Funding Strategies for Defined Benefit Pension Schemes

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Declaration

Abbreviations and Glossary

AC_x	Accrued Liability for a member aged x
$adj(t)$	Supplemental cost at time t
$AL(t)$	Actuarial Liability at time t
α	Minimum entry age
b	Last salary benefit fraction
$B(t)$	Total Benefit paid at time t
B_y	Benefit to retired members aged y
B_T	Value of the bond asset at time t
$\beta_r(t)$	Proportion of fund invested in the short term asset at time t
$\beta_B(t)$	Proportion of fund invested in the bond asset
$\beta_S(t)$	Proportion of fund invested in the equity asset
$c(t)$	Contribution paid at time t
CP	Constant premium
CVaR	Conditional value at risk
γ	Intensity of default ($1/\gamma$ is the expected number of years before default)
γ_x	Random indicator variable of sponsor's survival after $x - \alpha$ years
δ	Constant force of interest (valuation)
δ_t	Force of interest (valuation) at time t
δ_g	Constant force of population growth
${}_xE_\alpha$	Pure endowment term
$\mathbf{E}$	Vector of pure endowment term
EA	Entry age method
ε_n	Deviation from the expected number of members

ε_p	Deviation from the expected proportion of survivors
ε_q	Deviation from the expected eliminated members
ε_g	Deviation from the expected number of new entrants
$f(t)$	Fund value at time t
ζ	Proportion of replaced members
$g(x,t)$	Random number of new entrants aged x at time t
$g_x(t)$	Expected number of new entrants aged x at time t
$\dot{g}(x,t)$	Observed number of new entrants aged x at time t
GLG	General liability growth equation (2.2)
HT	Half time funding method
i_t	Valuation rate for the year $(t,\ t+1)$
IF	Initial funding method
k	Proportion of spread ul
K	Expected cost of a contribution strategy
κ	Number of years of past service
L_{AF}	Level of advance funding
LGE	Liability growth equation (3.1)
LGP	Liability growth process equation (3.6)
λ	Lagrangian multiplier
λ_Q	Market price for risk
λ_r	Bond's market price for risk
$\lambda_{r\|S}$	Equity's market price for risk
m	Spreading period
$m(x)$	Continuous accrual density function at age x
m_x	Discrete accrual density function at age x
$M(x)$	Accrual cumulative function at age x
MS	Mean shortfall
$\mu(x,t)$	Observed force of elimination at age x and at time t
$\mu_x(t)$	Expected force of elimination at age x and at time t
$n(x,t)$	Random variable number of members aged x at time t
$n_x(t)$	Expected number of members aged x at time t

$\dot{n}(x,t)$	Observed number of members aged x at time t
$N = 1/\gamma$	Expected number of years before defaulting
$NC(t)$	Total normal cost contributed at the beginning of year t
NC_x	Individual normal cost at age x
NC	Vector of individual normal costs
ν	Cost scaling factor
p	Parameter of power function
$p(x,t)$	Random variable proportion of survivors aged x at time t
$p_x(t)$	Expected proportion of survivors aged x at time t
$\dot{p}(x,t)$	Observed proportion of survivors aged x at time t
P	Vector of probability of surviving
$PVFB_x$	Present value of future laibilities at age x
π	Probability of sponsor's default over one period
$Q(r(t))$	Probability of sponsor's default over one period, dependent on the investment realisations
$r(t)$	Rate of return from investments during the year $(t,\ t+1)$
$r_s(t)$	Value of the short term rate at time t
R	Retirement age
ρ	Discounted deteminsitic mismatch between rate of return and valuation rate
s_x	Salary received at age x
$S(t)$	Value of the equity asset at time t
SF	Split funding method
Σ	General variance / covariance matrix
TCE	Tail conditional expectation
TF	Terminal funding method
τ	Maximum working lifetime
UC	Unit credit method
$ul(t)$	Unfunded Liability
$X(t),\ X_t$	General stochastic process
Y	General control variable

$\phi(h,t)$	Measure of compounded mismatch between random rates of return and the valuation rate
Φ	General risk measure
ψ	Vector of covariances
VaR	Value at risk
Var_t	Variance conditioned to the filtration at time t
w	Compounded proportion of not spread annual ul
ω	Extreme age

Abstract

The main area of concern of this thesis is the development of the area of pension mathematics dealing with the funding of Defined Benefit pension schemes.

Particular attention is directed to the modelling of a stochastically evolving structure, whereby the demographic and financial variables may differ from the expectations, according to specific probability distributions.

In such a framework, we investigate how to efficiently combine exogenous variables, such as the level of contributions and the asset allocation, with the goal of devising an optimal risk management of pension funds.

The development of a stochastic model for the demographic evolution of the scheme is central for describing the dynamics of a pension scheme. Thus, the population plan theory, as presented in the literature, is extended, allowing for a random evolution of the membership population. Furthermore, the impact of this uncertainty is measured with and without the coexistence of a randomly evolving financial world.

For a pension scheme, the main sources of funding are the contribution paid by the sponsor and the returns from investing the available funds. The way of combining these two sources of income is a key issue in the determination of the risk profile and the costs of implementing a pension plan.

Using mathematical models and numerical algorithms, several contribution strategies are investigated, emphasising the aspects related to the risk and cost borne by the pension scheme. Specifically, the following intuitive insight, that a higher security is achieved by spending more, is found in analysing the profiles of different contribution strategies. Moreover, the impact on this tradeoff between risk and cost is illustrated by separating the different effects of several sources of uncertainty. Finally, optimal

contribution strategies are found analytically and numerically.

The allocation of the available funds is also taken into account, with the specific aim of identifying the optimal proportions of investment in a range of three possible assets. This issue, which is part of a broader discussion on the fundamentals of pension funding, is also considered together with the choice of the contribution strategy. The main result is that, when the two strategies are contextually developed, optimality is reached when there is support between the two strategies. In other words, within certain boundaries, it is optimal to combine a contribution strategy, which extensively relies on investment returns, with a high proportion of risky assets in the investment portfolio.

Avenues for further research are also suggested.

Chapter 1

Introduction

Defined Benefit (DB) pensions schemes are structures which aim to provide workers at retirement with a benefit. This benefit is normally linked to the final pre-retirement salary (or an average of salaries in a short period before retirement). In order to meet this future and uncertain liability, contributions are paid by the scheme's sponsor into a fund, which is subsequently invested with the aim of adding investment income to the already mentioned contribution income.

Pension mathematics provides a scientific approach to the funding of a DB pension scheme, leading to several tools for calculating contributions and valuing both assets and liabilities.

The aim of this thesis is contributing to the development of this area of pension mathematics, through the understanding of the laws governing the funding process of a DB pension scheme.

In order to do so, we benefit from the use of mathematical models, which, loosely speaking, have the advantage of enabling us to derive universally valid solutions to the problems being posed. Such a mathematical approach is of great value, because it helps in providing a deeper understanding of the object of study, as well as insights which at times are difficult to achieve by simple intuition.

However, this comes at the cost of a necessary simplification of the reality. In fact, while trying to capture the fundamental aspects of the investigated phenomenon, mathematical models seldom describe the reality in its entire complexity, but usually focus on a simplified version of it.

However, there are some situations where more realism may be desirable, such as when practical applications are required; or when there is the need to test whether the simplifications required by the mathematical model distort reality, and thus, the results. In these situations a mathematical approach may be not convenient, because it may be too cumbersome, and sometimes not viable at all. Therefore, when we need to work with this higher degree of complexity, we integrate the mathematical approach with numerical routines, which simulate a more realistic description of a DB pension scheme.

Simulation techniques allow us to push further the results derived from the mathematical approach, and hence, to gather extra information and insight.

Although realistic and intricate assumptions can be incorporated in a numerical exercise, this approach can only focus on an area of investigation which is narrower than the most general description of the phenomenon. Hence, computer simulation is not the panacea for all the limitations of a mathematical approach, as it cannot cope with the complexity of the world.

Being aware of these limitations, in this thesis we direct the attention to the subject of pension funding, using both mathematical and numerical approaches

The thesis is organised in seven chapters, which are briefly introduced below.

Chapter 2 - The Defined Benefit pension model. Chapter 2 introduces the equations of the mathematical model describing an idealised DB pension scheme.

Specifically, there are a number of quantities which are characteristic of a pension fund: such as, contributions and investment returns, that are the main source of income; the benefits which are the main source of outflow of a pension fund. These cash-flows are the elementary amounts from which it is possible to compose the assets and the liabilities of the scheme. Furthermore, since these components vary according to the current economic and demographic situations, the dynamics of the assets and liabilities depend on the way all of these factors evolve and interact with each other.

For each of these elementary components, the literature provides a wide range of models which are introduced and discussed in chapter 2, together with some issues related to employing such models.

Chapter 3 - Population Plan Theory. This chapter is dedicated to extending the so-called population plan theory, introduced by Trowbridge (1952), and subsequently studied in Bowers *et al* (1976), in order to describe the evolution of the membership population of a pension scheme.

The importance of this theory is that a direct link does exist between the dynamics of the population and the dynamics of the liabilities of the pension scheme.

The proposed model is based on a deterministic stable population, as described in Keyfitz (1985), and it is currently used in most of the actuarial literature on pension funding.

By using this model, several results relating to the maturity of a pension scheme have been obtained in the actuarial literature. The contribution of this chapter is to extend this theory including the addition of the stochastic evolution of the membership population.

Sections 3.1 and 3.2 introduce the well established idea of stable and stationary population. Sections 3.3 and 3.4 extend the existing theory to the case of stochastic new entrants and stochastic membership, respectively. Section 3.5 analyses the risk of a mismatch between assets and liabilities of the whole scheme when both the demographic evolution and the investment returns are stochastic.

Further, the two cases of independence and dependence between demographic and financial risks are considered.

Chapter 4 - Contribution strategies. Chapter 4 focuses on studying the general issue of the classification of normal cost methods (also called contribution strategies). A contribution strategy is the set of contributions to be paid in order to fund a pre-defined benefit at retirement. A comprehensive, risk-based, classification of these strategies would provide a valuable tool for the risk management of a pension scheme.

Several classifications have been proposed in the literature, mainly based on the distinction between individual and aggregated methods, and between projected and non-projected methods; see for instance Winklevoss (1993) or Aitken (1996).

As an alternative, a wide family of normal cost methods can be classified according to the corresponding accrual density function, as introduced in Cooper and Hickman (1967).

By employing this second approach (introduced in §4.1), in section 4.2 we study the contribution strategies from a disaggregated point of view, i.e. for each member of the scheme. Specifically, we proceed to classify the contribution strategies by measuring the risk of mismatch between the value of the accrued fund at a specific time and its expected value. In detail, we separately model the effects of the financial risk, the demographic risk and the risk of default of the sponsor.

Further, we include the cost of pension provision of a contribution strategy. This has the effect of creating a tradeoff in the decision of which normal cost method should be used. Specifically, a tradeoff between cost and risk of contribution strategies arises. This aspect is investigated in section 4.3.

Chapter 5 - Optimal contribution strategies. Starting from the tradeoff between risk and cost, we aim to devise a methodology to find optimal contribution strategies, by means of identifying an efficient frontier in the cost-risk plane.

In section 5.1, we set the optimal problem in a general form, without specifying whether the risk to be minimised refers to the mismatch between the value of an individual member's accrued fund at a specific time and its expected value (disaggregated approach); or whether it refers to the mismatch between the scheme's assets and liabilities (aggregated approach).

By using optimisation techniques for constrained quadratic problems, we show that a solution exists under some fairly general assumptions.

In section 5.2, the methodology devised is applied to the disaggregated case, and a numerical application is then implemented in order to display the optimality of the solution when rates of return and the sponsor's risk of default are the sources of uncertainty.

Similarly, the aggregated case is analysed in section 5.3, focusing on the particular case of a random number of annual new entrants in the pension scheme, where a systematic, deterministic, mismatch exists between the expected rate of return and the valuation rate.

Chapter 6 - Optimal funding strategies. In this chapter we focus on investigating the asset allocation of a pension scheme, with the aim of identifying optimal strategies. In fact, changing the proportion of risky asset alters the risk/reward profile of an investment portfolio. Thus, the asset allocation is a tool for controlling the flow of investment income.

Moreover, results from the previous chapters on contribution strategies are further extended to the more general case of managing a DB pension scheme when the rates of investment return are random and evolve according to stochastic processes.

In §6.2 we introduce several models proposed in the literature for describing the dynamics of the rates of return of three different assets: namely, a short term asset (cash), a fixed income security (bond) and a stock market index (equity). Furthermore, we find conditions on the spreading period such that the stability of the unfunded liability is assured in the long term.

A number of issues arise in the choice of the valuation rate when the asset allocation is allowed to change. Two different approaches - a "classical actuarial" and a "financial economic" one - are discussed in §6.3.

The implications of the two schools of thought on the "equities vs bonds" dispute are illustrated in §6.4. In this section, optimal investment strategies are identified in the cases of two and three available assets.

In section 6.5, we consider random demographic evolution for different sizes of the scheme's membership population.

Section 6.6 deals with the more general problem of optimal funding strategies. Specifically, we illustrate how to combine efficiently contribution and investment incomes, using results from chapter 5 together with the devised optimal investment strategies.

Throughout chapter 6, the work is developed following the classical actuarial approach. However, in section 6.7, we briefly implement the same analysis following an approach consistent with the financial-economic school.

Chapter 7 - Conclusions. The last chapter summarises the main findings and the conclusions in each of the previous chapters. Furthermore, avenues for possible extensions are presented, with respect to all the topics discussed.

Chapter 2

The Defined Benefit pension model

The pension scheme is modelled using a discrete-time stochastic model, first described and investigated in Dufresne (1986, 1988, 1989, 1990). We choose to work with this discrete model, believing that, while offering a good description of the reality, it has shown in the literature the potential to lead to results of interest.

Nonetheless, a continuous time model in the fashion of Bowers (1976, 1979, 1982) is also used in chapter 4, because of mathematical convenience and consistency with parts of the literature.

In this chapter we introduce the fundamental equations and we discuss the main issues related to the use of them.

2.1 Fund level.

The fund level is the value of the assets belonging to the scheme at a specified time t. The literature provides a number of different methodologies to evaluate the assets, among which the Discounted Cash Flow and the market-based methods are the most used. It is not the aim of this work to analyse the features of these methodologies, a comprehensive review of which can be consulted in Exley *et al* (1997) and Owadally and Haberman (2003).

Whatever the method of valuation, the dynamics of the fund value may be described by the following recursive equation:

$$f(t+1) = [f(t) + c(t) - B(t)](1 + r(t)) \quad (2.1)$$

where

- $r(t)$ is the rate of return of the investment portfolio gained during the year$(t, t+1)$.
- $c(t) = NC(t) + adj(t)$ is the annual contribution at time t and it is computed by adding an adjustment, *adj(t)*, to the normal cost, *NC(t)*.
- $B(t)$ the benefit paid to the pensioners at the beginning of year t.

All this quantities are due at the beginning of each year and are also evaluated net of price inflation [1].

2.2 Actuarial Liability.

The actuarial liability (AL), also known as the reserve, is the difference between the actuarial discounted values of liabilities and future contributions for current members of the scheme.

Similarly to the fund level, also the AL might be calculated in different ways. An important concern lies in the choice of the valuation discount rate applied at time t and in the methodology used to determine it. Here, we define the discount rates as a constant quantity i, although dynamic discount rates $\{i_t\}$ could be considered in a more complex model. A more detailed discussion about the choice of the valuation discount rate is in Chapter 6.

The dynamic of AL is dependent on the assumptions regarding the demographic evolution of the membership population. As proposed in Winklevoss (1993), the following equation provide a general, but not unique, way of describing this dynamic:

[1] The assumption on inflation might actually overestimate the risk due to the benefit variations. This happens because benefits are usually adjusted according to price inflation; while all of the other amounts vary according to salary inflation. According to classical economics, the rate of wage inflation is equal to the price inflation rate plus the rate of productivity growth (usually nonnegative), see Samuelson (1989). Hence, salary inflation is normally higher than price inflation.

$$E[AL(t+1)|\mathcal{F}_t] = [AL(t) + NC(t) - B(t)](1+i) \tag{2.2}$$

Analogously to the notation belonging to stochastic calculus, $\{\mathcal{F}_t\}$ indicates the filtration generated by the whole process of the pension scheme. Specifically, an interpretation of this filtration would suggest that $\mathcal{F}_t$ represents the information available at time t and therefore, in equation (2.2), the value of the reserve at that specific point in time, $AL(t)$, is known. Equation (2.2), to which we will refer as the general liability growth (**GLG**) equation, states that if the assumptions on the demographic evolution are actually borne out by experience, then the AL dynamic can be described by a recursive equation.

2.3 Unfunded Liability

From the previous two quantities an amount is computed, which provides valuable information about the financial status of the scheme: the unfunded liability (*ul*). At any time *ul* is defined by the difference between the actuarial liability and the fund value; in formula:

$$ul(t) = AL(t) - f(t) \tag{2.3}$$

It is straightforward to notice that a positive value happens only when the fund at time t, *f(t)*, is not sufficient to cover the actuarial liability, i.e. the current and future liability towards the existing scheme members. Conversely, a negative *ul* indicates that the value of the assets is higher than the actuarial liability.

Thus, it is common practice to refer to *ul*, when evaluating the financial status of a DB pension scheme.

Mathematically, the model allows *ul* to diverge to ($\pm$) infinity, and thus, the parameters must assume appropriate values in order to avoid such inconvenience. Practically, the appropriate settings of parameters avoid the case of systematic growth and assure the convergence of the unfunded liability.

From the model perspective, a situation of perfect funding happens when the *ul* is null; i.e. when no surplus nor deficit exist. Thus, specific methods are developed in order to assure this condition in the long run and on average. In actual practice, a pension scheme may be required by the regulating authority to be over-funded; i.e. the fund level should be higher than the AL, and hence, the ul should converge to a finite value lower than 0.

2.4 Individual amounts.

The quantities introduced above, AL, NC and B, are aggregated amounts; i.e. they refer to the whole scheme. However, the previous quantities can be disaggregated by age and written as the corresponding individual amount multiply by the number of existing members. Further decompositions are also possible.

For instance, if an individual funding method is employed, *NC(t)* can be broken down by age and expressed in a formula as:

$$NC(t) = \sum_{x=0}^{\tau} NC_{\alpha+x}\, n(\alpha + x, t) \tag{2.4}$$

where $n(\alpha+x, t)$ is the membership function, which represents the number of members aged $\alpha + x$ at time t; α is the minimum age and τ is the length of the working life time ($\tau = R - \alpha - 1$, with R fixed retirement age). $NC_{\alpha+x}$ is the contribution paid by all the members aged $\alpha + x$. In section 2.6.1, classical normal cost methods are introduced, while in chapter 4 a general cost method is introduced.

In chapter 3 we analysis potential models for the membership function, including the case of a randomly evolving population.

2.4.1 Past service.

Similarly to the Normal Cost, the annual total Benefit and the Actuarial Liability can be disaggregated by age. However, for these quantities it is necessary to take into account the number of years of past service; i.e. the number of years during which each active member has contributed into the fund.

It is reasonable to assume that all new members, older than α, transfer into the scheme a previously accrued fund. The size of this fund would depend on the period of contributions, as well as on the underlying actuarial assumptions used by the former scheme.

Differently, in the case of elimination from the scheme before retirement, a monetary transfer out of the fund is not allowed. This assumption is made for the sake of simplicity, in order to limit the outflows to only the retired population. Nonetheless, a more comprehensive structure could be considered, by allowing for the payment of special benefits (perhaps on the form of lump sums) to those active members eliminated prior to retirement. However, it is important to note that the extra fund derived from these early eliminations is consistently accounted for by the actuarial assumptions. Thus, the contributions are expected to grow in time by means of both financial and demographic compounding.

If the size of transferred fund is equal to the AL accrued by members of same age who joined the scheme in α, then the plan sponsor could grant to the new members a full past service. In this way, the new scheme would effectively make no distinction between new and old members and there would be no need to decompose the membership function in order to take into account the past service of each member.

However, this assumption is not likely to be accurate for two main reasons: first, young workers are probably in their first job, therefore may have not accrued any fund; second, usually, pension schemes apply a penalty on withdrawal, which has the effect of decreasing the value of the transferable fund and, consequently, the number of years granted by the new sponsor.

Therefore, it seems reasonable to allow for members joining the scheme at an age z older than α and transferring into the scheme a previously accrued fund of any size. According to the size of the transferred fund, a number of years of past service between 0 and $z - \alpha$ are credited to the new member.

In the light of this, the membership function is further decomposed, as in the following formula:

$$n(z,t) = \sum_{\kappa=0}^{M} n(z,t,\kappa), \quad , z = \alpha, \ldots, \omega$$

where $n(z,t,\kappa)$ is the number of members aged z, at time t and with κ years of past service. Summing $n(z,t,\kappa)$ by κ determines the aggregated membership function as previously defined, i.e. regardless of the years of past service. The upper extreme M varies according to the age z: for ages in retirement $(z \geq R)$, M is the longest possible past service, $M = R - \alpha$; while in ages of working activity $(z < R)$ the past service cannot be longer than the age z less the minimum age α, so $M = z - \alpha$.

In order to ease the notation, instead of using a varying upper extreme, we assume that the number of active members with a past service longer than their age less minimum age α is zero. Mathematically, this means

$$n(z,t,\kappa) = 0 \quad \forall \kappa > z - \alpha \tag{2.5}$$

Henceforth, the Actuarial Liability and the annual total Benefit at time t have the following expressions:

$$AL(t) = \sum_{x=0}^{\omega-\alpha} \sum_{\kappa=0}^{R-\alpha} AL_{\alpha+x,\kappa} \; n(\alpha + x, t, \kappa) \tag{2.6}$$

$$B(t) = \sum_{y=0}^{\omega-R} \sum_{\kappa=0}^{R-\alpha} B_{R+y,\kappa} \; n(R + y, t, \kappa) \tag{2.7}$$

where ω is the extreme age; $AL_{\alpha+x,\kappa}$ is the Actuarial Liability corresponding to a member aged $\alpha + x$ with κ years of past service; and, finally, $B_{R+y,\kappa}$ is the annual benefit received by a pensioner, aged $R+y$, who contributed for κ years into the fund.

For the purpose of this work, there is no need to go in to further detail in analysing the composition of AL and B by past service. It is sufficient to note that this model implicitly assumes that all the members having the same age and past service have accrued the same actuarial liability, and are thus entitled to an equal benefit.

Disaggregating the membership function by years of past service allows the model to account for liabilities and benefits, that vary according to how long each member has actually contributed into the fund. Moreover, the membership function does not need to be necessarily decreasing, thus allowing for new entrants at any age before retirement.

Specifically, the following relation between the new entrants function g and the membership function n exists:

$$n(z,t,\kappa) = g(z-\kappa, t-\kappa)\ {}_{\kappa}p_{z-\kappa}(t-\kappa) \tag{2.8}$$

where $g(x,t)$ is the number of new entrant aged x, at time t; and ${}_{\kappa}p_z(t)$ is the probability that an individual aged z at time t remains in the scheme for κ years. These probabilities are actually assumed to be time-invariant, thus excluding from the analysis the evolution of the decrement process in time. A more realistic model would allow for time-varying probabilities; however, such an extension would come at the cost of mathematical tractability. In this instance, we prefer to work with a less realistic but more manageable model. Nonetheless, in chapter 3, when focusing exclusively on a randomly varying retired population, time evolution of these probabilities is taken into account. Moreover, some of the results achieved in Chapter 6 are tested using different life tables in a sensitivity analysis fashion.

2.5 Benefit

In a DB scheme at retirement each individual member receives a benefit in the form of periodical payments until the age of death. Usually, the amount of such a benefit is linked to the last received salary (or to an average of some of the last salaries) and to the number of years of contributions.

Without loss of generality, we assume that payments are made once per year, and that the amount of pension remains constant year after year.

Specifically in this work, a pension equal to a fraction b of the last received salary s_R is paid for each year of contribution at the fixed retirement age R.

Therefore, the pension B_{R+y} paid to all the retirees - used in equation (2.7) - has the following expression, also known as the benefit formula[2]:

$$B_{R+y,\kappa} = \kappa \cdot b \cdot s_R, \quad \forall y \in (0,\ \omega - R) \tag{2.9}$$

where, consistently with the previous section, κ is the number of years of past service.

A benefit provision paying a pension equal to (2.9) at the beginning of each year to a retired member, who contributed for κ years, has a present value at retirement age R given by the following:

$$PVFB_R = \sum_{y=0}^{\omega-1} B_{R+y,\kappa}\ {}_yP_R\ v^y = \kappa \cdot b \cdot s_R \cdot \ddot{a}_R \tag{2.10}$$

where $\ddot{a}_R$ is the present value of an annuity paid in advance as long as the pensioner is alive. The discount rate is i and the post retirement probabilities of surviving are assumed to be static.

This present value can be computed at different ages as well. Specifically, distinguishing between pre and post retirement ages, the present value of future benefits ($PVFB$) at age x is given by the following equation:

$$PVFB_x = \begin{cases} \kappa \cdot b \cdot s_R \cdot {}_{R-x/}\ \ddot{a}_x & x < R \\ \\ \kappa \cdot b \cdot s_R \cdot \ddot{a}_x & x \geq R \end{cases} \tag{2.11}$$

where ${}_{R-x/}\ddot{a}_R$ is the present value of a deferred annuity of $R - x$ years (i.e. until retirement age) and paid in advance as long as the pensioner is alive. Hence, the following relationship holds:

$$PVFB_x = v^{R-x}\ {}_{R-x}p_x PVFB_R \qquad \text{for} \qquad x \leq R \tag{2.12}$$

[2]Refer to Winklevoss (1993) and Owadally (1999) for similar definitions of the benefit.

2.6 Contributions.

In order to fund the pension benefit to the the retirees, contributions are periodically paid over the working-life into a fund. This fund is then invested in a variety of assets with the aim of maintaining, and possibly increasing, its real value.

Two parts give the annual contribution: the normal cost and the supplemental cost or adjustment.

The normal cost is the amount that provides the adequate funding for future liabilities, under the assumption that the actual experience of the scheme would follow the actuarial expectations. Hence, if the assumptions are borne out by experience, the set of normal costs and the returns from the investments will match the retirement benefit.

However, deviations from the assumptions are likely to happen, thus generating a surplus or a deficit. According to the methodology implemented for dealing with this mismatch, contributions are adjusted in order to keep the scheme balanced.

This adjustment is usually referred to as the supplemental cost, and it could be positive or negative depending on the financial status of the scheme.

2.6.1 Normal cost.

Pension funding (or normal cost) methods determine the set of contributions should be periodically paid to provide retirement benefits. In the development of this thesis the sequence $\{NC_k,\ k = \alpha, \dots, R\}$, already introduced in section 2.4, will indicate this set of normal contributions. Specifically, NC_k is the normal cost paid at age k, where k can assume the values between the minimum entry age α and the retirement age R.

Each method generates a characteristic path of contributions and returns from the investments, which together will eventually match the retirement benefit.

Several methods exist, derived from different principles, in order to compute contribution strategies. In fact, the number of possible methods are infinite, but only those satisfying some requirements are considered acceptable, and can thus be implemented in practice. The rationale of these requirements lies in the fairness of the actuarial equilibrium, as well as strict bounds and characteristics which may be imposed by an appropriate supervisory or regulatory authority[3].

According to Sharp (1996) a reasonable funding method should produce no experience gains or losses, if the actuarial assumptions are exactly matched.

In other words, a reasonable funding method should satisfy the "actuarial principle of equivalence". This principle states that at any time the actuarial present value of the benefits should equal the total of the actuarial present value of future contributions and the already accrued fund.

It is common practice to distinguish between methods seeking equilibrium between inflows and outflows at an individual member or at a whole scheme level.

Individual methods provide the contributions to be paid by the active members at each age and are based on the idea that the principle of equivalence must hold for each individual plan participant.

Conversely, *aggregated methods* seek to satisfy the principle of equivalence at a scheme level, and thus provide a contribution rate to be applied to each member's annual salary.

In this work, individual methods are implemented, as they provide a convenient framework within which to study the evolution of the membership population.

In Winklevoss (1993) the following convenient representation of the normal cost at age x is given:

$$NC_x = k_x PVFB_x \tag{2.13}$$

Specifically, the normal cost is defined as a proportion of the $PVFB_x$, where the fraction k_x varies according to the chosen actuarial cost method and to age x.

Table 2.1 concisely displays some of the most common individual normal cost methods:

- Accrued benefit method: under this method the proportion is given by the size of the annual accrual benefit divided by the accrued benefit at retirement. The path of normal costs consequently depends on the choice of the benefit formula.

[3]See Aitken (1996), Sharp (1996) for a fuller discussion.

Table 2.1: Table of actuarial cost methods, readapted from Winklevoss (1993).

k_x	*Actuarial cost method*
$\frac{b_x}{B_R}$	Accrued benefit method
$\frac{s_x}{S_R}$	Unit credit method as a salary percent
$\frac{1}{R-\alpha}$	Unit credit method as a currency-unit
$\frac{s_x \; {}_{x-\alpha}p_\alpha v^{x-\alpha}}{s_\alpha \; {}^s\ddot{a}_{\alpha:\overline{R-\alpha}\vert}}$	Entry age method or individual projected benefit as a salary percent
$\frac{{}_{x-\alpha}p_\alpha v^{x-\alpha}}{\ddot{a}_{\alpha:\overline{R-\alpha}\vert}}$	Entry age method or individual projected benefit as a currency-unit

- Unit credit methods: in both cases, the annual normal cost is a proportion of future benefit. Such a proportion may be constant and in particular may be the inverse of the length of working lifetime (unit currency case). Alternatively, this proportion may vary according to the actual employee's annual salary (salary percent case).

- Entry age methods: according to these methods, at each age the normal cost is the fair premium to be paid for a benefit of value $PVFB_R$. NC might be either a percentage of salary or a constant monetary unit.

As is noted in Winklevoss (1993), these methods produce normal cost values that usually are relatively dispersed near the employee's entry and retirement ages, while they are reasonably close midway through his career. Under certain assumptions it is possible that some of the methods (accrued benefit and entry age) would provide undesirably high or low values. In particular, the actuarial assumptions on the financial and demographic expectations and on the composition of the membership population are the most significant variables for explaining the differences between the resulting normal costs related to each method.

Furthermore, in a scheme characterised by a large population with relatively stable age and past service distributions, each method will produce a reasonably constant normal cost percentage for the entire plan.

Specifically, amongst the considered methods the unit credit ones "will produce the lowest costs for a relatively under-mature active employee population and vice versa for a relatively over-mature", Winklevoss (1993). However, in this context, the cost is intended as the total amount of contribution paid in a specific year from a purely static standpoint, i.e. without taking into account the future evolution of the membership population.

Moreover, in Khorasanee (2002) the unit credit method is proved to be "an appropriate funding strategy for a plan with stationary population of active members". Conversely, if the entry age is fixed, entry age methods seem to be the most appropriate.

In chapter 4, the generalised expression (2.13) is reintroduced following the approach developed in Bowers *et al* (1976). Furthermore a comprehensive analysis of the risk and cost (in a dynamic sense) of contribution strategies is carried out.

2.6.2 Supplemental cost.

Fund valuations are periodically run in order to control the financial status of the scheme. UK legislation requires that, at least every three years, such a valuation is carried out and supplemental costs are evaluated. In this model, we assume that adjustments are computed at end of each year and, specifically, at these dates the fund value is compared to the actuarial liability.

Mismatches between the fund level and the AL are likely to happen for many reasons, of which we list the most common ones:

- Experience variations: it is likely that the experience of the fund will differ from the underlining actuarial assumption.
- Assumption changes: it is possible that the actuarial valuation assumptions will be changed from time to time.
- Benefit changes: different benefit formula might be set up by the plan, with change frequently being retroactive.

- Contribution variances: it may not be the case that the sponsor will pay exactly the estimated costs suggested by the actuary.

In order to deal with the unfunded liability and hence with experience deviations from the actuarial valuation assumptions, several adjustment (or supplemental cost) methods exist.

In this work we use the method of spreading surpluses and deficits over a moving term, which is widely used in the British actuarial practice.

According to this method, the adjustments are calculated as a proportion of the ul, with the aim of spreading the existing surplus or deficit over a moving term of (say) m years. Specifically, the annual supplemental cost is given by the following formula:

$$adj(t) = \frac{ul(t)}{\ddot{a}_{\overline{m}|}} \tag{2.14}$$

This method is extensively studied in Dufresne (1986, 1988, 1989), Haberman (1994a), Haberman and Wong (1997), Owadally (1999) and Owadally and Haberman (1999), where particular attention is directed to the problem of choosing the spreading period.

In fact, the period m affects the solvency profile of the scheme as well as the stability of the contributions. As stated by Dufresne (1986), it is evident that when security is the most important issue, a lower m should be chosen. Nevertheless, a low m would imply a higher adjustment, which in turn might determine a significant shift in the level of contributions, and thus ultimately reducing the stability of contributions.

Hence, the choice of the spread period m creates a tradeoff between the stability and the security of a pension scheme.

Dufresne (1988, 1989) comprehensively analyses this tradeoff, finding an efficient range for the spreading period m. Specifically, under current economic conditions he suggests that m should assume a value in a interval of time between 1 and 10 years.

Dufresne (1989) also describes another method whereby the adjustment is computed in order to amortise the gains and losses emerged during two valuation periods.

This method consists in amortising directly and over a fixed term the actuarial gains and losses, by explicitly computing the annual adjustment as following:

$$adj(t) = \sum_{j=0}^{m-1} \frac{l(t-j)}{\ddot{a}_{\overline{m}|}} \tag{2.15}$$

where the annual gain/loss is defined as follows:

$$l(t) = ul(t) - \big[ul(t-1) - adj(t-1)\big](1+i) \tag{2.16}$$

i.e., the one-period gain/loss arising from the experience deviations is determined as the increment of the unfunded liability in the time-interval $(t-1, t)$. An extra factor $P(t)$ can be added to the adjustment in order to amortise the initial unfunded liability. The same method is also described in Winklevoss (1993) as a widely used alternative, particularly in USA and Canada.

Similarly to the previous method, also in this case the choice of the period of amortisation affects the stability of the contributions, as well as the solvency of the fund. However, it is possible to define an efficient frontier for the period of amortisation. Specifically, in Owadally and Haberman (1999) it is proved that increasing the amortisation period over a certain threshold has the effect of increasing the contribution variability.

Furthermore, in Owadally (1999) numerical investigations based on current economic conditions suggest the efficiency of amortising in 5 or less years.

As proved by Owadally and Haberman (1999), amortising gains/losses is more efficient than spreading deficits/surpluses in terms of security of the scheme. In fact, for the same periods m, amortising leads - in the long run - to a lower variance of the fund level, than spreading.

However, the same authors also show that "spreading surpluses/deficits may be regarded as more efficient than amortising gains/losses", when the efficiency criterion is to minimise the variance of both fund value and contribution level.

Other methods can also be implemented. For instance, deficits could be considered more severe deviations than surpluses, thus inspiring methods of asymmetric spreading of these mismatches, as studied in Haberman and Smith (1997) and Owadally (1999).

