HOCHSCHILD COHOMOLOGY OF SYMMETRIC GROUPS AND GENERATING FUNCTIONS

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ABSTRACT. We compute the dimensions of the Hochschild cohomology of symmetric groups over prime fields in low degrees. This involves us in studying some partition identities and generating functions of the dimensions in any fixed degree of the Hochschild cohomology of symmetric groups. We show that the generating function of the dimensions of the Hochschild cohomology in any fixed degree of the symmetric groups differs from that in degree 0 by a rational function.

1. INTRODUCTION

The purpose of this note is to compute the dimensions of the Hochschild cohomology of symmetric groups over prime fields in low degrees. We relate this to some partition identities.

For $n \geqslant 0$ we denote by $p(n)$ the number of partitions of n, with the convention $p(0) = 1$. We write $P(t)$ for the generating function of $p(n)$, which is given by Euler's identity as

$$P(t) = \sum_{n=0}^{\infty} p(n)\, t^n = \prod_{m=1}^{\infty} \frac{1}{1-t^m} = 1 \Big/ \sum_{n=-\infty}^{\infty} (-1)^n t^{n(3n+1)/2}. \tag{1.1}$$

Throughout the paper p is a prime.

Theorem 1.2. *The generating functions for the dimensions of the Hochschild cohomology of the group algebra of the symmetric group $\mathfrak{S}_n$ on n letters in low degrees are given by*

(i) $$\sum_{n=0}^{\infty} \dim_{\mathbb{F}_p}(HH^0(\mathbb{F}_p\mathfrak{S}_n))\, t^n = P(t)$$

(ii) $$\sum_{n=0}^{\infty} \dim_{\mathbb{F}_p}(HH^1(\mathbb{F}_p\mathfrak{S}_n))\, t^n = \begin{cases} \dfrac{2t^2}{1-t^2}\, P(t) & p = 2, \\ \dfrac{t^p}{1-t^p}\, P(t) & p \geqslant 3. \end{cases}$$

(iii) $$\sum_{n=0}^{\infty} \dim_{\mathbb{F}_p}(HH^2(\mathbb{F}_p\mathfrak{S}_n))\, t^n = \begin{cases} \dfrac{2t^2+3t^4-t^6}{(1-t^2)(1-t^4)}\, P(t) & p = 2, \\ \dfrac{t^p}{(1-t^p)(1-t^{2p})}\, P(t) & p \geqslant 3. \end{cases}$$

For degree zero, this is well known: $p(n)$ is equal to the dimension of the centre of $\mathbb{F}_p\mathfrak{S}_n$. For degrees one and two, the proof of the theorem is given in Sections 4 and 5, where we give explicit combinatorial descriptions of the dimensions of $HH^1(\mathbb{F}_p\mathfrak{S}_n)$ and $HH^2(\mathbb{F}_p\mathfrak{S}_n)$ in

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Corollary 4.7 and Corollary 5.9, respectively. The formula for degree one has independently been obtained by Briggs and Rubio y Degrassi [3] In general, we have the following result.

Theorem 1.3. *For any integer $r \geqslant 0$ there exists a rational function $R_{p,r}(t)$ with integer coefficients such that*

$$\sum_{n=0}^{\infty} \dim_{\mathbb{F}_p}(HH^r(\mathbb{F}_p\mathfrak{S}_n))\, t^n = R_{p,r}(t)P(t).$$

The theorem is proved in Section 6. The analysis in Section 6 could also be used for an alternative description of $R_{p,1}(t)$ and $R_{p,2}(t)$ in Theorem 1.2.

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2. Some partition identities

In this section, we discuss some partition identities that will appear in the proofs in later sections. For $n \geqslant 0$ we write $\mathcal{P}(n)$ for the set of partitions of n, with the standard convention that $\mathcal{P}(0)$ contains the empty partition as unique element. We set $p(n) = |\mathcal{P}(n)|$, the number of partitions of n, and we write $P(t)$ for the generating function, as described in (1.1). If λ is a partition of n, we write $\lambda \vdash n$. We write λ_k for the number of parts of λ of length k, and we use the notation $\lambda = (n^{\lambda_n} \dots 2^{\lambda_2} 1^{\lambda_1})$.

Definition 2.1. Let n, k be integers such that $n \geqslant 0$ and $k \geqslant 1$. Define $F_k(n) = \sum_{\lambda \vdash n} \lambda_k$. Thus $F_k(n)$ is the total number of parts of length k in all partitions of n.

Lemma 2.2. *Let n, k be integers such that $n \geqslant k \geqslant 1$. We have $F_k(n) = F_k(n-k) + p(n-k)$.*

Proof. Removing a part of length k gives a bijection between the partitions of n with a part of length k and the partitions of $n-k$, which reduces by one the number of parts of length k. □

Proposition 2.3. *Let n, k be integers such that $n \geqslant 0$ and $k \geqslant 1$. We have*

$$F_k(n) = \sum_{i=1}^{\lfloor \frac{n}{k} \rfloor} p(n-ik) = p(n-k) + p(n-2k) + \cdots$$

In particular, $F_k(n) = 0$ for $n < k$. The generating function for $F_k(n)$ is given by

$$\sum_{n=1}^{\infty} F_k(n)\, t^n = \frac{t^k}{1-t^k}\, P(t).$$

Proof. The first part follows by induction from Lemma 2.2. Thus

$$\sum_{n=1}^{\infty} F_k(n)\, t^n = \sum_{n=1}^{\infty} \left(\sum_{i=1}^{\lfloor \frac{n}{k} \rfloor} p(n-ik) \right) t^n = \sum_{i=1}^{\infty} \sum_{n=ik}^{\infty} p(n-ik)\, t^n$$
$$= \sum_{i=1}^{\infty} \sum_{n=0}^{\infty} p(n)\, t^{n+ik} = \sum_{i=1}^{\infty} P(t)\, t^{ik} = \frac{t^k}{1-t^k}\, P(t).$$
□

For $1 \leqslant k \leqslant 10$, the sequences $F_k(n)$ can be found in Sloane's online encyclopedia of integer sequences as sequence number A000070 for $k = 1$, and A024784 + k for $2 \leqslant k \leqslant 10$.

Next, we relate $F_k(n)$ to some other functions defined in terms of partitions. These are the functions that we associate with Hochschild cohomology of symmetric groups, through the centraliser decomposition.

