowen L-fuzzy compact.

Proof

Necessity:

By theorem 4.1.6., $(X,\delta)$ compact implies that the L-fts $(X,\omega(\delta))$ is compact. Hence by proposition 4.6.7.

$(X,\omega(\delta))$ is Lowen L-fuzzy compact.

Sufficiency:

Let $(A_i)_{i\in J}$ be an open cover of $(X,\delta)$ and let $p\in pr(L)$. Then, by proposition 3.2.10, $(\chi_{A_i})_{i\in J}$ is a family of open L-fuzzy sets in $(X,\omega(\delta))$ and we also have $\left(\bigvee_{i\in J} \chi_{A_i}\right)(x)=1\nleq\gamma$ for all $x\in X$ and for all $\gamma\in L$ such that $\gamma'\in\beta^*(p')$. From the Lowen L-fuzzy compactness of $(X,\omega(\delta))$ and theorem 4.6.7., there is a finite subset F of J with $\left(\bigvee_{i\in F} \chi_{A_i}\right)(x)\nleq p$, i.e., $\left(\bigvee_{i\in F} \chi_{A_i}\right)(x)=1$ for all $x\in X$. Thus $\bigcup_{i\in F} A_i = X$ and hence $(X,\delta)$ is compact.

Remark 4.6.9.

Let $(X,\mathcal{T})$ be an L-fts where L is a fuzzy lattice and let $i_p(f) = \{x\in X;\ f(x)\nleq p\}$ where $f\in\mathcal{T}$ and $p\in pr(L)$. Then the collection $\varphi(\mathcal{T}) = \{i_p(f);\ p\in p[r(L)$ and $f\in\mathcal{T}\} \cup\{X\}$ is a subbase for some ordinary topology, $i_L(\mathcal{T})$, on X.

Definition 4.6.10. Lowen [56]

Let $(X,\mathcal{T})$ be a [0,1]-fuzzy topological space. $(X,\mathcal{T})$ is said to be ultra-fuzzy compact if and only if the topological space $(X,i_{[0,1]}(\mathcal{T}))$ is compact.

Definition 4.6.11. Wang [67]

Let $(X,\mathcal{T})$ be an L-fts. $(X,\mathcal{T})$ is said to be ultra-L-fuzzy compact if and only if the topological space $(X,\ i_L(\mathcal{T}))$ is compact.

Proposition 4.6.12

Let $(X,\mathcal{T})$ be an ultra-L-fuzzy compact L-fts. Then

$(X,\mathcal{T})$ is N-L-compact.

<u>Proof</u>

Let $(S_m)_{m\in D}$ be an $\alpha$-net in $(X,\mathcal{T})$. Let $x^m$ be the support of $S_m$ for each $m\in D$.

So, $(x^m)_{m\in D}$ is a net in the ordinary topological space $(X, i_L(\mathcal{T}))$. Since $(X,\mathcal{T})$ is ultra-L-fuzzy compact we have that $(X, i_L(\mathcal{T}))$ is compact. Thus, $(x^m)_{m\in D}$ has a subnet $(x^{m_i})_{i\in E}$ converging to some $x\in X$.

So, $(S_{m_i})_{i\in E}$ is a subnet of $(S_m)_{m\in D}$ converging to $x_\alpha\in M(L^X)$. In fact, if f is a closed L-fuzzy set with $f(x)\not\geq\alpha$, since by proposition 1.2.7. $\beta^*(\alpha)$ is a minimal set relative to $\alpha$, there is $\alpha_0\in\beta^*(\alpha)$ such that $f(x)\not\geq\alpha_0\leq\alpha$. Let $U = \{t\in X;\ f'(t)\not\leq\alpha_0'\}$. Since $\alpha_0'\in pr(L)$ and $f'\in\mathcal{T}$, we have that $U\in i_L(\mathcal{T})$. We also have that $x\in U$. Thus, because $x^{m_i}\to x$ in $(X, i_L(\mathcal{T}))$, there exists $i_0\in E$ such that $i\geq i_0$ implies that $x^{m_i}\in U$, that is, $f(x^{m_i})\not\geq\alpha_0$ for every $i\geq i_0$. Moreover, since $(S_{m_i})_{i\in E}$ is an $\alpha$-net and $\alpha_0\in\beta^*(\alpha)$, there is $i_1\in E$ such that $h(S_{m_i})\geq\alpha_0$ for every $i\geq i_1$. Take $i_2\in E$ such that $i_2\geq \max\{i_0, i_1\}$. So, $f(x^{m_i})\not\geq\alpha_0\leq h(S_{m_i})$ for every $i\geq i_2$. Hence $S_{m_i}\to x_\alpha$.

Therefore, by proposition 3.1.10., $x_\alpha$ is a cluster point of $(S_m)_{m\in D}$.

Hence, by proposition 4.5.5., $(X,\mathcal{T})$ is N-L-compact.

<u>Theorem</u> <u>4.6.13.</u> (The goodness of ultra-L-fuzzy compactness)

Let $(X,\delta)$ be a topological space. Then $(X,\delta)$ is compact if and only if the L-fts $(X,\omega(\delta))$ is ultra-L-fuzzy compact.

Proof

Necessity:

If $(X,\delta)$ is compact then we need to prove that $(X, i_L(\omega(\delta)))$ is a compact topological space.

Let $(A_i)_{i\in J}$ be a subbasic open cover of $(X, i_L(\omega(\delta)))$. Thus each $A_i$ is of the form $\{x\in X;\ f(x)\nleq p\}$ for some $f\in\omega(\delta)$ and $p\in pr(L)$ or $A_i=X$. By proposition 3.2.9., $f\in\omega(\delta)$ implies that for each $p\in pr(L)$, $\{x\in X; f(x)\nleq p\}$ $\in\delta$. So, $(A_i)_{i\in J}$ is an open cover of $(X,\delta)$. By the compactness of $(X,\delta)$, there exists a finite subset F of J such that $\bigcup_{i\in F} A_i = X$.

Hence $(X, i_L(\omega(\delta)))$ is compact.

Sufficiency:

If $(X, \omega(\delta))$ is ultra-L-fuzzy compact then by proposition 4.6.12. $(X, \omega(\delta))$ is N-L-compact. From theorem 4.5.8., $(X, \omega(\delta))$ N-L-compact implies that $(X,\delta)$ is compact.

Definition 4.6.14. Xu [105]

Let $(X,\mathcal{T})$ be an L-fts where L is a fuzzy lattice and let $f\in L^X$. The L-fuzzy set f is called X-compact if and only if for any $\alpha\in M(L)$, each family $\mathcal{A}$ of $\alpha^-$-R-Neighbourhoods of f has a finite subfamily $\mathcal{B}$ such that $\mathcal{B}$ is a family of $\alpha^-$-R-Neighbourhoods of f.

Proposition 4.6.15.

Let $(X,\mathcal{T})$ be an L-fts and let $f \in L^X$. The L-fuzzy set $f$ is X-compact if and only if for each prime $p \in L$ and every collection $(f_i)_{i \in J}$ of open L-fuzzy sets such that there is $\gamma \in L$ with $\gamma' \in \beta^*(p')$ and $\left(\bigvee_{i \in J} f_i\right)(x) \not\geq \gamma$ for all $x \in X$ with $f(x) \geq \gamma'$, there exist a finite subset $F$ of $J$ and $\gamma_1 \in L$ with $\gamma_1' \in \beta^*(p')$ such that $\left(\bigvee_{i \in F} f_i\right)(x) \not\geq \gamma_1$ for all $x \in X$ with $f(x) \geq \gamma_1'$.

Proof

By definition 4.6.14. we have that $f$ is X-compact if and only if for every $\alpha \in M(L)$ and every family $(f_i)_{i \in J}$ of closed L-fuzzy sets such that there is $\delta \in \beta^*(\alpha)$ with $\left(\bigwedge_{i \in J} f_i\right)(x) \not\geq \delta$ for all $x \in X$ with $f(x) \geq \delta$, there exist a finite subset $F$ of $J$ and $\delta_1 \in \beta^*(\alpha)$ such that $\left(\bigwedge_{i \in F} f_i\right)(x) \not\geq \delta_1$ for all $x \in X$ with $f(x) \geq \delta_1$. And this is clearly equivalent to our result.

Proposition 4.6.16.

Let $(X,\mathcal{T})$ be an L-fts and let $f$ be a compact L-fuzzy set of $(X,\mathcal{T})$. Then $f$ is X-compact.

Proof

Let $p \in pr(L)$ and let $(f_i)_{i \in J}$ be a family of open L-fuzzy sets such that there is $\gamma \in L$ with $\gamma' \in \beta^*(p')$ and $\left(\bigvee_{i \in J} f_i\right)(x) \not\geq \gamma$ for all $x \in X$ with $f(x) \geq \gamma'$. Since $\gamma' \in \beta^*(p')$, $\gamma \in pr(L)$. By the compactness of $f$, there is a finite subset $F$ of $J$ such that $\left(\bigvee_{i \in F} f_i\right)(x) \not\geq \gamma$ for all $x \in X$ with $f(x) \geq \gamma'$.

Hence f is X-compact by proposition 4.6.15.

<u>Theorem</u> <u>4.6.17.</u> (The goodness of X-compactness)

Let $(X,\delta)$ be a topological space. Then $(X,\delta)$ is compact if and only if the L-fts $(X, \omega(\delta))$ is X-compact.

<u>Proof</u>

<u>Necessity</u>:

If $(X,\delta)$ is compact, by theorem 4.1.6., the L-fts $(X,\omega(\delta))$ is compact. Then from proposition 4.6.16. $(X,\omega(\delta))$ is X-compact.

<u>Sufficiency</u>:

This is similar to the sufficiency of theorem 4.6.8.

<u>Theorem</u> <u>4.6.18.</u> (The goodness of $\alpha$-compactness for $\alpha\neq1$)

Let $(X,\delta)$ be a topological space and let $\alpha\in L$ with $\alpha\neq1$. Then $(X,\delta)$ is compact if and only if $(X, \omega(\delta))$ is $\alpha$-compact.

<u>Proof</u>

<u>Necessity</u>:

Let $\mathcal{A} = (f_i^{e_iU_i})_{i\in J}$ be a family of basic open L-fuzzy sets in $(X, \omega(\delta))$ such that for each $x\in X$ there is $i\in J$ with $f_i(x)>\alpha$. By proposition 3.2.11., we consider

$f_i^{e_iU_i}(x) = \begin{cases} e_i & \text{if } x\in U_i\in\delta \\ 0 & \text{otherwise} \end{cases}$ for all $x\in X$ and each $i\in J$.

Let $\mathcal{C} = \{U_i$, there is $i\in J$ with $\alpha<e_i$ and $f_i^{e_iU_i} \in\mathcal{A}\}$.

Therefore $\bigcup_{i\in J} U_i = X$.

By the compactness of $(X,\delta)$, there exists a finite subset F of J with $\bigcup_{i\in F} U_i = X$.

Hence for each $x\in X$ there is $i\in F$ with $f_i^{e_i U_i}(x)>\alpha$ and $(X, \omega(\delta))$ is $\alpha$-compact.

<u>Sufficiency</u>:

Let $(A_i)_{i\in J}$ be an open cover of $(X,\delta)$. Then by proposition 3.2.10. $(\chi_{A_i})_{i\in J}$ is a family of open L-fuzzy sets in $(X, \omega(\delta))$. Since $\bigcup_{i\in J} A_i = X$, for each $x\in X$ there is $i\in J$ with $\chi_{A_i}(x) = 1>\alpha$. So, by the $\alpha$-compactness of $(X, \omega(\delta))$ there exists a finite subset F of J such that for each $x\in X$ there is $i\in F$ with $\chi_{A_i}(x)>\alpha$, i.e., $\chi_{A_i}(x) =1$.

Hence $\bigcup_{i\in F} A_i = X$ and $(X,\delta)$ is compact.

<u>Remark</u> <u>4.6.19.</u>

1-compactness is not a good extension.

In fact, it is immediate from definition 4.1.1. that every L-fts is 1-compact. So, by considering $X=[0,1]=L$ with the discrete ordinary topology $\delta$, we have that $(X,\delta)$ is not compact but $(X,\omega(\delta))$ is 1-compact.

<u>Remark</u> <u>4.6.20.</u>

By proving that if X is infinite and $\delta$ is the finite complement topology on X then $(X, \omega(\delta))$ is not $\alpha^*$-compact for any $\alpha\in(0,1]$, Lowen [54] showed that $\alpha^*$-compactness is not a good extension.

## 7. Relations between the good definitions of compactness in L-fuzzy topological spaces

Proposition 4.7.1.

Let $(X,\mathcal{T})$ be an X-compact L-fts. Then $(X,\mathcal{T})$ is Lowen L-fuzzy compact.

Proof

Let $p\in pr(L)$, $\gamma\in L$ such that $\gamma'\in\beta^*(p')$ and let $(f_i)_{i\in J}$ be a family of open L-fuzzy sets with $\left(\bigvee_{i\in J} f_i\right)(x)\not\geq\gamma$ for all $x\in X$. By the X-compactness of $(X,\mathcal{T})$ there exists a finite subset F of J and $\gamma_1' \in\beta^*(p')$ such that $\left(\bigvee_{i\in F} f_i\right)(x)$ $\not\geq\gamma_1$ for all $x\in X$. Since by proposition 1.2.7. $\vee\beta^*(p') =$ $p'$ we have $\gamma_1'\leq p'$, i.e., $\gamma_1\geq p$. So, $\left(\bigvee_{i\in F} f_i\right)(x)\not\geq p$ for all $x\in X$.

Hence, by proposition 4.6.6., $(X,\mathcal{T})$ is Lowen L-fuzzy compact.

Remark 4.7.2.

We can have an N-L-compact L-fts $(X,\mathcal{T})$ which is not ultra-L-fuzzy compact.

Consider $L = [0,1]$, $X = \mathbb{N}$.

For each $\alpha\in(0,1)$ there exists $m\in\mathbb{N}$ such that $(m-1)/m<\alpha\leq m/(m+1)$.

Let $\alpha_i \in[(m-1)/m, \alpha)$, $i\in\{1, ..., m\}$ and let

$$f(\alpha, \alpha_1, ..., \alpha_m)(x) = \begin{cases}\alpha & \text{if } x>m \\ \alpha_i & \text{if } x=i \quad i\in\{1, ..., m\}\end{cases}.$$

Let $\mathcal{T}$ consist of $\phi$,X and all those L-fuzzy sets that are complements of the L-fuzzy sets above.

We have that $(X,\mathcal{T})$ is N-L-compact and it is not ultra-L-fuzzy compact (see example 5.2. [95]).

Remark 4.7.3.

We can have a compact L-fts $(X,\mathcal{T})$ which is not N-L-compact.

Consider $X = (0,1)$, $L = [0,1]$, $f:\underset{x\to x}{X\to L}$ and $\mathcal{T} = \{X, \phi, f'\}$.

By remark 4.5.4. $(X,\mathcal{T})$ is not N-L-compact and by proposition 4.1.7. it is compact.

Remark 4.7.4.

We can have an X-compact L-fts $(X,\mathcal{T})$ which is not compact.

Consider $X = [0,1] = L$ and $\mathcal{T}$ the fuzzy topology with a subbase consisting of $X,\phi$ and all the L-fuzzy sets $f_x^t(y) = \begin{cases} 1/2 & \text{if } y = x \\ t & \text{if } y \neq x \end{cases}$ for each $x\in X$ and $t<1/2$.

(i) $(X,\mathcal{T})$ is not compact.

In fact, by taking $p<1/2$ and considering the family $\mathcal{A} = \left(f_x^{t_0}\right)_{x\in X}$ where $t_0<p$, we have that $\left(\bigvee_{x\in X} f_x^{t_0}\right)(y) = \frac{1}{2}>p$ for all $y\in X$ and there is no finite subfamily $\mathcal{B}$ of $\mathcal{A}$ with $\left(\bigvee_{f\in\mathcal{B}} f\right)(y)>p$ for all $y\in X$ because if $\mathcal{B} = \left\{f_{x_1}^{t_0}, \ldots, f_{x_m}^{t_0}\right\}$ then $\left(\bigvee_{f\in\mathcal{B}} f\right)(y) = \begin{cases} 1/2 & \text{if } y \in \{x_1, \ldots, x_m\} \\ t_0 & \text{if } y \notin \{x_1, \ldots, x_m\} \end{cases}$ and $t_0 <p$.

(ii) $(X,\mathcal{T})$ is X-compact.

In fact:

If $p\in[1/2, 1)$ and $\mathcal{A}$ is a family of subbasic open L-fuzzy sets with $\left(\bigvee_{f\in\mathcal{A}} f\right)(y)>\gamma$ for all $y\in X$ for some $\gamma>p$ then $X\in\mathcal{A}$. Therefore there exists a finite subfamily $\mathcal{B}$ of $\mathcal{A}$, $\mathcal{B} = \{X\}$, and $\gamma_1>p$ such that $\left(\bigvee_{f\in\mathcal{B}} f\right)(y) > \gamma_1$ for all $y\in X$.

If $p\in[0,1/2)$ and $\mathcal{A}$ is a family of subbasic open L-fuzzy sets with $\left(\bigvee_{f\in\mathcal{A}} f\right)(y)>\gamma$ for $y\in X$ for some $\gamma>p$ and if $X\in\mathcal{A}$, there exists a finite subfamily $\mathcal{B}$ of $\mathcal{A}$, $\mathcal{B} = \{X\}$ and $\gamma_1>p$ such that $\left(\bigvee_{f\in\mathcal{B}} f\right)(y)>\gamma_1$ for all $y\in X$. If $X\notin\mathcal{A}$ then $\mathcal{A} = (f_x^t)_{\substack{t\in T\subset[0,1/2)\\ x\in Z\subset X}}$.

So, $\left(\bigvee_{f\in\mathcal{A}} f\right)(y) = \begin{cases} 1/2 & \text{if } y\in Z \\ \vee T & \text{if } y\notin Z \end{cases}$.

Take $T_1$ any finite subset of T, $Z_1$ and finite subset of Z and $\gamma_1$ such that $\max T_1>\gamma_1>p$.

Let $\mathcal{B} = \left(f_x^t\right)_{\substack{t\in T_1\\ x\in Z_1}}$.

Therefore $\left(\bigvee_{f\in\mathcal{B}} f\right)(y) = \begin{cases} 1/2 & \text{if } y \in Z_1 \\ \max T_1 & \text{if } y\notin Z_1 \end{cases} > \gamma_1$ for all $y\in X$.

Hence $(X,\mathcal{T})$ is X-compact since Alexander's subbase theorem is valid for X-compactness (see this remark in [105].

<u>Remark</u> <u>4.7.5.</u>

We can have a Lowen L-fuzzy compact L-fts which is not X-compact.

Consider $X = L = [0,1]$.

For all $x\in X\cap\mathbb{Q}$ let $x = p/q$ in smallest terms and then put $f_x^s = \frac{s}{q} + \frac{1}{q}\chi_{\{x\}}$ for all $s\in\mathbb{N}$, $0\le s\le q-1$.

Let $\mathcal{T}_1 = \left\{\chi_{\{x\}};\ x\in X,\ x \text{ irrational}\right\}$ and let $\mathcal{T}_2 = \left\{f_x^s;\ x\in X\cap\mathbb{Q},\ x=p/q,\ s\in\mathbb{N},\ 0\le s\le q-1\right\}$.

Let $\mathcal{T}$ be the fuzzy topology on X generated by $\mathcal{B} = (f^t)_{t\in[0,1]} \cup\mathcal{T}_1\cup\mathcal{T}_2$ where $f^t(y) = t$ for all $y\in X$.

We have that $i_p(\mathcal{T}) = \{i_p(f);\ f\in\mathcal{T}\}$ is generated by

$(f^{-1}(p,L])_{f\in\mathcal{T}_1\cup\mathcal{T}_2}$ and is the discrete ordinary topology on X for all $p\in[0,1)$. (See counterexamples pp 451 in [56]).

So, $(X,\mathcal{T})$ is not X-compact.

In fact:

Take $p = 1/2$ and consider the family $\mathcal{A}=(f)_{f\in\mathcal{T}_1\cup\mathcal{T}_2}$ of open L-fuzzy sets. Then $\left(\bigvee_{f\in\mathcal{A}} f\right)(x)=1>\gamma$ for all $x\in X$ where $\gamma\in(1/2,\ 1)$ and since $i_p(\mathcal{T}) = (\{x\in X;\ f(x)>p\})_{f\in\mathcal{T}}$ is the discrete topology for every $p\in[0,1)$ there is no finite subfamily $\mathcal{B}$ of $\mathcal{A}$ and $\gamma_1\in(1/2,1)$ with $\left(\bigvee_{f\in\mathcal{B}} f\right)(x) > \gamma_1$ for all $x\in X$.

Hence $(X,\mathcal{T})$ is not X-compact.

But we have that $(X,\mathcal{T})$ is Lowen L-fuzzy compact (see pp 451 in [56]).

<u>Theorem</u> <u>4.7.6.</u>

The relations we have established between ultra-L-fuzzy compactness, N-L-compactness, compactness, X-compactness and Lowen L-fuzzy compactness are the following:

ultra-L-fuzzy compactness $\overset{\nLeftarrow}{\Rightarrow}$ N-L-compactness $\overset{\nLeftarrow}{\Rightarrow}$ compactness $\overset{\nLeftarrow}{\Rightarrow}$

X-compactness $\overset{\nLeftarrow}{\Rightarrow}$ Lowen L-fuzzy compactness

<u>Proposition</u> <u>4.7.7.</u>

Let $(X,\mathcal{T})$ be a Hausdorff Lowen L-fuzzy compact L-fts. Then $(X,\mathcal{T})$ is compact.

<u>Proof</u>

Let $\alpha\in M(L)$ and let $(S_m)_{m\in D}$ be a constant $\alpha$-net in

$(X,\mathcal{T})$ where supp $S_m = x^m$.

We want to prove that $(S_m)_{m\in D}$ has a cluster point $z_\alpha$ in X with height $\alpha$, i.e., there is $z\in X$ such that for each closed L-fuzzy set f with $f(z)\not\geq\alpha$ we have that for all $j\in D$ there is $m\in D$ such that $m\geq j$ and $f(x^m)\not\geq\alpha$.

By the Lowen L-fuzzy compactness of $(X,\mathcal{T})$ we have that given $\gamma\in\beta^*(\alpha)$, $(S_m)_{m\in D}$ has a cluster point $x^\gamma_\gamma \in M(L^X)$ with height $\gamma$ and support $x^\gamma$. So, by proposition 3.1.10., $(S_m)_{m\in D}$ has a subnet $(P_q)_{q\in D_1}$ converging to $x^\gamma_\gamma$.

Let f be a closed L-fuzzy set with $f(x^\gamma)\not\geq\alpha$. So, since $\beta^*(\alpha)$ is a minimal set relative to $\alpha$ by proposition 1.2.7., there is $\gamma_1\in\beta^*(\alpha)$ such that $f(x^\gamma)\not\geq\gamma_1\leq\alpha$.

Since $(P_q)_{q\in D_1}$ is an $\alpha$-net, by the Lowen L-fuzzy compactness of $(X,\mathcal{T})$, $(P_q)_{q\in D_1}$ has a cluster point $y^{\gamma_1}_{\gamma_1} \in M(L^X)$ with height $\gamma_1$ and support $y^{\gamma_1}$. So, by proposition 3.1.10., $(P_q)_{q\in D_1}$ has a subnet converging to $y^{\gamma_1}_{\gamma_1}$.

We also have that this subnet of $(P_q)_{q\in D_1}$ converges to $x^\gamma_\gamma$ because $P_m\to x^\gamma_\gamma$. By proposition 3.4.7. we have $x^\gamma=y^{\gamma_1}$.

Therefore $x^\gamma_{\gamma_1}$ is a cluster point of $(S_m)_{m\in D}$. As f is a closed L-fuzzy set with $f(x^\gamma)\not\geq\gamma_1$ and $x^\gamma_{\gamma_1}$ is a cluster point of $(S_m)_{m\in D}$, for all $j\in D$ there is $m\in D$ such that $m\geq j$ and $f(x^m)\not\geq\alpha$.

<u>Proposition</u> <u>4.7.8.</u>

Let $(X,\mathcal{T})$ be a Hausdorff compact L-fts. Then $(X,\mathcal{T})$ is ultra-L-fuzzy compact.

Proof

We need to prove that $(X, i_L(\mathcal{T}))$ is compact.

Let $(A_i)_{i\in J}$ be a family of subbasic open sets in $(X, i_L(\mathcal{T}))$ such that $X = \bigcup_{i\in J} A_i$

So, each $A_i$ is of the form $A_i = \{x\in X;\ f(x) \nleq p,\ f\in\mathcal{T}$ and $p\in pr(L)\}$ and $\left(\bigvee_{i\in J} \chi_{A_i}\right)(x) = 1 \nleq p$ for all $x\in X$ and all $p\in pr(L)$.

By theorem 4.1.9. we have $\mathcal{T}\subset\omega(\delta)$ where $\delta = \{A\subset X;\ \chi_A \in\mathcal{T}\}$. Therefore we have that if $f\in\mathcal{T}$ then $f\in\omega(\delta)$ which, by proposition 3.2.9.(i), implies that $\{x\in X;\ f(x)\nleq p\}\in\delta$ for all $p\in pr(L)$. So, $\chi_{\{x\in X;\ f(x)\nleq p\}} \in \mathcal{T}$ for every $p\in pr(L)$.

By the compactness of the L-fts $(X,\mathcal{T})$, there is a finite subset F of J such that $\left(\bigvee_{i\in F} \chi_{A_i}\right)(x) \nleq p$ for all $x\in X$, i.e., $\left(\bigvee_{i\in F} \chi_{A_i}\right)(x) = 1$ for all $x\in X$. Therefore $X = \bigcup_{i\in F} A_i$.

Hence $(X, i_L(\mathcal{T}))$ is compact.

So we have the following:

Theorem 4.7.9.

In a Hausdorff L-fts compactness, ultra-L-fuzzy compactness, N-L-compactness, X-compactness and Lowen L-fuzzy compactness are equivalent.

Remark 4.7.10.

Since we have heard of X-compactness and Lowen L-fuzzy compactness only very recently and they seem to have been studied only in works in Chinese, we do not know what work has been done on them. We do not even know if Lowen L-fuzzy compactness has been defined for arbitrary L-fuzzy sets or not.

# Chapter V

## Countable compactness, sequential compactness and Lindelöfness

In this chapter we introduce good definitions of L-fuzzy countable and sequential compactness and the Lindelöf property for L a fuzzy lattice. Each of them is defined on arbitrary L-fuzzy sets and their properties studied.

The first attempt to define the Lindelöf property, sequential and countable compactness was done by Wong [102]. But his definitions are not good.

Following the lines of Gantner, Steinlage and Warren's $\alpha$-compactness (definition 4.1.1.), Malghan and Benchalli defined countable compactness and the Lindelöf property in [61].

Sequential compactness was also studied by Xuan [107]. His definition is based on N-compactness (definition 4.5.1.).

In [1], Abd El-Hakeim introduced and studied N-compactness and N-sequential compactness in fuzzy neighbourhood spaces.

All of these works were developed on [0,1]-fuzzy topological spaces. As far as we know, none of these notions has been introduced in L-fuzzy topological spaces for $L \neq [0,1]$.

We divide this chapter into 3 sections.

In the first section we present our definitions of countable compactness, Lindelöfness and sequential compactness. In this section we also prove their goodness.

The second section is reserved for other characterizations of countable compactness.

In the third section we study some of their properties.

From this chapter on, L will be always a fuzzy lattice.

## 1. Proposed definitions and their goodness theorems

Definition 5.1.1.

Let $(X,\mathcal{T})$ be an L-fts and $g \in L^X$. The L-fuzzy set g is said to be countably compact if and only if for every prime $p \in L$ and every countable collection $(f_i)_{i\in J}$ of open L-fuzzy sets with $\left(\bigvee_{i\in J} f_i\right)(x) \nleq p$ for all $x \in X$ with $g(x) \geq p'$, there exists a finite subset F of J with $\left(\bigvee_{i\in F} f_i\right)(x) \nleq p$ for all $x \in X$ with $g(x) \geq p'$.

If g is the whole space, then we say that the L-fts $(X,\mathcal{T})$ is countably compact.

Theorem 5.1.2. (The goodness of countable compactness)

Let $(X,\delta)$ be a topological space. The L-fts $(X,\omega(\delta))$ is countably compact if and only if $(X,\delta)$ is countably compact.

Proof

Necessity:

This is similar to the proof of sufficiency in our theorem 4.6.8.

Sufficiency:

Let $p \in pr(L)$ and let $(f_i)_{i\in K}$ be a countable family of open L-fuzzy sets in $(X, \omega(\delta))$ with $\left(\bigvee_{i\in K} f_i\right)(x) \nleq p$ for all $x \in X$.

Therefore $(f_i^{-1}(\{t \in L;\ t \nleq p\}))_{i\in K}$ is an open cover of $(X,\delta)$. In fact, since $f_i \in \omega(\delta)$ for all $i \in K$ and $\{t \in L;\ t \nleq p\}$ is Scott open, $f_i^{-1}(\{t \in L;\ t \nleq p\}) \in \delta$ for every $i \in K$. We also have $\bigcup_{i\in\mathbb{N}} f_i^{-1}(\{t\in L;\ t\nleq p\}) = X$ because for each $x\in X$ there is $i\in K$ with $f_i(x)\nleq p$, i.e., for every $x\in X$ there is $i\in K$ with $x\in f_i^{-1}(\{t\in L;\ t\nleq p\})$.