In Kleynen (1997), the possibility to set a buffer capital to be used as an extra investment reserve is tested by means of a Monte Carlo simulation of a Dutch pension scheme. Specifically, this buffer would increase in response to favourable financial realisations, while it would decrease during difficult economic periods.

Furthermore, a "corridor approach" could be implemented, according to which deficits and surpluses are being ignored as long as they lie within a given interval (or corridor). Such a method, investigated in Dufresne (1993), relies on the fact that valuation assumptions should on average be correct. Intuitively, this methodology should provide a reasonable degree of stability of the contribution process, within, however, an acceptable level of security.

In this work, we choose to use the spreading method mainly for three reasons: because of the desirable efficiency in terms of minimum variability of fund level and contributions; for its extensive use in the UK; and for its mathematical tractability. The spreading period m is assumed to satisfy the conditions developed in Dufresne (1988, 1989) and in Owadally (1999). However, appropriate conditions are developed when different assumptions, from those used in the literature, are implemented.

2.7 Risk measures.

Although not strictly related to the pension model, this section is dedicated to introduce briefly some risk measures and their characteristic way of quantifying risk, in order to facilitate the choice of which risk measure should be employed for analysing the risk.

We have already introduced ul as an indicator of the financial status of the scheme. In the development of this thesis, we need to provide a measure of risk of ul, as well as of other random variables. Thus, in this section we refer to a general underlying process, whose riskiness we wish to measure, rather than referring to a specific variable, such as ul.

Several measures exist, each of them conveying valuable but different information, because there are different methods for weighting the same adverse outcomes.

The first measure that we consider is the variance of the underlying process at a given point in time. The mathematical tractability of this measure, and the large number of results available in the literature, are the main advantages when using this measure. However, giving the same weight to positive and negative deviations from the mean is not always desirable, because relevant patterns may remain undiscovered.

For this reason we also direct the attention to "downside" risk measures, which have the desirable feature of being asymmetric; i.e. to focus on the negative side of the distribution. Amongst the many measures, we take into account the Value at Risk, as well as coherent risk measures.

2.7.1 Value at Risk.

Value at Risk (**VaR**) is a percentile of a loss distribution and so measures the maximum loss within a specified confidence level. This measure has found a vast consensus in several applications in finance and it is currently employed in the determination of minimum solvency capital in both banks and insurance companies. See for instance Basle Committee on Banking Supervision (1998) and the Risk Based Capital system implemented in North American Countries[4].

Let us indicate with $\{X(t)\}$ a general loss process, as it could be the *ul* or any other stochastic process informative of the performance of a given strategy. When the time is fixed, say $t = \tau$, then the value of the loss $X(\tau)$ is a random variable in a space of probability Ω_τ, which (we assume) has cumulative probability function $F_{X(\tau)}(x)$. Then the **VaR** at level α is defined as:

$$\mathbf{VaR}_\alpha = \min\left\{x \in \mathcal{R} : 1 - F_{X(\tau)}(x) = \alpha\right\} \tag{2.17}$$

where the equality sign in the set (2.17) holds only for continuous functions F.

The **VaR** conveys information on the risk of loss, providing also a measure easily interpretable in terms of the allocation of security capital. However, the peculiar

[4]Refer to Webb and Lilly (1995) for a comprehensive analysis of the RBC system.

dependence on the level of confidence causes the analytical computation of the **VaR** to be very complicated. Therefore, several techniques have been developed to estimate the **VaR**, among which the so called Δ-norm is one of the most commonly used.

This technique consists in assuming that the variable $X(\tau)$ is normally distributed. Under this assumption an estimate of the mean and of the volatility is sufficient to determine the maximum loss with a specified confidence level.

However, if the baseline process is not normally distributed and tends to be skewed and platokurtic, the Δ-norm technique systematically underestimate the risk.

In order to improve the estimate of the **VaR**, the inclusion of information conveyed in higher order moments may avoid, or at least smooth, the effects of non-normality. A potential avenue is illustrated in Li (1999), where applications of this approach to foreign exchange spot rates shows remarkable result in catching the extreme tail peculiarities.

However, taking into account the complexity of the pension scheme dynamics, an analytical approach is likely to lead to complex results, and therefore a simulation approach is also considered.

Using a Monte Carlo methodology, the **VaR** at a confidence level $\alpha \in (0,1)$ is defined as the α-percentile of the distribution of the loss, which is derived by simulation.

2.7.2 Coherent risk measures.

More recently, the literature on risk measures has vastly expanded following a strong criticism of the **VaR**.

The rationale behind this criticism is that the **VaR** fails to satisfy the property of subadditivity. Therefore, an analysis based on **VaR** may lead to make conclusions or to back strategies which actually contradict well established economic laws.

In particular, the lack of subadditivity is not coherent with the principle that a diversification of investments lead to a reduction of the overall risk. This has been proved in Artzner *et al* (1999) and Uryasev (2000) among the others.

In the light of this, a new class of risk measures has been recently introduced in the literature. In a seminal paper, Artzner *et al* (1999) propose an axiomatic definition of coherent risk measure, providing also several examples. Further applications of coherent risk measures in the specific field of insurance are in Artzner (2000).

According to Artzner *et al* (1999), a risk measure, which here we indicate with Ψ, is said to be coherent if it satisfies the following four axioms:

Axiom PH. Positive homogeneity: for all $\lambda \geq 0$ and all $X(\tau) \in \Omega$, $\Psi\big(\lambda X(\tau)\big) = \lambda\Psi\big(X(\tau)\big)$. It can be seen that the variance fails to satisfy this axiom.

Axiom T. Translation invariance: for all $X(\tau) \in \Omega$ and all real numbers C, we have $\Psi\big(X(\tau) + C \cdot r\big) = \Psi\big(X(\tau)\big) - C$. If a solvency capital is held and invested at a risk free rate r, the risk is reduced by exactly the solvency capital. As it is noted in Artzner *et al* (1999) Axiom **T** ensures that if the solvency capital is equal to the risk measure, then there is no risk involved in the current position. As a matter of fact, for each $X(\tau)$, $\Psi\big(X(\tau) + \Psi\big(X(\tau)\big) \cdot r\big) = 0$.

Axiom M. Monotonicity: for all $X(\tau) \text{and} Y(\tau) \in \Omega$ with $X(\tau) \leq Y(\tau)$, we have $\Psi\big(X(\tau)\big) \leq \Psi\big(Y(\tau)\big)$.

Axiom S. Subadditivity: for all $X_1(\tau) \text{and} X_2(\tau) \in \Omega$, $\Psi\big(X_1(\tau)+X_2(\tau)\big) \leq \Psi\big(X_1(\tau)\big)+\Psi\big(X_1(\tau)\big)$. Artzner asserts that this property is a natural requirement in several situations, especially when such a measure is employed in computing extra capital requirement.

These four axioms rule out the variance, the **VaR** and other measures, such as the "semi-variance" type of measure, that do not satisfy Axiom **M**. The following measures have thus been proposed by Embrechts *et al* (2003) (among the others) and Artzner *et al* (1999), respectively:

MS. Mean shortfall:

$$\mathrm{MS}_\alpha\big(X(\tau)\big) = E\Big[\max\big(X(\tau) - VaR_\alpha\big(X(\tau)\big),\ 0\big)\Big] \tag{2.18}$$

TCE. Tail conditional expectation:

$$\text{TCE}_{\alpha}\big(X(\tau)\big) = E\Big[X(\tau)|X(\tau) \geq VaR_{\alpha}\big(X(\tau)\big)\Big] \tag{2.19}$$

These measures, as well as several derivatives [5] provide information of the kind "how bad is bad", and thus, they actually recall the work done on the conditional distribution within the field of reinsurance; see, for instance, Hogg and Klugman (1984).

Furthermore, it is shown in Acerbi and Tasche (2002) that these measures are coherent, when the loss distribution is continuous. Moreover, a comprehensive analysis of the mean excessive functions, core of these measures, is in Embrechts *et al* (2003). Closed formulae for the **MS** of standard distributions, as well as methods of estimation of the whole conditional distribution, are also included there.

In this thesis we will use a general risk measure $\boldsymbol{\Phi}$ which is coherent in the sense of Artzner. However, when mathematical tractability will be required, simpler measures such as the variance will be used to quantify the risk.

[5] See for instance Artzner *et al* (1999), Rockafellar and Uryasev(2000) and Uryasev (2000) for the definitions of worst conditional expectation, conditional VaR and others.

Chapter 3

Population plan theory

This chapter deals with the study and the development of population plan theory and of potential models which aim to describe the factors that generate the evolution of the membership population of a pension scheme.

Population plan theory is related to the wider area of study of mathematical demography. As such, most of the research is focused on adapting general models to the specific case of the membership population of pension schemes. Thus, it is of interest to study the situation whereby the population grows because of new entrants in the scheme, rather than birth; and it decreases because of eliminations due to mortality, disability, unemployment and other factors which will be subsequently considered in further detail.

Recalling the classification of population theory presented in Keyfitz (1985), the attention is focused on the rates of growth and of elimination, both of which may be fixed, or changing in time; as well as being either deterministic or stochastic.

Furthermore, it is the main scope of this theory to analyse the consequences of different demographic structures on the current and future liabilities of the scheme.

In fact, a direct link does exist between the dynamics of the population and the dynamics of the liabilities of the plan. Specifically, the membership function $n(\alpha + x, t)$, providing the number of members for each age at any time, describes the demographic structure (age and past service distribution) and, as such, directly affects the amount of the liabilities.

Moreover, for some particular types of population, several results have been achieved in the classical actuarial literature relating to specific equilibria between contributions and benefits.

Another characteristic feature of pension plan populations is the existence of distinctive sub-populations. Active members can be identified and separated from pensioners; likewise, other sub-populations exist, such as those who are temporarily disabled or unemployed, spouses and dependants. Hence, models separately dealing with these populations, and thus accounting for the differences among those populations, can be developed.

The chapter is structured in the following way: §3.1 and §3.2 introduce the models of stable and stationary population, respectively, as well as illustrate standard results achieved in the classical actuarial literature. Section 3.3 extends these standard results to the more generic case of a population characterised by random new entrants. The demographic evolution is analysed in its full complexity in §3.4, which separately deals with the active and retired populations. Section 3.5 directs the attention to the effects of randomly evolving population on the deficits and surpluses of a pension scheme. In particular, the case of a demographic evolution correlated to the current economic conditions is compared to the more standard case of independence between these two sources of uncertainty.

3.1 Stability.

As introduced in section 2.4, the membership function $n(\alpha+x, t)$ provides the number of members in each cohort at any time. Therefore, it is through this function that we analyse different demographic structures. Specifically, at a fixed point in time t, $n(\alpha + x, t)$, as a function of x, describes the age distribution of the population.

This distribution can be further broken down by taking into account the past service; i.e. the number of years of membership of each member. Thus, in this work, past service is accounted for.

If the age distribution remains constant year after year, a stability condition arises in the population. This will eventually happen, when the population decrement rates remain constant and the increment rate grows at a constant rate. Refer to Keyfitz (1985), chapter 4, for a full analysis of the properties of stable populations.

Part of the actuarial literature (see Winklevoss (1993), for instance) refers to this population as *mature*, in contrast to the definitions of under-mature and over-mature populations.

A population is *under-mature* if the age distribution changes in time and it is characterised by a high proportion of young cohorts. This is the case when, although the increment rate has been constant for some time, a sufficient number of years has not yet passed and the age distribution has not yet reached stability. In this situation, the resulting population is usually characterised by a large proportion of younger cohorts. According to Winklevoss, this is particular to growing industries, which are characterised by firms having under-mature populations.

On the contrary, a population is said to be over-mature when the age distribution tends to an increasing proportion of older generations. This happens when the increment rate decelerates or when the decrements reduce. The first case is typical of declining industries, in which case the rate of hiring decreases. The second case has recently become particularly relevant, as the general population has experienced a systematic decrease in mortality, especially at older cohorts.

It is important to note, that the definitions of under-mature and over-mature population are based on the idea that populations are dynamically evolving, respectively tending to stability, or deviating from it.

Under the assumption of a deterministic evolution of the membership population and using a deterministic and constant discount rate i, the AL has the following resulting dynamics:

$$AL(t+1) = [AL(t) + NC(t) - B(t)](1+i) \tag{3.1}$$

This expression, firstly introduced in Bowers et al(1976), is known as the *liability growth equation* (**LGE**) and describes the dynamical system in which AL evolves. These dynamics are consistent with the general liability growth equation **GLG** (2.2), since if the population is deterministic, then AL is deterministic too.

Let us assume, in accordance with Keyfitz's model, that a population grows exponentially at a logarithmic rate δ_g, i.e.: $n(z,t+1,\kappa) = e^{\delta_g} n(z,t,\kappa)$. It is simple enough to see that equation (2.6) implies that $AL(t+1) = e^{\delta_g} AL(t)$ and hence the following equilibrium holds:

$$AL(t)\left[e^{\delta_g - \delta} - 1\right] = NC(t) - B(t) \tag{3.2}$$

where $\delta = \log(1+i)$.

This equation recreates a set of well known results, that when the rate of population growth is equal to the rate of economic growth, the annual contributions match the annual benefit and therefore any funding method is equivalent in terms of contributions to a "pay as you go" scheme.

This result can be found in different forms in Owadally (1999), Bowers et al. (1976), and as early as Cantelli (1926).

3.2 Stationarity.

As a special case of stability, consider the case where the rate of population growth remains constant, $\delta_g = 0$: the resulting population is then said to be stationary.

Moreover, as stated in Winklevoss(1993), if a constant flow of new entrants annually joins the scheme, and decrement rates do not change over time, *"a stationary condition will exist after* n *years, where* n *equals the oldest age in the population less the youngest."*

In terms of the membership function, stationarity is equivalent to assuming that $n(\alpha + x, t, \kappa) = n_{\alpha+x,\kappa}$. This assumption implies that stationarity holds for the aggregated membership function as well:

$$n(\alpha + x, t) = \sum_{\kappa=0}^{R-\alpha} n(\alpha + x, t, \kappa) = \sum_{\kappa=0}^{R-\alpha} n_{\alpha+x,\kappa} = n_{\alpha+x} \tag{3.3}$$

So, for instance, the Actuarial Liability broken down by generations, as expressed in equation (2.6) would become constant:

$$AL(t) = \sum_{x=0}^{\omega-\alpha} \sum_{\kappa=0}^{R-\alpha} AL_{\alpha+x,\kappa}\, n_{\alpha+x,\kappa} = AL \tag{3.4}$$

The main source of interest of this simple model is that stationarity in the population evolution leads to a remarkable equilibrium between the total contributions, the returns from investments and the payment of benefits. Specifically, the liability growth equation (3.1) can be rewritten in the following way:

$$AL = (AL + NC - B)(1 + i) \tag{3.5}$$

which states that total annual benefit is constant, and it is equal to the annual constant contributions plus the (discounted) returns from investing the AL; i.e.,

$$B = NC + dAL$$

As stated in Trowbridge (1952), we can refer to (3.5) as the *equation of maturity*; since, when the membership population eventually becomes stationary, the pension scheme reaches the status of maturity, and an equilibrium will exist between inflows and outflows.

3.3 Stochastic new entrants.

A first extension of the existing standard model consists of assuming a randomly evolving population. In order to do so, we shall consider the membership function $n(\alpha + x, t, \kappa)$ as a stochastic process.

In detail, we assume that the random nature of this process is due to stochastic new entrants, while the assumptions regarding the decrements from the scheme population are consistently borne out by experience. Thus, we assume that the probabilities in equation (2.8) are deterministic.

One of the first extension in this direction has been implemented in Mandl and Mazurova (1996), where stochastic new entrants are modelled by means of a process which is stationary in the wide sense [1].

Similarly, in Owadally (1999) a sequence of *iid* random variables describes the new entrants process.

Chang *et al* (2002) consider a random demographic evolution by modelling the normal costs with a standard Brownian motion. Josa-Fombellida and Rincón-Zapatero (2004) further extend this assumption by describing the dynamics also of the benefit flows with a standard Brownian motion.

Although using different techniques, the authors focus in the cited works on deriving expressions for the variance of the fund value and of the contributions.

Under this assumption, the equilibrium displayed in equation (3.5) clearly does not hold any more. However, we are now going to show that under equivalent assumptions regarding the stochastic membership process, an equivalent, and more general, form of equilibrium does exist.

Liability growth process. Deterministic decrements allow us to write equation the **GLG** equation (2.2) as the following:

$$AL(t+1) = [AL(t) + NC(t) - B(t)](1+i) \tag{3.6}$$

Equation (3.6) looks identical to the **LGE** - equation (3.1) - however, random new entrants make $AL(t)$ a stochastic process, and therefore, it is more appropriate to refer to (3.6) as the *liability growth process* (**LGP**).

This happens because, although having a random size, each generation of members evolves with certainty. The mathematical counterpart of this assertion is the rationale behind the proof provided in appendix A.1.

As we shall see in the next chapter, this random process does not give rise to a demographic risk *per se*, but it has the effect of amplifying existing risks.

The advantage is that this model allows the analysis of the liability of a pension scheme, with a randomly evolving population, to be conducted by using equations

[1] A stochastic process is said to be stationary in the wide sense (or weakly stationary) if the first moment is constant and the autocorrelation for any time lag is time-invariant; see, for instance Rosenblatt (1974), or Mills (1999).

cited in established literature. However, it needs to be borne in mind that, in this context, these equations describe stochastic dynamics.

Stochastic stationary population. We direct the attention specifically to the case that the number of members is random, but where the evolution is not driven by any drift. In another words, $n(\alpha + x, t, \kappa)$ is assumed to be a stationary stochastic process, and its mean, by the definition of stationarity, is independent of time, i.e. $E[n(x, t, \kappa)] = n_{x,\kappa}$.

Under this assumption the AL at time t is a sum of random variables with constant expectation, and therefore, its expectation is constant as well:

$$\begin{aligned} E\big[AL(t)\big] &= E\left[\sum_{x=0}^{\omega-\alpha}\sum_{\kappa=0}^{R-\alpha} AL_{\alpha+x,\kappa}n(\alpha+x,t,\kappa)\right] \\ &= \sum_{x=0}^{\omega-\alpha}\sum_{\kappa=0}^{R-\alpha} AL_{\alpha+x,\kappa}E\big[n(\alpha+x,t,\kappa)\big] \\ &= \sum_{x=0}^{\omega-\alpha}\sum_{\kappa=0}^{R-\alpha} AL_{\alpha+x,\kappa}n_{\alpha+x,\kappa} = AL, \quad \text{say.} \end{aligned} \tag{3.7}$$

In light of this, the result is that the **LGP** in equation (3.6) leads to the Trowbridge equation of maturity on average:

$$\begin{aligned} AL = {} & E\big[AL(t)\big] = E\big[AL(t-1) + NC(t-1) - B(t)\big](1+i) \\ = {} & \big[AL + NC - B\big](1+i) \end{aligned} \tag{3.8}$$

Equation (3.8) is the equivalent - in a stochastic framework - of the equation of maturity (3.5). In fact, equation (3.5) represents a special case of the more general result in (3.8), and hence, we can refer to (3.8) as a *weak equation of maturity.*

3.4 Stochastic membership.

In order to extend further the standard model, we shall consider the demographic evolution in its full complexity. Specifically, several factors simultaneously interact, with the ultimate effect of driving this evolution.

These factors are likely to be randomly perturbated, thus generating potential mismatches between the actual observations and the expectations.

The following is an attempt to classify these factors according to the effect that they have on the membership population.

We define *positive* components, those leading to an increment of the number of members. Vice versa, components are said to be *negative* if they lead to a decrement of the total population.

We consider the *positive* components as follows:

New members - first job ; who join the scheme at same stage and who have never had a job before. In the most general case, there are no restrictions on the entry age.

New members - previous job ; who join the scheme and contextually transfer a previously accrued fund. Even in this case there are restrictions on the entry age.

New pensioners ; in a general classification we include the possibility of a transfer of a pensioner to a different scheme, which is expected to deal with the pensioner decumulation phase.

These positive components are strictly related to the specific rules of each scheme. In fact, the transfer of funds might be allowed or not by the regulations of the scheme. The transfer of funds might be easily encouraged (or discouraged) by applying favourable (unfavourable) rules to convert the transferred funds into accrued funds in the new scheme. As a consequence of this, for a member who is moving from an old scheme to a new one, it might be financially advantageous (or not) to transfer the previously accrued fund.

Another important factor that affects the positive fluctuations is the demographic evolution of the general population. Since the membership population is a subset of the general population, it seems obvious that "booms and busts" in national fertility affect the membership of pensions schemes, as well as they affect other social services such as national education, health service and others. Similarly, immigration and emigration might influence the membership size and structure, since they affect the age distribution of the national population.

Finally, it is important to mention that positive components (and especially the new entrants class) may be related to the general economic trend as well. It is not unreasonable to believe that during favourable economic periods, business growth would bring new employment, and hence, an increase in the number of new members of pension scheme. On the contrary, phases of economic contraction may generate unemployment, which, in turn, would reduce the number of new members.

Moreover, many economic theories have proposed a direct link between inflation and employment. For instance Phillips' theory states that a natural rate of unemployment exists in the long run; however, in the short term, the relationship between inflation and unemployment is such that a reduction in the unemployment leads to an increase in the inflation rate [2]. Henceforth, the correlation between inflation (and perhaps the market returns) and the evolution of a scheme membership will be considered in a later section.

There are two different approaches to modelling the evolution of a pension plan population: either (1) modelling the new entrants; or (2) following a size-constrained population approach.

In the first case, the number of annual new entrants might be described as an independent process or linked to an underlying process; i.e. to inflation, to the general economic growth (GDP, market returns), or to the specific economic sector performances. In any case, the total size of the membership is the resulting dependent variable. However, according to Winklevoss (1993, p. 61) this may not be an appropriate assumption, because the number of the employees - and in turns, the scheme membership population - depends on business considerations and plans.

[2] Samuelson (1989) is a standard reference for introduction to economics.

So the alternative approach (2) would suggest allowing the membership population to vary according to one (or more) of the aforementioned economic trends and consequently deriving the number of annual new entrants. However, such an approach would focus only on the uncertainties deriving from the internal evolution of a population, due to the negative factors described below. For this purpose, an interesting model to describe the internal evolution is proposed in Janssen and Manca (1997). In this paper and in subsequent works, the authors apply a semi-Markov model to describe the movements of the members from one state to another.

As mentioned above, *negative* components decrease the number of members. Specifically, the following factors have a predominant impact:

Mortality ; this factor affects both active members and pensioners. The case of decease of an active member leads to a permanent suspension of contribution and perhaps to the payment of a benefit (related to the accrued fund) to his/her family, if any. The death of a pensioners leads to the permanent suspension of payments, unless a reversionary pension has been initially included in the pensioner benefit.

Mortality is usually described by a life table, which indicates the probability of death at each age. Specific tables exist for describing the characteristic mortality of pensioners. Nevertheless, it is not necessary to assume a stationary life table and therefore it would be of interest to analyse the effects of dynamic mortality, both in a deterministic and stochastic case.

Elimination of active members ; employees may move into other companies and this could lead to a permanent suspension of the contribution into the scheme. Specific service tables deal with the probability of elimination due to transfer. Moreover, a multiple state model could be considered, in order to explicitly describe different forces of eliminations, as well as distinguish between active and retired members.

The aforementioned factors affect the size of the membership and therefore affect the annual cash-flow by increasing or decreasing contributions and benefits. However there is another possible event, which, although does not affect the size of the membership, still affects the cash-flow.

Morbidity ; employees might transfer into a state of disability, which implies the inability to work and therefore to contribute. It is common practice, but not necessary, to provide the disabled member with a benefit in terms of a pension. Likewise an increased benefit might be paid to a newly disabled pensioner, if such extra benefit has been initially included. In either case, morbidity has a negative impact on the scheme, as either it reduces the contribution or it increases the total pay-out.

The membership function actually incorporates all of the factors described. In fact, it could be disaggregated in a number of more elementary components, as mortality, disability, withdrawals, new entrants and so on.

In fact, these forces may be combined in a multiple state model, describing the evolution of the active membership population through a variety of possible states.

However, although all of these factors interact in the development of a specific population, it is possible to synthesise this evolution as the result of two forces: a positive force, or force of increment, mainly due to the process of new entrants; and a negative force, or force of decrement, which is mainly due to the process of withdrawals (in the active membership) and the process of mortality (in the population of pensioners).

Hence the membership process can be described by the following equation:

$$n(x, t+1) = n(x-1, t)p(x-1, t) + g(x, t+1) \tag{3.9}$$

which states that the number of members aged x at time $t+1$ is naturally given by the number of survived (not eliminated) members aged $x-1$ in t, plus those aged x who joined the scheme at the beginning of the year $t+1$.

As mentioned above, we include the possibility that new pensioners join the scheme; however, in order to describe the case in which this is not allowed, it is sufficient to set $g(x,t) = 0$, for $x \geq R$ and $\forall t$.

The gain/loss generated by the demographic uncertainty is thus proportional to the difference of the actual number of members, given by (3.9), and its expectation. The following equation expresses this difference with respect to the cohort aged x and time $t+1$ and evaluated at time t:

$$\begin{aligned}\varepsilon_n(x,t+1) &= n(x,t+1) - E[n(x,t+1)|\mathcal{F}_t] \\ &= n(x-1,t)[p(x-1,t) - p_{x-1}(t)] + g(x,t+1) - g_x(t+1) \\ &= n(x-1,t)\varepsilon_p(x-1,t) + \varepsilon_g(x,t+1) \qquad (3.10)\end{aligned}$$

In order to focus on the mismatch that develops between times t and $t+1$, the deviation is expressed as a function of the number of members in the cohort aged $x-1$ at time t. Whether this number is known or not (in this case it would be a random variable) depends on the time of valuation: at time t, the quantity $n(x-1,t)$ is known.

The expected number of member aged x in $t+1$ is derivable from equation (3.9). If the evaluation happens a time t, this expectation is conditioned on the knowledge available at that time, $(\mathcal{F}_t)$, and therefore, $n(x-1,t)$ is known.

If the mismatches are due to accidental perturbations, it is straightforward to see that the expected deviations are equal to zero. The variance is instead given by the following:

$$\begin{aligned}\mathrm{Var}\Big[\varepsilon_n(x,t+1)|\mathcal{F}_t\Big] &= n(x-1,t)^2\,\mathrm{Var}[p(x-1,t)\Big] + \mathrm{Var}[g(x,t+1)\Big] \\ &\quad +2n(x-1,t)\,\mathrm{Cov}\Big[p(x-1,t);\ g(x,t+1)\Big] \qquad (3.11)\end{aligned}$$

Hence, the variance of the mismatch is given by the rescaled variances of the two process of new entrants and eliminations; plus a component which takes into account the covariance between the two. This last component reflects whether the number of new entrants is correlated to the number of eliminations. Specifically, a hiring policy could be adopted in order to replace partially those employees who left. This possibility is considered in the following section.

3.4.1 Active membership.

Let us focus exclusively on the case of active workers, aged between α and R. Deviations from the expected value can be positive or negative, and according to the type, consequences and possible reactions may differ.

On one hand, in a particular year an unexpectedly high number of members may be eliminated. As a consequence, the employer may adopt an appropriate hiring policy in order to reduce the impact of an unexpectedly high number of eliminated members. However, replacing the eliminations with new members is not sufficient in order to maintain the liabilities unchanged. In fact, new members may differ from the eliminated ones because of age and accrued funds.

On the other hand, it is possible that actual eliminations are less than expected. In such a case, it is unlikely to imagine that a plan sponsor could implement a policy to reduce the number of members.

Let us assume that the hiring policy is such that a proportion ζ of the unexpectedly eliminated members is replaced. This proportion is assumed to be constant with respect to age. This may be not very realistic, as different cohorts may be not equally easy to replace. Hence, an extension including different proportions for each age may be considered for further research, and especially in the studies in section 3.4.3 regarding the effects of random demographic evolution on the liabilities.

If the deviation $\varepsilon_p(t)$ is negative, then the subsequent year the initially predicted number of new entrants $g_x(t+1)$ is increased according to the deviation $\varepsilon_p(t)$ and the proportion ζ, whom we are willing to replace. Conversely, if $\varepsilon_p(t)$ is nonnegative, then no action is taken.

Thus, the deviation in new entrants has the following form:

$$\varepsilon_g(x, t+1) = \max\left[0;\ -\zeta\ n(x-1, t)\varepsilon_p(x-1, t)\right] \tag{3.12}$$

In order to express the expected value and the variability of the demographic variation, let us split the deviation ε_p into its positive and negative part:

$$\varepsilon_p = \varepsilon_p^+ + \varepsilon_p^- = \max\left(0;\ \varepsilon_p\right) + \min\left(0;\ \varepsilon_p\right)$$

Equation (3.12) suggests that the deviation ε_g in the new entrants is positive only if ε_p is negative. Thus, ε_g also has the following mathematical representation:

$$\varepsilon_g(x, t+1) = -\zeta\ n(x-1, t)\varepsilon_p^-(x-1, t) \tag{3.13}$$

From equations (3.10) and (3.13) it can be derived that the demographic risk, allocated to age x at time t, has the following expectation and variance[3]:

$$E\Big[\varepsilon_n(x, t+1)|\mathcal{F}_t\Big] = n\ E[\varepsilon_p^+] + n\ (1-\zeta)E[\varepsilon_p^-] \tag{3.14}$$

$$\begin{aligned}\mathrm{Var}\Big[\varepsilon_n(x, t+1)|\mathcal{F}_t\Big] &= n^2\ \mathrm{Var}(\varepsilon_p^+) + n^2\ (1-\zeta)^2\ \mathrm{Var}(\varepsilon_p^-) \\ &\quad -2(1-\zeta)\ n^2\ \mathrm{Cov}\big[\varepsilon_p^+; \varepsilon_p^-\big]\end{aligned} \tag{3.15}$$

The expected value of the deviation in the number of members from the cohort x at time t is null, if the proportion ζ is set equal to $1 + \frac{E[\varepsilon_p^+]}{E[\varepsilon_p^-]}$, or, equivalently, to $\frac{E[\varepsilon_p]}{E[\varepsilon_p^-]}$.

Hence, if the deviations from the process of elimination have expectation equal to zero, ζ is nil. This means that if the noise disturbing the number of annual eliminations in each cohort has expectation equal to zero, there is no need of a special hiring policy, in order to keep the number of scheme's members in line with the expectations.

In addition to this intuitive result, equation (3.14) also suggests that, if there is a systematic (negative) mismatch in ε_p, then a hiring policy seeking for a balance in the number of scheme's member would call for a fixed proportion of replacements. Moreover, such a proportion, which is equal to $\frac{E[\varepsilon_p]}{E[\varepsilon_p^-]}$, tends to 1 for a sufficiently large negative mismatch.

[3]Redundant notation has been excluded, in order to ease the reading.

As far as the variance of the deviation ε_n is concerned, equation (3.15) shows that a minimum exists when [4]:

$$\zeta^* = 1 + \frac{\mathrm{Cov}\left(\varepsilon_p^+ ; \varepsilon_p^-\right)}{\mathrm{Var}\left(\varepsilon_p^-\right)}$$

Since the $\mathrm{Cov}\left(\varepsilon_p^+ ; \varepsilon_p^-\right)$ is positive, the variance has its minimum when ζ is larger than 1. Hence, in order to minimise the variance in the deviation of the membership function, the number of replacement should be higher than the number of eliminations.

However, if only a proportion between 0 and 1 can be considered - $\zeta \in (0, 1)$; and bearing in mind that in this interval the variance is a monotonic decreasing function, we can state that the higher is the proportion of replaced eliminated members, the lower is the variance of the deviations ε_n.

Furthermore, if the deviations ε_p are symmetrically distributed, and hence the positive and negative parts have the same distribution, the ζ^* is equal to the following

$$\zeta^* = 1 + \frac{\left[\mathrm{E}\ \varepsilon_p^-\right]^2}{\mathrm{Var}\left(\varepsilon_p^-\right)}$$

where the last term is the reciprocal of the square of the coefficient of variation of ε_p^- (or, equivalently ε_p^+).

3.4.2 Retired membership.

In the case of retirement ages ($x > 65$) the number of new entrants could reasonably be assumed to be equal to zero. In such a case, the difference between the expectations and the actual observed number of eliminations is likely to be due to the effect of mortality. Specifically, the mismatch may be of two types: an accidental type of mismatch, which is unlikely to be particularly substantial. The second is a systematic type of mismatch, which instead arises when the actuarial assumptions fail to adequately take into account all of the significant factors.

[4]Note that $\mathrm{Cov}\left(\varepsilon_p^+ ; \varepsilon_p^-\right) = -\mathrm{E}[\varepsilon_p^+] \cdot \mathrm{E}[\varepsilon_p^-]$, because $\mathrm{E}[\varepsilon_p^+ \cdot \varepsilon_p^-] = \mathrm{E}[\max(0, \varepsilon_p) \cdot \min(0, \varepsilon_p)] = 0$. Moreover, since $\mathrm{E}[\varepsilon_p^-] < 0$ and $\mathrm{E}[\varepsilon_p^+] > 0$, the covariance is positive.

The second type is often referred to as longevity risk, since actuarial assumptions have been shown in the past to have underestimated the decrements in mortality recorded in the past, particularly for older cohorts[5]. An extensive literature on the topic exists, focusing on modelling issues, as well as on practical fitting problems and solutions.

A classical approach to modelling the mortality risk consists in distinguishing the two factors above described [6]. A random noise with expectation equal to zero and finite variance (perhaps decreasing with the number of members) might adequately describe the deviations [7]. Adding a reduction factor, a function of age and time, to the probability of being eliminated could provide a reasonable approach to model the longevity risk. Thus, the observed proportion of member aged $x-1$ at time t can be modelled as following:

$$p(x-1,t) = 1 - q_{x-1}(t)RF(t)(1+\varepsilon_q(x-1,t))$$

where $q_{x-1}(t)$ is the probability that an individual aged $x-1$ at time t is eliminated between $t-1$ and t.

Hence, the random deviation between observed and expected number of members aged x at time $t+1$ has the following expression:

$$\begin{aligned}\varepsilon_n(x,t+1) &= n(x-1,t)\Big(p(x-1,t) - p_{x-1}\Big)\\ &= n(x-1,t)\big(1-p_{x-1}(t)\big)\Big[1 - RF(t)\big(1+\varepsilon_q(x-1,t)\big)\Big] \end{aligned} \quad (3.16)$$

and, it has expectation and variance given by the following equations:

[5]Refer to CMIB for a comprehensive analysis of past and recent trends in UK mortality.

[6]See for instance Pitacco (2002).