Definition 2.4. Let n be a non-negative integer. If $\lambda \vdash n$ and k, ℓ are positive integers, let $g_{k,\ell}(\lambda)$ be the number of distinct lengths divisible by k of parts of λ that are repeated at least ℓ times. More precisely, if $\lambda = (n^{\lambda_n} \dots 2^{\lambda_2} 1^{\lambda_1})$, then $g_{k,\ell}(\lambda)$ is the number of indices i, $1 \leqslant i \leqslant n$, such that k divides i and $\lambda_i \geqslant \ell$. Let $G_{k,\ell}(n) = \displaystyle\sum_{\lambda \vdash n} g_{k,\ell}(\lambda)$.

Lemma 2.5. *For n a non-negative integer and k, ℓ positive integers we have*

$$G_{k,\ell}(n) = \sum_{i=1}^{\lfloor \frac{n}{k} \rfloor} p(n - ik\ell).$$

Proof. Consider the set $\mathcal{G}_{k,\ell}(n)$ of ordered pairs (λ, i) where $\lambda \vdash n$ is a partition containing a part of length ik repeated at least ℓ times. Then $G_{k,\ell}(n) = |\mathcal{G}_{k,\ell}(n)|$. We have a bijection from $\mathcal{G}_{k,\ell}(n)$ to $\displaystyle\bigcup_{i=1}^{\lfloor \frac{n}{k} \rfloor} \mathcal{P}(n - ik\ell)$ given by removing ℓ parts of length ik from λ. □

Theorem 2.6. *For n a non-negative integer and k, ℓ positive integers we have*

$$G_{k,\ell}(n) = F_{k\ell}(n).$$

Proof. This follows from Proposition 2.3 and Lemma 2.5. □

Remark 2.7. This identity with $k = 1$ can be found in Exercise 80 in Chapter 1 of Stanley's book [10]. This is commonly known as Elder's theorem, and the case $k = 1$, $\ell = 1$ is known as Stanley's theorem. In the solution to the exercise in [10], Stanley explains that he discovered the result for $k = 1$ and all $\ell \geqslant 1$ in 1972 and submitted it to the Problems and Solutions section of Amer. Math. Monthly, where it was rejected with the comment "A bit on the easy side, and using only a standard argument." The statement was reproved by Elder in 1984, and other proofs can be found in Hoare [7] and Kirdar and Skyrme [8]. But as pointed out by Gilbert [5], all these proofs were predated by more than a decade in a 1959 paper of Fine [4]. A different generalisation of Elder's theorem was proved by Andrews and Deutsch [1].

Definition 2.8. For a non-negative integer n and a positive integer r let $q(n, r)$ be the number of partitions of n into r distinct parts.

The following is well known.

Lemma 2.9. *Let r be a positive integer. The generating function for $q(n, r)$, $n \geqslant 0$, is given by*

$$Q_r(t) := \sum_{n=0}^{\infty} q(n, r)\, t^n = \prod_{i=1}^{r} \frac{t^i}{1 - t^i}.$$

Proof. By going over to conjugate partitions, one sees that $q(n,r)$ is also equal to the number of partitions of n with each of $1,2,\ldots,r$ occurring as a part and with r as largest part. Thus, the generating function equals

$$(t+t^2+\cdots)(t^2+t^4+\cdots)\cdots(t^r+t^{2r}+\cdots)=\prod_{i=1}^{r}\frac{t^i}{1-t^i}. \qquad \square$$

Definition 2.10. Let n be a non-negative integer. If $\lambda\vdash n$ and k, r are positive integers, let $c_{k,r}(\lambda)$ be the number of unordered r-tuples of distinct part lengths of λ divisible by k. Let $C_{k,r}(n)=\sum_{\lambda\vdash n}c_{k,r}(\lambda)$ and let $D_{k,r}(n)=\sum_{\lambda\vdash n}\binom{g_{1,k}(\lambda)}{r}$.

Theorem 2.11. *Let n be a non-negative integer and let r,k be positive integers. Then*

$$C_{k,r}(n)=D_{k,r}(n).$$

The generating function for $C_{k,r}(n)$, $n\geqslant 0$ is

$$\left(\prod_{i=1}^{r}\frac{t^{ik}}{1-t^{ik}}\right)P(t).$$

Proof. Let $\{i_1,\ldots,i_r\}$ be a set of distinct positive numbers. The set of partitions of n having $ki_1,\ldots,ki_r$ as parts is in bijection with the set of partitions of $n-k(i_1+\cdots+i_r)$ (where as usual the set of partitions of a negative integer is taken to be the empty set). Similarly, the set of partitions of n with each of the parts $i_1,\ldots,i_r$ occurring at least k times is in bijection with the set of partitions of $n-k(i_1+\cdots+i_r)$. Thus, the generating functions of both $C_{k,r}(n)$ and of $D_{k,r}(n)$ equals $Q_r(t^k)P(t)$. Now the result follows from Lemma 2.9. $\square$

Definition 2.12. Let n be a non-negative integer. If $\lambda\vdash n$, let $e(\lambda)$ denote the number of ordered pairs (m,m') of positive integers such that m and m' are parts of λ, $m\neq m'$, $\lambda_m\geqslant 2$, and m' is even. Let $E(n)=\sum_{\lambda\vdash n}e(\lambda)$.

Theorem 2.13. *The generating function for $E(n)$, $n\geqslant 0$ is*

$$\frac{t^4+2t^8}{(1-t^2)(1-t^6)}P(t).$$

Proof. Let m be a positive integer and let $m'=2m''\neq m$ be an even positive integer. The set of partitions of n having both m and m' as parts and with the multiplicity of m as a part being at least 2 is in bijection with the partitions of $n-2(m+m'')$. Thus the generating function for $E(n)$, $n\geqslant 0$ is

$$R(t^2)P(t)$$

where $R(t)=\sum_{i\geqslant 0}^{\infty}R_it^i$ and R_i equals the number of ordered pairs (m,m'') of positive integers such that $i=m+m''$ and $m\neq 2m''$. We have

$$R_i=\begin{cases} i-1 & i \text{ not a multiple of } 3\\ i-2 & i \text{ a multiple of } 3.\end{cases}$$

Thus,

$$R(t)=\frac{t^2}{(1-t)^2}-\frac{t^3}{1-t^3}=\frac{t^2+2t^4}{(1-t)(1-t^3)},$$

proving the result. $\square$

3. On the Hochschild cohomology of symmetric groups

If R is a commutative ring of coefficients and A is an R-algebra which is projective as an R-module, then the Hochschild cohomology $HH^*(A)$ of A is the graded-commutative Ext-algebra $\mathsf{Ext}^*_{A^e}(A, A)$, where $A^e = A \otimes_R A^{\mathrm{op}}$ acts on A via left and right multiplication.