From the countable compactness of $(X,\delta)$, there exists a finite subset F of K with $\bigcup_{i\in F} f_i^{-1}(\{t\in L;\ t\nleq p\}) = X$. Therefore, $\left(\bigvee_{i\in F} f_i\right)(x)\nleq p$ for every $x\in X$ because for each $x\in X$ there is $i\in F$ such that $f_i(x)\nleq p$.

Hence $(X, \omega(\delta))$ is countably compact.

<u>Definition</u> <u>5.1.3.</u>

Let $(X,\mathcal{T})$ be an L-fts and let $g\in L^X$. The L-fuzzy set g is said to be <u>Lindelöf</u> if and only if for every prime $p\in L$ and every collection $(f_i)_{i\in J}$ of open L-fuzzy sets with $\left(\bigvee_{i\in J} f_i\right)(x)\nleq p$ for all $x\in X$ with $g(x)\geq p'$, there is a countable subcollection of $(f_i)_{i\in J}$ with this property.

If g is the whole space, then we say that the L-fts $(X,\mathcal{T})$ is Lindelöf.

<u>Theorem</u> <u>5.1.4.</u> (The goodness of Lindelöfness)

Let $(X,\delta)$ be a topological space. The L-fts $(X,\omega(\delta))$ is Lindelöf if and only if $(X,\delta)$ is Lindelöf.

<u>Proof</u>

This is similar to the proof of theorem 5.1.2.

<u>Definition</u> <u>5.1.5.</u>

Let $(X,\mathcal{T})$ be an L-fts and let $g\in L^X$. The L-fuzzy set g is said to be <u>sequentially</u> <u>compact</u> if and only if each constant $\alpha$-sequence $(S_m)_{m\in\mathbb{N}}$ contained in g (i.e., $S_m\leq g$

for every $m\in\mathbb{N}$) has a subsequence converging to an L-fuzzy point $x_\alpha\in M(L^X)$, with height $\alpha$, contained in g (i.e., $x_\alpha\leq g$), for each $\alpha\in M(L)$.

If g is the whole space, then we say that the L-fts $(X,\mathcal{T})$ is sequentially compact.

<u>Theorem</u> <u>5.1.6.</u> (The goodness of sequential compactness)

Let $(X,\delta)$ be a topological space. The L-fts $(X,\omega(\delta))$ is sequentially compact if and only if $(X,\delta)$ is sequentially compact.

<u>Proof</u>

<u>Necessity</u>:

Let $(x^m)_{m\in\mathbb{N}}$ be a sequence in $(X,\delta)$ and $\alpha\in M(L)$. Then $(x^m_\alpha)_{m\in\mathbb{N}}$ is a constant $\alpha$-sequence (remark 2.3.9.) in $(X,\omega(\delta))$. From the sequential compactness of $(X, \omega(\delta))$, $(x^m_\alpha)_{m\in\mathbb{N}}$ has a subsequence $(x^{m_i}_\alpha)_{i\in\mathbb{N}}$ converging to some $x_\alpha\in M(L^X)$.

Let P be a closed set in $(X,\delta)$ with $x\notin P$. Then $\chi_P$ is a closed L-fuzzy set in $(X, \omega(\delta))$ by proposition 3.2.10. and since $x\notin P$, $\chi_P(x)=0\ngeq\alpha$.

Since $x^{m_i}_\alpha\to x_\alpha$, by definition 3.1.9.(i), there is $m_0\in\mathbb{N}$ such that $i\geq m_0$ implies that $x^{m_i}_\alpha\nleq\chi_P$, i.e., $\chi_P(x^{m_i})\ngeq\alpha$ for every $i\geq m_0$, that is, $x^{m_i}\notin P$ for every $i\geq m_0$. Then $(x^{m_i})_{i\in\mathbb{N}}$ is a subsequence of $(x^m)_{m\in\mathbb{N}}$ and $x^{m_i}\to x$.

Hence $(X,\delta)$ is sequentially compact.

<u>Sufficiency</u>:

Let $\alpha\in M(L)$ and let $(S_m)_{m\in\mathbb{N}}$ be a constant $\alpha$-sequence with supp $(S_m) = x^m$ for each $m\in\mathbb{N}$.

Then $(x^m)_{m\in\mathbb{N}}$ is a sequence in $(X,\delta)$. Since $(X,\delta)$ is sequentially compact, the sequence $(x^m)_{m\in\mathbb{N}}$ has a subsequence $(x^{m_i})_{i\in\mathbb{N}}$ converging to some $x\in X$.

We shall prove that $(x_\alpha^{m_i})_{i\in\mathbb{N}}$ converges to $x_\alpha$, i.e., for each closed L-fuzzy set f with $f(x)\not\leq\alpha$ there exists $m_0\in\mathbb{N}$ such that $i\geq m_0$ implies that $f(x^{m_i})\not\leq\alpha$.

Let f be a closed L-fuzzy set in $(X, \omega(\delta))$ with $f(x)\not\leq\alpha$.

Let $H = \{t\in X;\ f'(t)\not\geq\alpha'\}$. Since $f'\in\omega(\delta)$, by proposition 3.2.9. we have $H\in\delta$. We also have that $x\in H$. Since x is a limit point of $(x^{m_i})_{i\in\mathbb{N}}$, there exists $m_0\in\mathbb{N}$ such that $i\geq m_0$ implies that $x^{m_i}\in H$, i.e., $f(x^{m_i})\not\leq\alpha$ for every $i\geq m_0$.

## 2. Other characterizations of countable compactness

Proposition 5.2.1.

Let $(X,\mathcal{T})$ be an L-fts. Then $g\in L^X$ is countably compact if and only if for every $\alpha\in M(L)$ and every countable collection $(f_i)_{i\in J}$ of closed L-fuzzy sets with $\left(\bigwedge_{i\in J} f_i\right)(x)\not\geq\alpha$ for all $x\in X$ with $g(x)\geq\alpha$, there exists a finite subset F of J with $\left(\bigwedge_{i\in F} f_i\right)(x)\not\geq\alpha$ for all $x\in X$ with $g(x)\geq\alpha$.

Proof

This immediately follows from definition 5.1.1.

Theorem 5.2.2.

Let $(X,\mathcal{T})$ be an L-fts. Then $g\in L^X$ is countably compact if and only if every constant $\alpha$-sequence $(S_m)_{m\in\mathbb{N}}$ contained in g has a cluster point $x_\alpha\in M(L^X)$ with height $\alpha$, contained in g, for each $\alpha\in M(L)$.

Proof

Necessity:

Let $\alpha\in M(L)$ and let $(S_m)_{m\in\mathbb{N}}$, where support and height of $S_m$ are respectively supp $S_m = x^m$ and $h(S_m) = \alpha$, be a constant $\alpha$-sequence contained in g without any cluster point with height $\alpha$ contained in g.

For each $m\in\mathbb{N}$ define a closed L-fuzzy set $f_m = \wedge\{f;\ f$ is closed in $(X,\mathcal{T})$ and $f(x^i)\geq\alpha$ for all $i\in\mathbb{N}$ with $i\geq m\}$.

Thus, $\mathcal{A} = (f_m)_{m\in\mathbb{N}}$ is a countable family of closed L-fuzzy sets with $\left(\bigwedge_{m\in\mathbb{N}} f_m\right)(x)\not\geq\alpha$ for all $x\in X$ with $g(x)\geq\alpha$.

In fact, if $x \in X$ and $g(x) \geq \alpha$, then $x_\alpha \in M(L^X)$ is not a cluster point of $(S_m)_{m \in \mathbb{N}}$, i.e., there are $j \in \mathbb{N}$ and a closed L-fuzzy set $g^*$ with $g^*(x) \not\geq \alpha$ and $S_i \leq g^*$ $(g^*(x^i) \geq \alpha)$ for each $i \geq j$. Thus, $f_j \leq g^*$ and $f_j(x) \leq g^*(x) \not\geq \alpha$. So, $f_j(x) \not\geq \alpha$, which implies $\left(\bigwedge_{m \in \mathbb{N}} f_m\right)(x) \not\geq \alpha$.

We also have that for any finite subfamily $\mathcal{B} \subset \mathcal{A}$ there is $x \in X$ with $g(x) \geq \alpha$ and $\left(\bigwedge_{f \in \mathcal{B}} f\right)(x) \geq \alpha$. In fact, if $\mathcal{B} = \{f_{j_1}, \ldots, f_{j_k}\}$ since $f_m(x^i) \geq \alpha$ for every $i \in \mathbb{N}$ with $i \geq m$, we have $\left(\bigwedge_{m=j_1}^{j_k} f_m\right)(x^i) \geq \alpha$ for every $i \geq \max\{j_1, \ldots, j_k\}$.

Hence g is not countably compact.

<u>Sufficiency</u>:

Suppose that g is not countably compact. Then there exist $\alpha \in M(L)$ and a countable collection $\mathcal{A} = (f_i)_{i \in \mathbb{N}}$ of closed L-fuzzy sets with $\left(\bigwedge_{i \in \mathbb{N}} f_i\right)(x) \not\geq \alpha$ for all $x \in X$ with $g(x) \geq \alpha$ but for any finite subfamily $\mathcal{B}$ of $\mathcal{A}$ there is $x_\alpha \in M(L^X)$ with $x_\alpha \leq g$ and $\left(\bigwedge_{f_i \in \mathcal{B}} f_i\right)(x) \geq \alpha$.

Thus, for each $m \in \mathbb{N}$ there is $x^m \in X$ with $g(x^m) \geq \alpha$ and $\left(\bigwedge_{i=1}^{m} f_i\right)(x^m) \geq \alpha$.

Therefore $(S_m)_{m \in \mathbb{N}}$, where $\operatorname{supp} S_m = x^m$ and $h(S_m) = \alpha$, is a constant $\alpha$-sequence contained in g with no cluster point with height $\alpha$ contained in g. In fact, if $y \in X$ and $g(y) \geq \alpha$ then $\left(\bigwedge_{i \in \mathbb{N}} f_i\right)(y) \not\geq \alpha$. Thus, there exists $j \in \mathbb{N}$ with $f_j(y) \not\geq \alpha$. We also have $\alpha \leq \left(\bigwedge_{i=1}^{m} f_i\right)(x^m) \leq f_j(x^m)$ for all $m \geq j$ Hence, $y_\alpha$ is not a cluster point of $(S_m)_{m \in \mathbb{N}}$ for all $y \in X$ with $g(y) \geq \alpha$.

## 3. Some properties

Proposition 5.3.1.

Let $(X,\mathcal{T})$ be an L-fts. Every compact L-fuzzy set g is countably compact and Lindelöf.

Proof

This is immediate from the definitions.

Theorem 5.3.2.

Let $(X,\mathcal{T})$ be a second countable $(C_2)$ L-fts. Then $g\in L^X$ is compact if and only if g is countably compact.

Proof

Necessity:

This follows from proposition 5.3.1.

Sufficiency:

Let $p\in pr(L)$ and let $(f_i)_{i\in J}$ be a family of open L-fuzzy sets with $\left(\bigvee_{i\in J} f_i\right)(x)\nleq p$ for all $x\in X$ with $g(x)\geq p'$.

Since $(X,\mathcal{T})$ is $C_2$ (definition 3.4.2.), $\mathcal{T}$ has a countable base $\mathcal{B} = (b_m)_{m\in\mathbb{N}}$. Then $f_i = \bigvee_{k=1}^{i_0} b_{ik}$ where $i_0$ may be infinity.

Thus, $\mathcal{B}_0 = (b_{ik})_{\substack{i\in J \\ k\in\{1,2,\ldots,i_0\}}}$ is a countable family of open L-fuzzy sets with $\left(\bigvee_{b_{ik}\in\mathcal{B}_0} b_{ik}\right)(x)\nleq p$ for all $x\in X$ with $g(x)\geq p'$. By the countable compactness of g, there exists a finite subcollection $\mathcal{B}_1$ of $\mathcal{B}_0$ with this property. Each member $h\in\mathcal{B}_1$ satisfies $h\leq f_i$ for some $i\in J$.

In fact, if $h \in \mathcal{B}_1$, then $h = b_{i',k'}$ for some $i' \in J$ and $k' \in \{1, \ldots, i_0\}$ because $\mathcal{B}_1 \subset \mathcal{B}_0$ and $h = b_{i',k'} \leq \bigvee_{k=1}^{i_0} b_{i',k} = f_{i'}$.

Let $\mathcal{S} = \{f_i;\ h \leq f_i,\ b \in \mathcal{B}_1\}$.

Therefore $\mathcal{S}$ is a finite subcollection of $(f_i)_{i \in J}$ with $\left(\bigvee_{f_i \in \mathcal{S}} f_i\right)(x) \not\geq p$ for all $x \in X$ with $g(x) \geq p'$ because $\mathcal{B}_1$ satisfies this property.

Hence g is compact.

<u>Proposition</u> <u>5.3.3.</u>

Let $(X, \mathcal{T})$ be an L-fts. If g is a sequentially compact L-fuzzy set, then g is countably compact.

<u>Proof</u>

Let g be a sequentially compact L-fuzzy set and let $(S_m)_{m \in \mathbb{N}}$ be a constant $\alpha$-sequence contained in g.

Thus, by the sequential compactness of g, $(S_m)_{m \in \mathbb{N}}$ has a subsequence converging to some $x_\alpha \in M(L^X)$, contained in g. From proposition 3.1.10. $x_\alpha$ is a cluster point of $(S_m)_{m \in \mathbb{N}}$.

Hence g is countably compact by theorem 5.2.2.

<u>Theorem</u> <u>5.3.4.</u>

Let $(X, \mathcal{T})$ be a $C_1$ L-fts. If g is a countably compact L-fuzzy set, then g is sequentially compact.

<u>Proof</u>

Let g be a countably compact L-fuzzy set and let

$(S_m)_{m\in\mathbb{N}}$ be a constant $\alpha$-sequence contained in g. Thus, by theorem 5.3.2., $(S_m)_{m\in\mathbb{N}}$ has a cluster point $x_\alpha \in M(L^X)$, with height $\alpha$, contained in g.

Since $(X,\mathcal{T})$ is $C_1$ (definition 3.4.1.), by proposition 3.4.4., $(S_m)_{m\in\mathbb{N}}$ has a subsequence converging to $x_\alpha$.

Hence g is sequentially compact.

<u>Proposition 5.3.5.</u>

Let $(X,\mathcal{T})$ be an L-fts and let g be a sequentially compact L-fuzzy set. Then each closed L-fuzzy set h contained in g is sequentially compact as well.

<u>Proof</u>

Let $\alpha\in M(L)$ and let h be a closed L-fuzzy set contained in g. Let $(S_m)_{m\in\mathbb{N}}$ be a constant $\alpha$-sequence contained in h.

Thus, $(S_m)_{m\in\mathbb{N}}$ is contained in g as well and from the sequential compactness of g, there exists a subsequence $(T_i)_{i\in\mathbb{N}}$ of $(S_m)_{m\in\mathbb{N}}$ converging to an L-fuzzy point $x_\alpha \in M(L^X)$ with $x_\alpha \le g$, i.e., for each closed L-fuzzy set f with $f(x) \not\geq \alpha$ there is $m_0 \in \mathbb{N}$ such that $i \ge m_0$ implies that $T_i \not\leq f$. But $T_i \le h$ for every $i\in\mathbb{N}$ and since h is a closed L-fuzzy set, $h(x) \ge \alpha$, i.e., $x_\alpha \le h$.

Hence h is sequentially compact.

<u>Proposition 5.3.6.</u>

Let $(X,\mathcal{T})$ be a $C_1$ L-fts and let g be a compact L-fuzzy set. Then g is sequentially compact.

Proof

Let $\alpha \in M(L)$ and let $(S_m)_{m \in \mathbb{N}}$ be a constant $\alpha$-sequence contained in g.

From theorem 4.4.2., $(S_m)_{m \in \mathbb{N}}$ has a cluster point $x_\alpha \in M(L^X)$ contained in g.

Since $(X, \mathcal{T})$ is $C_1$, by proposition 3.4.4. $(S_m)_{m \in \mathbb{N}}$ has a subsequence converging to $x_\alpha$.

Hence g is sequentially compact.

Proposition 5.3.7.

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be L-fts's, let $f: (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ be a continuous mapping such that $f^{-1}(y)$ is finite for every $y \in Y$ and let g be a sequentially compact L-fuzzy set of $(X, \mathcal{T}_X)$. Then f(g) is a sequentially compact L-fuzzy set of $(Y, \mathcal{T}_Y)$.

Proof

Let $\alpha \in M(L)$ and let $(S_m)_{m \in \mathbb{N}}$ be a constant $\alpha$-sequence contained in f(g) with supp $S_m = y^m$. Then $f(g)(y^m) \geq \alpha$ for every $m \in \mathbb{N}$, i.e., $\vee\{g(z);\ z \in X,\ f(z) = y^m\} \geq \alpha$ for every $m \in \mathbb{N}$. Thus, for each $m \in \mathbb{N}$ there is $x^m \in X$ with $g(x^m) \geq \alpha$ and $f(x^m) = y^m$. Therefore $(x^m_\alpha)_{m \in \mathbb{N}}$ is a constant $\alpha$-sequence contained in g.

From the sequential compactness of g, $(x^m_\alpha)_{m \in \mathbb{N}}$ has a subsequence $(x^{m_i}_\alpha)_{i \in \mathbb{N}}$ converging to some $x_\alpha \in M(L^X)$ contained in g.

Now we are going to prove that the subsequence $(y^{m_i}_\alpha)_{i \in \mathbb{N}}$ of $(S_m)_{m \in \mathbb{N}}$ converges to $y_\alpha = f(x)_\alpha$.

In fact, for each closed L-fuzzy set h in $(Y, \mathcal{T}_Y)$ with $h(y) \not\geq \alpha$ we have that $f^{-1}(h)$ is closed in $(X, \mathcal{T}_X)$ and

$f^{-1}(h)(x) = h(y) \nleq \alpha$, i.e., $x_\alpha \nleq f^{-1}(h)$. Since $(x_\alpha^{m_i})_{i\in\mathbb{N}}$ converges to $x_\alpha$, there exists $m_0\in\mathbb{N}$ such that $i\geq m_0$ implies that $x_\alpha^{m_i} \nleq f^{-1}(h)$, i.e., $h(y^{m_i}) = f^{-1}(h)(x^{m_i}) \nleq \alpha$ for all $i\geq m_0$, that is, $y_\alpha^{m_i} \nleq h$ for every $i\geq m_0$.

Hence f(g) is sequentially compact.

When $g=\chi_A$ for some $A\subseteq X$, $f^{-1}(y)$ does not need to be finite.

Theorem 5.3.8.

Let $\left[(X_m, \mathcal{T}_{X_m})\right]_{m\in\mathbb{N}}$ be a countable family of L-fts's and let X be their product. X is sequentially compact if and only if $(X_m, \mathcal{T}_{X_m})$ is sequentially compact for each $m\in\mathbb{N}$.

Proof

Necessity:

This follows from proposition 5.3.7. and the fact that the projection maps $\pi_m : X\to X_m$ are onto, continuous and X by hypothesis is sequentially compact.

Sufficiency:

Let $\alpha\in M(L)$ and let $(S_m)_{m\in\mathbb{N}}$ be a constant $\alpha$-sequence in X, where supp $S_m = x^m$ for each $m\in\mathbb{N}$. Then $x^m$ is of the form $x^m = (x^{jm})_{j\in\mathbb{N}}$ for each $m\in\mathbb{N}$.

Thus, $(S_m)_{m\in\mathbb{N}} = \left[(x_\alpha^{11}, x_\alpha^{21}, \ldots, x_\alpha^{j1}, \ldots), \ldots, (x_\alpha^{1m}, \ldots, x^{jm}, \ldots), \ldots\right]$.

We have that $(x_\alpha^{1m})_{m\in\mathbb{N}} = (x_\alpha^{11}, x_\alpha^{12}, \ldots, x_\alpha^{1m}, \ldots)$ is a constant $\alpha$-sequence in $X_1$. Since $(X_1, \mathcal{T}_{X_1})$ is sequentially compact, the constant $\alpha$-sequence $(x_\alpha^{1m})_{m\in\mathbb{N}}$ has a subsequence $(x_\alpha^{1k_1^1}, \ldots, x_\alpha^{1k_m^1}, \ldots)$ converging to $x_\alpha^1$ in $(X_1, \mathcal{T}_{X_1})$. As $(X_2, \mathcal{T}_{X_2})$ is also sequentially compact, the constant $\alpha$-sequence $(x_\alpha^{2k_1^1}, x_\alpha^{2k_2^1}, \ldots, x_\alpha^{2k_m^1}, \ldots)$ in $X_2$ has a subsequence $(x_\alpha^{2k_1^2}, \ldots, x_\alpha^{2k_m^2}, \ldots)$ converging to $x_\alpha^2$ in $(X_2, \mathcal{T}_{X_2})$. So, by induction we obtain for every $r\in\mathbb{N}$ $(x_\alpha^{rk_1^r}, \ldots, x_\alpha^{rk_m^r}, \ldots)$ subsequence of the $\alpha$-sequence $(x_\alpha^{rk_1^{r-1}}, \ldots, x_\alpha^{rk_m^{r-1}}, \ldots)$ in the sequentially compact L-fts $(X_r, \mathcal{T}_{X_r})$, converging to $x_\alpha^r$ in $X_r$.

Since $(x_\alpha^{rk_1^r}, \ldots, x_\alpha^{rk_m^r}, \ldots)$ is a subsequence of the $\alpha$-sequence $(x_\alpha^{rk_1^{r-1}}, \ldots, x_\alpha^{rk_m^{r-1}}, \ldots)$, we have that the sequence $(k_1^r, \ldots, k_m^r, \ldots)$ is a subsequence of the monotone increasing sequence $(k_1^{r-1}, \ldots, k_m^{r-1}, \ldots)$. So, $k_{r-1}^{r-1} < k_r^{r-1} \leq k_r^r$ for each $r\in\mathbb{N}-\{1\}$.

Hence $(x_\alpha^{k_m^m})_{m\in\mathbb{N}} = \left((x_\alpha^{1k_1^1}, \ldots, x_\alpha^{jk_1^1}, \ldots), \ldots, (x_\alpha^{1k_m^m}, \ldots, x_\alpha^{jk_m^m}, \ldots), \ldots\right)$ is a subsequence of the constant $\alpha$-sequence $(S_m)_{m\in\mathbb{N}}$ and we are going to show that this subsequence converges to $x_\alpha$ where $x = (x^1, \ldots, x^r, \ldots)$ in the product X, that is, for each closed L-fuzzy set h with $h(x)\nleq\alpha$ there exists $m_0\in\mathbb{N}$ such that $m\geq m_0$ implies that $h(x^{k_m^m})\nleq\alpha$.

Firstly we want to remark that except for the first r-1 terms, the sequence $(x_\alpha^{rk_1^1}, \ldots, x_\alpha^{rk_m^m}, \ldots)$ is a subsequence of the $\alpha$-sequence $(x_\alpha^{rk_1^r}, \ldots, x_\alpha^{rk_r^r}, x_\alpha^{rk_{r+1}^r},$ $\ldots)$ that converges to $x_\alpha^r$. So $(x_\alpha^{rk_1^1}, \ldots, x_\alpha^{rk_m^m}, \ldots)$ converges to $x_\alpha^r$ for each $r \in \mathbb{N}$.

Since $h = \bigwedge \bigvee_{e=1}^{q} \pi_{j_e}^{-1}(g_{j_e})$ where $g_{j_e}$ is closed in $(X_{j_e}, \mathcal{T}_{X_{j_e}})$ for each $e \in \{1, \ldots, q\}$, it is sufficient to prove that for each closed L-fuzzy set $f = \bigvee_{e=1}^{q} \pi_{j_e}^{-1}(g_{j_e})$ with $f(x) \not\geq \alpha$ there exists $m_0 \in \mathbb{N}$ such that $m \geq m_0$ implies that $f(x^{k_m^m}) \not\geq \alpha$.

Let $f = \bigvee_{e=1}^{q} \pi_{j_e}^{-1}(g_{j_e})$ ($g_{j_e}$ closed in $(X_{j_e}, \mathcal{T}_{X_{j_e}})$) with $f(x) \not\geq \alpha$. Then $\bigvee_{e=1}^{q} g_{j_e}(x^{j_e}) \not\geq \alpha$. Thus, $g_{j_e}(x^{j_e}) \not\geq \alpha$ for each $e \in \{1, \ldots, q\}$. Since $x_\alpha^{j_e k_m^m} \to x_\alpha^{j_e}$, there exists $m_e \in \mathbb{N}$ such that $m \geq m_e$ implies that $g(x^{j_e k_m^m}) \not\geq \alpha$ for each $e \in \{1, \ldots, q\}$. Let $m_0 = \max\{m_e;\ e \in \{1, \ldots, q\}\}$.

Therefore, $f(x^{k_m^m}) = \left(\bigvee_{e=1}^{q} \pi_{j_e}^{-1}(g_{j_e})\right)(x^{k_m^m}) \not\geq \alpha$ for each $m \geq m_0$ $(\alpha \in M(L))$.

Hence X is sequentially compact.

<u>Proposition</u> <u>5.3.9.</u>

Let $(X, \mathcal{T})$ be a $C_2$ L-fts. Then $(X, \mathcal{T})$ is Lindelöf.

<u>Proof</u>

Let $p \in pr(L)$ and let $(f_i)_{i \in J}$ be a collection of open L-fuzzy sets with $\left(\bigvee_{i \in J} f_i\right)(x) \not\geq p$ for all $x \in X$.

Since $(X, \mathcal{T})$ is $C_2$, $\mathcal{T}$ has a countable base $\mathcal{B}$ =

$(b_m)_{m\in\mathbb{N}}$. Then $f_i = \bigvee_{k=1}^{i_0} b_{ik}$ where $i_0$ may be infinity.

Thus, $\mathcal{B}_0 = (b_{ik})_{\substack{i\in J \\ k\in\{1,\ \dots,\ i_0\}}}$ is a countable family of open L-fuzzy sets with $\left(\bigvee_{b_{ik}\in\mathcal{B}} b_{ik}\right)(x) \not\leq p$ for all $x\in X$. We have that each member $b_{ik}$ of $\mathcal{B}_0$ is less than or equal to $f_i$.

Let $\mathcal{S} = \{f_i;\ b_{ik} \leq f_i,\ b_{ik} \in\mathcal{B}_0\}$.

Thus, $\mathcal{S}$ is a countable subcollection of $(f_i)_{i\in J}$ with $\left(\bigvee_{f_i\in\mathcal{S}} f_i\right)(x) \not\leq p$ for all $x\in X$ because $\mathcal{B}_0$ satisfies this property.

Hence $(X,\mathcal{T})$ is Lindelöf.

<u>Theorem 5.3.10.</u>

Let $(X,\mathcal{T})$ be an L-fts and let g be a Lindelöf L-fuzzy set. Then g is countably compact if and only if g is compact.

<u>Proof</u>

<u>Necessity</u>:

Let $p\in pr(L)$ and let $(f_i)_{i\in J}$ be a family of open L-fuzzy sets with $\left(\bigvee_{i\in J} f_i\right)(x) \not\leq p$ for all $x\in X$ with $g(x) \geq p'$.

Since g is Lindelöf, there exists a countable subfamily $(f_i)_{i\in K}$ with $\left(\bigvee_{i\in K} f_i\right)(x) \not\leq p$.

From the countable compactness of g, there is a finite subfamily $(f_i)_{i\in\{1,\ \dots,\ k\}}$ with $\left(\bigvee_{i\in\{1,\ \dots,k\}} f_i\right)(x) \not\leq p$ for all $x\in X$ with $g(x) \geq p'$.

Hence g is compact.

Sufficiency:

This follows from proposition 5.3.1.

Proposition 5.3.11.

Let $(X,\mathcal{T})$ be an L-fts and let h and g be countably compact (Lindelöf) L-fuzzy sets. Then $h \vee g$ is countably compact (Lindelöf) as well.

Proof

This is similar to the proof of proposition 4.1.10.

Proposition 5.3.12.

Let $(X,\mathcal{T})$ be an L-fts and let g be a countably compact (Lindelöf) L-fuzzy set. Then for each closed L-fuzzy set h, $h \wedge g$ is countably compact (Lindelöf).

Proof

This is similar to the proof of proposition 4.1.12.

Proposition 5.3.13.

Let $(X,\mathcal{T}_X)$ and $(Y,\mathcal{T}_Y)$ be L-fts's, let g be a countably compact (Lindelöf) L-fuzzy set of $(X,\mathcal{T}_X)$ and let $f:(X,\mathcal{T}_X)\rightarrow(Y,\mathcal{T}_Y)$ be a continuous mapping such that $f^{-1}(y)$ is finite for every $y \in Y$. Then $f(g)$ is a countably compact (Lindelöf) L-fuzzy set of $(Y,\mathcal{T}_Y)$.

Proof

This is similar to the proof of proposition 4.1.14.

When $g=\chi_A$ for some $A \subseteq X$, $f^{-1}(y)$ does not need to be finite.

# Chapter VI

## Local compactness in L-fuzzy topological spaces

In this chapter we focus on local compactness in L-fuzzy topological spaces.

In ordinary topology there are two ways to define local compactness. One of them, more truly "local" in nature than the other, implies the continuity of the locale of open sets.