[7]Recent literature, such as Lee and Carter (1992), Sithole *et al* (2000), Haberman and Renshaw (2003), Milevsky and Promislow (2001) and Ballotta and Haberman (2004), prefers to apply a reduction factor to the rate of mortality, rather than to the probability of elimination. However, in this context modifying the probabilities simplifies the mathematics. Nonetheless, more sophisticated models can be applied in this work as well.

$$
\begin{aligned}
E\Big[\varepsilon_n(x,t+1)|\mathcal{F}_t\Big] &= n(x-1,t)\big(1-p_{x-1}(t)\big)\Big[1-RF(t)\Big] \\
\operatorname{Var}\Big[\varepsilon_n(x,t+1)|\mathcal{F}_t\Big] &= \Big[n(x-1,t)\big(1-p_{x-1}(t)\big)RF(t)\Big]^2\sigma_q^2
\end{aligned}
\tag{3.17}
$$

These expressions highlight the effect of a systematically decreasing mortality on the age distribution of the membership population, and as such, are used in the following section where we analyse the effect of mortality risk on the liability of a DB pension scheme.

3.4.3 The effect on the liability.

When the population is assumed to be random, and particularly the decrements are random, then the **GLG** equation (2.2) cannot be expressed as the **LGP** in equation (3.6). However, it is still possible to derive a number of results for the distribution of the liability.

Trivially, we can express the value of the reserve at time $t+1$ as the difference between its expectation and the deviation from it:

$$
AL(t+1) = E\Big[AL(t+1)|\mathcal{F}_t\Big] + \Delta AL(t+1) \tag{3.18}
$$

The annual deviation from the expectation, $\Delta AL(t)$, can be expressed as a function of the mismatches in the number of members in each cohort. In detail, from equation (2.6) (assuming the same past service for all of the members) and first part of equation (3.10) we can write the following:

$$\begin{aligned}\Delta AL(t+1) &= AL(t+1) - E\Big[AL(t+1)|\mathcal{F}_t\Big] \\ &= \sum_{x=\alpha}^{\omega} AL_x n(x,t+1) - \sum_{x=\alpha}^{\omega} AL_x E\Big[n(x,t+1)|\mathcal{F}_t\Big] \\ &= \sum_{x=\alpha}^{\omega} AL_x n(x,t+1) - \sum_{x=\alpha}^{\omega} AL_x n_x(t+1) \\ &= \sum_{x=\alpha}^{\omega} AL_x \varepsilon_n(x,t+1) \qquad (3.19)\end{aligned}$$

According to the **GLG** equation (2.2) and from (3.19) and (3.18), we can write the following:

$$AL(t+1) = \Big[AL(t) + NC(t) - B(t)\Big](1+i) + \sum_{x=\alpha}^{\omega} AL_x \varepsilon_n(x,t+1) \qquad (3.20)$$

or equivalently,

$$AL(t+1) = \Delta AL(t+1) + \sum_{k=0}^{t} \Big[\Delta AL(k) + CF(k)\Big](1+i)^{t+1-k} \qquad (3.21)$$

where $CF(t)$ is the cash-flow at time t, $CF(t) = NC(t) - B(t)$.

The **GLG** equation (2.2) lets us write the expected value of the reserve at time $t+1$ as a function of the previous year values of the reserve and of the cash-flow. Applying the same equation recursively, it is possible to describe the dynamics of AL as in equation (3.21), where the mismatch and cash-flow from each year are identified.

The number of random variables composing $AL(t+1)$ is dependent on the time of valuation. When the expectation is conditioned to the filtration at time t, then only the deviation happening between t and $t+1$ is random. Instead, if the valuation happens at a time $s < t$, then all the deviations between the two years, s and $t+1$, are unknown.

Let us assume that the new entrants replace a proportion ζ of the unexpected exits, and that there is no correlation among different cohorts of members. As mentioned before, the condition that the replacements have the same demographic characteristic of the exited members is not sufficient to lead to an equilibrium. In fact, it is also required that these new entrants have the same past service. Let us also assume that longevity risk exists, and thus, the scheme's retired members experience a systematic decline in their mortality.

Hence, from equations (3.15), (3.17) and (3.19), we derive the following expression of the variance of the actuarial liability [8] :

$$\begin{aligned}
\mathrm{Var}\Big[AL(t+1)|\mathcal{F}_t\Big] &= \mathrm{Var}_t\Big[\Delta AL(t+1)\Big] = \\
&= \mathrm{Var}_t\Big[\sum_{x=\alpha}^{\omega} AL_x \varepsilon_n(x,t+1)\Big] = \sum_{x=\alpha}^{\omega} AL_x^2 \mathrm{Var}_t\Big[\varepsilon_n(x,t+1)\Big] \\
&= \sum_{x=\alpha}^{R-1} AL_x^2 \mathrm{Var}\Big[\varepsilon_n(x,t+1)\Big] + \sum_{x=R}^{\omega} AL_x^2 \mathrm{Var}\Big[\varepsilon_n(x,t+1)\Big] \\
&= \sum_{x=\alpha}^{R-1} AL_x^2 n(x-1,t)^2 \Big[\mathrm{Var}(\varepsilon_p^+(x-1,t)) + \big(1-\zeta\big)^2 \mathrm{Var}(\varepsilon_p^-(x-1,t)) \\
&\quad -2\big(1-\zeta\big)\mathrm{Cov}\big(\varepsilon_p^-(x-1,t), \varepsilon_p^+(x-1,t)\big)\Big] \\
&\quad \sum_{x=R}^{\omega} AL_x^2 \Big[n(x-1,t)(1-p_{x-1}(t))RF(t)\Big]^2 \sigma_q^2 \qquad (3.22)
\end{aligned}$$

Equation (3.22) displays separately the effects of both the active and retired membership variability on the variance of AL. Hence, we can compute the proportion of eliminated members to be replaced, in order to minimise this variance. This proportion is given by the following equation:

$$\zeta^* = 1 - \frac{V_2}{V_1} \qquad (3.23)$$

[8] Note that the notation Var_t indicates the conditional variance to the filtration $\mathcal{F}_t$ at time t.

where V_1 and V_2 are, respectively, the variance and covariance of negative and positive deviations of eliminated active members from the expectations and are given by the following:

$$V_1 = \sum_{x=0}^{\omega} AL_x^2 n^2(x-1,t) \mathrm{Var}\Big[\varepsilon_p(x-1,t)^-\Big]$$
$$V_2 = \sum_{x=0}^{\omega} AL_x^2 n^2(x-1,t) \mathrm{Cov}\Big[\varepsilon_p(x-1,t)^-,\ \varepsilon_p(x-1,t)^+\Big]$$

3.5 Surpluses and deficits.

The interest in analysing the impact of random population on AL mainly lies in understanding what are the effects on the evolution of the liabilities. Therefore, it is also of interest to study to what extent a random population can affect the unfunded part of this liability. With this scope, we direct our attention to the equations describing the unfunded liability.

Specifically, a demographic model with stochastic membership affects the number of eliminations from the scheme. Thus, part of ul is generated by this variability, and hence, this model incorporates a demographic risk.

On the contrary, a demographic model with stochastic new entrants does not generate an extra contribution to ul, as long as funding is provided for the random number of new members. Nonetheless, the randomness included in this model amplifies the already existing risks.

Bearing this in mind we deal with the two models in separate subsections. Specifically, we first consider the case of a general stochastic membership, where it is assumed that no other risk exists. Then, we consider the more specific case of a demographic model with stochastic new entrants, but in presence of a financial risk as well; i.e. when the rates of return from the investments are allowed to randomly vary.

3.5.1 Stochastic membership.

In a manner similar to the **GLG** equation, we express the expected dynamics of ul as in the following equation:

$$E[ul(t+1)|\mathcal{F}_t] = ul(t)(1+i)(1-k) \tag{3.24}$$

Recalling section 2.6.2, k is the proportion of the annually amortised ul, which is added to the normal contributions in order to re-balance the scheme. Specifically, if the scheme is under-funded ($ul > 0$), the supplemental cost is positive and determines an extra contribution to be paid by the sponsor. On the contrary, if the scheme is over-funded ($ul < 0$), the adjustment is translated into a reduction in the annual contribution. However, the actual behaviour of the mathematical model depends on the parameter k; in detail,

1. If $k = 1$, all the existing ul is amortised and the resulting expected ul is null at any time.
2. If $k = 0$, none of the current ul is amortised, and hence, the liability is expected to grow at the anticipated interest rate.
3. If $|(1+i)(1-k)| < 1$, then the expectation of ul at time $t+1$ is, in absolute value, lower than the absolute value of ul at time t.

This methodology for dealing with surpluses and deficits is commonly implemented in UK, but it is not the only one. Currently, in USA and Canada the annual gains and losses are instead spread over a fixing term, as illustrated in section 2.6.2.

Both methods can make the expected value of ul vanish in the long term (infinite time), if a number of assumptions are met, namely: the rates of return from financial investments have to be independent one year from the next one and must have expectation equal to the discount rate i; moreover, condition 3 from the previous list must be met; and the demographic assumptions have to be borne out by experience.

In Owadally (2003) an alternative method is proposed, which assures convergence to zero, also in the case of systematic mismatches between the deterministic rates of return and the discount rate, i.e. $r(t) = r \neq i$.

In chapter 6, we analyse the effect of relaxing the assumption of independence. In particular, sufficient conditions on k are found in order to assure that the expected ul converges to zero, when $\{r(t)\}$ is a bounded stochastic process.

In this section, it is of interest to focus on the demographic risk only. Hence, we require that the rates of return are deterministic and equal to the discount rate; i.e. $r(t) = i\ \forall t$. In addition, we assume that the annual random ul increment (or decrement) is due to demographic variations only. Therefore, recalling equation (3.20) we assume that $\Delta ul(t) = \Delta AL(t)$.

Hence, in analogy with equation (3.18), it results that

$$\begin{aligned} ul(t+1) &= E[ul(t+1)|\mathcal{F}_t] + \Delta ul(t+1) \\ &= ul(t)(1+i)(1-k) + \Delta AL(t+1) \\ &= \sum_{j=0}^{t+1} \Delta AL(j) w^{t+1-j} \quad \text{where } w = (1+i)(1-k) \end{aligned} \tag{3.25}$$

which holds if the scheme is assumed to be initially funded, i.e. $ul(0) = 0$.

From equation (3.25), we can see that the expectation of ul depends on the amortisation proportion k and on the expectation of deviations of the number of members ε_n; i.e.,

$$\mathrm{E}[ul(t+1)] = \sum_{j=0}^{t+1} w^{t+1-j}\mathrm{E}\big[\Delta AL(j)\big] \tag{3.26}$$

which is nil if the expected deviations in the membership population are nil [9].

3.5.2 Stochastic new entrants.

In this section we recall the demographic model based on stochastic new entrants, introduced in section 3.3. This simplified demographic model allows us to relax the assumptions on the financial risk. Specifically, we assume that the investment rates

[9]In order for the expectation of ul to be finite, the deviations ε_n must have a finite expectation as well.

of return randomly evolve in time.

Interesting aspects can be studied by allowing the rates of return to be random. However, this step further can be done at the expenses of narrowing the assumption on the demographic evolution. Specifically, under the assumptions that the eliminations from the population of the scheme follow exactly the expectations, the *ul* is given by the following recursive equation, the derivation of which is illustrated in Appendix A.2:

$$ul(t) = -\sum_{h=1}^{t} AL(h)\ \phi(h,t) \tag{3.27}$$

where the function $\phi(h,t)$ has the following form:

$$\phi(h,t) = \begin{cases} v(1-k)^{t-h}(r(h-1)-i)\prod_{j=h}^{t-1}(1+r(j)) & h \in (1,\ t-1) \\ v(r(t-1)-i) & h = t \end{cases} \tag{3.28}$$

As is shown in equation (3.27), *ul* is a sum of products of random variables. In particular, the variables $\phi(h,t)$ describe the actual mismatch between the rates of return and the valuation rate, compounding their effect up to the time of valuation t. In a certain sense, $\phi(h,t)$ plays a similar role to the parameter w in (3.26) and in previous equations, but extending it to the case of random rates of return.

The value of the reserve is random due to the stochastic number of members. We have seen in the previous sections that a stochastic demographic evolution is at the core of the variability of the reserve. Since the assumptions on eliminations are consistently borne out by experience, the demographic random evolution does not create a risk "per se", but amplifies (or reduces) the impact of the financial risk.

From equation (3.27), the derivation of the variance of *ul* trivially follows:

$$\begin{aligned}\text{Var}\Big(ul(t)\Big) &= \text{Var}\Big[\sum_{h=1}^{t} AL(h)\ \phi(h,t)\Big] \\ &= \sum_{h=1}^{t}\sum_{j=1}^{t}\text{Cov}\Big[AL(h)\phi(h,t), AL(j)\phi(j,t)\Big] \end{aligned} \tag{3.29}$$

Once the reserves in equation (3.29) are broken down by the number of members at each age[10], the variance has also the alternative form:

$$\begin{aligned}\text{Var}\Big(ul(t)\Big) &= \\ &= \sum_{x=0}^{T}\sum_{y=0}^{T} AL_{\alpha+x}AL_{\alpha+y}\sum_{h=1}^{t}\sum_{j=1}^{t}\text{Cov}\Big[n(\alpha+x,h)\phi(h,t), n(\alpha+y,j)\phi(j,t)\Big]\end{aligned} \tag{3.30}$$

which shows the direct link between the demographic evolution (number of members) and the financial risk.

The vast majority of scientific works in the literature of actuarial and life insurance mathematics is based on the assumption that financial and demographic risks are independent. Although this assumption is not necessarily appropriate, it is often made for mathematical convenience. However, several authors have investigated a number of potential avenues for modeling dependent insurance and financial risks: see, for instance, Dhaene *et al* (2001).

With respect to the model here considered, it is not clear whether a dependence between the returns from investments and the number of new entrants exists. It is a possibility that the underlying economic cycles do affect both the recruitment process of a company (and thus the entrants in its pension scheme) and the investment returns achieved by the management of the pension fund. However, the actual relation

[10]A more detailed expression would take into account the past service as well. This can be done by involving one more sum in the equation; hence further complications would not arise but the resulting notation would be more cumbersome.

between the two sources of risk is not yet clear and data and statistical evidence are not sufficient to support any conjecture.

Nevertheless, it is of great interest to investigate how those two sources of uncertainty give rise to the variability of the mismatch between assets and liabilities. Furthermore, particular attention is focused on how the two assumptions - independence and dependence - differently affect the overall variability of *ul*.

Recalling the decomposition of the variance of *ul* in equations (3.29) and (3.30), in this thesis we investigate the case of dependence as measured by the covariance. Hence, we do not consider the case of lagged dependence between the labour market and the capital market. In other words, we implicitly assume that any fluctuation in the rate of investment return has an immediate effect on the number of new employees, and thus, of new members in the scheme.

3.5.3 Independence.

Independence between investment returns and number of new members means that the actual realisations in the financial markets (or the more general economic growth) do not affect in any way the number of annual new entrants in the pension scheme, and vice versa. Mathematically, it implies that the covariance among those two processes is equal to 0, so $\text{Cov}[\phi(t), AL(t)] = 0 \; \forall t$. Hence, from equation (3.29) we can derive the following result, as it is illustrated in appendix A.3.1 (3.31):

$$\begin{aligned}\text{Var}\Big(ul(t)\Big) = \sum_{h=1}^{t}\sum_{j=1}^{t}\Big\{ & \text{Cov}\Big(AL(h), AL(j)\Big)\text{Cov}\Big(\phi(h,t), \phi(j,t)\Big) \\ & + \text{Cov}\Big(\phi(h,t), \phi(j,t)\Big) \cdot E[AL(h)]\; E[AL(j)] \\ & + \text{Cov}\Big(AL(h), AL(j)\Big)E[\phi(h,t)]\; E[\phi(j,t)]\Big\} \qquad (3.31)\end{aligned}$$

According to equation (3.31), the variance of *ul* is fully described by three fundamental components:

1. $\text{Cov}\Big(AL(h), AL(j)\Big)\text{Cov}\Big(\phi(h,t), \phi(j,t)\Big)$, which is given by the product of the covariances of both the random processes;

2. $\text{Cov}\Big(\phi(h,t), \phi(j,t)\Big) \cdot E[AL(h)]\ E[AL(j)]$, which summarise the covariance of the financial risk, rescaled taking into account the expected value of the reserve at the corresponding points in time;

3. $\text{Cov}\Big(AL(h), AL(j)\Big)E[\phi(h,t)]\ E[\phi(j,t)]$, which summarises the covariance of the reserve, rescaled with the expected financial mismatches at the corresponding points in time.

The effect on the variance of ul of specific assumptions regarding the demographic and financial evolutions are separately considered in the following paragraphs.

Deterministic stationary population.

If the pension plan population is assumed to be stationary and deterministic, then the financial risk is the only source of uncertainty in the model. Therefore, since at any time h the auto-covariance of the process $AL(h)$ is equal to zero and its expectation is constant, $E[AL(t)] = AL\ \forall t$, the magnitude of the resulting variance is proportional to the square of the value of the reserve.

$$\text{Var}[ul(t)] = AL^2 \text{Var}\left[\sum_{h=1}^{t} \phi(h,t)\right] \tag{3.32}$$

This results holds for any process of the investment returns $\{r(t)\}$. Furthermore, it is consistent with Colombo and Haberman (2004), where the expression of the variance of ul has been used to derive an optimal contribution strategy under a deterministic model of a stationary population.

Stochastic stationary population.

Adding another source of randomness, by assuming that the membership function is a stationary stochastic process as modelled in section 3.3, the resulting variance of

$ul(t)$ has the following form:

$$\mathrm{Var}\Big(ul(t)\Big) = AL^2\mathrm{Var}\Big(\sum_{h=1}^{t}\phi(h,t)\Big) + \sum_{h=1}^{t}\sum_{j=1}^{t}\mathrm{Cov}\Big(AL(h), AL(k)\Big)\mathrm{Cov}\Big(\phi(h,t), \phi(k,t)\Big)$$
$$+ \sum_{h=1}^{t}\sum_{j=1}^{t}\mathrm{Cov}\Big(AL(h), AL(k)\Big)\ E[\phi(h,t)]\ E[\phi(k,t)] \qquad (3.33)$$

Comparing equation (3.33) to equation (3.32) shows that the demographic risk has an additive effect on the variance of *ul*. Specifically, the second term in the right hand side of equation (3.33) summarises the joint variability of the demographic and financial risks, by means of multiplying the covariances of the corresponding processes.

This particular structure holds only under the assumptions of independence between the demographic evolution and the financial risk. This assumption is also reflected in the fact that equation (3.33) contains information only about the dependency *within*, but not *between*, the two sources of variability.

***IID* rates of return.** Let us assume that a sequence of *iid* random variables, having same expectations equal to the valuation discount rate $\mathrm{E}[r(t)] = r = i$ describes the dynamics of the rates of return from the investments.

Such an assumption, on the one hand, is consistent with most of the literature in the field of DB pension schemes modelling, as in Dufresne (1989) and in Owadally and Haberman (1999). On the other hand, it is likely that it may not be entirely realistic, as returns from investments show a certain degree of autocorrelation[11] and prudent actuarial approaches tend to set conservatively a valuation rate lower than the actual expected rate of return. The issue of prudence in the actuarial estimate of the discount rate is discussed in §6.3.

Under this assumption on the distribution of the rates of return, the random variable $\phi(h,t)$ has nil expectation $\forall h$. This results for $h < t$ because:

[11]Refer to Wilkie (1987, 1995) and Haberman (1994) for models and applications of autoregressive processes in actuarial science.

$$\begin{aligned}\mathrm{E}[\phi(h,t)] &= \mathrm{E}\Big[v(1-k)^{t-h}(r(h-1)-i)\prod_{j=h}^{t-1}(1+r(j))\Big]\\ &= v(1-k)^{t-h}\prod_{j=h}^{t-1}\mathrm{E}\big[(1+r(j))\big]\mathrm{E}\big[(r(h-1)-i)\big]=0 \end{aligned}\tag{3.34}$$

because $\phi(h,t)$ is the product of $t-h+1$ independent random variables, and $\mathrm{E}\big[(r(h-1)-i)\big]=0$ by assumption.

We separately deal with the two cases of a stochastic stationary population, and of any random population.

Stochastic stationary population. If the population is stochastic stationary, the following result holds:

$$\mathrm{Var}\Big(ul(t)\Big) = AL^2\sum_{h=1}^{t}\mathrm{Var}\Big(\phi(h,t)\Big)+\sum_{h=1}^{t}\mathrm{Var}\Big(AL(h)\Big)\mathrm{Var}\Big(\phi(h,t)\Big) \tag{3.35}$$

It is worth noting that the variance of the financial risk, times both the (square of the) expected reserve and its variance, fully explain the variability of *ul* [12]. So when the population is stationary (on average) and the returns are *iid*, the financial risk is amplified by the amount of the reserve (which is informative of the liability position of the fund) and by the reserve variability due to the random population evolution.

Therefore, the actuarial method used for the reserving process affects the *ul* variability through the (square) expectation and the variance of the reserve. So a method leading to a smaller reserve does not necessarily lead to a higher risk.

[12]Alternatively, the equation could be written as the product of the variance of $\phi(h,t)$ and the second moment of the reserve at time h. This alternative expression is used in the following case.

Any random population. The previous result can be slightly generalised, by letting the membership process assume any form, not necessarily specified. Specifically, holding the assumption of *iid* returns from investments the following relation holds for any distribution of the demographic evolution:

$$\mathrm{Var}\Big(ul(t)\Big) = \sum_{h=1}^{t} E\big[AL(h)^2\big]\mathrm{Var}\Big(\phi(h,t)\Big) \tag{3.36}$$

Equation (3.36) suggests that the impact of a specific actuarial method on the variance of *ul* should be evaluated on the basis of the second moment of the distribution of the reserve.

3.5.4 Dependence.

As already mentioned, the assumption of independence between the demographic and financial risks is made mainly for mathematical convenience. Although there is no evidence to support any conjecture in favor or against dependence, it is a worthwhile exercise to analyse the variability of *ul*, once the assumption of independence is relaxed.

Recalling the variance of *ul* as expressed in equation (3.29), it can be seen that the structure of the variability is ultimately given by the covariances between the number of members in the scheme and the measure of the financial mismatch. However, it must be noted that this way of including dependence between the demographic and financial uncertainties does not allow for lagged dependence.

In the previous section 3.5.3, it is assumed that the processes $\{\phi(t)\}$ and $n(z,t)$ are independent and such an assumption allows to derive the result in equation (3.31). That decomposition of the variance no longer holds if the two processes are dependent on each other.

However, referring to the covariances in equation (3.30), the following result is proved in appendix A.3.2:

$$
\begin{aligned}
&\mathrm{Cov}\Big(n(x,h)\phi(h,t),\ n(y,j)\phi(j,t)\Big) \\
&= \mathrm{Cov}_{\perp}\Big(n(x,h)\phi(h,t),\ n(y,j)\phi(j,t)\Big) - \mathrm{Cov}\Big(n(x,h),\phi(h,t)\Big) \\
&\quad \cdot \mathrm{Cov}\Big(n(y,j),\phi(j,t)\Big) - \mathrm{Cov}\Big(n(x,h),\phi(h,t)\Big)\ E[n(y,j)]\ E[\phi(j,t)] \\
&\quad - \mathrm{Cov}\Big(n(y,j),\ \phi(j,t)\Big)\ E[n(x,h)]\ E[\phi(h,t)] \qquad (3.37)
\end{aligned}
$$

where $\mathrm{Cov}_{\perp}$ is the covariance between the two products computed as if the demographic and financial risks were independent (see equation (3.31)).

Equation (3.37) provides a way to extend any model based on demographic and financial independent risks, without modifying the underlying assumptions concerning the dependence within each source of uncertainty.

The dependence adds some extra terms to the covariance as computed for the case of independence. Specifically, the added terms are the covariances *between* the two risks [13] and contain no information about the dependence *within* the two risks. As we have seen in the previous section, Cov$\perp$ is the measure describing the covariance within each source of uncertainty, and ultimately each source's variability. Therefore, by giving an expression to the other covariances in equation (3.37), it is possible to describe different type of dependence between the financial risk and the number of annual new entrants.

Applying the decomposition in equation (3.37), such covariances lead to the following expression for the variance of *ul*:

$$
\begin{aligned}
\mathrm{Var}\Big(ul(t)\Big) &= \mathrm{Var}_{\perp}\Big(ul(t)\Big) - \Big(\sum_{h=1}^{t}\mathrm{Cov}\Big(AL(h),\ \phi(h,t)\Big)\Big)^2 \\
&\quad -2AL\sum_{j=1}^{t}E\Big[\phi(j,t)\Big]\sum_{h=1}^{t}\mathrm{Cov}\Big(AL(h),\ \phi(h,t)\Big) \qquad (3.38)
\end{aligned}
$$

[13] In fact, those covariances are combined and amplified in different ways.

***IID* rates of return.** In order to gain some more information about the nature of this structure, it is worthwhile to analyse the simplified case described by *iid* investment returns, with expectation equal to the discount rate. Bearing in mind the limits of this assumption as described in section 3.5.3, it has to be said that in this case the assumption of independence is unrealistic. However, it might be the case that the rates of returns, although not independent of each other, are uncorrelated, which would imply that the expected value of $\phi(h,t)$ is equal to zero at any time h. Hence, equation (3.37) becomes:

$$\begin{aligned} &\mathrm{Cov}\Big(n(x,h)\phi(h,t),\ n(y,j)\phi(j,t)\Big) \qquad (3.39)\\ &= \mathrm{Cov}_{\perp}\Big(n(x,h)\phi(h,t),\ n(y,j)\phi(j,t)\Big) - \mathrm{Cov}\Big(n(x,h),\phi(h,t)\Big)\mathrm{Cov}\Big(n(y,j),\phi(j,t)\Big) \end{aligned}$$

which, in the light of equations (3.29), (3.30) and (3.36), leads to the following expression of the variance of ul [14]:

$$\mathrm{Var}\Big(ul(t)\Big) = \mathrm{Var}_{\perp}\Big(ul(t)\Big) - \Bigg(\sum_{h=1}^{t}\mathrm{Cov}\Big(AL(h),\ \phi(h,t)\Big)\Bigg)^2 \qquad (3.40)$$

As anticipated by the result in equation (3.39), equation (3.40) shows that the variance is higher in the case of independence than when financial returns and demographic evolution are somehow dependent and henceforth it provides an upper-bound.

[14] The result in equation (3.40) can be directly derived from equation (3.38) as well, since $E[\phi(h,t)] = 0$.

3.6 Summary.

This Chapter deals with the evolution of the membership population of a defined benefit pension scheme.

By introducing the classical population plan theory and the results in the liability dynamics deriving from it, we show the direct link existing between the structure of the membership population and the liabilities.

The standard deterministic model of a stable population, core of the classical pension plan theory, is then extended to a more general stochastic model. In section 3.3, we demonstrate that, when the number of annual new entrants is randomly perturbated, while the decrements from the scheme are assumed to deterministic, a set of generalised results exist.

Specifically, using this demographic model, we identify the liability growth process, in analogy to the Bowers' liability growth equation, which holds for deterministic stable population. Similarly, the stochastic counterpart of a stationary population leads to the Trowbridge equation of maturity - or maturity equilibrium - on average.

In section 3.4, a more general demographic model with stochastic membership is analysed. In detail, the complex phenomenon of the demographic evolution is decomposed into two streams of forces: positive factors, which increase the number of members; and negative factors, which decrease the number of members.

Specific models for describing how the demographic risk affect the variability of the membership population are considered, for both the active and retired populations. Furthermore, the effect of the demographic risk on the liability of the scheme is analysed as well.

The demographic models are then used to illustrate how the demographic risk generates unfunded liability.

Specifically, using the model with stochastic new entrants, the dynamics of the ul is studied, particularly focusing on its variance.

The risk is analysed from a general standpoint, as two sources of uncertainty affect the dynamics of the pension scheme: in fact, the new entrants process and the financial realisations are assumed to be randomly perturbated.

In classical actuarial science, it is common practice to assume independence between the two sources of uncertainty. Although this assumption is not necessarily appropriate, it is often made for mathematical convenience. In this section, we analyse the risk of mismatches (described by the variance of *ul*), under both the assumptions of independence and dependence between the two risks. However, using covariances in order to measure the degree of dependence between the two sources if uncertainty does not allow for dependence over time. Thus, this model extends previous research by taking into account a simplified case of dependence, which includes independence as a special case, but which does not capture the full complexity of the phenomenon.

Thus, it is possible to provide closed expressions of the variance of *ul* for a number of specific cases. In addition, we show how it is possible to identify and separate the variables describing the dependence structure *within* the two risks, from those describing the dependence *between* the two risks.

Under certain conditions, we find that the independence assumption leads to an upper-bound of the risk for the case of dependence.

Chapter 4

Contribution strategies

In DB pension schemes, the capital accrued at the time of retirement should be sufficient to finance the payment of an annual pension to each pensioner. As illustrated in §2.5, the final cost of such a benefit is known only a posteriori, after the death of the pensioner, since several components make this final cost random. Among these, there are: the final salary, which determines the amount of the pension; price inflation, to which pension benefits are often linked; the number of payments, which is clearly dependent on the length of the pensioner's lifetime, as well as the lifetime of the spouse, or dependants, if any. Further, the expected present value of such a benefit can be evaluated at different points in time. Let $PVFB_R$ indicate this present value at retirement age (say R), as expressed in section 2.5.

In order to accrue this amount, contribution strategies (or normal cost methods) rely on the annual contributions and on the returns from investments.

Specifically, each contribution strategy is characterised by a level of advance funding; i.e. the proportion of benefit funded by the returns from investments. In order to define this level we shall make use of the so-called accrual density function introduced in the next section.

This chapter focuses on the study of contribution strategies, proposing a classification of normal cost methods based on the level of advance funding. The advantage of this classification is that it leads to natural measures of risk and cost of each contribution strategy under stochastic rates of return, risk of the sponsor going bankrupt and random demographic evolution.

For computational convenience and for consistency with the existing literature, this chapter presents the pension model in continuous time. Indeed, it is straightforward to derive a discrete version of the model, which will be used when consistency with previous chapters is required.

Thus, in this continuous time model, the valuation rate i_t is substituted with an instantaneous valuation rate δ_t. Likewise, the process of demographic elimination is described by using an expected elimination rate μ_x. Hence, in this model, it is possible to compute the $PVFB_x$ (as defined in section 2.9) by actuarially discounting $PVFB_R$ for $R - x$ years; i.e., in this case, according to the following formula:

$$PVFB_x = PVFB_R e^{-\int_x^R \left[\delta_u + \mu_u\right] du} \tag{4.1}$$

4.1 Accrual (of liability) density function.

First introduced in Cooper and Hickman (1967) and therein described as the pension purchase function, the accrual density function is a convenient tool in order to analyse the level of advance funding characterising a given contribution strategy.

This function can be thought of as the speed with which the fund is being accrued, and we shall see that this is a proxy for the level of advance funding.

Using the same notation and interpretation of Bowers *et al* (1976) - the accrual density function $m(x)$ is defined to correspond to a probability density function, hence:

$$m(x) \geq 0,\ \alpha \leq x \leq R \text{ and } \int_\alpha^R m(x)dx = 1 \tag{4.2}$$

In analogy to a probability density function, it is possible to define the (cumulative) accrual function $M(x)$, which indicates how much of the fund has been accrued until an age x, as follows:

$$M(x) = \int_\alpha^x m(u)du \tag{4.3}$$

Having defined these two functions, it is clear that the set of potential accrual density functions is extensive. We shall see that to each accrual density function corresponds a specific normal cost method (or contribution strategy), and therefore the set of normal costs is at least as wide as the set of these functions.

The following equation provide a mathematical definition of the level of advance funding (L_{AF}):

$$L_{AF} = \int_{\alpha}^{R} e^{\int_x^R \delta_u du} m(x) dx \tag{4.4}$$

Thus, we can interpret L_{AF} as a weighted mean of compounding factors, where the accrual density function identifies the set of weights. Alternatively, equation (4.4) may express a measure of pension purchasing, which accounts for the annual purchased proportion of pension, proportionally rescaled by compounding factors, which give importance according to how long each proportion is invested in the market.

The mathematical representation of L_{AF} in equation (4.4) does not include any compounding due to the elimination of members, as we have defined the L_{AF} as the proportion of final benefit funded by the investment returns. However, mortality could be included, in order to consider the L_{AF} as a measure of how much the normal cost are compounded up to retirement.

Remark. It is of interest for the development of this study to highlight the link between the L_{AF} and the timing of the contributions: an accrual function with high weights at early ages determines an early accrual of the reserve. Therefore, the corresponding normal cost method is characterised by a high level of advance funding, since it substantially relies on investment returns.

Adopting a similar interpretation to the one given in Cooper and Hickman (1967) and in Economou (2003), the following classification of accrual functions is possible. We can define an actuarial cost method as increasing or decreasing according to the following rules:

Decreasing If $M''(x) = m'(x) < 0, \quad \alpha \leq x \leq R$, the resulting cost method is said to be *decreasing*, i.e. the accrual density function is decreasing with age and the cumulative accrual function decelerates. A method of this type quickly accrues the reserve, and so it heavily relies on the returns from investment. Therefore, a decreasing method is characterised by a high L_{AF} .

Increasing If $M''(x) = m'(x) > 0, \quad \alpha \leq x \leq R$, the resulting cost method is said to be *increasing*, i.e. the accrual density function is increasing with age and the cumulative accrual function accelerates. Such a method determines a late accruing of the reserve and so it has a low L_{AF} .

Further distinctions are clearly possible by recursively differentiating the accrual density function. In this specific case, it is of interest to take into account the second derivative and extend the previous classification.

Indeed, an *increasing* method can be classified as

Accelerating if $m''(x) > 0$, i.e. if the corresponding accrual density function is increasing and convex.

Decelerating if $m''(x) < 0$, i.e. if the corresponding accrual density function is increasing and concave.

In the case of a *decreasing* method ($m'(x) < 0$), the classification above is inverted: a decreasing convex (concave) accrual function determines a decreasing and decelerating (accelerating) normal cost method.

This classification extends the one given in Economou (2003), where an actuarial cost method is defined as accelerating if the density function satisfies the first derivative condition.

4.1.1 Power accrual function.

As an example, let us assume that the accrual density function $m(x)$ follows a power distribution of parameter p. According to the definition given in Johnson *et al* (1995), $m(x)$ and $M(x)$ have the following expressions, respectively:

$$m(x) = \begin{cases} p\frac{(x-\alpha)^{p-1}}{(R-\alpha)^p} & \text{if } \alpha \leq x \leq R \text{ and } p > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$M(x) = \begin{cases} \frac{(x-\alpha)^p}{(R-\alpha)^p} & \text{if } \alpha \leq x \leq R \text{ and } p > 0 \\ 0 & \text{otherwise} \end{cases}$$

By varying the parameter p, the resulting accrual density function belongs to one of the different classes introduced before:

- For $p < 1$, the resulting accrual density function is decelerating-decreasing and quickly builds up a reserve.
- For $p = 1$, the power distribution collapses on to the uniform distribution and therefore $m(x)$ is constant throughout the interval.
- For $1 < p < 2$, the resulting density function is increasing and decelerating.
- For $p = 2$, the function $m(x)$ increases linearly, and therefore the speed of increase (first derivative) is constant.
- For $p > 2$, the accrual density function is accelerating and increasing, and therefore the reserve is mainly built up with the latest contributions.

