STRUCTURE OF BLOCKS WITH NORMAL DEFECT AND ABELIAN p' INERTIAL QUOTIENT

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Abstract. Let k be an algebraically closed field of prime characteristic p. Let kGe be a block of a group algebra of a finite group G, with normal defect group P and abelian p' inertial quotient L. Then we show that kGe is a matrix algebra over a quantised version of the group algebra of a semidirect product of P with a certain subgroup of L. To do this, we first examine the associated graded algebra, using a Jennings–Quillen style theorem.

As an example, we calculate the associated graded of the basic algebra of the non-principal block in the case of a semidirect product of an extraspecial p-group P of exponent p and order p^3 with a quaternion group of order eight with the centre acting trivially. In the case $p = 3$ we give explicit generators and relations for the basic algebra as a quantised version of kP. As a second example, we give explicit generators and relations in the case of a group of shape $2^{1+4} : 3^{1+2}$ in characteristic two.

1. Introduction

Throughout this paper p is a prime and k is an algebraically closed field of characteristic p. The study of blocks with normal defect groups has a long history, starting with the work of Brauer [6], and continuing with Reynolds [13], Dade [7], and Külshammer [11]. In the case of abelian normal defect, abelian inertial quotient and one simple module, explicit descriptions of the basic algebra were given by Benson and Green [3], and Holloway and Kessar [8]. Dropping the hypothesis of one simple module led to our paper [4]. The main structural feature of the basic algebras calculated in these papers is that they appear to be quantised versions of the group algebras of semidirect products of a defect group and a subgroup of the inertial quotient.

The purpose of this paper is to generalise the results from [4] to blocks of group algebras over k of finite groups that have a normal defect group P which is no longer necessarily abelian, but still with abelian p' inertial quotient L. By a theorem of Külshammer [11], any such block is isomorphic to a matrix algebra over a twisted group algebra $k_\alpha(P \rtimes L)$ of the semidirect product $P \rtimes L$, for some $\alpha \in H^2(L, k^\times)$, inflated to $P \rtimes L$. So there is a central p'-extension

$$1 \to Z \to H \to L \to 1$$

and an idempotent e in kZ, such that $k_\alpha(P \rtimes L) \cong kGe$, where $G = P \rtimes H$.

Theorem 1.1. *With the notation and hypotheses above, let $\tilde{\mathfrak{A}}$ be a basic algebra of the twisted group algebra $k_\alpha(P \rtimes L)$. Then $k_\alpha(P \rtimes L)$ is a matrix algebra over $\tilde{\mathfrak{A}}$ and $\tilde{\mathfrak{A}}$ has an explicit presentation as a quantised version of the group algebra $k(P \rtimes Z(H)/Z)$.*

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For the precise presentation and the proof, see Section 4. There are several new ingredients required to extend the results from [4] to nonabelian defect groups. We first consider the associated graded $\mathsf{gr}_*(kGe) = \bigoplus_{n\geqslant 0} J^n(kGe)/J^{n+1}(kGe)$ of kGe, briefly reviewed in the next section, and make use of the Jennings–Quillen theorem [10, 12] and Semmen [14]. We show that $\mathsf{gr}_*(kGe)$ is isomorphic to a matrix algebra over a quantised version of the associated graded of the group algebra of the group $P \rtimes (Z(H)/Z)$. Specialising to the case $\alpha = 0$, we get a presentation of $\mathsf{gr}_*(k(P \rtimes L))$ which we have not seen before in the literature (see Remark 3.12). The exact relations are stated in Theorem 3.8; see also Theorems 3.11 and 3.14 and Corollary 3.15. We then show that this may be ungraded to exhibit the basic algebra of kGe as a quantised version of the group algebra of $P \rtimes (Z(H)/Z)$, see Section 4.

In Section 5, in order to illustrate the main results, we explicitly calculate the following examples of blocks with a normal extraspecial defect group of order p^3 and exponent p having a single isomorphism class of simple modules.

Theorem 1.2. *Suppose that p is odd. Let P be an extraspecial group of order p^3 and exponent p, let H be a quaternion group of order 8 acting on P with $Z(H)$ acting trivially, and with the two generators of H inverting the two generators of P. Set $G = P \rtimes H$. The basic algebra of the associated graded $\mathsf{gr}_*(kP)$ of kP is given by generators x, y, z, subject to the relations*

$$x^p = 0, \quad y^p = 0, \quad xy - yx = z, \quad xz - zx = 0, \quad yz - zy = 0$$

(these imply $z^p = 0$) while the basic algebra of the associated graded $\mathsf{gr}_*(kGe)$ of the non-principal block e of kG is given by generators x, y, z, subject to the relations

$$\mathsf{x}^p = 0, \quad \mathsf{y}^p = 0, \quad \mathsf{xy} + \mathsf{yx} = \mathsf{z}, \quad \mathsf{xz} + \mathsf{zx} = 0, \quad \mathsf{yz} + \mathsf{zy} = 0$$

(these imply $\mathsf{z}^p = 0$).

In the case $p = 3$, we can be more precise and explicitly describe the algebra kP and a basic algebra of kGe by 'ungrading' the previous Theorem.

Theorem 1.3. *With the notation of the previous theorem, assume that $p = 3$. The algebra kP is given by generators $\tilde{x}$, $\tilde{y}$, $\tilde{z}$, subject to the relations*

$$\tilde{x}^3 = 0, \quad \tilde{y}^3 = 0, \quad \tilde{x}\tilde{y} - \tilde{y}\tilde{x} = \tilde{z}, \quad \tilde{x}\tilde{z} - \tilde{z}\tilde{x} = \tilde{z}\tilde{y}\tilde{z}, \quad \tilde{y}\tilde{z} - \tilde{z}\tilde{y} = -\tilde{z}\tilde{x}\tilde{z}$$

(these imply $\tilde{z}^3 = 0$) while a basic algebra of kGe is given by generators $\tilde{\mathsf{x}}$, $\tilde{\mathsf{y}}$, $\tilde{\mathsf{z}}$, subject to the relations

$$\tilde{\mathsf{x}}^3 = 0, \quad \tilde{\mathsf{y}}^3 = 0, \quad \tilde{\mathsf{x}}\tilde{\mathsf{y}} + \tilde{\mathsf{y}}\tilde{\mathsf{x}} = \tilde{\mathsf{z}}, \quad \tilde{\mathsf{x}}\tilde{\mathsf{z}} + \tilde{\mathsf{z}}\tilde{\mathsf{x}} = -\tilde{\mathsf{z}}\tilde{\mathsf{y}}\tilde{\mathsf{z}}, \quad \tilde{\mathsf{y}}\tilde{\mathsf{z}} + \tilde{\mathsf{z}}\tilde{\mathsf{y}} = -\tilde{\mathsf{z}}\tilde{\mathsf{x}}\tilde{\mathsf{z}}$$

(these imply $\tilde{\mathsf{z}}^3 = 0$).

In Section 6 we give an example in characteristic two, with P an extraspecial group of order 2^{1+4} and H an extraspecial group of order 3^{1+2}.

Finally, the appendix contains some corrections to the calculations in [4].

Notation. The bracket $[-,-]$ is used in three different ways, depending on the context: as commutator $[g,h] = ghg^{-1}h^{-1}$ for elements g, h in a multiplicatively written group, as Lie bracket in a Lie algebra, and as additive commutator $[a,b] = ab - ba$ for elements a, b in an associative algebra.

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2. The associated graded

The associated graded of a finite-dimensional k-algebra A is the graded algebra

$$\mathsf{gr}_*(A) = \bigoplus_{n\geqslant 0} J^n(A)/J^{n+1}(A),$$

with the summands $J^n(A)/J^{n+1}(A)$ in degree n, where we adopt the convention $J^0(A) = A$. The image in $A/J(A)$ of a block idempotent of A is a block idempotent of $\mathsf{gr}_*(A)$, and this induces a bijection between the blocks of A and the blocks of $\mathsf{gr}_*(A)$. Similarly, the image in $A/J(A)$ of a primitive idempotent of A is a primitive idempotent in $\mathsf{gr}_*(A)$. It follows that A and $\mathsf{gr}_*(A)$ have the same quiver.

Let P be a finite p-group, L an abelian p'-subgroup of $\mathsf{Aut}(P)$, and let $\alpha \in H^2(L, k^\times)$. Since k is algebraically closed, the canonical group homomorphism $Z^2(G, k^\times) \to H^2(G, k^\times)$ splits (see for example Theorem 11.15 of Isaacs [9]). Thus we may represent α by a 2-cocycle having the same order in $Z^2(G, k^\times)$ as its image in $H^2(G, k^\times)$, still denoted by α. Such a choice of α yields a central p'-extension

$$1 \to Z \to H \to L \to 1$$

and a faithful character $\chi : Z \to k^\times$ such that $Z = [H, H]$ and such that for some choice of inverse images $\hat{x}$ in H for all x, we have

$$\alpha(x, y) = \chi(\hat{x}\hat{y}\widehat{xy}^{-1})$$

for all x, $y \in L$. Moreover, $|Z|$ is equal to the order of α in $H^2(L, k^\times)$; that is, the subgroup of $k^\times$ generated by the values of α is equal to $\chi(Z)$.

Set $G = P \rtimes H$, where H acts on P via the canonical map $H \to L$, so that $Z = C_H(P) \leqslant Z(H)$, and hence $Z \leqslant Z(G)$. Thus the idempotent

$$e = \frac{1}{|Z|} \sum_{z \in Z} \chi(z^{-1})z,$$

is a non-principal block of kG, and the canonical surjection $G \to P \rtimes L$ with kernel Z induces an algebra isomorphism

$$kGe \xrightarrow{\cong} k_\alpha(P \rtimes L),$$

where α is inflated to $P \rtimes L$ via the canonical surjection $P \rtimes L \to L$.

We wish to describe kGe. This being difficult, we tackle first the associated graded algebra

$$\mathsf{gr}_*(kGe) = \bigoplus_{n\geqslant 0} J^n(kGe)/J^{n+1}(kGe).$$

Our goal is to give an explicit presentation of this as a quantum deformation of the corresponding associated graded for the (untwisted) group algebra $k(P \rtimes Z(H)/Z)$.

First, we recall the Jennings–Quillen theorem [10, 12] for the associated graded of kP. Our treatment follows Section 3.14 of [2]. For $r \geqslant 1$, we have dimension subgroups

$$F_r(P) = \{g \in P \mid g - 1 \in J^r(kP)\}.$$

Thus $F_1(P) = P$, $F_2(P) = \Phi(P)$, $[F_r(P), F_s(P)] \subseteq F_{r+s}(P)$, and if $g \in F_r(P)$ then $g^p \in F_{pr}(P)$. Furthermore, $F_r(P)$ is the most rapidly descending central series with these properties. Define

$$\mathsf{Jen}_*(P) = \bigoplus_{r \geqslant 1} k \otimes_{\mathbb{F}_p} F_r(P)/F_{r+1}(P).$$

Then $\mathsf{Jen}_*(P)$ is a p-restricted Lie algebra with Lie bracket induced by taking commutators in P and pth power map coming from taking pth powers in P. As a restricted Lie algebra, $\mathsf{Jen}_*(P)$ is generated by its degree one elements because the subgroups $F_r(P)$ form the lowest central series with the properties mentioned above. Let $\mathcal{U}\mathsf{Jen}_*(P)$ be the restricted universal enveloping algebra of $\mathsf{Jen}_*(P)$ over k. As an associative algebra, $\mathcal{U}\mathsf{Jen}_*(P)$ is generated by its degree one elements. The commutator $[g, h]$ of two elements g, $h \in P$ becomes the Lie bracket of the images of g, h in $\mathsf{Jen}_*(P)$, and the image of that Lie bracket in $\mathcal{U}\mathsf{Jen}_*(P)$ is in turn equal to the additive commutator of the images of g, h in the associative algebra $\mathcal{U}\mathsf{Jen}_*(P)$.

The Jennings–Quillen theorem states that there is a k-algebra isomorphism

$$\mathcal{U}\mathsf{Jen}_*(P) \to \mathsf{gr}_*(kP)$$

which for any r and any $g \in F_r(P)$ sends the image of g in $F_r(P)/F_{r+1}(P)$ to the image of $g - 1$ in $\mathsf{gr}_*(kP)$.