Here we present the two corresponding formulations of L-fuzzy local compactness, both of which are good extensions of those in ordinary topology and study their properties. We prove that one of them, in the case L = [0,1], implies the continuity of the locale of fuzzy open sets. We also obtain a one point compactification.

Local compactness was introduced in [0,1]-fuzzy topological spaces by Wong [103]. In [22], Christoph weakened Wong's definition. Both worked with Chang's compactness (remark 4.6.2.).

By using $\alpha$-compactness (definition 4.1.1.), Gantner, Steinlage and Warren defined local $\alpha$-compactness in L-fuzzy topological spaces [35] and obtained, with some restrictions on $\alpha$, a one point compactification. Rodabaugh [83], also using $\alpha$-compactness ($\alpha^*$-compactness) defined local $\alpha$-compactness (local $\alpha^*$-compactness) and introducing $\alpha$-Hausdorffness obtained an extension of the one point compactification proposed by Gantner, Steinlage and Warren [35].

Malghan and Benchalli [106], weakening the definition of local $\alpha$-compactness given in [35], introduced another definition of local $\alpha$-compactness (local $\alpha^*$-compactness), but in [0,1]-fuzzy topological spaces.

Each of these local compactnesses failed to produce fuzzy versions of important results, such as the regularity of locally compact Hausdorff spaces.

This chapter is divided in six sections.

In the first section we present the proposed definitions and prove their goodness.

The second section is reserved for the proof of the fact that, in the case of L = [0,1], one of the proposed definitions implies the continuity of the locale of fuzzy open sets.

In the third section we establish some properties of the proposed definitions. We obtain fuzzy versions of the main classical properties of local compactness, but leaving to section five the regularity of locally compact Hausdorff L-fts's.

In the fourth section we present a comparison between the proposed L-fuzzy local compactnesses.

The fifth section contains the proof of the regularity of weakly locally compact Hausdorff L-fts's and a fuzzy version of k-spaces as well as the proof that every weakly locally compact L-fts is a k-space.

The last section is devoted to the one point compactification.

Remark

If (X,$\delta$) is a topological space then we shall say that (X,$\delta$) is locally compact if and only if each $x \in X$ has a base of compact neighbourhoods. And we shall say that (X,$\delta$) is weakly locally compact if and only if each $x \in X$ has a compact neighbourhood.

## 1. Proposed definitions and their goodness theorems

Definition 6.1.1.

Let $(X,\mathcal{T})$ be an L-fuzzy topological space. An L-fuzzy set k is very compact if and only if for some $e \in L$ it is of the form $k(x) = \begin{cases} e & \text{if } x \in D \subseteq X \\ 0 & \text{otherwise} \end{cases}$ and for every prime p of L and every collection $(f_i)_{i \in J}$ of open L-fuzzy sets with $\left(\bigvee_{i \in J} f_i\right)(x) \nleq p$ for all $x \in D$, there is a finite subset F of J with $\left(\bigvee_{i \in F} f_i\right)(x) \nleq p$ for all $x \in D$.

Remark 6.1.2.

In definition 6.1.1. D is the support of k and it is simply required that $\chi_D$ be compact.

Remark 6.1.3.

Clearly a very compact L-fuzzy set is compact.

Definition 6.1.4.

An L-fuzzy topological space $(X,\mathcal{T})$ is locally compact if and only if for all $x \in X$, for every $p \in pr(L)$ and for every open L-fuzzy set g with $g(x) \nleq p$, there exists a very compact L-fuzzy set k and $f \in \mathcal{T}$ such that $g \geq k \geq f$ and $f(x) \nleq p$.

Theorem 6.1.5. (The goodness of local compactness)

Let $(X,\delta)$ be a topological space. Then $(X,\delta)$ is locally compact if and only if the L-fts $(X,\omega(\delta))$ is locally compact.

Proof

Necessity:

Let $x_0 \in X$, $p \in pr(L)$ and let f be a basic open L-fuzzy set in $(X,\omega(\delta))$ with $f(x) = \begin{cases} b & \text{if } x \in U \in \delta \\ 0 & \text{otherwise} \end{cases}$ and $f(x_0) \nleq p$.

Thus, $x_0 \in U \in \delta$ and by the local compactness of $(X,\delta)$, there are C compact in $(X,\delta)$ and $V \in \delta$ such that $x_0 \in V \subseteq C \subseteq U$.

By defining $g(x) = \begin{cases} b & \text{if } x \in V \\ 0 & \text{otherwise} \end{cases}$ and $k(x) = \begin{cases} b & \text{if } x \in C \\ 0 & \text{otherwise} \end{cases}$ we have $g \leq k \leq f$, $g \in \omega(\delta)$, $g(x_0) = b \nleq p$ and $\chi_C$ compact in $(X,\omega(\delta))$. In fact, obviously $g \leq k \leq f$, $g \in \omega(\delta)$ and $g(x_0) \nleq p$ and from the compactness of the subspace $(C, \delta_C)$ of $(X,\delta)$ we have by the goodness of compactness (theorem 4.1.6.) that $(C,\omega(\delta_C))$ is compact. Hence $\chi_C$ is compact in $(X,\omega(\delta))$.

Sufficiency:

Let $x_0 \in X$, $p \in pr(L)$ and let $U \in \delta$ with $x_0 \in U$.

By considering $f(x) = \begin{cases} b & \text{if } x \in U \\ 0 & \text{otherwise} \end{cases}$ where $b \nleq p$, we have $f \in \omega(\delta)$ and $f(x_0) \nleq p$. By the local compactness of $(X,\omega(\delta))$, there are a very compact L-fuzzy set k and a $g \in \omega(\delta)$ such that $g \leq k \leq f$ and $g(x_0) \nleq p$ where $k(x) = \begin{cases} e \in L & \text{if } x \in D \subseteq X \\ 0 & \text{otherwise} \end{cases}$ and $\chi_D$ is compact in $(X,\omega(\delta))$. Since $g \in \omega(\delta)$ and $g(x_0) \nleq p$, there is a basic open L-fuzzy set $h(x) = \begin{cases} d & \text{if } x \in V \in \delta \\ 0 & \text{otherwise} \end{cases}$ such that $h \leq g \leq k \leq f$ and $h(x_0) \nleq p$. Then $V \subseteq D \subseteq U$, $x_0 \in V \in \delta$ and D is compact in $(X,\delta)$ because since $\chi_D$ is compact in $(X,\omega(\delta))$ the L-fuzzy subspace $(D,\omega(\delta_D))$ of $(X,\omega(\delta))$ is compact and by lemma 4.1.21. $D \times \{p\}$ is a compact subspace of $(D \times pr(L), \delta_D \times \phi(L))$, thus projection from $D \times \{p\}$ onto D give the compactness of D in $(X,\delta)$.

Definition 6.1.6.

Let $(X,\mathcal{T})$ be an L-fts. We say that $(X,\mathcal{T})$ is weakly locally compact if and only if for all $x \in X$ and for every $p \in pr(L)$ there exist a very compact L-fuzzy set $k$ and $f \in \mathcal{T}$ such that $k \geq f$ and $f(x) \nleq p$.

Theorem 6.1.7. (The goodness of weak local compactness)

Let $(X,\delta)$ be a topological space. Then $(X,\delta)$ is weakly locally compact if and only if the L-fts $(X,\omega(\delta))$ is weakly locally compact.

Proof

This is similar to the proof of theorem 6.1.5.

## 2. The continuity of the locale

Theorem 6.2.1.

Let $(X,\mathcal{T})$ be a fully stratified locally compact $[0,1]$-fuzzy topological space. Then the locale $\mathcal{T}$ is continuous.

Proof

We know that $\mathcal{T}$ is isomorphic to the topology $\phi(\mathcal{T})$ on $X\times pr(L)$ (proposition 3.2.19) and in this case on $X\times pr([0,1]) = X\times[0,1)$.

Since if a topological space $(X,\delta)$ is locally compact then $\delta$ is a continuous lattice [45], it is only necessary to prove that the topological space $(X\times pr(L), \phi(\mathcal{T}))$ is locally compact in order to deduce the continuity of the locale $\phi(\mathcal{T})$ and therefore of $\mathcal{T}$.

Let $(x_0, p_0) \in X\times[0,1)$ and $\phi(f) \in \phi(\mathcal{T})$ such that $(x_0, p_0) \in \phi(f)$.

Then $f(x_0)>p$ and by the local compactness of $(X,\mathcal{T})$, there are a $g\in\mathcal{T}$ and a very compact L-fuzzy set k with $g\le k\le f$ and $g(x_0)>p_0$ where $k(x) = \begin{cases} e\in L & \text{if } x\in D\subseteq X \\ 0 & \text{otherwise} \end{cases}$.

Therefore $(x_0, p_0) \in \phi(g)\subseteq\phi(k)\subseteq\phi(f)$ in $X\times[0,1)$. As $\phi(k) = \left\{(x,p) \in X\times pr(L);\ x\in D,\ e>p\right\}$ and since $\chi_D$ is compact in $(X,\mathcal{T})$, the L-fuzzy subspace $(D,\mathcal{T}_D)$ of $(X,\mathcal{T})$ is compact and by lemma 4.1.21. $C=D\times\{p\in pr(L);\ p\le q\}$ is compact in $X\times[0,1)$ for all $q\in[0,1)$.

Take $q\in[0,1)$ such that $p_0<q<e$.

Then $V=\phi(g\wedge q) \in \phi(\mathcal{T})$ where q denotes here the constant fuzzy set with value q. We also have $(x_0, p_0) \in$

$V \subseteq C \subseteq \phi(f)$. In fact, since $g(x_0) > p_0 < q$, $(g \wedge q)(x_0) > p_0$, i.e., $(x_0, p_0) \in V$ and $C \subseteq \phi(f)$ because for every $(x,p) \in D \times \{p \in pr(L);\ p \leq q\}$, $f(x) \geq e > q \geq p$. Now we are going to check that $V \subseteq C$.

$$V = \phi(g) \cap \phi(q) = \left\{(x,p) \in X \times [0,1);\ g(x) > p\right\} \cap \left\{(x,p) \in X \times [0,1);\ p < q\right\} \subseteq D \times [0,e) \cap X \times [0,q) = D \times [0,q) \subseteq D \times [0,q] = C$$

Hence $(X \times [0,1), \phi(\mathcal{T}))$ is locally compact.

<u>Remark</u> <u>6.2.2.</u>

We were unable to extend this result to L-fuzzy topological spaces.

## 3. Some properties

Proposition 6.3.1.

A fully stratified compact Hausdorff L-fts $(X,\mathcal{T})$ is locally compact.

Proof

By theorem 4.1.9. we have that $\mathcal{T} = \omega(\delta)$ where $\delta = \{U \in \mathbb{P}(X);\ \chi_U \in \mathcal{T}\}$. Since compactness and Hausdorffness are good extensions (theorems 4.1.6. and 3.4.6. respectively), $(X,\delta)$ is compact Hausdorff, so is locally compact. Hence, by the goodness of local compactness (theorem 6.1.5.), $(X,\mathcal{T})$ is locally compact.

Proposition 6.3.2.

A compact L-fts $(X,\mathcal{T})$ is weakly locally compact.

Proof

Given $x \in X$ and $p \in pr(L)$ just take X as the required very compact L-fuzzy set k and the open L-fuzzy set f. Then $k \geq f$ and $1 = f(x) \not\leq p$.

Proposition 6.3.3.

Let $(X,\mathcal{T}_X)$ and $(Y,\mathcal{T}_Y)$ be L-fts's and let $h:(X:\mathcal{T}_X) \rightarrow (Y,\mathcal{T}_Y)$ be a continuous open surjection. If $(X,\mathcal{T}_X)$ is locally compact then $(Y,\mathcal{T}_Y)$ is also locally compact.

Proof

Let $p \in pr(L)$, $y \in Y$ with $h(x) = y$ and let $f \in \mathcal{T}_Y$ such that

$f(y) \nleq p$.

Then $h^{-1}(f) \in \mathcal{T}_X$ and $h^{-1}(f)(x) = f(h(x)) = f(y) \nleq p$. Since $(X, \mathcal{T}_X)$ is locally compact, there are a very compact k in $(X, \mathcal{T}_X)$ and $g \in \mathcal{T}_X$ with $g(x) \nleq p$ such that $g \leq k \leq h^{-1}(f)$. Since h is an open mapping, $h(g) \in \mathcal{T}_Y$ and $h(g)(y) = \vee\{g(z);\ z \in h^{-1}(y)\} \nleq p$ because $h(x) = y$ and $g(x) \nleq p$. We also have $h(g) \leq h(k) \leq h(h^{-1}(f)) = f$ and h(k) very compact in $(Y, \mathcal{T}_Y)$ because k is very compact in $(X, \mathcal{T}_X)$, i.e,. $k(x) = \begin{cases} e \in L & \text{if } x \in D \subseteq X \\ 0 & \text{otherwise} \end{cases}$ and $\chi_D$ is compact in $(X, \mathcal{T}_X)$; we have $h(k)(u) = \vee\left\{k(z);\ z \in h^{-1}(u)\right\} = \vee_{z \in h^{-1}(u)} \begin{cases} e & \text{if } z \in D \subseteq X \\ 0 & \text{otherwise} \end{cases} = \begin{cases} e & \text{if } u \in h(D) \\ 0 & \text{otherwise} \end{cases}$ and since $\chi_D$ is compact in $(X, \mathcal{T}_X)$, h is continuous and $h(\chi_D) = \chi_{h(D)}$, by proposition 4.1.14. $\chi_{h(D)}$ is compact in $(Y, \mathcal{T}_Y)$.

Hence $(Y, \mathcal{T}_Y)$ is locally compact.

<u>Proposition</u> <u>6.3.4.</u>

Let $(X, \mathcal{T}_X)$ be an L-fts and let $F \subset X$ such that $\chi_F$ is closed in $(X, \mathcal{T}_X)$. If $(X, \mathcal{T}_X)$ is locally compact then the subspace $(F, \mathcal{T}_F)$ is locally compact.

<u>Proof</u>

Let $p \in pr(L)$, $x \in F$ and $f_F \in \mathcal{T}_F$ such that $f_F(x) \nleq p$.

Then there is $f \in \mathcal{T}_X$ such that $f_F = f|_F$, so $f(x) \nleq p$. From the local compactness of $(X, \mathcal{T}_X)$, there are a very compact L-fuzzy set $k(z) = \begin{cases} e \in L & \text{if } z \in D \subseteq X \\ 0 & \text{otherwise} \end{cases}$ and $g \in \mathcal{T}_X$ such that $g \leq k \leq f$ and $g(x) \nleq p$. Therefore $g|_F = g_F \in \mathcal{T}_F$ and $g_F(x) \nleq p$. We also have $g_F \leq k_F \leq f_F$ with $\chi_{D \cap F}$ compact in $(F, \mathcal{T}_F)$ where $k_F(z) = \begin{cases} e & \text{if } z \in D \cap F \\ 0 & \text{otherwise} \end{cases}$. In fact, since $\chi_F$ is closed and

$\chi_D$ is compact in $(X,\mathcal{T}_X)$, by proposition 4.1.12. $\chi_{D\cap F} = \chi_D \wedge \chi_F$ is compact in $(X,\mathcal{T}_X)$. Then $\chi_{D\cap F}$ is compact in $(F,\mathcal{T}_F)$.

Hence $(F,\mathcal{T}_F)$ is locally compact.

Theorem 6.3.5.

Let $\left\{(X_\lambda,\mathcal{T}_{X_\lambda})\right\}_{\lambda\in J}$ be a family of fully stratified L-fts's. Then the product L-fuzzy topological space X is locally compact if and only if each $X_\lambda$ is locally compact and all but finitely many $X_\lambda$ are compact.

Proof

Necessity:

Since, the $\lambda$-th projection, $\pi_\lambda : X \to X_\lambda$, is a continuous open surjection (proposition 3.3.3.) and X is locally compact, by proposition 6.3.3., $X_\lambda$ is locally compact for each $\lambda\in J$.

Now let $p\in pr(L)$, $x\in X$ and let f be an open L-fuzzy set in X with $f(x)\nleq p$.

Thus, by the local compactness of X, there are an open L-fuzzy set g in X with $g(x)\nleq p$ and a very compact L-fuzzy set $k(z) = \begin{cases} e & \text{if } z\in D\subseteq X \\ 0 & \text{otherwise} \end{cases}$ such that $g\leq k\leq f$. Then, there is a basic open L-fuzzy set $\bigwedge_{i=1}^{m} \pi_{\lambda_i}^{-1}(g_{\lambda_i})$ such that $\bigwedge_{i=1}^{m}\pi_{\lambda_i}^{-1}(g_{\lambda_i})\leq g\leq k\leq f$. Therefore $\chi_D = \chi_{\text{supp}k} \geq$

$\chi_{\text{supp}\bigwedge_{i=1}^{m}\pi_{\lambda_i}^{-1}(g_{\lambda_i})} = \chi_{\bigcap_{i=1}^{m}\text{supp}\,\pi_{\lambda_i}^{-1}(g_{\lambda_i})} =$

$\bigwedge_{i=1}^{m}\chi_{\text{supp}\,\pi_{\lambda_i}^{-1}(g_{\lambda_i})} = \bigwedge_{i=1}^{m}\chi_{\pi_{\lambda_i}^{-1}(\text{supp}(g_{\lambda_i}))} = \bigwedge_{i=1}^{m}\pi_{\lambda_i}^{-1}(\chi_{\text{supp}g_{\lambda_i}})$.

Thus, $\pi_\lambda(\chi_{\mathrm{supp}k}) \geq \pi_\lambda(\bigwedge_{i=1}^{m} \pi_{\lambda_i}^{-1} (\chi_{\mathrm{supp}g_{\lambda_i}})) = X_\lambda$ for all $\lambda \notin \{\lambda_1, \ldots, \lambda_m\}$. Since $\pi_\lambda$ is continuous, $\chi_{\mathrm{supp}k}$ is compact in X and $\pi_\lambda(\chi_{\mathrm{supp}k}) = X_\lambda$, we have by proposition 4.1.14. $X_\lambda$ compact for each $\lambda$ except possibly $\lambda \in \{\lambda_1, \ldots, \lambda_m\}$.

Sufficiency:

Let $p \in pr(L)$, $x \in X$ and let $f = \bigwedge_{i=1}^{m} \pi_{\lambda_i}^{-1}(f_{\lambda_i})$ be a basic open L-fuzzy set in the product L-fts X such that $f(x) \nleq p$, where $f_{\lambda_i}$ is an open L-fuzzy set in $X_{\lambda_i}$. We assume that the set $\{\lambda_1, \ldots, \lambda_m\}$ is expanded to include all $\lambda$ for which $X_\lambda$ is not compact.

We have that $f(x) = \bigwedge_{i=1}^{m} f_{\lambda_i}(x_{\lambda_i}) \nleq p$ implies for all $i \in \{1, \ldots, m\}$ $f_{\lambda_i}(x_{\lambda_i}) \nleq p$. From the local compactness of each $X_{\lambda_i}$, there are $g_{\lambda_i}$ open in $X_{\lambda_i}$ and a very compact $k_{\lambda_i}(z_{\lambda_i}) = \begin{cases} e_{\lambda_i} & \text{if } z_{\lambda_i} \in D_{\lambda_i} \subseteq X_{\lambda_i} \\ 0 & \text{otherwise} \end{cases}$ in $X_{\lambda_i}$ with $g_{\lambda_i}(x_{\lambda_i}) \nleq p$ and $g_{\lambda_i} \leq k_{\lambda_i} \leq f_{\lambda_i}$. Thus, the L-fuzzy set $g = \bigwedge_{i=1}^{m} \pi_{\lambda_i}^{-1}(g_{\lambda_i}) = g_{\lambda_1} \times \ldots \times g_{\lambda_m} \times \prod_{\lambda \notin \{\lambda_1, \ldots, \lambda_m\}} X_\lambda$ is an open L-fuzzy set in X and $g(x) = \left(\bigwedge_{i=1}^{m} \pi_{\lambda_i}^{-1} (g_{\lambda_i})\right)(x) = \bigwedge_{i=1}^{m} g_{\lambda_i}(\pi_{\lambda_i}(x)) = \bigwedge_{i=1}^{m} g_{\lambda_i}(x_{\lambda_i}) \nleq p$ $(p \in pr(L))$. We also have

$$g = \bigwedge_{i=1}^{m} \pi_{\lambda_i}^{-1} (g_{\lambda_i}) \leq k = \bigwedge_{i=1}^{m} \pi_{\lambda_i}^{-1} (k_{\lambda_i}) \leq f,$$

$$k(z) = \left(\bigwedge_{i=1}^{m} \pi_{\lambda_i}^{-1}(k_{\lambda_i})\right)(z) = \bigwedge_{i=1}^{m} k_{\lambda_i}(z_{\lambda_i}) =$$

$$\bigwedge_{i=1}^{m} \begin{cases} e_{\lambda_i} & \text{if } z_{\lambda_i} \in D_{\lambda_i} \subseteq X_{\lambda_i} \\ 0 & \text{otherwise} \end{cases} =$$

$$\begin{cases} \bigwedge_{i=1}^{m} e_{\lambda_i} & \text{if } z \in \bigcap_{i=1}^{m} \pi_{\lambda_i}^{-1}(D_{\lambda_i}) \subseteq X \\ 0 & \text{otherwise} \end{cases} \quad \text{and}$$

$$\chi_{\text{supp}k} = \chi_{\text{supp}\ \bigwedge_{i=1}^{m} \pi_{\lambda_i}^{-1}(k_{\lambda_i})} = \chi_{\bigcap_{i=1}^{m} \text{supp}\ \pi_{\lambda_i}^{-1}(k_{\lambda_i})}$$

$$= \bigwedge_{i=1}^{m} \chi_{\text{supp}\ \pi_{\lambda_i}^{-1}(k_{\lambda_i})} = \bigwedge_{i=1}^{m} \pi_{\lambda_i}^{-1}(\chi_{\text{supp}\ k_{\lambda_i}}) =$$

$$\chi_{\text{supp}\ k_{\lambda_i}} \times \dots \times \chi_{\text{supp}\ k_{\lambda_i}} \times \prod_{\lambda \notin \{\lambda_1, \dots, \lambda_m\}} X_\lambda \quad \text{is compact}$$

by theorem 4.2.2. since $\chi_{\text{supp}\ k_{\lambda_i}}$ is compact for every $i \in \{1, \dots, m\}$ and $X_\lambda$ is compact for every $\lambda \notin \{\lambda_1, \dots, \lambda_m\}$.

Remark 6.3.6.

By considering weak local compactness instead of local compactness we can obtain, in the same way, the results obtained in propositions 6.3.3. and 6.3.4. and in theorem 6.3.5.

## 4. A comparison of our proposed L-fuzzy local compactnesses

Theorem 6.4.1.

Let $(X,\mathcal{T})$ be a fully stratified Hausdorff L-fts. Then $(X,\mathcal{T})$ is locally compact if and only if $(X,\mathcal{T})$ is weakly locally compact.

Proof

Necessity:

This immediately follows from the definitions.

Sufficiency:

Let $x\in X$, $p\in pr(L)$ and let $g\in\mathcal{T}$ such that $g(x)\nleq p$.

We want to show that there are a very compact L-fuzzy set c in $(X,\mathcal{T})$ and $j\in\mathcal{T}$ such that $g\geq c\geq j$ with $j(x)\nleq p$.

By the weakly local compactness of $(X,\mathcal{T})$ there are a very compact L-fuzzy set $k(y) = \begin{cases} e & \text{if } y\in D\subseteq X \\ 0 & \text{otherwise} \end{cases}$ and $f\in\mathcal{T}$ such that $k\geq f$ and $f(x)\nleq p$.

Since $f(x)\nleq p$ and $k\geq f$, we have $k(x)\nleq p$ which implies that $x\in D$.

From the fact that k is a very compact L-fuzzy set in $(X,\mathcal{T})$, we have that $\chi_D$ is compact. So, the L-fuzzy subspace $(D,\mathcal{T}_D)$ is compact. As $(X,\mathcal{T})$ is a fully stratified Hausdorff L-fts, we have $(D,\mathcal{T}_D)$ fully stratified Hausdorff.

Thus, $(D,\mathcal{T}_D)$ is a fully stratified Hausdorff compact L-fts. Therefore, by proposition 6.3.1., $(D,\mathcal{T}_D)$ is locally compact.

We have $g_D = g|_D \in \mathcal{T}_D$, $x \in D$ and $g_D(x) \nleq p$. Then, by the local compactness of $(D, \mathcal{T}_D)$ there are a very compact L-fuzzy set $c_D(y) = \begin{cases} b & \text{if } y \in V \subseteq D \\ 0 & \text{otherwise} \end{cases}$ in $(D, \mathcal{T}_D)$ and $h_D \in \mathcal{T}_D$ such that $g_D \geq c_D \geq h_D$ with $h_D(x) \nleq p$.

Since $h_D \in \mathcal{T}_D$, there exists $h^* \in \mathcal{T}$ such that $h_D = h^*|_D$. Let $c \in L^X$ such that $c(y) = \begin{cases} b & \text{if } y \in V \subseteq X \\ 0 & \text{if } y \in X-V \end{cases}$ and $j = h^* \wedge f$.

Thus, $g \geq c \geq j$, c is a very compact L-fuzzy set in $(X, \mathcal{T})$, $j \in \mathcal{T}$ and $j(x) \nleq p$.

In fact, because $h^*$ and $f \in \mathcal{T}$, $h^* \wedge f = j \in \mathcal{T}$. Since $h^*(x) = h_D(x) \nleq p$ $f(x) \nleq p$ and $p \in pr(L)$ we have $(h^* \wedge f)(x) = j(x) \nleq p$. As $c_D$ is a very compact L-fuzzy set in $(D, \mathcal{T}_D)$, $\chi_V$ is compact in $(D, \mathcal{T}_D)$. So $\chi_V$ is compact in $(X, \mathcal{T})$ and then c is a very compact L-fuzzy set in $(X, \mathcal{T})$. Now we are going to show that $g \geq c \geq j$.

Since $g_D \geq c_D$, $g_D(y) \geq b$ for all $y \in V$. So $g \geq c$. From $k \geq f$ we have $f(y) = 0$ for all $y \in X-D$. Since for every $y \in D$ $h^*(y) = h_D(y) \leq c_D(y)$, we have $h^*(y) = 0$ for all $y \in D-V$ and $h^*(y) \leq b$ for all $y \in V$. Then $j(y) = (h^* \wedge f)(y) \leq f(y) = 0$ for all $y \in X-D$ and $j(y) \leq h^*(y) = 0$ for all $y \in D-V$ and $j(y) \leq h^*(y) \leq b$ for all $y \in V$. Therefore $j \leq c$.

## 5. Further properties

Theorem 6.5.1.

Let $(X,\mathcal{T})$ be a weakly locally compact Hausdorff L-fts. Then $(X,\mathcal{T})$ is regular.

Proof

Let $x\in X$, $p\in pr(L)$ and let h be a closed L-fuzzy set such that $h(x)=0$ and there is $y\in X$ with $h'(y)\leq p$.

We want to show that there exist u, $v\in\mathcal{T}$ with $u(x)\nleq p$, $v(y)\nleq p$ for every $y\in X$ with $h'(y)\leq p$ and $(\forall z\in X)$ $u(z)=0$ or $v(z)=0$.

Since $(X,\mathcal{T})$ is weakly locally compact, there exist a very compact L-fuzzy set k and an open L-fuzzy set f such that $k\geq f$ and $f(x)\nleq p$ where for each $y\in X$ $k(y) = \begin{cases} e & \text{if } y\in D\subseteq X \\ 0 & \text{otherwise} \end{cases}$. Because k is a very compact L-fuzzy set, $\chi_D$ is compact. Thus, the L-fuzzy subspace $(D,\mathcal{T}_D)$ is compact. From the Hausdorffness of $(X,\mathcal{T})$, we have the Hausdorffness of $(D,\mathcal{T}_D)$. So, by theorem 4.3.1., $(D,\mathcal{T}_D)$ is regular.

From the fact that $(X,\mathcal{T})$ is Hausdorff and $\chi_D$ is compact in $(X,\mathcal{T})$, we have by proposition 4.1.16. that $\chi_D$ is a closed L-fuzzy set in $(X,\mathcal{T})$.

Since $k\geq f$ and $f(x)\nleq p$, we have $k(x)\nleq p$. Thus $k(x)\neq 0$ and $x\in D$. We also have $f(y)=0$ for every $y\notin D$.

Case 1:

Since $(D,\mathcal{T}_D)$ is regular and $x\in D$ if there is $y\in D$ such that $h'(y)\leq p$, then there are $u_D$, $v_D\in\mathcal{T}_D$ with $u_D(x)\nleq p$,

$v_D(y) \nleq p$ for every $y \in D$ with $h'(y) \leq p$ and $(\forall z \in D)$ $u_D(z)=0$ or $v_D(z)=0$.

Because $u_D, v_D \in \mathcal{T}_D$, there exist $u^*, v^* \in \mathcal{T}$ such that $u_D = u^*|_D$ and $v_D = v^*|_D$.

Take $u = u^* \wedge f$ and $v = v^* \vee \chi_{D'}$.

Thus $u, v \in \mathcal{T}$ and we are going to show that they satisfy what we want.