If G is a finite group, recall from [2, Theorem 2.11.2] the centraliser decomposition of Hochschild cohomology

$$HH^*(RG) \cong \bigoplus_g H^*(C_G(g), R).$$

where in the sum g runs over a set of representatives of the conjugacy classes in G. This is an isomorphism of graded R-modules but not in general of graded R-algebras.

For notational convenience, we adopt the convention that $\mathfrak{S}_0$ is the trivial group and that wreath products of the form $G \wr \mathfrak{S}_0$ are trivial, for any group G. For m a positive integer, we denote by $\mathbb{Z}/m$ a cyclic group of order m.

Proposition 3.1. *Let n be a positive integer. We have an isomorphism of graded vector spaces*

$$HH^*(\mathbb{F}_p\mathfrak{S}_n) \cong \bigoplus_{\lambda \vdash n} \bigotimes_{m=1}^{n} H^*(\mathbb{Z}/m \wr \mathfrak{S}_{\lambda_m}, \mathbb{F}_p).$$

Proof. The conjugacy classes of $\mathfrak{S}_n$ correspond to partitions of n. If $\lambda = (n^{\lambda_n} \dots 2^{\lambda_2} 1^{\lambda_1}) \vdash n$, then an element g of cycle type λ has as centraliser the direct product of wreath products

$$C_G(g) = \prod_{m=1}^{n} (\mathbb{Z}/m \wr \mathfrak{S}_{\lambda_m}).$$

Applying the Künneth formula to this direct product of groups yields the result. $\square$

To compute the dimension of the expression given in Proposition 3.1, we use Nakaoka's description of the cohomology of a wreath product of finite groups over a field.

Proposition 3.2. *Let ℓ, m be positive integers. We have an isomorphism of graded vector spaces*

$$H^*(\mathbb{Z}/m \wr \mathfrak{S}_\ell, \mathbb{F}_p) \cong H^*(\mathfrak{S}_\ell, H^*((\mathbb{Z}/m)^\ell, \mathbb{F}_p)).$$

Proof. This follows from Theorem 3.3 of Nakaoka [9]. $\square$

This is in fact an isomorphism of graded k-algebras with the appropriate algebra structure on the right side, but this will not be needed.

4. Degree one

For G a finite group, we have $H^0(G, \mathbb{F}_p) \cong \mathbb{F}_p$ and a canonical identification $H^1(G, \mathbb{F}_p) = \mathsf{Hom}(G, \mathbb{F}_p)$. In particular, $H^1(G, \mathbb{F}_p) = 0$ unless G has a nontrivial quotient of p-power order. In order to calculate the dimension of $HH^1(\mathbb{F}_p\mathfrak{S}_n)$, the formulae in Proposition 3.1 and Proposition 3.2 imply that we need to calculate the dimensions of $H^i(\mathfrak{S}_\ell, H^j((\mathbb{Z}/m)^\ell, \mathbb{F}_p))$, where (i, j) is either $(0, 1)$ or $(1, 0)$. In the case $(0, 1)$ we will need to identify the action of $\mathfrak{S}_\ell$ on $H^1((\mathbb{Z}/m)^\ell, \mathbb{F}_p)$.

For the remainder of the paper, we write M_ℓ for the canonical ℓ-dimensional permutation module for $\mathfrak{S}_\ell$ over $\mathbb{F}_p$, where ℓ is a positive integer. This is induced from the trivial module for $\mathfrak{S}_{\ell-1}$ over $\mathbb{F}_p$.

Lemma 4.1. *For ℓ a positive integer, we have $H^0(\mathfrak{S}_\ell, M_\ell) \cong \mathbb{F}_p$.*

Proof. By Frobenius reciprocity, we have $H^0(\mathfrak{S}_\ell, M_\ell) \cong H^0(\mathfrak{S}_{\ell-1}, \mathbb{F}_p) \cong \mathbb{F}_p$. □

Lemma 4.2. *Let ℓ, m be positive integers. As modules for $\mathfrak{S}_\ell$, we have*

$$H^1((\mathbb{Z}/m)^\ell, \mathbb{F}_p) \cong \begin{cases} 0 & p \nmid m, \\ M_\ell & p \mid m. \end{cases}$$

Proof. If $p \nmid m$ then the cohomology of $(\mathbb{Z}/m)^m$ with coefficients in $\mathbb{F}_p$ vanishes in all positive degrees. If $p \mid m$, then $\mathsf{Hom}(\mathbb{Z}/m, \mathbb{F}_p) \cong \mathsf{Hom}(\mathbb{Z}/p, \mathbb{F}_p) \cong \mathbb{F}_p$. The result follows by looking at the action of $\mathfrak{S}_\ell$ on the two sides of the isomorphism

$$H^1((\mathbb{Z}/m)^\ell, \mathbb{F}_p) \cong \mathsf{Hom}((\mathbb{Z}/m)^\ell, \mathbb{F}_p).$$ □

Lemma 4.3. *Let ℓ, m be positive integers. We have*

$$\dim_{\mathbb{F}_p} H^0(\mathfrak{S}_\ell, H^1((\mathbb{Z}/m)^\ell, \mathbb{F}_p)) = \begin{cases} 0 & p \nmid m, \\ 1 & p \mid m. \end{cases}$$

Proof. This follows from Lemma 4.1 and Lemma 4.2. □

Lemma 4.4. *Let n be a positive integer and λ a partition of n. We have*

$$\sum_{m=1}^{n} \dim_{\mathbb{F}_p} H^0(\mathfrak{S}_{\lambda_m}, H^1((\mathbb{Z}/m)^{\lambda_m}, \mathbb{F}_p)) = g_{p,1}(\lambda).$$

Proof. Only summands with $\lambda_m \geqslant 1$ contribute to the left side. It follows from Lemma 4.3 that the left side counts the number of parts of λ of distinct lengths divisible by p, and this is $g_{p,1}(\lambda)$. □

Lemma 4.5. *Let n be a positive integer and λ a partition of n. We have*

$$\sum_{m=1}^{n} \dim_{\mathbb{F}_p} H^1(\mathfrak{S}_{\lambda_m}, \mathbb{F}_p) = \begin{cases} g_{1,2}(\lambda) & p = 2, \\ 0 & p \geqslant 3. \end{cases}$$

Proof. Only summands with $\lambda_m \geqslant 1$ contribute to the left side. For ℓ a positive integer we have $H^1(\mathfrak{S}_\ell, \mathbb{F}_p) = \mathsf{Hom}(\mathfrak{S}_\ell, \mathbb{F}_p) = 0$ for $p \geqslant 3$ or $\ell = 1$, and $\dim_{\mathbb{F}_2} H^1(\mathfrak{S}_\ell, \mathbb{F}_2) = \dim_{\mathbb{F}_2} \mathsf{Hom}(\mathfrak{S}_\ell, \mathbb{F}_2) = 1$ if $\ell \geqslant 2$. Thus the left side in the statement is zero for $p \geqslant 3$. For $p = 2$ this counts the number of distinct part lengths of λ that appear at least twice, and this is $g_{1,2}(\lambda)$. □

Combining the above results yields the following combinatorial descriptions of the dimension of $HH^1(\mathbb{F}_p\mathfrak{S}_n)$.