The group action of H on P induces an action of H on $\mathsf{Jen}_*(P)$ as a restricted Lie algebra, because the Lie bracket in $\mathsf{Jen}_*(P)$ is induced by taking commutators in P and the p-power map in $\mathsf{Jen}_*(P)$ is induced by taking p-th powers in P. The Jennings–Quillen map is equivariant with respect to H, and therefore extends to an isomorphism

$$\mathcal{U}\mathsf{Jen}_*(P) \rtimes H \xrightarrow{\cong} \mathsf{gr}_*(kP) \rtimes H \xrightarrow{\cong} \mathsf{gr}_*(kG),$$

where the second isomorphism uses the fact that $J(kG) = J(kP)kG = kGJ(kP)$ since H is a p'-group (cf. [14, Theorem 4]). Since we have a canonical bijection between the blocks of kG and the blocks of $\mathsf{gr}_*(kG)$ as described at the beginning of this section, it follows that the blocks of both kG and $\mathsf{gr}_*(kG)$ are the idempotents in kZ.

Remark 2.1. If e is an idempotent in kH, then the restriction of the projective module kGe to P is a direct sum of $\dim_k(kHe)$ copies of kP. Furthermore, the radical layers of kGe as a kG-module are the same as the radical layers as a kP-module. So we have

$$\begin{aligned}\sum_{i \geqslant 0} \dim_k J^i(kGe)/J^{i+1}(kGe) &= \dim_k(kHe). \sum_{i \geqslant 0} \dim_k J^i(kP)/J^{i+1}(kP) \\ &= \dim_k(kHe). \prod_r \left(\frac{1 - t^{pr}}{1 - t^r}\right)^{\dim_k \mathsf{Jen}_r(P)}.\end{aligned}$$

It can also be seen by restriction to P that if e is a central idempotent in kH then the associated graded $\mathsf{gr}_*(kGe)$ of the algebra kGe is generated by its degree zero and degree one elements.

Remark 2.2. The algebra $\mathcal{U}\mathsf{Jen}_*(P)$ is a finite dimensional cocommutative Hopf algebra, which defines a connected unipotent finite group scheme $\mathcal{P}$ whose group algebra is $k\mathcal{P} \cong \mathcal{U}\mathsf{Jen}_*(P)$. The finite group H acts as automorphism on $\mathcal{P}$, so we may form the semidirect product $\mathcal{G} = \mathcal{P} \rtimes H$, which is again a finite group scheme.

3. The quantum relations

In this section, we define an algebra $\mathfrak{A}$, which will turn out to be a basic algebra for $\mathsf{gr}_*(kGe)$. The quantum commutation rules for $\mathfrak{A}$ are given in Theorem 3.8, and the fact that $\mathfrak{A}$ is indeed a basic algebra is shown in Corollary 3.15.

By [4, Proposition 3.1] we have a bijection

$$\mathsf{Irr}(Z(H)|\chi) \xrightarrow{\cong} \mathsf{Irr}(H|\chi), \qquad \phi \mapsto \tau_\phi$$

between one-dimensional characters of $Z(H)$ lying over χ and irreducible characters of H lying over χ, such that τ_ϕ lies over ϕ. The central idempotent corresponding to τ_ϕ is

$$e_\phi = \frac{1}{|Z(H)|} \sum_{h \in Z(H)} \phi(h^{-1})h.$$

Then $e = \sum_{\phi \in \mathsf{Irr}(Z(H)|\chi)} e_\phi$, and hence

$$kHe = \prod_{\phi \in \mathsf{Irr}(Z(H)|\chi)} kHe_\phi \ .$$

The factors kHe_ϕ are matrix algebras, corresponding to τ_ϕ, all of the same dimension. An element ξ of $\mathsf{Hom}(H/Z, k^\times)$ induces an algebra automorphism of kHe sending he to $\xi(h)^{-1}he$. This yields an action of $\mathsf{Hom}(H/Z, k^\times)$ on kHe by algebra automorphisms which in turn induces a permutation action of $\mathsf{Hom}(H/Z, k^\times)$ on the set of factors kHe_ϕ. The stabiliser of any factor is the subgroup $\mathsf{Irr}(H/Z(H))$ of $\mathsf{Irr}(H/Z)$ and elements of $\mathsf{Irr}(H/Z(H))$ act as inner automorphisms on each factor.

Choose $\phi_0 \in \mathsf{Irr}(Z(H)|\chi)$, and set $\tau = \tau_{\phi_0}$. For each $\phi \in \mathsf{Irr}(Z(H)|\chi)$, choose a one dimensional representation $\xi_\phi \in \mathsf{Irr}(H/Z)$ inflated to H whose restriction to $Z(H)$ is $\phi\phi_0^{-1}$, and so that $\xi_{\phi_0} = 1$. The ξ_ϕ form a set of coset representatives of $\mathsf{Irr}(H/Z(H))$ in $\mathsf{Irr}(H/Z)$. The algebra automorphism induced by ξ_ϕ sends e_{ϕ_0} to e_ϕ, hence restricts to an algebra isomorphism

$$kHe_{\phi_0} \cong kHe_\phi$$

sending he_{ϕ_0} to $\xi_\phi(h)^{-1}ee_\phi$. Taking the product over all ϕ yields a unital injective algebra homomorphism

$$kHe_{\phi_0} \to kHe$$

sending he_{ϕ_0} to $\sum_{\phi \in \mathsf{Irr}(Z(H)|\chi)} \xi_\phi(h)^{-1}he_\phi$. By the above, this homomorphism depends on the choice of the ξ_ϕ, but only up to inner automorphisms of kHe. We write $\mathfrak{M}$ for the image in kHe of the matrix algebra kHe_{ϕ_0} under this algebra homomorphism. This is a unital matrix subalgebra in kHe.

We have a canonical homomorphism $\rho\colon H \to \mathsf{Hom}(H, k^\times)$ sending g to $\rho(g)\colon h \mapsto \chi([h, g])$. The kernel of this homomorphism is $Z(H)$ and its image is $\mathsf{Hom}(H/Z(H), k^\times)$. For each

$\psi \in \mathsf{Irr}(H/Z)$ and each $\phi \in \mathsf{Irr}(Z(H)|\chi)$, we write for simplicity $\phi\psi$ instead of $\phi\,(\psi|_{Z(H)})$. Then $\xi_{\phi\psi}\xi_\phi^{-1}\psi^{-1}$ is trivial on $Z(H)$. So there exists an element $g_{\psi,\phi} \in H$ such that

$$\rho(g_{\psi,\phi}) = \xi_{\phi\psi}\xi_\phi^{-1}\psi^{-1},$$

or equivalently, such that

$$\chi([h, g_{\psi,\phi}]) = \xi_{\phi\psi}(h)\xi_\phi(h)^{-1}\psi(h)^{-1}$$

for all $h \in H$. We choose such elements $g_{\psi,\phi}$, one for each ψ and ϕ. Note that these elements are unique up to multiplication by elements in $Z(H)$.

For any $\psi, \eta \in \mathsf{Irr}(H/Z)$ and any $\phi \in \mathsf{Irr}(Z(H)|\chi)$, we have

$$\rho(g_{\eta,\phi\psi}g_{\psi,\phi}) = \rho(g_{\eta,\phi\psi})\rho(g_{\psi,\phi}) = \xi_{\phi\psi\eta}\xi_\phi^{-1}\eta^{-1}\psi^{-1} = \rho(g_{\eta\psi,\phi}).$$

Lemma 3.1. *Let $\psi_i, \eta_j \in \mathsf{Irr}(H/Z)$, $\phi_i, \zeta_j \in \mathsf{Irr}(Z(H)|\chi)$, $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$. Suppose that $\phi_i = \phi_{i-1}\psi_{i-1}$ for all $2 \leqslant i \leqslant m$. Then*

(i) *$g_{\psi_m,\phi_m}\dots g_{\psi_1,\phi_1} = g_{\psi_m\cdots\psi_1,\phi_1}z$ for some $z \in Z(H)$.*

(ii) *Suppose further that $\zeta_j = \zeta_{j-1}\psi_{j-1}$ for all $2 \leqslant j \leqslant n$, $\phi_1 = \zeta_1$ and $\psi_m \dots \psi_1 = \eta_n \dots \eta_1$. Then $g_{\psi_m,\phi_m}\dots g_{\psi_1,\phi_1} = g_{\eta_n,\zeta_n}\dots g_{\eta_1,\zeta_1}z'$ for some $z' \in Z(H)$.*

Proof. Since $Z = \mathsf{Ker}(\rho)$, (i) follows by repeated application of the equation displayed above the lemma. Part (ii) follows from (i) applied to both $g_{\psi_m,\phi_m}\dots g_{\psi_1,\phi_1}$ and $g_{\eta_n,\zeta_n}\dots g_{\eta_1,\zeta_1}$. □

3.2. Since k is algebraically closed, we may choose a k-basis $w_1, \dots, w_m$ of $\mathsf{Jen}_*(P)$, where $p^m = |P|$, consisting of homogeneous eigenvectors of the action of H. We arrange the indices in such a way that if $i \leqslant j$ then $\deg(w_i) \leqslant \deg(w_j)$. Then for each w_i there is a character ψ_i of L, inflated to H, such that

$${}^g w_i = \psi_i(g)w_i$$

for $g \in H$. Define structure constants $c_{i,j,k}$ and $d_{i,k}$ for $\mathsf{Jen}_*(P)$ via

$$[w_i, w_j] = \sum_k c_{i,j,k}w_k, \qquad w_i^{[p]} = \sum_k d_{i,k}w_k.$$

Here, $[w_i, w_j]$ denotes the Lie bracket and $w_i^{[p]}$ the p-restriction map in $\mathsf{Jen}_*(P)$. We have

$${}^g[w_i, w_j] = [{}^g w_i, {}^g w_j] = \psi_i(g)\psi_j(g)[w_i, w_j],$$

$${}^g(w_i^{[p]}) = ({}^g w_i)^{[p]} = (\psi_i(g)w_i)^{[p]} = \psi_i(g)^p w_i^{[p]}.$$

It follows that if $c_{i,j,k} \neq 0$ then $\psi_i\psi_j = \psi_k$, and if $d_{i,k} \neq 0$ then $\psi_i^p = \psi_k$.

By the Poincaré–Birkhoff–Witt (PBW) theorem for restricted Lie algebras (Jacobson [**?**], page 190), the algebra $\mathcal{U}\mathsf{Jen}_*(P) \cong \mathsf{gr}_*(kP)$ has a basis $\mathcal{B}$ consisting of words $w_{i_1}\dots w_{i_r}$ where $i_1 \leqslant \dots \leqslant i_r$, and each index is repeated at most $p-1$ times (so we are writing w_i^a as $w_i \dots w_i$). We follow the convention is that the empty word denotes the identity element in degree zero. The element $w_{i_1}\dots w_{i_r}$ is an eigenvector for the conjugation action of H, with character $\psi_{i_1}\dots\psi_{i_r}$.

In what follows we identify $\mathsf{Jen}_*(P)$ with its image in $\mathcal{U}\mathsf{Jen}_*(P) \rtimes H$. The calculations that follow are similar to those in Section 4 of [4] (with the corrections described in Section 7 below). For any $\phi \in \mathsf{Irr}(Z(H)|\chi)$, w_i a basis element of $\mathsf{Jen}_*(P)$, with associated linear characters ψ_i, we write $g_{i,\phi}$ for the element $g_{\psi_i,\phi}$.

Lemma 3.3. *With the notation above, the following equations in $(\mathcal{U}\mathsf{Jen}_*(P) \rtimes H)e$ hold for all $h \in H$, all basis elements w_i of $\mathsf{Jen}_*(P)$, the associated linear characters $\psi_i \in \mathsf{Hom}(H, k^\times)$ and all $\phi \in \mathsf{Irr}(Z(H)|\chi)$.*

(i) $w_i e_\phi = e_{\phi\psi_i} w_i$.
(ii) $(g_{i,\phi} w_i)(\xi_\phi(h)^{-1} e_\phi\, h) = (\xi_{\phi\psi_i}(h)^{-1} e_{\phi\psi_i}\, h)(g_{i,\phi} w_i)$.
(iii) $g_{i,\phi} w_i e_\phi = e_{\phi\psi_i} g_{i,\phi} w_i$ *commutes with* $\mathfrak{M}$.