These five possibilities are considered and the corresponding accrual density and cumulative functions are displayed in the following figures 4.1. Specifically, the green line corresponds to $p = 0.5$, the red line to $p = 1$, the black line to $p = 1.5$, the yellow line to $p = 2$ and the blue line to $p = 5$:

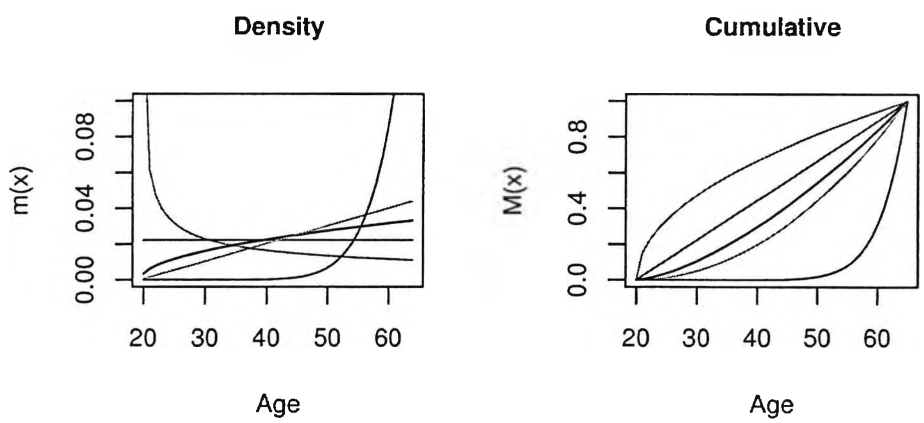
It is of interest to note that the lowest p produces a high density at early stages, which quickly reduces. The corresponding cumulative function, instead, rapidly increases and then slows down until reaching the limit equal to 1.

By contrast, the highest p determines an extremely low density, which rapidly increases by the end of the period, and therefore, the corresponding cumulative function has a similar behaviour.

4.1.2 Normal cost methods.

Through the pension purchase density function it is possible to derive a wide class of cost methods, as defined in section 2.6.1. Specifically, the normal cost to be paid

Figure 4.1: Power accrual function.



at age x and the corresponding actuarial liability accrued up to that age, can be expressed in the following way:

$$NC_x = m(x)\ PVFB_x \qquad AL_x = M(x)PVFB_x \tag{4.5}$$

These definitions are equivalent to those given in Winklevoss (1993), in which the normal cost and the actuarial liability at age x are expressed as a proportion k_x and K_x, respectively, of the corresponding $PVFB_x$[1].

Moreover, we have seen that when the parameter p is equal to 1, the power distribution describes a uniform distribution and thus the resulting density function is constant. In this case, the power distribution leads to the method known in actuarial practice as the unit credit method, see Table 2.1.

As already mentioned in section 2.6, contributions and investment returns are the sources of income funding the retirement benefit, and that the actuarial principle of equivalence needs to hold at any time. This means that in this specific model the following expression - derived from equations (4.1) and (4.2) - must hold:

[1] See Winklevoss (1993) chapter 5, page 73, for the actuarial liability; and chapter 6, page 89, for the normal costs. Similar formulae are given in Bowers et al (1976), p.183-184, as well.

$$PVFB_R = \int_{\alpha}^{R} PVFB_x m(x) e^{\int_x^R \left[\delta_u + \mu_u\right] du} dx \tag{4.6}$$

4.2 Risk of strategies.

For each member of the scheme her/his fund accrued after x years matches the theoretical value of AL_x, only if (a) the returns from the investments have exactly replicated the expectations; and (b) the demographic assumptions have been borne out by experience [2]. If at least one of the two assumptions fails, a mismatch between the fund held at time x and the $PVFB_x$ will arise.

This section focuses on describing the risk of contribution strategies, by means of analysing the distribution of this mismatch and the individual effect of different sources of uncertainty.

4.2.1 Sources of uncertainty and the accrued function.

In the previous chapter we have described the sources of uncertainty that affect the pension scheme. We have paid particular attention to the nature of the demographic variations, whilst the financial risk has been described as the risk of mismatch between the valuation rate and the rates of return.

In the continuous time model, a force of elimination μ_x describes the expected behaviour of the membership population aged x, while $\mu(x)$ represents the corresponding observed force of elimination.

In other words, μ_x and $\mu(x)$ describe the theoretical and the observed elimination due to demographic causes, such as mortality, disability and unemployment.

A model including the eliminations as a source of uncertainty will then consider $\{\mu(x)\}$ as a stochastic process. For this purpose, the model in Section 3.4 can easily be extended to the case of continuous time.

Similar considerations can be made with regard to the rates of return from investments. In particular, it is noted that no model has been specified for describing

[2] In a subsequent section we also introduce the possibility that the sponsor stops contributing in to the pension fund, as another source of uncertainty.

these returns. Chapter 6 focuses on several potential models, in order to describe the returns from investing in a portfolio of different types of assets. Moreover, the allocation of the capital into different assets adds one more degree of complexity to the investigation of the risk profile of contribution strategies. Hence, in this chapter, we consider a fixed allocation of the capital, and hence, we do not allow for using asset allocation as a tool to control the financial risk.

The sources of uncertainty affect the risk level of the contribution strategies. This risk varies according to the way in which the payments have been structured, i.e. in terms of both amounts and timing of payments.

It is of interest to illustrate two different viewpoints from which perceiving the risk of a contribution strategy: a classical actuarial view and an alternative one.

In the *classical actuarial* view, a decreasing cost method is believed to be safer than an increasing cost method. According to Collinson (2001), this is because a decreasing method builds up a substantial amount in a relatively short period of time, through payments of large amounts at the early stages of the working career of an employee [3]. In this way, a substantial reserve is quickly built up.

The risk profile of a contribution strategy is thus linked to its characteristic rate of accrual and we can summarise this *classical view* with the following rule of thumb: the earlier the funding of the reserve, the safer the strategy.

Recalling the classification of accrual density functions illustrated in section 4.1 from the classical point of view, a decreasing method is safer than an increasing one, because it determines an early accrual of the reserve. Furthermore, among the decreasing methods, the decelerating strategies are even safer than the accelerating ones.

This view of riskiness is exclusively concerned with the risk of the plan's sponsor defaulting on her/his payments and stopping contributing in to the pension scheme. Therefore, the risk of default is the only source of uncertainty accounted for, whilst valuating the risk of contribution strategies.

However, there is a pitfall in this logic. Specifically, a decreasing method (or early contribution strategy) particularly relies on the returns from investments in order to

[3]See, for instance, the definition of security provided in Collinson (2001), App 2-10.

fund the retirement benefit. Thus, if these returns are not guaranteed, as is likely to be the case, then the more a method relies on these returns, the riskier it is.

Henceforth, we propose an *alternative* view of the risk, which takes into account the stochastic nature of the investment returns. The corresponding intuitive rule is: the earlier the funding of the reserve, the riskier the strategy.

This alternative view focuses on the effects of the financial risk on the distribution of the accrued fund at retirement. Clearly, a strategy which relies on investment returns exposes the actual value of the accrued contribution to the volatility of the markets, and therefore, the resulting final accrued fund is random. Trivially, a strategy which does not involve investing in the market, does not expose any capital to the volatility of the returns, and therefore it is completely safe.

Thus, from this standpoint, an increasing method is less risky than a decreasing method, since the former builds up the reserve at a later time, relying less on investment returns. Moreover, among increasing strategies accelerating methods are the safest.

The fact that the two views of risk are in complete contrast does not mean that one is wrong and the other is correct. These two rules lead to opposite strategies, because they pay attention to two different sources of uncertainty.

Actually, the alternative view is not better than the classical one, since while attempting to account for a specific source of risk, it actually disregards the other. In fact, this view of the risk is correct, if later payments will certainly be made. This is maybe true in the case of public-sector pension schemes, where the probability of default is almost nil.

Therefore, the alternative view fails to account for the risk that the plan's sponsor may default in her/his payments. Likewise, the classical view fails to account for the risk that the actual returns from investments may be lower than expected.

In actual practice, the scheme manager is likely to find herself/himself in a situation where both risks exist. In such a situation, the risk profile of a strategy varies according to the default risk of the sponsor and to the investment strategy implemented by the fund manager.

Therefore, there is the need of a measure of risk which accounts for both the sources of uncertainty [4].

In chapter 6, the issues related to the choice of suitable financial models are considered in detail; whereas, in this chapter, the returns from investments $\{r(t)\}$ are regarded as a general stochastic process.

In the following section, two models are considered as potential approaches to the modelling of the default risk of the plan sponsor.

4.2.2 The default probability model.

As previously illustrated, the default risk plays a crucial role in the determination of the overall risk of a strategy, and for such a reason, this risk is included in the current analysis.

Let γ_x be a random indicator variable, which is equal to 1 if the sponsor is still in business at time x, while it is equal to 0 if the sponsor goes bankrupt before that time.

In order to compute the probability of such an event, we assume that the time to default is distributed exponentially with a constant parameter γ. Such a model is an extreme simplification of the hazard models introduced in Cox and Oakes (1984), which have been subsequently extended for the analysis of credit risk in recent years, see for instance Duffie and Singleton (2003). Here, we do not aim to develop an alternative model, but consider the simplest case of a constant hazard function.

Moreover, the inverse of a constant intensity is the expected number of years before defaulting, thus providing a useful interpretation of a parameter which may be hard to estimate.

So, the the probability that the sponsor defaults between time $x-1$ and x, given that at time $x-1$ she/he is still in business, is given by the following:

$$\Pr\big[\gamma_x = 0 | \gamma_{x-1} = 1\big] = 1 - e^{-\gamma} \tag{4.7}$$

Furthermore, the distribution of the default depends uniquely on the intensity of

[4]The demographic risk could be considered as well, by using in equation 4.4 a general compounding factor which includes both the force of investment and the force of elimination.

default γ and so it is independent of any other factor.

Of course, this assumption may not be entirely accurate, because the risk profile of a sponsor (in terms of probability of default) is likely to depend on its financial status and on the current economic conditions.

A potential model to represent the effect on the time to default of economic conditions is proposed in Ngwira and Wright (2004). Specifically they introduce the deviations of the annual market return from its expectation as an explanatory variable of the probability of default. Quoting the authors, the underlying idea is that *"... weak financial markets could lead to large pension schemes deficits at a time when the sponsor is financially weak and thus unable to meet with large additional scheme contributions."*

The model in Ngwira and Wright (2004) is a log logistic accelerated life model as defined in Cox and Oakes (1984), where the probability of default over one year is formulated as the following [5]:

$$Q\big(r(t)\big) = \frac{1}{1 + exp\{\pi + \lambda_Q \ \big(r(t) - \delta_t\big)\}} \tag{4.8}$$

where λ_Q is the risk premium in the market; and π is such that if the market return exactly matches δ_t, the probability of default is exactly $(1 + e^{\pi})^{-1}$.

In fact, in the Ngwira and Wright (2004) model (4.8) is the probability of sponsor's default, conditional on the event that the pension scheme is currently under-funded. Differently from the cited authors, here we consider (4.8) as the probability of sponsor's default independently of the financial status of the pension scheme, since we are interested in modelling the possibility that the sponsor stops contributing to the pension scheme.

Furthermore, here we require that the probability of default (when not affected by the financial market) is equal to the probability (4.7). Hence, the following is assumed to hold:

$$\pi = -\log\left(e^{\frac{1}{N}} - 1\right) \tag{4.9}$$

[5] Here the simple case, with no business specific components relating to the sponsor, is considered.

where $N = 1/\gamma$ is the expected number of years before default. In addition, when this condition holds, the probability (4.7) becomes the special case of probability (4.8) when λ_Q is set equal to 0.

4.2.3 Measuring the risk.

The mismatch between the accrued fund value and its expectation can be defined in several ways. In Chapter 3, we have seen how the unfunded liability provides a natural process, from which it is possible to compute a measure of risk for the whole scheme. However, in this specific case a mismatch happens at the level of an individual member (i.e. related to her/his accrued fund), and so it is convenient to identify an individual member-based underlying process, which, in consideration of the previous discussion, should also include the probability of sponsor's default.

In light of this, we consider that, at a general age y, the accrued value of normal costs (AC_y) is given by the contributions paid up to time y, and then compounded at the earned rates of return and at the observed elimination rate. In the most general case, this value is random, because the financial and demographic realisations are stochastic processes. In addition, we consider that the contribution to be paid at the beginning of the general year x may not be paid, according to whether the sponsor defaulted or not.

Specifically, the probability that each payment is made is equal to the probability that the sponsor does not default on its payments. Such a probability is expected to decrease with time.

Let us describe the accrued value at age y with the following expression:

$$AC_y = \int_{\alpha}^{y} NC_x e^{\int_x^y [\mu(u)+r(u)]du} \gamma_x dx \tag{4.10}$$

In order to evaluate the wealth of the fund at any given time, we compare this process to a benchmark provided by the level of reserve, which is the theoretical value of the member's fund; i.e. the value of the fund if the actuarial assumptions are actually realised.

For mathematical convenience we consider the ratio between the accrued value at a given age y and the corresponding AL_y [6]. Using (4.1) and (4.5), we can show that:

$$\frac{AC_y}{AL_y} = \frac{\int_\alpha^y e^{\int_x^y [\mu(u)-\mu_u]du + \int_x^y [r(u)-\delta_u]du} \ m(x) \ \gamma_x dx}{M(y)} \tag{4.11}$$

According to the terminology of life insurance actuarial mathematics, the above member-based underlying process is defined as the ratio between the actual and the expected retrospective reserves. The two stochastic processes - financial and demographic - are represented in terms of instantaneous deviations from the (logarithmic) expected values.

Fixing the time of valuation - say at time of retirement R - process (4.11) becomes a random variable. Hence, we can measure the risk of mismatch, by means of studying the distribution of this variable as a function of the implemented contribution strategy identified by the accrual density function $m(x)$.

Leaving aside the demographic risk (so $\mu(t) = \mu_t \ \forall t$) [7], we can compute a general risk measure Φ at retirement:

$$\Phi\left[\frac{AC_R}{PVFB_R}\right] = \Phi\left(\int_\alpha^R e^{\int_x^R [r(u)-\delta_u]du} \ m(x) \ \gamma_x dx\right) \tag{4.12}$$

An example of a coherent measure

In order to gain a better insight into the study of the risk of normal cost methods, it is a useful exercise to compute this measure for some basic contribution strategies. Let us assume that the measure Φ is a coherent measure in the sense of Artzner [8], without specifying the functional form of Ψ.

[6] The convenience lies in the fact that the model is in continuous time, thus using forces of interest, rather than rates. In a discrete time model, the difference between the accrued value and its benchmark may be a more convenient quantity to consider.

[7] The extension to the case of existing demographic risk is relatively straightforward and it could be implemented using the model described in Chapter 3.

[8] As illustrated in section 2.7.2, subadditivity, positive homogeneity, monotonicity and translation invariance are the properties characterising a coherent measure. For more details refer to Artzner *et al* (1999).

Terminal Funding. A terminal funding (TF) strategy requires the funding of the final benefit with the payment of a lump sum at retirement; see, for instance, Lee (1984). The value of this payment is equal to the value of $PVFB_R$, so the following is the corresponding accrual density function:

$$m_{TF}(x) = \begin{cases} 0 & \text{if } x < R \\ 1 & \text{if } x = R \end{cases}$$

Since the accrual density function is non zero only at time R, and at that time the exponential function in equation (4.12) is equal to 1, the default risk is the only source of uncertainty that can affect this strategy.

Hence, the risk measure assumes the following value:

$$\Phi\left[\frac{AC_R}{PVFB_R}\right]_{TF} = \Phi(\gamma_R) \tag{4.13}$$

Loosely speaking, the risk of this strategy is equivalent to the risk that on the day of the member's retirement the sponsor will make the payment of the lump sum in order to finance the promised benefit.

Initial Funding. The so-called initial funding (IF) strategy requires the funding of the retirement benefit with the payment of a lump sum at the beginning of the employee's working lifetime (say age α). The value of this payment is clearly equal to the value of $PVFB_\alpha$, so the following is the corresponding accrual density function:

$$m_{IF}(x) = \begin{cases} 1 & \text{if } x = \alpha \\ 0 & \text{if } x > \alpha \end{cases}$$

The risk of this strategy is dependent only on the financial risk, as the sponsor is assumed to be solvent at the beginning of the employee's working lifetime; i.e. $\gamma_\alpha = 1$. Hence, the risk of the IF strategy is given by the following:

$$\Phi\left[\frac{AC_R}{PVFB_R}\right]_{IF} = \Phi\left[e^{\int_\alpha^R [r(u)-\delta_u]du}\right] \tag{4.14}$$

Trivially, the risk of this extreme strategy is that the returns from the investments do not match the expectations, with the result that the fund accrued at retirement differs from the required amount.

Remark. It is of interest to investigate how the order of dominance changes while varying the initial assumption regarding the sources of uncertainty. Specifically, let us assume the two following extreme situations:

(1) Suppose that there is no financial risk, but the default risk exists; then the ranking anticipated by the classical view is satisfied.

In fact, if the process $\{r(t)\}$ is deterministic (and, without loss of generality, equal to the instantaneous valuation rate [9]), then in equation (4.14) the resulting exponential is equal to 1; while, the resulting default indicator function is a random variable. Therefore, the risk measure Φ would be expected to provide the following relation of dominance, because of the property of a coherent risk measure:

$$\Phi_{IF} = \Phi(1) \leq \Phi(\gamma_R) = \Phi_{TF}$$

The IF strategy is thus safer than the TF strategy.

(2) On the contrary, if there is no risk of default, but the force of return is random, than the ranking anticipated by the alternative view is satisfied.

In fact, if there is no default, γ_R is certainly equal to 1, whereas the financial risk leads the exponential in equation (4.14) to be a random variable.

Thus, the following order holds:

$$\Phi_{IF} = \Phi\left[e^{\int_a^R [r(u)-\delta_u]du}\right] \geq \Phi(1) = \Phi_{TF}$$

and according to this the TF strategy dominates the IF strategy. These two results are consistent with the intuitive arguments discussed in section 4.2.1.

[9]This assumption is made for the sake of simplicity. In fact, if $\{r(t)\}$ is deterministic but different from $\{\delta_t\}$, then the exponential in equation (4.14) would result in a constant. Hence, for the property of positive homogeneity, the analysis of the risk measure as a function of the contribution strategy $m(x)$ would be insensitive to that constant.

Half Time Funding. Let us consider a "half time" (HT) strategy, which relies on the payment of a lump sum at exactly half time between the entry age α and the retirement age R. The amount of such a payment is equal to the present value of the benefit at that time and therefore the corresponding accrual density function is given by the following:

$$m_{HT}(x) = \begin{cases} 0 & \text{if } \alpha \leq x < H \\ 1 & \text{if } x = H \\ 0 & \text{if } H < x \leq R \end{cases}$$

where $H = \frac{R+\alpha}{2}$ is half time. The corresponding risk measure is:

$$\Phi\left[\frac{AC_R}{PVFB_R}\right]_{HT} = \Phi\left[e^{\int_H^R [r(u)-\delta_u]du}\gamma_H\right] \tag{4.15}$$

and it reduces to the following values in the two cases of **(1)** no financial risk but default risk and **(2)** financial risk but not default, respectively:

$$\textbf{(1)}: \quad \Phi_{HT} = \Phi(\gamma_H) \quad \text{and } \textbf{(2)}: \quad \Phi_{HT} = \Phi\left[e^{\int_H^R [r(u)-\delta_u]du}\right] \tag{4.16}$$

Split Funding. Finally, let us consider a "split funding" (**SF**) contribution strategy, which requires the payments of half of the benefit at the beginning of the employee's career and the other half at retirement. Basically, this strategy is a combination of the (**IF**) and (**TF**) strategies, and the corresponding accrual density function has the following form:

$$m_{SF}(x) = \begin{cases} 1/2 & \text{if } x = \alpha \\ 0 & \text{if } \alpha < x < R \\ 1/2 & \text{if } x = R \end{cases}$$

The risk measure is then given by:

$$\Phi\left[\frac{AC_R}{PVFB_R}\right]_{SF} = \Phi\left[\frac{1}{2}e^{\int_\alpha^R [r(u)-\delta_u]du} + \frac{\gamma_R}{2}\right] \tag{4.17}$$

If the property of subadditivity of the risk measure is satisfied, the split strategy is less risky than a combination of the (**IF**) and (**TF**) strategies.

This risk measure assume the following values in the two cases of (**1**) no financial risk but default, and (**2**) financial risk but not default:

$$
\begin{aligned}
(\mathbf{1}):\quad \Phi_{SF} &= \frac{1}{2}\Phi\left(\gamma_{R-\alpha}\right) + \frac{1}{2}\Phi(1) \\
\text{and } (\mathbf{2}):\quad \Phi_{SF} &= \frac{1}{2}\Phi\left[e^{\int_{\alpha}^{R}[r(u)-\delta_u]du}\right] + \frac{1}{2}\Phi(1)
\end{aligned}
\tag{4.18}
$$

In both equations (4.18), the sign of equality holds (instead of the $\leq$ required by the property of subadditivity), because only one source of uncertainty exists in each case.

Remark. Assuming that only the financial risk exists, a coherent risk measure determines the following ranking:

$$\Phi_{TF} < \Phi_{SF} < \Phi_{HT} < \Phi_{IF} \tag{4.19}$$

Not surprisingly, the IF strategy is the riskiest one, since the capital is exposed to the financial risk for the longest time.

By contrast, if there is no financial risk, but the sponsor is subject to the risk of sponsor default (i.e. bankruptcy), the following ranking holds:

$$\Phi_{IF} < \Phi_{SF} < \Phi_{HT} < \Phi_{TF} \tag{4.20}$$

It is of interest to note that the SF strategy dominates in both cases the HT strategy. This relation between magnitudes of risk is expected, because the former strategy guarantees the payment of half of the value of the benefit. As a matter of fact, if there is no risk of default the final payment is guaranteed; similarly, if the return from investment is certain, the initial payment, compounded up to retirement, will constitute half of the benefit. Conversely, the HT strategy will certainly expose the fund to at least one of the two sources of uncertainty.

However, if both the risks exist, this ranking does not necessarily hold.

Simulation based analysis.

Assuming the model in §4.2.2, we aim to compare the case in which the two sources of risk act independently of each other, against the case in which they are somehow related. In order to do so, a simulations based approach is used. Details of the assumptions and the numerical figures for the parameters are in Appendix B.

As illustrated in section 4.2.2, in order to describe the dependence between the realisations in the financial market and the probability of default, we employ the model proposed in Ngwira and Wright (2004). When, instead, we do not need to include such a dependence, we assume that the time before default follows an exponential distribution, as described in section 4.2.2.

The forces of returns $\{r(t)\}$ are modelled as a sequence of *iid* random variables following a normal distribution with expectation equal to the (constant) force of discount δ. Potential approaches to more structured models of the rates of return are considered in chapter 6.

According to these assumptions, a risk measure of the loss is computed on 10,000 generated scenarios. Specifically, we use a risk measure, coherent in the sense of Artzner, of the following form:

$$\Phi(X) = \mathrm{E}\Big[1 - X|X < 1\Big] \cdot \Pr\Big[X < 1\Big] \tag{4.21}$$

where $X = \frac{AC_R}{PVFB_R}$. This measure is similar to the **MS** and the **TCE** introduced in section 2.7 (see equations (2.18) and (2.19)), however it differs from them, because it maps the ratio only if it takes a value lower than a fixed threshold (equal to 1, and which can be interpreted as the *"loss"*), instead of being defined in terms of a quantile. For this reason, the conditional expectation is subsequently multiplied by the probability that a "loss" actually occurs.

Moreover, this measure has a direct economic interpretation as the expected capital required to recover from the loss. Clearly, the higher is the measure the riskier is the strategy [10].

[10] Whether this measure really provides a reliable ordering may be subject to criticism. In fact, measure (4.21) can assume the same value in both of the two opposite situations, where the losses are (a) large but unlikely; or (b) frequent but small. In such a situation, it is not clear whether a measure

Sensitivity to financial volatility and probability of default. The following set of four graphs display the surfaces generated by the risk measure computed against the probability of no default and the volatility of returns when the two sources are independent of each other. Specifically, the cases of four contribution strategies, derived by setting $p = 0.1,\ 0.25,\ 1$ and 2 are considered [11], when the volatility varies from 0 to 10% and the probability of not defaulting varies roughly between 1% and 90% [12].

In all of these figures, the risk has its minimum in correspondence to the lowest probability of default and the lowest volatility of returns, independently of the employed strategy. Clearly the risk increases as the sources of uncertainty strengthen.

The main information conveyed by these figures is that, as the L_{AF} increases (i.e. earlier funding), the risk becomes **more** sensitive to the volatility of the returns, and **less** sensitive to the probability of default of the sponsor. These results are expected in the sense that they are consistent with the intuitive arguments considered in section 4.2.3.

In fact, for $p = 0.1$ - Figure 4.2(a), the surface increases with the volatility, regardless of the value of the probability of not defaulting. In Figure 4.2(b), this increasing trend is not as marked as in the previous figure, especially when the default is very likely to happen.

Furthermore, as the level of advance funding decreases (p increases) below a certain threshold, the risk measure becomes more sensitive to the risk of default than to the risk of negative financial performances. In particular, for $p = 1$, an increase in the volatility of the returns is not necessarily followed by an increase in the risk measure; in fact, the risk measure can actually decrease when the probability of default is extremely high.

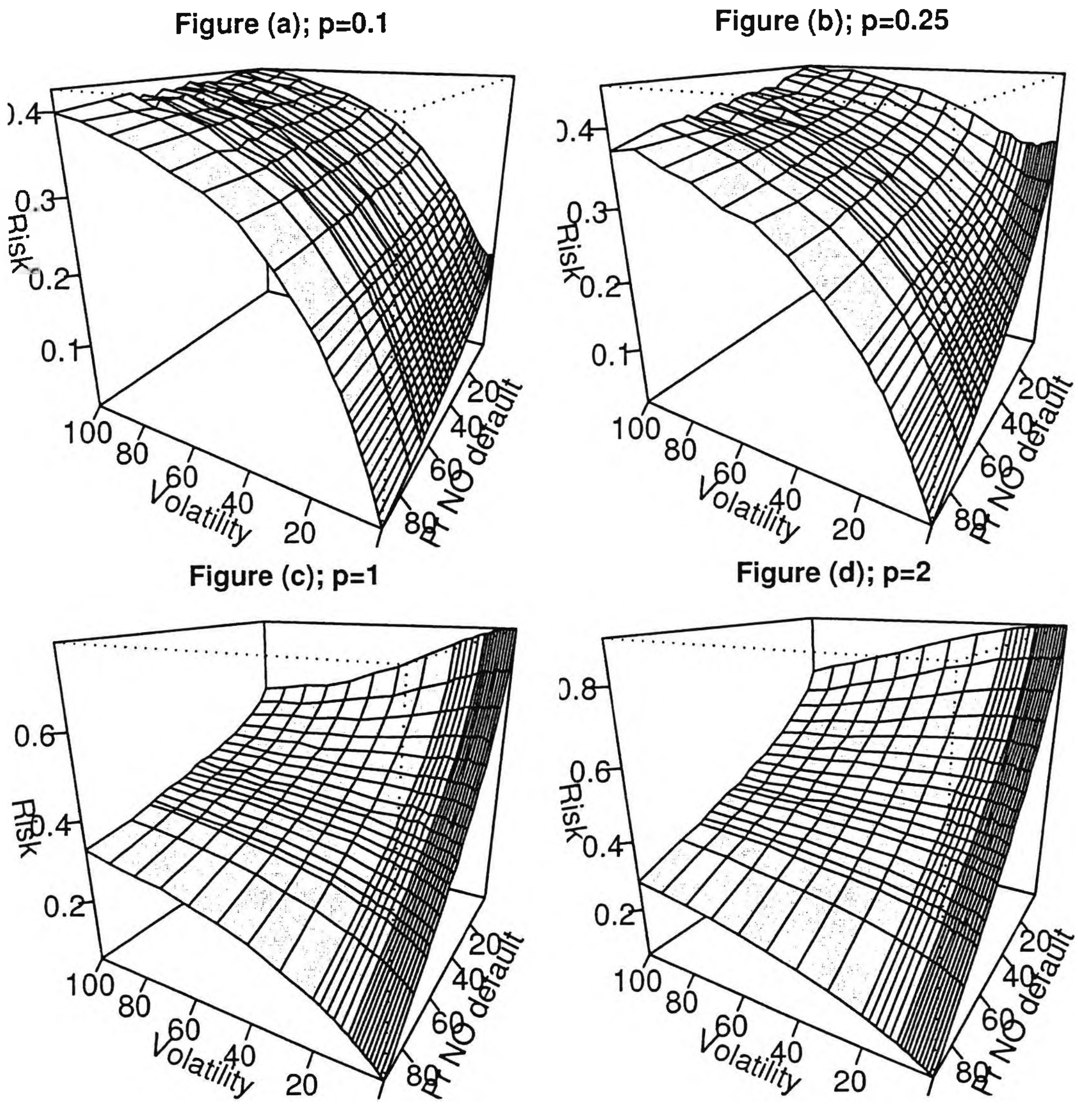
---

based on expectations - as in this case - really reflects the risk preferences of an economic agent. An alternative measure, overcoming this problem, is the semi-variance as defined in Markowitz (1991). Such a measure, however, is mathematically intractable and has a difficult interpretation.

[11] For $p > 2$, that is for accelerating increasing methods, the resulting surfaces look very similar to the case $p = 2$, although they are shifted to different values on the $y-$axis. For this reason, cases for $p > 2$ are not considered.

[12] In an exponential model, these probabilities correspond to the range from 1 to 500 expected years before default.

Figure 4.2: Risk sensitivity to sources of uncertainty. Probability are in percentages and volatility is expressed per thousands.



This apparently counterintuitive result has a logical explanation. In this instance, the volatility of the financial market is modelled by means of assigning a specific value to the standard deviation of the log-normal distribution used to simulate the rates of returns. Thus, the larger is the standard deviation, the higher is the volatility and, as a consequence of that, the more likely are both high and low returns.

Therefore, if the value of p is sufficiently large (say $p \geq 1$) and the probability of default is extremely high ($\geq$ 95%), then increasing the volatility of returns has a two fold effect on the risk measure (4.21): on one hand, it *reduces the probability* of a "loss" (value less than 1), because very high returns are possible although very unlikely; on the other hand, high volatility *increases the conditional expectation* of this "loss", i.e. the capital required to make up for the missing fund.

When we aggregate these two reactions, we realise that a more volatile financial market has the overall effect of reducing the risk measure, because the reduction in the probability overwhelms the increase in the conditional expectation.

For instance, in graph (d) ($p = 2$, low L_{AF}) corresponding to a high probability of default (95%), the surface declines from the top right corner (0.83) to the left top corner (0.66) as the volatility increases. This happens because the probability of a "loss" decreases from 95% to roughly 76% when the volatility increases from 0 to 10%.

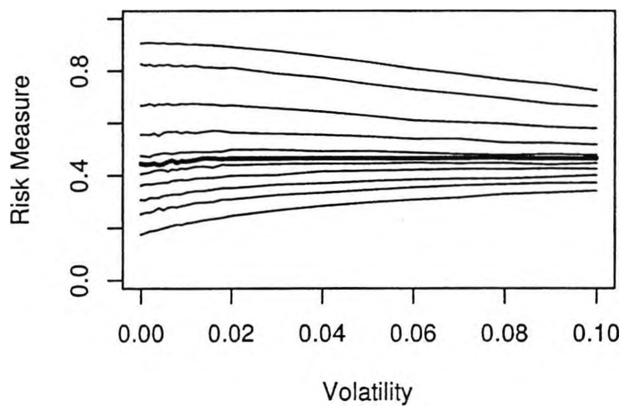
Bearing these considerations in mind, it is worthwhile asking whether - given the probability of default of the sponsor - a level of advance funding exists such that the risk measure is insensitive to the financial volatility.

The next figure displays 4.2(d) projected on to a two-dimensional plane.

The L_{AF} corresponding to $p = 2$ (red thick line) is such that, when the probability of default is equal to 63% (red thick line), the risk measure is (almost) insensitive to the volatility of the returns (the red thick line is almost a straight horizontal line). A linear regression of the risk measure against the values of the volatility returns a gradient not significantly different from 0 (-0.003756), with a p-value roughly equal to 94%.

The following table 4.1 displays, for a set of values of the probability of default, the levels of advance funding that make the risk measure insensitive to the financial

Figure 4.3: Risk insensitivity: volatility *vs* risk measure. L_{AF} $p = 2$.



risk [13].

Table 4.1: Risk insensitivity.

Prob default	LAF ~ p
99%	0.375
95%	0.475
83%	0.725
72%	1
68%	1.4
63%	2

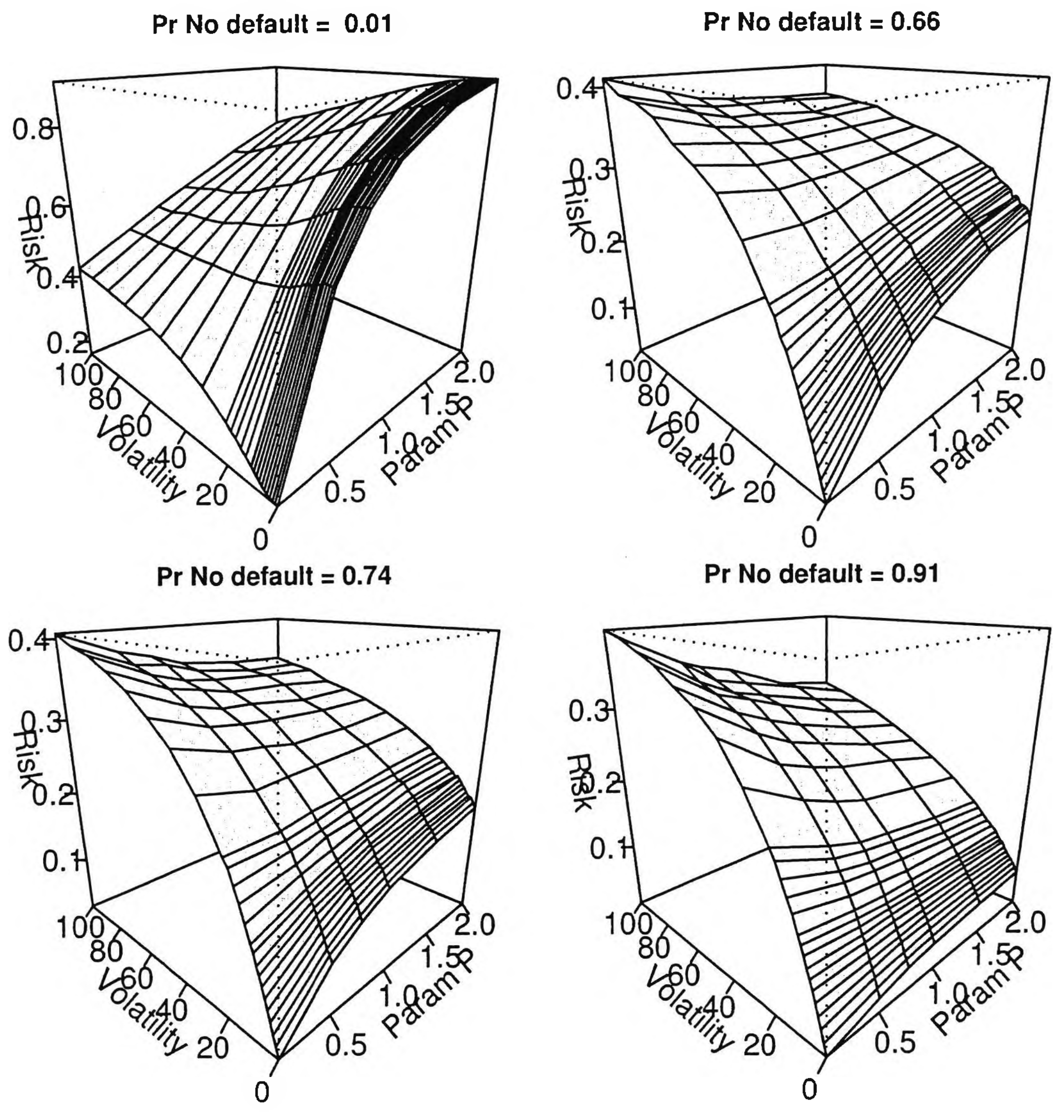
According to table 4.1, if, for instance, the probability of default is estimated around 68%, the strategy with $p = 1.4$ is almost insensitive to the financial volatility.