Theorem 4.6. *Let n be a positive integer. The dimension of the degree one Hochschild cohomology of the group algebra of the symmetric group $\mathfrak{S}_n$ on n letters is given by*

$$\dim_{\mathbb{F}_p} HH^1(\mathbb{F}_p\mathfrak{S}_n) = \begin{cases} G_{2,1}(n) + G_{1,2}(n) & p = 2, \\ G_{p,1}(n) & p \geqslant 3. \end{cases}$$

Proof. Let λ be a partition of n. Note that the degree one part of the tensor product on the right side in Proposition 3.1 corresponding to the summand indexed by λ is isomorphic to

$$\bigoplus_{m=1}^{n} H^1(\mathbb{Z}/m \wr \mathfrak{S}_{\lambda_m}, \mathbb{F}_p).$$

Fix an integer m such that $1 \leqslant m \leqslant n$. Applying Proposition 3.2 in degree one, we have

$$H^1(\mathbb{Z}/m \wr \mathfrak{S}_{\lambda_m}, \mathbb{F}_p) \cong H^0(\mathfrak{S}_{\lambda_m}, H^1((\mathbb{Z}/m)^{\lambda_m}, \mathbb{F}_p)) \oplus H^1(\mathfrak{S}_{\lambda_m}, \mathbb{F}_p).$$

Taking the sum over all partitions of n and all m such that $1 \leqslant m \leqslant n$ we obtain

$$HH^1(\mathbb{F}_p\mathfrak{S}_n) = \bigoplus_{\lambda \vdash n} \bigoplus_{m=1}^{n} \left(H^0(\mathfrak{S}_{\lambda_m}, H^1((\mathbb{Z}/m)^{\lambda_m}, \mathbb{F}_p)) \oplus H^1(\mathfrak{S}_{\lambda_m}, \mathbb{F}_p) \right).$$

Adding up dimensions, using Lemma 4.4 and Lemma 4.5 yields the result. $\square$

Corollary 4.7. *For any positive integer n we have*

$$\dim_{\mathbb{F}_p}(HH^1(\mathbb{F}_p\mathfrak{S}_n)) = \begin{cases} 2F_2(n) & p = 2, \\ F_p(n) & p \geqslant 3. \end{cases}$$

Proof. This follows from combining Theorem 4.6 and Theorem 2.6. $\square$

Proof of Theorem 1.2 (ii). Combining Corollary 4.7 with Proposition 2.3 yields the result. $\square$

5. Degree two

In order to calculate the dimension of $HH^2(\mathbb{F}_p\mathfrak{S}_n)$, the formulae in Proposition 3.1 and Proposition 3.2 imply that we need to calculate the dimensions of $H^i(\mathfrak{S}_\ell, H^j((\mathbb{Z}/m)^\ell, \mathbb{F}_p))$, where (i, j) is either $(0, 2)$ or $(1, 1)$ or $(2, 0)$. In the case $(0, 2)$, we will need to identify the action of $\mathfrak{S}_\ell$ on $H^2((\mathbb{Z}/m)^\ell, \mathbb{F}_p)$.

Lemma 5.1. *Let ℓ be a positive integer. We have*

$$\dim_{\mathbb{F}_p} H^0(\mathfrak{S}_\ell, \Lambda^2(M_\ell)) = \begin{cases} 0 & p \neq 2 \text{ or } \ell = 1, \\ 1 & p = 2 \text{ and } \ell \geqslant 2, \end{cases}$$

$$\dim_{\mathbb{F}_p} H^1(\mathfrak{S}_\ell, \Lambda^2(M_\ell)) = \begin{cases} 0 & p \neq 2 \text{ or } \ell = 1, \\ 1 & p = 2 \text{ and } \ell = 2, 3, \\ 2 & p = 2 \text{ and } \ell \geqslant 4. \end{cases}$$

Proof. The module $\Lambda^2(M_\ell)$ is zero for $\ell = 1$. For $\ell \geqslant 2$, this module is induced from the trivial tensor sign representation $\mathbb{F}_p \otimes \mathsf{sgn}$ of $\mathfrak{S}_{\ell-2} \times \mathfrak{S}_2$, so by Frobenius reciprocity, we have

$$H^i(\mathfrak{S}_\ell, \Lambda^2(M_\ell)) \cong H^i(\mathfrak{S}_{\ell-2} \times \mathfrak{S}_2, \mathbb{F}_p \otimes \mathsf{sgn}).$$

If $p \neq 2$, the module $\mathbb{F}_p \otimes \mathsf{sgn}$ is not in the principal block, so there is no cohomology in any degree. If $p = 2$, $\mathbb{F}_p \otimes \mathsf{sgn}$ is the trivial module, and the computation is straightforward. $\square$

Lemma 5.2. *Let ℓ, m be positive integers. As an $\mathbb{F}_p\mathfrak{S}_\ell$-module, we have*

$$H^2((\mathbb{Z}/m)^\ell, \mathbb{F}_p) \cong \begin{cases} 0 & p \nmid m \\ M_\ell \oplus \Lambda^2(M_\ell) & p \mid m. \end{cases}$$