Since $u_D(x) \nleq p$ and $x \in D$, $u^*(x) = u_D(x) \nleq p$. We also have $f(x) \nleq p$. From the fact that p is prime we conclude that $u(x) = u^*(x) \wedge f(x) \nleq p$.

If $y \in D$ and $h'(y) \leq p$, then $v_D(y) \nleq p$. So, $v^*(y) \nleq p$ and hence $v(y) \nleq p$. If $y \in D'$ and $h'(y) \leq p$, then $\chi_{D'}(y) = 1 \nleq p$ and hence $v(y) \nleq p$. Therefore for every $y \in X$ with $h'(y) \leq p$ we have $v(y) \nleq p$.

If $z \in X$ and $u(z) \neq 0$ then $u^*(z) \neq 0$ and $f(z) \neq 0$. From $f(z) \neq 0$ we conclude that $z \in D$. Thus from $u^*(z) \neq 0$ we have $u_D(z) \neq 0$ which implies $v^*(z) = v_D(z) = 0$. And since $z \in D$, $\chi_{D'}(z) = 0$. Therefore $v(z) = 0$.

<u>Case 2</u>:

If there is no $y \in D$ with $h'(y) \leq p$, then take $u = f$ and $v = \chi_{D'}$.

Thus $u, v \in \mathcal{T}$, $u(x) = f(x) \nleq p$ and for every $y \in X$ with $h'(y) \leq p$ we have $v(y) \nleq p$ because in this case $h'(y) \leq p$ implies $y \notin D$ and so $v(y) = \chi_{D'}(y) = 1 \nleq p$. We also have that $(\forall z \in X)$ $u(z)=0$ or $v(z)=0$. In fact, if $u(z) = f(z) \neq 0$ then $z \in D$ and so $v(z) = \chi_{D'}(z) = 0$.

Hence $(X, \mathcal{T})$ is regular.

Theorem 6.5.2.

Let $(X,\mathcal{T})$ be a locally compact Hausdorff L-fts. Then $(X,\mathcal{T})$ is regular.

Proof

Since every locally compact L-fts is weakly locally compact we have this result from theorem 6.5.1.

Definition 6.5.3.

An L-fts $(X,\mathcal{T})$ is said to be a k-space if and only if the closed L-fuzzy sets are those $f \in L^X$ for which $f|_F$ is closed in $(F,\mathcal{T}_F)$ for each compact L-fuzzy subspace $(F,\mathcal{T}_F)$.

Theorem 6.5.4.

Let $(X,\mathcal{T})$ be an L-fts. Then $(X,\mathcal{T})$ is a k-space if and only if the open L-fuzzy sets are those $f \in L^X$ for which $f|_F \in \mathcal{T}_F$ for each compact L-fuzzy subspace $(F,\mathcal{T}_F)$.

Proof

Necessity:

We always have that $f|_F \in \mathcal{T}_F$ for every $F \subseteq X$ if $f \in \mathcal{T}$.

Now let $f \in L^X$ be an L-fuzzy set such that $f|_F \in \mathcal{T}_F$ for each compact L-fuzzy subspace $(F,\mathcal{T}_F)$.

Then $f'|_F$ is a closed L-fuzzy set in $(F,\mathcal{T}_F)$ for each compact L-fuzzy subspace $(F,\mathcal{T}_F)$. Since $(X,\mathcal{T})$ is a k-space, $f'$ is closed in $(X,\mathcal{T})$. So, $f \in \mathcal{T}$.

Sufficiency:

We always have that $f|_F$ is closed in $(F,\mathcal{T}_F)$ for every $F\subseteq X$ if f is closed in $(X,\mathcal{T})$.

Now let $f\in L^X$ be an L-fuzzy set such that $f|_F$ is closed in $(F,\mathcal{T}_F)$ for each compact L-fuzzy subspace $(F,\mathcal{T}_F)$.

Then $f'|_F \in\mathcal{T}_F$ for each compact L-fuzzy subspace $(F,\mathcal{T}_F)$. Thus, by hypothesis, $f'\in\mathcal{T}$. Therefore f is closed in $(X,\mathcal{T})$.

Theorem 6.5.5.

Let $(X,\delta)$ be a topological space. Then $(X,\delta)$ is a k-space if and only if $(X,\omega(\delta))$ is a k-space.

Proof

Necessity:

By theorem 6.5.4. it is sufficient to prove that for every $f\in L^X$ such that $f|_F$ is an open L-fuzzy set in $(F,\omega(\delta_F))$ for each compact L-fuzzy subspace $(F,\omega(\delta_F))$ we have $f\in\omega(\delta)$.

Let $f\in L^X$ be such that $f|_F$ is open in $(F,\omega(\delta_F))$ for each compact L-fuzzy subspace $(F,\omega(\delta_F))$.

Thus, by proposition 3.2.9., $H = \{x\in F;\ f(x)\nleq p\}\in\delta_F$ for each $p\in pr(L)$ and for each compact L-fuzzy subspace $(F,\ \omega(\delta_F))$.

Therefore, by the goodness of compactness (theorem 4.1.6.), $H = F\cap\{x\in X;\ f(x)\nleq p\}$ is open in $(F,\delta_F)$ for each compact subspace $(F,\delta_F)$ and each $p\in pr(L)$. So, by the fact that $(X,\delta)$ is a k-space, $\{x\in X;\ f(x)\nleq p\}$ is open in

$(X,\delta)$ for every $p\in pr(L)$. Then, by proposition 3.2.9., $f\in\omega(\delta)$.

Sufficiency:

Let $U\subseteq X$ such that $U\cap C$ is open in C for each compact subspace C of $(X,\delta)$.

Thus, by theorem 4.1.6. and proposition 3.2.10., $\chi_{U\cap C}|_C = \chi_U|_C \in\omega(\delta_C)$ for each compact L-fuzzy subspace $(F, \omega(\delta_F))$. So, by the fact that $(X,\omega(\delta))$ is a k-space and by theorem 6.5.4., $\chi_U\in\omega(\delta)$.

Hence $U\in\delta$.

Theorem 6.5.6.

Let $(X,\mathcal{T})$ be a weakly locally compact L-fts. Then $(X,\mathcal{T})$ is a k-space.

Proof

By theorem 6.5.4. we only need to prove that if $f\in L^X$ is such that $f|_F$ is open in $(F,\mathcal{T}_F)$ for each compact L-fuzzy subspace $(F,\mathcal{T}_F)$ then f is open in $(X,\mathcal{T})$.

Let $f\in L^X$ be such that $f|_F$ is open in $(F,\mathcal{T}_F)$ for each compact L-fuzzy subspace $(F,\mathcal{T}_F)$.

We want to prove that $f\in\mathcal{T}$. For this, let $p\in pr(L)$, $x\in X$ with $f(x)\not\leq p$. By proposition 3.1.4., it is sufficient to show that there is $g\in\mathcal{T}$ such that $g\leq f$ and $g(x)\not\leq p$.

By the weakly local compactness of $(X,\mathcal{T})$, there are a very compact L-fuzzy set $k(x) = \begin{cases} e & \text{if } x\in D\subseteq X \\ 0 & \text{otherwise} \end{cases}$ and $h\in\mathcal{T}$ such that $k\geq h$ and $h(x)\not\leq p$.

Since k is a very compact L-fuzzy set, $\chi_D$ is compact in $(X,\mathcal{T})$, so $(D,\mathcal{T}_D)$ is a compact subspace.

Thus, by our assumptions on f, $f|_D \in\mathcal{T}_D$. Then there exists $f^*\in\mathcal{T}$ such that $f|_D = f^*|_D$.

Take $g=f^*\wedge h$.

Therefore, $g\in\mathcal{T}$, $g\leq f$ and $g(x)\nleq p$. In fact, since $f^*$ and $h\in\mathcal{T}$ we have $g\in\mathcal{T}$. As $h(x)\nleq p$ and $k\geq h$ we have $k(x)\nleq p$ which implies that $x\in D$. Because $f(x)\nleq p$ and $x\in D$ we also have $f^*(x) = f(x)\nleq p$. Thus $g(x)\nleq p$ since $p\in pr(L)$. Now we are going to prove that $g\leq f$.

Since $k\geq h$ and $k(x)=0$ for every $x\notin D$, $h(x)=0$ for every $x\notin D$. So, $g(x)=0$ for every $x\notin D$ and then $g(x)\leq f(x)$ for every $x\notin D$. If $x\in D$ then $f(x)=f^*(x)$ which implies that $g(x)=(f^*\wedge h)(x)\leq f^*(x)=f(x)$ for every $x\in D$. Hence $g\leq f$.

<u>Theorem 6.5.7.</u>

Let $(X,\mathcal{T})$ be a locally compact L-fts. Then $(X,\mathcal{T})$ is a k-space.

<u>Proof</u>

Since every locally compact L-fts is weakly locally compact we have this result from theorem 6.5.6.

## 6. <u>One point compactfication</u>

Definition 6.6.1.

Let $(X,\mathcal{T}_X)$ be an L-fts which is not compact, but is locally compact and Hausdorff. Take some object outside X, denoted by the symbol $\infty$ for convenience and adjoin it to X forming the set $Y=X\cup\{\infty\}$. Fuzzy topologize Y by defining as a subbase the collection

$\mathcal{S} = \{f_1;\ f\in\mathcal{T}_X\}\cup\{\chi_{B_\infty};\ \chi_B \in \mathcal{C}\}$ where:

(i) $f_1 \in L^Y$ defined by $f_1(x) = \begin{cases} f(x) & \text{if } x\in X \\ 0 & \text{if } x=\infty \end{cases}$ for each $f\in\mathcal{T}_X$.

(ii) $\mathcal{C} = \left\{\chi_B;\ B\subset X;\ \chi_B \text{ is compact in } (X,\mathcal{T}_X)\right\}$.

(iii) For $\chi_B \in\mathcal{C}$, $B_\infty = \{\infty\} \cup (X-B)$, $\chi_{B_\infty}(x) = \begin{cases} 0 & \text{if } x\in B \\ 1 & \text{if } x\in B_\infty \end{cases}$.

Let $\mathcal{T}_Y$ be the L-fuzzy topology on Y having $\mathcal{S}$ as a subbase.

The L-fuzzy space $(Y,\mathcal{T}_Y)$ is called the <u>one point compactification of $(X,\mathcal{T}_X)$.</u>

Theorem 6.6.2.

Let $(X,\mathcal{T}_X)$ be a locally compact Hausdorff space which is not compact and let $(Y,\mathcal{T}_Y)$ be its one point compactification. Then $(Y,\mathcal{T}_Y)$ is a compact Hausdorff L-fts, $(X,\mathcal{T}_X)$ is a subspace of $(Y,\mathcal{T}_Y)$ and $cl(X) = Y$.

Proof

(i) Clearly $(X,\mathcal{T}_X)$ is a subspace of $(Y,\mathcal{T}_Y)$.

(ii) $cl(X)=Y$. In fact, if $cl(X)\neq Y$ in $(Y,\mathcal{T}_Y)$ then $cl(X)$

is an L-fuzzy set of the form:

$$cl(X)(x) = \begin{cases} 1 & \text{if } x\in X \\ e\neq 1 & \text{if } x=\infty \end{cases}.$$

The complement of cl(X) is then the open L-fuzzy set

$$(cl(X))'(x) = \begin{cases} 0 & \text{if } x\in X \\ e'\neq 0 & \text{if } x=\infty \end{cases}.$$

Since $\mathcal{S}$ is a subbase for $\mathcal{T}_Y$ and $(cl(X))'\in\mathcal{T}_Y$, there are $B_1, \ldots, B_m$ in X with $\chi_{B_1}, \ldots, \chi_{B_m} \in \mathcal{C}$ and $\chi_{B_{1\infty}} \wedge \ldots \wedge \chi_{B_{m\infty}} \leq (cl(X))'$.

Thus $e'=1$, so $e=0$ and $X = \chi_{B_1} \vee \ldots \vee \chi_{B_m}$. Hence $(X,\mathcal{T}_X)$ is compact, yielding a contradiction.

(iii) $(Y,\mathcal{T}_Y)$ is compact. In fact:

Let $p\in pr(L)$ and let $\mathcal{A} = (f_i)_{i\in J}$ be a collection of $\mathcal{T}_Y$-subbasic open L-fuzzy sets with $\left(\bigvee_{i\in J} f_i\right)(y)\not\leq p$ for all $y\in Y$.

This collection must contain an element of the type $\chi_{B_\infty}$. Then $\chi_{B_\infty}(x)=1\not\leq p$ for all $x\in X-B$.

By taking all the members of $\mathcal{A}$ different from $\chi_{B_\infty}$ and restricting them to X, we have a collection $\mathcal{A}^*$ of $\mathcal{T}_X$ open L-fuzzy sets with $\left(\bigvee_{h\in\mathcal{A}^*} h\right)(y)\not\leq p$ for all $y\in B$. Since $\chi_B$ is compact, there is a finite subcollection of $\mathcal{A}^*$, say $\{h_1, \ldots, h_m\}$, with $\left(\bigvee_{i=1}^{m} h_i\right)(y)\not\leq p$ for all $y\in B$. Then $\left(\left(\bigvee_{i=1}^{m} h_i\right)\vee\chi_{B_\infty}\right)(y)\not\leq p$ for all $y\in Y$.

Hence, by theorem 4.2.1., $(Y,\mathcal{T}_Y)$ is compact.

(iv) $(Y,\mathcal{T}_Y)$ is Hausdorff. In fact:

Since $(X,\mathcal{T}_X)$ is Hausdorff (definition 3.4.5.), given $p,q \in pr(L)$ and $x,y \in X$ with $x\neq y$, there exist $f_p, g_q \in \mathcal{T}_X$ with $f_p(x)\not\leq p$, $g_q(y)\not\leq q$ and $(\forall z\in X)$ $f_p(z)=0$ or $g_q(z)=0$.

Hence $(f_p)_1(x) \nleq p$, $(g_q)_1(y) \nleq q$ and $(\forall z \in Y)$ $(f_p)_1(z)=0$ or $(g_q)_1(z)=0$ where $(f_p)_1$ and $(g_q)_1$ are defined as in definition 6.6.1.(i).

Suppose now that $x \in X$ and $y=\infty$.

Since $x \in X$ and $(X,\mathcal{T}_X)$ is locally compact, given $p,q \in pr(L)$ and $f \in \mathcal{T}_X$ with $f(x) \nleq p$; there are k very compact and $g \in \mathcal{T}_X$ such that $g \leq k \leq f$ and $g(x) \nleq p$. Then $\chi_{suppk}$ is compact and since $(X,\mathcal{T}_X)$ is Hausdorff, we have by proposition 4.1.16. $\chi_{suppk}$ closed in $(X,\mathcal{T}_X)$. Thus, $\chi_{suppk} \in \mathcal{C}$ and $\chi_{(suppk)_\infty} \in \mathcal{T}_Y$. Therefore $g_1(x) \nleq p$ where $g_1 : Y \to L$ is defined by $g_1(z) = \begin{cases} 0 & \text{if } z=\infty \\ g(z) & \text{if } z \in X \end{cases}$, $\chi_{(suppk)_\infty}(y) = 1 \nleq q$ where $\chi_{(suppk)_\infty} : Y \to L$ is defined by $\chi_{(suppk)_\infty}(z) = \begin{cases} 0 \text{ if } z \in suppk \\ 1 \text{ if } z \in \{\infty\} \cup (X-suppk) \end{cases}$ and $g_1 \in \mathcal{T}_Y$.

We also have $(\forall z \in Y)$ $g_1(z)=0$ or $\chi_{(suppk)_\infty}(z)=0$ because:

If $z=\infty$ then $g_1(z)=0$.

If $z \in suppk$ then $\chi_{(suppk)_\infty}(z) = 0$.

If $z \in X-suppk$, $g(z)=0$ and then $g_1(z)=0$.

Hence $(Y,\mathcal{T}_Y)$ is Hausdorff.

Remark 6.6.3.

In this section, considering weak local compactness instead of local compactness, we can obtain the same results in the same way.

# Chapter VII

## Paracompactness in L-fuzzy topological spaces

In this chapter we suggest a good definition of L-fuzzy paracompactness and study some of its properties.

Some definitions of paracompactness were presented in [0,1]-fuzzy topological spaces by Malghan and Benchalli [61], Luo [59], Abd El-Monsef and al [2] and Bülbül and Warner [13]. The works [13] and [2] are based on fuzzy compactness (definition 4.6.3.) and [61] and [59] are based on $\alpha$-compactness (definition 4.1.1.).

In L-fuzzy topological spaces, paracompactness was studied by Chen [21] and Xu [106]. The first one based on N-L-compactness (definition 4.5.3.) and the latter based on X-compactness (definition 4.6.14.).

By introducing a new definition of a locally finite family of L-fuzzy sets and combining it with our definition of compactness for arbitrary L-fuzzy sets, we propose a different L-fuzzy paracompactness which is defined on arbitrary L-fuzzy sets. We also study some of its properties such as:- paracompactness is a good extension; is inherited by closed L-fuzzy subsets; the product of a compact L-fts and a paracompact space is paracompact and prove that a Hausdorff paracompact space is regular.

This chapter is divided in three sections.

In the first section we present the proposed definition and prove its goodness.

In the second section we study some of its properties.

The third section is devoted to the regularity of a paracompact Hausdorff L-fts and to the paracompactness of the product of a paracompact L-fts with a compact L-fts.

## 1. Proposed definition and its goodness

Definition 7.1.1.

A family $(f_i)_{i\in J}$ of L-fuzzy sets in an L-fts is said to be locally finite in an L-fuzzy set g if and only if for each $p\in pr(L)$ and for each $x\in X$ with $g(x)\geq p'$, there are an open L-fuzzy set r with $r(x)\nleq p$ and a finite subset $J_0$ of J such that $(\forall z\in X)$ $f_i(z)=0$ or $r(z)=0$ for every $i\in J-J_0$.

When g is the whole space X, we shall directly say locally finite, omitting "in an L-fuzzy set g".

Definition 7.1.2.

Let $(X,\mathcal{T})$ be an L-fts. A family $(f_i)_{i\in I}$ of L-fuzzy sets is said to be a refinement of the family $(g_j)_{j\in J}$ of L-fuzzy sets if and only if for each $i\in I$ there is $j\in J$ with $f_i\leq g_j$.

Definition 7.1.3.

An L-fuzzy set g in an L-fts $(X,\mathcal{T})$ is said to be paracompact if and only if for every $p\in pr(L)$ and every family $(f_i)_{i\in I}$ of open L-fuzzy sets with $\left(\bigvee_{i\in I} f_i\right)(x)\nleq p$ for all $x\in X$ with $g(x)\geq p'$, there exists a family $(g_j)_{j\in J}$ of open L-fuzzy sets that is a refinement of $(f_i)_{i\in I}$, locally finite in g and $\left(\bigvee_{j\in J} g_j\right)(x)\nleq p$ for all $x\in X$ with $g(x)\geq p'$.

If the L-fuzzy set g is the whole space X, we say that the L-fts $(X,\mathcal{T})$ is paracompact.

Theorem 7.1.4. (The goodness of paracompactness)

Let $(X,\delta)$ be a topological space. Then $(X,\delta)$ is paracompact if and only if $(X,\omega(\delta))$ is a paracompact L-fts.

Proof

Necessity:

Let $p\in pr(L)$ and let $\mathcal{A} = (f_i)_{i\in I}$ be a family of open L-fuzzy sets in $(X,\omega(\delta))$ with $\left(\bigvee_{i\in I} f_i\right)(x)\nleq p$ for all $x\in X$.

Thus, $\mathcal{B} = (\{x\in X;\ f_i(x)\nleq p\})_{i\in I}$ is an open cover of $(X,\delta)$. In fact, since $f_i\in \omega(\delta)$ for every $i\in I$, by proposition 3.2.9. $\{x\in X;\ f_i(x)\nleq p\}\in\delta$ for all $i\in I$. We also have that for each $x\in X$ there is $i\in I$ such that $f_i(x)\nleq p$, so $\mathcal{B}$ is an open cover of $(X,\delta)$.

From the paracompactness of $(X,\delta)$, $\mathcal{B}$ has a locally finite open refinement $\mathcal{C}$ that covers X. Since $\mathcal{C}$ is a refinement of $\mathcal{B}$, for each $C\in\mathcal{C}$ we can take $f_{iC}\in\mathcal{A}$ such that $C\subset\{x\in X;\ f_{iC}(x)\nleq p\}$.

Therefore $\mathcal{D} = (\chi_C\wedge f_{iC})_{C\in\mathcal{C}}$ is a family of open L-fuzzy sets in $(X,\omega(\delta))$ with $\left(\bigvee_{h\in\mathcal{D}} h\right)(x)\nleq p$ for every $x\in X$ and is a refinement of $\mathcal{A}$. In fact, evidently $\mathcal{D}$ is a refinement of $\mathcal{A}$ because for each $h=\chi_C\wedge f_{iC}\in\mathcal{D}$ there is $g=f_{iC}\in\mathcal{A}$ such that $h\leq g$. We also have $\left(\bigvee_{h\in\mathcal{D}} h\right)(x)\nleq p$ for every $x\in X$ because for every $x\in X$ there is $C^*\in\mathcal{C}$ such that $x\in C^*$ which implies that for all $x\in X$ there is $C^*\in\mathcal{C}$ such that $\chi_{C^*}(x)= 1\nleq p$ and $f_{iC^*}(x)\nleq p$ since $C^*\subset\{x\in X;\ f_{iC^*}(x)\nleq p\}$, so $\left(\chi_{C^*}\wedge f_{iC^*}\right)(x)\nleq p$ (p is prime). Hence $\left(\bigvee_{h\in\mathcal{D}} h\right)(x)\nleq p$ for every $x\in X$.

We now prove that $\mathcal{D}$ is locally finite.

For each $x \in X$, take $B_x \in \delta$ with $x \in B_x$ such that $B_x$ intersects only a finite number of members of $\mathcal{C}$, which is possible because $\mathcal{C}$ is locally finite. So, for each $x \in X$ and for each $p \in pr(L)$, there exists an open L-fuzzy set $r = \chi_{B_x}$ with $r(x) = 1 \nleq p$ such that $(\forall z \in X)$ $h(z) = 0$ or $r(z) = 0$ for all but finitely many $h \in \mathcal{D}$. In fact, $r(x) \nleq p$ and $r \in \omega(\delta)$ and we also have that $(\forall z \in X)$ $h(z) = 0$ or $r(z) = 0$ for all but finitely many $h \in \mathcal{D}$ because $h(z) = \left(\chi_C \wedge f_{iC}\right)(z) \neq 0$ and $r(z) = \chi_{B_x} \neq 0$ if and only if $z \in C \cap B_x$ and we have that $B \cap C \neq \phi$ only for a finite number of $C \in \mathcal{C}$. Thus $\mathcal{D}$ is also locally finite.

Hence $(X, \omega(\delta))$ is paracompact.

<u>Sufficiency</u>:

Let $\mathcal{A}$ be an open cover of $(X, \delta)$.

Thus, $(\chi_U)_{U \in \mathcal{A}}$ is a family of open L-fuzzy sets in $(X, \omega(\delta))$ with $\left(\bigvee_{U \in \mathcal{A}} \chi_U\right)(x) = 1 \nleq p$ for all $x \in X$ and for all $p \in pr(L)$.

From the paracompactness of $(X, \omega(\delta))$, there exists a locally finite open refinement $\mathcal{C}$ with $\left(\bigvee_{f \in \mathcal{C}} f\right)(x) \nleq p$ for all $x \in X$ and for all $p \in pr(L)$.

Let $\mathcal{B} = (\{x \in X;\ f(x) \nleq p\})_{f \in \mathcal{C}}$.

Therefore $\mathcal{B}$ is a refinement of $\mathcal{A}$ and an open cover of $(X, \delta)$. In fact, since $\mathcal{C}$ is a refinement of $(\chi_U)_{U \in \mathcal{A}}$, for each $f \in \mathcal{C}$ there is $U \in \mathcal{A}$ such that $f \leq \chi_U$, so $\{x \in X; f(x) \nleq p\} \subset \{x \in X;\ \chi_U(x) \nleq p\} = U$ and $\mathcal{B}$ is a refinement of $\mathcal{A}$.
We also have that for all $x \in X$ there exists $f \in \mathcal{C}$ with $f(x) \nleq p$, so $\mathcal{B}$ is a cover of $(X, \delta)$. Actually $\mathcal{B}$ is an open cover of $(X, \delta)$ because each $f \in \mathcal{C}$ belongs to $\omega(\delta)$ and by

proposition 3.2.9. each $\{x\in X;\ f(x)\nleq p\}\in\delta$.

Now we are going to prove that $\mathcal{B}$ is locally finite, that is, for all $x\in X$ there exists $K\in\delta$ with $x\in K$ such that $\{x\in X;\ f(x)\nleq p\}\cap K\neq\phi$ for at most finitely many $f\in\mathcal{C}$.

Since $\mathcal{C}$ is locally finite, let $x_0\in X$ and $r\in\omega(\delta)$ with $r(x_0)\nleq p$ such that $(\forall z\in X)$ $f(z)=0$ or $r(z)=0$ for all but finitely many $f\in\mathcal{C}$, say $f_1, \ldots, f_m$.

Let $K = \{x\in X;\ r(x)\nleq p\}$.

Thus, by proposition 3.2.9. $K\in\delta$ and since $r(x_0)\nleq p$, we have $x_0\in K$. We also have $\{x\in X;\ f(x)\nleq p\}\cap K\neq\phi$ for at most finitely many $f\in\mathcal{C}$ because if $y\in\{x\in X;\ f(x)\nleq p\}\cap K$ then $r(y)\nleq p$ and $f(y)\nleq p$ which implies that $f\in\{f_1, \ldots, f_m\}$. Therefore $\mathcal{B}$ is locally finite.

Hence $(X,\delta)$ is paracompact.

## 2. Some properties

Proposition 7.2.1.

Every compact L-fuzzy set in an L-fts is paracompact.

Proof

This immediately follows from the definitions.

Proposition 7.2.2.

Let $(X,\mathcal{T})$ be an L-fts, let h be a paracompact L-fuzzy set and g a closed L-fuzzy set. Then $g\wedge h$ is a paracompact L-fuzzy set.

Proof

Let $p\in pr(L)$ and let $(f_i)_{i\in J}=\mathcal{A}$ be a family of open L-fuzzy sets with $\left(\bigvee_{i\in J} f_i\right)(x)\nleq p$ for all $x\in X$ with $(g\wedge h)(x)\geq p'$.

Thus, $\mathcal{B}=\mathcal{A}\cup\{g'\}$ is a family of open L-fuzzy sets with $\left(\bigvee_{k\in\mathcal{B}} k\right)(x)\nleq p$ for all $x\in X$ with $h(x)\geq p'$. In fact, for each $x\in X$ with $h(x)\geq p'$, if $g(x)\geq p'$ then $(g\wedge h)(x)\geq p'$ which implies that $\left(\bigvee_{i\in J} f_i\right)(x)\nleq p$, thus $\left(\bigvee_{k\in\mathcal{B}} k\right)(x)\nleq p$. If $g(x)\ngeq p'$ then $g'(x)\nleq p$ which implies that $\left(\bigvee_{k\in\mathcal{B}} k\right)(x)\nleq p$.

From the paracompactness of h, there exists a family $\mathcal{C}$ of open L-fuzzy sets that is a refinement of $\mathcal{B}$, locally finite in h and $\left(\bigvee_{k\in\mathcal{C}} k\right)(x)\nleq p$ for all $x\in X$ with $h(x)\geq p'$.

Let $\mathcal{C}^*=\{k\in\mathcal{C};$ there is $f_i\in\mathcal{A}$, $k\leq f_i\}$.

Evidently $\mathcal{C}^*$ is an open refinement of $\mathcal{A}$ and also is locally finite in $g\wedge h$. We also have that $\left(\bigvee_{k\in\mathcal{C}^*} k\right)(x)\nleq p$ for all $x \in X$ with $(g \wedge h)(x) \geq p'$. In fact, otherwise there is $x \in X$ with $(g \wedge h)(x) \geq p'$ and $\left(\bigvee_{k \in \mathcal{C}^*} k\right)(x) \leq p$. But we have $\left(\bigvee_{k \in \mathcal{C}} k\right)(x) \nleq p$ which implies that there exists $k_1 \in \mathcal{C}$ such that $k_1(x) \nleq p$. Since $\mathcal{C}$ is a refinement of $\mathcal{B}$, there is $k_2 \in \mathcal{B}$ such that $k_1 \leq k_2$ and since $k_1 \notin \mathcal{C}^*$ because $\left(\bigvee_{k \in \mathcal{C}^*} k\right)(x) \leq p$, $k_2 = g'$. Therefore, $p \nleq k_1(x) \leq k_2(x) = g'(x) \leq p$, yielding a contradiction.

Hence $g \wedge h$ is paracompact.

<u>Corollary</u> <u>7.2.3.</u>

Let $(X, \mathcal{T})$ be an L-fts. IF g is a paracompact L-fuzzy set, then each closed L-fuzzy set contained in g is paracompact as well.

<u>Proof</u>

This immediately follows from proposition 7.2.2.

## 3. **Further properties**

Lemma 7.3.1.

Let $(X,\mathcal{T})$ be a paracompact L-fts, $p\in pr(L)$ and let $\mathcal{A} = (f_i)_{i\in I}$ be a family of open L-fuzzy sets with $\left(\bigvee_{i\in I} f_i\right)(x)\nleq p$ for all $x\in X$. Then, there exists a family $(g_i)_{i\in I} = \mathcal{B}$ of open L-fuzzy sets with $\left(\bigvee_{i\in I} g_i\right)(x)\nleq p$ for all $x\in X$ and $\mathcal{B}$ locally finite, such that $g_i\leq f_i$ for all $i\in I$.