The set of graphs in Figure 4.4 display the sensitivity of risk when the probability of default is fixed. Specifically, we choose 4 possible values for this probability, in order to display how the surface changes when the probability of default decreases. From this perspective, it is possible to illustrate how to combine a specific L_{AF} with

[13] In particular, if the risk measure plotted against the volatility has a gradient non significantly different from zero with a p-value greater than 50%, the strategy is considered insensitive to the volatility.

the volatility of the financial uncertainty, in order to achieve an acceptable level of risk, given the sponsor's probability of default.

Figure 4.4: Risk sensitivity to financial volatility and p



The risk surface rotates around the diagonal from the origin to the coordinates of maximum volatility and maximum p (lowest L_{AF}).

It can be seen also from this perspective that, when the default is almost certain (probability of no default = 1%), the risk measure tends to be indifferent to the volatility of the investments.

However, in the determination of the risk measure, the lower is the probability of default, the more significant is the volatility of returns.

In fact, when the probability of no default is very low (9%), the volatility of returns plays the major role. As previously illustrated, in this situation it is possible that a higher volatility may imply a lower risk measure.

Sensitivity to the dependence between investment return and probability of default. In order to investigate the difference between the results obtained when the probability of sponsor's default is dependent to the investment returns, and when is not, the following analysis is carried out. The two figures 4.5(a) and 4.5(b) display the surface of the ratio of the two risk measures generated when the two sources of uncertainty are dependent and independent.

The model described in section (4.2.2) is considered for the two cases of $\lambda_Q = 4.5\%$ (as used in Ngwira and Wright (2004)) and $\lambda_Q = 0$. By construction, when the model of dependence determines more risk than the the model of independence, the resulting ratio is higher than 1, and hence, the surface (red area) lies above this level.

The two cases of high and low L_{AF} are considered in Figure 4.5 (a) and (b), respectively, when the volatility of the returns varies between 0 and 10% and the probability of not defaulting lies in the range 10% and 99%.

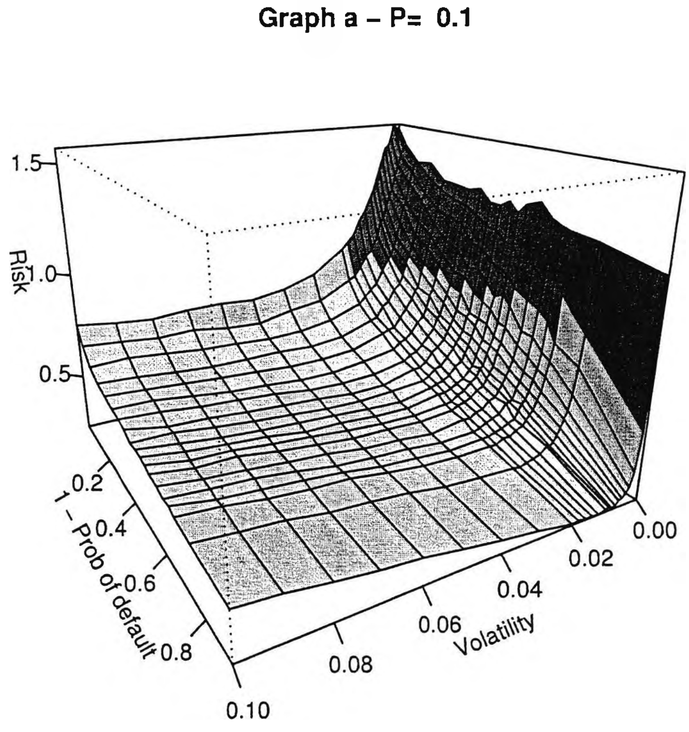
Based on these cases we make the following remarks

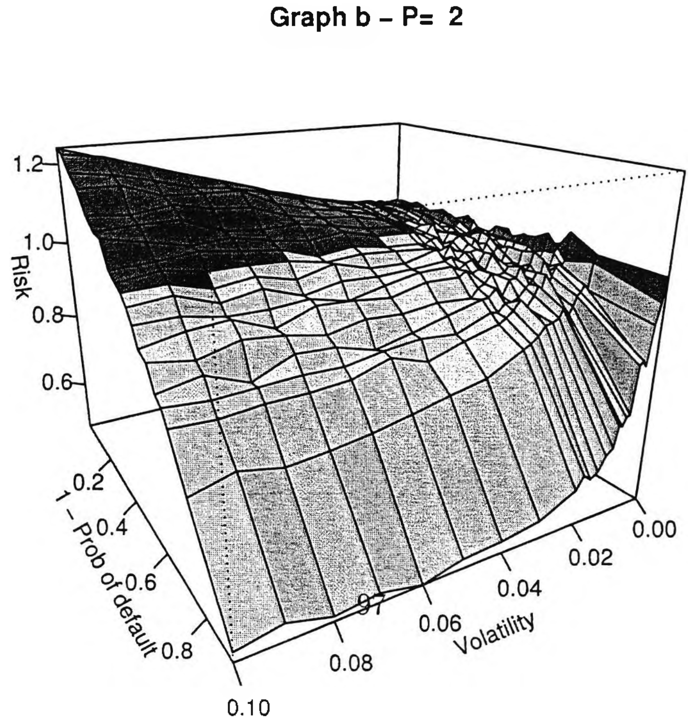
- In graph (a) the surface is not very sensitive to the probability of default. This happens because $p = 0.1$ generates an early contribution strategy heavily relying on investment returns and fairly insensitive to the probability of default.

  Conversely, graph (b) is much more dependent on the default probability than on the volatility of the market. Again, this is due to the level of advance funding implied by the parameter p.

- In graph (a) the maximum occurs at the origin for volatility. Moreover, when

Figure 4.5: Ratio of risk measures from the dependent model to the independent model.





this volatility is low the surface is constantly over 1, regardless of the probability of default. This suggests that when a safe financial strategy is implemented, the dependence between the two sources of uncertainties generates some extra risk. Hence, if an early contribution strategy is implemented, and thus, the financial risk is the most relevant, then volatility is more dangerous when there is independence, and not dependence, between the two sources of uncertainty.

In graph (b) the maximum is at the point of maximum volatility and highest probability of default, thus showing that a late contribution strategy is even riskier if the sponsor is likely to default in a world where the investment returns are going to affect (or are correlated with) the probability of such an event. Hence, if a late contribution strategy is implemented, and thus, the default risk is the most relevant, then dependence, rather than independence, makes the default risk even more dangerous.

- In general, this analysis corroborates the hypothesis that when the two sources of uncertainty act independently of each other, the overall level of risk that results may be extremely different from the risk generated by two correlated sources. However, Figure 4.5 suggests that whether this level of risk results higher or lower depends on the combination of contribution and investment strategies implemented.

4.3 Cost of strategies.

In this section we argue that a comprehensive analysis has to take into account both measures of risk as well as the actual cost of a sound risk management strategy.

As we have seen, it is possible to derive a risk-based ordering of contribution strategies. However, such a ranking does not necessarily match the preferences of the sponsor.

In fact, when choosing the contribution strategy to adopt, the sponsor takes into account other aspects in addition to its characteristic risk. One of these aspects is the actual cost of implementation - Owadally (1999) refers to this as the long term cost of pension provision; see also Daykin (1999).

In the environment described by the current model, inflation is fixed and all of the amounts are expressed net of price inflation. Hence, the sum of the normal costs may provide a suitable measure of the cost of a contribution strategy. Thus, the cost of a strategy is the proportion of the final benefit funded by the contributions.

However, since not all the members are expected to survive up to the retirement age, the expected cost is instead taken into account:

$$\begin{aligned}\mathrm{E[Cost]} &= \int_{\alpha}^{R} NC_x e^{-\int_{\alpha}^{x} \mu_u du} dx = \\ &= PVFB_{\alpha} \int_{\alpha}^{R} m(x) e^{\int_{\alpha}^{x} \delta_u du} dx \end{aligned} \tag{4.22}$$

By substituting the normal costs NC_x, as in equation (4.5), we derive the second expression for the expected cost of a strategy. Equation (4.22) leads to the natural interpretation of the expected cost as the complement of the level of advance funding.

The equivalent discrete version of the cost is given by the following:

$$E\big[\mathrm{Cost}\big] = \sum_{x=0}^{T} NC_{\alpha+x}\ {}_{x}p_{\alpha} \tag{4.23}$$

where ${}_{x}p_{\alpha}$ is the probability that an individual aged α survives for x years, ${}_{x}p_{\alpha} = e^{-\int_{\alpha}^{x} \mu_u du}$, and the annual premium NC_y is computed as following:

$$NC_y = \int_{y}^{y+1} PVFB_x m(x) dx \tag{4.24}$$

Clearly, by artificially setting the survival probability equal to 1 at all ages, the measure given by the expected cost would collapse to the simple total cost.

Financial risk. In the previous section we have seen that, if the financial risk is the only source of uncertainty, the higher is the level of advance funding of a strategy, the higher is its risk.

In this specific case, we can state the following intuitive rule: the more expensive is the strategy, the lower is the risk.

This is because of the timing of a contribution strategy: the earlier is the funding of the pension (which means that the pension is mainly funded with the contributions at early ages), the less expensive is the strategy. This happens because an earlier contribution strategy relies more on investment returns, than a later one would do. On the other hand, an earlier strategy would expose the accrued fund to the volatility of all the existing risks for a longer period.

Therefore, the relationship between the level of funding and the cost leads the risk and the cost itself to be inversely related. So, even though a strategy may optimally reduce the risk, its cost might be excessively high for the plan sponsor. Hence the plan sponsor might not want to achieve the safest position. Nevertheless, it might prefer to implement a more convenient strategy, for which the resulting level of risk is still acceptable. Clearly, in the decision-making process of choosing a suitable contribution strategy, a trade off between cost and risk arises.

Default risk. If the risk of default is the only source of uncertainty, then the higher the level of funding of a strategy, the lower its risk. In this specific case, a trade off between risk and cost does not arise and the safest strategy dominates all the others in term of risk/cost efficiency, since it is the cheapest strategy as well.

In the most general case, where both sources of uncertainty exist (and perhaps interact), it is likely that a risk/cost trade off arises.

Hence, with the objective of exploring tools to reduce a DB pension scheme risk, it appears crucial to include in the model a variable which takes into account the cost of contribution strategies. In the following chapter, we devise a methodology to derive optimal contribution strategies, in the sense that they minimises a give risk measure, taking also into account the cost of pension provision.

4.4 Summary.

In this chapter the classification of normal cost methods based on the accrual density function as introduced in Cooper and Hickman (1967) is adapted to define the level of advance funding of a contribution strategy. Using this function, we also interpret the normal cost methods in terms of the risk that, for each member of the scheme, the fund accrued at retirement differs from the expected value of the benefit.

With the aim of proposing a ranking of the normal cost methods, we separate the effects of the financial risk and of the risk of the sponsor defaulting in its payments, as well as showing the way to include the demographic risk as a third source of uncertainty. We illustrate how the two risks have contrasting effects on the level of risk of a given contribution strategy. Moreover, the normal cost methods are ranked according to an alternative view of risk, which takes into account the stochastic nature of the returns from investments.

The accrual density function is also used to define the cost of implementation of any given contribution strategy. Furthermore, the tradeoff between risk and cost is identified in the case of existing financial risk. Conversely, we show that, when the risk of default is the only source of uncertainty, the cheapest strategy is also the safest one and therefore the above mentioned tradeoff does not arise.

The case in which the source of uncertainty are correlated is also explicitly taken into account. Specifically, we show that the presence of dependence may determine, in some cases, a lower risk if an appropriate contribution strategies is implemented.

Chapter 5

Optimal contribution strategies

Chapter 4 provides definitions of the risk and the cost of a contribution strategy, and illustrates that a tradeoff between the two exists. In this chapter, we investigate whether an efficient frontier can be identified.

Since the cost and the level of advance funding are complementary characteristics of a contribution strategy (knowing one of these is equivalent to know the other); and since risk and level of advance funding do not share the same relation; strategies having the same cost may have different risk [1].

Starting from this point here, we aim in this chapter to develop a methodology to identify, amongst the strategies having the same cost, the contribution strategy which minimises this risk.

Ideally, the devised methodology needs to be able to deal with both the risk of individual contribution strategies (as defined in chapter 4); and with the more general case of a scheme-aggregated risk measure.

Therefore, the intention is also to derive the optimal contribution strategies, which minimise the risk of mismatch between the assets and the liabilities of the whole scheme, for different level of cost of pension provision.

Among the several potential measures that can be used to quantify the risk, we choose to use the variance mainly because of mathematical convenience. In fact, it is possible to derive a general closed expression for this measure, thus allowing

[1] Refer to appendix A.4 for an example.

the implementation of optimisation techniques in order to provide analytical and numerical solutions. Further, since the variance is a quadratic function in the normal costs, the problem of identifying optimal strategies can be solved with constrained quadratic programming.

The major drawback of this risk measure is indeed its quadratic form, which makes no distinctions between positive and negative deviations from the mean.

From this point of view, downside risk measures could be more appropriate than the variance. In fact, these measures give more importance to the losses than to the gains, and, especially those satisfying Artzner's axioms of coherence, have desirable properties as well as a convenient economic interpretation. In chapter 4 we have made full use of these measures in order to illustrate the risk of a contribution strategy. Instead, since in this chapter we aim to derive analytical results, mathematical tractability is an important criterion to take into account while choosing the risk measure to implement. From this perspective, coherent measures often lead to problems which are rather complicated to deal with and, in some cases, not mathematically tractable. It is of interest to note that Rockafellar and Uryasev (2000) (and following papers) have used the CVaR to measure the risks in problems of optimal asset allocation, by means of including this measure in numerical routines. [2].

5.1 The optimal problem.

The optimal problem consists in identifying the contribution strategy which minimises the variance of a random variable, which will shortly be introduced. Specifically, this random variable shall describe either the accruing process of an individual member retirement benefit (as described in chapter 4); or the funding process of the pension benefit of the whole membership population (as described in section 3.5). Moreover, the optimal problem needs to take into account the cost of pension provision.

[2]Refer to the original paper Uryasev (2000) for the definition of the Conditional Value at Risk (CVaR).

Setting the objective function. This section aims to set this optimal problem in a general framework, where the underlying process is not specified. Hence, we let $\{X_t\}$ indicate a general underlying process, which we aim to control through the choice of a suitable set of parameters $\mathbf{Y}' = ([Y_1, ..., Y_R])$. In the development of this work, the process $\{X_t\}$ will identify the (individual or aggregate) pension funding process and the set of parameters $\mathbf{Y}$ will represent the set of normal costs, or contribution strategy.

Since $\{X_t\}$ is meant to be representative of the funding status of a (individual or aggregate) pension scheme, such a process is affected by the inflows of the contributions and outflows of the benefits. In order to define the objective function, we focus on the variance of the process $\{X_t\}$ at a specified point in time. In addition, we proceed to decomposing this variance into terms representing the variability of the two main components of inflow and outflow streams, as follows:

$$\begin{aligned} \text{Var}\big[X\big] = & \text{ Var}\Big\{\text{Inflows - Outflows}\Big\} = \\ = & \text{ Var}\Big\{\text{Inflows}\Big\} - 2\text{Cov}\Big\{\text{Inflows, Outflows}\Big\} + \text{Var}\Big\{\text{Outflows}\Big\} \end{aligned} \quad (5.1)$$

Since the process $\{X_t\}$ is to be controlled through the contribution strategy, we factorise the set of variables $\mathbf{Y}$ from the first two components in equation (5.1). The third component, $\text{Var}\Big\{\text{Outflows}\Big\}$, is not affected by the contribution strategy, as it exclusively describes the outflow process. For the same reason, the objective function shall only contain those components of the variance of X which are affected by the contribution strategy. More formally, let us assume the following:

Assumption. The variance of the underlying process X can be expressed in the following quadratic form:

$$\text{Var}\big[X\big] = \mathbf{Y}'\Sigma_X\mathbf{Y} - 2\mathbf{Y}'\psi \quad (5.2)$$

where Σ_X is the variance/covariance matrix of the process of inflows; ψ is a vector summarising the covariance in equation (5.1) and, $\mathbf{Y}$ is the vector of the normal costs $[NC_\alpha, ..., NC_{R-1}]'$ paid at each age.

Equation (5.2) is the objective function to be minimised, with respect to the contribution strategy identified by the set of variables $([NC_\alpha, ..., NC_{R-1}])$. Moreover, the optimal solution minimises the variance of X as well.

Constraints on the control variables. The minimisation of (5.2) is however bounded by a set of constraints that the optimal solution must satisfy. First, it is required that, if the financial and demographic expectations are actually realised, the contribution strategy must fund the defined retirement benefit. In other words, the space of the feasible solutions is restricted to the strategies that satisfy the principle of actuarial equivalence. Second, it seems reasonable to assume that the contributions are non-negative.

This requirements find their mathematical representation in the form of the following linear constraints:

$$\mathbf{NC'E} = PVFB_\alpha \tag{5.3}$$

$$\mathrm{NC}_k \geq 0, \quad \forall k = \alpha, \ldots, \alpha + \tau = R - 1 \tag{5.4}$$

where the k^{th} component of the vector $\mathbf{E}$ is the pure endowment term ${}_{k-\alpha}E_\alpha = e^{-\int_\alpha^k (\delta_u + \mu_u) du}$, which actuarially discounts for $k - \alpha$ years the contribution paid at age k.

Since a direct relation between the normal costs and the accrual density function exists (refer to equation (4.5)), these two constraints have equivalent counterparts in terms of the accrual function. Specifically, here we consider the discrete case, whereby the (discrete) accrual factors are obtained by integrating the the accrual function in each age interval. Thus, mathematically:

$$m_k = M(k+1) - M(k) = \int_k^{k+1} m(z)dz \geq 0, \quad \forall k = \alpha, \ldots, R-1 \tag{5.5}$$

$$\mathbf{m}'\mathbf{1} = 1 \tag{5.6}$$

where $\mathbf{1}$ is a vector of ones and is of the same dimension of the vector $\mathbf{m}$, which is defined as $\mathbf{m} = [m_\alpha, \ \ldots, \ m_{R-1}]$.

Including the cost. As illustrated in section 4.3, it is necessary to include the cost in this analysis, and this can be done in several ways. One of these would require to include the cost as a strict constraint to the minimisation problem.

Alternatively, it is possible to define and subsequently minimise a penalty function, which gives increasing importance to the cost measure according to a scaling factor.

Finally, a multiple objective function minimisation could be carried out, with the effect of simultaneously minimise risk and cost.

The work developed here is mainly based on the first approach, although the second one is also introduced. In fact, we show that the two approaches require the same techniques and that, under particular assumptions, they are equivalent and lead to the same results.

The third approach would indeed require different techniques and it could be an interesting extension for further research [3].

As illustrated in the previous section 4.3, the expected cost is a suitable measure and so we included it in the minimisation problem as the following strict constraint:

$$K = \mathrm{E}\Big[\mathrm{Cost}(\mathbf{NC})\Big] = \mathbf{NC}'\mathbf{P} \tag{5.7}$$

where vector $\mathbf{P}$ indicates the probabilities of surviving up to the retirement age: so that $\mathbf{P} = [{}_{\tau}p_\alpha, \ {}_{\tau-1}p_{\alpha+1}, \ \ldots, \ {}_{1}p_{\alpha+\tau-1}]'$. If all the elements of the vector $\mathbf{P}$ are

[3] A comprehensive review of some of the potential algorithms is illustrated in Rustem (1998). Particularly interesting is chapter IV, which deals with optimisation problems in uncertainty. An interesting application has been developed in Guo and Huang (1996) in the field of asset allocation within a fuzzy set theory framework.

artificially set equal to 1, then the resulting measure is the simple total cost of the contribution strategy.

Henceforth, the optimisation problem consists in finding the minimum of equation (5.2), subject to the constraints (5.3), (5.4) (or equivalently (5.5) and (5.6)) and (5.7).

5.1.1 Lagrangian conditions and solution.

Unfortunately, including the non-negativity constraints increases significantly the computational complexity of the system to be solved, leading the constrained quadratic problem to be, in the first place, a combinatorial problem.

For this reason we approach the problem by temporarily excluding the non-negativity constraints, on the ground of the following (non-rigorous) remark.

Remark. One consideration concerns the sign of the elements involved in the minimisation. We have assumed that the quadratic form in equation (5.2) is the variance of a general underlying process X and hence the value in equation (5.2) is always non-negative.

Hence, during the process of minimisation, *if all the elements of* ψ *are non-negative*, the quadratic term will push the variables to be zero, while the term of order 1 should push them to be positive. Since the minimum value of the variance of X must be positive, it seems reasonable to assume that the optimal solution will lie in the non-negative subspace.

In the light of this discussion, this alternative approach could be worthwhile.

Hence, if the assumption on the elements of ψ is satisfied, the non-negativity constraints can be ignored.

Thus, associated the Lagrangian function associated to the above constrained optimal problem has the following expression:

$$L(\mathbf{NC}, \lambda, \nu) = \mathbf{NC}'\Sigma_X\mathbf{NC} - 2\mathbf{NC}'\psi - \lambda\Big(\mathbf{NC}'\,\mathbf{E} - PVFB\Big) - \nu\Big(\mathbf{NC}'\mathbf{P} - K\Big) \quad (5.8)$$

Hence, the optimal normal costs sequence can be found as the solution of the following system:

$$\begin{aligned}
\frac{\partial L(\mathbf{NC})}{\partial \mathbf{NC}} &= 2\Sigma_X \mathbf{NC} - 2\psi - \lambda \mathbf{E} - \nu \mathbf{P} = 0 \\
\frac{\partial L(\mathbf{NC})}{\partial \lambda} &= \mathbf{NC}'\ \mathbf{E} - PVFB = 0 \\
\frac{\partial L(\mathbf{NC})}{\partial \nu} &= \mathbf{NC}'\ \mathbf{P} - K = 0
\end{aligned} \tag{5.9}$$

This problem consists of $(\tau+3)$ equations in $(\tau+3)$ variables and has the following unique solution:

$$\mathbf{NC}^* = \Sigma_X^{-1}\psi + \frac{\lambda^*}{2}\Sigma_X^{-1}\mathbf{E} + \frac{\nu^*}{2}\Sigma_X^{-1}\mathbf{P} \tag{5.10}$$

with multipliers

$$\lambda^* = \frac{\mathbf{E}'\Sigma_X^{-1}\mathbf{P}\left(\psi'\Sigma_X^{-1}\mathbf{P} - K\right) - \mathbf{p}'\Sigma_X^{-1}\mathbf{P}\left(\psi'\Sigma_X^{-1}\mathbf{E} - PVFB\right)}{\mathbf{E}'\Sigma_X^{-1}\mathbf{E}\ \mathbf{p}'\Sigma_X^{-1}\mathbf{P} - \left(\mathbf{E}'\Sigma_X^{-1}\mathbf{P}\right)^2} \tag{5.11}$$

$$\nu^* = K - \lambda^* \frac{\mathbf{E}'\Sigma_X^{-1}\mathbf{P}}{\mathbf{p}'\Sigma_X^{-1}\mathbf{P}} - \psi'\Sigma_X^{-1}\mathbf{P} \tag{5.12}$$

Similar problems are tackled and solved in Schmidt (2003).

Remark. An interesting remark arises from the interpretation of the Lagrangian multiplier λ^*. If the right hand side of equation (5.3) is increased by a small amount Δb, the minimum value of (5.2) will increase by $\lambda^*\Delta b$. This means that the effects on the variability of X due to changes in the benefit policy, and/or in the actuarial assumptions, can be exactly quantified.

5.1.2 Penalty function approach.

As an alternative to the illustrated approach, it is possible to identify the optimal contribution strategy as the solution which minimises an "ad hoc" objective function.

The rationale behind this alternative approach is that, instead of bounding the space of the possible solutions, it is possible to minimise an objective function consisting of the original risk measure, and a number of functions which assign penalties for the unfeasibility of the solution.

A new objective function is thus created by adding two penalty functions, Λ and N, for the two constraints of the actuarial equivalence of the cost of contribution strategy [4].

The resulting optimal problem is given by the following:

$$\min_{\mathbf{NC}} \mathbf{NC}'\Sigma_X\mathbf{NC} - 2\mathbf{NC}'\psi + \Lambda(\mathbf{NC},\ \mathbf{E}) + N(\mathbf{NC},\ \mathbf{P}) \tag{5.13}$$

As it is noted in the literature, such an approach may have considerable computational drawbacks[5]. The main pitfall is that the two penalty functions (Λ and N) need to be adequately calibrated in order to ensure that sufficient penalty is given for unfeasibility. In this specific case, the actuarial equivalence constraint is a strict requirement and the unfeasibility of the solution is not admissible. Therefore, once the function Λ is chosen, an exact calibration is needed in order to make sure that the constraint is satisfied.

On the other hand, this approach has some conceptual advantages. First, using a penalty function approach allows for nonlinear non-convex constraints. Thus, a wide range of measures of variability of contributions can easily be added to the optimal problem.

Second, this approach allows for a convenient interpretation when it comes to taking into account the cost of a strategy. In fact, increasing the cost penalty is equivalent to giving, in the minimisation process, more importance to the cost and less to the risk. This means that, for example, a higher cost penalty leads to a riskier

[4]In light of the previous discussion, the non-negativity constraints are here ignored, although they could be included in this approach as well.

[5]Refer to Gill *et al* (1981) for a fuller discussion on practical optimisation techniques.

and cheaper optimal contribution strategy. Thus, in order to identify an efficient frontier in the tradeoff between cost and risk, there is no need to specify a set of acceptable values for the cost, but it is sufficient to let the cost penalty vary between two extremes.

Moreover, for particular penalty functions, the first and second approaches are equivalent, in the sense that they lead to the same optimal solution. Let us consider the case that the two penalty functions have the following linear form:

$$\Lambda(\mathbf{NC},\ \mathbf{E}) = \lambda\, \mathbf{NC}'\mathbf{E} \quad \text{and} \quad N(\mathbf{NC},\ \mathbf{P}) = \nu\, \mathbf{NC}'\mathbf{P} \tag{5.14}$$

where the parameters λ and ν, acting as scaling factors, emphasise the importance of the present value and of the cost during the minimisation. As mentioned before, the function Λ must be adequately calibrated in order to satisfy the constraint on the present value. Specifically, if the parameter λ is set equal to the opposite of the Lagrangian λ^* in (5.11) (so $\lambda = -\lambda^*$, which is uniquely determined once ν has been chosen), the optimal solution strictly satisfies the principle of actuarial equivalence.

With regard to the cost, for $\nu = 0$ the problem collapses to minimising the variance of X. On the contrary, for ν sufficiently large ($\nu \to \infty$) the problem will eventually minimise the cost, regardless of the risk.

Furthermore, for each value of ν there is a value of K such that the solution of problem (5.13), when penalty functions in 5.14 is also solution of the Lagrangian problem in equation (5.8). This means that the two approaches are equivalent and they lead to the same optimal solution. This K is uniquely determined from equation (5.12) and taking into account the positive sign in equation (5.13), once the value of the cost penalty factor ν has been fixed. In formula:

$$K = \lambda^* \frac{\mathbf{E}'\Sigma_X^{-1}\mathbf{P}}{\mathbf{p}'\Sigma_X^{-1}\mathbf{P}} + \psi'\Sigma_X^{-1}\mathbf{P} - \nu \tag{5.15}$$

This expression illustrates how fixing a high value for ν in the problem with the penalty function is equivalent to set a small value for the expected cost K in the original problem. In both cases, the minimum variance will be fairly high, since the contribution strategy is not expensive.

Equation (5.15) also suggests that, in order to increase to importance of cost in the minimisation there is no need to let ν up to infinity. In fact, the expected cost K must be positive and has a minimum equal to the expected cost of the cheapest strategy, i.e. the **IF** strategy. Hence, ν must vary between 0 and $\lambda^* \frac{\mathbf{E}'\Sigma_X^{-1}\mathbf{P}}{\mathbf{P}'\Sigma_X^{-1}\mathbf{P}} + \psi'\Sigma_X^{-1}\mathbf{P} - K_{IF}$.

In appendix A.5, the equivalence between the two approaches is demonstrated in the simpler case of only one constraint. Although, in this specific case of an optimal strategy problem we have two constraints, the peculiarities of the problem are such that the two approaches are still equivalent and the linear relationship between ν and K holds. In fact, the penalty factor λ has to be set equal to the Lagrangian λ^* (equation (5.11)), in order to satisfy strictly the constraint represented by the principle of actuarial equivalence. Therefore the case considered in appendix A.5 is equivalent to the current problem.

Further, in the more general case, where the parameter λ is free to assume any value, the two approaches still lead to the same solution, but the equivalent cost K is not uniquely determined by the the cost penalty factor ν, but varies according to the values of both the penalty factors ν and λ.

In conclusion, under the assumption of linear penalty functions and setting the penalty factor λ equal to the Lagrangian coefficient λ^*, the two approaches are equivalent.

5.1.3 Non-negativity and Kuhn-Tucker conditions.

As mentioned above, it is reasonable to assume that optimal contributions are all positive. Hence, we can add the non-negativity constraints to the optimal problem solved by finding the solution to the Lagrangian conditions. The Lagrangian function including the non-negativity constraints has the following expression:

$$\begin{aligned} L(\mathbf{NC}, \lambda, \mu) &= \mathbf{NC}'\Sigma_X\mathbf{NC} - 2\mathbf{NC}'\psi + \\ &\quad -\lambda\Big(\mathbf{NC}'\,\mathbf{E} - PVFB\Big) - \nu\Big(\mathbf{NC}'\mathbf{P} - K\Big) - \mathbf{NC}'\mu \end{aligned} \tag{5.16}$$

The Kuhn-Tucker (**K**|**T**) theorem provides necessary and sufficient conditions for a point to be optimal while satisfying the imposed constraints [6]. In this specific case, if an optimal solution $\mathbf{NC}^*$ exists, then there must exist multipliers λ, ν and $\mu_0, \ldots, \mu_\tau$ satisfying the following K|T conditions.

$$\frac{\partial L(\mathbf{NC}^*)}{\partial \mathbf{NC}} = 0 \tag{5.17}$$

$$\mathbf{NC}^{*\,\prime}\,\mathbf{E} - PVFB = 0 \tag{5.18}$$

$$\mathbf{NC}^{*\,\prime}\,\mathbf{P} - K = 0 \tag{5.19}$$

$$\mu_k\, NC^*_{\alpha+k} = 0 \qquad k = 0, \ldots, \tau \tag{5.20}$$

$$\lambda \geq 0 \tag{5.21}$$

$$\mu_k \geq 0 \qquad k = 0, \ldots, \tau \tag{5.22}$$

In addition, since the variance is a convex function of the normal costs, and so are the constraints (actually linear), the K|T theorem state that any point $\mathbf{NC}^*$ satisfying the above conditions is an optimal solution of the problem.

Remark. The system of equation above shows how the optimal problem including non-negativity constraints is a combinatorial problem in the first place. Indeed, we must consider all of the possible $2^{\tau+1} - 1$ cases, in which alternatively some or all the multipliers μ_k $(k = 0, \ldots, \tau)$ might be null.

A possible way to deal with the problem consists in numerically finding the minimum by choosing an algorithm which systematically excludes negative points. This approach has been implemented for a specific case, and details and results are shown in the subsequent sections 5.2.2 and 5.3.3.

[6]See the original paper by Kuhn and Tucker (1951), or Chiang (1984) for a full introduction to the theory and some applications.

5.2 Optimal individual contributions.

In this section the optimal contribution strategy is derived when X describes the process of funding an individual member's retirement benefit. In analogy to chapter 4, the variance of the ratio $\frac{AC_R}{PVFB_R}$ is used to measure the risk implied by the implemented contributions strategy.

5.2.1 The optimal problem.

In order to apply the procedure developed in section 5.1, first we have to express the risk measure in a quadratic form as in equation (5.2); second, we have to derive explicitly the cost of a general strategy; and finally, set the optimal problem. In a discrete-time version of the problem, the variance of $\frac{AC_R}{PVFB_R}$ can be expressed in the following quadratic form:

$$
\begin{aligned}
\mathrm{Var}\left[\frac{AC_R}{PVFB_R}\right] &= \mathrm{Var}\left[\sum_{x=\alpha}^{R} e^{\sum_{z=x}^{R}\left(r(z)-\delta_z\right)}\gamma_x m_x\right] = \\
&= \sum_{x=\alpha}^{R}\sum_{y=\alpha}^{R} \mathrm{Cov}\left[m_x e^{\sum_{z=x}^{R}\left(r(z)-\delta_z\right)}\gamma_x;\ m_y e^{\sum_{w=y}^{R}\left(r(w)-\delta_w\right)}\gamma_y\right] \\
&= \sum_{x=\alpha}^{R}\sum_{y=\alpha}^{R} \mathrm{Cov}\left[\phi_x\gamma_x;\ \phi_y\gamma_y\right] m_x m_y \qquad (5.23) \\
&= \mathbf{m}'\Sigma_{\phi,\gamma}\mathbf{m} \qquad (5.24)
\end{aligned}
$$

In (5.23), the variance is broken down in order to highlight the covariances between each combination of cohorts. Equation (4.12) explains the first equality sign and function ϕ_x summarises the financial risk, and it is equal to the exponential $e^{\sum_{z=x}^{R}\left(r(z)-\delta_z\right)}$.

In equation (5.24) the variance is instead expressed in a more convenient notation, where $\Sigma_{\phi,\gamma}$ is the variance/covariance matrix containing the covariances in equation (5.23).

The expected cost of a contribution strategy, identified by the path of accrual functions $\mathbf{m} = \{m_x\}$, is computed as the following:

$$\mathrm{E}\Big[\mathrm{Cost}(\mathbf{m})\Big] = \sum_{x=\alpha}^{R} e^{\sum_{z=\alpha}^{x}\delta_z} m_x \tag{5.25}$$

Differently from equation (4.22) in section 4.3, such an expression of the cost does not include the term $PVFB_\alpha$, and therefore it does not return the real cost of a strategy.