Proof. For $p \nmid m$ this is clear, so suppose that $p \mid m$. If p is odd, the cohomology is a tensor product of an exterior algebra on m generators in degree one, tensored with a polynomial ring on their Bocksteins in degree two. So in degree two we have the polynomial generators, which give a copy of M_ℓ, and the products of two exterior generators, which give a copy of $\Lambda^2(M_\ell)$. If $p = 2$ and $4 \mid m$, the computation is the same. If $p = 2$ and $m \equiv 2 \pmod 4$ then the cohomology is a polynomial ring on the degree one generators, so in degree two we have $S^2(M_\ell)$. As a module for $\mathfrak{S}_\ell$, this is isomorphic to $M_\ell \oplus \Lambda^2(M_\ell)$. The squares of degree one generators give a copy of M_ℓ and the products of distinct degree one generators give a copy of $\Lambda^2(M_\ell)$. □

Lemma 5.3. *Let ℓ, m be positive integers. We have*

$$\dim_{\mathbb{F}_p} H^0(\mathfrak{S}_\ell, H^2((\mathbb{Z}/m)^\ell, \mathbb{F}_p)) = \begin{cases} 0 & p \nmid m, \\ 1 & 2 \neq p \mid m, \\ 1 & 2 = p \mid m \text{ and } \ell = 1, \\ 2 & 2 = p \mid m \text{ and } \ell \geqslant 2. \end{cases}$$

Proof. By Lemma 5.2, we get zero if $p \nmid m$ and two terms when $p \mid m$. The computation of $H^0(\mathfrak{S}_\ell, M_\ell)$ is given in Lemma 4.1, and the computation of $H^0(\mathfrak{S}_\ell, \Lambda^2(M_\ell))$ is given in Lemma 5.1. □

Lemma 5.4. *Let ℓ be a positive integer. We have*

$$\dim_{\mathbb{F}_p} H^1(\mathfrak{S}_\ell, M_\ell) = \begin{cases} 1 & p = 2,\ \ell \geqslant 3, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By Frobenius reciprocity, we have $H^1(\mathfrak{S}_\ell, M_\ell) \cong H^1(\mathfrak{S}_{\ell-1}, \mathbb{F}_p)$. □

Lemma 5.5. *Let ℓ, m be positive integers. We have*

$$\dim_{\mathbb{F}_p} H^1(\mathfrak{S}_\ell, H^1((\mathbb{Z}/m)^\ell, \mathbb{F}_p)) = \begin{cases} 1 & 2 = p \mid m \text{ and } \ell \geqslant 3, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By Lemma 4.2, $H^1((\mathbb{Z}/m)^\ell, \mathbb{F}_p)$ is equal to M_ℓ if $p \mid m$ and zero otherwise. Now use Lemma 5.4. □

Lemma 5.6. *Let ℓ be a positive integer. We have*

$$\dim_{\mathbb{F}_2} H^2(\mathfrak{S}_\ell, \mathbb{F}_2) = \begin{cases} 0 & \ell = 1, \\ 1 & \ell = 2, 3, \\ 2 & \ell \geqslant 4, \end{cases}$$

and $H^2(\mathfrak{S}_\ell, \mathbb{F}_p) = 0$ for $p \geqslant 3$.

Proof. The group $H^1(\mathfrak{S}_\ell, \mathbb{C}^\times)$ is trivial for $\ell = 1$ and isomorphic to $\mathbb{Z}/2$ for $\ell \geqslant 2$. The Schur multiplier $H^2(\mathfrak{S}_n, \mathbb{C}^\times)$ is trivial for $\ell \leqslant 3$ and isomorphic to $\mathbb{Z}/2$ for $\ell \geqslant 4$. Raising complex numbers to their p-th powers induces a surjective group homomorphism $\varphi : \mathbb{C}^\times \to \mathbb{C}^\times$ with cyclic kernel of order p, which we identify with the additive group of $\mathbb{F}_p$. The associated long exact cohomology sequence in low degrees takes the form

$$H^1(\mathfrak{S}_\ell, \mathbb{C}^\times) \longrightarrow H^1(\mathfrak{S}_\ell, \mathbb{C}^\times) \longrightarrow H^2(\mathfrak{S}_\ell, \mathbb{F}_p) \longrightarrow H^2(\mathfrak{S}_\ell, \mathbb{C}^\times) \longrightarrow H^2(\mathfrak{S}_\ell, \mathbb{C}^\times)$$

The maps induced by φ on $H^1(\mathfrak{S}_\ell, \mathbb{C}^\times)$ and on $H^2(\mathfrak{S}_\ell, \mathbb{C}^\times)$ are isomorphisms if p is odd and trivial if $p = 2$. The result follows in all cases from the exactness of the above sequence. $\square$

Theorem 5.7. *The dimension of the degree two Hochschild cohomology of the symmetric groups is given by*

$$\dim_{\mathbb{F}_p} HH^2(\mathbb{F}_p\mathfrak{S}_n) = \begin{cases} G_{2,1}(n) + G_{2,2}(n) + G_{2,3}(n) + G_{1,2}(n) & \\ \qquad\qquad + G_{1,4}(n) + C_{2,2}(n) + E(n) + D_{2,2}(n) & p = 2, \\ G_{p,1}(n) + C_{p,2}(n) & p \geqslant 3. \end{cases}$$

Proof. By Proposition 3.1,

$$(5.8) \quad \dim_{\mathbb{F}_p} HH^2(\mathbb{F}_p\mathfrak{S}_n) = \sum_{\lambda \vdash n} \sum_{m=1}^{n} \dim_{\mathbb{F}_p} H^2(\mathbb{Z}/m \wr \mathfrak{S}_{\lambda_m}, \mathbb{F}_p) + \sum_{\lambda \vdash n} \sum_{\{m,m'\}} (\dim_{\mathbb{F}_p} H^1(\mathbb{Z}/m \wr \mathfrak{S}_{\lambda_m}, \mathbb{F}_p))(\dim_{\mathbb{F}_p} H^1(\mathbb{Z}/m' \wr \mathfrak{S}_{\lambda_{m'}}, \mathbb{F}_p))$$

where the indices $\{m, m'\}$ run over unordered pairs of distinct positive integers between 1 and n.

Using Proposition 3.2, we have

$$H^2(\mathbb{Z}/m \wr \mathfrak{S}_{\lambda_m}, \mathbb{F}_p) \cong H^0(\mathfrak{S}_{\lambda_m}, H^2((\mathbb{Z}/m)^{\lambda_m}, \mathbb{F}_p)) \oplus H^1(\mathfrak{S}_{\lambda_m}, H^1((\mathbb{Z}/m)^{\lambda_m}, \mathbb{F}_p)) \oplus H^2(\mathfrak{S}_{\lambda_m}, \mathbb{F}_p).$$

Using Lemmas 5.3, 5.5, and 5.6, we get that for $p \geqslant 3$, the first term on the right contributes $G_{p,1}(n)$, the second and third contribute zero to the sum first sum on the right of Equation 5.8. For $p = 2$, the first term contributes $G_{2,1}(n) + G_{2,2}(n)$, the second contributes $G_{2,3}(n)$, and the third contributes $G_{1,2}(n) + G_{1,4}(n)$.