Proof

From the paracompactness of $(X,\mathcal{T})$, $\mathcal{A}$ has a locally finite refinement $\mathcal{C} = (h_j)_{j\in J}$ with each $h_j\in\mathcal{T}$ and $\left(\bigvee_{j\in J} h_j\right)(x)\nleq p$ for all $x\in X$. Every $h_j$ is associated with a containing set from $\mathcal{A}$. Hence there is defined a mapping $\vartheta:J\to I$ such that $h_j\leq f_{\vartheta(j)}$ for all $j\in J$. For $i\in I$ we put $g_i = \bigvee_{\vartheta(j)=i} h_j$ where $g_i = \phi$ if there is no $j$ with $\vartheta(j) = i$. By the construction of $\vartheta$ we have that each $g_i\in\mathcal{T}$, $g_i\leq f_i$ and $(g_i)_{i\in I}$ satisfies $\left(\bigvee_{i\in I} g_i\right)(x)\nleq p$ for all $x\in X$. For $x\in X$ there are an open L-fuzzy set $r$ with $r(x)\nleq p$ and a finite subset $J_0$ of $J$ such that $(\forall z\in X)\, h_j(z)=0$ or $r(z)=0$ for every $j\in J-J_0$. So, we have $g_i(z)\neq 0$ and $r(z)\neq 0$ only when $i=\vartheta(j)$ for $j\in J_0$.

Hence $\mathcal{B}$ is locally finite.

Theorem 7.3.2.

Let $(X,\mathcal{T})$ be a paracompact Hausdorff L-fts. Then $(X,\mathcal{T})$ is regular (definition 3.4.8.).

Proof

Let $p \in pr(L)$, $x \in X$, and let f be a closed L-fuzzy set in $(X,\mathcal{T})$ such that there is $y \in X$ with $f(y) \geq p'$ and $f(x)=0$.

Let $F = \{t \in X;\ f(t) \geq p'\}$.

We have that $x \notin F$. Since $(X,\mathcal{T})$ is Hausdorff (definition 3.4.5.) for each $y \in F$ there exist $f_y$, $g_y \in \mathcal{T}$ with $f_y(x) \nleq p$, $g_y(y) \nleq p$ and $(\forall z \in X)$ $f_y(z)=0$ or $g_y(z)=0$.

Let $\mathcal{A} = (g_y)_{y \in F} \cup \{f'\}$.

We have $\left(\bigvee_{h \in \mathcal{A}} h\right)(z) \nleq p$ for all $z \in X$. In fact, if $z \in F$ then $g_z(z) \nleq p$ and if $z \notin F$ then $f'(z) \nleq p$, so $\left(\bigvee_{h \in \mathcal{A}} h\right)(z) \nleq p$ for every $z \in X$. From the paracompactness of $(X,\mathcal{T})$ and from lemma 7.3.1., there is a family $\mathcal{B} = (k_y)_{y \in F} \cup \{k_0\}$ of open L-fuzzy sets that is a refinement of $\mathcal{A}$, locally finite and $\left(\bigvee_{y \in F} k_y \vee k_0\right)(z) \nleq p$ for all $z \in X$, where $k_y \leq g_y$ for each $y \in F$ and $k_0 \leq f'$.

Thus, for our point x in X and our p in pr(L), there are $r \in \mathcal{T}$ with $r(x) \nleq p$ and a finite subfamily $\mathcal{B}_0$ of $\mathcal{B}$ such that $(\forall z \in X)$ $b(z)=0$ or $r(z)=0$ for every $b \in \mathcal{B}-\mathcal{B}_0$. Therefore, there are $r \in \mathcal{T}$ with $r(x) \nleq p$ and a finite subset $F_0$ of F such that $(\forall z \in X)$ $k_y(z)=0$ or $r(z)=0$ for every $y \in F-F_0$.

Since for each $y \in F$ $k_y \leq g_y$ and because $f_y(z)=0$ or $g_y(z)=0$ for all $z \in X$, we have that $k_y(z)=0$ or $f_y(z)=0$ for all $z \in X$ and for all $y \in F$.

Let $u = r \wedge \left(\bigwedge_{y \in F_0} f_y\right)$ and $v = \bigvee_{y \in F} k_y$.

We have $u,v \in \mathcal{T}$, $u(x) \nleq p$; for every $z \in X$ with $f(z) \geq p'$, $v(z) \nleq p$ and $(\forall z \in X)$ $u(z)=0$ or $v(z)=0$. In fact since r, each $f_y$ and $k_y$ are open L-fuzzy sets we have $u,v\in\mathcal{T}$. As each $f_y$ satisfies $f_y(x)\nleq p$ and $r(x)\nleq p$, from the fact that p is prime we have $u(x)\nleq p$. For every $z\in X$ with $f(z)\geq p'$, that is, $z\in F$, we have $v(z)\nleq p$ because $v(z)=\left(\bigvee_{y\in F} k_y\right)(z)\nleq p$ for all $z\in F$. We also have $u(z)=0$ or $v(z)=0$ because if $z\in X$ and $u(z)\neq 0$ then $r(z)\neq 0$ and $f_y(z)\neq 0$ for every $y\in F_0$. Thus, from $r(z)\neq 0$, we have $k_y(z)=0$ for every $y\in F-F_0$ and from $f_y(z)\neq 0$ for every $y\in F_0$, we have $k_y(z)=0$ for every $y\in F_0$. So, $v(z)=\left(\bigvee_{y\in F} k_y\right)(z)=0$.

Hence $(X,\mathcal{T})$ is regular.

Theorem 7.3.3.

Let $(X,\mathcal{T})$ be a paracompact L-fts and let $(Y,\mathcal{T}_Y)$ be a compact L-fts. Then the product $X\times Y$ is a paracompact L-fts.

Proof

Let $p\in pr(L)$ and let $\mathcal{A}$ be a family of open L-fuzzy sets in the product space $X\times Y$ with $\left(\bigvee_{f\in\mathcal{A}} f\right)(x,y)\nleq p$ for all $(x,y)\in X\times Y$. Select for each $(x,y)\in X\times Y$, $g_{xy}\in\mathcal{T}_X$, $h_{xy}\in\mathcal{T}_Y$ with $g_{xy}(x)\nleq p$ and $h_{xy}(y)\nleq p$ such that $\mathcal{A}\ni f_{xy}\geq$ $\pi_1^{-1}(g_{xy})\wedge\pi_2^{-1}(h_{xy})$ where $\pi_1$, $\pi_2$ are the projection maps. This is possible because for every $(x,y)\in X\times Y$ there is $f\in\mathcal{A}$ such that $f(x,y)\nleq p$ and since f is an open L-fuzzy set in the product space $X\times Y$ (definition 3.2.4.),

$f = \bigvee_{\substack{g_{xy}\in\mathcal{T}_X \\ h_{xy}\in\mathcal{T}_Y}} \pi_1^{-1}(g_{xy})\wedge\pi_2^{-1}(h_{xy})$. So there are $g_{xy}\in\mathcal{T}_X$ and $h_{xy}\in\mathcal{T}_Y$ with $\left(\pi_1^{-1}(g_{xy})\wedge\pi_2^{-1}(h_{xy})\right)(x,y)\nleq p$ which implies

$g_{xy}(x) \nleq p$ and $h_{xy}(y) \nleq p$.

Therefore, for a given $x \in X$, $(h_{xy})_{y \in Y} = \mathcal{C}$ is a family of $\mathcal{T}_Y$-open L-fuzzy sets with $\left(\bigvee_{c \in \mathcal{C}} c\right)(z_2) \nleq p$ for all $z_2 \in Y$.

Thus, by the compactness of $(Y, \mathcal{T}_Y)$, there exists a finite subfamily $\mathcal{C}_1$ of $\mathcal{C}$, say $(h_{xy_i(x)})_{i \in \{1, \ldots, m(x)\}}$, with $\left(\bigvee_{i=1}^{m(x)} h_{xy_i(x)}\right)(z_2) \nleq p$ for all $z_2 \in Y$.

Let $g_x = \bigwedge_{i=1}^{m(x)} g_{xy_i(x)}$ .

Thus, since each $g_{xy_i(x)} \in \mathcal{T}_X$, $g_{xy_i(x)}(x) \nleq p$ for every $i \in \{1, \ldots, m(x)\}$ and $p \in pr(L)$ we have $g_x \in \mathcal{T}_X$ and $g_x(x) \nleq p$. Therefore $(g_x)_{x \in X}$ is a family of $\mathcal{T}_X$-open L-fuzzy sets with $\left(\bigvee_{x \in X} g_x\right)(z_1) \nleq p$ for all $z_1 \in X$.

From the paracompactness of $(X, \mathcal{T}_X)$, there is a family $\mathcal{D}$ of $\mathcal{T}_X$-open L-fuzzy sets that is a refinement of $(g_x)_{x \in X}$, locally finite and $\left(\bigvee_{d \in \mathcal{D}} d\right)(z_1) \nleq p$ for all $z_1 \in X$.

For each $d \in \mathcal{D}$ take $x_d \in X$ with $d \le g_{x_d}$, which is possible because $\mathcal{D}$ is a refinement of $(g_x)_{x \in X}$ .

Let $\mathcal{H} = \left(\pi_1^{-1}(d) \wedge \pi_2^{-1}(h_{x_d y_i(x_d)})\right)_{\substack{d \in \mathcal{D} \\ i \in \{1, \ldots, m(x_d)\}}}$ .

Thus, $\mathcal{H}$ is a family of open L-fuzzy sets in the product space with $\left(\bigvee_{h \in \mathcal{H}} h\right)(z) \nleq p$ for all $z \in X \times Y$ because, for each $z = (z_1, z_2) \in X \times Y$ there is $d_1 \in \mathcal{D}$ with $d_1(z_1) \nleq p$ and corresponding to it there exists $h_{x_{d_1} y_1(x_{d_1})} \in \mathcal{C}_1$ with $h_{x_{d_1} y_i(x_{d_1})}(z_2) \nleq p$. Further from the choice of $g_{x_d y_i(x_d)}$

and

from $\pi_1^{-1}(d) \wedge \pi_2^{-1}(h_{x_d y_i(x_d)}) \le \pi_1^{-1}(g_{x_d y_i(x_d)}) \wedge \pi_2^{-1}(h_{x_d y_i(x_d)})$

it follows that $\mathcal{H}$ is a refinement of $\mathcal{A}$.

Now we are going to prove that $\mathcal{H}$ is locally finite.

Let $(x_0,y_0) \in X\times Y$.

Since $\mathcal{D}$ is locally finite, there are a $\mathcal{T}_X$-open L-fuzzy set g with $g(x_0) \nleq p$ and a finite subfamily $\mathcal{D}_0$ of $\mathcal{D}$ such that $(\forall z_1 \in X)$ $d(z_1) = 0$ or $g(z_1)=0$ for every $d\in\mathcal{D}-\mathcal{D}_0$.

Let $\mathcal{H}_0 = \left(\pi_1^{-1}(d)\wedge\pi_2^{-1}(h_{x_d y_i(x_d)})\right)_{\substack{d\in\mathcal{D}_0 \\ i\in\{1,\dots,m(x_d)\}}}$

and let $r = \pi_1^{-1}(g)$.

We have $r(x_0,y_0) \nleq p$ and $(\forall z\in X\times Y) h(z)=0$ or $r(z)=0$ for every $h\in\mathcal{H}-\mathcal{H}_0$. In fact, $r(z_0,y_0) = \pi_1^{-1}(g)(x_0,y_0) = g(x_0) \nleq p$. We also have that if $z=(z_1,z_2)\in X\times Y$ and $h(z)\neq 0$ then $h(z) = \left(\pi_1^{-1}(d)\wedge\pi_2^{-1}(h_{x_d y_i(x_d)})\right)(z) =$

$d(z_1)\wedge h_{x_d y_i(x_d)}(z_2)\neq 0$. So, $d(z_1)\neq 0$ and $h_{x_d y_i(x_d)}(z_2)$ $\neq 0$ which implies that $g(z_1)=0$ for every $d\in\mathcal{D}-\mathcal{D}_0$, hence $r(z)=0$ for every $h\in\mathcal{H}-\mathcal{H}_0$. Thus, $\mathcal{H}$ is also locally finite.

Hence $X\times Y$ is paracompact.

# Chapter VIII

## Some weaker forms of compactness

The aim of this chapter is to introduce good definitions of almost and near compactness in L-fuzzy topological spaces. These weak compactnesses are defined for arbitrary L-fuzzy sets and their properties studied.

In ordinary topology, near compactness was introduced by Singal and Mathur in [89] and almost compactness was studied by many authors such as Cameron [14]. A topological space $(X,\delta)$ is said to be almost compact (nearly compact) if and only if for every open cover $(A_i)_{i\in J}$ of X, there is a finite subset F of J with $\bigcup_{i\in F} cl(A_i) = X$ ($\bigcup_{i\in F} int(cl(A_i)) = X$).

Almost and near compactness were introduced and studied by several authors in [0,1]-fuzzy topological spaces. Some of them, such as Di Concilio and Gerla [28], Es [33], Mukherjee and Sinha [72] and Mukherjee and Ghosh [73], based their work on Chang's compactness (remark 4.6.2.), which is not a good extension of compactness [54]. Some others, such as Allam and Zahran [3] and Mashhour and al. [64], adopted $\alpha$-compactness (definition 4.1.1.). In [12], Bülbül and Warner used fuzzy compactness (definition 4.6.3.) to produce in [0,1]-fts good extensions of these dilutions.

In L-fuzzy topological spaces, where L is a fuzzy lattice, almost compactness was defined by Chen [20], by means of $\alpha$-nets.

This chapter is divided in three sections.

The first section contains the proposed definitions and their goodness.

The second section is reserved for other characterizations of these weak compactnesses.

And lastly, the third section is devoted to some properties.

## 1. Proposed definitions and their goodness

Definition 8.1.1.

Let $(X,\mathcal{T})$ be an L-fts and let $g \in L^X$. The L-fuzzy set g is said to be almost compact if and only if for every $p \in pr(L)$ and every collection $(f_i)_{i\in J}$ of open L-fuzzy sets with $\left(\bigvee_{i\in J} f_i\right)(x) \nleq p$ for all $x \in X$ with $g(x) \geq p'$, there exists a finite subset F of J with $\left(\bigvee_{i\in F} cl(f_i)\right)(x) \nleq p$ for all $x \in X$ with $g(x) \geq p'$.

If g is the whole space, then we say that the L-fts $(X,\mathcal{T})$ is almost compact.

Definition 8.1.2.

Let $(X,\mathcal{T})$ be an L-fts and let $g \in L^X$. The L-fuzzy set g is said to be nearly compact if and only if for every $p \in pr(L)$ and every collection $(f_i)_{i\in J}$ of open L-fuzzy sets with $\left(\bigvee_{i\in J} f_i\right)(x) \nleq p$ for all $x \in X$ with $g(x) \geq p'$, there exists a finite subset F of J with $\left(\bigvee_{i\in F} int\ (cl(f_i))\right)(x) \nleq p$ for all $x \in X$ with $g(x) \geq p'$.

If g is the whole space, then we say that the L-fts $(X,\mathcal{T})$ is nearly compact.

Theorem 8.1.3. (The goodness of almost compactness)

Let $(X,\delta)$ be a topological space. Then $(X,\delta)$ is almost compact if and only if $(X,\omega(\delta))$ is an almost compact L-fts.

Proof

Necessity:

Let $p \in pr(L)$ and $(f_i)_{i \in J} = \mathcal{A}$ be a family of basic open L-fuzzy sets in $(X,\omega(\delta))$ with $\left(\bigvee_{i \in J} f_i\right)(x) \not\leq p$ for all $x \in X$. Thus, by proposition 3.2.11., for each $i \in J$, consider $f_i(x) = f_i^{e_i U_i}(x) = \begin{cases} e_i & \text{if } x \in U_i \in \mathcal{T} \\ 0 & \text{otherwise} \end{cases}$.

Since $\left(\bigvee_{i \in J} f_i\right)(x) \not\leq p$ for all $x \in X$, for each $x \in X$ there is $i \in J$ such that $f_i^{e_i U_i}(x) \not\leq p$, i.e., $e_i \not\leq p$.

Let $\mathcal{C} = \{U_i;$ there is $i \in J$ with $p \not\geq e_i$ and $f_i^{e_i U_i} \in \mathcal{A}\}$.

Thus, $\mathcal{C}$ is a family of open sets covering $(X,\delta)$.

From the almost compactness of $(X,\delta)$, there is a finite subfamily $\mathcal{B}$ of $\mathcal{C}$, say $\{U_1, \ldots, U_m\}$ such that $\bigcup_{i \in \{1, \ldots, m\}} cl(U_i) = X$.

Since, by proposition 3.2.13., $cl(f_i)(x) = \begin{cases} e_i & \text{if } x \in cl(U_i) \\ 0 & \text{otherwise} \end{cases}$, we have $\left(\bigvee_{i \in \{1, \ldots, m\}} cl(f_i)\right)(x) \not\leq p$ for all $x \in X$.

Hence $(X,\omega(\delta))$ is almost compact.

Sufficiency:

Let $(A_i)_{i \in J}$ be an open cover of $(X,\delta)$.

Thus, by proposition 3.2.10., $(\chi_{A_i})_{i \in J}$ is a family of open L-fuzzy sets in $(X,\omega(\delta))$. We also have $\left(\bigvee_{i \in J} \chi_{A_i}\right)(x) = 1 \not\leq p$ for all $x \in X$ and for all $p \in pr(L)$.

From the almost compactness of $(X,\omega(\delta))$ there exists a finite subset F of J with $\left(\bigvee_{i \in F} cl(\chi_{A_i})\right)(x) = 1 \not\leq p$ for all $x \in X$ and for all $p \in pr(L)$.

Since, by proposition 3.2.13. $cl(\chi_{A_i}) = \chi_{cl(A_i)}$, we have $\left(\bigvee_{i \in F} \chi_{cl(A_i)}\right)(x) = 1$ for all $x \in X$. So, $\bigcup_{i \in F} cl(A_i) = X$.

Hence $(X,\delta)$ is almost compact.

Theorem 8.1.4. (The goodness of near compactness)

Let $(X,\delta)$ be a topological space. Then $(X,\delta)$ is nearly compact if and only if the L-fts $(X,\omega(\delta))$ is nearly compact.

Proof

By using corollary 3.2.14., this is similar to the proof of theorem 8.1.3.

## 2. Other characterizations

Proposition 8.2.1.

Let $(X,\mathcal{T})$ be an L-fts. Then $g\in L^X$ is almost compact if and only if for every $\alpha\in M(L)$ and every family $(f_i)_{i\in J}$ of closed L-fuzzy sets with $\left(\bigwedge_{i\in J} f_i\right)(x)\not\geq\alpha$ for all $x\in X$ with $g(x)\geq\alpha$, there exists a finite subset F of J with $\left(\bigwedge_{i\in F} \mathrm{int}(f_i)\right)(x)\not\geq\alpha$ for all $x\in X$ with $g(x)\geq\alpha$.

Proof

This immediately follows from the definition and remark 3.1.6.

Theorem 8.2.2.

Let $(X,\mathcal{T})$ be an L-fts. Then $g\in L^X$ is almost compact if and only if every constant $\alpha$-net $(S_m)_{m\in D}$ contained in g, has a $\theta$-cluster point (definition 3.1.9.(iii) $x_\alpha\in M(L^X)$, with height $\alpha$, contained in g, for each $\alpha\in M(L)$.

Proof

This is similar to the proof of theorem 4.4.2.

Theorem 8.2.3.

Let $(X,\mathcal{T})$ be an L-fts. Then $g\in L^X$ is almost compact if and only if for every $p\in pr(L)$ and every collection $(f_i)_{i\in J}$ of regularly open L-fuzzy sets with $\left(\bigvee_{i\in J} f_i\right)(x)\not\leq p$ for all $x\in X$ with $g(x)\geq p'$, there is a finite subset F of J with $\left(\bigvee_{i\in F} \mathrm{cl}(f_i)\right)(x)\not\leq p$ for all $x\in X$ with $g(x)\geq p'$.

Proof

Necessity:

Let $p \in pr(L)$ and let $(f_i)_{i \in J}$ be a collection of regularly open L-fuzzy sets (definition 3.1.7.(i)) with $\left(\bigvee_{i \in J} f_i\right)(x) \nleq p$ for all $x \in X$ with $g(x) \geq p'$.

Since each $f_i$ is a regularly open L-fuzzy set $f_i \in \mathcal{T}$ for each $i \in J$. Then by the almost compactness of g, there is a finite subset F of J with $\left(\bigvee_{i \in F} cl(f_i)\right)(x) \nleq p$ for all $x \in X$ with $g(x) \geq p'$.

Sufficiency:

Let $p \in pr(L)$ and let $(f_i)_{i \in F}$ be a collection of open L-fuzzy sets with $\left(\bigvee_{i \in J} f_i\right)(x) \nleq p$ for all $x \in X$ with $g(x) \geq p'$.

From remark 3.1.8. (iv), $int(cl(f_i))$ is a regularly open L-fuzzy set for each $i \in J$. Then, by our hypothesis, there exists a finite subset F of J with $\left(\bigvee_{i \in F} cl(int(cl(f_i)))\right)(x) \nleq p$ for all $x \in X$ with $g(x) \geq p'$. From remark 3.1.8. (iii) $cl(f_i)$ is a regularly closed L-fuzzy set, so $cl(f_i) = cl(int(cl(f_i)))$. Therefore $\left(\bigvee_{i \in F} cl(f_i)\right)(x) \nleq p$ for all $x \in X$ with $g(x) \geq p'$.

Hence $(X, \mathcal{T})$ is almost compact.

Proposition 8.2.4.

Let $(X, \mathcal{T})$ be an L-fts. Then $g \in L^X$ is nearly compact if and only if for every $\alpha \in M(L)$ and every family $(f_i)_{i \in J}$ of closed L-fuzzy sets with $\left(\bigwedge_{i \in J} f_i\right)(x) \ngeq \alpha$ for all $x \in X$ with $g(x) \geq \alpha$, there exists a finite subset F of J with $\left(\bigwedge_{i \in F} cl(int(f_i))\right)(x) \ngeq \alpha$ for all $x \in X$ with $g(x) \geq \alpha$.

Proof

This immediately follows from the definition and remark 3.1.6.

Theorem 8.2.5.

Let $(X,\mathcal{T})$ be an L-fts. Then $g \in L^X$ is nearly compact if and only if every constant $\alpha$-net $(S_m)_{m\in D}$ contained in g has a $\delta$-cluster point (definition 3.1.9. (iv)) $x_\alpha \in M(L^X)$, with height $\alpha$, contained in g, for each $\alpha \in M(L)$.

Proof

This is similar to the proof of theorem 4.4.2.

Theorem 8.2.6.

Let $(X,\mathcal{T})$ be an L-fts. Then $g \in L^X$ is nearly compact if and only if for all $p \in pr(L)$ and every collection $(f_i)_{i\in J}$ of regularly open L-fuzzy sets with $\left(\bigvee_{i\in J} f_i\right)(x) \nleq p$ for all $x \in X$ with $g(x) \geq p'$, there exists a finite subset F of J with $\left(\bigvee_{i\in F} f_i\right)(x) \nleq p$ for all $x \in X$ with $g(x) \geq p'$.

Proof

This is similar to the proof of theorem 8.2.3.

## 3. Some properties

Proposition 8.3.1.

Let $(X,\mathcal{T})$ be an L-fts and let g,h be almost compact L-fuzzy sets. Then $g\vee h$ is almost compact.

Proof

This is similar to the proof of proposition 4.1.10.

Proposition 8.3.2.

Let $(X,\mathcal{T})$ be an L-fts, let g be an almost compact L-fuzzy set and h be a clopen L-fuzzy set. Then $h\wedge g$ is almost compact.

Proof

This is similar to the proof of proposition 4.1.12.

Corollary 8.3.3.

Let $(X.\mathcal{T})$ be an almost compact L-fts. Then each clopen L-fuzzy set is almost compact in $(X,\mathcal{T})$.

Proof

This immediately follows from proposition 8.3.2.

Proposition 8.3.4.

Let $(X,\mathcal{T}_X)$ and $(Y,\mathcal{T}_Y)$ be L-fts's; let $f:(X,\mathcal{T}_X)\rightarrow(Y,\mathcal{T}_Y)$ be an almost continuous mapping such that $f^{-1}(y)$ is finite for every $y\in Y$ and let g be an almost compact L-fuzzy set of $(X,\mathcal{T}_X)$. Then f(g) is an almost compact L-fuzzy set of $(Y,\mathcal{T}_Y)$.

Proof

Let $p \in pr(L)$ and let $(f_i)_{i\in J}$ be a family of regularly open L-fuzzy sets of $(Y,\mathcal{T}_Y)$ with $\left(\bigvee_{i\in J} f_i\right)(y) \not\geq p$ for all $y\in Y$ with $f(g)(y) \geq p'$.

Thus, from the almost continuity of f, $(f^{-1}(f_i))_{i\in J}$ is a family of open L-fuzzy sets in $(X,\mathcal{T}_X)$. We also have $\left(\bigvee_{i\in J} f^{-1}(f_i)\right)(x) \not\geq p$ for all $x\in X$ with $g(x)\geq p'$ because if $g(x)\geq p'$ then $f(g)(f(x))\geq p'$, so $\left(\bigvee_{i\in J} f^{-1}(f_i)\right)(x) = \left(\bigvee_{i\in J} f_i\right)(f(x)) \not\geq p$.

From the almost compactness of g in $(X,\mathcal{T}_X)$, there exists a finite subset F of J with $\left(\bigvee_{i\in F} cl(f^{-1}(f_i)\right)(x) \not\geq p$ for all $x\in X$ with $g(x)\geq p'$.

Therefore $\left(\bigvee_{i\in F} cl(f_i)\right)(y) \not\geq p$ for all $y\in Y$ with $f(g)(y)\geq p'$. In fact, if $f(g)(y)\geq p'$ then $\bigvee_{x\in f^{-1}(y)} \{g(x)\} \geq p'$ which implies that there is $x\in X$ with $g(x)\geq p'$ and $f(x)=y$. So, $\left(\bigvee_{i\in F} cl(f_i)\right)(y) = \left(\bigvee_{i\in F} cl(f_i)\right)(f(x)) = \left(\bigvee_{i\in F} f^{-1}(cl(f_i))\right)(x) = \left(\bigvee_{i\in F} cl(f^{-1}(cl(f_i)))\right)(x) \geq \left(\bigvee_{i\in F} cl(f^{-1}(f_i))\right)(x) \not\geq p$ where the last equality is due to the fact that $cl(f_i)$ is regularly closed by remark 3.1.8. (iii) and then $f^{-1}(cl(f_i))$ is closed by the almost continuity of f.

Hence f(g) is almost compact by theorem 8.2.3.

When $g=\chi_A$ for some $A\subseteq X$, $f^{-1}(y)$ does not need to be finite.

Proposition 8.3.5.

Let $(X,\mathcal{T}_X)$ and $(Y,\mathcal{T}_Y)$ be L-fts's, let $f:(X,\mathcal{T}_X)\rightarrow(Y,\mathcal{T}_Y)$ be a weakly continuous mapping such that

$f^{-1}(y)$ is finite for every $y \in Y$ and let g be a compact L-fuzzy set of $(X, \mathcal{T}_X)$. Then $f(g)$ is almost compact in $(Y, \mathcal{T}_Y)$.

<u>Proof</u>

Let $p \in pr(L)$ and let $(f_i)_{i \in J}$ be a family of open L-fuzzy sets in $(Y, \mathcal{T}_Y)$ with $\left(\bigvee_{i \in J} f_i\right)(y) \nleq p$ for all $y \in Y$ with $f(g)(y) \geq p'$.

Thus, $\left(\bigvee_{i \in J} f^{-1}(f_i)\right)(x) \nleq p$ for all $x \in X$ with $g(x) \geq p'$.

By the weak continuity of f (definition 3.3.1. (iv)), $f^{-1}(f_i) \leq int(f^{-1}(cl(f_i)))$. Then $\left(\bigvee_{i \in J} int(f^{-1}(cl(f_i)))\right)(x) \nleq p$ for all $x \in X$ with $g(x) \geq p'$. Since g is compact, there is a finite subset F of J with $\left(\bigvee_{i \in F} int(f^{-1}(cl(f_i)))\right)(x) \nleq p$ for all $x \in X$ with $g(x) \geq p'$.

Therefore $\left(\bigvee_{i \in F} cl(f_i)\right)(y) \nleq p$ for all $y \in Y$ with $f(g)(y) \geq p'$. In fact, if $f(g)(y) \geq p'$ then $\bigvee_{x \in f^{-1}(y)} \{g(x)\} \geq p'$ which implies that there is $x \in X$ with $f(x) = y$ and $g(x) \geq p'$. So, $\left(\bigvee_{i \in F} cl(f_i)\right)(y) = \left(\bigvee_{i \in F} cl(f_i)\right)(f(x)) = \left(\bigvee_{i \in F} f^{-1}(cl(f_i))\right)(x) \geq \left(\bigvee_{i \in F} int(f^{-1}(cl(f_i)))\right)(x) \nleq p$.