However, when comparing the costs of different strategies, this measure does not affect the outcome of the comparison, as long as the valuation rates $\{\delta_t\}$ do not change. This happens because $PVFB_\alpha$ is computed by actuarially discounting the future benefits and, hence, changing the sequence $\{\delta_t\}$ has the effect of modifying the value of the $PVFB$, which, in turns, affects the cost of a strategy. The expected cost measure, as is defined in (5.25), would not account for this change.

The optimal problem described in section 5.1 consists in identifying, among the contribution strategies which have the same expected cost, the discrete accrual function which minimises the variance (5.24) under the constraints that the discrete accrual factors $\mathbf{m}$ are non negative and add up to 1. In addition the cost constraint requires that the measure of expected cost is equal to a predefined value K. In formula:

$$\min_{m_x} \mathrm{Var}\Big[\frac{AC_R}{PVFB_R}\Big] = \min_{m_x} \mathbf{m}'\Sigma_{\phi,\gamma}\mathbf{m} \tag{5.26}$$

$$\tag{5.27}$$

$$\text{subject to} \quad \mathbf{m}'\mathbf{e} = K \tag{5.28}$$

$$\mathbf{m}'\mathbf{1} = 1 \tag{5.29}$$

$$\mathbf{m} \geq 0 \tag{5.30}$$

where the j^{th} element of the vector $\mathbf{e}$ is $e^{\sum_{z=\alpha}^{j}\delta_z}$; $\mathbf{1}$ is a vector of ones, whose length is $R - \alpha$.

In this particular case, the process of funding the retirement benefit for an individual member of the pension scheme is not affected by the outflows, as we focus only on the accruing process with random rate of returns and the risk of sponsor's default. Hence, comparing equation (5.2) to equation (5.26), it can be noted that, in this case, $\Sigma_X = \Sigma_{\phi,\gamma}$ and $\psi = 0$.

5.2.2 Numerical application

Let us assume that the deviations of the rates of returns $(r(u)-\delta_u)$ from the valuation rates are independent from one year to next one and are normally distributed with mean equal to $\bar{r}$ and variance $\sigma_{\bar{r}}^2$. The financial random variable ϕ_{x+1} defined in (5.23) is thus the exponential of the sum of $R-x$ normal random variables, and so it is log-normally distributed with parameters $(R-x)\bar{r}$ and $(R-x)\sigma_{\bar{r}}^2$. [7]

Therefore, the expectation and the variance of ϕ_{x+1} are given by the following, respectively:

$$\mathrm{E}[\phi_{x+1}] = e^{(R-x)\left(\bar{r}+\frac{1}{2}\sigma_{\bar{r}}^2\right)} \tag{5.31}$$

$$\mathrm{Var}[\phi_{x+1}] = e^{2(R-x)\left(\bar{r}+\frac{1}{2}\sigma_{\bar{r}}^2\right)}\left(e^{(R-x)\sigma_{\bar{r}}^2}-1\right) \tag{5.32}$$

Further, let us assume that the probability of default is independent of the financial realisations. In appendix A.6 it is shown that, under these assumptions and without specifying a model for the probability of default, the general (x^{th}, y^{th}) element of the variance/covariance matrix $\Sigma_{\phi,\gamma}$ is given by the following [8]:

$$\Sigma_{\phi,\gamma}[x,y] = e^{(2R-y-x+2)\left(\bar{r}+\frac{1}{2}\sigma_{\bar{r}}^2\right)}\Pr\Big(\gamma_y = 1\Big)\left(e^{(R-y+1)\sigma_{\bar{r}}^2} - \Pr\Big(\gamma_x = 1\Big)\right) \tag{5.33}$$

Therefore, the variance of the ratio $\frac{AC_R}{PVFB_R}$ is the weighted mean of the elements (5.33) $\forall x, y = \alpha, \ldots, R$, with weights identified by the density accrual function.

In a simpler model, where the risk of default is ignored, the probability of defaulting would then be set equal to zero. In that case, all the probabilities in equation

[7] Refer to Aitchison and Brown (1957) for a monograph on the log-normal distribution.

[8] Since the matrix $\Sigma_{\phi,\gamma}$ is symmetric, without losing generality we display the case of $x \leq y$.

(5.33) would clearly be equal to 1, and thus, equation (5.33) would still be the expression for the general (x^{th}, y^{th}) element of $\Sigma_{\phi,\gamma}$.

Section 4.2.2 considers the possibility that the time to default is an exponentially distributed random variable. In such a case, the probabilities in equation (5.33) would have the form expressed in equation (4.7) and the general (x^{th}, y^{th}) element of $\Sigma_{\phi,\gamma}$ would be equal to the following:

$$\Sigma_{\phi,\gamma}[x, y] = e^{(2R-y-x+2)\left(\bar{r}+\frac{1}{2}\sigma_{\bar{r}}^2\right)-\gamma(y-\alpha)}\left(e^{(R-y+1)\sigma_{\bar{r}}^2} - e^{-\gamma(x-\alpha)}\right) \tag{5.34}$$

Therefore, a lower variance (5.24) corresponds to a lower expected time to default (and so a higher γ). In particular, for $\gamma \rightarrow \infty$ the variance tends to zero. According to this feature, it would seem that the higher is the probability of default, the lower is the risk.

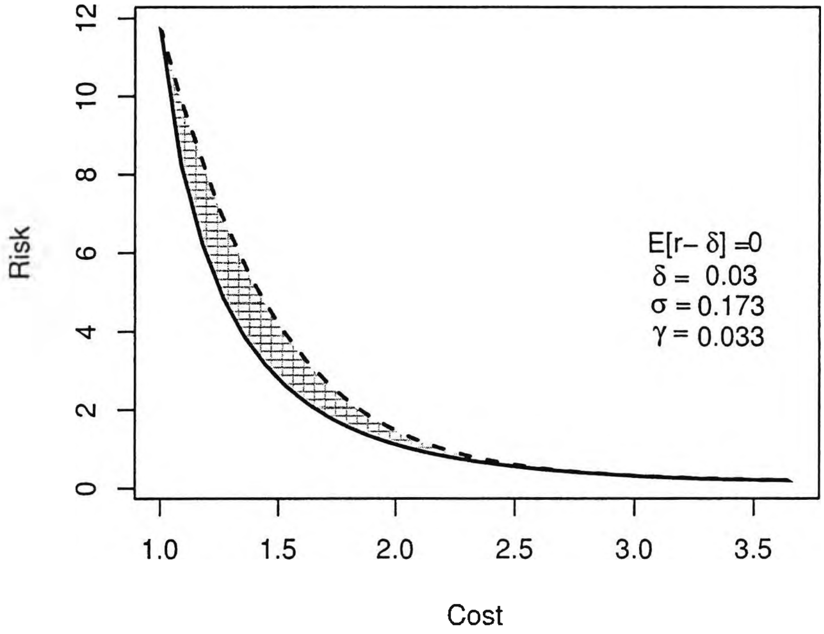
This apparently illogical feature is due to a characteristic feature of this risk measure. The variance measures the variability around the mean of the random variable being considered. Hence, for $\gamma \rightarrow \infty$ the probability of surviving is equal to zero, and no more variability exists as the default is certain. Hence this risk measure should only be used to evaluate the risk and to compare two or more contribution strategies "ceteris paribus"; i.e. fixing the parameters describing the financial and economic world in which the pension plan's sponsor is operating. As it stands, the variance should not be used to compare the performance of the same strategy under different economic conditions. For that purpose, a risk measure like (4.21) provides a consistent way of measuring; because the magnitude of the "loss" would compensate the null variability. This happens since measures like (4.21) separately deal with the probability and with the expected value of a "loss".

The following figure 5.1 displays the efficient frontier obtained by running a numerical minimisation routine in order to solve problem (5.26)-(5.30) with (5.34) and to identify the optimal contribution strategy for a large set of acceptable costs:

The continuous line identifies the efficient frontier in the cost/risk plan, hence each point on this line is efficient in the sense that it minimises the risk and the cost.

Furthermore, each point on the right hand side of the continuous line is inefficient

Figure 5.1: Efficient frontier: optimal contributions *vs* power accrual function.



in the sense that either a cheaper strategy is possible, leaving the risk at the same level; or a safer strategy can be implemented at the same cost.

The dashed line is the frontier generated by the set of the "power accrual function", introduced in section 4.1.1, for a wide range of values for the parameter p. It can be seen that a gap exists between the two lines, thus suggesting the inefficiency (in this cost/risk plan) of currently employed normal cost methods [9].

5.3 Optimal contributions - scheme level.

Instead of looking at the risk at the individual member level, in this section we focus on the aggregated risk relative to the whole pension scheme.

The risk is thus computed on the basis of the process of ul, in contrast to the process in equation (4.11) used in the previous section.

By doing so, we seek optimal strategies which minimise (a measure of) the risk of mismatch between assets and liabilities of the whole pension scheme (thus no longer related to only an individual member's fund).

Analogously to the previous sections, the variance of ul is used to measure the risk implied by the implemented contributions strategy. In this aggregate case, using the variance allows us to place the optimal problem into the theoretical framework described in chapter 3. In such a way, the demographic variations can be included as a source of uncertainty, using the discrete model already introduced.

5.3.1 The variance of *ul*.

The variance of ul has been extensively studied in section 3.5, and in this instance we refer in particular to the specific case of a randomly evolving population due to stochastic new entrants described in section 3.5.2.

From equation (3.29), it is possible to derive the required quadratic form of the variance of ul, highlighting the role of the contribution strategy in the determination

[9] In section 4.1.2, it has been noted that, for $p = 1$, the power accrual function generates the unit credit method.

of the overall risk [10].

Recalling the general equation (5.1), we need to find a mathematical expression for the inflows and the outflows from the fund. Differently from the previous section, the outflows must now be included explicitly. Thus, we introduce the following function $\widehat{B}(t)$, which summarises the compounded history of benefits paid to the pensioners.

$$\widehat{B}(t) = \sum_{h=1}^{t} \phi(h,t) \sum_{j=0}^{h-1} u^{h-j} \sum_{y=0}^{\omega-r} \sum_{\kappa=0}^{r-\alpha} B_{r+y,\kappa}\ n(r+y,j,\kappa)$$

Hence, the variance of ul at time t, which formally is derived in Appendix A.7, has the following expression:

$$\mathrm{Var}\big[ul(t)\big] = \mathbf{NC}'\Sigma_{G(t)}\mathbf{NC} - 2\mathbf{NC}'\mathbf{C}(t) + \mathrm{Var}\Big[\widehat{B}(t)\Big] \tag{5.35}$$

where the general $(i,j)^{th}$ element of the square matrix $\Sigma_{G(t)}$ is the covariance $\mathrm{Cov}\big\{G(i,t),\ G(j,t)\big\}$, and where the function $G(x,t)$ summarises the weights of present and past generations of active members combined to the realisations of the process of investment returns; in formula:

$$G(x,t) = \sum_{h=x+1}^{t} \phi(h,t) \sum_{j=x}^{h-1} u^{h-j} \sum_{\kappa=0}^{r-\alpha} n(\alpha+x,j,\kappa)$$

and $\phi(h,t)$, defined in equation (3.28), contains the returns from the investments.

Similarly, the vector $\mathbf{C}(t)$ contains the covariances between the inflow and the outflow processes, i.e. $\mathrm{Cov}\big\{G(i,t),\ \widehat{B}(t)\big\}$ is the i^{th} element of the vector.

As illustrated in §5.1, when the underlying process X is equal to ul, we can ignore the variability of the outflows represented by $\mathrm{Var}\Big[\widehat{B}(t)\Big]$, since it cannot be controlled through $\mathbf{NC}$. Moreover, comparing equation (5.2) to equation (5.35), it can be noted that in this specific case $\Sigma_X = \Sigma_{G(t)}$ and $\psi = \mathbf{C}(t) \neq 0$.

[10] The current state of research does not take into account the probability of default of the sponsor.

5.3.2 The optimal problem.

Equation (5.35) allows us to define an optimal problem aiming to reduce the variance within the boundaries implied by the maximum acceptable cost.

In this case, we state the minimisation problem, using the penalty function approach for the cost, while the actuarial equivalence principle is included as a strict linear constraint.

$$
\begin{aligned}
Z(\mathbf{NC}^*, \nu) &= \min_{NC_k} \mathrm{Var}\big[ul(t)\big] + \nu \cdot E[\mathrm{Cost}(t)] \\
&= \min_{NC_k} \ \mathbf{NC}'\Sigma_{G(t)}\mathbf{NC} - 2\mathbf{NC}'\Big(\mathbf{C}(t) - \nu\mathbf{P}\Big) \qquad (5.36) \\
\text{subject to} \quad & \sum_{x=0}^{\tau} NC_{\alpha+x} \ {}_xE_\alpha = PVFB \leftrightarrow \mathbf{NC}'\mathbf{E} = PVFB \qquad (5.37) \\
& NC_{\alpha+k} \geq 0, \qquad k = 0, \ldots, \tau \qquad (5.38)
\end{aligned}
$$

For the above problem, the K|T conditions are given by the following:

$$
\begin{aligned}
\frac{\partial L(\mathbf{NC}^*)}{\partial \mathbf{NC}} &= 0 && && (5.39) \\
\mathbf{NC}^{*\prime}\,\mathbf{E} - PVFB &= 0 && && (5.40) \\
\mu_k \ NC^*_{\alpha+k} &= 0 && k = 0, \ldots, \tau && (5.41) \\
\lambda &\geq 0 && && (5.42) \\
\mu_k &\geq 0 && k = 0, \ldots, \tau && (5.43)
\end{aligned}
$$

In order to ignore the non-negativity constraints we need to check when the coefficients of order 1 are non-negative. The components of both the matrix Σ and vector $\mathbf{C}$ in equation (5.36) are covariances. Since such covariances are ultimately given by the recursive structure of the model, it appears reasonable that such elements should all be positive. Further, the larger is the value of AL at time $t-1$ the higher it is expected to be at time t. On the other hand, the "cost factor" ν is assumed to be non-negative. Hence, the coefficients of the term of order 1 will be all definitely positive, if the "cost penalty factor" is such that: ${}_xp_\alpha \cdot \nu \leq C_x(t), \quad \forall x \in (0, \tau)$.

Applying the Lagrangian methods, the following unique solution is identified:

$$\mathbf{NC} = PVFB \left(\mathbf{E}'\Sigma_{G(t)}^{-1}\mathbf{E}\right)^{-1}\Sigma_{G(t)}^{-1}\mathbf{E} - \Sigma_{G(t)}^{-1}\left(\mathbf{C}(t) - \nu\mathbf{P}\right) + \left(\mathbf{C}(t) - \nu\mathbf{P}\right)'\Sigma_{G(t)}^{-1}\mathbf{E}\left(\mathbf{E}'\Sigma_{G(t)}^{-1}\mathbf{E}\right)^{-1}\Sigma_{G(t)}^{-1}\mathbf{E} \tag{5.44}$$

with multiplier

$$\lambda = PVFB \left(\mathbf{E}'\Sigma_{G(t)}^{-1}\mathbf{E}\right)^{-1} + \left(\mathbf{C}(t) - \nu\mathbf{P}\right)'\Sigma_{G(t)}^{-1}\mathbf{E}\left(\mathbf{E}'\Sigma_{G(t)}^{-1}\mathbf{E}\right)^{-1}$$

5.3.3 Numerical application.

In order to display the effect of using optimal contribution strategies, numerical calculations have been carried out for a specific set of simplifying assumptions.

The case of stochastic new entrants.

In this section we focus on the specific case that the number of new entrants is random and that this process is the only source of uncertainty.

In chapter 3, we have introduced a potential model to describe random perturbations to the new entrants in a DB pension scheme. In addition, it has been mentioned that this process for new entrants does not give rise to a risk *per se*, but it amplifies the existing risks.

In fact, as long as the rates of return from investments exactly match the liability valuation rate, perturbations in the number of new entrants do not affect the variability of $ul(t)$. This is easily proved: from equation (2.3), if $1 + r(t) = 1 + i$ then the resulting $ul(t)$ is independent of the demographic variations, and its dynamics is described by the following recursive relationship[11]:

[11] Refer to Dufresne (1986) or Owadally and Haberman (2003)

$$ul(t) = AL(t) - f(t) = ul(t-1)(1-k)(1+i)$$

where the adjustment at time $t-1$ is a proportion of the current unfunded liability, $adj(t-1) = k \cdot ul(t-1)$, as in the case of spreading the surpluses and deficits over a moving term. Moreover, if at time 0 the scheme is fully funded, $ul(0) = 0$, then

$$ul(t) = 0 \quad \forall\, t \geq 0$$

However, it is well known that the financial realisations are very likely to differ from the expectations and, as a consequence of this, an unfunded liability may arise, being either positive or negative. In fact, we analyse the case in which the investment rate is deterministic and constant, but different from the valuation rate [12].

Let r denote the deterministic annual rate of return achieved from the investments. Let us also introduce the following parameter $\rho = \frac{r-i}{1+i}$, which summarises the degree of mismatching between the return and the valuation rates. Thus the following holds, from (2.1), (2.3) and (3.1):

$$ul(t+1) \;=\; ul(t)\;(1-k)\;(1+r) \;-\; \rho\, AL(t+1) \tag{5.45}$$

Furthermore, if the scheme is initially fully funded (i.e.: $ul(0) = 0$), we obtain:

$$ul(t) \;=\; -\,\rho \sum_{h=1}^{t} AL(h) \cdot w^{t-h}\;; \qquad w = \big[(1-k)(1+r)\big] \tag{5.46}$$

This equation can also be derived from equations (3.27) and (3.28), by substituting the process of rates of return, $\{r(t)\}$, with the deterministic value r. In fact, if $r(t)$ is equal to a constant r, then equation (3.28) suggests that $\phi(h,t) = \rho\, w^{t-h}$.

A suitable expression for the membership stochastic process is now needed. Specifically, the membership process has to lead the population to stationarity on average, as required in section 3.3.

[12]The extension to consider stochastic rates of return is possible. However, several difficulties would arise in the computation of the closed solution derived in the following section

The assumption of annual new entrants joining the scheme at a fixed aged independently one year after the other satisfies the required condition. The new entrants are thus modeled by means of a stochastic process $\{g_t\}$ of *iid* random variables with the same mean g and variance σ_g^2. Hence, the membership function has the following form:

$$n(\alpha + x, t) = g(t - x)\ {}_x p_\alpha \tag{5.47}$$

The variance of ul is given in equation (5.35) and the number of members varies according to (5.47). In this particular case, the resulting function $G(x,t)$, when $\phi(h,t) = \rho\ w^{t-h}$, is equal to the following:

$$G(x,t) = \sum_{h=x+1}^{t} w^{t-h} \sum_{j=x}^{h-1} u^{h-j} g(j-x)_x p_\alpha$$

In a similar manner, we can derive an expression for $\widehat{B}(t)$.

Under these assumptions, it is possible to obtain closed form expressions for the covariances introduced in the previous section and therefore implement a numerical routine in order to identify the efficient strategies [13].

The results.

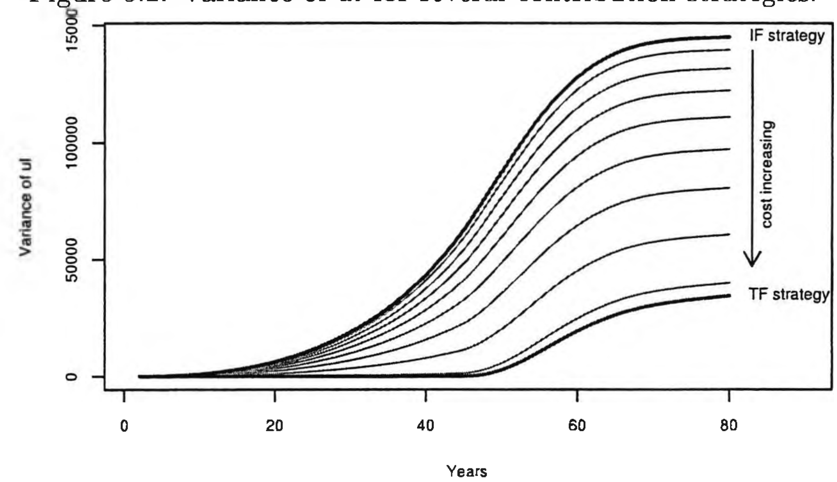
In this section, we display the results obtained according to the described assumptions and using a set of reasonable values for the model parameters, as is summarised in Appendix B.

Figure 5.2 displays the exact value of the variance of ul during the years $0 - 80$ for different contribution strategies. In detail, setting the cost factor ν equal to zero leads us to find the safest strategy. Since there is no risk of default, the resulting safest contribution strategy is, as expected, the **TF** strategy. By implementing this strategy, the variance of ul is the lowest possible, thus generating in the graph the lowest line.

Conversely, by setting ν sufficiently large ($\nu \to \nu_{\max}$, as shown in §5.1.2), the

[13]Refer to Appendix A.8 for the closed form formulae of the covariances.

Figure 5.2: Variance of ul for several contribution strategies.



cost assumes primary importance in the minimisation, thus leading to the cheapest, and clearly the riskiest, contribution strategy. As anticipated by the alternative view discussed in section 4.2.1, the resulting riskiest strategy is the **IF** strategy. If this strategy is adopted, the risk measured by variance of ul is the largest possible (the top line in the graph).

For intermediate levels of the cost factor ν, the minimisation problem leads to contribution strategies that are riskier as the cost of pension provision decreases.

It is worthwhile noting that the contribution strategy does reduce (or increase) the variance of ul, but does not change its fundamental trend in time. All the lines displayed show a sigmoidal shape, which is due to the basic characteristics of the demographic risk.

As argued in section 4.3, the risk of a strategy is directly related to its cost. In fact, a trade off exists between the risk and the cost of a strategy [14].

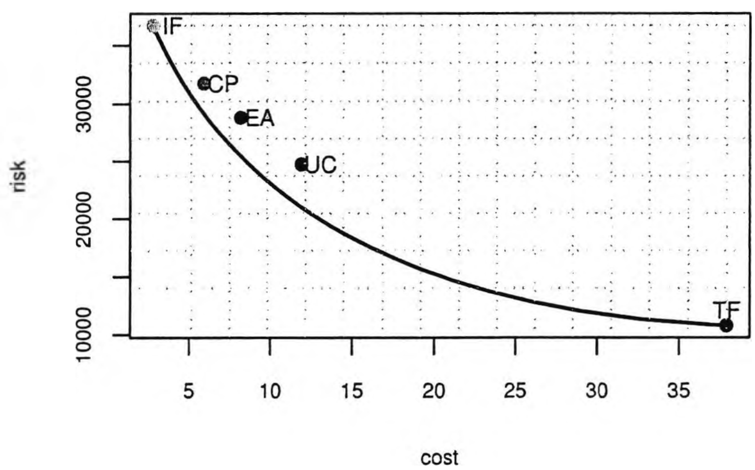
Figure 5.3 displays such a trade off, whereby the efficient frontier is identified.

[14] We have seen in section 4.2.1 that this is true as long as the financial risk exists. However, when the risk of the sponsor going bankrupt is the only source of uncertainty this tradeoff does not arise.

Specifically, the case of time $t = 80$ and amortisation period $m = 4$ has been considered [15].

As predicted by the intuitive rule, as the cost increases the variance decreases and vice versa.

Figure 5.3: Efficient cost/risk frontier.



A detailed examination reveals that, **TF** is the safest and the most expensive strategy. Therefore, the resulting point for this method is displayed on the right hand end of the graph. Conversely, the strategy **IF** is the cheapest and the riskiest possible strategy, and so the corresponding point is displayed on the left hand end of the graph. The black line identifies the efficient frontier, which represent the trade off between cost and risk of a contribution strategy.

For the sake of comparison, a set of classical normal costs methods has been selected. It is interesting to note that all of them turn out to be inefficient. In fact, the unit credit (**UC**), fixed entry age (**EA**) and constant premiums (**CP**) methods lie above the efficient line. Accepting the level of risk that each of the classical methods

[15]Time and spread period seem to affect the magnitude of the results, but not its meaning. Since the spread period is one of the tools commonly used to calibrate convenient strategies with respect to the solvency and stability of the scheme (see Haberman and Sung (1994)), a sensitivity analysis on the parameter m has been carried out. The results are illustrated at the end of this section.

implies, it is always possible to find a less expensive contribution strategy. Conversely, given the cost of each of the classical methods, it is possible to find a strategy which implies lower riskiness.

As briefly mentioned in the note 15, it is of interest to investigate the degree to which the efficient line is sensitive to the spread period. The longer this period is, the more variable is the resulting unfunded liability. Bearing in mind that the covariances between AL at different points in time are positive, it can be shown that the variance of ul is an increasing function of w. Since $w = (1-k)(1+r)$ and $k = 1/a_{\overline{m}|} = \frac{i}{(1-(1+i)^{-m})(1+i)}$, the variance of ul increases with the length of the spread period (m).

In order to illustrate the effect of this parameter on the efficient frontier, we have computed the minimum variance of ul for a set of spread periods and for different acceptable levels of cost. The resulting variances have been displayed in Figure 5.4 [16].

The lines displayed are consistent with the expected behaviour that the larger is m, the higher is the variance of ul. Specifically, the increments in the spread period have a significant effect on the risk. In fact, it can be seen that for high value of m (say $m = 8$ or 9) the most expensive contribution strategy (strategy **TF** and cost= 1) yields a level of risk which is higher than the variance achievable from combining a cheaper strategy with a shorter spread period (say $m \leq 5$ and cost= 0.2).

However, the risk can be maintained on the same level by choosing adequate combinations of cost and m. Table 5.1 displays the cost necessary to achieve a given level of risk, when combined with a set of possible spread periods.

These figures highlight the strong impact that the spread period has on the variability of ul. In order to have a variance equal to 20,000 (4^{th} column), a very cheap contribution strategy (12% of **TF** cost) is needed when the surpluses and deficits are spread over 4 years (3^{rd} row). In order to achieve the same position when $m = 5$ or 6, the cost of the optimal strategy is respectively 34% and 60% the cost of the **TF** strategy. Increasing the spread by one further year, $m = 7$, a strategy almost as expensive

[16] In this graph and in the following tables the actual cost of each contribution strategy has been rescaled in the interval (0, 1), by dividing by the cost of the TF strategy, which is the maximum cost.

Figure 5.4: Efficient cost/risk frontier and spreading period.

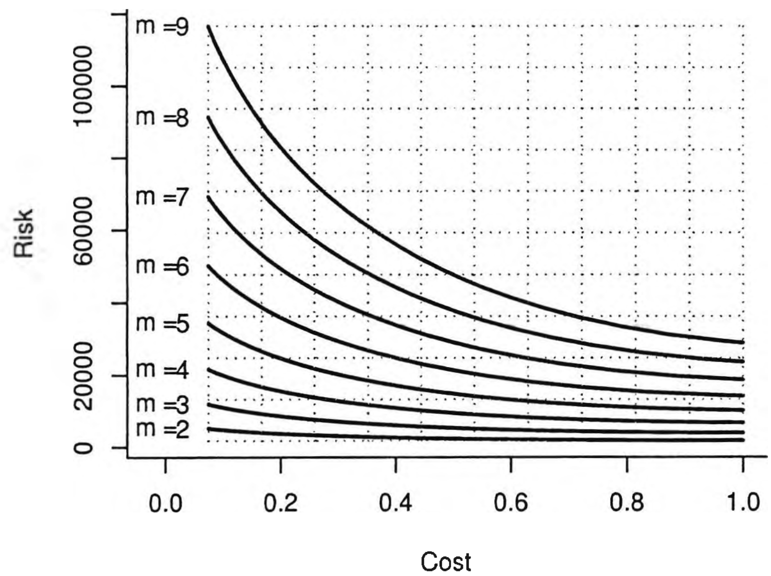


Table 5.1: Relative cost (as percentage of TF cost) for different levels of variance of *ul* (in *,000s*) and spread periods.

m/var	5	11	16	20	26	31	38
2	11%	-	-	-	-	-	-
3	62%	12%	-	-	-	-	-
4	-	44%	22%	12%	-	-	-
5	-	94%	49%	34%	21%	13%	-
6	-	-	86%	60%	41%	31%	21%
7	-	-	-	98%	64%	50%	37%
8	-	-	-	-	93%	71%	54%
9	-	-	-	-	-	97%	72%
10	-	-	-	-	-	-	93%

as **TF** is needed to maintain the variance at the chosen level.

Finally, it is worthwhile to compare again those figures with those determined by the classical methods considered.

Table 5.2: Cost and variance for classical methods.

	Cost	Risk
CP	15.5%	31,700
EA	21.6%	28,800
UC	31.5%	24,700

In Table 5.2 the relative cost and the variance of ul, corresponding to each classical method, when $m = 5$ and $t = 80$ are displayed. The inefficiency of those methods is still evident. For instance, the Entry Age method requires a cost slightly higher than 21% and implies a variance higher than 26,000, which is achievable by implementing the optimal strategy for a comparable cost. Similar conclusions apply for the other two methods.

5.4 Summary.

In this chapter, we have shown how constrained nonlinear programming can be applied to derive optimal contribution strategies. Under specific conditions it has also been possible to derive an analytical solution. When this has not been possible, a numerical algorithm have been proved to be a suitable alternative for solving the optimisation problem.

Specifically, optimal contribution strategies have been identified in two different situations: when the objective function involves the optimisation of (a) a risk measure which is computed over an individual member-based process; (b) when the risk measure is computed over a whole scheme-based risk process.

In case (a), we have considered the two sources of uncertainty (stochastic rates of return and random remaining lifetime of the plan's sponsor) described in chapter 4. Optimal strategies have been numerically found, and the efficient frontier on the cost/risk plane has been identified. Moreover, we have shown that the normal cost methods generated by the power accrual density function are inefficient in the considered plane.

With regard to case (b), we have used the variance of the unfunded liability as a measure of risk for the whole scheme. In particular, we have used an expression of this variance in order to highlight the role of the contribution strategy in the determination of the risk of mismatching between the assets and the liabilities of the whole scheme. In this case, financial and demographic risks have been included as sources of uncertainty.

A numerical application has focused on the way in which stochastic new entrants amplify the effects of a deterministic mismatch between the expected and actual returns from investments.

With regard to the specific case of *iid* new entrants at a fixed age, optimal strategies have been found numerically. Furthermore, the cost of a strategy has been taken into account, since the resulting optimal strategy might be too expensive the plan's sponsor. Hence, the tradeoff between cost and risk of a contribution strategy has been solved, by means of finding an efficient frontier.

In addition, classical normal methods have been compared to the optimal strategies, showing the inefficiency of these methods in terms of cost and risk.

Chapter 6

Optimal funding strategies

6.1 Introduction.

In this chapter we focus on extending previously derived results to the more generic case of managing a DB pension scheme when the rates of investment return are random and evolve according to stochastic processes. In such a situation, the allocation of the assets is a tool that can be used to exercise control over the process of funding the retirement benefit.

From this perspective, this chapter deals with the development of optimal funding strategies, i.e. optimal combination of investment and contribution incomes, through the choice of the asset allocation and the contribution strategy.

In order to do so, numerical methods and computer simulations are the main technical tools to be used. Being aware of and concerned with the limitation of such an approach, but nonetheless aiming to provide a comprehensive range of results, we employ conceptually different financial models. The intention is to illustrate how some results are not exclusive to one model.

As a sufficiently general case, we are interested in a portfolio composed of three assets, each of a different nature in terms of risk-reward profile and characteristic dynamics. Specifically, we include:

a) a risk free rate asset r_s;

b) a bond asset B_T, with maturity T;

c) and a stock asset S.

Further assets, such as a variety of bonds and equities, as well as property investments, could be included in order to generalise the financial model. However, the three considered assets provide the main financial blocks for the purpose of investigating the investment strategy.

The fund is thus invested in a combination of these assets with proportions β_r, β_B and β_S $(= 1 - \beta_r - \beta_B)$ respectively. Hence, the fund value, $f(t)$, at time t is given by the following equation:

$$f(t) = \beta_r(t)\ r_s(t) + \beta_B(t)\ B_T(t) + \beta_S(t)\ S(t) \tag{6.1}$$

So, in order to finance the provision of a retirement benefit to all the employees, the capital accrued is invested in a collection of assets, and thus, its value is a linear combination of the values of the elementary assets.

The choice of the technique of valuation of these elementary assets is a controversial issue widely dealt within the pension literature. In fact, the controversy lies in the tradeoff between the accuracy of each valuation and the stability of subsequent valuations.

This tradeoff arises because the funding of retirement benefits is an issue, which may be considered from different perspectives. As illustrated in Ezra (1980), fund managers, as well as accountants, are mainly concerned with the valuation of short term investment performances and of the current wealth of a pension scheme. On the contrary, actuaries are concerned with the level of funding of a pension scheme, i.e. whether the assets *currently held* will be sufficient *in the future* to meet the plan's liabilities.

A *market value* approach, as the name suggests, sets the fund value equal to the market value of plan assets as at the valuation date.

An alternative method consists of computing an *actuarial value* of assets. As Winklevoss (1993) states *"an* [actuarial] *asset valuation method is designed to smooth the year to year fluctuations in market value".*

Several asset valuation methods exist for accomplishing this smoothing, and a comprehensive analysis can be found in Winklevoss (1993) and more recently in Owadally and Haberman (2004*b*). In this work, we use these random returns for computing the fund value without applying any smoothing, and thus implicitly use a market value approach for the valuation of the assets.

The chapter is organised in the following way: the dynamics of stochastic rates of return are presented in section 6.2, where additional issues, such as the choice of the valuation rate and of the spreading period, are also taken into account. §6.4 deals with identifying optimal investment strategies, with two and three assets. Random demographic variations, as well as schemes of different sizes, are analysed in §6.5. Optimal funding strategies, by means of optimal combinations of investment and contribution strategies, are developed in §6.6. Section 6.7 aims to answer questions from the previous sections, where a financial economic approach to pension funding is implemented.

6.2 Financial models.

This section briefly introduces the models used for describing the dynamics of the three assets. Although theoretically the number of financial models is infinite, we believe that testing with a limited number of structurally different models allows us to gain some additional information, and to avoid being reliant on conclusions made on the basis of the results from only one model.

6.2.1 Multivariate normal model.

The first approach we consider consists of modelling the three assets with a multivariate normal distribution. This model assumes that the returns are independent one year from the other and that the unconditional first and second moments of the assets return are stationary. Moreover, these assets are pairwise correlated.

In Blake *et al* (2001) expectations, standard deviations and correlations for UK T-bills, UK equities, UK bonds, UK property, US equities and US bonds are estimated from the data relative to the period 1947-1998. In this work we include the first three of these assets, where the UK T-bill is considered the short term asset.