Next we consider the contribution of the second sum on the right hand side of Equation 5.8. Suppose first that $p \geqslant 3$. As in the proof of Theorem 4.6, for any positive integer m

$$\dim_{\mathbb{F}_p} H^1(\mathbb{Z}/m \wr \mathfrak{S}_{\lambda_m}, \mathbb{F}_p) = \begin{cases} 0 & p \nmid m \text{ or } \lambda_m = 0 \\ 1 & p \mid m \text{ and } \lambda_m \geqslant 1. \end{cases}$$

Thus the contribution of a partition $\lambda \vdash n$ to the second sum of Equation 5.8 equals $c_{p,2}(\lambda)$ and the second sum equals $C_{p,2}(n)$. This completes the proof for $p \geqslant 3$. Now suppose that $p = 2$. Again as in the proof of Theorem 4.6 for any positive integer m

$$\dim_{\mathbb{F}_p} H^1(\mathbb{Z}/m \wr \mathfrak{S}_{\lambda_m}, \mathbb{F}_p) = \begin{cases} 0 & \lambda_m = 0 \\ 0 & 2 \nmid m \text{ and } \lambda_m = 1 \\ 1 & 2 \mid m \text{ and } \lambda_m = 1 \\ 1 & 2 \nmid m \text{ and } \lambda_m \geqslant 2 \\ 2 & 2 \mid m \text{ and } \lambda_m \geqslant 2. \end{cases}$$

Thus the contribution of a partition $\lambda \vdash n$ to the second sum of Equation 5.8 equals

$$c_{2,2,}(\lambda) + e(\lambda) + \binom{g_{1,2}(\lambda)}{2}$$

and the second sum equals

$$C_{2,2}(n) + E(n) + D_{2,2}(n),$$

as required. $\square$

Corollary 5.9. *For any positive integer n we have*

$$\dim_{\mathbb{F}_p} HH^2(\mathbb{F}_p\mathfrak{S}_n) = \begin{cases} 2F_2(n) + 2F_4(n) + F_6(n) + 2C_{2,2}(n) + E(n) & p = 2, \\ F_p(n) + C_{p,2}(n) & p \geqslant 3. \end{cases}$$

Proof. This follows from combining Theorem 5.7 with Theorem 2.6 and the first equality in Theorem 2.11. $\square$

Proof of Theorem 1.2 (iii). Applying Proposition 2.3, Theorem 2.11 and Theorem 2.13 to the terms in the formula in Corollary 5.9, we obtain

$$\sum_{n=0}^{\infty} \dim_{\mathbb{F}_p} HH^2(\mathbb{F}_p\mathfrak{S}_n)\, t^n =$$

$$\begin{cases} \left(\frac{2t^2}{1-t^2} + \frac{2t^4}{1-t^4} + \frac{t^6}{1-t^6} + \frac{2t^6}{(1-t^2)(1-t^4)} + \frac{t^4+2t^8}{(1-t^2)(1-t^6)} \right) P(t) & p = 2, \\ \frac{t^p}{(1-t^p)(1-t^{2p})}\, P(t) & p \geqslant 3. \end{cases}$$

Simplifying the expression for $p = 2$ completes the proof. $\square$

6. Proof of Theorem 1.3.

6.1. **More combinatorics.** If $f(t)$ and $g(t)$ are monic polynomials and α is a non-zero constant, then we say that α is the leading coefficient of the rational function $\frac{\alpha f(t)}{g(t)}$ and the difference of the degree of $f(t)$ and $g(t)$ is referred to as the total degree of $\frac{\alpha f(t)}{g(t)}$. For a positive integer m, set

$$E_m(t) := \frac{t^m}{1-t^m}.$$

We fix a positive integer r and let $\mathcal{V} = \{1, \ldots, r\}$. We let $\mathcal{E}$ be the set of two element subsets of $\mathcal{V}$ and let $\mathcal{G}$ be the set of undirected simple graphs with vertex set $\mathcal{V}$. Then $\mathcal{G}$ may be identified with the power set of $\mathcal{E}$ via the map which sends a graph Y to the subset of $\mathcal{E}$ consisting of those $\{i, j\}$ such that i and j are connected by an edge in Y. We refer to the elements of $\mathcal{E}$ as edges.

For each Y in $\mathcal{G}$, let $e(Y)$ denote the number of edges of Y and let $c(Y)$ denote the number of connected components of Y. Let C_Y be the set partition of $\mathcal{V}$ induced by the equivalence relation corresponding to the connected components of Y. In other words, $i, j \in \mathcal{V}$ are in the same element of C_Y iff i and j belong to the same connected component of Y.

Lemma 6.1. *With the above notation,* $\sum_{Y\in\mathcal{G}} (-1)^{c(Y)+e(Y)} = (-1)^r r!$.

Proof. Let Γ be the complete graph on $\mathcal{V}$. By Tutte [11, IX.2.2]

$$\sum_{Y\in\mathcal{G}} (-1)^{c(Y)+e(Y)} = P_\Gamma(-1)$$

where $P_\Gamma(x)$ is the chromatic polynomial of Γ. By [11, Thm. IX.26], we have

$$P_\Gamma(x) = x(x-1)\dots(x-(r-1)),$$

proving the result. $\square$

Let k_i, ℓ_i, $1 \leqslant i \leqslant r$ be positive integers. For each non-empty subset A of $\mathcal{V}$, set

$$k_A := \mathsf{lcm}(k_i \colon i \in A)$$

$$\ell_A := \sum_{i\in A} \ell_i$$

$$E_A(t) := E_{k_A\ell_A}(t) = \frac{t^{\ell_A k_A}}{1-t^{\ell_A k_A}}.$$

For a graph Y in $\mathcal{G}$, set

$$E_Y(t) := \prod_{A\in C_Y} E_A(t) = \prod_{A\in C_Y} \frac{t^{\ell_A k_A}}{1-t^{\ell_A k_A}}.$$

Lemma 6.2. *Let r, k_i, ℓ_i, k_A, ℓ_A be as above and let $Q(t) = \sum_{n=1}^{\infty} q(n)t^n$, where for a positive integer n, $q(n)$ equals the number of ordered r-tuples $(u_1,\dots,u_r)$ of positive integers such that $n = \sum_{i=1}^{n} \ell_i k_i u_i$ and such that $k_i u_i \neq k_j u_j$ if $i \neq j$. Then*

$$Q(t) = \sum_{Y\in\mathcal{G}} (-1)^{e(Y)} E_Y(t).$$

Consequently, $Q(t)$ is a rational function of total degree 0 and leading coefficient $(-1)^r r!$.