Proof. We have $hw_ih^{-1} = \psi_i(h)w_i$, hence $w_ih = \psi_i(h)^{-1}hw_i$. Thus if $h \in Z(H)$, then $\phi(h)^{-1}w_ih = \phi(h)^{-1}\psi_i(h)^{-1}hw_i$. Taking the sum over all $h \in Z(H)$ and dividing by $|Z(H)|$ shows (i). Note that $[g, h]e = \chi([g, h])e$ for all g, $h \in H$. Thus $g_{i,\phi}he = \chi([h, g_{i,\phi}])^{-1}hg_{i,\phi}e$. It follows that

$$\xi_\phi(h)^{-1}g_{i\phi}w_ihe_\phi = \xi_\phi(h)^{-1}\psi_i(h)^{-1}g_{i,\phi}hw_ie_\phi = \xi_\phi(h)^{-1}\psi_i(h)^{-1}\chi([h, g_{i,\phi}])^{-1}hg_{i,\phi}e_{\phi\psi_i}w_i \ ,$$

where we have used (i). Note that $e_{\phi\psi_i}$ is central in kH. Using the definition of ρ, the scalar in the last expression is $\xi_{\phi\psi_i}(h)^{-1}$. This shows (ii). The equality in (iii) is the special case of (ii) applied with $h = 1$. For the commutation with $\mathfrak{M}$, we need to check that the elements in the statement commute with expressions of the form $\sum_{\phi'} \xi_{\phi'}(h)^{-1}he_{\phi'}$. This follows easily using (ii) and the fact that the e_ϕ are pairwise orthogonal. □

Definition 3.4. We define $\mathsf{w}_{i,\phi} = g_{i,\phi}w_ie_\phi$, and let $\mathfrak{A}$ be the subalgebra of $(\mathcal{U}\mathsf{Jen}_*(P) \rtimes H)e$ generated by the elements e_ϕ and $\mathsf{w}_{i,\phi}$.

By Lemma 3.3, the subalgebras $\mathfrak{A}$ and $\mathfrak{M}$ of $(\mathcal{U}\mathsf{Jen}_*(P) \rtimes H)e$ commute.

Lemma 3.5. *The algebra $\mathfrak{A}$ is generated by the elements e_ϕ and $\mathsf{w}_{i,\phi}$ for those i such that the element w_i of $\mathsf{Jen}_*(P)$ has degree one.*

Proof. Since $\mathsf{Jen}_*(P)$ is generated by elements in degree one, there exists a basis $\mathcal{V}$ of $\mathcal{U}\mathsf{Jen}_*(P)$ consisting of a subset of the set of monomials in the degree one w_i's. Let w_t be an arbitrary element of the chosen basis of $\mathsf{Jen}_*(P)$ and write

$$w_t = \sum_{v\in\mathcal{V}} \alpha_v v.$$

If $u, u' \in \mathcal{U}\mathsf{Jen}_*(P)$ are eigenvectors for the H action corresponding to characters ψ and ψ' respectively, then uu' is an H-eigenvector with corresponding character $\psi\psi'$. From this it follows that if a monomial $v = w_{i_m} \dots w_{i_1}$ in degree one elements w_{i_j} is an element of $\mathcal{V}$ such that $\alpha_v \neq 0$, then $\psi_t = \psi_{i_m} \dots \psi_{i_1}$, where for each j, $1 \leqslant j \leqslant m$, $\psi_{i_j} \in \mathsf{Irr}(H/Z)$ is the character of H corresponding to the action on w_{i_j}. Let $\zeta \in \mathsf{Irr}(Z(H)|\chi)$ and let v be as above. By Lemma 3.1,

$$g_{\psi,\zeta} = g_{i_m,\phi_m} \cdots g_{i_1,\phi_1} z$$

where $z \in Z(H)$, $\phi_1 = \zeta$ and $\phi_j = \phi_{j-1}\psi_{i_{j-1}}$, $2 \leqslant j \leqslant m$. On the other hand, since every w_{i_j} is an eigenvector for the H action,

$$vg_{i_m,\phi_m} \cdots g_{i_1,\phi_1} = \beta_v w_{i_m} g_{i_m,\phi_m} \cdots w_{i_1} g_{i_1,\phi_1}$$

for some $\beta_v \in k^\times$. Since ze_ζ is a non-zero scalar multiple of e_ζ, the above equation and Lemma 3.3 (iii) give that

$$vg_{\psi,\zeta}e_\zeta = q_v \mathsf{w}_{i_m,\phi_m} \cdots \mathsf{w}_{i_1,\phi_1}$$

for some non-zero scalar q_v. Since all w_{i_j} are in degree one, it follows that

$$\mathsf{w}_{t,\zeta} = w g_{\psi,\zeta} e_\phi = \sum_{v\in\mathcal{V}} \alpha_v v g_{\psi,\zeta} e_\zeta$$

is a linear combination of monomials in the $\mathsf{w}_{i,\phi}$ for those i such that w_i has degree one.

□

Definition 3.6. We define elements $z_{i,j,\phi}$, $z'_{i,j,k,\phi}$ and $z''_{i,k,\phi}$ in $Z(H)$ as follows. By Lemma 3.1 we have

$$g_{j,\phi\psi_i} g_{i,\phi} = g_{i,\phi\psi_j} g_{j,\phi} z_{i,j,\phi}$$

for some $z_{i,j,\phi} \in Z(H)$. If $c_{i,j,k} \neq 0$ then

$$g_{j,\phi\psi_i} g_{i,\phi} = g_{k,\phi} z'_{i,j,k,\phi}$$

for some $z'_{i,j,k,\phi} \in Z(H)$. If $d_{i,k} \neq 0$ then

$$g_{i,\phi\psi_i^{p-1}} \dots g_{i,\phi\psi_i} g_{i,\phi} = g_{k,\phi} z''_{i,k,\phi}$$

for some $z''_{i,k,\phi} \in Z(H)$.

Remark 3.7. We have

$$z_{i,j,\phi} e_\phi = \phi(z_{i,j,\phi}) e_\phi, \qquad z'_{i,j,k,\phi} e_\phi = \phi(z'_{i,j,k,\phi}) e_\phi, \qquad z''_{i,k,\phi} e_\phi = \phi(z''_{i,k,\phi}) e_\phi.$$

Also, we have $z_{i,j,\phi} z_{j,i,\phi} = 1$, and if $c_{i,j,k} \neq 0$ then $z'_{i,j,k,\phi} = z'_{j,i,k,\phi} z_{i,j,\phi}$.

Theorem 3.8. *Defining constants*

$$\begin{aligned} q_{i,j,\phi} &= \psi_i(g_{j,\phi} z_{i,j,\phi}) \psi_j(g_{i,\phi}^{-1} z_{i,j,\phi}) \phi(z_{i,j,\phi}), \\ q'_{i,j,k,\phi} &= \psi_j(g_{i,\phi})^{-1} \psi_k(z'_{i,j,k,\phi}) \phi(z'_{i,j,k,\phi}), \\ q''_{i,k,\phi} &= \psi_i(g_{i,\phi\psi_i^{p-2}})^{-1} \dots \psi_i(g_{i,\phi\psi_i})^{-p+2} \psi_i(g_{i,\phi})^{-p+1} \psi_k(z''_{i,k,\phi}) \phi(z''_{i,k,\phi}), \end{aligned}$$

we have

$$\mathsf{w}_{j,\phi\psi_i} \mathsf{w}_{i,\phi} - q_{i,j,\phi} \mathsf{w}_{i,\phi\psi_j} \mathsf{w}_{j,\phi} = \sum_k c_{i,j,k} q'_{i,j,k,\phi} \mathsf{w}_{k,\phi} \tag{3.9}$$

$$\mathsf{w}_{i,\phi\psi_i^{p-1}} \dots \mathsf{w}_{i,\phi\psi_i} \mathsf{w}_{i,\phi} = \sum_k d_{i,k,\phi}\, q''_{i,k,\phi} \mathsf{w}_{k,\phi}. \tag{3.10}$$

By changing the choices of $g_{i,\phi}$ by elements of $Z(H)$, we may ensure that $z_{i,j,\phi} \in Z$, and then the formula for the parameters $q_{i,j,\phi}$ simplifies to

$$q_{i,j,\phi} = \psi_i(g_{j,\phi}) \psi_j(g_{i,\phi}^{-1}) \chi(z_{i,j,\phi}).$$

Proof. We have

$$\begin{aligned} \mathsf{w}_{j,\phi\psi_i} \mathsf{w}_{i,\phi} &= (g_{j,\phi\psi_i} w_j e_{\phi\psi_i})(g_{i,\phi} w_i e_\phi) \\ &= g_{j,\phi\psi_i} w_j g_{i,\phi} w_i e_\phi \\ &= \psi_j(g_{i,\phi})^{-1} g_{j,\phi\psi_i} g_{i,\phi} w_j w_i e_\phi \\ &= \psi_j(g_{i,\phi})^{-1} g_{j,\phi\psi_i} g_{i,\phi} (w_i w_j + [w_i, w_j]) e_\phi \\ &= \psi_j(g_{i,\phi})^{-1} g_{i,\phi\psi_j} g_{j,\phi} z_{i,j,\phi} w_i w_j e_\phi \\ &\quad + \psi_j(g_{i,\phi})^{-1} g_{j,\phi\psi_i} g_{i,\phi} [w_i, w_j] e_\phi \end{aligned}$$

$$
\begin{aligned}
&= \psi_j(g_{i,\phi})^{-1}\psi_i(z_{i,j,\phi})\psi_j(z_{i,j,\phi})g_{i,\phi\psi_j}g_{j,\phi}w_iw_jz_{i,j,\phi}e_\phi \\
&\quad + \sum_k c_{i,j,k}\psi_j(g_{i,\phi})^{-1}g_{k,\phi}z'_{i,j,k,\phi}w_ke_\phi \\
&= \psi_j(g_{i,\phi})^{-1}\psi_i(z_{i,j,\phi})\psi_j(z_{i,j,\phi})\psi_i(g_{j,\phi})\phi(z_{i,j,\phi})g_{i,\phi\psi_j}w_ig_{j,\phi}w_je_\phi \\
&\quad + \sum_k c_{i,j,k}\psi_j(g_{i,\phi})^{-1}\psi_k(z'_{i,j,k,\phi})g_{k,\phi}w_kz'_{i,j,k,\phi}e_\phi \\
&= \psi_i(g_{j,\phi}z_{i,j,\phi})\psi_j(g_{i,\phi}^{-1}z_{i,j,\phi})\phi(z_{i,j,\phi})(g_{i,\phi\psi_j}w_ie_{\phi\psi_j})(g_{j,\phi}w_je_\phi) \\
&\quad + \sum_k c_{i,j,k}\psi_j(g_{i,\phi})^{-1}\psi_k(z'_{i,j,k,\phi})\phi(z'_{i,j,k,\phi})g_{k,\phi}w_ke_\phi \\
&= q_{i,j,\phi}\mathsf{w}_{i,\phi\psi_j}\mathsf{w}_{j,\phi} + \sum_k c_{i,j,k}q'_{i,j,k,\phi}\mathsf{w}_{k,\phi}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\mathsf{w}_{i,\phi\psi_i^{p-1}}\dots\mathsf{w}_{i,\phi\psi_i}\mathsf{w}_{i,\phi} \\
&\qquad = (g_{i\phi\psi_i^{p-1}}w_ie_{\phi\psi_i^{p-1}})\dots(g_{i,\phi\psi_i}w_ie_{\phi\psi_i})(g_{i,\phi}w_ie_\phi) \\
&\qquad = g_{i,\phi\psi_i^{p-1}}w_i\dots g_{i,\phi\psi_i}w_ig_{i,\phi}w_ie_\phi \\
&\qquad = \psi_i(g_{i,\phi\psi_i^{p-2}})^{-1}\dots\psi_i(g_{i,\phi\psi_i})^{-p+2}\psi_i(g_{i,\phi})^{-p+1}(g_{i,\phi\psi_i^{p-1}}\dots g_{i,\phi\psi_i}g_{i,\phi})w_i^pe_\phi \\
&\qquad = \sum_k d_{i,k}\psi_i(g_{i,\phi\psi_i^{p-2}})^{-1}\dots\psi_i(g_{i,\phi\psi_i})^{-p+2}\psi_i(g_{i,\phi})^{-p+1}g_{k,\phi}z''_{i,k,\phi}w_ke_\phi \\
&\qquad = \sum_k d_{i,k}\psi_i(g_{i,\phi\psi_i^{p-2}})^{-1}\dots\psi_i(g_{i,\phi\psi_i})^{-p+2}\psi_i(g_{i,\phi})^{-p+1}\psi_k(z''_{i,k,\phi})g_{k,\phi}w_kz''_{i,k,\phi}e_\phi \\
&\qquad = \sum_k d_{i,k}\psi_i(g_{i,\phi\psi_i^{p-2}})^{-1}\dots\psi_i(g_{i,\phi\psi_i})^{-p+2}\psi_i(g_{i,\phi})^{-p+1}\psi_k(z''_{i,k,\phi})\phi(z''_{i,k,\phi})g_{k,\phi}w_ke_\phi \\
&\qquad = \sum_k d_{i,k}q''_{i,k,\phi}\mathsf{w}_{k,\phi}.
\end{aligned}
$$