The generation of a random normal vector $\mathbf{x} = (r_s,\ B_T,\ S)$ with given mean vector $\hat{\mu}$ and variance-covariance matrix $\mathbf{V}$ utilises a theorem, which states that if $\mathbf{z}$ is a standard normal vector, there exist a unique lower triangular matrix $\mathbf{C}$ such that:

$$\mathbf{x} = \mathbf{C}\mathbf{z} + \hat{\mu}$$

Furthermore, the vector $(\mathbf{x} - \hat{\mu})$ has the variance-covariance matrix

$$\mathbf{V} = \mathbf{C} \cdot \mathbf{C}'$$

In order to obtain $\mathbf{C}$ from $\mathbf{V}$, the so-called "square root method" can be used; which provides a set of recursive formulae for the computation of the elements of $\mathbf{C}$. [1]

On one hand, such a model has the advantage of simplicity in both analytical and numerical computations. On the other hand, it has severe limitations in providing a realistic description of investment markets. In fact, the lack of market shocks, independence of returns from one year to another and underestimation of tails are among the most significant weaknesses in the assumptions in this model.

6.2.2 Regime-switching model.

Among the many limitations of the previous model, the lack of market shocks can be overcome by explicitly including structural breaks in the parameters of the model.

Specifically, the recent literature advocates the simulation of investment returns from a mixture of distributions with regime-independent transition probabilities. In this way, it is possible to represent different environments, in which the behaviour of each asset changes.

[1] Refer to Naylor et al (1966, pp. 90-99) for an introduction to computer simulation of continuous probability distributions and to Kenney and Keeping (1951, pp. 298-300) for a review of the "square root method".

In a general Markov regime-switching model the parameters are stochastic processes satisfying the Markov property of no memory. Such an approach allows for describing parameter structural breaks, whereby the value of these parameters changes in time according to the current state.

Regime switching was first introduced by Hamilton (1989), who describes an autoregressive regime-switching model. Hardy (2001) proposes the modelling of the log-return with mixture of regime-switching normal distributions.

In order to implement such an approach in this work, we model the returns from the risk-less asset with a normal distribution, as in the previous model. Specifically, we take into account two possible states for the riskier assets B_T and S: in the first state assets behave as in a *standard* situation; while in the second state a *high uncertainty* situation is described, whereby the volatility of the assets is largely increased.

Hardy (2001) finds no significant improvement in fitting 3 different states for the log-returns. Similarly, Harris (1997) implements a regime-switching vector **AR** for modelling the real economic growth, change in the rate of price inflation, share price return and change in the 10 year bond yield, using Monte Carlo Markov Chain estimation procedure on Australian data. The estimation procedure identifies two clearly distinct regimes, thus providing other evidence that 2 states should adequately describe the variability of the market.

Blake *et al* (2001) introduce this model as a mixed multivariate normal model and specific estimates of the rates of return and volatilities are thereby suggested for the same financial components (r_s,B_T,S) listed in the previous section. From Blake *et al* (2001) it is possible to compute directly the transition matrix by recognising that in their model the probability of remaining in the high risk state is equal to the simple probability of switching into it. In fact, the probability of a structural break is independent of the current state.

It is noted that, under the specifications set in Blake *et al* (2001), log-returns maintain a symmetrical distribution independent of previous realisations, even in the high-risk state. However, this model, allowing for stochastic volatility, leads to a higher variability than the multivariate normal distribution.

6.2.3 Wilkie model.

The Wilkie model is a very common stochastic projection asset model, especially in actuarial applications and analyses. It is a stochastic parametric model first proposed in Wilkie (1987), and subsequently extended and updated in Wilkie (1995).

The asset rates of return are assumed to be autoregressive processes, and hence, dependent on their previous values.

Several variables are modelled, including retail price index, price inflation, wage inflation, equity dividend yield, equity dividends and Consols yield; short-term interest rates, property yields based on income and price; as well as yield on index-linked bonds.

The characteristic feature of the Wilkie model is that it is composed of connected models, one for each of the listed variables. The connections are such that, for instance, the price inflation model includes the retail price index; the equity dividend yield model is dependent on price inflation, and so on. Moreover, the residuals are normal independent and identically distributed and least square estimates (calculated over the period 1919-1994) are used for the parameters.

However, the literature recognises that this model contains some notable pitfalls from statistical viewpoint and financial economic viewpoint.

It is stated in Huber (1995) that "Wilkie's stochastic model does not provide a particularly good description of the data", since the residuals seems to be not normally distributed. As Huber suggests, this may be due to the fact that the model is over parametrised and ill-conditioned.

It is also noted in Ong (1994) that the Wilkie's model "does not provide a term structure for interest rates, a crucial concept in most actuarial applications".

Moreover, the financial economic literature criticises the model as it fails to satisfy the hypothesis of no arbitrage.

In this work, the dynamics of the returns from the assets r_s, B_T and S are derived from the Wilkie models for the short term interest rate, the Consols yield and the equity, respectively, as described in Blake *et al* (2001).

6.2.4 Financial Economic model.

In light of the last consideration, we include a model satisfying this requirement of no arbitrage drawn from financial economics. Specifically, here we separately model the three different assets according to models proposed in the mathematical finance literature, in order to build up an efficient and complete market where no arbitrage is possible [2].

Short rate.

In order to model the value of money in time, we include a one-factor model (the short rate model) dealing with the short rate as the only driving factor. Although more complex models including more factors are less restrictive, the attraction of simpler models is the tractability and (in some cases) the availability of closed formulae for the prices (and returns) of bonds.

The purpose of this section is to describe the probabilistic model implemented for the simulation of the interest rates and of the yield to maturity. Hence, a formal and mathematical introduction of these models is beyond the scope of this work. Standard references for interest rate models are James and Webber (2000), Brigo and Mercurio (2001), and Zagst (2002).

The short rate is modelled by means of a diffusion process, described by a stochastic differential equation of the family:

$$\mathrm{d}r_s(t) = \mu_r(t,r)\mathrm{d}t + \sigma_r(t,r)\mathrm{d}W_r(t) \tag{6.2}$$

Function $\mu_r(t,r)$ is the drift of the process, i.e. the infinitesimal mean; while $\sigma_r(t,r)$ is the diffusion coefficient, i.e. the infinitesimal variance of the process. Furthermore, $W_r(t)$ is a standard Brownian motion, thus continuous and with independent and normally distributed increments.

A number of models presented in the literature belongs to this family of processes, for instance, the models included in the Heath-Jarrow-Morton framework (see Heath,

[2] When mentioning the market completeness, we refer only to *financial* sources of uncertainties, thus excluding demographic variations.

Jarrow and Morton (1992)). As far as one-factor models are considered, some of the most used models are *Vasicek*'s model presented in Vasicek (1977), *CIR*'s model in Cox, Ingersoll and Ross (1978), *Brennan and Schwartz*'s model in Brennan and Schwartz (1982).

In this work, we use the *Vasicek* model, where the drift is assumed to be mean reverting and the instantaneous volatility $\sigma_r(t, r)$ is constant and so independent of time and of the current level of $r_s(t)$. We choose to employ the *Vasicek* model because it is the simplest among those mentioned, although maintaining desirable features thereafter described.

Thus, the short term rate is modelled with an Ornstein-Uhlenbeck process, first proposed in Merton (1971), the dynamics of which are described by the following stochastic differential equation:

$$\mathrm{d}r_s(t) = (\bar{\gamma} - a\ r(t))\mathrm{d}t + \sigma_r \mathrm{d}W_r(t) \tag{6.3}$$

This process has the desirable feature of possessing a stationary distribution, and hence, it does not diverge to infinity. In fact, the drift term $(\bar{\gamma} - a\ r_s(t))$ is a force pulling the process towards its long term expectation $\bar{\gamma}$, with an increasing magnitude as the processes moves away from $\bar{\gamma}$.

In contrast, undesirable features of this model are the possibility of negative interest rates and the assumptions of constant variance. The CIR model overcomes these pitfalls, by proposing the following dynamics for the short term rate:

$$\mathrm{d}r(t) = (\bar{\gamma} - a\ r_s(t))\mathrm{d}t + \sigma_r \sqrt{r_s(t)}\mathrm{d}W_r(t) \tag{6.4}$$

According to the dynamics (6.4), the short term rate is distributed as a non centred χ-square random variable.

However, for the purposes of this thesis, the more realistic CIR model does not provide results significantly different from the Vasicek model, as some preliminary results (not presented here) have shown.

Bond.

According to the theory of interest rates and term structure, in an efficient market the prices of bonds with various maturities are uniquely determined by the *dynamics of the short term rate* and by the (unique) *market price of risk*, see Björk (1998, pp. 242-250).

Following Boulier *et al* (2001) we introduce a rolling zero coupon bond with constant maturity T. The dynamics in continuous time of this bond are given by the following equation:

$$\frac{\mathrm{d}B_T(t)}{B_T(t)} = \big(r_s(t) + \lambda_r \sigma_T\big)\mathrm{d}t + \sigma_T \mathrm{d}W_r(t) \tag{6.5}$$

where, according to Vasicek's short interest rate model (where the market price of risk λ_r is constant), the volatility is constant and given by[3]

$$\sigma_T = \frac{1 - e^{-aT}}{a}\sigma_r \tag{6.9}$$

Equation 6.5 (discretised in order to fit the current framework) provides the return of bond investments assuming that at the end of each year the bonds (now with maturity $T - 1$) are replaced by bonds with maturity T. Whether this is a realistic strategy is debatable; particularly in light of the fact that the disinvestments occur independently of whether a gain or a loss will be realised. However, a zero coupon bond with constant maturity would reasonably match the liabilities, if the mean term

[3] For the case of the CIR model, according to Cox *et al* (1978) and following Deelstra *et al* (2003) the volatility is given by

$$\sigma_T = h(T)\sigma_r\sqrt{r_s(t)} \tag{6.6}$$

where

$$h(T) = \frac{2\big(e^{cT} - 1\big)}{c - (a - \lambda_r\sigma_r^2) + e^{cT}(c + a - \lambda_r\sigma_r^2)} \tag{6.7}$$

and

$$c = \sqrt{\left(a - \lambda_r\sigma_r^2\right)^2 + 2\sigma_r^2} \tag{6.8}$$

were approximately constant.

Equity.

In this work, we also include a high risk and high reward asset, which can be interpreted as an equity. In fact, instead of modelling a variety of equities, we model only one asset which describes the stock market by means of a stock index.

This equity price is modelled by means of a *geometric Brownian* motion (GMB). This process, known also as *economic Brownian* motion, is widely used in the financial literature for describing the dynamics of the price of an equity, because of the characteristic log-normal distribution, which leads to right skewed non negative prices. For further and deeper discussion on the suitability of the GMB refer to Cootner (1964) and Samuelson (1965).

Furthermore, we assume that the behaviour of the equity is correlated to the short term rate. Henceforth, the dynamics of the log-return of this asset is described by the following stochastic differential equation:

$$\begin{aligned}\frac{\mathrm{d}S(t)}{S(t)} &= \mu_S(t,r)\mathrm{d}t + \sigma_S(t,r)\mathrm{d}W_S(t) = \\ &= \mu_S(t,r)\mathrm{d}t + \sigma(t,r)\mathrm{d}W_r(t) + \sigma_{S|r}(t,r)\mathrm{d}W_{S|r}(t), \qquad (6.10)\\ S(0) &= S_0 \qquad (6.11)\end{aligned}$$

where the two Brownian motions $W_r(t)$ and $W_{S|r}(t)$ are independent; the instantaneous mean μ_S is a function of the time and of the short term rate $r(t)$ and is given by the following

$$\mu_S(t,r) = r(t) + \lambda_r\sigma(t,r) + \lambda_{S|r}\sigma_{S|r}(t,r) \qquad (6.12)$$

where, $\sigma_{S|r} \neq 0$ is the stock's own volatility and $\lambda_{S|r}$ is the market price for holding this risk; and σ is the volatility linked to the short term rate: $\sigma^2 = \nu_{r,S} \cdot \sigma_r^2$ where $\nu_{r,S}$ is a scaling factor measuring how the interest rate volatility affects the stock volatility.

Moreover, the correlation coefficient between W_S and W_r is given by

$$\rho_{r,S} = \frac{\sigma}{\sqrt{\sigma^2 + \sigma^2_{S|r}}} = \frac{\sigma_r \sqrt{\nu_{r,S}}}{\sqrt{\sigma_r^2 + \sigma^2_{S|r}}} \tag{6.13}$$

6.2.5 Spreading period.

In this chapter, rates of return are simulated according to a number of models reflecting different assumptions regarding the financial markets. The reason why a Monte Carlo approach is of interest is that it allows working with complicated models, which are otherwise intractable from an analytic perspective.

However, extreme care is needed when a work develops from the particular to the general. Looking at the bigger picture, in this chapter we want to investigate the effect of random rates of returns, which evolve according to specific probabilistic models.

In particular, a problem arises when the financial economic and the Wilkie models are used. In fact, both models assume that the annual returns show some degree of autocorrelation. In other words, the rates of return are no longer independent year after year. If this assumption of independence does not hold, then the adjustment method is no longer guaranteed to remove surplus/deficits, not even in the long term.

Dufresne (1988) derives conditions which assure that spreading surplus/deficits over a moving term efficiently leads the expectation of ul to tend to 0. Haberman (1994) finds similar conditions when rates of return show an autoregressive behaviour, while in Haberman and Wong (1997) the case of returns following a moving average process is also considered.

However, assuming that either the financial economic model, or the Wilkie model hold, the distribution of the portfolio of assets is not known, and thus, general conditions are needed in order to assure the convergence of the expected value of ul, when the rates of return are described by a non-specified stochastic process.

6.3 Asset allocation strategy and valuation rate.

An important assumption concerns the rate which the actuary should use for discounting future liabilities. There are two (or maybe more) main schools of thought: an *actuarial school*, which is in favour of discounting the liabilities at a rate which is linked to the expected rate of investment return. Then there is an *economic school*, which asserts that the discounting rate should instead be independent of the investment policy.

6.3.1 Actuarial approach.

In the previous chapters the valuation rate has been set equal to the expected rate of returns from the investment. In the classical actuarial literature, this approach is justified by the requirement that the pension funding process should not lead to gains or losses if the pension plan experience exactly matches the expectations. Hence, if all of the assumptions set by the actuary - regarding the unknown future on demographic, financial and economic evolutions - are borne out by experience, then the contribution and investment income should match the benefit outgo. This requirement is also known as the principle of actuarial equivalence.

For instance, as illustrated in §2.6.1, according to Aitken (1996) and Sharp (1996), acceptable normal cost methods must be devised in such a way that no gains/losses arise if the experience exactly matches the expectations. Setting the valuation rate different from the expected rate of return would inevitably lead to a technical surplus/deficit.

In Bowers *et al* (1976, 1979, 1982) the liabilities are discounted at the same force of interest which is expected to be earned, where these investment returns are assumed to be constant and deterministic. More recent papers have assumed that rates of return are instead random, and that the dynamics of these are described by stochastic processes. In this situation, the condition satisfying the principle of actuarial equivalence and imposed in Bowers *et al* (1976) is substituted by its intuitive stochastic extension of requiring that the valuation rate is equal to the *expected* rate of return. See, among others, Dufresne (1986, 1988, 1989), Haberman (1994), Owadally and

Haberman (1999, 2004), Cairns (2000), and Khorasanee (2002).

In actual practice, liabilities are often valued using what is *prudently* expected to be the return from future investments. Indeed, from an actuarial perspective there is a conceptual difference between the valuation rate and the expected rate of investment returns. To some extent, the valuation rate should be thought of as the prudent/safe estimate, rather than the best estimate, of the rate of return. This conceptual difference between the two rates is thus translated in a numerical difference, with the effect of creating a margin against the risk of mismatch, i.e. against the risk that the pension fund earns a return different from the expected one. See Thornton and Wilson (1992), Wright (1998), Cairns (2000) and Owadally (2003) for different aspects and consequences of assuming a prudent margin in the valuation rate[4].

As Brownlee and Daskais (1991) conclude, the expected rate of investment returns should be used for discounting liabilities. However, because of the "*conservative*" nature of the actuary, often the discount rate is set equal to a lower value. Nonetheless, it is still the authors' opinion that the higher is the expected return from the investment policy, the higher should be the discounting rate used by the actuary. This assertion suggests that a link between the valuation rate and the expected investment return should exist.

Since, in this thesis, the rate of returns from investment is not directly modelled, but is derived by modelling three elementary assets aggregated in a portfolio, the resulting valuation rate depends on the asset allocation. Therefore, a more aggressive strategy expecting to earn a higher return than a more prudent strategy should be combined with a higher discount rate. Specifically, the extra premium for bearing risk is accounted for whilst evaluating future liabilities.

The dependence of the valuation rate on the asset allocation does not create computational problems as long as static investment strategies are implemented. More sophisticated strategies allowing for dynamics allocation of the fund may complicate

[4]The current aim is analysing under which circumstances the valuation rate should be considered dependent on the investment policy, and what consequences should we expect from such an assumption. For a comprehensive analysis of the more general issue of actuarial prudence in pension funding the reader can refer to Owadally (1999, chapter 5).

the computation of the expected returns and hence a correct valuation rate. However, a more elaborate model could allow for a dynamic valuation rate.

The main criticism of this actuarial approach is that the liabilities hide away the risk and volatility by using a fixed discount rate. Economists do not agree with the principle that a high discount rate is justified by high equity allocation in the asset policy. In fact, these high returns are only anticipated and whether they will be realised is a matter of chance; hence, the actuary should not account for them.

6.3.2 Financial economic approach.

As an alternative to the actuarial school of thinking, a financial economic approach to pension funding requires the liability to be independent from the asset allocation. Bader and Gold (2003) point out this difference:

> "Financial economics measures a liability by using a discount rate curve embedded in a reference portfolio - a portfolio that matches the liability. [...] The actuarial pension model discounts liabilities at expected return on the assets held to fund those liabilities; it ignores risk."

In particular, Bader and Gold (2003) refer to a market method for discounting liabilities governed by the principle that, if no arbitrage is allowed, then the value of two identical cash-flows must be equal. Hence, the valuation rate to be used for discounting liabilities should be the implied rate of the portfolio of assets that matches the liabilities.

This financial economic approach goes further and interprets the pension as the asset which will provide retirement income to each individual employee; see Ralfe *et al* (2003), for instance. Therefore, pension is a debt owed by the sponsor to the pension fund members, and as such, should be dealt with as a bond-like debt. According to this, Bader and Gold (2003) propose valuing the liabilities as the price of the portfolio of Treasuries that replicates the benefit pay out assuming that the scheme is terminating. This approach finds agreement in the previously established Accounting Standard's FRS17, which requires the use of a AA-rated bond-based valuation rate for discounting the pension liabilities.

The idea of replicating pension debts with a portfolio of bonds arises from the self evident similarities between the two streams of payments. However, there are a number of features that make the pension debt more complex than conventional bonds:

- the number of payments, although certainly not infinite, is not known and depends on the mortality rates specific to each pension plan population. Hence, two separate schemes are likely to experience different mortality, due to the accidental random differences as well as structurally different populations. In addition, regulations often allow the retired member to choose the preferred form of payments out of a limited selection of possibilities (retirement pension, partial lump sum, deferral annuities, etc). Furthermore, benefits might be paid in case of premature death and under other unfortunate circumstances.

- defined benefit pension schemes provide the pensioners with benefits which depend on their final salaries. In addition, the amount of the pension is usually linked to some indexes (most likely the price inflation), in order to avoid erosion of value over time.

- the pension plan sponsor has a walk-away option which allows temporary suspension of the payments and liability, in the case of scheme termination, being limited to the Minimum Funding Requirement (**MFR**).

Hence, substantial differences exist between the two streams of payments and these differences are usually acknowledged in the financial economic literature; see for instance Ralfe *et al* (2003).

Therefore, attention should be directed to investigating whether these differences are significant or not. In particular, as Carne (2004) points out, there is the need to understand whether pension benefits are "sufficiently bond-like" to justify the use of conventional bond to derive the market value of the sponsor's liabilities.

Thus, the financial economic school's contribution provides insight into the pension funding matter, coming from a perspective different from the classical actuarial one. This approach is particularly valuable because, if the scheme were to wind up, it would provide at the time of valuation a (market) consistent measure of the pending liability on the sponsor. Unfortunately, as pointed out in Mindlin (2003), McCrory and Bartel (2003) and Carne (2004), the assumption of termination is critical. Hence, this measure cannot help in setting up the funding and investment strategy for long term implementation and it may be of limited use if the sponsor is not contemplating termination.

From the financial economic standpoint, only the benefit promised to the current members would be accounted for in the valuation process, making no allowances for future salary growth, wage inflation and other uncertain events that may modify the benefit structure of the pension plan. As Smith (1998) illustrates, the current unit method, which consists of buying a series of deferred annuities linked to the current salary, is the correct method to estimate economic pension cost, because it does not require estimation of salary growth and of decrements of active members. However, no Accounting Standards adopt such a method for valuing DB pension scheme liabilities, because these depend on future salaries.

6.3.3 The area of intersection.

The two schools of thought seem to be in complete contrast with each other and room for reconciliation appears to be very narrow. Bader and Gold (2003) hope for the actuarial profession opening towards this financial economic science and "recognize where it must be applied".

With the risk of sounding trivial, Carne (2004) identifies the area of intersection between the two schools: If the investment policy requires that the fund value is invested entirely in Treasury bonds, then the correct discount rate would be the implied return of such an investment portfolio. In this situation also the actuarial approach would suggest the adoption of the implied interest rate for discounting the liabilities, since a prudent margin would not be needed whilst discounting at a risk-free rate [5].

[5]However, under this specific assumption, the two cases - whether the valuation rate is independent of the asset allocation or not - are indistinguishable. Moreover, it is not entirely accurate to define the Treasury bonds as risk-less assets. In fact, they do reward the investors with a (small)

In addition, it is worth recalling the stated concept that a bond-based valuation is informative of the cost of winding up the scheme, independently of the asset allocation employed.

6.4 Optimal investment strategies.

6.4.1 Bonds vs equities.

Whether bond based or equity based portfolio provides the best tool for funding a defined benefit pension scheme is an open debate in the actuarial and financial-economic literature.

Equities.

The main reasons in favour of investing in equities are as follows:

- It is broadly recognised that equities provide a good match for salary-linked pension liabilities. This belief arises from the following argument: Since both equity dividend growth and salary inflation are linked to the firm's productivity and to price inflation, and since the equity price is the expected present value of future dividends, so it may be shown that the returns from equity investments are somehow related to salary inflation. This argument is particularly supported by the UK actuarial profession, as illustrated in Thornton and Wilson (1992) and Wilkie (1995).

  Not everybody shares the same view. In particular, Exley *et al* (1997) criticise this economic link, arguing that the wage inflation is driven by the price inflation plus a real salary inflation variable. Furthermore, Smith (1998) uses basic statistical analysis to demonstrate that the relationship between equities and salary-linked benefits does not hold in the UK. However, Cardinale (2003), using more sophisticated techniques, illustrates statistical evidence of cointegration between the labour market and equity returns. Similarly, Khorasanee

premium for risk. However, their risk of default is generally considered as negligible.

(2004) finds that future wage inflation should be positively correlated to the current equity returns.

- It is also accepted that equities outperform bonds in the long run, and this statement tends to hold particularly when referring to highly rated bonds. The extra premium is the reward paid by equities to the investors for accepting variable returns, in contrast to fixed-income products, which, as their name suggest, guarantee a specified return. Although equity performances in recent years have suffered because of long economic stagnation and bear markets, this view seems to be still widely accepted.

  However, a controversy exists about whether these extra premia should be accounted for, when discounting the liabilities. Supporters of a traditional actuarial view suggest that while setting a strategy for funding a retirement benefit, these expected extra premia should be included. See for instance, Carne (2004).

  Putting this in another way, in the actuarial approach each funding strategy is characterised by a unique way of combining contribution and investment income in order to finance a retirement benefit in a far future.

  Promoters of a financial economic view support the argument that the equity risk premium cannot be included in discount rates. Gordon (1999) asserts that allowing for equity premium is "double counting". Chapman *et al* (2001), Shuttleworth (2002), Ralfe *et al* (2003), among others, endorse this argument.

Bonds and the Modigliani-Miller proposition.

In contrast, the financial economic school supports the view that the pension funds should be entirely in fixed income investments. This view originated from the '*irrelevancy proposition*' set in Modigliani and Miller (1958), which states that "the market value of any firm is independent of the capital structure". This proposition has been subsequently applied to corporate pension funding in a number of works as Topper and Affleck (1974), Black (1980), Smith (1996), Exley *et al* (1997), Bader and Gold (2003) and Ralfe *et al* (2003), among others.

Black (1980) asserts that, because of the MM proposition, combining a change in the pension fund investments with a change in the firm's capital structure is harmless for the firm. The argument is that, by issuing more debts and buying back the firm's own stocks, eventually the firm will benefit from its capital structure (because stocks outperform bonds in the long term) offsetting the losses emerging from switching the pension fund investments from stocks to bonds.

In addition, the changes in the pension fund and in the capital structure do not affect the firm's leverage. In fact, by increasing the proportion of equities in the capital structure, the firm's performances will improve when the economic conditions are good, and will worsen when these conditions are bad. In economics terms, this means that the firm's leverage is increased. At the same time, shifting from equities to bonds in the pension fund has the contrary effect of reducing this sensitivity. Hence, in the Black plan the two capital shifts would compensate for each other, ultimately leaving the firm's leverage unchanged.

Having said that, the advantage of such a strategy lies in the *second order* effects, and in particular, in the special - advantageous - taxation regime in which a pension fund operates. Interest expense for the corporation is tax deductible and interest income for the fund is tax free. Hence, what effectively happens is: by directly borrowing at the tax deductible rate and lending through the pension fund at a tax free rate, the firm would benefit from the spread between the pre-tax and after-tax rates.

Black (1980) does recognise that an aggressive investment strategy has the effect of reducing the cost of pension provision by reducing the expected contributions, all of which is in exchange for an increase in the risk of not meeting the liability and in the variability of the surplus/deficit [6].

However, it is also Black (1980)'s opinion that:

> "Just as an increase in the risk of a firm's assets that doesn't change the firm's value will make the bondholders less secure, so an increase in the risk of the pension fund assets that doesn't change the fund's value will

[6] This view is in agreement with the thesis illustrated in chapter 4.

> make the beneficiaries less secure. Since investing pension fund assets in bonds makes the beneficiaries more secure than investing in stocks, it should make the trustees more secure too."

The same view is expressed by Bader and Gold (2003), where the authors simplified the concept by playing with the tautology that "$ 1 million of bonds has the same value at $ 1 million of equities". In the authors' view, this statement holds regardless of the expected value of these amounts in a future point in time, because once these are discounted, taking into account the different risks, the current value does not change.

Henceforth, the financial-economists strategy has the additional value of reducing the risk of insolvency by funding the pension liabilities exclusively with fixed income products.

However, Regulatory Authorities often require pension funds to insure their liabilities against the risk of insolvency. In the US, the Pension Benefit Guaranty Corporation charges a fixed premium for insurance coverage to each firm sponsoring DB pension scheme. Similar programmes exist in UK and in the EU.

This aspect is not considered in the financial economist framework, and it has important consequences on the optimality of asset allocations. In fact, Sharpe (1976) shows that, if taxation is not considered, the *insurance effect* calls for a "mini-max" strategy; e.g. reducing the funding to the minimum level, and investing the fund in risky asset, in order to maximise the value of the insurance.

This strategy is the exact opposite of Black's one, which is also called a "maxi-min" strategy, because it calls for maximum funding and minimum investing in risky assets, in order to maximise the benefits from taxation.

Furthermore, Harrison and Sharpe (1983) take into account both the insurance and the taxation effects, demonstrating that, in absence of market imperfections, the two opposite strategies mini-max and maxi-min are both optimal. Thus, the sponsoring firm should choose to implement one of the two *corner solutions.*

Bickler and Chen (1985) extend such research by including the cost of plan termination as a market imperfection. The results are interesting and show that, under these more general assumptions, corner solutions are not necessarily optimal. In fact, a mixture of equities and bonds can maximise the total firm value.

Another critical assumption in the financial-economics view is the fact that the actual pension fund is an integrated part of the firm. Thus, it is assumed that the firm has complete control in managing the pension fund and decides the asset allocation aiming to maximise the firm value. Hence, pension funding is one of the issues for the firm's management, which operates from the point of view of the shareholders.

Legally, this is not the case as a firm's pension fund is a body separated from the firm. Particularly in the UK, the fund's Trustees act primarily in the interest of the pension plan's member [7].

This aspect has importance consequences on the validity of the "irrelevancy proposition". Modigliani and Miller (1958) states that if the proposition is not valid: "an investor could buy and sell stocks and bonds in such a way as to exchange one income stream for another stream, identical in all relevant respects but selling at a lower price". Once this arbitrage opportunity is exploited the market would adjust the mispricing, thus eliminating the discrepancies between the market values of the firm.

However, pension plan members do not have the possibility to trade the promised benefit in the market. Hence, from their perspective - or alternatively from the Trustees' perspective - there are no financial operations which can re-balance a change in the pension fund asset allocation. Hence, plan members are not indifferent to the pension fund asset allocation, notwithstanding the capital structure of the sponsoring firm.

In this chapter, we contribute to the discussion by identifying optimal asset allocations through the simulation of a DB pension scheme, according to the mathematical models developed in the previous chapters.

In conclusion, it is necessary to acknowledge that the resulting optimal strategy depends on which of the two schools of thinking we decide to follow. The crucial assumption relies on the discount rate(s), in particular whether it is related to the

[7] Indeed, it is a classical argument in favor of the independently funded DB pension scheme that the pension fund is separated legally from the sponsoring employing: see, for example, Lee (1984).

investment policy. Bearing this in mind, in the following paragraphs we adopt the traditional approach endorsed all along in this thesis and we investigate how the investment and contribution strategies should be combined in order to obtain efficient funding strategies. For completeness, in §6.7 similar questions are addressed when a financial economic approach is implemented.

6.4.2 Two assets optimal allocation.

Our attention is initially directed to the choice of the proportion of the fund should be invested in equities, where bonds are the only alternative. Thus, cash is temporarily discarded from the range of choices and will be reconsidered in §6.4.3.

In order to identify the optimal investment strategy we set the following decision criterion: the optimal allocation is that combination of the two assets (bonds and equities) which minimises a pre-specified risk measure.

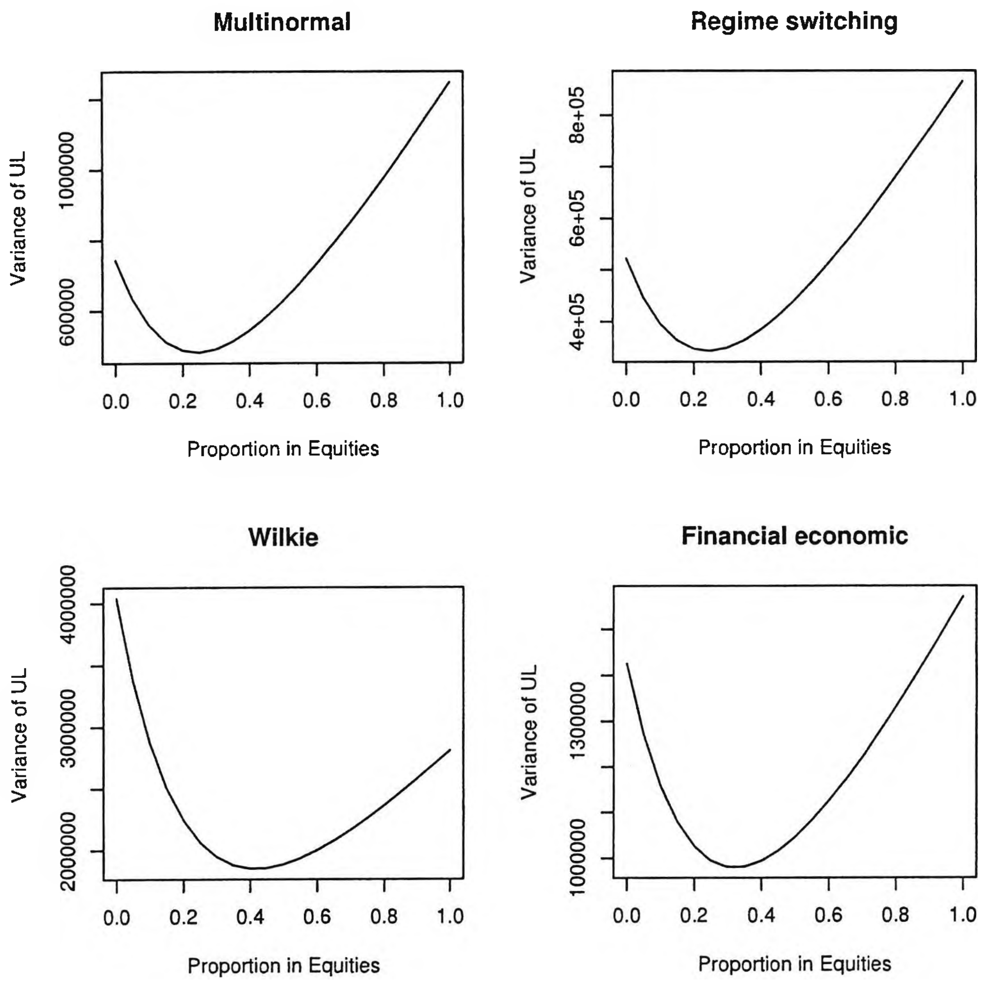
In a manner similar to the previous chapter, the variance of ul is the risk measure mainly used as the quantitative basis of the decision criterion. However, other measures have also been considered; namely, the **TCE** and the **MS**. This has been done with the intention of avoiding results which hold for only one measure and are of limited applicability.

Figure 6.1 displays the variance of ul at time $t = 90$ corresponding to an increasing proportion invested in equities. The time t is set equal to 90, because after $\omega - \alpha = 85$ years the first cohort of random new entrants is certainly eliminated, and thus, the number of members is random throughout the spectrum of ages. We perform the same calculations with all of the four models previously introduced.

The shape of the figures suggest that the variance has its minimum when the proportion invested in equities lies between 20% and 35% of the fund value. This result holds independently of the asset model considered.

The reason why this minimum exists comes from the mathematical form of the variance of ul, which, for this particular case, is expressed in equation (3.32). Let us recall it for convenience:

Figure 6.1: Risk *vs* proportion in equities.



$$\mathrm{Var}[ul(t)] = AL^2 \mathrm{Var}\left[\sum_{h=1}^{t} \phi(h)\right] \tag{6.14}$$

Function $\phi(h)$ summarises the mismatches between the rates of return $\{r(t)\}$ and the valuation rate (i) during the scheme operating life. In detail, $\phi(h)$ has the following form:

$$\phi(h) = \begin{cases} v(1-k)^{t-h}(r(h-1)-i)\prod_{j=h}^{t-1}(1+r(j)) & h \in (1,\ t-1) \\ v(r(t-1)-i) & h = t \end{cases} \tag{6.15}$$

Hence, the variance of *ul* is a function of the variance of the rates of returns (summarised by function ϕ), as well as of the value of the current AL [8].

Since equities are the high risk and high reward asset, an increase in the proportion of the fund value invested in this asset has the effect of increasing the variance of the rates of returns. This effect is in turn passed on to the variance of ϕ and finally to the variance of *ul*.