Hence $f(g)$ is almost compact.

When $g = \chi_A$ for some $A \subseteq X$, $f^{-1}(y)$ does not need to be finite.

<u>Proposition 8.3.6.</u>

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be L-fts's, let $f: (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ be a strongly continuous mapping such that $f^{-1}(y)$ is finite for every $y \in Y$ and let g be an almost compact L-fuzzy set of $(X, \mathcal{T}_X)$. Then $f(g)$ is compact in $(Y,\mathcal{T}_Y)$.

Proof

Let $p\in pr(L)$ and let $(f_i)_{i\in J}$ be a family of open L-fuzzy sets in $(Y,\mathcal{T}_Y)$ with $\left(\bigvee_{i\in J} f_i\right)(y)\not\leq p$ for all $y\in Y$ with $f(g)(y)\geq p'$.

Since f is strongly continuous (definition 3.3.1. (vii)), it is continuous as well. So, $(f^{-1}(f_i))_{i\in J}$ is a family of open L-fuzzy sets in $(X,\mathcal{T}_X)$ and we also have $\left(\bigvee_{i\in J} f^{-1}(f_i)\right)(x)\not\leq p$ for all $x\in X$ with $g(x)\geq p'$.

From the almost compactness of g, there is a finite subset F of J with $\left(\bigvee_{i\in F} cl(f^{-1}(f_i))\right)(x)\not\leq p$ for all $x\in X$ with $g(x)\geq p'$. Then $p\not\geq f\left(\bigvee_{i\in F} cl(f^{-1}(f_i))\right)(y) =$
$\left(\bigvee_{i\in F} f(cl(f^{-1}(f_i)))\right)(y) \leq \left(\bigvee_{i\in F} f(f^{-1}(f_i))\right)(y) \leq \left(\bigvee_{i\in F} f_i\right)(y)$
for all $y\in Y$ with $f(g)(y)\geq p'$, where the inequality before the last one is due to the strong continuity of f.

Hence f(g) is compact in $(Y,\mathcal{T}_Y)$.

When $g=\chi_A$ for some $A\subseteq X$, $f^{-1}(y)$ does not need to be finite.

Proposition 8.3.7.

Let $(X,\mathcal{T})$ be an L-fts and let g and h be nearly compact L-fuzzy sets. Then $g\vee h$ is nearly compact.

Proof

This is similar to the proof of proposition 4.1.10.

Proposition 8.3.8.

Let $(X,\mathcal{T})$ be an L-fts and let g be a nearly compact L-fuzzy set and let h be a regularly closed (definition 3.1.7. (ii)) L-fuzzy set. Then $h \wedge g$ is nearly compact.

Proof

This is similar to the proof of proposition 4.1.12.

Corollary 8.3.9.

Let $(X,\mathcal{T})$ be a nearly compact L-fts. Then each regularly closed L-fuzzy set is nearly compact.

Proof

This immediately follows from proposition 8.3.8.

Corollary 8.3.10.

Let $(X,\mathcal{T})$ be an L-fts and let g be nearly compact L-fuzzy set and let h be a clopen L-fuzzy set. Then $h \wedge g$ is nearly compact.

Proof

This immediately follows from proposition 8.3.8. since h clopen implies h regularly closed.

Corollary 8.3.11.

Let $(X,\mathcal{T})$ be a nearly compact L-fts. Then each clopen L-fuzzy set is nearly compact.

Proof

This immediately follows from corollary 8.3.10.

Proposition 8.3.12.

Let $(X,\mathcal{T}_X)$ and $(Y,\mathcal{T}_Y)$ be L-fts's, let $f:(X,\mathcal{T}_X)\to(Y,\mathcal{T}_Y)$ be an almost continuous (definition 3.3.1.(iii)), almost open (definition 3.3.1.(vi)) mapping such that $f^{-1}(y)$ is finite for every $y\in Y$ and let g be a nearly compact L-fuzzy set of $(X,\mathcal{T}_X)$. Then f(g) is nearly compact in $(Y,\mathcal{T}_Y)$.

Proof

Let $p\in pr(L)$ and $(f_i)_{i\in J}$ be a family of regularly open L-fuzzy sets in $(Y,\mathcal{T}_Y)$ with $\left(\bigvee_{i\in J} f_i\right)(y)\nleq p$ for all $y\in Y$ with $f(g)(y)\geq p'$.

By proposition 3.3.4., $(f^{-1}(f_i))_{i\in J}$ is a family of regularly open L-fuzzy sets in $(X,\mathcal{T}_X)$. We also have $\left(\bigvee_{i\in J} f^{-1}(f_i)\right)(x)\nleq p$ for all $x\in X$ with $g(x)\geq p'$.

From the near compactness of g and theorem 8.2.6., there exists a finite subset F of J with $\left(\bigvee_{i\in F} f^{-1}(f_i)\right)(x)\nleq p$ for all $x\in X$ with $g(x)\geq p'$.

Thus, $\left(\bigvee_{i\in F} f_i\right)(y)\nleq p$ for all $y\in Y$ with $f(g)(y)\geq p'$. In fact, if $f(g)(y)\geq p'$ then $\bigvee_{x\in\{f^{-1}(y)\}}\{g(x)\}\geq p'$ which implies that there is $x\in X$ with $g(x)\geq p'$ and $f(x)=y$. So, $\left(\bigvee_{i\in F} f_i\right)(y) = \left(\bigvee_{i\in F} f_i\right)(f(x)) = \left(\bigvee_{i\in F} f^{-1}(f_i)\right)(x)\nleq p$.

Hence by theorem 8.2.6. f(g) is nearly compact.

When $g=\chi_A$ for some $A\subseteq X$, $f^{-1}(y)$ does not need to be finite.

Proposition 8.3.13.

Let $(X,\mathcal{T}_X)$ and $(Y,\mathcal{T}_Y)$ be L-fts's, let $f:(X,\mathcal{T}_X)\to(Y,\mathcal{T}_Y)$ be an almost continuous mapping with

$f^{-1}(cl(h)) \leq cl(f^{-1}(h))$ for all $h \in \mathcal{T}_Y$ such that $f^{-1}(y)$ is finite for every $y \in Y$ and let g be a nearly compact L-fuzzy set of $(X, \mathcal{T}_X)$. Then f(g) is nearly compact in $(Y, \mathcal{T}_Y)$.

Proof

By using theorem 8.2.6. and proposition 3.3.5., this follows as in proposition 8.3.12.

When $g = \chi_A$ for some $A \subseteq X$, $f^{-1}(y)$ does not need to be finite.

Proposition 8.3.14.

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be L-fts's, let $f:(X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ be a weakly continuous mapping such that $f^{-1}(y)$ is finite for every $y \in Y$, with $f^{-1}(cl(h)) \leq cl(f^{-1}(h))$ for every regularly open L-fuzzy sets h in $(Y, \mathcal{T}_Y)$ and let g be a nearly compact L-fuzzy set of $(X, \mathcal{T}_X)$. Then f(g) is nearly compact in $(Y, \mathcal{T}_Y)$.

Proof

By using theorem 8.2.6. and proposition 3.3.6., this follows as in proposition 8.3.12.

When $g = \chi_A$ for some $A \subseteq X$, $f^{-1}(y)$ does not need to be finite.

Proposition 8.3.15.

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be L-fts's, let

$f:(X,\mathcal{T}_X)\to(Y,\mathcal{T}_Y)$ be an almost continuous mapping such that $f^{-1}(y)$ is finite for every $y\in Y$ and let g be a compact L-fuzzy set of $(X,\mathcal{T}_X)$. Then f(g) is nearly compact in $(Y,\mathcal{T}_Y)$.

<u>Proof</u>

Let $p\in pr(L)$ and let $(f_i)_{i\in J}$ be a family of regularly open L-fuzzy sets in $(Y,\mathcal{T}_Y)$ with $\left(\bigvee_{i\in J} f_i\right)(y)\not\leq p$ for all $y\in Y$ with $f(g)(y)\geq p'$.

From the almost continuity of f, $(f^{-1}(f_i))_{i\in J}$ is a family of open L-fuzzy sets in $(X,\mathcal{T}_X)$. We also have $\left(\bigvee_{i\in J} f^{-1}(f_i)\right)(x)\not\leq p$ for all $x\in X$ with $g(x)\geq p'$ because if $g(x)\geq p'$ then $f(g)(f(x))\geq p'$, so $\left(\bigvee_{i\in J} f^{-1}(f_i)\right)(x) = \left(\bigvee_{i\in J} f_i\right)(f(x))\not\leq p$.

By the compactness of g, there is a finite subset F of J with $\left(\bigvee_{i\in F} f^{-1}(f_i)\right)(x)\not\leq p$ for all $x\in X$ with $g(x)\geq p'$.

Then $\left(\bigvee_{i\in F} f_i\right)(y)\not\leq p$ for all $y\in Y$ with $f(g)(y)\geq p'$.

Hence by theorem 8.2.6. f(g) is nearly compact.

When $g=\chi_A$ for some $A\subseteq X$, $f^{-1}(y)$ does not need to be finite.

# Chapter IX

## S-closedness in L-fuzzy topological spaces

In this chapter we introduce S-closedness in L-fuzzy topological spaces. S-closedness is defined for arbitrary L-fuzzy sets and is a good extension. We give other characterizations of S-closedness and study some of its properties.

In ordinary topology, S-closedness was introduced by Thompson in [93] and also studied by several authors [14, 46, 63, 76, 94]. A topological space $(X,\delta)$ is said to be S-closed if and only if for every semiopen cover $(A_i)_{i\in J}$ of X, there is a finite subset F of J with $\bigcup_{i\in F} cl(A_i)=X$.

S-closedness was introduced and studied in [0,1]-fuzzy topological spaces by Mashhour, Ghanim and Fath Alla in [64]. In their work they adopted $\alpha$-compactness (definition 4.1.1.) and defined $\alpha$S-closed fuzzy spaces. In [4], Allam and Zahran extended $\alpha$S-closedness to arbitrary fuzzy sets.

In [23], Coker and Es, considering Chang's compactness (remark 4.6.2.), defined S-closed [0,1]-fuzzy topological spaces.

Bülbül and Warner in [12], using fuzzy compactness (definition 4.6.3.), presented a good definition of S-closed [0,1]-fuzzy topological spaces.

This chapter contains three sections.

In the first section we present our definition and establish its goodness.

In the second section we obtain some other characterizations of the proposed S-closedness.

The third section focuses on some properties.

## 1. **Proposed definition and its goodness**

Definition 9.1.1.

Let $(X,\mathcal{T})$ be an L-fts and let $g \in L^X$. The L-fuzzy set $g$ is said to be S-closed if and only if for every $p \in pr(L)$ and every collection $(f_i)_{i\in J}$ of semiopen L-fuzzy sets (definition 3.1.7. (iii)) with $\left(\bigvee_{i\in J} f_i\right)(x) \not\leq p$ for all $x \in X$ with $g(x) \geq p'$, there exists a finite subset $F$ of $J$ with $\left(\bigvee_{i\in F} cl(f_i)\right)(x) \not\leq p$ for all $x \in X$ with $g(x) \geq p'$.

If $g$ is the whole space, then we say that the L-fts $(X,\mathcal{T})$ is S-closed.

Theorem 9.1.2. (The goodness of S-closedness)

Let $(X,\delta)$ be a topological space. Then $(X,\delta)$ is S-closed if and only if the L-fts $(X,\omega(\delta))$ is S-closed.

Proof

Necessity:

By using proposition 3.2.18. and proposition 3.2.13., this is similar to the proof of the necessity of theorem 8.1.3.

Sufficiency:

By using proposition 3.2.15. and proposition 3.2.13. this is similar to the proof of the sufficiency of theorem 8.1.3.

## 2. Other characterizations

Theorem 9.2.1.

Let $(X,\mathcal{T})$ be an L-fts and let $g\in L^X$. Then g is S-closed if and only if for all $p\in pr(L)$ and every collection $(f_i)_{i\in J}$ of regularly closed L-fuzzy sets with $\left(\bigvee_{i\in J} f_i\right)(x)\not\leq p$ for all $x\in X$ with $g(x)\geq p'$, there is a finite subset F of J with $\left(\bigvee_{i\in F} f_i\right)(x)\not\leq p$ for all $x\in X$ with $g(x)\geq p'$.

Proof

Necessity:

Let $p\in pr(L)$ and let $(f_i)_{i\in J}$ be a collection of regularly closed L-fuzzy sets (definition 3.1.7. (ii)) with $\left(\bigvee_{i\in J} f_i\right)(x)\not\leq p$ for all $x\in X$ with $g(x)\geq p'$.

Since every regularly closed L-fuzzy set is a regularly semiopen L-fuzzy set because if $f=cl(int(f_i))$ then $int(f_i)\leq f_i\leq cl(int(f_i))$, we have that $f_i$ is a regularly semiopen L-fuzzy set for each $i\in J$.

Therefore, from the S-closedness of g there exists a finite subset F of J with $\left(\bigvee_{i\in F} cl(f_i)\right)(x)\not\leq p$ for all $x\in X$ with $g(x)\geq p'$. Since every regularly closed L-fuzzy set is closed, then $\left(\bigvee_{i\in F} f_i\right)(x)\not\leq p$ for all $x\in X$ with $g(x)\geq p'$.

Sufficiency:

Let $p\in pr(L)$ and let $(f_i)_{i\in J}$ be a collection of semiopen L-fuzzy sets with $\left(\bigvee_{i\in J} f_i\right)(x)\not\leq p$ for all $x\in X$ with $g(x)\geq p'$.

Thus, for each $i\in J$ there exists $h_i\in\mathcal{T}$ such that $h_i\leq f_i\leq cl(h_i)$. So we have $cl(f_i)=cl(h_i)$ and by remark

3.1.8. (iii) $cl(h_i)$ is regularly closed.

Since $f_i \leq cl(h_i)$ for every $i \in J$ and $\left(\bigvee_{i \in J} f_i\right)(x) \not\geq p$ for all $x \in X$ with $g(x) \geq P'$, we have $\left(\bigvee_{i \in J} cl(h_i)\right)(x) \not\geq p$ for all $x \in X$ with $g(x) \geq p'$.

Thus, by hypothesis, there exists a finite subset F of J with $\left(\bigvee_{i \in F} cl(h_i)\right)(x) \not\geq p$ for all $x \in X$ with $g(x) \geq p'$, i.e., $\left(\bigvee_{i \in F} cl(f_i)\right)(x) \not\geq p$ for all $x \in X$ with $g(x) \geq p'$.

Hence g is S-closed.

Theorem 9.2.2.

Let $(X, \mathcal{T})$ be an L-fts and let $g \in L^X$. The L-fuzzy set g is S-closed if and only if for all $p \in pr(L)$ and every collection $(f_i)_{i \in J}$ of regularly semiopen L-fuzzy sets with $\left(\bigvee_{i \in J} f_i\right)(x) \not\geq p$ for all $x \in X$ with $g(x) \geq p'$, there exists a finite subset F of J with $\left(\bigvee_{i \in J} cl(f_i)\right)(x) \not\geq p$ for all $x \in X$ with $g(x) \geq p'$.

Proof

Necessity:

Since every regularly open L-fuzzy set is an open L-fuzzy set, we have that every regularly semiopen L-fuzzy set is a semiopen L-fuzzy set. Hence the result follows immediately.

Sufficiency:

Let $p \in pr(L)$ and let $(f_i)_{i \in J}$ be a collection of semiopen L-fuzzy sets with $\left(\bigvee_{i \in J} f_i\right)(x) \not\geq p$ for all $x \in X$ with $g(x) \geq p'$.

Thus, for each $i \in J$ there exists $h_i \in \mathcal{T}$ such that

$h_i \leq f_i \leq cl(h_i)$. Then, $int(cl(h_i)) \leq int(cl(f_i)) \leq int(cl(h_i)) \leq cl(int(cl(h_i))) = cl(h_i)$, the last equality is due to the fact that from remark 3.1.8. (iii) $cl(h_i)$ is a regularly closed L-fuzzy set. Thus, from $f_i \leq cl(h_i)$ and $int(cl(f_i)) \leq cl(h_i)$ we have $(int(cl(f_i))) \vee f_i \leq cl(h_i)$ and from $int(cl(h_i)) \leq int(cl(f_i)) \leq (int(cl(f_i))) \vee f_i$ we have $int(cl(h_i)) \leq (int(cl(f_i))) \vee f_i \leq cl(h_i) = cl(int(cl(h_i)))$. Then, $(int(cl(f_i))) \vee f_i$ is a regularly semiopen L-fuzzy set for every $i \in J$ with $\left( \bigvee_{i \in J} (int(cl(f_i))) \vee f_i \right)(x) \not\leq p$ for all $x \in X$ such that $g(x) \geq p'$. So, by hypothesis, there is a finite subset F of J with $\left( \bigvee_{i \in F} cl(int(cl(f_i))) \vee f_i \right)(x) \not\leq p$ for all $x \in X$ such that $g(x) \geq p'$. Then $\left( \bigvee_{i \in F} cl(int(cl(f_i))) \vee cl(f_i) \right)(x) \not\leq p$ for all $x \in X$ with $g(x) \geq p'$. Since $cl(int(cl(f_i))) \vee cl(f_i) = cl(f_i)$, we have $\left( \bigvee_{i \in F} cl(f_i) \right)(x) \not\leq p$ for all $x \in X$ with $g(x) \geq p'$.

Hence g is S-closed.

<u>Proposition 9.2.3.</u>

Let $(X, \mathcal{T})$ be an L-fts. Then $g \in L^X$ is an S-closed L-fuzzy set if and only if for all $\alpha \in M(L)$ and for every collection $(f_i)_{i \in J}$ of semiclosed (definition 3.1.7. (iv)) L-fuzzy sets with $\left( \bigwedge_{i \in J} f_i \right)(x) \geq \alpha$ for all $x \in X$ with $g(x) \geq \alpha$, there exists a finite subset F of J with $\left( \bigwedge_{i \in F} int(f_i) \right)(x) \not\geq \alpha$ for all $x \in X$ with $g(x) \geq \alpha$.

<u>Proof</u>

This immediately follows from the definition.

Theorem 9.2.4.

Let $(X,\mathcal{T})$ be an L-fts. Then $g\in L^X$ is an S-closed L-fuzzy set if and only if every constant $\alpha$-net $(S_m)_{m\in D}$ contained in g, has a semi-$\theta$-cluster point (definition 3.1.9. (vi)) $x_\alpha \in M(L^X)$, with height $\alpha$, contained in g, for each $\alpha\in M(L)$.

Proof

This is similar to the proof of theorem 4.4.2.

## 3. Some properties

Proposition 9.3.1.

Let $(X,\mathcal{T})$ be an L-fts and let g and h be S-closed L-fuzzy sets. Then $h \vee g$ is S-closed as well.

Proof

This is similar to the proof of proposition 4.1.10.

Corollary 9.3.2.

Let $(X,\mathcal{T})$ be an L-fts. Every L-fuzzy set g with finite support is S-closed.

Proof

By using proposition 9.3.1., this is similar to the proof of corollary 4.1.11.

Proposition 9.3.3.

Let $(X,\mathcal{T})$ be an L-fts and let g be an S-closed L-fuzzy set. Then for each regularly open L-fuzzy set h, $h \wedge g$ is S-closed.

Proof

By using theorem 9.2.1., this is similar to the proof of proposition 4.1.12.

Corollary 9.3.4.

Let $(X,\mathcal{T})$ be an S-closed L-fts. Then each regularly open L-fuzzy set is S-closed in $(X,\mathcal{T})$.

Proof

This immediately follows from proposition 9.3.3.

Corollary 9.3.5.

Let $(X,\mathcal{T})$ be an L-fts. If g is an S-closed L-fuzzy set, then for each clopen L-fuzzy set h, $h \wedge g$ is S-closed.

Proof

This follows from proposition 9.3.3. because if h is clopen, h=int(h)=cl(h) then, int(cl(h))=h, i.e., h is regularly open.

Proposition 9.3.6.

Let $(X,\mathcal{T}_X)$ and $(Y,\mathcal{T}_Y)$ be L-fts's. Let $f:(X,\mathcal{T}_X)\rightarrow(Y,\mathcal{T}_Y)$ be an almost continuous (definition 3.3.1. (iii)), almost open (definition 3.3.1. (vi)) mapping such that $f^{-1}(y)$ is finite for every $y \in Y$ and let g be an S-closed L-fuzzy set in $(X,\mathcal{T}_X)$. Then f(g) is an S-closed L-fuzzy set in $(Y,\mathcal{T}_Y)$.

Proof

By using theorem 9.2.1. and proposition 3.3.4., this is similar to the proof of proposition 4.1.14.

When $g=\chi_A$ for some $A \subseteq X$, $f^{-1}(y)$ does not need to be finite.

Proposition 9.3.7.

Let $(X,\mathcal{T}_X)$ and $(Y,\mathcal{T}_Y)$ be L-fts's. Let

$f:(X,\mathcal{T}_X)\to(Y,\mathcal{T}_Y)$ be a weakly continuous mapping (definition 3.3.1. (iv)), such that $f^{-1}(y)$ is finite for every $y\in Y$, with $f^{-1}(cl(h))\le cl(f^{-1}(h))$ for every regularly open L-fuzzy set h in $(Y,\mathcal{T}_Y)$ and let g be an S-closed L-fuzzy set in $(X,\mathcal{T}_X)$. Then f(g) is S-closed in $(Y,\mathcal{T}_Y)$.

Proof

By using proposition 3.3.6. and theorem 9.2., this is similar to the proof of proposition 4.1.14.

When $g=\chi_A$ for some $A\subseteq X$, $f^{-1}(y)$ does not need to be finite.

Proposition 9.3.8.

Let $(X,\mathcal{T}_X)$ and $(Y,\mathcal{T}_Y)$ be L-fts's. Let $f:(X,\mathcal{T}_X)\to(Y,\mathcal{T}_Y)$ be an irresolute almost continuous (definition 3.3.1. (iii)(viii)) mapping such that $f^{-1}(y)$ is finite for every $y\in Y$ and let g be an S-closed L-fuzzy set in $(X,\mathcal{T}_X)$. Then f(g) is S-closed in $(Y,\mathcal{T}_Y)$.

Proof

Let $p\in pr(L)$ and let $(f_i)_{i\in J}$ be a family of semiopen L-fuzzy sets in $(Y,\mathcal{T}_Y)$ with $\left(\bigvee_{i\in J} f_i\right)(y)\nleq p$ for all $y\in Y$ with $f(g)(y)\ge p'$.

Since f is irresolute, $(f^{-1}(f_i))_{i\in J}$ is a family of semiopen L-fuzzy sets in $(X,\mathcal{T}_X)$. We also have $\left(\bigvee_{i\in J} f^{-1}(f_i)\right)(x)\nleq p$ for all $x\in X$ with $g(x)\ge p'$ because if $g(x)\ge p'$ then $f(g)(f(x)\ge p'$. So $\left(\bigvee_{i\in J} f^{-1}(f_i)\right)(x)$ $=\left(\bigvee_{i\in J} f_i\right)(f(x))\nleq p$.

From the S-closedness of g in $(X,\mathcal{T}_X)$, there exists a finite subset F of J with $\left(\bigvee_{i\in F} cl(f^{-1}(f_i))\right)(x) \not\geq p$ for all $x\in X$ with $g(x)\geq p'$.

Therefore, $\left(\bigvee_{i\in F} cl(f_i)\right)(y) \not\geq p$ for all $y\in Y$ with $f(g)(y)\geq p'$. In fact, if $f(g)(y)\geq p'$ then $\bigvee_{x\in f^{-1}(y)}\{g(x)\}\geq p'$ which implies that there is $x\in X$ with $g(x)\geq p'$ and $f(x)=y$. So, $\left(\bigvee_{i\in F} cl(f_i)\right)(y) = \left(\bigvee_{i\in F} cl(f_i)\right)(f(x))$ $= \left(\bigvee_{i\in F} f^{-1}(cl(f_i))\right)(x) \geq \left(\bigvee_{i\in F} cl(f^{-1}(f_i))\right)(x) \not\geq p$ where the last inequality is due to proposition 3.3.11.

Hence $f(g)$ is S-closed.

When $g=\chi_A$ for some $A\subseteq X$, $f^{-1}(y)$ does not need to be finite.

# Chapter X

## RS-compactness in L-fuzzy topological spaces

This chapter is reserved for RS-compactness in L-fuzzy topological spaces. We prove the goodness of the proposed definition, obtain different characterizations of it and study some of its properties.

In ordinary topology, RS-compactness has been introduced by Hong [43] and has also been studied by Noiri [77]. A topological space $(X,\delta)$ is said to be RS-compact if and only if every regularly semiopen cover of X has a finite subfamily whose interiors cover X [43]. In [77], Noiri claimed without proving that this is equivalent to every regularly closed cover of X has a finite subfamily whose interiors cover X. Similarly to our proof of theorem 10.2.2. one can prove that these are also equivalent to every semiopen cover of X has a finite subfamily whose interiors of closures cover X. Here we use this last characterization of RS-compactness to prove the goodness of our definition. In the same way as we proved our theorem 10.2.3. we can prove that ordinary RS-compactness is also equivalent to every regularly semiopen cover of X has a finite subfamily whose interiors of closures cover X.

In [0,1]-fuzzy topological spaces, RS-compactness was studied by Coker and Es [24]. Their definition is along the lines of Chang's compactness (remark 4.6.2.). Also in [0,1]-fuzzy topological spaces, Allam and Zahran, using the concept of $\alpha$-shading (definition 4.1.1.), suggested another version of this concept.

This chapter is divided in three sections.

In section 1 we introduce our definition and establish its goodness.

The second section contains other characterizations of RS-compactness.

In the third section we focus on some properties.

## 1. Proposed definition and its goodness

Definition 10.1.1.

Let $(X,\mathcal{T})$ be an L-fts and let $g \in L^X$. The L-fuzzy set g is said to be RS-compact if and only if for all $p \in pr(L)$ and every collection $(f_i)_{i \in J}$ of semiopen L-fuzzy sets (definition 3.1.7. (iii)) with $\left(\bigvee_{i \in J} f_i\right)(x) \not\leq p$ for all $x \in X$ with $g(x) \geq p'$, there is a finite subset F of J with $\left(\bigvee_{i \in F} int(cl(f_i))\right)(x) \not\leq p$ for all $x \in X$ with $g(x) \geq p'$.

If g is the whole space, then we say that the L-fts $(X,\mathcal{T})$ is RS-compact.

Theorem 10.1.2. (The goodness of RS-compactness)

Let $(X,\delta)$ be a topological space. Then $(X,\delta)$ is RS-compact if and only if the L-fts $(X,\omega(\delta))$ is RS-compact.

Proof

Necessity:

By using proposition 3.2.18. and corollary 3.2.14., this is similar to the proof of the necessity of theorem 8.1.3.

Sufficiency:

By using proposition 3.2.15. and corollary 3.2.14. this is similar to the proof of the sufficiency of theorem 8.1.3.

## 2. Other characterizations

Theorem 10.2.1.

Let $(X,\mathcal{T})$ be an L-fts and let $g \in L^X$. Then g is RS-compact if and only if for all $p \in pr(L)$ and every collection $(f_i)_{i \in J}$ of regularly closed L-fuzzy sets (definition 3.1.7. (ii)) with $\left(\bigvee_{i \in J} f_i\right)(x) \nleq p$ for all $x \in X$ with $g(x) \geq p'$, there is a finite subset F of J with $\left(\bigvee_{i \in F} int(f_i)\right)(x) \nleq p$ for all $x \in X$ with $g(x) \geq p'$.

Proof

This is similar to the proof of theorem 9.2.1..

Theorem 10.2.2.

Let $(X,\mathcal{T})$ be an L-fts and let $g \in L^X$. The L-fuzzy set g is RS-compact if and only if for all $p \in pr(L)$ and every collection $(f_i)_{i \in J}$ of regularly semiopen L-fuzzy sets with $\left(\bigvee_{i \in J} f_i\right)(x) \nleq p$ for all $x \in X$ with $g(x) \geq p'$, there exists a finite subset F of J with $\left(\bigvee_{i \in F} int(f_i)\right)(x) \nleq p$ for all $x \in X$ with $g(x) \geq p'$.

Proof

Necessity:

Let $p \in pr(L)$ and let $(f_i)_{i \in J}$ be a family of regularly semiopen L-fuzzy sets (definition 3.1.7. (v)) with $\left(\bigvee_{i \in J} f_i\right)(x) \nleq p$ for every $x \in X$ with $g(x) \geq p'$.

Thus, for each $i \in J$ there exists a regularly open L-fuzzy set $h_i$ such that $h_i \leq f_i \leq cl(h_i)$. So, by definition 3.1.7. (i), $h_i = int(cl(h_i))$ and we also have $cl(f_i) = cl(h_i)$

and that $f_i$ is a semiopen L-fuzzy set.

Therefore, by the RS-compactness of g, there is a finite subset F of J with $\left(\bigvee_{i\in F} \text{int}(\text{cl}(f_i))\right)(x) \not\geq p$ for all $x\in X$ with $g(x)\geq p'$; i.e., $\left(\bigvee_{i\in F} h_i\right)(x) \not\geq p$. Since $h_i \leq \text{int}(f_i)$, $\left(\bigvee_{i\in F} \text{int}(f_i)\right)(x) \not\geq p$ for all $x\in X$ with $g(x)\geq p'$.