For the final remark, just as in Lemma 4.12 (3) of [4], we may change the choices of $g_{i,\phi}$ by elements of $Z(H)$ to ensure that $z_{i,j,\phi} \in Z$, with the same argument. Then the characters ψ_i take value one on these elements, leading to the given simplifications of the constants. □

Recall that by Lemma 3.5, $\mathfrak{A}$ is generated by the e_ϕ and the $\mathsf{w}_{i,\phi}$ for those i such that the element w_i of $\mathsf{Jen}_*(P)$ has degree one.

Theorem 3.11. *The algebra $\mathfrak{A}$ is given as a quiver with relations kQ/I, where Q is the quiver with $|Z(H) : Z|$ vertices labelled $[\phi]$ corresponding to the idempotents $e_\phi \in kZ(H)$ lying over χ and directed edges*

$$[\phi] \xrightarrow{\ i\ } [\phi\psi_i]$$

corresponding to the element

$$\mathsf{w}_{i,\phi} = g_{i,\phi}w_ie_\phi = e_{\phi\psi_i}g_{i,\phi}w_i = e_{\phi\psi_i}g_{i,\phi}w_ie_\phi$$

for those i such that the element w_i of $\mathsf{Jen}_*(P)$ has degree one. The relations are those that follow from the structure constant relations of Theorem 3.8, where for each k such that w_k is in degree greater than or equal to 2, any $\mathsf{w}_{k,\zeta}$ appearing in Theorem 3.8 is replaced by *an element in kQ corresponding via Lemma 3.5 to an expression for $\mathsf{w}_{k,\zeta}$ in terms of the $\mathsf{w}_{i,\phi}$ such that w_i has degree one. There is a PBW style basis $\mathcal{B}'$ for $\mathfrak{A}$ (described below in the proof), consisting of composable monomials in the $\mathsf{w}_{i,\phi}$, giving* $\dim(kQ/I) = \dim(\mathfrak{A}) = |Z(H) : Z| \cdot |P|$.

Proof. By Lemma 3.5, $\mathfrak{A}$ is generated by the idempotents e_ϕ and the elements $\mathsf{w}_{i,\phi}$. By Lemma 3.3 and Theorem 3.8 they satisfy the given relations. Thus we have a surjective homomorphism from kQ/I to $\mathfrak{A}$ taking $[\phi]$ to e_ϕ and $[\phi] \xrightarrow{\ i\ } [\phi\psi_i]$ to $\mathsf{w}_{i,\phi}$.

The relations holding in kQ/I allow us to write every element of $\mathfrak{A}$ as a linear combination of elements of the set $\mathcal{B}'$ consisting of the e_ϕ and composable monomials in the $\mathsf{w}_{i,\phi}$ where the indices i are in order, and each index i is repeated at most $p-1$ times. The number of such monomials (including the e_ϕ) is $|Z(H) : Z| \cdot |P|$. Replacing $[\phi]$ by e_ϕ, $\mathsf{w}_{i,\phi}$ by $[\phi] \xrightarrow{\ i\ } [\phi\psi_i]$ for those i such that w_i has degree one and $\mathsf{w}_{i,\phi}$ by their chosen lifts in kQ for those i such that w_i has degree greater than or equal to two, we see by the same reasoning that $\dim_k Q/I$ is at most $|Z(H) : Z| \cdot |P|$.

If there were a linear relation in $\mathfrak{A}$ between the monomials in $\mathcal{B}'$, then there would be a linear relation between the ones of maximal length, namely length $m(p-1)$. There is one of these for each ϕ, and they are linearly independent elements of the socle of kG because they are non-zero elements of different projective summands kGe_ϕ. Thus $\dim(\mathfrak{A})$ is equal to $|Z(H) : Z| \cdot |P|$ and $kQ/I \to \mathfrak{A}$ is an isomorphism. □

Remark 3.12. The group algebra of the semidirect product $P \rtimes Z(H)/Z$, with the action given by restricting the action of H/Z on P, has only one block. We can perform the computations above for this group, and the results look similar, except that the factors of $\phi(z_{i,j,\phi})$, $\phi(z'_{i,j,k,\phi})$, and $\phi(z''_{i,k,\phi})$ in the definitions of $q_{i,j,\phi}$, $q'_{i,j,k,\phi}$, and $q''_{i,k,\phi}$ are missing in Theorem 3.11. So removing these factors, the relations in Theorem 3.8 are the relations in $\mathsf{gr}_*(k(P \rtimes Z(H)/Z)) \cong \mathcal{U}\mathsf{Jen}_*(P) \rtimes Z(H)/Z$. Thus we can see $\mathfrak{A}$ as a quantum deformation of the algebra $\mathcal{U}\mathsf{Jen}_*(P) \rtimes Z(H)/Z$. Also, we note that in the case that $\alpha = 0$, we have $Z = 1$, $H = L$ and Theorem 3.11 provides an explicit presentation of $\mathsf{gr}_*(k(P \rtimes L))$.

As in [4], we now make use of the following lemma (see Chapter 3, Corollary 4.3 in Bass [1]).

Lemma 3.13. *Let $A \leqslant B$ be k-algebras with A an Azumaya algebra (that is, a finite-dimensional central separable k-algebra). Then the map $A \otimes_k C_B(A) \to B$ is an isomorphism.* □

Theorem 3.14. *The multiplication in $(\mathcal{U}\mathsf{Jen}_*(P) \rtimes H)e$ induces an isomorphism*

$$\mathfrak{A} \otimes_k \mathfrak{M} \xrightarrow{\cong} (\mathcal{U}\mathsf{Jen}_*(P) \rtimes H)e.$$

Proof. The proof is similar to that of Theorem 4.15 of [4]. Applying Lemma 3.13 with $A = \mathfrak{M}$ and B the subalgebra generated by $\mathfrak{A}$ and $\mathfrak{M}$, we see that the given map is injective. The dimensions are given by $\dim(\mathfrak{A}) = |Z(H) : Z| \cdot |P|$, $\dim(\mathfrak{M}) = |H : Z(H)|$ and $\dim((\mathcal{U}\mathsf{Jen}_*(P) \rtimes H)e) = \dim(kGe) = |G : Z|$, so $\dim((\mathcal{U}\mathsf{Jen}_*(P) \rtimes H)e) = \dim(\mathfrak{A}) \cdot \dim(\mathfrak{M})$ and the map is an isomorphism. □

Corollary 3.15. *We have*

$$\mathsf{gr}_*(kGe) \cong (\mathcal{U}\mathsf{Jen}_*(P) \rtimes H)e \cong \mathsf{Mat}_m(\mathfrak{A}),$$

where $m = \sqrt{|H : Z(H)|}$. In particular, $\mathfrak{A}$ is a basic algebra of $\mathsf{gr}_*(kGe)$. □

Corollary 3.16. *The algebra $\mathfrak{A}$ is generated by its degree zero and degree one elements.*

Proof. This follows from Corollary 3.15 and Remark 2.1. □

4. Ungrading the relations

We saw in the last section that the relations for the basic algebra of $\mathsf{gr}_*(kGe)$ are a quantised version of the relations for $\mathsf{gr}_*(kP \rtimes (Z(H)/Z))$. In this section, we show that the same holds without taking the associated graded.

Since $|H|$ is coprime to p, the characteristic of k, we can choose invariant complements to $J^{n+1}(kP)$ in $J^n(kP)$ for each $n \geqslant 0$. Let $w_1, \dots, w_m$ be the basis of $\mathsf{Jen}_*(P)$ chosen in Section 3.2, and let $\mathcal{B}$ be the resulting PBW basis of $\mathcal{U}\mathsf{Jen}_*(P) \cong \mathsf{gr}_*(kP)$ described there. Regarding $\mathsf{Jen}_*(P)$ as an k-linear subspace of $\mathsf{gr}_*(kP)$, this enables us to choose representatives $\tilde{w}_i$ in kP of the w_i in such a way that

$$g\tilde{w}_i g^{-1} = \psi_i(g)\tilde{w}_i.$$

Let $\tilde{\mathcal{B}}$ be the corresponding basis of kP consisting of monomials in the $\tilde{w}_i$. That is, if $w_{i_1} \dots w_{i_r}$ is an element of $\mathcal{B}$ then the corresponding element of $\tilde{\mathcal{B}}$ is $\tilde{w}_{i_1} \dots \tilde{w}_{i_r}$. An element $\tilde{w}_{i_1} \dots \tilde{w}_{i_r}$ of $\tilde{\mathcal{B}}$ is an eigenvector for the action of H for the character $\psi_{i_1} \dots \psi_{i_r}$.

When we ungrade a relation of the form $[w_i, w_j] = \sum_k c_{i,j,k} w_k$, we obtain a relation of the form

$$[\tilde{w}_i, \tilde{w}_j] = \sum_k c_{i,j,k}\tilde{w}_k + y_{i,j} \tag{4.1}$$

in kP, where $y_{i,j}$ is a linear combination of elements of $\tilde{\mathcal{B}}$ in a higher power of the radical than $\deg(w_i) + \deg(w_j)$. Moreover, each basis monomial $\tilde{w}_{i_1} \dots \tilde{w}_{i_r}$ that occurs in y_{ij} is an eigenvector for the character $\psi_i\psi_j$, and consequently

$$\psi_{i_1} \dots \psi_{i_r} = \psi_i\psi_j. \tag{4.2}$$

Similarly, when we ungrade a relation of the form $w_i^p = \sum_k d_{i,k} w_k$, we obtain a relation of the form

$$\tilde{w}_i^p = \sum_k d_{i,k}\tilde{w}_k + y_i'' \tag{4.3}$$

in kP, where y_i'' is a linear combination of monomial basis elements in a higher power of the radical than $p.\deg(w_i)$. Each basis monomial $\tilde{w}_{i_1} \dots \tilde{w}_{i_r}$ that occurs in y_i'' is an eigenvector for the character ψ_i^p and consequently

$$\psi_{i_r} \dots \psi_{i_1} = \psi_i^p. \tag{4.4}$$

Definition 4.5. As in Definition 3.4, we define $\tilde{\mathsf{w}}_{i,\phi} = g_{i,\phi}\tilde{w}_i e_\phi$. Then $\tilde{\mathsf{w}}_{i,\phi}$ commutes with $\mathfrak{M}$. We define $\tilde{\mathfrak{A}}$ to be the subalgebra of kGe generated by the elements e_ϕ and $\tilde{\mathsf{w}}_{i,\phi}$. For an element $\tilde{w} = \tilde{w}_{i_1} \dots \tilde{w}_{i_r}$ of $\tilde{\mathcal{B}}$ and a character $\phi \in \mathsf{Irr}(Z(H)|\chi)$, set $\tilde{\mathsf{w}}_\phi = \tilde{\mathsf{w}}_{i_1,\phi\psi_1\dots\psi_r} \dots \tilde{\mathsf{w}}_{i_{r-1},\phi\psi_r}\tilde{\mathsf{w}}_{i_r,\phi}$. Denote by $\tilde{\mathcal{B}}'$ the subset of $\tilde{\mathfrak{A}}$ consisting of the elements $\tilde{x}_\phi$ for $\tilde{w}$ in $\tilde{\mathcal{B}}$ and $\phi \in \mathsf{Irr}(Z(H)|\chi)$. We shall see below in Theorem 4.7 that $\tilde{\mathcal{B}}'$ is a basis for $\tilde{\mathfrak{A}}$.