Proof. Let $U(n)$ be the set of ordered tuples r-tuples $(u_1,\dots,u_r)$ of positive integers such that $n = \sum_{i=1}^{n} \ell_i k_i u_i$. For $Y \in \mathcal{G}$, let $U_Y(n)$ be the subset of $U(n)$ such that $k_i u_i = k_j u_j$ whenever i and j are in the same connected component of Y. So, $U(n) = U_\emptyset(n)$ where $\emptyset$ denotes the graph without any edges. We claim that

$$q(n) = \sum_{Y\in\mathcal{G}} (-1)^{e(Y)} |U_Y(n)|.$$

Indeed, let $U'(n)$ be the subset of $U(n)$ consisting of those r-tuples such that $k_i u_i = k_j u_j$ for some $i \neq j$. For an edge $e = \{i,j\} \in \mathcal{E}$, let $U^e(n)$ be the subset of $U(n)$ consisting of those r-tuples such that $k_i u_i = k_j u_j$. Then $U'(n)$ is the union of the $U^e(n)$ as e runs over the edge set $\mathcal{E}$ and

$$q(n) = |U(n)| - |U'(n)|.$$

Now let $Y \subseteq \mathcal{E}$ be a graph. Then $\bigcap_{e\in Y} U^e(n) = U_Y(n)$. The claim now follows by the above equation and the inclusion-exclusion principle.

Next, we claim that

$$\sum_{n=1}^{\infty} |U_Y(n)| t^n = E_Y(t).$$

To see this, note that if $(u_1,\dots,u_r) \in U_Y(n)$, then for any $A \in C_Y$ there exists a positive number u'_A such that $k_i u_i = k_A u'_A$ for all $i \in A$. The assignment $(u_1,\dots,u_r) \mapsto (u'_A)_{A\in C_Y}$ induces a bijection between $U_Y(n)$ and the set of tuples $(u'_A)_{A\in C_Y}$ of positive integers u'_A such that $\sum_{A\in C_Y} \ell_A k_A u'_A = n$. The claim follows.

The first assertion is a consequence of the two claims. By definition, $E_Y(t)$ is a rational function of total degree 0 with leading coefficient $(-1)^{c(Y)}$. Taking common denominators in $Q(t)$ and applying Lemma 6.1 yields the result. $\square$

Remark. In the case $r = 1$, $Q(t) = \frac{t^{k_1\ell_1}}{1-t^{k_1\ell_1}}$ is the generating function of the sequence $(G_{k_1,\ell_1}(n) = F_{k_1 l_1}(n))$, $n \geqslant 1$.

Definition 6.3. Let $k > 0$ and $\ell \geqslant 0$ be integers. Let $\theta_{k,\ell} \colon \mathbb{N} \times \mathbb{N}_0 \to \mathbb{Z}$ be defined by setting $\theta_{k,\ell}(x, y) = 1$ if $k \mid x$ and $y \geqslant \ell$ and $\theta_{k,\ell}(x, y) = 0$ otherwise.

Proposition 6.4. *Let r be a positive integer, and for $1 \leqslant i \leqslant r$ let k_i, ℓ_i be positive integers, and let $Q(t)$ be as in Lemma 6.2. Let*

$$G(n) = \sum_{\lambda \vdash n} \sum_{(m_1,\ldots,m_r)} \prod_{i=1}^{r} \theta_{k_i,\ell_i}(m_i, \lambda_{m_i}),$$

where the inner sum runs over ordered r-tuples of distinct part lengths $m_1, \ldots, m_r$ of λ. Then

$$\sum_{n=1}^{\infty} G(n)t^n = Q(t)P(t).$$

Proof. Let u_i, $1 \leqslant i \leqslant r$ be positive integers such that setting $m_i := k_i u_i \neq m_j := k_j u_j$, for any $i \neq j$, $1 \leqslant i, j \leqslant r$. Then the set of partitions of n with m_i as a part with multiplicity at least ℓ_i, $i = 1, \ldots, r$ is in bijection with the set of partitions of $n - (k_1\ell_1 u_1 + \cdots + k_r\ell_r u_r)$. The result follows, using Lemma 6.2. $\square$

6.2. **Proof of Theorem 1.3.**

Lemma 6.5. *Let $d \geqslant 1$ and let $h \colon \mathbb{N} \times \mathbb{N}_0 \to \mathbb{Z}$ be defined by*

$$h(m, n) = \dim_{\mathbb{F}_p}(H^d(\mathbb{Z}/m \wr \mathfrak{S}_n, \mathbb{F}_p)),\ m \in \mathbb{N},\ n \in \mathbb{N}_0.$$

Then h is a $\mathbb{Z}$-linear combination of finitely many functions θ_{k_i,ℓ_i} indexed by some positive integers k_i, ℓ_i, $1 \leqslant i \leqslant s$, with the property that if θ_{k_i,ℓ_i} appears with a non-zero coefficient, then $k_i \in \{1, p\}$ when p is odd, and $k_i \in \{1, 2, 4\}$ when $p = 2$, for all i such that $1 \leqslant i \leqslant s$.

Proof. For a positive integer k, let $\epsilon_k \colon \mathbb{N} \to \mathbb{Z}$ be the function defined by $\epsilon_k(m) = 1$ if $k \mid m$ and $\epsilon_k(m) = 0$ otherwise. For a non-negative integer ℓ, let $\gamma_\ell \colon \mathbb{N}_0 \to \mathbb{Z}$ be defined by $\gamma_\ell(n) = 1$ if $n \geqslant \ell$ and $\gamma_\ell(n) = 0$ otherwise. So, $\theta_{k,\ell}(m, n) = \epsilon_k(m)\gamma_\ell(n)$ for all $k, m > 0$ and all $\ell, n \geqslant 0$.