<u>Sufficiency</u>:

Let $p\in pr(L)$ and let $(f_i)_{i\in J}$ be a family of regularly closed L-fuzzy sets with $\left(\bigvee_{i\in J} f_i\right)(x) \not\geq p$ for all $x\in X$ with $g(x)\geq p'$.

Since each $f_i$ is a regularly closed L-fuzzy set $f_i$ is a regularly semiopen L-fuzzy set (proved in the necessity of theorem 9.2.1.).

Therefore, by our hypothesis, there exists a finite subset F of J with $\left(\bigvee_{i\in F} \text{int}(f_i)\right)(x) \not\geq p$ for all $x\in X$ with $g(x)\geq p'$.

Hence, by theorem 10.2.1., g is RS-compact.

<u>Theorem 10.2.3.</u>

Let $(X,\mathcal{T})$ be an L-fts and let $g\in L^X$. The L-fuzzy set g is RS-compact if and only if for all $p\in pr(L)$ and every collection $(f_i)_{i\in J}$ of regularly semiopen L-fuzzy sets with $\left(\bigvee_{i\in J} f_i\right)(x) \not\geq p$ for all $x\in X$ with $g(x)\geq p'$, there exists a finite subset F of J with $\left(\bigvee_{i\in F} \text{int}(\text{cl}(f_i))\right)(x) \not\geq p$ for all $x\in X$ with $g(x)\geq p'$.

<u>Proof</u>

<u>Necessity</u>:

Let $p\in pr(L)$ and let $(f_i)_{i\in J}$ be a family of regularly semiopen L-fuzzy sets with $\left(\bigvee_{i\in J} f_i\right)(x)\nleq p$ for all $x\in X$ with $g(x)\geq p'$.

Thus, by theorem 10.2.2. and the RS-compactness of $g$, there is a finite subset $F$ of $J$ with $\left(\bigvee_{i\in F} int(f_i)\right)(x)\nleq p$ for all $x\in X$ with $g(x)\geq p'$. Therefore, $\left(\bigvee_{i\in F} int(cl(f_i))\right)(x)\nleq p$ for all $x\in X$ with $g(x)\geq p'$ because $int(f_i)\leq int(cl(f_i))$.

Sufficiency:

Let $p\in pr(L)$ and let $(f_i)_{i\in J}$ be a family of regularly closed L-fuzzy sets with $\left(\bigvee_{i\in J} f_i\right)(x)\nleq p$ for all $x\in X$ with $g(x)\geq p'$.

As showed in the necessity of theorem 9.2.1., each $f_i$ is a regularly semiopen L-fuzzy set. So by our hypothesis there exists a finite subset $F$ of $J$ with $\left(\bigvee_{i\in F} int(cl(f_i))\right)(x)\nleq p$ for all $x\in X$ with $g(x)\geq p'$. But each $f_i$ is regularly closed, so is closed, i.e., $cl(f_i)=f_i$. Thus $\left(\bigvee_{i\in F} int(f_i)\right)(x)\nleq p$ for all $x\in X$ with $g(x)\geq p'$.

Hence, by theorem 10.2.1., $g$ is RS-compact.

Proposition 10.2.4.

Let $(X,\mathcal{T})$ be an L-fts. Then $g\in L^X$ is an RS-compact L-fuzzy set if and only if for all $\alpha\in M(L)$ and for every collection $(f_i)_{i\in J}$ of semiclosed L-fuzzy sets with $\left(\bigwedge_{i\in J} f_i\right)(x)\ngeq\alpha$ for all $x\in X$ with $g(x)\geq\alpha$, there exists a finite subset $F$ of $J$ with $\left(\bigwedge_{i\in F} cl(int(f_i))\right)(x)\ngeq\alpha$ for all $x\in X$ with $g(x)\geq\alpha$.

Proof

This immediately follows from the definition.

Theorem 10.2.5.

Let $(X,\mathcal{T})$ be an L-fts. Then $g \in L^X$ is an RS-compact L-fuzzy set if and only if every constant $\alpha$-net $(S_m)_{m \in D}$ contained in g has a semi-$\delta$-cluster point (definition 3.1.9. (v)) $x_\alpha \in M(L^X)$, with height $\alpha$, contained in g, for each $\alpha \in M(L)$.

Proof

This is similar to the proof of theorem 4.4.2.

## 3. Some properties

Proposition 10.3.1.

Let $(X,\mathcal{T})$ be an L-fts and let g and h be RS-compact L-fuzzy sets. Then $h \vee g$ is RS-compact as well.

Proof

This is similar to the proof of proposition 4.1.10.

Proposition 10.3.2.

Let $(X,\mathcal{T})$ be an L-fts and let g be an RS-compact L-fuzzy set and h a regularly semiopen L-fuzzy set. Then $h \wedge g$ is RS-compact.

Proof

By using theorem 10.2.2. and the fact that if h is a regularly semiopen L-fuzzy set (definition 3.1.7. (v)) then $h'$ is also regularly semiopen, this is similar to the proof of proposition 4.1.12..

Proposition 10.3.3.

Let $(X,\mathcal{T})$ be an L-fts and let g be an RS-compact L-fuzzy set and h a regularly closed L-fuzzy set. Then $h \wedge g$ is RS-compact.

Proof

Since h regularly closed L-fuzzy set implies h regularly semiopen (proved in the necessity of theorem 9.2.1.), by proposition 10.3.2. $h \wedge g$ is RS-compact.

Proposition 10.3.4.

Let $(X,\mathcal{T}_X)$ and $(Y,\mathcal{T}_Y)$ be L-fts's. Let $f:(X,\mathcal{T}_X)\to(Y,\mathcal{T}_Y)$ be an almost continuous (definition 3.3.1. (iii)), almost open (definition 3.3.1. (vi)) mapping such that $f^{-1}(y)$ is finite for every $y\in Y$ and let $g$ be an RS-compact L-fuzzy set in $(X,\mathcal{T}_X)$. Then $f(g)$ is an RS-compact L-fuzzy set in $(Y,\mathcal{T}_Y)$.

Proof

Let $p\in pr(L)$ and let $(f_i)_{i\in J}$ be a family of regularly closed L-fuzzy sets with $\left(\bigvee_{i\in J} f_i\right)(y)\not\geq p$ for all $y\in Y$ with $f(g)(y)\geq p'$.

Thus, by proposition 3.3.4., $(f^{-1}(f_i))_{i\in J}$ is a family of regularly closed L-fuzzy sets in $(X,\mathcal{T}_X)$. We also have $\left(\bigvee_{i\in J} f^{-1}(f_i)\right)(x)\not\geq p$ for all $x\in X$ with $g(x)\geq p'$ because if $g(x)\geq p'$ then $f(g)(f(x))\geq p'$, so $\left(\bigvee_{i\in J} f^{-1}(f_i)\right)(x)=\left(\bigvee_{i\in J} f_i\right)(f(x))\not\geq p$.

From the RS-compactness of $g$ in $(X,\mathcal{T}_X)$ and by theorem 10.2.1., there is a finite subset $F$ of $J$ with $\left(\bigvee_{i\in F} int(f^{-1}(f_i))\right)(x)\not\geq p$ for all $x\in X$ with $g(x)\geq p'$.

Thus, $p\not\geq f\left(\bigvee_{i\in F} int(f^{-1})(f_i))\right)(y)$ = $\left(\bigvee_{i\in F} f(int(f^{-1}(f_i)))\right)(y)\leq \left(\bigvee_{i\in F} int(cl(f(int(f^{-1}(f_i)))))\right)(y)$ $\leq\left(\bigvee_{i\in F} int(cl(f(f^{-1}(f_i))))\right)(y)\leq\left(\bigvee_{i\in F} int(cl(f_i))\right)(y)$ = $\left(\bigvee_{i\in F} int(f_i)\right)(y)$ for all $y\in Y$ with $f(g)(y)\geq p'$, where first inequality is due to the fact that since each $f^{-1}(f_i)$ is closed we have, by remark 3.1.8. (iv), $int(f^{-1}(f_i))$ is regularly open and so by the almost openness of $f$ we have $f(int(f^{-1}(f_i)))\in\mathcal{T}_Y$. And last equality is due to the closedness of each $f_i$.

Hence, by theorem 10.2.1., f(g) is RS-compact.

When $g=\chi_A$ for some $A \subseteq X$, $f^{-1}(y)$ does not need to be finite.

<u>Proposition</u> <u>10.3.5.</u>

Let $(X,\mathcal{T}_X)$ and $(Y,\mathcal{T}_Y)$ be L-fts's. Let $f:(X,\mathcal{T}_X) \to (Y,\mathcal{T}_Y)$ be a weakly continuous mapping (definition 3.3.1. (iv)) such that $\text{int}(f^{-1}(h)) \le f^{-1}(\text{int}(h))$ for each regularly semiopen L-fuzzy set h in $(Y,\mathcal{T}_Y)$ with $f^{-1}(y)$ finite for every $y \in Y$ and let g be an RS-compact L-fuzzy set in $(X,\mathcal{T}_X)$. Then f(g) is an RS-compact L-fuzzy set in $(Y,\mathcal{T}_Y)$.

<u>Proof</u>

Let $p \in pr(L)$ and let $(f_i)_{i \in J}$ be a family of regularly closed L-fuzzy sets with $\left(\bigvee_{i \in J} f_i\right)(y) \not\ge p$ for all $y \in Y$ with $f(g)(y) \ge p'$.

Thus, by proposition 3.3.8., $(f^{-1}(f_i))_{i \in J}$ is a family of regularly closed L-fuzzy sets in $(X,\mathcal{T}_X)$. We also have $\left(\bigvee_{i \in J} f^{-1}(f_i)\right)(x) \not\ge p$ for all $x \in X$ with $g(x) \ge p'$ because if $g(x) \ge p'$ then $f(g)(f(x)) \ge p'$, so $\left(\bigvee_{i \in J} f^{-1}(f_i)\right)(x)$ $= \left(\bigvee_{i \in J} f_i(f(x))\right) \not\ge p$.

From the RS-compactness of g in $(X,\mathcal{T}_X)$ and by theorem 10.2.1., there is a finite subset F of J with $\left(\bigvee_{i \in F} \text{int}(f^{-1}(f_i))\right)(x) \not\ge p$ for all $x \in X$ with $g(x) \ge p'$.

Since every regularly closed L-fuzzy set is regularly semiopen (proved in the necessity of theorem

9.2.1.) we have that each $f_i$ is a regularly semiopen. So, by hypothesis, $int(f^{-1}(f_i)) \leq f^{-1}(int(f_i))$ for each $i \in F$ and then $\left( \bigvee_{i\in F} f^{-1}(int(f_i)) \right)(x) \not\geq p$ for all $x \in X$ with $g(x) \geq p'$.

For $y \in Y$ with $\bigvee_{x \in f^{-1}(y)} \{g(x)\} = f(g)(y) \geq p'$, we have $f(x)=y$ for some $x \in X$ with $g(x) \geq p'$. So, $\left( \bigvee_{i\in F} int(f_i) \right)(y) = \left( \bigvee_{i\in F} int(f_i) \right)(f(x)) = \left( \bigvee_{i\in F} f^{-1}(int(f_i)) \right)(x) \not\geq p$ for all $y \in Y$ with $f(g)(y) \geq p'$.

Hence, by theorem 10.2.1., $f(g)$ is RS-compact.

When $g=\chi_A$ for some $A \subseteq X$, $f^{-1}(y)$ does not need to be finite.

<u>Proposition</u> <u>10.3.6.</u>

Let $(X,\mathcal{T}_X)$ and $(Y,\mathcal{T}_Y)$ be L-fts's. Let $f:(X,\mathcal{T}_X) \to (Y,\mathcal{T}_Y)$ be a weakly continuous open mapping such that $f^{-1}(y)$ is finite for every $y \in Y$ and let $g$ be an RS-compact L-fuzzy set in $(X,\mathcal{T}_X)$. Then $f(g)$ is an RS-compact L-fuzzy set in $(Y,\mathcal{T}_Y)$.

<u>Proof</u>

By proposition 3.3.9., $f$ is also almost continuous. So this follows from proposition 10.3.4. since $f$ open mapping implies $f$ almost open mapping (definition 3.3.1.(v)(vi)).

When $g=\chi_A$ for some $A \subseteq X$, $f^{-1}(y)$ does not need to be finite.

<u>Proposition</u> <u>10.3.7.</u>

Let $(X,\mathcal{T}_X)$ and $(Y,\mathcal{T}_Y)$ be L-fts's. Let

$f:(X,\mathcal{T}_X)\to(Y,\mathcal{T}_Y)$ be a semicontinuous (definition 3.3.1. (ii)) mapping such that $f^{-1}(y)$ is finite for every $y\in Y$ and let $g$ be an RS-compact L-fuzzy set in $(X,\mathcal{T}_X)$. Then $f(g)$ is almost compact in $(Y,\mathcal{T}_Y)$ (definition 8.1.1.).

<u>Proof</u>

Let $p\in pr(L)$ and let $(f_i)_{i\in J}$ be a collection of open L-fuzzy sets with $\left(\bigvee_{i\in J} f_i\right)(y)\not\geq p$ for all $y\in Y$ with $f(g)(y)\geq p'$. Thus, by the semicontinuity of $f$, $(f^{-1}(f_i))_{i\in J}$ is a family of semiopen L-fuzzy sets in $(X,\mathcal{T}_X)$. We also have $\left(\bigvee_{i\in J} f^{-1}(f_i)\right)(x)\not\geq p$ for all $x\in X$ with $g(x)\geq p'$ because if $g(x)\geq p'$ then $f(g)(f(x))\geq p'$, so $\left(\bigvee_{i\in J} f^{-1}(f_i)\right)(x) = \left(\bigvee_{i\in J} f_i\right)(f(x))\not\geq p$.

Thus, by the RS-compactness of $g$, there is a finite subset $F$ of $J$ with $\left(\bigvee_{i\in F} int(cl(f^{-1}(f_i)))\right)(x)\not\geq p$ for all $x\in X$ with $g(x)\geq p'$.

For all $y\in Y$ with $\bigvee_{x\in f^{-1}(y)}\{g(x)\} = f(g)(y)\geq p'$, we have $f(x)=y$ for some $x\in X$ with $g(x)\geq p'$. So,

$$\left(\bigvee_{i\in F} cl(f_i)\right)(y) = \left(\bigvee_{i\in F} cl(f_i)\right)(f(x)) = \left(\bigvee_{i\in F} f^{-1}(cl(f_i))\right)(x)$$
$$\overset{*}{\geq}\left(\bigvee_{i\in F} int(cl(f^{-1}(cl(f_i))))\right)(x)\geq\left(\bigvee_{i\in F} int(cl(f^{-1}(f_i)))\right)(x)\not\geq p$$

where the inequality * is due to the fact that since $f$ is semicontinuous and $cl(f_i)$ is closed, $f^{-1}(cl(f_i))$ is semiclosed, so by remark 3.1.8.(i) we have $int(cl(f^{-1}(cl(f_i))))\leq f^{-1}(cl(f_i))$.

Hence $f(g)$ is almost compact.

When $g=\chi_A$ for some $A\subseteq X$, $f^{-1}(y)$ does not need to be finite.

Proposition 10.3.8.

Let $(X,\mathcal{T}_X)$ and $(Y,\mathcal{T}_Y)$ be L-fts's. Let $f:(X,\mathcal{T}_X)\to(Y,\mathcal{T}_Y)$ be an irresolute mapping (definition 3.3.1.(viii)) such that $f^{-1}(y)$ is finite for every $y\in Y$ and let g be an RS-compact L-fuzzy set in $(X,\mathcal{T}_X)$. Then f(g) is S-closed in $(Y,\mathcal{T}_Y)$.

Proof

Let $p\in pr(L)$ and let $(f_i)_{i\in J}$ be a collection of semiopen L-fuzzy sets with $\left(\bigvee_{i\in J} f_i\right)(y)\nleq p$ for all $y\in Y$ with $f(g)(y)\geq p'$.

Thus, by the irresoluteness of f, $(f^{-1}(f_i))_{i\in J}$ is a family of semiopen L-fuzzy sets in $(X,\mathcal{T}_X)$. As in the proof of proposition 10.3.7., we have $\left(\bigvee_{i\in J} f^{-1}(f_i)\right)(x)\nleq p$ for all $x\in X$ with $g(x)\geq p'$. Now the proof follows exactly as in our proof of proposition 10.3.7. where here the justification of the inequality * is the fact that since f is irresolute and $cl(f_i)$ is semiclosed, $f^{-1}(cl(f_i))$ is semiclosed, so by remark 3.1.8. (i) we have $int(cl(f^{-1}(cl(f_i))))\leq f^{-1}(cl(f_i))$.

Hence f(g) is S-closed.

When $g=\chi_A$ for some $A\subseteq X$, $f^{-1}(y)$ does not need to be finite.

# Chapter XI

## S-compactness in L-fuzzy topological spaces

In ordinary topology strong compactness was discussed by Atia et al. [6]. A topological space $(X,\delta)$ is called strongly compact if and only if every pre-open cover of X has a finite subcover.

In [0,1]-fuzzy topological spaces, strong compactness has been introduced by Nanda [75]. His definition is based on Chang's compactness (remark 4.6.2.).

In this chapter a good definition of strong compactness is introduced in L-fuzzy topological spaces. To avoid confusion between this strong compactness and the strong fuzzy compactness introduced by Lowen (definition 4.1.2.), we shall call it here S-compactness, in ordinary topology as well as in L-fuzzy topology. We define S-compactness for arbitrary L-fuzzy sets and study its properties.

This chapter is divided in three sections.

In the first section we present our S-compactness and prove its goodness.

The second section contains other characterizations of S-compactness.

The third section is reserved for some properties.

## 1. Proposed definition and its goodness

Definition 11.1.1.

Let $(X,\mathcal{T})$ be an L-fts and let $g\in L^X$. The L-fuzzy set g is said to be S-compact if and only if for every prime $p\in L$ and every collection $(f_i)_{i\in J}$ of pre-open L-fuzzy sets (definition 3.1.7. (viii)) with $\left(\bigvee_{i\in J} f_i\right)(x)\not\leq p$ for all $x\in X$ with $g(x)\geq p'$, there exists a finite subset F of J with $\left(\bigvee_{i\in F} f_i\right)(x)\not\leq p$ for all $x\in X$ with $g(x)\geq p'$.

If g is the whole space, then we say that the L-fts $(X,\mathcal{T})$ is S-compact.

Theorem 11.1.2. (The goodness of S-compactness)

Let $(X,\delta)$ be a topological space. Then $(X,\delta)$ is S-compact if and only if the L-fts $(X,\omega(\delta))$ is S-compact.

Proof

Necessity:

Let $p\in pr(L)$ and let $(f_i)_{i\in J}$ be a family of pre-open L-fuzzy sets in $(X,\omega(\delta))$ with $\left(\bigvee_{i\in J} f_i\right)(x)\not\leq p$ for all $x\in X$.

Thus, $(f_i^{-1}(\{t\in L;\ t\not\leq p\}))_{i\in J}$ is a family of pre-open sets in $(X,\delta)$ that covers X. In fact, since for each $x\in X$ there exists $i\in J$ with $f_i(x)\not\leq p$, for each $x\in X$ there is $i\in J$ with $x\in f_i^{-1}(\{t\in L;\ t\not\leq p\})$. Then $\bigcup_{i\in J} f_i^{-1}(\{t\in L;\ p\not\leq t\})\supseteq X$. We also have that, for each $i\in J$, $f_i^{-1}(\{t\in L;\ t\not\leq p\})$ is pre-open in $(X,\delta)$ because for every $i\in J$, $f_i\leq int(cl(f_i))$ which implies $f_i^{-1}(\{t\in L;\ t\not\leq p\})\subseteq(int(cl(f_i)))^{-1}(\{t\in L;\ t\not\leq p\})$. Since $int(g)\leq g$ and $int(g)\in\omega(\delta)$ for every $g\in L^X$ and

$\{t\in L;\ t\nleq p\})$ is Scott open, we have, by proposition 3.2.9., $(\text{int}(g))^{-1}(\{t\in L;t\nleq p\})\in\delta$, so $(\text{int}(g))^{-1}(\{t\in L;p\nleq t\})$ $\leq \text{int}(g^{-1}(\{t\in L;\ t\nleq p\}))$. Therefore, by considering $g=\text{cl}(f_i)$, we have $f_i^{-1}(\{t\in L;1\ t\nleq p\})\leq$ $(\text{int}(\text{cl}(f_i)))^{-1}\ (\{t\in L;\ t\nleq p\})\leq \text{int}\left((\text{cl}(f_i))^{-1}(\{t\in L;\ t\nleq p\})\right)$. From lemma 3.2.12. we obtain $f_i^{-1}(\{t\in L;t\nleq p\})\leq$ $\text{int}(\text{cl}(f_i^{-1}(\{t\in L;t\nleq p\})))$ and then $f_i^{-1}(\{t\in L;\ t\nleq p\})$ is pre-open for every $i\in J$.

From the S-compactness of $(X,\delta)$, there is a finite subset F of J with $\bigcup_{i\in F} f_i^{-1}(\{t\in L;\ t\nleq p\})\supseteq X$. So, for every $x\in X$ there is $i\in F$ such that $f_i(x)\nleq p$, i.e., $\left(\bigvee_{i\in F} f_i\right)(x)\nleq p$ for all $x\in X$.

Hence $(X,\omega(\delta))$ is S-compact.

Sufficiency:

Let $(A_i)_{i\in J}$ be a pre-open cover of $(X,\delta)$.

Thus, $(\chi_{A_i})_{i\in J}$ is a family of pre-open L-fuzzy sets in $(X,\omega(\delta))$ with $\left(\bigvee_{i\in J} \chi_{A_i}\right)(x)=1\nleq p$ for all $x\in X$ and for all $p\in pr(L)$. In fact, since $A_i$ is pre-open for every $i\in J$, $A_i\leq \text{int}(\text{cl}(A_i))$ and then $\chi_{A_i}\leq\chi_{\text{int}(\text{cl}(A_i))}$. Since, by corollary 3.2.14., $\chi_{\text{int}(\text{cl}(A_i))} = \text{int}(\text{cl}(\chi_{A_i}))$, we have $\chi_{A_i}\leq \text{int}(\text{cl}(\chi_{A_i})$, so $\chi_{A_i}$ is pre-open in $(X,\omega(\delta))$ for all $i\in J$. We also have that for all $x\in X$ there is $i\in J$ such that $x\in A_i$. So, for all $x\in X$ there exists $i\in J$ with $\chi_{A_i}(x)=1$ which implies that $\left(\bigvee_{i\in J} \chi_{A_i}\right)(x)=1\ \nleq p$ for all $x\in X$ and for all $p\in pr(L)$.

From the S-compactness of $(X,\omega(\delta))$, there exists a finite subset F of J with $\left(\bigvee_{i\in F} \chi_{A_i}\right)(x) \nleq p$ for all $x\in X$ and for every $p\in pr(L)$, thus $\left(\bigvee_{i\in F} \chi_{A_i}\right)(x)=1$ for all $x\in X$.

Therefore $\bigcup_{i\in F} A_i \supseteq X$.

Hence $(X,\delta)$ is S-compact.

## 2. Other characterizations

Proposition 11.2.1.

Let $(X,\mathcal{T})$ be an L-fts. Then $g\in L^X$ is S-compact if and only if for all $\alpha\in M(L)$ and every collection $(f_i)_{i\in J}$ of pre-closed L-fuzzy sets (definition 3.1.7. (viii)) with $\left(\bigwedge_{i\in J} f_i\right)(x)\not\geq\alpha$ for all $x\in X$ with $g(x)\geq\alpha$, there exists a finite subset F of J with $\left(\bigwedge_{i\in F} f_i\right)(x)\not\geq\alpha$ for all $x\in X$ with $g(x)\geq\alpha$.

Proof

This immediately follows from the definition.

Theorem 11.2.2.

Let $(X,\mathcal{T})$ be an L-fts. Then $g\in L^X$ is S-compact if and only if every constant $\alpha$-net $(S_m)_{m\in D}$ contained in g, has a pre-cluster point (definition 3.1.9.(vii)) $x_\alpha\in M(L^X)$, with height $\alpha$, contained in g, for each $\alpha\in M(L)$.

Proof

This is similar to the proof of theorem 4.4.2.

## 3. **Some properties**

Proposition 11.3.1.

Let $(X,\mathcal{T})$ be an L-fts. If h and g are S-compact L-fuzzy sets, then $h \vee g$ is S-compact as well.

Proof

This is similar to the proof of proposition 4.1.10.

Corollary 11.3.2.

Let $(X,\mathcal{T})$ be an L-fts. Every L-fuzzy set g with finite support is S-compact.

Proof

This is similar to the proof of corollary 4.1.11.

Proposition 11.3.3.

Let $(X,\mathcal{T})$ be an L-fts. If g is an S-compact L-fuzzy set, then for each pre-closed L-fuzzy set h, $h \wedge g$ is S-compact.

Proof

This is similar to the proof of proposition 4.1.12.

Proposition 11.3.4.

Let $(X,\mathcal{T})$ be an L-fts and $\mathcal{T}_{\phi}$ as defined in 3.3.2. Then f is S-compact in $(X,\mathcal{T})$ if and only if f is compact in $(X,\mathcal{T}_{\phi})$.

Proof

Necessity:

Let $p \in pr(L)$ and let $(f_i)_{i \in J}$ be a collection of subbasic $\mathcal{T}_\phi$-open L-fuzzy sets with $\left(\bigvee_{i \in J} f_i\right)(x) \not\geq p$ for all $x \in X$ such that $f(x) \geq p'$.

Thus, each $f_i$ is a pre-open L-fuzzy set in $(X, \mathcal{T})$ and, by the S-compactness of f, there is a finite subset F of J with $\left(\bigvee_{i \in F} f_i\right)(x) \not\geq p$ for all $x \in X$ such that $f(x) \geq p'$.

Hence, by theorem 4.2.1., f is compact in $(X, \mathcal{T}_\phi)$.

Sufficiency:

Let $p \in pr(L)$ and let $(f_i)_{i \in J}$ be a collection of pre-open L-fuzzy sets in $(X, \mathcal{T})$ with $\left(\bigvee_{i \in J} f_i\right)(x) \not\geq p$ for all $x \in X$ with $f(x) \geq p'$.

Since every pre-open L-fuzzy set in $(X, \mathcal{T})$ is an $\mathcal{T}_\phi$, by the compactness of f in $(X, \mathcal{T}_\phi)$, there exists a finite subset F of J with $\left(\bigvee_{i \in F} f_i\right)(x) \not\geq p$ for all $x \in X$ such that $f(x) \geq p'$.

Hence f is S-compact in $(X, \mathcal{T})$.

Proposition 11.3.5.

Let $(X, \mathcal{T})$ be an L-fts. If g is an S-compact L-fuzzy set in $(X, \mathcal{T})$, then for each closed L-fuzzy set h in $(X, \mathcal{T}_\phi)$, $h \wedge g$ is S-compact in $(X, \mathcal{T})$.

Proof

By proposition 11.3.4. g is compact in $(X, \mathcal{T}_\phi)$ and since h is closed in $(X, \mathcal{T}_\phi)$, from proposition 4.1.12., $h \wedge g$ is compact in $(X, \mathcal{T}_\phi)$.

Hence, by proposition 11.3.4., $h \wedge g$ is S-compact in $(X, \mathcal{T})$.

<u>Proposition</u> <u>11.3.6.</u>

Let $(X,\mathcal{T})$ and $(Y,\mathcal{T}^*)$ be L-fts's and let $f:(X,\mathcal{T}) \to (Y,\mathcal{T}^*)$ be a $\phi'$-continuous mapping (definition 3.3.2.) such that $f^{-1}(y)$ is finite for every $y \in Y$. Let g be an S-compact L-fuzzy set in $(X,\mathcal{T})$. Then f(g) is S-compact in $(Y,\mathcal{T}^*)$.

<u>Proof</u>

Let $p \in pr(L)$ and let $(f_i)_{i\in J}$ be a collection of pre-open L-fuzzy sets in $(Y,\mathcal{T}^*)$, that is, each $f_i$ is a $\mathcal{T}^*_\phi$-subbasic open L-fuzzy set; with $\left(\bigvee_{i\in J} f_i\right)(x) \not\leq p$ for all $y \in Y$ such that $f(g)(y) \geq p'$.

Since, by definition 3.3.2., $f:(X,\mathcal{T}_\phi) \to (Y,\mathcal{T}^*_\phi)$ is continuous, each $f^{-1}(f_i) \in \mathcal{T}_\phi$. We also have that $\left(\bigvee_{i\in J} f^{-1}(f_i)\right)(x) \not\leq p$ for all $x \in X$ such that $g(x) \geq p'$ because if $g(x) \geq p'$ then $f(g)(f(x)) = \bigvee_{z \in f^{-1}(f(x))} \{g(z)\} \geq p'$, so $\left(\bigvee_{i\in J} f^{-1}(f_i)\right)(x) = \left(\bigvee_{i\in J} f_i\right)(f(x)) \not\leq p$.

By the S-compactness of g in $(X,\mathcal{T})$, from proposition 11.3.4., g is compact in $(X,\mathcal{T}_\phi)$.

Thus, there exists a finite subset F of J with $\left(\bigvee_{i\in F} f^{-1}(f_i)\right)(x) \not\leq p$ for all $x \in X$ such that $g(x) \geq p'$. Then, $\left(\bigvee_{i\in F} f_i\right)(y) \not\leq p$ for all $y \in Y$ with $f(g)(y) \geq p'$ because $\left(\bigvee_{i\in F} f(f^{-1}(f_i))\right)(y) \not\leq p$ for all $y \in Y$ such that $f(g)(y) \geq p'$.