Proposition 4.6. *The elements $\tilde{\mathsf{w}}_{i,\phi}$ satisfy the following relations.*

$$\tilde{\mathsf{w}}_{j,\phi\psi_i}\tilde{\mathsf{w}}_{i,\phi} - q_{i,j,\phi}\tilde{\mathsf{w}}_{i,\phi}\tilde{\mathsf{w}}_{j,\phi\psi_i} = \sum_k c_{i,j,k}q'_{i,j,k,\phi}\tilde{\mathsf{w}}_{k,\phi} + \psi_j(g_{i,\phi})^{-1}g_{j,\phi\psi_i}g_{i,\phi}y_{i,j}e_\phi.$$

$$\tilde{\mathsf{w}}_{i,\phi\psi_i^{p-1}} \dots \tilde{\mathsf{w}}_{i,\phi\psi_i}\tilde{\mathsf{w}}_{i,\phi} = \sum_k d_{i,k} q''_{i,k,\phi} \tilde{\mathsf{w}}_{k,\phi} + \psi_i(g_{i,\phi\psi_i^{p-2}})^{-1} \dots$$
$$\dots \psi_i(g_{i,\phi\psi_i})^{-p+2} \psi_i(g_{i,\phi})^{-p+1} (g_{i,\phi\psi_i^{p-1}} \dots g_{i,\phi\psi_i} g_{i,\phi}) y''_i e_\phi.$$

Moreover, suppose that $y_{i,j} = \sum_{\tilde{w}\in\tilde{\mathcal{B}}} c_{i,j,\tilde{w}} \tilde{w}$ *and* $y''_i = \sum_{\tilde{w}\in\tilde{\mathcal{B}}} d_{i,\tilde{w}} \tilde{w}$*. For each* $\tilde{w} \in \tilde{\mathcal{B}}$*, there exist elements* $q'_{i,j,\tilde{w},\phi}$ *and* $q''_{i,\tilde{w},\phi}$ *of* $k^\times$ *such that*

$$g_{j,\phi\psi_i} g_{i,\phi} y_{i,j} e_\phi = \sum_{\tilde{w}\in\tilde{\mathcal{B}}} q'_{i,j,\tilde{w},\phi} c_{i,j,\tilde{w}} \tilde{\mathsf{w}}_\phi.$$
$$(g_{i,\phi\psi_i^{p-1}} \dots g_{i,\phi\psi_i} g_{i,\phi}) y''_i e_\phi = \sum_{\tilde{w}\in\tilde{\mathcal{B}}} q''_{i,\tilde{w},\phi} d_{i,\tilde{w}} \tilde{\mathsf{w}}_\phi.$$

Proof. Following through the proof of relation (3.9), we can replace each w with w until the sixth line, where we have to use (4.1) for the commutator. At this point, the extra term is

$$\psi_j(g_{i,\phi})^{-1} g_{j,\phi\psi_i} g_{i,\phi} y_{i,j} e_\phi.$$

Similarly, following through the proof of relation (3.10), we can replace each w with w until the fourth line, where we have to use (4.3). At this point, the extra term is

$$\psi_i(g_{i,\phi\psi_i^{p-2}})^{-1} \dots \psi_i(g_{i,\phi\psi_i})^{-p+2} \psi_i(g_{i,\phi})^{-p+1} (g_{i,\phi\psi_i^{p-1}} \dots g_{i,\phi\psi_i} g_{i,\phi}) y''_i e_\phi.$$

By Lemma 3.1 and Equation 4.2, for each $\mathsf{w} = \mathsf{w}_{i_r} \dots \mathsf{w}_{i_1} \in \tilde{\mathcal{B}}$ such that $c_{i,j,\tilde{w},\phi} \neq 0$,

$$g_{j,\phi\psi_i} g_{i,\phi} = g_{i_r,\phi\psi_1\dots\psi_r} \cdots g_{i_2,\phi\psi_1} g_{i_1,\phi} z$$

for some $z \in Z(H)$. The second assertion follows from this by the fact that for any $g \in H$, any $z \in Z(H)$, any $\tilde{w}_i$, and any $\zeta \in \mathsf{Irr}(Z(H)|\chi)$, $g\tilde{w}_i = \psi_i(g)\tilde{w}_i g$ is a scalar multiple of $\tilde{w}_i g$, $ze_\zeta = \zeta(z)e_\zeta$ is a scalar multiple of e_ζ and $g\tilde{w}_i e_\zeta = e_{\zeta\psi_i} g\tilde{w}_i e_\phi$. The last assertion follows in a similar fashion from Lemma 3.1 and Equation 4.4. □

Theorem 4.7. *The algebra* $\tilde{\mathfrak{A}}$ *is given as a quiver with relations* $kQ/\tilde{I}$*, where* Q *is as in Theorem 3.11, but with edges corresponding to the lifts* $\tilde{\mathsf{w}}_{i,\phi}$ *of the* $\mathsf{w}_{i,\phi}$ *given there. The relations are those that follow from the structure constant relations of Proposition 4.6, together with relations saying that every composite of at least* s *arrows is zero, where* s *is the radical length of* kP*.*

The set $\tilde{\mathcal{B}}'$ *is a PBW style basis of* $\tilde{\mathfrak{A}}$*, giving* $\dim \tilde{\mathfrak{A}} = |Z(H) : H|$*. There is a natural isomorphism* $\mathsf{gr}_*(\tilde{\mathfrak{A}}) \cong \mathfrak{A}$*, sending each* $\tilde{\mathsf{w}}_{i,\phi}$ *to* $\mathsf{w}_{i,\phi}$*.*

Proof. It follows from the relations in Proposition 4.6 that the linear span of $\tilde{\mathcal{B}}'$ is closed under multiplication modulo a large enough power of the arrow ideal. The zero relations for composites of s arrows then show that this ideal is zero, and therefore that $\tilde{\mathcal{B}}'$ linearly spans $\tilde{\mathfrak{A}}$. The image of an element $\tilde{\mathsf{w}}_{i,\phi}$ in $\mathsf{gr}_*(kGe)$ is equal to $\mathsf{w}_{i,\phi}$, which lies in $\mathfrak{A}$. Since the elements $\mathsf{w}_{i,\phi}$ of $\mathcal{B}'$ are linearly independent, it follows that the elements $\tilde{\mathsf{w}}_{i,\phi}$ of $\tilde{\mathcal{B}}'$ are linearly independent, and therefore form a basis for $\tilde{\mathfrak{A}}$. This therefore induces a natural isomorphism $\mathsf{gr}_*(\tilde{\mathfrak{A}}) \cong \mathfrak{A}$. Since $\mathfrak{A}$ is generated by its degree one elements, $\tilde{\mathfrak{A}}$ has the same quiver, with the lifts of the relations. □

Theorem 4.8. *The multiplication in* kGe *induces an isomorphism* $\tilde{\mathfrak{A}} \otimes_k \mathfrak{M} \to kGe$*.*

Proof. By Theorem 4.7 we have $\dim(\tilde{\mathfrak{A}}) = |Z(H) : Z| \cdot |P|$. So this is now proved in the same way as Theorem 3.14. $\square$

Corollary 4.9. *We have* $kGe \cong \mathsf{Mat}_m(\tilde{\mathfrak{A}})$*, where* $m = \sqrt{|H : Z(H)|}$*, so that* $\tilde{\mathfrak{A}}$ *is the basic algebra of* kGe*.* $\square$

Remark 4.10. As in Remark 3.12, if we perform the computations of this section with the group algebra of the semidirect product $P \rtimes Z(H)/Z$ instead of kGe, the results look similar except with different scalars. So we can see $\tilde{\mathfrak{A}}$ as a quantum deformation of the algebra $k(P \rtimes Z(H)/Z)$. This observation, together with Theorems 4.7 and 4.8, complete the proof of Theorem 1.1.

We shall see some explicit examples of the ungrading of the relations in Section 5.

5. Example: P extraspecial of order p^3 and exponent p

Let k have characteristic p, an odd prime, and P be an extraspecial p-group of order p^3 and exponent p, with presentation

$$P = \langle g, h, c \mid g^p = h^p = c^p = 1,\ [g,h] = c,\ [g,c] = [h,c] = 1\rangle.$$

We denote by H the quaternion group of order 8, given by a presentation

$$H = \langle s, t \mid s^4 = 1,\ s^2 = t^2,\ ts = s^{-1}t\rangle \cong Q_8.$$

Set $Z = \langle s^2 \rangle$; this is the centre of H. We consider the following action of H on P, and set $G = P \rtimes H$.

$$g^s = g^{-1}, \qquad g^t = g, \qquad h^s = h, \qquad h^t = h^{-1}$$

It follows that $c^s = c^t = c^{-1}$, and Z acts trivially on P. This action lifts the action of $C_2 \times C_2$ on $C_p \times C_p \cong P/\langle c \rangle$, where here the nontrivial element of each copy of C_2 acts as inversion on the corresponding copy C_p. The group algebra kG has two blocks, namely the principal block $e_0 = \frac{1}{2}(1 + s^2)$ and the nonprincipal block $e = \frac{1}{2}(1 - s^2)$ corresponding to the faithful central character $\chi\colon Z \to k^\times$ given by $\chi(s^2) = -1$. We shall be interested in kGe.

Remark 5.1. Let $x = g - 1$, $y = h - 1$, $z = c - 1$ in kP. Then

$$z = (xy - yx)(1 + x)^{-1}(1 + y)^{-1} \tag{5.2}$$

and a presentation for kP is given by generators x and y, and relations saying that $x^p = 0$, $y^p = 0$, and the element z defined by (5.2) is central with pth power equal to zero. Note that the element $(1 + x)^{-1}(1 + y)^{-1}$ is congruent to 1 modulo $J(kP)$, and so in the associated graded $\mathsf{gr}_*(kP)$ this term in (5.2) may be ignored. This is used in the proof of Theorem 1.2 that follows.

Proof of Theorem 1.2. Denote by x, y, z the images of g, h, c in $\mathsf{Jen}_*(P)$, respectively. (These elements are mapped to the images of $g - 1$, $h - 1$, $c - 1$ in $\mathsf{gr}_*(kP)$ under the canonical map $\mathsf{Jen}_*(P) \to \mathsf{gr}_*(kP)$). The three dimensional p-restricted Lie algebra $\mathsf{Jen}_*(P)$ is spanned by the elements x, y in degree one together with $z = [x, y]$ (by the previous Remark) in degree two, satisfying $[x, y] = [y, z] = 0$. The p-restriction map given by $x^{[p]} = y^{[p]} = z^{[p]} = 0$. Its p^3 dimensional universal enveloping algebra $\mathcal{U}\mathsf{Jen}_*(P)$ is isomorphic to $\mathsf{gr}_*(kP)$. This shows the first part of Theorem 1.2.

The action of H on $\mathsf{Jen}_*(P)$ is given by

$$x^s = -x, \qquad x^t = x, \qquad\qquad y^s = y, \qquad y^t = -y, \qquad\qquad z^s = -z, \qquad z^t = -z.$$

The elements x, y and z are eigenvectors for H on $\mathsf{Jen}_*(P)$. So we set $w_1 = x$, $w_2 = y$, $w_3 = z$. The characters ψ_i of H satisfying $gw_ig^{-1} = \psi_i(g)w_i$ for $g \in H$ are given as follows.

$$\psi_1(s) = -1, \quad \psi_1(t) = 1, \qquad \psi_2(s) = 1, \quad \psi_2(t) = -1, \qquad \psi_3(s) = -1, \quad \psi_3(t) = -1.$$

Note that the relation $[x, y] = z$ in $\mathsf{Jen}_*(P)$ implies $\psi_1\psi_2 = \psi_3$.