Suppose that p is odd. By Proposition 3.2 and the structure of the cohomology of cyclic groups, $h(m, -) = h(p, -)$ if $p \mid m$ and $h(m, -) = h(1, -)$ if $p \nmid m$. Thus, for all $n \geqslant 0$,

$$h(m, n) = \epsilon_1(m)h(1, n) - \epsilon_p(m)h(1, n) + \epsilon_p(m)h(p, n).$$

Similarly, if $p = 2$, then $h(m, -) = h(4, -)$ if $4 \mid m$, $h(m, -) = h(2, -)$ if $m \equiv 2 \pmod 4$ and $h(m, -) = h(1, -)$ if m is odd. Hence,

$$h(m, n) = \epsilon_1(m)h(1, n) - \epsilon_2(m)h(1, n) + \epsilon_2(m)h(2, n) - \epsilon_4(m)h(2, n) + \epsilon_4(m)h(4, n).$$

Let $k \in \{1, p, 4\}$. By [6, Prop. 1.6], the sequence $h(k, n), n \geqslant 0$ eventually stabilises, that is, there exists $s_k > 0$ such that $h(k, n) = h(k, s_k)$ for all $n \geqslant s_k$. Here, we note that for

$k=1$, this is Nakaoka's result on the stability of the cohomology of symmetric groups. Since $d \geqslant 1$, $h(k,0)=0$. It follows that there exist integers $u_{i,k}$, $1 \leqslant i \leqslant n_k$ such that

$$h(k,n) = \sum_{\ell=1}^{s_k} u_{i,k}\gamma_\ell(n)$$

for all $n \geqslant 0$. Combining this with the previous displayed equations yields the result. $\square$

Theorem 1.3 is a consequence of the above lemma and Lemma 6.2 as we now show. For each $d \geqslant 1$, we fix a $\mathbb{Z}$-linear combination of $\theta_{k,l}$ representing the function h as in Lemma 6.5.

Proof of Theorem 1.3. For positive integers d and n let $A_{d,n}$ be the set of ordered tuples $(d_1,\ldots,d_n)$ of non-negative integers d_m such that $\sum_{m=1}^n d_m = d$, considered as $\mathfrak{S}_n$-set via place permutation. Let $A'(d,n)$ be the ($\mathfrak{S}_n$-invariant) subset of $A_{d,n}$ consisting of tuples in which each component is strictly positive.

By Proposition 3.1,

$$\dim_{\mathbb{F}_p}(HH^d(\mathbb{F}_p\mathfrak{S}_n)) = \sum_{\lambda\vdash n}\sum_{(d_1,\ldots,d_n)\in A_{d,n}}\prod_{m=1}^n \dim_{\mathbb{F}_p}(H^{d_m}(\mathbb{Z}/m \wr \mathfrak{S}_{\lambda_m},\mathbb{F}_p)).$$

Thus in order to prove Theorem 1.3 it suffices to show that for any $\mathfrak{S}_n$-orbit $\mathcal{O}$ of $A_{d,n}$ the generating function of the sequence $(G_n)_{n\geqslant 1}$ where

$$G_n := \sum_{\lambda\vdash n}\sum_{(d_1,\ldots,d_n)\in \mathcal{O}}\prod_{m=1}^n \dim_{\mathbb{F}_p}(H^{d_m}(\mathbb{Z}/m \wr \mathfrak{S}_{\lambda_m},\mathbb{F}_p))$$

is of the form $S(t)P(t)$ for some rational function $S(t)$.

Let $\mathcal{O}$ be as above and let $\mathcal{O}_0$ be the subset of $\mathcal{O}$ consisting of the tuples $(d_1,\ldots,d_n)$ such that if $d_i \geqslant 1$ then $\lambda_i > 0$. Since $H^i(Z/m \wr S_0,\mathbb{F}_p)=0$ for all $m \geqslant 1$ and all $i \geqslant 1$, only elements of $\mathcal{O}_0$ contribute to G_n. Now for each $r>0$, and each size r subset M of the set of distinct part lengths of λ, let $\mathcal{O}_{0,M}$ be the subset of $\mathcal{O}_0$ consisting of those tuples $(d_1,\ldots,d_m)$ such that $d_m > 0$ if and only if $m \in M$. Then $\mathcal{O}_{0,M}$ is in bijection with a union of $\mathfrak{S}_r$-orbits of $A'_{d,r}$ via the map which sends an n-tuple $(d_1,\ldots,d_n)$ to the subtuple of positive d_m's. Further, note that $H^0(G,\mathbb{F}_p)$ is one-dimensional for all finite groups G. Hence, for any $(d_1,\ldots,d_n)\in\mathcal{O}$, the product $\prod_{m=1}^n \dim_{\mathbb{F}_p}(H^{d_m}(\mathbb{Z}/m \wr \mathfrak{S}_{\lambda_m},\mathbb{F}_p))$ equals the subproduct ranging over the m's such that $d_m>0$. Thus, running over all size r subsets of part lengths of λ and over all $r>0$, it follows that G_n may be written as

$$\sum_{\lambda\vdash n}\sum_{(e_1,\ldots,e_r)\in\mathcal{O}'}\sum_{(m_1<\cdots<m_r)}\prod_{i=1}^r \dim_{\mathbb{F}_p}(H^{e_i}(\mathbb{Z}/m_i \wr \mathfrak{S}_{\lambda_{m_i}},\mathbb{F}_p))$$

where $\mathcal{O}'$ is an $\mathfrak{S}_r$-orbit of $A'_{d,r}$ for some r, and where in the inner sum $(m_1<\cdots<m_r)$ runs over collections of r distinct part lengths of λ.

By the above and Lemma 6.5 G_n is a sum of terms of the form

$$G'_n = \sum_{\lambda\vdash n}\sum_{((a_1,b_1),\ldots,(a_r,b_r))\in\mathcal{X}}\sum_{(m_1<\cdots<m_r)}\theta_{a_1,b_1}(m_1,\lambda_1)\ldots\theta_{a_r,b_r}(m_r,\lambda_r)$$

where $\mathcal{X}$ is an $\mathfrak{S}_r$-orbit of r-tuples of the form $((a_1,b_1),\ldots(a_r,b_r))$ with $\mathfrak{S}_r$ acting again by place permutations. It therefore suffices to show that the generating function of (G'_n), $n\geqslant 1$

is the product of a rational function with $P(t)$. But this generating function is $\frac{1}{c}Q(t)$, where $Q(t)$ is as in Lemma 6.2 and where $c = \frac{r!}{|\mathcal{X}|}$ is the order of the $\mathfrak{S}_r$-stabiliser of an element $((a_1, b_1), \ldots, (a_r, b_r))$ of $\mathcal{X}$. $\square$

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