Therefore, by proposition 4.2.1., f(g) is compact in $(Y,\mathcal{T}^*_\phi)$.

Hence, from proposition 11.3.4., f(g) is S-compact in $(Y, \mathcal{T}^{*})$.

When $g=\chi_A$ for some $A \subseteq X$, $f^{-1}(y)$ does not need to be finite.

<u>Proposition</u> <u>11.3.7.</u>

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be L-fts's and let $f:(X, \mathcal{T}_X) \rightarrow (Y, \mathcal{T}_Y)$ be an M-pre-continuous mapping (definition 3.3.1. (x)) such that $f^{-1}(y)$ is finite for every $y \in Y$. If g is S-compact in $(X, \mathcal{T}_X)$, then f(g) is S-compact in $(Y, \mathcal{T}_Y)$.

<u>Proof</u>

Since, by proposition 3.3.7., f is $\phi'$-continuous; this follows from proposition 11.3.6.

When $g=\chi_A$ for some $A \subseteq X$, $f^{-1}(y)$ does not need to be finite.

# Chapter XII

## A comparison between the concepts introduced in chapters VIII, IX, X and XI and some related properties

This chapter is devoted to a comparison between S-compactness, compactness, almost compactness, S-closedness, near compactness and RS-compactness, as well as, to some properties related to extremally disconnected L-fuzzy topological spaces.

This chapter is divided in two sections.

Section one contains a comparison between these compactness related concepts and a condition for almost, near, RS-compactness and S-closedness to be equivalent.

In section two we obtain some properties that follow from this comparison.

## 1. A comparison between compactness; almost, near, RS and S-compactness and S-closedness

Theorem 12.1.1.

For an L-fuzzy topological space $(X,\mathcal{T})$ and $g\in L^X$ the following implications hold:

g nearly compact

(ii) ⇗     ⇓(iv)     ⇖ (vi)

g S-compact $\overset{(i)}{\Rightarrow}$ g compact $\overset{(iii)}{\Rightarrow}$ g almost compact $\overset{(vii)}{\Leftarrow}$ g RS-compact

⇑(v)     ⇙(viii)

g S-closed

Proof

(i) Let $p\in pr(L)$ and let $(f_i)_{i\in J}$ be a family of open L-fuzzy sets such that $\left(\bigvee_{i\in J} f_i\right)(x)\not\leq p$ for all $x\in X$ with $g(x)\geq p'$.

Since each $f_i\in\mathcal{T}$, $f_i$ is pre-open (definition 3.1.7. (vii)). So, by the S-compactness of g (definition 11.1.1.), there is a finite subset F of J such that $\left(\bigvee_{i\in F} f_i\right)(x)\not\leq p$ for all $x\in X$ with $g(x)\geq p'$.

Hence g is compact (definition 4.1.4.).

(ii) and (iii) Let $p\in pr(L)$ and let $(f_i)_{i\in J}$ be a family of open L-fuzzy sets such that $\left(\bigvee_{i\in J} f_i\right)(x)\not\leq p$ for all $x\in X$ with $g(x)\geq p'$.

By the compactness of g there is a finite subset F of J such that $\left(\bigvee_{i\in F} f_i\right)(x)\not\leq p$ for all $x\in X$ with $g(x)\geq p'$. So, $\left(\bigvee_{i\in F} cl(f_i)\right)(x)\not\leq p$ and $\left(\bigvee_{i\in F} int(cl(f_i))\right)(x)\not\leq p$ for all $x\in X$ with $g(x)\geq p'$ because $intf_i=f_i\leq int(cl(f_i))\leq cl(f_i)$.

Hence g is nearly compact and almost compact (definitions 8.1.1., 8.1.2.).

(iv) Let $p \in pr(L)$ and let $(f_i)_{i\in J}$ be a family as above.

By the near compactness of g there is a finite subset F of J such that $\left(\bigvee_{i\in F} int(cl(f_i))\right)(x) \not\leq p$ for all $x \in X$ with $g(x) \geq p'$. So $\left(\bigvee_{i\in F} cl(f_i)\right)(x) \not\leq p$ for all $x \in X$ with $g(x) \geq p'$.

Hence g is almost compact.

(v) Let $p \in pr(L)$ and let $(f_i)_{i\in J}$ be a family as above.

Since each $f_i \in \mathcal{T}$, f is semiopen (definition 3.1.7. (iii)) as well. So, by the S-closedness of g (definition 9.1.1.), there is a finite subset F of J such that $\left(\bigvee_{i\in F} cl(f_i)\right)(x) \not\leq p$ for all $x \in X$ with $g(x) \geq p'$.

Hence g is almost compact.

(vi) and (vii) Let $p \in pr(L)$ and let $(f_i)_{i\in J}$ be a family as above.

Since each $f_i \in \mathcal{T}$, $f_i$ is semiopen as well. So, by the RS-compactness of g (definition 10.1.1.), there is a finite subset F of J such that $\left(\bigvee_{i\in F} int(cl(f_i))\right)(x) \not\leq p$ for all $x \in X$ with $g(x) \geq p'$.

Hence g is nearly compact and almost compact.

(viii) Let $p \in pr(L)$ and let $(f_i)_{i\in J}$ be a family of semiopen L-fuzzy sets such that $\left(\bigvee_{i\in J} f_i\right)(x) \not\leq p$ for all $x \in X$ with $g(x) \geq p'$.

By the RS-compactness of g there is a finite subset F of J such that $\left(\bigvee_{i\in F} int(cl(f_i))\right)(x) \not\leq p$ for all $x \in X$ with $g(x) \geq p'$. So, $\left(\bigvee_{i\in F} cl(f_i)\right)(x) \not\leq p$ for all $x \in X$ with $g(x) \geq p'$.

Hence g is S-closed.

Remark 12.1.2.

The implications in theorem 12.1.1. are the only ones that are valid, since the others are not valid in ordinary topology [43,77] and all these concepts are good extensions.

Theorem 12.1.3.

For an extremally disconnected L-fts $(X,\mathcal{T})$ (definition 3.2.2.), the following are equivalent:

(i) $(X,\mathcal{T})$ is almost compact
(ii) $(X,\mathcal{T})$ is nearly compact
(iii) $(X,\mathcal{T})$ is S-closed
(iv) $(X,\mathcal{T})$ is RS-compact

Proof

Considering theorem 12.1.1., to prove these equivalences, it is sufficient to prove that $(X,\mathcal{T})$ almost compact extremally disconnected L-fts implies that $(X,\mathcal{T})$ is RS-compact.

Let $(X,\mathcal{T})$ be an almost compact extremally disconnected L-fts. Let $p \in pr(L)$ and let $(f_i)_{i\in J}$ be a family of semiopen L-fuzzy sets with $\left(\bigvee_{i\in J} f_i\right)(x) \nleq p$ for all $x \in X$.

Thus, for each $i \in J$ there is $g_i \in \mathcal{T}$ such that $g_i \le f_i \le cl(g_i)$. Therefore $\left(\bigvee_{i\in J} cl(g_i)\right)(x) \nleq p$ for all $x \in X$. Since $(X,\mathcal{T})$ is extremally disconnected, $cl(g_i) \in \mathcal{T}$ for each $i \in J$. So, by the almost compactness of $(X,\mathcal{T})$ there is a finite subset F of J with $\left(\bigvee_{i\in F} cl(g_i)\right)(x) \nleq p$ for all $x \in X$. Because $cl(f_i) = cl(g_i)$ and $cl(g_i) \in \mathcal{T}$, we have $int(cl(f_i)) = cl(g_i)$. Then $\left(\bigvee_{i\in F} int(cl(f_i))\right)(x) \nleq p$ for all $x \in X$.

Hence $(X,\mathcal{T})$ is RS-compact.

## 2. Some properties

Proposition 12.2.1.

Let $(X,\mathcal{T})$ be an L-fts and let g be an L-fuzzy set with finite support. Then g is nearly and almost compact.

Proof

By corollary 4.1.11. g is compact. Thus, by theorem 12.1.1. this result follows.

Proposition 12.2.2.

Let $(X,\mathcal{T}_X)$ and $(Y,\mathcal{T}_Y)$ be L-fts's where $(Y,\mathcal{T}_Y)$ is extremally disconnected. Let $f:(X,\mathcal{T}_X)\rightarrow(Y,\mathcal{T}_Y)$ be an almost continuous mapping such that $f^{-1}(y)$ is finite for every $y\in Y$ and let g be an S-closed (nearly compact) [RS-compact] L-fuzzy set in $(X,\mathcal{T}_X)$. Then f(g) is an S-closed (nearly compact) [RS-compact] L-fuzzy set in $(Y,\mathcal{T}_Y)$.

Proof

Since g is an S-closed (nearly compact) [RS-compact] L-fuzzy set, by theorem 12.1.1., g is also almost compact. So, by proposition 8.3.4., f(g) is almost compact in $(Y,\mathcal{T}_Y)$. Because $(Y,\mathcal{T}_Y)$ is extremally disconnected, by theorem 12.1.3. f(g) is S-closed (nearly compact) [RS-compact] in $(Y,\mathcal{T}_Y)$.

When $g=\chi_A$ for some $A\subseteq X$, $f^{-1}(y)$ does not need to be finite.

Proposition 12.2.3.

Let $(X,\mathcal{T}_X)$ and $(Y,\mathcal{T}_Y)$ be L-fts's with $(Y,\mathcal{T}_Y)$ extremally disconnected. Let $f:(X,\mathcal{T}_X)\to(Y,\mathcal{T}_Y)$ be a weakly continuous mapping such that $f^{-1}(y)$ is finite for every $y\in Y$ and let g be an S-closed (nearly compact) [RS-compact] L-fuzzy set in $(X,\mathcal{T}_X)$. Then f(g) is an S-closed (nearly compact) [RS-compact] L-fuzzy set in $(Y,\mathcal{T}_Y)$.

Proof

Since f is weakly continuous and $(Y,\mathcal{T}_Y)$ is extremally disconnected, f is almost continuous. In fact, if h is a regularly open L-fuzzy set in $(Y,\mathcal{T}_Y)$ then $h=int(cl(h))$ and since $(Y,\mathcal{T}_Y)$ is extremally disconnected we have $cl(h)\in\mathcal{T}_Y$, thus $h=cl(h)$. Then by the weak continuity of f, $f^{-1}(h)\le int(f^{-1}(cl(h)))=int(f^{-1}(h))$. Hence $f^{-1}(h)\in\mathcal{T}_X$ and f is almost continuous.

Therefore we have our result from proposition 12.2.2.

When $g=\chi_A$ for some $A\subseteq X$, $f^{-1}(y)$ does not need to be finite.

Proposition 12.2.4.

Let $(X,\mathcal{T}_X)$ and $(Y,\mathcal{T}_Y)$ be L-fts's with $(Y,\mathcal{T}_Y)$ extremally disconnected. Let $f:(X,\mathcal{T}_X)\to(Y,\mathcal{T}_Y)$ be a semicontinuous mapping such that $f^{-1}(y)$ is finite for every $y\in Y$ and let g be an RS-compact L-fuzzy set in $(X,\mathcal{T}_X)$. Then f(g) is RS-compact in $(Y,\mathcal{T}_Y)$.

When $g=\chi_A$ for some $A \subseteq X$, $f^{-1}(y)$ does not need to be finite.

<u>Proof</u>

By proposition 10.3.7. f(g) is almost compact in $(Y,\mathcal{T}_Y)$. Since $(Y,\mathcal{T}_Y)$ is extremally disconnected, by theorem 12.1.3., f(g) is RS-compact.

<u>Proposition 12.2.5.</u>

Let $(X,\mathcal{T}_X)$ and $(Y,\mathcal{T}_Y)$ be L-fts's with $(Y,\mathcal{T}_Y)$ extremally disconnected. Let $f:(X,\mathcal{T}_X)\to(Y,\mathcal{T}_Y)$ be an irresolute mapping such that $f^{-1}(y)$ is finite for every $y\in Y$ and let g be an RS-compact L-fuzzy set in $(X,\mathcal{T}_X)$. Then f(g) is RS-compact in $(Y,\mathcal{T}_Y)$.

<u>Proof</u>

By proposition 10.3.8. f(g) is S-closed in $(Y,\mathcal{T}_Y)$. Since $(Y,\mathcal{T}_Y)$ is extremally disconnected, by theorem 12.1.3., f(g) is RS-compact.

When $g=\chi_A$ for some $A \subseteq X$, $f^{-1}(y)$ does not need to be finite.

## REFERENCES


[1] K.M. Abd El-Hakeim, *Sequential compactness in fuzzy neighbourhood spaces*, Fuzzy Sets and Systems 60 (1993) 225-228.

[2] M.E. Abd El-Monsef, F.M. Zeyada, S.N. El Deeb and I.M. Hanafy, *Good extensions of paracompactness*, Math•Japonica 37 No. 1 (1992) 195-200.

[3] A.A. Allam and A.M. Zahran, *On fuzzy δ-continuity and α-near compactness in fuzzy topological spaces*, Fuzzy Sets and Systems 50 (1992) 103-112.

[4] A.A. Allam and A.M. Zahran, *Functions and αS-closedness in fuzzy topological space*, Fuzzy Sets and Systems 52 (1992) 103-111.

[5] A.A. Allam and A.M. Zahran, *Functions and αI-compactness on fuzzy topological spaces*, Fuzzy Sets and Systems 57. (1993) 249-256.

[6] R.H. Atia, S.N. El Deeb and I.A. Hasamin, *A note on strong compactness and S-closedness*, Math.Vesnik 6 (19)(34) (1982) 23-28.

[7] K.K. Azad, *On fuzzy semicontinuity, fuzzy almost continuity and fuzzy weakly continuity*, J.Math.Anal and Appl. 82 (1981) 14-32.

[8] G. Birkhoff, *Lattice theory*, Amer. Math. Soc.

Colloq. Publ., vol. 25, Providence, R.I., 1984.

[9] T.S. Blyth and M.F. Janowitz, *Residuation Theory*, Pergamon Press, Oxford, 1972.

[10] A. Bülbül, *A note on the characterization of F-continuous functions on $C_1$ fuzzy topological spaces*, Hacettepe Bull. Nat. Sci. Eng. 13 (1984) 53-58.

[11] A. Bülbül, *On connection between the countability properties of $(X,\omega(\mathcal{T}))$ and $(X,\mathcal{T})$*, Fuzzy Sets and Systems 15 (1985) 91-97.

[12] A. Bülbül and M.W. Warner, *Good dilutions of fuzzy compactness*, Fuzzy Sets and Systems 51 (1992) 111-115.

[13] A. Bülbül and M.W. Warner, *On the goodness of some types of fuzzy paracompactness*, Fuzzy Sets and Systems 55 (1993) 187-191.

[14] D.E. Cameron, *Properties of S-closed spaces*, Proc. of the Amer. Math. Soc. 72 (1978) 581-586.

[15] J.J. Chadwick, *A generalized form of compactness in fuzzy topological spaces*, J.Math. Anal. Appl. 162 (1991) 92-110.

[16] M.K. Chakraborty and T.M.G. Ahsanullah, *Fuzzy topology on fuzzy sets and tolerance topology*, Fuzzy Sets and Systems 45 (1992) 103-108.

[17] M.K. Chakraborty and Mohua Banerjee, *A new category for fuzzy topological spaces*, Fuzzy Sets and Systems 51

(1992) 227-233.

[18] C.L. Chang, *Fuzzy topological spaces*, J.Math.Anal. Appl. 24 (1968) 182-190.

[19] K.C. Chattopadhyay, R.N. Hazra and S.K. Samanta, *Gradation of openness: fuzzy topology*, Fuzzy Sets and Systems 49 (1992) 237-242.

[20] S.K. Chen, *Theory of L-fuzzy H-sets*, Fuzzy Sets and Systems 51 (1992) 89-94.

[21] Y. Chen, *Paracompactness on L-fuzzy topological spaces*, Fuzzy Sets and Systems 53 (1993) 335-342.

[22] F.T. Christoph, *Quotient fuzzy topology and local compactness*, J.Math.Anal.Appl. 57 (1977) 497-504.

[23] D. Coker and A.H. Es, *On fuzzy S-closed spaces*, Doga Mat 11 (1987) 145-152.

[24] D. Coker and A.H. Es, *On fuzzy RS-compact spaces*, Doga TU J.Math. 13 (1989) 34-42.

[25] S. Dang, A. Behera and S. Nanda, *Some results on fuzzy supratopological spaces*, Fuzzy Sets and Systems 62 (1994) 333-339.

[26] Z. Deng, *Fuzzy pseudo metric spaces*, J.Math.Anal. Appl. 86 (1982) 74-95.

[27] Z. Deng, *m-Uniformization and metrization of fuzzy topological spaces*, J.Math.Anal.Appl. 112 (1985) 471-486.

[28] A. Di Concilio and G. Gerla, *Almost compactness in*

*fuzzy topological spaces*, Fuzzy Sets and Systems 13 (1984) 187-192.

[29] J. Dugundji, *Topology*, Allyn and Bacon, INC., Boston London Sydney Toronto, 1966.

[30] P. Eklund, *Category theoretic properties of fuzzy topological spaces*, Fuzzy Sets and Systems 13 (1984) 303-310.

[31] P.Eklund and W. Gähler, *Basic notions for fuzzy topology I*, Fuzzy Sets and Systems 26 (1988) 333-356.

[32] P. Eklund and W. Gähler, *Basic notions for fuzzy topology II*, Fuzzy Sets and Systems 27 (1988) 171-195.

[33] A.H. Es, *Almost compactness and near compactness in fuzzy topological spaces*, Fuzzy Sets and Systems 22 (1987) 289-295.

[34] S. Ganguly and S. Saha, *A note on compactness in a fuzzy setting*, Fuzzy Sets and Systems 34 (1990) 117-124.

[35] T.E. Gantner, R.C. Steinlage and R.H. Warren, *Compactness in fuzzy topological spaces*, J.Math.Anal. Appl. (1978) 547-562.

[36] M.H. Ghanim, E.E. Kerre and A.S. Mashhour, *Separation axioms, subspaces and sums in fuzzy topology*, J.Math.Anal.Appl. 102 (1984) 189-202.

[37] G. Gierz et al., *A Compendium of Continuous Lattices*, Springer-Verlag, Berlin Heidelberg New York,

1980.

[38] J.A. Goguen, *L-fuzzy sets*, J.Math.Anal.Appl. 18 (1967), 145-174.

[39] J.A. Goguen, *The fuzzy Tychonoff theorem*, J.Math. Anal.Appl. 43 (1973) 734-742.

[40] S. Gottwald, *Fuzzy points and local properties of fuzzy topological spaces*, Fuzzy Sets and Systems 5 (1981) 199-201.

[41] R.N. Hazra, S.K. Samanta and K.C. Chattopadhyay, *Fuzzy topology redefined*, Fuzzy Sets and Systems 45 (1992) 79-82.

[42] U. Höhle, *Compact $\mathcal{L}$-fuzzy topological spaces*, Fuzzy Sets and Systems 13 (1984) 39-63.

[43] W.C. Hong, *RS-compact spaces*, J.Korean Math.Soc. 17 (1980) 39-43.

[44] B. Hutton, *Products of fuzzy topological spaces*, Topology and its Applications 11 (1980) 59-67.

[45] P.T. Johnstone, *Stone Spaces*, Cambridge University Press, 1982.

[46] J.E. Joseph, M.H. Kwack, *On S-closed spaces*, Proc. of the Amer. Math.Soc. 80 (1980) 341-348.

[47] E.E. Kerre, *Fuzzy Sierpinski space and its generalization*, J.Math.Anal.Appl. 74 (1980) 318-324.

[48] E.E. Kerre and P.L. Ottoy, *On the different notions*

*of neighbourhood in Chang-Goguen fuzzy topological spaces*, Simon Stevin 61 (1987) 131-146.

[49] Z. Li, *Compactness in fuzzy topological spaces*, Kexue Tongbao vol. 29 No. 5 (1984) 582-585.

[50] Y. Liu and M. Luo, *Induced Spaces and fuzzy Stone-Cech compactification*, Scientia Sinica (Series A) vol. xxx no. 10 (1987) 1034-1044.

[51] Y. Liu and M. Luo, *Lattice-valued mappings, completely distributive law and induced spaces*, Fuzzy Sets and Systems 42 (1991) 43-56.

[52] E. Lowen and R. Lowen, *On Measures of compactness in fuzzy topological spaces*, J.Math.Anal.Appl. 131 (1988) 329-340.

[53] R.Lowen, *Topologies floues*, C.R. Acad. Sc. Paris, 278 Série A (1974) 925-928.

[54] R. Lowen, *Fuzzy topological spaces and fuzzy compactness*, J.Math.Anal. Appl. 56 (1976) 621-633.

[55] R. Lowen, *Initial and final fuzzy topologies and the fuzzy Tychonoff theorem*, J.Math.Anal.Appl. 58 (1977) 11-21.

[56] R. Lowen, *A comparison of different compactness notion in fuzzy topological spaces*, J.Math.Anal.Appl. 64 (1978) 446-454.

[57] R. Lowen, *On the existence of natural non-topological, fuzzy topological spaces*, Helderman,

Berlin, 1985.

[58] R. Lowen and P. Wuyts, *Concerning the constants in fuzzy topology*, J.Math.Anal.Appl. 129 (1988) 256-268.

[59] M. Luo, *Paracompactness in fuzzy topological spaces*, J.Math.Anal.Appl. 130 (1988) 55-77.

[60] M. Macho Stadler and M.A. de Prada Vicente, *On fuzzy subspaces*, Fuzzy Sets and Systems (1993) 365-373.

[61] S.R. Malghan and S.S. Benchalli, *On fuzzy topological spaces*, Glasnik Matematicki vol. 16 (36) (1981) 313-325.

[62] S.R. Malghan and S.S. Benchalli, *Open maps, closed maps and local compactness in fuzzy topological spaces*, J.Math.Anal.Appl. 99 (1984) 338-349.

[63] A.S. Mashhour and I.A. Hasanein, *On S-closed spaces and almost (nearly) strong paracompactness*, Annales de la Société Scientifique de Bruxelles, 95(I) (1981) 35-44.

[64] A.S. Mashhour, M.H. Ghanim, M.A. Fath Alla, *α-separation axioms and α-compactness in fuzzy topological spaces*, Rocky Mountain Journal of Mathematics vol.16 No.2 (1986) 591-600.

[65] R.G. McLean and M.W. Warner, *Locale theory and fuzzy topology*, Fuzzy Sets and Systems 54 (1993) 91-97.

[66] G. Meng, *Lowen's compactness in L-fuzzy topological spaces*, Fuzzy Sets and Systems 53 (1993) 329-335.

[67] G. Meng, *On the sum of L-fuzzy topological spaces*, Fuzzy Sets and Systems 59 (1993) 65-77.

[68] Y. Mingsheng, *A new approach for fuzzy topology I*, Fuzzy Sets and Systems 39 (1991) 303-321.

[69] C. de Mitri and E. Pascali, *Characterization of fuzzy topologies from neighbourhoods of fuzzy points*, J.Math.Anal.Appl. 93 (1983) 324-327.

[70] M.N. Mukherjee and S.P. Sinha, *Irresolute and almost open functions between fuzzy topological spaces*, Fuzzy Sets and Systems 29 (1989) 381-388.

[71] M.N. Mukherjee and S.P. Sinha, *On some weaker forms of fuzzy continuous and fuzzy open mappings on fuzzy topological spaces*, Fuzzy Sets and Systems 32 (1989) 103-114.

[72] M.N. Mukherjee and S.P. Sinha, *Almost compact fuzzy sets in fuzzy topological spaces*, Fuzzy Sets and Systems 38 (1990) 389-396.

[73] M.N. Mukherjee and B. Ghosh, *On nearly compact and θ-rigid fuzzy sets in fuzzy topological spaces*, Fuzzy Sets and Systems 43 (1991) 57-68.

[74] S. Nanda, *On fuzzy topological spaces*, Fuzzy Sets and Systems 19 (1986) 193-197.

[75] S. Nanda, *Strongly compact fuzzy topological spaces*, Fuzzy Sets and Systems 42 (1991) 259-262.

[76] T. Noiri, *Properties of S-closed spaces*, Acta

Mathematica Acad. Sci Hungaricae 35 (1980) 431-436.

[77] T. Noiri, *On RS-compact spaces*, J. Korean Math. Soc. 22 (1985) 19-34.

[78] Y.W. Peng, *The N-compactness in L-fuzzy topological spaces*, Acta Math. Sinicia 29 (1986) 555-558 (translation in Physical Science).

[79] M.A. de Prada Vicente and M. Macho Stadler, *t-pre-filter theory*, Fuzzy Sets and Systems 38 (1990) 115-124.

[80] P.M. Pu and Y.M. Liu, *Fuzzy topology I, Neighbourhood structure of a fuzzy point and Moore-Smith convergence*, J.Math.Anal.Appl. 76 (1980) 571-599.

[81] P.M. Piu and Y.M. Liu, *Fuzzy topology II, Product and Quotient Spaces*, J.Math.Anal.Appl. 77 (1980) 20-37.

[82] A.A. Ramadan, *Smooth topological spaces*, Fuzzy Sets and Systems 48 (1992) 371-375.

[83] S.E. Rodabaugh, *The Hausdorff separation axiom for fuzzy topological spaces*, Topology and its Appl. 11 (1980) 319-334.

[84] S.E. Rodabaugh, *A categorical accommodation of various notions of fuzzy topology*, Fuzzy Sets and Systems 9 (1983) 241-265.

[85] S.E. Rodabaugh, *Point-Set lattice-theoretic topology*, Fuzzy Sets and Systems 40 (1991) 297-345.

[86] M. Sarkar, *On fuzzy topological spaces*, J.Math. Anal.Appl. 79 (1981) 384-394.

[87] M. Sarkar, *On L-fuzzy topological spaces*, J.Math. Anal.Appl. 84 (1981) 431-442.

[88] R.D. Sarma and N. Ajmal, *A new approach to fuzzy topological spaces and fuzzy perfect mappings*, unpublished.

[89] M.K. Singal and Asha Mathur, *On nearly-compact spaces*, Bull.Un.Mat.Ital. 4(6) (1969) 702-710.

[90] M.K. Singal and N. Rajvanshi, *Regularly open sets in fuzzy topological spaces*, Fuzzy Sets and Systems 50 (1992) 343-353.

[91] R. Srivastava, S.N. Lal and A.K. Srivastava, *Fuzzy Hausdorff topological spaces*, J.Math.Anal.Appl. 81 (1981) 497-506.

[92] R. Srivastava, S.N. Lal and A.K. Srivastava, *Fuzzy* $T_1$*-topological spaces*, J.Math.Anal.Appl. 102 (1984) 442-448.

[93] T. Thompson, *S-closed spaces*, Proceedings of the American Mathematical Society 60 (1976) 335-338.

[94] T. Thompson, *Semicontinuous and Irresolute Images of S-closed spaces*, Amer. Math. Soc. 66 (1977) 359-362.

[95] G.J. Wang, *A new fuzzy compactness defined by fuzzy nets*, J.Math.Anal.Appl. 94 (1983) 1-23.

[96] G. Wang, *Generalised topological molecular lattices*, Sci.Sinica 8 (1984) 785-798.

[97] G. Wang, *Theory of topological molecular lattices*, Fuzzy Sets and Systems 47 (1992) 351-376.

[98] M.W. Warner, *Fuzzy topology with respect to continuous lattice*, Fuzzy Sets and Systems 35 (1990) 85-91.

[99] M.W. Warner, *Frame-fuzzy points and membership*, Fuzzy Sets and Systems 42 (1991) 335-344.

[100] M.W. Warner and R.G. McLean, *On compact Hausdorff L-fuzzy spaces*, Fuzzy Sets and Systems 56 (1993) 103-110.

[101] M.D. Weiss, *Fixed points, separation and induced topologies for fuzzy sets*, J.Math.Anal.Appl. 50 (1975) 142-150.

[102] C.K. Wong, *Covering properties of fuzzy topological spaces*, J.Math.Anal.Appl. 43 (1973) 697-704.

[103] C.K. Wong, *Fuzzy points and local properties of fuzzy topology*, J.Math.Anal.Appl. 46 (1974) 316-328.

[104] P. Wuyts and R. Lowen, *On local and global measures of separation in fuzzy topological spaces*, Fuzzy Sets and Systems 19 (1986) 51-80.

[105] J. Xu, *On the I-F Paracompactness concerning the F-compactness in L-fuzzy topological spaces*, Proceedings of Fifth IFSA World Congress (1993) 344-345.

[106] J. Xu, *On the II-F paracompactness concerning the F-compactness in L-fuzzy topological spaces*, Proceedings of Fifth IFSA World Congress (1993) 346-347.

[107] L.X. Xuan, *Sequential compactness*, Fuzzy Sets and Systems 35 (1990) 93-100.

[108] L.A. Zadeh, *Fuzzy Sets*, Inform.Contr. 8 (1965) 338-353.

[109] L.A. Zadeh, *A fuzzy-set-theoretic interpretation of linguistic hedges*, J. Cybernet 2(3) (1972) 4-34.

[110] D. Zhao, *The N-compactness in L-fuzzy topological spaces*, J.Math.Anal.Appl. 128 (1987) 64-79.