Denoting as above by $e = \frac{1}{2}(1 - s^2)$ the nonprincipal block of kG, the block algebra kGe has a unique isomorphism class of simple modules. Indeed, e corresponds to the unique 2-dimensional simple kH-module, and hence the semisimple quotient of kGe is the matrix algebra $\mathfrak{M} = kHe \cong \mathsf{Mat}_2(k)$.

Since $Z = Z(H)$, there is only one central character of $Z(H)$ lying above χ, namely $\phi = \chi$, and $\xi_\phi = 1$. The map $\rho\colon H/Z(H) \to \mathsf{Hom}(H/Z(H), k^\times)$ takes s to ϕ_2, t to ϕ_1 and st to ϕ_3. Thus $g_{1,\phi} = t$, $g_{2,\phi} = s$ and $g_{3,\phi} = st$; these are only well defined up to multiplication by $Z(H)$.

The block algebra $\mathsf{gr}_*(kGe)$ of $\mathsf{gr}_*(kG)$ also has one isomorphism class of simple modules, namely the same 2-dimensional simple kH-module as above, and by Theorem 3.14 and Corollary 3.15 we have

$$\mathsf{gr}_*(kGe) \cong \mathfrak{A} \otimes_k \mathfrak{M} \cong \mathsf{Mat}_2(\mathfrak{A}),$$

where $\mathfrak{M} = kHe \cong \mathsf{Mat}_2(k)$ and $\mathfrak{A} = (\mathsf{gr}_*(kGe))^H$. The algebra $\mathfrak{A}$ contains elements $g_{1,\phi}w_1e = txe$, $g_{2,\phi}w_2e = sye$ and $g_{3,\phi}w_3e = stze$. The constants are given by $q_{1,2,\phi} = -1$ and $q'_{1,2,3,\phi} = -1$, so these satisfy the following relation:

$$(txe)(sye) + (sye)(txe) = -tsxye - styxe = st(xy - yx)e = stze$$

Similar computations give

$$(txe)(stze) + (stze)(txe) = 0, \qquad (sye)(stze) + (stze)(sye) = 0.$$

Writing $\mathsf{x} = txe$, $\mathsf{y} = sye$ and $\mathsf{z} = stze$, we therefore have

$$\mathsf{xy} + \mathsf{yx} = \mathsf{z}, \quad \mathsf{xz} + \mathsf{zx} = 0, \quad \mathsf{yz} + \mathsf{zy} = 0, \quad \mathsf{x}^p = 0, \quad \mathsf{y}^p = 0, \quad \mathsf{z}^p = 0.$$

This is a presentation for the basic algebra $\mathfrak{A}$ of $\mathsf{gr}_*(kGe)$, with generators x and y, and with z defined as $\mathsf{xy} + \mathsf{yx}$. This proves Theorem 1.2. □

Remark 5.3. The first part of the above proof shows that $\mathsf{Jen}_*(P)$ is isomorphic to the p-restricted Lie algebra of 3×3 matrices of the form $\left(\begin{smallmatrix} 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{smallmatrix}\right)$.

In order to prove Theorem 1.3, ungrading the algebra is our next task. The problem is that the generators $g - 1$ and $h - 1$ of kP are not well suited to dealing with automorphisms. We have an action of $\mathbb{F}_p^\times \times \mathbb{F}_p^\times$ on P where (i, j) sends g to g^i and h to h^j. The commutator $c = [g, h]$ is sent to c^{ij}. Set

$$\tilde{x} = -\sum_{i=1}^{p-1} g^i/i, \qquad \tilde{y} = -\sum_{j=1}^{p-1} h^j/j.$$

Lemma 5.4. *We have* $\tilde{x} \equiv g - 1 \pmod{J^2(kP)}$ *and* $\tilde{y} \equiv h - 1 \pmod{J^2(kP)}$.

Proof. Since p is odd, $\sum_{i=1}^{p-1} 1/i = \sum_{i=1}^{p-1} i = 0$ in k whence $\tilde{x} = -\sum_{i=1}^{p-1}(g^i - 1)/i$. Now the assertion for $\tilde{x}$ follows since $(g^i-1)/i \equiv g-1 \pmod{J^2(kP)}$ for any i, $1 \leqslant i \leqslant p-1$. The proof for $\tilde{y}$ is similar. □

Note that $\tilde{x}$ an eigenvector in the $(1,0)$ eigenspace and $\tilde{y}$ an eigenvector in the $(0,1)$ eigenspace of $\mathbb{F}_p^\times \times \mathbb{F}_p^\times$. Then we set $\tilde{z} = [\tilde{x}, \tilde{y}] = \tilde{x}\tilde{y} - \tilde{y}\tilde{x}$, an eigenvector in the $(1,1)$ eigenspace. By Lemma 5.4 and the proof of Theorem 1.2, kP has a PBW basis consisting of monomials in the $\tilde{x}$, $\tilde{y}$ and $\tilde{z}$. Moreover, a PBW basis element $\tilde{x}^i\tilde{y}^j\tilde{z}^k$ of kP with $0 \leqslant i, j, k < p$, is an eigenvector in the $(i+k, j+k)$ eigenspace, where $i+k$ and $j+k$ are read modulo $p-1$.

Lemma 5.5. *We have* $\tilde{z}^p = 0$.

Proof. The element $\tilde{z}^p$ is an eigenvector in the $(1,1)$ eigenspace. Further, $\tilde{z}^p$ has image $z^p = 0 \in \mathsf{gr}_{2p}(kP)$, hence $\tilde{z}^p$ is in $J^{2p+1}(kP)$. The PBW basis elements in this eigenspace have $i+k$ and $j+k$ congruent to one modulo $p-1$ and at most $2p-2$, and hence at most p, but then $i+j+2k \leqslant 2p$, so the basis element is not in $J^{2p+1}(kP)$. It follows that the $(1,1)$ eigenspace in $J^{2p}(kP+1)$ is zero and so $\tilde{z}^p = 0$. □

Lemma 5.6. *The element* $[\tilde{x}, \tilde{z}]$ *is a linear combination of the elements* $\tilde{z}\tilde{y}^{p-2}\tilde{z} \in J^{p+2}(kP)$ *and* $\tilde{x}^i\tilde{y}^{2i-1}\tilde{x}^i\tilde{z}^{p+1-2i} \in J^{2p+1}(kP)$ *with* $1 \leqslant i \leqslant (p-1)/2$. *Similarly,* $[\tilde{y}, \tilde{z}]$ *is a linear combination of the elements* $\tilde{z}\tilde{x}^{p-2}\tilde{z} \in J^{p+2}(kP)$ *and* $\tilde{y}^i\tilde{x}^{2i-1}\tilde{y}^i\tilde{z}^{p+1-2i} \in J^{2p+1}(kP)$ *with* $1 \leqslant i \leqslant (p-1)/2$.

Proof. We prove the first statement. The proof of the second is identical, with the roles of $\tilde{x}$ and $\tilde{y}$ reversed.

The element $[\tilde{x}, \tilde{z}]$ has image $[x, z] = 0$ in $\mathsf{gr}_3(kP)$, and hence lies in $J^4(kP)$. It is in the $(2,1)$ eigenspace, so we start by identifying the PBW basis elements of $J^4(kP)$ in this eigenspace. These are $\tilde{y}^{p-2}\tilde{z}^2$ and $\tilde{x}\tilde{y}^{p-1}\tilde{z} \in J^{p+2}(kP)$ and $\tilde{x}^{i+1}\tilde{y}^i\tilde{z}^{p-i} \in J^{2p+1}(kP)$ with $1 \leqslant i \leqslant p-2$.

However, we also need to make use of symmetry. Let σ be the composition of the automorphism of kP which inverts g and h (and hence fixes c) with the anti-automorphism of kP which inverts all elements of P. Then σ fixes $\tilde{x}$ and $\tilde{y}$, reverses multiplication in kP, and negates $\tilde{z}$. The point is that $[\tilde{x}, \tilde{z}] = \tilde{x}^2\tilde{y} - 2\tilde{x}\tilde{y}\tilde{x} + \tilde{y}\tilde{x}^2$ is fixed by σ, whereas σ does not fix all elements of the $(2,1)$ eigenspace. With this in mind, we modify the PBW basis of this eigenspace so that the action of σ is more transparent.

The element $\tilde{y}^{p-2}\tilde{z}^2$, for example, is not fixed by σ, even though it's fixed modulo $J^{p+3}(kP)$. So instead, we use the element $\tilde{z}\tilde{y}^{p-2}\tilde{z}$, which is equivalent to it modulo $J^{p+3}(kP)$, and therefore just as good as part of a PBW basis of kP, but is fixed by σ. Since $\sigma(\tilde{x}\tilde{y}^{p-1}\tilde{z}) \equiv -\tilde{x}\tilde{y}^{p-1}\tilde{z} - \tilde{y}^{p-2}\tilde{z}^2 \pmod{J^{p+3}(kP)}$, the element $\tilde{x}\tilde{y}^{p-1}\tilde{z}$ is not involved in the expression for $[\tilde{x}, \tilde{z}]$. So $[\tilde{x}, \tilde{z}]$ is congruent to a multiple of $\tilde{z}\tilde{y}^{p-2}\tilde{z}$ modulo $J^{2p+1}(kP)$.

For the linear span of the elements $\tilde{x}^{i+1}\tilde{y}^i\tilde{z}^{p-i}$, since there are no $(2,1)$ eigenvectors lower in the radical series, reordering the terms in a monomial has the same effect as in $\mathcal{U}\mathsf{Jen}_*(kP)$. So we can choose a basis consisting of the elements $\tilde{x}^i\tilde{y}^{2i-1}\tilde{x}^i\tilde{z}^{p+1-2i}$ $(1 \leqslant i \leqslant (p-1)/2)$ and the elements $\tilde{y}^i\tilde{x}^{2i+1}\tilde{y}^i\tilde{z}^{p-2i}$ $(1 \leqslant i \leqslant (p-3)/2)$. The former are $+1$ eigenvectors of σ, while the latter are -1 eigenvectors. So the expression for $[\tilde{x}, \tilde{z}]$ only involves the former. □

By Lemma 5.6, we can write

$$[\tilde{x}, \tilde{z}] = a_0\tilde{z}\tilde{y}^{p-2}\tilde{z} + a_1\tilde{x}\tilde{y}\tilde{x}\tilde{z}^{p-1} + a_2\tilde{x}^2\tilde{y}^3\tilde{x}^2\tilde{z}^{p-3} + \cdots + a_{(p-1)/2}\tilde{x}^{\frac{p-1}{2}}\tilde{y}^{p-2}\tilde{x}^{\frac{p-1}{2}}\tilde{z}^2, \tag{5.7}$$

$$(5.8)\qquad [\tilde{y},\tilde{z}] = -a_0\tilde{z}\tilde{x}^{p-2}\tilde{z} - a_1\tilde{y}\tilde{x}\tilde{y}\tilde{z}^{p-1} - a_2\tilde{y}^2\tilde{x}^3\tilde{y}^2\tilde{z}^{p-3} - \cdots - a_{(p-1)/2}\tilde{x}^{\frac{p-1}{2}}\tilde{y}^{p-2}\tilde{x}^{\frac{p-1}{2}}\tilde{z}^2.$$

Here, we have used the symmetry of kP which swaps $\tilde{x}$ and $\tilde{y}$, and negates $\tilde{z}$, to compare the coefficients in (5.7) and those in (5.8).

Remark 5.9. With the aid of the computer algebra system MAGMA [5] we have determined the relation (5.7) for small p as follows:

$$\begin{aligned} p=3: &\qquad [\tilde{x},\tilde{z}] = \tilde{z}\tilde{y}\tilde{z},\\ p=5: &\qquad [\tilde{x},\tilde{z}] = \tilde{z}\tilde{y}^3\tilde{z} + 2\tilde{x}\tilde{y}\tilde{x}\tilde{z}^4,\\ p=7: &\qquad [\tilde{x},\tilde{z}] = \tilde{z}\tilde{y}^5\tilde{z} + 4\tilde{x}\tilde{y}\tilde{x}\tilde{z}^6 + 2\tilde{x}^2\tilde{y}^3\tilde{x}^2\tilde{z}^4. \end{aligned}$$

One might surmise that $a_0 = 1$ and $a_{(p-1)/2} = 0$, but we have not proved that. Nor have we spotted the general pattern of the coefficients.

Theorem 5.10. *A presentation for kP is given by generators $\tilde{x}$, $\tilde{y}$, $\tilde{z}$ with the relations* (5.7) *and* (5.8) *together with*

$$\tilde{x}^p = \tilde{y}^p = \tilde{z}^p = 0, \qquad [\tilde{x},\tilde{y}] = \tilde{z},$$

and relations saying that all words of length at least $4p-3$ in $\tilde{x}$ and $\tilde{y}$ are equal to zero.

Proof. These relations hold in kP by Lemmas 5.5 and 5.6, and the fact that $J^{4p-3}(kP) = 0$. Let $\mathbf{A}$ be the algebra defined by these generators and relations. Then we have a surjective map $\mathbf{A} \to kP$ taking $\tilde{x}$, $\tilde{y}$ and $\tilde{z}$ to the elements with the same names. This induces a map $\mathsf{gr}_*\mathbf{A} \to \mathsf{gr}_* kP$. The relations (5.7) and (5.8) imply that the images x, y and z in $\mathsf{gr}_*\mathbf{A}$ of $\tilde{x}$, $\tilde{y}$ and $\tilde{z}$ in $\mathbf{A}$ satisfy $[x,z] = 0$ and $[y,z] = 0$. Thus all the relations in $\mathcal{U}\mathsf{Jen}_*(P)$ hold in $\mathsf{gr}_*\mathbf{A}$, and $\mathsf{gr}_*\mathbf{A} \to \mathsf{gr}_* kP$ is an isomorphism. Since the radical of $\mathbf{A}$ is nilpotent, this implies that $\mathbf{A} \to kP$ is an isomorphism. □

Recall from the proof of Theorem 1.2, that setting $\mathsf{x} = txe$, $\mathsf{y} = sye$ and $\mathsf{z} = stze$ in $\mathsf{gr}_*(kGe)$, we have that the algebra $\mathfrak{A}$ is generated by x, y and z centralises $\mathfrak{M}$ in $\mathcal{U}\mathsf{Jen}_*(P) \rtimes kH$. Further, these elements satisfy the relations

$$\mathsf{x}^p = 0, \quad \mathsf{y}^p = 0, \quad \mathsf{xy} + \mathsf{yx} = \mathsf{z}, \quad \mathsf{xz} + \mathsf{zx} = 0, \quad \mathsf{yz} + \mathsf{zy} = 0$$

(and these imply that $\mathsf{z}^p = 0$).

In kGe, we set $\tilde{\mathsf{x}} = t\tilde{x}e$, $\tilde{\mathsf{y}} = s\tilde{y}e$ and $\tilde{\mathsf{z}} = st\tilde{z}e$. The algebra $\tilde{\mathfrak{A}}$ generated by $\tilde{\mathsf{x}}$, $\tilde{\mathsf{y}}$ and $\tilde{\mathsf{z}}$ centralises $\mathfrak{M}$ in kGe. These elements satisfy the relations

$$\tilde{\mathsf{x}}^p = \tilde{\mathsf{y}}^p = \tilde{\mathsf{z}}^p = 0, \qquad \tilde{\mathsf{x}}\tilde{\mathsf{y}} + \tilde{\mathsf{y}}\tilde{\mathsf{x}} = \tilde{\mathsf{z}}$$

together with the following quantised versions of (5.7) and (5.8)

$$\begin{aligned} \tilde{\mathsf{x}}\tilde{\mathsf{z}} + \tilde{\mathsf{z}}\tilde{\mathsf{x}} &= (-1)^{\frac{p-1}{2}}\big(a_0\tilde{\mathsf{z}}\tilde{\mathsf{y}}^{p-2}\tilde{\mathsf{z}} - a_1\tilde{\mathsf{x}}\tilde{\mathsf{y}}\tilde{\mathsf{x}}\tilde{\mathsf{z}}^{p-1} - a_2\tilde{\mathsf{x}}^2\tilde{\mathsf{y}}^3\tilde{\mathsf{x}}^2\tilde{\mathsf{z}}^{p-3} - \cdots - a_{(p-1)/2}\tilde{\mathsf{x}}^{\frac{p-1}{2}}\tilde{\mathsf{y}}^{p-2}\tilde{\mathsf{x}}^{\frac{p-1}{2}}\tilde{\mathsf{z}}^2\big),\\ \tilde{\mathsf{y}}\tilde{\mathsf{z}} + \tilde{\mathsf{z}}\tilde{\mathsf{y}} &= (-1)^{\frac{p-1}{2}}\big(a_0\tilde{\mathsf{z}}\tilde{\mathsf{x}}^{p-2}\tilde{\mathsf{z}} - a_1\tilde{\mathsf{y}}\tilde{\mathsf{x}}\tilde{\mathsf{y}}\tilde{\mathsf{z}}^{p-1} - a_2\tilde{\mathsf{y}}^2\tilde{\mathsf{x}}^3\tilde{\mathsf{y}}^2\tilde{\mathsf{z}}^{p-3} - \cdots - a_{(p-1)/2}\tilde{\mathsf{y}}^{\frac{p-1}{2}}\tilde{\mathsf{x}}^{p-2}\tilde{\mathsf{y}}^{\frac{p-1}{2}}\tilde{\mathsf{z}}^2\big), \end{aligned}$$

together with relations saying that all words of length at least $4p-3$ in $\tilde{\mathsf{x}}$ and $\tilde{\mathsf{y}}$ are equal to zero.

Using MAGMA [5], in the case $p = 3$ we have succeeded in finding a short presentation for kP in terms of the generators $\tilde{x}$ and $\tilde{y}$. In this case, we have $\tilde{x} = g^{-1} - g$ and $\tilde{y} = h^{-1} - h$. Defining $\tilde{z} = [\tilde{x},\tilde{y}]$, the following relations hold in kP.

$$(5.11)\qquad \tilde{x}^3 = 0, \qquad \tilde{y}^3 = 0, \qquad [\tilde{x},\tilde{y}] = \tilde{z}, \qquad [\tilde{x},\tilde{z}] = \tilde{z}\tilde{y}\tilde{z}, \qquad [\tilde{y},\tilde{z}] = -\tilde{z}\tilde{x}\tilde{z}.$$

It follows from these relations that $\tilde{z}^3 = 0$, so it is not necessary to include this in the relations, and hence that the algebra defined by these relations has dimension 27, and is isomorphic to kP. This is the content of the next theorem. Note, however, that the proof is difficult, so for some purposes it is better to adjoin $\tilde{z}^3 = 0$ to the above presentation. We restate and prove the first part of Theorem 1.3.

Theorem 5.12. *Suppose that $p = 3$. The generators $\tilde{x}$, $\tilde{y}$ and $\tilde{z}$ and the relations* (5.11) *give a presentation for kP.*

Proof. Since the given elements of kP satisfy these relations, it suffices to prove that the algebra defined by the relations has dimension at most 27. The crucial point is to prove that $\tilde{z}^3 = 0$.

It is more convenient to extend the field so that it has a square root of -1, which we denote i. Then we set $a = \tilde{x} + \mathsf{i}\,\tilde{y}$, $b = \tilde{x} - \mathsf{i}\,\tilde{y}$, $c = \mathsf{i}\,\tilde{z}$, and the presentation becomes

$$a^3 = [b, c] = cbc, \qquad b^3 = -[a, c] = cac, \qquad [a, b] = c,$$

and we must show that $c^3 = 0$.

We have

$$(1 + c)a(1 - c) = a,$$
$$(1 - c)b(1 + c) = b,$$

and so

$$(1 + c)ab = (1 + c)a(1 - c)b(1 + c) = ab(1 + c).$$

Therefore c commutes with ab and with ba.

Next,

$$cab = acb + cacb = abc - acbc + cacb,$$

and since $cab = abc$ it follows that c commutes with acb. Thus we have

$$cbca = a^4 = acbc = cacb = b^4 = bcac. \tag{5.13}$$

Since we are in characteristic three, we also have

$$[a, [a, c]] = -[a, cac] = -[a, c]ac - c[a, a]c - ca[a, c] = cacac + cacac = -cacac$$

and so

$$[a^3, c] = [a, [a, [a, c]]] = -[a, cacac] = -[a, c]acac - ca[a, c]ac - caca[a, c] = -3cacacac = 0.$$

Thus c also commutes with a^3:

$$a^3c = ca^3. \tag{5.14}$$

Next, using (5.13) we have

$$\begin{aligned} c^3 &= cabc - cbac = acbc + cacbc - cbca + cbcac = a^4 + b^4c - b^4 + a^4c \\ &= -a^4c = -b^4c = -cacbc = -cbcac = -ca^4 = -cb^4. \end{aligned} \tag{5.15}$$

Using (5.14) and (5.15), we have

$$a^4c = ca^4 = a^3ca = a^4c + a^3cac = a^4c + ca^4c = a^4c - c^4$$

and so $c^4 = 0$.

The fact that $c^4 = 0$ enables us to write

$$ca = ac + cac = ac + ac^2 + cac^2 = ac + ac^2 + ac^3 + cac^3 = a(c + c^2 + c^3) \tag{5.16}$$

$$cb = bc - cbc = bc - bc^2 + cbc^2 = bc - bc^2 + bc^3 - cbc^3 = b(c - c^2 + c^3). \tag{5.17}$$

We can use these to move copies of c to the end of expressions, at the expense of accumulating higher powers of c. Applying this to $a^6 = cbc^2bc$, we get $b^2(c^4 + \text{higher powers of } c)$, so we get $a^6 = 0$. Similarly, we get $b^6 = 0$. Then when we do the same with $(ab)^9$, moving b past a using $ba = ab - c$, we get

$$a^9b^9 + a^8b^8cf_1(c) + a^7b^7c^2f_2(c) + a^6b^6c^3f_3(c) + a^5b^5c^4f_4(c) + \dots$$

for suitable polynomials $f_i(c)$. Since $a^6 = b^6 = c^4 = 0$, every term here is zero, and so $(ab)^9 = 0$.

Now using (5.15) and the same method, we have

$$c^3 = -cacbc = -cab(c - c^2 + c^3)c = -abc(c - c^2 + c^3)c = -abc^3,$$

and so $(1 + ab)c^3 = 0$. Since ab is nilpotent, $(1 + ab)$ is invertible, so this implies that $c^3 = 0$.

The original relations together with (5.16), (5.17) and $c^3 = 0$ allow us to rewrite every element as a linear combination of the elements $a^ib^jc^k$ with $0 \leqslant i, j, k < 3$, so the algebra has dimension at most 27, and we are done. □

Proof of Theorem 1.3. The relations for kP are proved in Theorem 5.12. As above, using $\tilde{x} = g^{-1} - g$, $\tilde{y} = h^{-1} - h$, and $\tilde{z} = [\tilde{x}, \tilde{y}]$, to obtain generators for the basic algebra for kGe, we set $\tilde{\mathsf{x}} = t\tilde{x}e$ and $\tilde{\mathsf{y}} = s\tilde{y}e$, $\tilde{\mathsf{z}} = st\tilde{z}e$. These satisfy

$$\tilde{\mathsf{x}}^3 = 0, \qquad \tilde{\mathsf{y}}^3 = 0, \qquad \tilde{\mathsf{x}}\tilde{\mathsf{y}} + \tilde{\mathsf{y}}\tilde{\mathsf{x}} = \tilde{\mathsf{z}}, \qquad \tilde{\mathsf{x}}\tilde{\mathsf{z}} + \tilde{\mathsf{z}}\tilde{\mathsf{x}} = -\tilde{\mathsf{z}}\tilde{\mathsf{y}}\tilde{\mathsf{z}}, \qquad \tilde{\mathsf{y}}\tilde{\mathsf{z}} + \tilde{\mathsf{z}}\tilde{\mathsf{y}} = -\tilde{\mathsf{z}}\tilde{\mathsf{x}}\tilde{\mathsf{z}}.$$

Furthermore, the algebra defined by these relations again has dimension 27, and is hence isomorphic to the basic algebra of kGe. □

Remark 5.18. The above relations for kGe are a quantised version of the relations for kP. These relations imply that $\tilde{\mathsf{z}}^3 = 0$, and adjoining this relation makes the presentation easier to work with if desired.

These presentations can be lifted to give integral presentations. The algebra $\mathcal{O}P$ has a presentation with corresponding generators $\hat{x}$, $\hat{y}$ and $\hat{z}$ subject to

$$\hat{x}^3 + 3\hat{x} = 0, \qquad \hat{y}^3 + 3\hat{y} = 0, \qquad [\hat{x}, \hat{y}] = \hat{z}, \qquad 2[\hat{x}, \hat{z}] = -\hat{z}\hat{y}\hat{z}, \qquad 2[\hat{y}, \hat{z}] = \hat{z}\hat{x}\hat{z},$$

while $\mathcal{O}Ge$ is generated by $\hat{\mathsf{x}}$, $\hat{\mathsf{y}}$ and $\hat{\mathsf{z}}$ subject to

$$\hat{\mathsf{x}}^3 = 3\hat{\mathsf{x}}, \qquad \hat{\mathsf{y}}^3 = 3\hat{\mathsf{y}}, \qquad \hat{\mathsf{x}}\hat{\mathsf{y}} + \hat{\mathsf{y}}\hat{\mathsf{x}} = \hat{\mathsf{z}}, \qquad 2\hat{\mathsf{x}}\hat{\mathsf{z}} + 2\hat{\mathsf{z}}\hat{\mathsf{x}} = \hat{\mathsf{z}}\hat{\mathsf{y}}\hat{\mathsf{z}}, \qquad 2\hat{\mathsf{y}}\hat{\mathsf{z}} + 2\hat{\mathsf{z}}\hat{\mathsf{y}} = \hat{\mathsf{z}}\hat{\mathsf{x}}\hat{\mathsf{z}}.$$

The expressions for $\hat{z}^3$ in $\mathcal{O}P$ and for $\hat{\mathsf{z}}^3$ in $\mathcal{O}Ge$, lifting the fact that they cube to zero modulo three, are ugly even though they follow from the presentations above.

6. Example: $2^{1+4}{:}\,3^{1+2}$ in characteristic two

The examples in the last section were at odd primes for extraspecial groups of order p^3. In this section we give an example in characteristic two with an extraspecial group of order 2^5.

Let P be an extraspecial group 2^{1+4} which is a central product of two copies of the quaternion group of order eight, and let H be an extraspecial group 3^{1+2} of exponent three. We let the centre $Z \cong \mathbb{Z}/3$ of H act trivially on P, and the elementary abelian quotient act as the automorphisms of order three on the two quaternion central factors of P, and we set $G = P \rtimes H$. Thus the quotient $G/Z \cong SL(2,3) \circ SL(2,3)$ is a central product of two copies of the group $SL(2,3)$ of order 24.

More precisely, we let

$$P = \langle g_1, g_2, h_1, h_2, c \mid g_1^2 = h_1^2 = [g_1, h_1] = g_2^2 = h_2^2 = [g_2, h_2] = c,$$
$$[g_1, c] = [g_2, c] = [h_1, c] = [h_2, c] = [g_1, g_2] = [g_1, h_2] = [h_1, g_2] = [h_1, h_2] = c^2 = 1\rangle,$$
$$H = \langle s_1, s_2, t \mid s_1^3 = s_2^3 = 1,\ [s_1, s_2] = t,\ [s_1, t] = [s_2, t] = t^3 = 1\rangle.$$

Let H act on P with $Z = \langle t\rangle$ acting trivially, and

$$s_1 g_1 s_1^{-1} = h_1, \qquad s_1 g_2 s_1^{-1} = g_2, \qquad s_1 c s_1^{-1} = c,$$
$$s_2 g_1 s_2^{-1} = g_1, \qquad s_2 g_2 s_2^{-1} = h_2, \qquad s_2 c s_2^{-1} = c.$$

Let k be a field of characteristic two containing $\mathbb{F}_4 = \{0, 1, \omega, \bar\omega\}$. A basis of eigenvectors in $\mathsf{gr}_1(kP)$ is given by

$$x_i = \bar\omega(g_i - 1) + \omega(h_i - 1), \qquad y_i = \omega(g_i - 1) + \bar\omega(h_i - 1) \qquad (i = 1, 2).$$

These give the following presentation for $\mathsf{gr}_*(kP)$.

$$x_i^2 = 0, \quad y_i^2 = 0, \quad [x_1, x_2] = [y_1, y_2] = [x_1, y_2] = [x_2, y_1] = 0, \quad [x_1, y_1] = [x_2, y_2].$$

A lift of x_i and y_i to eigenvectors complementing $J^2(kP)$ in $J(kP)$ is given by the elements

$$\tilde{x}_i = \omega g_i + \bar\omega h_i + g_i h_i, \qquad \tilde{y}_i = \bar\omega g_i + \omega h_i + g_i h_i \qquad (i = 1, 2).$$

The relations lift to

$$\tilde{x}_i^2 = \tilde{y}_i\tilde{x}_i\tilde{y}_i, \qquad \tilde{y}_i^2 = \tilde{x}_i\tilde{y}_i\tilde{x}_i, \qquad \tilde{x}_i^4 = 0 \qquad (i = 1, 2),$$
$$[\tilde{x}_1, \tilde{x}_2] = [\tilde{y}_1, \tilde{y}_2] = [\tilde{x}_1, \tilde{y}_2] = [\tilde{x}_2, \tilde{y}_1] = 0, \qquad [\tilde{x}_1, \tilde{y}_1] + \tilde{x}_1^3 = [\tilde{x}_2, \tilde{y}_2] + \tilde{x}_2^3$$

(both sides in the last relation are equal to $(1+c)$). Using the radical filtration, it is not hard to check that these relations define a k-algebra of dimension at most 32, which is therefore isomorphic to kP.

The action of H on kP with respect to these generators is given by

$$g\tilde{x}_i g^{-1} = \psi_i(g)^{-1}\tilde{x}_i, \qquad g\tilde{y}_i g^{-1} = \psi_i(g)\tilde{y}_i \qquad (g \in H)$$

where $\psi_1(s_1) = \psi_2(s_2) = \omega$, $\psi_1(s_2) = \psi_2(s_1) = 1$.

Set

$$e_0 = 1 + t + t^2, \qquad e = 1 + \bar\omega t + \omega t^2, \qquad \bar{e} = 1 + \omega t + \bar\omega t^2.$$

Then kG has three blocks, the principal block kGe_0, and two non-principal blocks kGe and $kG\bar{e}$. We examine the non-principal block kGe, the other is similar.

We have $s_1 s_2 e = \omega s_2 s_1 e$. We set

$$\tilde{\mathsf{x}}_1 = s_2\tilde{x}_1 e, \qquad \tilde{\mathsf{x}}_2 = s_1^{-1}\tilde{x}_2 e, \qquad \tilde{\mathsf{y}}_1 = s_2^{-1}\tilde{y}_1 e, \qquad \tilde{\mathsf{y}}_2 = s_1\tilde{y}_2 e.$$

These commute with $\mathfrak{M} = kHe$, and generate the subalgebra $\mathfrak{A}$, so that $kGe \cong \mathsf{Mat}_3(\mathfrak{A})$. They satisfy the relations:

$$\tilde{\mathsf{x}}_i^2 = \tilde{\mathsf{y}}_i\tilde{\mathsf{x}}_i\tilde{\mathsf{y}}_i, \qquad \tilde{\mathsf{y}}_i^2 = \tilde{\mathsf{x}}_i\tilde{\mathsf{y}}_i\tilde{\mathsf{x}}_i, \qquad \tilde{\mathsf{x}}_i^4 = 0 \qquad (i = 1, 2),$$
$$\tilde{\mathsf{x}}_1\tilde{\mathsf{x}}_2 = \bar\omega\tilde{\mathsf{x}}_2\tilde{\mathsf{x}}_1, \qquad \tilde{\mathsf{x}}_1\tilde{\mathsf{y}}_2 = \omega\tilde{\mathsf{y}}_2\tilde{\mathsf{x}}_1,$$
$$\tilde{\mathsf{y}}_1\tilde{\mathsf{x}}_2 = \omega\tilde{\mathsf{x}}_2\tilde{\mathsf{y}}_1, \qquad \tilde{\mathsf{y}}_1\tilde{\mathsf{y}}_2 = \bar\omega\tilde{\mathsf{y}}_2\tilde{\mathsf{y}}_1,$$

$$[\tilde{\mathsf{x}}_1, \tilde{\mathsf{y}}_1] + \tilde{\mathsf{x}}_1^3 = [\tilde{\mathsf{x}}_2, \tilde{\mathsf{y}}_2] + \tilde{\mathsf{x}}_2^3$$

(both sides in the last relation are equal to $(1+c)e$). These are identical to the relations for kP apart from the commutation relations, which have been quantised by the introduction of factors ω and $\bar{\omega}$.

7. Appendix: Errata

The present paper supersedes most of our previous paper [4]. In that paper there are a number of minor errors, mostly in the calculations in Section 4, which have been corrected in the present work. We give a list of those errors in [4].

In the statements of Theorem 1.2 and Corollary 1.3, it should read '...quantised version of $k(P \rtimes Z(H)/Z)$' (and not '...of $k(P \rtimes L)$.')

In the 3rd line of the proof of Proposition 3.1 insert the word 'abelian' between 'maximal' and 'subgroup' (as is done correctly in line 2 and line 4 of that proof).

On page 1441, third line from the bottom, insert *faithful*:

"... and a faithful linear character $\chi\colon Z \to k^\times \dots$"

On page 1443, in the first line, $\rho(g)\colon h \mapsto \chi([h,g])$. The display on the third line should read

$$\bar{\rho}\colon H/Z(H) \to \mathsf{Hom}(H/Z(H), k^\times)$$

On page 1444, line four should begin "where $\rho(g_{i,\phi})(h) = \chi([h, g_{i,\phi}])$." The third line of the proof of Lemma 4.8 should begin with $eg_{i,\phi}h = e\chi([h, g_{i,\phi}])^{-1}hg_{i,\phi}$. The displayed equation on the fourth line of the proof of Lemma 4.8 should read

$$\xi_\phi(h)^{-1}(g_{i,\phi}w_i)(e_\phi \cdot h) = \xi_\phi(h)^{-1}\psi_i(h)^{-1}\chi([h, g_{i,\phi}])^{-1}(e_{\phi\psi_i}h)(g_{i,\phi}w_i).$$

In Definition 4.9 and the four lines following, kHe should be $k\tilde{G}e$ four times. The dimension of $\mathfrak{A}$ should be given as $|P| \cdot |Z(H) : Z|$ and not $|P| \cdot |H : Z(H)|$.

On page 1445, in Lemma 4.12 (2), in the displayed equation the last w_i should be w_j. The scalar $q_{i,j,\phi}$ should equal $\psi_i(g_{j,\phi}z_{i,j,\phi})\psi_j(g_{i,\phi}^{-1}z_{i,j,\phi})\phi(z_{i,j,\phi})$ rather than $\phi(z_{i,j,\phi})$. Similarly, in Lemma 4.12 (3) the scalar $q_{i,j,\phi}$ should equal $\psi_i(g_{j,\phi})\psi_j(g_{i,\phi}^{-1})\chi(z_{i,j,\phi})$ rather than $\chi(z_{i,j,\phi})$. In the second line of the proof of Lemma 4.12 (1), $g_{j,\phi\phi_i}$ should be $g_{j,\phi\psi_i}$. The computation that was suppressed in the proof of Lemma 4.12 (2) uses (1) and equation (4.7). It is similar to the computation in Theorem 3.8 above, which we have spelled out in detail. There is a missing Z in the third to last line of the proof of Lemma 4.12 (3), and g_ji should be g_i in the second to last line.

On page 1447, in Theorem 4.15 and Corollary 4.16, kGe should be $k\tilde{G}e$ five times.

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