Hence, the higher the proportion of equities, the higher the volatility of *ul* due to the random nature of the investments. This result is absolutely consistent with economic theory, according to which a higher premium is paid to those willing to accept a higher risk.

In contrast, the value of AL depends on the valuation rate (i) used for discounting the future liabilities to the time of valuation: in particular, the higher this rate, the lower is the value of AL. Since the valuation rate is set equal to the expected rate of return, a higher proportion in equities determines a higher i, and hence, a lower value of the actuarial liability.

Adding up these two effects, we find out that a minimum exists and actually arises whatever model we use to simulate the rates of return. In fact, although there are some differences in terms of optimal asset allocation, it is of interest to note that this minimum lies in a rather narrow interval (20%—40%).

[8] In this case AL is deterministic and constant because of the assumptions of a deterministic and stationary pension plan population. Refer to chapter 3 for a fuller discussion.

6.4.3 Adding a third asset to the allocation problem.

As already mentioned, we now include a third option into the choice of how to allocate the available fund. Specifically, we consider the third asset - cash - as the lowest risk and lowest return asset. We refer to this asset also as a short term rate, providing a risk free rate [9].

The calibration of the parameters consistently with the current market conditions makes the short term rate an asset with a very low volatility and a low expected return [10]. As a consequence of this initialisation of parameters, we do expect this asset to be efficient because it reduces the variability of ul to almost zero.

The optimal allocation is computed and plotted, for the cases of cash and bond (Figure 6.2) and cash and equities (Figure 6.3) as the two only available assets, the dynamics of which are described by the financial economic model. Three risk measures (variance, **TCE** and **MS**) are displayed.

The low level of variability characterising the dynamics of the short term rate leads an $all - cash$ portfolio to be the optimal strategy, regardless of which risk measure is used (out of the three considered). Less trivial than it sounds, this result suggests that the safest investment strategy is investing in the safest asset [11].

From a different standpoint, we can say that the premium for accepting a higher risk asset is not worthwhile, as the reduction gained in the (expected value of the) future liabilities would not compensate for the higher variability of ul due to the more volatile portfolio. This means that $all - cash$ is the safest investment strategy possible.

However, as we have seen in chapter 4, such a strategy would have severe implications on the long term cost of the pensions provision, therefore whether $all - cash$ would provide a viable strategy has to be analysed in a broader context including the cost of pension provision as a decision variable.

It is of interest to remark that if a fixed proportion of the fund value would be

[9] To be precise, the short term rate $r(t)$ is risk-less only in an infinitesimal time interval; refer to Björk (1998) for a deeper discussion.

[10] Refer to appendix B for a summary of the used parameters.

[11] Similar results hold when the other financial models are used. In Appendix C.1 graphs displaying these results are shown.

Figure 6.2: Optimal allocation in bonds versus cash.

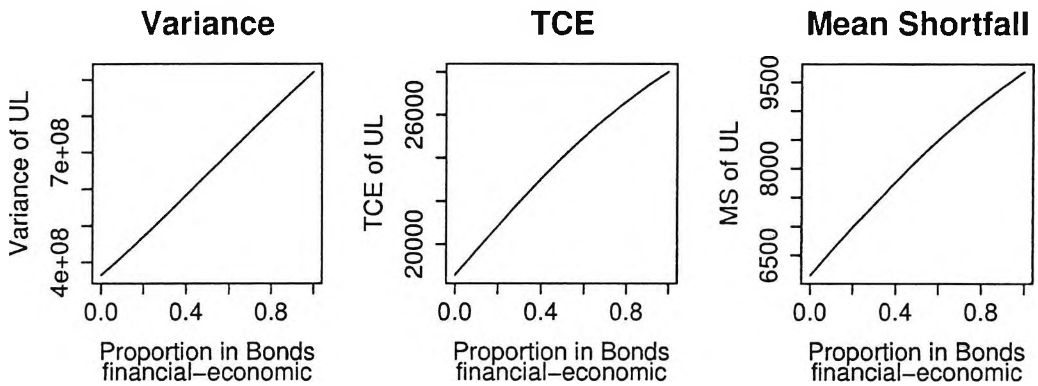
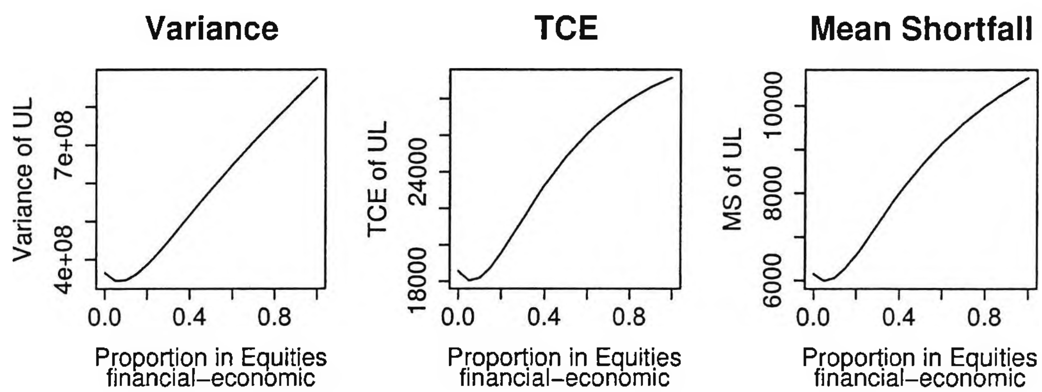


Figure 6.3: Optimal allocation in equities versus cash.



invested in cash, the optimal allocation of the remaining part of the assets between bonds and equities should be accordingly adjusted. According to Cairns (2000), in the UK, pension schemes only use cash for short term liquidity to cover immediate benefits, and not as an asset alternative to equities and bonds. Furthermore the cited author quotes 5% as a typical figure for the fixed proportion in cash.

The following table displays some of the numerical figures of this analysis. Specifically, β_r is the fixed proportion allocated in cash, β_S^* is the optimal allocation in equities, and $\beta_S^*/(1-\beta_r)$ is the optimal allocation in equities rescaled to that part non allocated in cash (i.e. the line in figure 6.4) [12].

Table 6.1: Table of optimal allocation in equities, with fixed cash.

β_r	β_S^*	$\beta_S^*/(1-\beta_r)$
0%	33.5%	33.5%
20%	24.8%	31.0%
40%	18.3%	30.5%
60%	12.1%	30.2%
80%	6.6%	32.8%

Figure 6.4 graphically illustrates these particular results, by showing the optimal allocation in equities of the fund not invested in cash.

It can be seen that an unpredicted effect arises. Up to a certain proportion ($\leq 60\%$) the higher the fixed allocation in cash, the lower the optimal proportion in equities.

This happens, because the reduction in variability - due to a higher proportion in cash - compensates for the increase in AL. So the more cash in the portfolio, the less the need to reduce AL by investing in a high risk - high return asset.

However, this rule is overturned when the proportion invested in cash exceeds the threshold of 60%. Apparently, an increasing part of the fund should be invested in equities if an already large amount has been allocated to the safest asset[13].

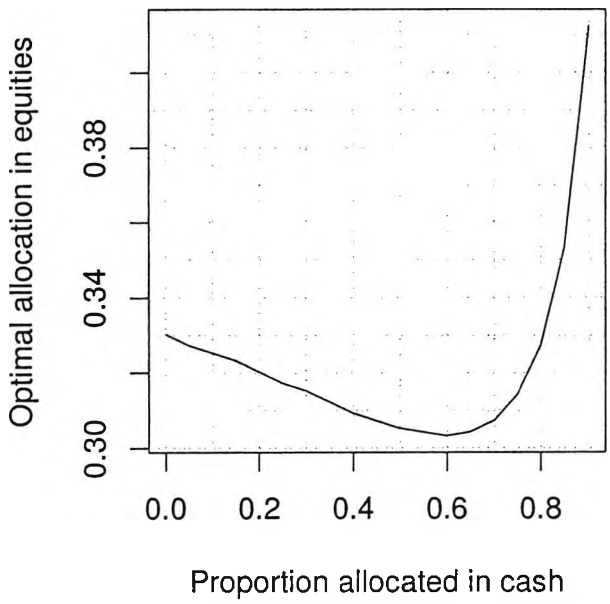
6.5 Adding a random demographic component.

As illustrated in chapter 3, the evolution of the pension plan population tends not to be deterministic. In this section, we are interested in investigating how a randomly evolving membership population would affect the asset allocation of the fund.

[12] For instance, if 20% of the fund is in cash, then 31.0% of the remaining 80% of the fund is the optimal allocation in equities. Hence, the final allocation is 30% cash, 55.2% bonds and 24.8% equities.

[13] Although with slightly different numerical figures, the other financial models lead to results conceptually consistent; refer to Appendix C.1 for additional figures.

Figure 6.4: Optimal allocation in equities, with fixed cash. Financial economic model



In particular, here we consider the situation where the number of annual new entrants into the scheme varies according to a stochastic process. The implications of such an assumption on the overall risk faced by a scheme are analysed in chapter 3.

The analysis is run according to the following methodology:

- Having fixed the year $t = 90$, we simulate the rates of return on the basis of one of the models proposed at the beginning of this chapter. Out of the three assets we take into account bonds and equities only, since cash is efficient in terms of risk.

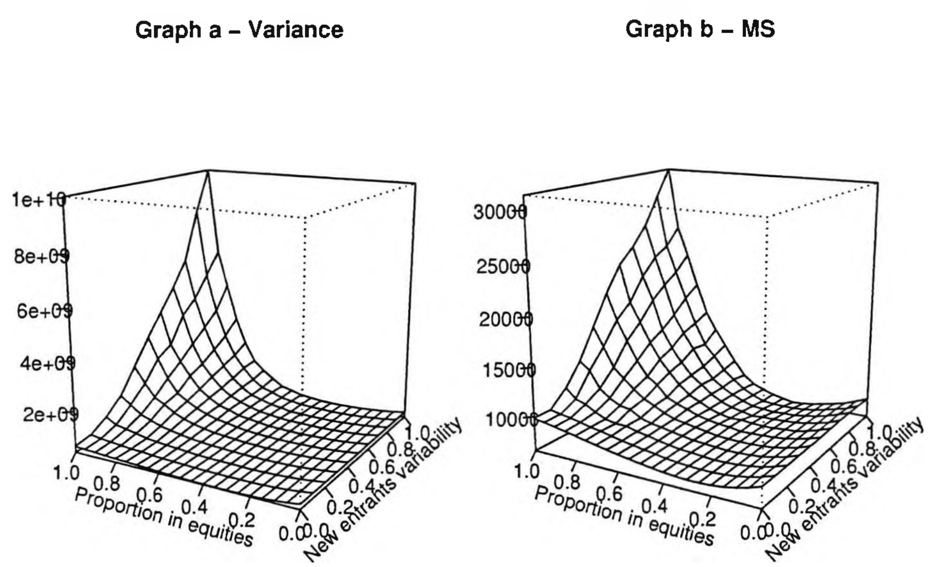
- Then we simulate the number of annual new entrants according to a normal distribution and we repeat the same operation increasing the variability of new entrants (using the same seeds, in order to eliminate the error due to accidental deviations) [14].

[14]Note that negative new entrants are not allowed by artificially setting negative numbers equal to

- we compute risk measures with respect to ul, corresponding to different asset allocations.

Figure 6.5 displays the result of such a procedure, by means of showing the surface generated by the risk measures. Thus, the variability of the new entrants process is on the z-axis; the asset allocation is on the x-axis; and finally, the value of the risk measures (variance in graph (a) and mean shortfall in graph (b)) are on the y-axis [15].

Figure 6.5: Optimal allocation in equities with increasingly variable random new entrants. Financial economic model.



It is of interest to note that the optimal allocation in equities lies in a proportion between 30% and 35%, independently of the variability of the process for new entrants.

zero. This procedure has the effect of creating a distortion in the normal distribution, by means of generating a higher concentration of no new entrants for the cases of higher variability. However, this distortion significantly affects the symmetry of the distribution only for a sufficiently large standard deviation (say at least 45% of the expected value).

[15]The variability is increased by choosing a standard deviation rising from 0 to one times the expected value of the number of new entrants. The asset allocation is expressed as proportion invested in equities.

Similarly, the figure provides evidence that a heavy investment in equities becomes relatively riskier as the demographic variability increases.

It is of comfort to note that effectively there is agreement in the response of the two risk measures in providing the optimal asset allocation. In fact, the **TCE** has also been computed, leading to consistent results. The results are available but are not shown.

Remarkably, it seems that the downside risk measures are more sensitive to the allocation of capital while identifying the optimal investment range. As a matter of fact, the surface in graph (b) is more convex than the one in graph (a).

In addition, these results also hold when the rates of return follow any of the other considered models [16].

As an extension of these results, we also analyse the value of those risk measures at different points in time, and the results suggest that the optimal asset allocation does not significantly change. However, the peak observed in both graph (a) and (b) of Figure 6.5 for the high proportion in equities appears only after the random perturbations start affecting the retired population. Specifically, in figure 6.6 it can be observed that the peak becomes evident after $80 - 85$ years of the scheme's life, thus suggesting that an over exposure of equities may cause a severe variability of the unfunded liability, when (almost) all of the retired population's age distribution is random.

6.5.1 Small and large schemes.

It is of interest here to address the following question: does the size of the pension scheme matter? Or, in other words, do large and small schemes share the same optimal allocation?

In order to answer these questions, we first take into account the case in which the rates of return are the only source of uncertainty. So by varying the deterministic (constant) size of the scheme, we focus on the issue as to whether small schemes should operate differently from large schemes.

[16]Refer to appendix C.2 for the complete collection of graphs for the other 3 asset models.

Figure 6.6: Optimal allocation in equities with increasingly variable random new entrants over time.

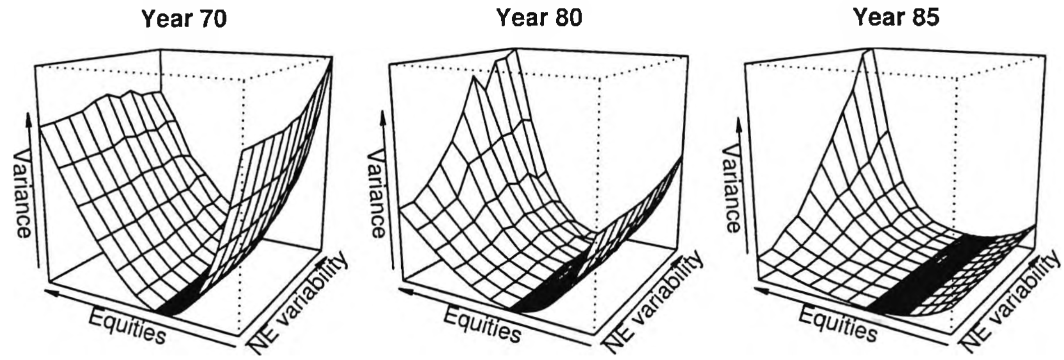


Figure 6.7 displays the variance of ul for different proportion of equities, in both the cases where the expected number of annual new entrants is equal to 1000 and to 50 [17]. The rates of return are simulated according to the financial-economic model, however, the other models lead to similar conclusions, as the figures in Appendix C.2.1 show.

These figures suggest that the population size does not matter to a material extent, and thus, the optimal allocation of the assets should be performed in exactly the same way regardless of the size of the scheme.

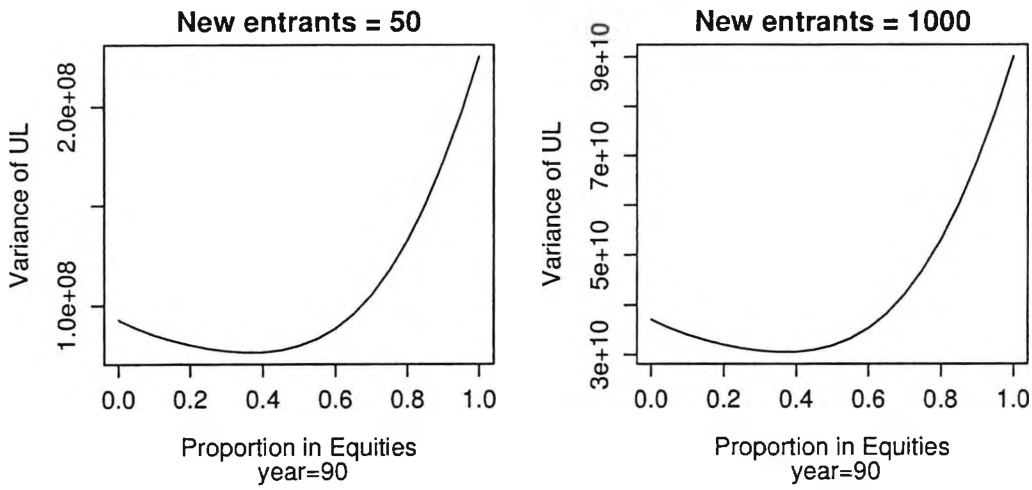
The current analysis is further extended by assuming that the number of annual new entrants randomly changes. In particular, we assume that the distribution of the new entrants is the same for both the schemes, whereas the expected number of new entrants in the large scheme is higher than in the small one. The coefficient of variation is the same for both the schemes [18]. Figures 6.8 illustrate the results of this investigation.

In such an environment, the optimal allocation does not seem to be affected by

[17]Since the number of eliminated members is deterministic, the large scheme reaches its maturity with a population of roughly 17,000 members (active and retired), whereas the level of stability of the small scheme is at about 900 members.

[18]In order to remove randomness from the results, we have used the same randomly generated rates of return, as well as the same seed in order to simulate the number of annual new entrants.

Figure 6.7: Optimal allocation in equities in small and large pension schemes.



the size of the scheme. Figures in Appendix C.2.1 show that this result also holds when different risk measures are considered; namely, the **TCE** and the **MS** measures.

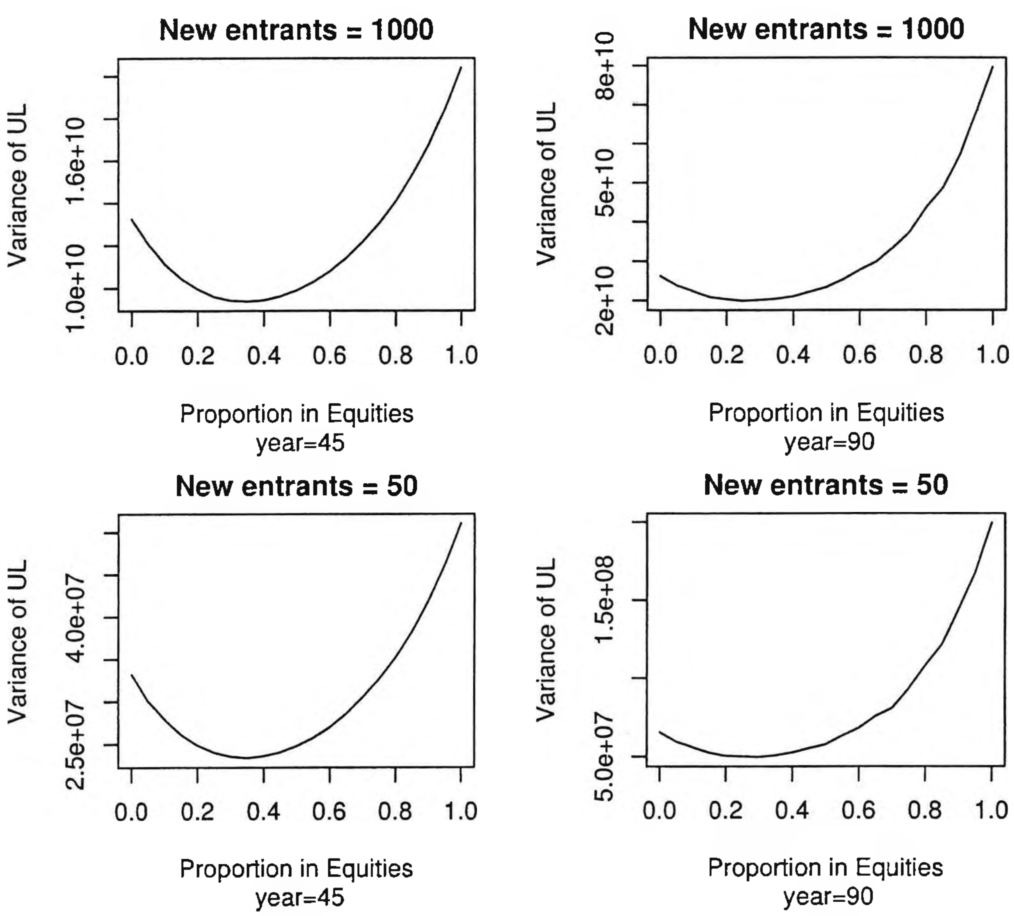
In addition, the graphs display the risk plane at the two particular years 45 and 90. The figures suggest that this result holds at any point in time.

6.6 Investment and contribution strategies.

Here we investigate the relationship between the asset allocation strategy and the contribution strategy, aiming to identify efficient strategies which take into account the overall risk (measured by the variance of *ul*), as well as the cost of implementation. Using the financial-economic model to generate random rates of return of bonds and equities only, a collection of graphs displaying the variance of *ul* against the proportion invested in equities is presented in the following sections.

Specifically, classical normal cost methods are considered before analysing the more general family of power function normal cost methods.

Figure 6.8: Optimal allocation in equities in small and large pension schemes, with random new entrants.



6.6.1 Classical normal cost methods and asset allocation.

Chapter 4 deals, among many issues, with four normal cost methods that we have defined as classical, namely: entry age (EA), unit credit (UC), initial funding (IF) and terminal funding (TF).

Each of these methods has a characteristic path of fund accumulation, determined by the payments schedule. Furthermore, this difference in amount and timing of the payments leads to a specific level of advance funding. Hence, each normal cost method has a distinctive way of substituting the contribution income with income from the investments. As illustrated in §5.3.3 (Figure 5.3), IF and TF are optimal strategies, whereas EA and UC show some degree of inefficiency, in the cost/risk plane.

Figure 6.9 displays the variance of ul plotted against the proportion of the fund invested in equities. Specifically, the four methods IF, EA, UC and TF are displayed in graph (a), (b), (c) and (d), respectively.

A first consideration concerns the range of variation of the minimum. As is evident from the figures, the minimum varies in a narrow interval (see table 6.2). Moreover, bearing in mind the classification presented in chapter 4, as the level of advance funding - L_{AF} - increases, so does the optimal amount to be invested in equities.

In fact, the IF is the strategy with the highest L_{AF} , followed by EA, UC and finally by TF which has the lowest L_{AF} . So looking at the four graphs, clockwise starting from (a), the optimal proportion in equities decreases. Table 6.2 displays the numerical figures. Specifically, the optimal proportion in equities is in the first column, the L_{AF} computed using a fixed force of interest $\delta = 0.03$ is in the second column, and finally the L_{AF} computed using the optimal allocation in equities is in the last column. The figures support the interpretation suggested by the lines in Figure 6.9. Moreover, Table 6.2 also shows that when contribution and investment strategies are combined, the resulting L_{AF} is much higher than when fixed δ is used.

This result suggests that the investment strategy should support the contribution strategy; i.e. the more a contribution strategy relies on the rates of return, the higher should be the proportion of assets invested in equities. However, this rule is limited to the narrow interval previously identified.

A possible explanation is based on the belief that the longer a portfolio holds risky

Figure 6.9: Optimal allocation in equities, for different contribution strategies.

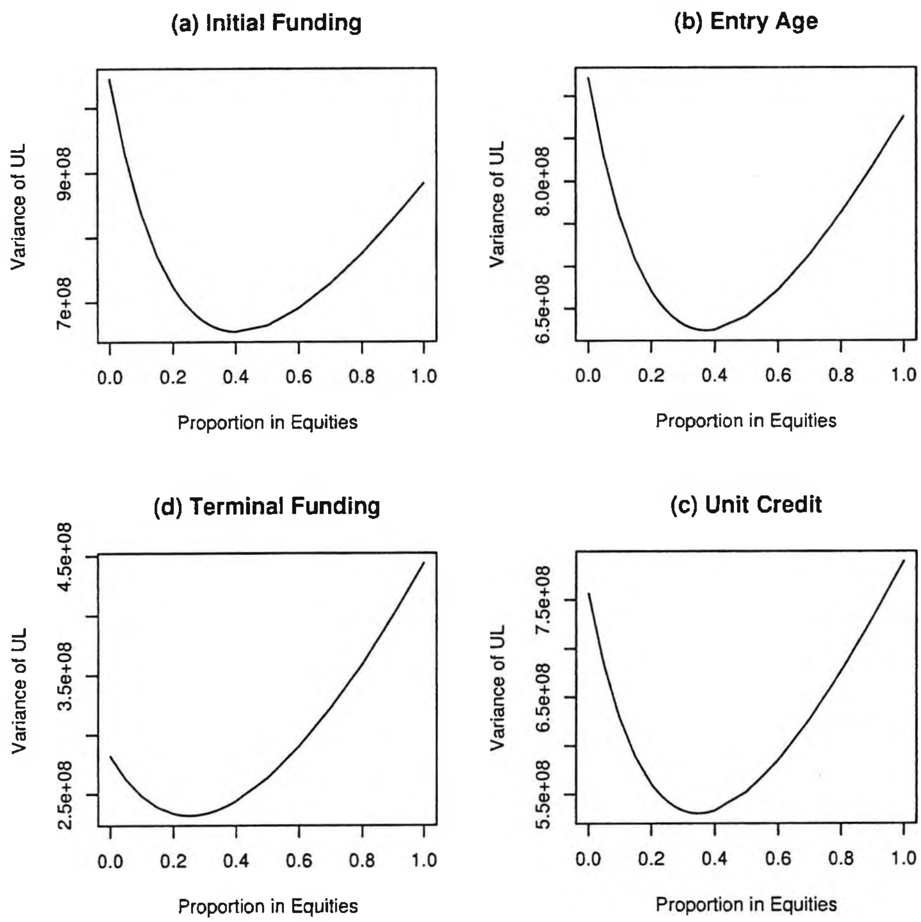


Table 6.2: Table of optimal equity allocation and L_{AF} for classic normal cost methods.

Normal cost method	β_S^*	L_{AF} (δ=0.03)	L_{AF} (optimal δ)
Initial Funding	34%	3.78	37.5
Entry Age	31.5%	3.34	24.2
Unit Credit	29.5%	2.12	9.50
Terminal Funding	22%	1.03	1.1

assets the more it may benefit from the higher expected returns.

As anticipated by the considerations illustrated in chapter 4, it can be seen from the scale on the y-axis that low L_{AF} contribution strategies lead to low values of risk. In addition, it is evident that, for the same strategies, the risk is less sensitive when it comes to implementing safer strategies, i.e. with proportions allocated to equities between 0 and 20%.

6.6.2 Power accrual function and asset allocation.

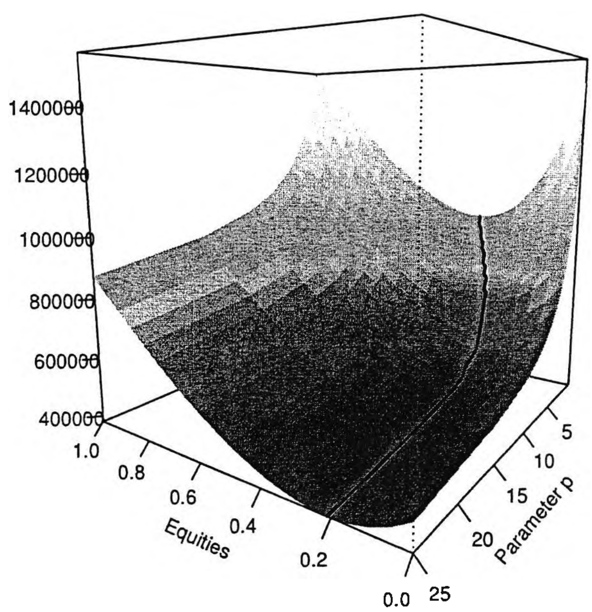
Let us extend this analysis to the wider class of normal cost methods deduced from assuming that the pension accrual density function is a power function as described in section 4.1.1. The financial assumptions are left unchanged: thus, we take into account bonds and equities as the two possible investments.

Risk of power accrual methods.

The following graph displays the variance of ul when various funding strategies are implemented. In particular, combinations of values for the parameter p and for the proportion in equities β_S are considered.

It can be seen that for each investment strategy, the variance decreases as the parameter p increases. The figure from this perspective replicates the analysis run in chapter 4.

Figure 6.10: Variance of ul, with power density normal costs.



More interesting is the fact that the peculiar concave shape of the surface suggests that for each level of the parameter p, there exists only one proportion (to be invested in equities) which minimises the variance. This proportion is highlighted by the black line and varies between 25% – 35% according to the level of the parameter p.

This is consistent with the previous analysis of well known normal cost methods, and the results suggest that the earlier the contributions, the higher the proportion in equities. For instance, the numerical analysis reports that for p very close to 0 (so the IF contribution strategy), 35% of the fund value should be invested in equities. When the parameter p is equal to 1 (unit credit method), the optimal allocation decreases to 30%. Finally, for $p = 25$ (roughly the TF contribution strategy), this analysis suggests 25% as the optimal proportion.

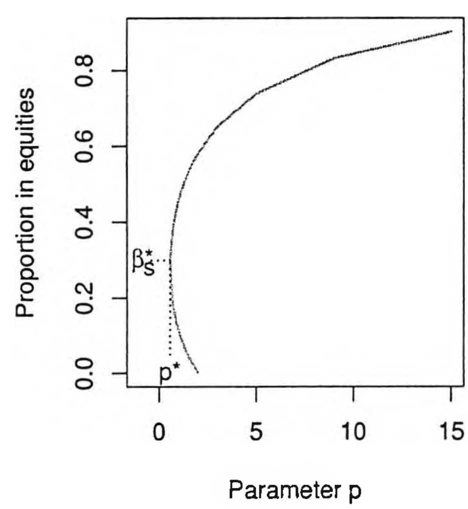
From a different perspective, we can observe that the space of strategies having the same risk is infinite. In other words, once the level of acceptable risk has been decided, there are an infinite number of combinations of contribution and investment strategies leading to this value of the variance. All these strategies identify a line of *isorisk* on the surface in figure 6.10.

The results of this kind of analysis are displayed in figure 6.11, where the variance of ul has been fixed at a defined level (depicted in yellow in Figure 6.10) and the equivalent combinations of p (on the x-axis) and β_S (on the y-axis) are derived.

The line suggests that there is a particular value of the power parameter (let us indicate it with p^*) which can be combined only with one β_S in order to satisfy the required condition that the variance of ul is equal to a pre-specified value. For all the other $p > p^*$, there are always two possible investment strategies leading to the same risk. The reason why this happens comes from the expression of the variance of ul and it is similar to the explanation given in §6.4.2 for the existence of a minimum variance. A low proportion in equities would have a two fold effect: it decreases the financial risk (in term of deviation from the mean) and it increases the total amount of the liabilities. Conversely, a higher valuation rate determines a lower value of the liabilities at the expenses of more volatile assets. This explain why two possible asset allocations lead to the same value of the variance of ul for each $p > p^*$.

Furthermore, the unique value ${\beta_S}^*$ associated with p^* is the optimal proportion of

Figure 6.11: Pairs of equivalent contribution and investment strategies, with respect to risk.



fund to invest in equities. The concave shape of the surface in Figure 6.10 supports this assertion. In fact, Figure 6.12, displaying the variance of ul for different asset allocations combined with p^*, shows that $\beta_S{}^*$ minimises the risk measure.

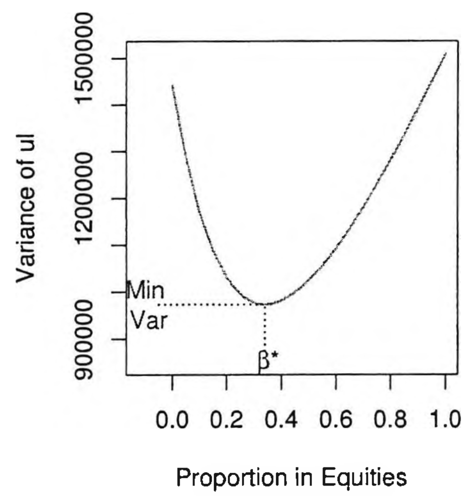
Thus, this combination of investment and contribution incomes is efficient, in the sense that employing a different β_S - together with the same p^* - would increase the risk, as measured by the variance of ul.

Figure 6.11 also illustrates that, for $p < p^*$, there is no proportion β_S which leads to the same level of risk. Bearing in mind the shape of the surface in Figure 6.10, it is evident that contribution strategies with $p < p^*$ necessarily lead to a higher risk.

This suggests an interesting result: having set the maximum level of acceptable risk, the efficient way of funding a retirement benefit is by contributing as early as possible. However, the level of risk excludes the case of "too early" funding, but identifies the unique optimal asset allocation.

If there is the need of reducing the maximum level of acceptable risk, the only way is by increasing the parameter p; e.g., by delaying the contributions. This approach would increase the cost of pension provision, thus recreating the tradeoff between risk

Figure 6.12: Variance of *ul* for different proportion in equities combined with p^*.



and cost. This aspect is investigated in the following section.

Cost of power accrual methods.

In chapter 4 the risk and cost of contribution strategies have been extensively analysed. However, in this analysis the discount rate was held fixed [19]. Thus the characteristic L_{AF} of a contribution strategy (to which risk and cost are linked) is explained exclusively in terms of the timing of the payments into the fund.

Specifically, the long term cost of a contribution strategy is low when L_{AF} is high; while L_{AF} increases when the contributions are paid at an early stage of the working life-time.

A natural extension of that work considers a variable valuation rate. Specifically, in this chapter the valuation rate varies according to the asset allocation. In fact, a higher proportion of fund invested in equities leads to a higher expected return at the expenses of a higher risk.

[19]In this case we refer to the general valuation rate, without specifying whether is it an annual effective valuation rate (i) or it is a continuous force of valuation (δ). In fact, the argument holds independently of the assumptions of continuous or discrete time.

The insight we want to exploit is that a higher expected returns should reduce the long term cost of a contribution strategy, and thus different combinations of p and β_S should lead to same L_{AF} and perhaps identify efficient funding strategies.

To start with, let us analyse the cost of a contribution strategy as a function of both p and β_S. Recalling the two expressions of the expected cost already given in equation (4.22), we can further extend it considering a more general case, as in the following equation in continuous time:

$$\mathrm{E}\big[\text{cost}\big] = Bs_R \; {}_{R-\alpha}a_{\alpha}^{(\delta_{\beta_S})} \cdot \int_{\alpha}^{R} m(x) e^{\delta_{\beta_S}(x-\alpha)} \mathrm{d}x \tag{6.16}$$

where the force of valuation δ_{β_S} is function of the asset allocation (i.e. the proportion β_S invested in equities).

Equation (6.16) suggests that the expected cost decreases when the density $m(x)$ is higher at older ages. On the contrary, it is not clear whether the cost should increase with δ_{β_S}, as the value of the annuity decreases and the value of the integral increases. However, intuition would suggest that the expected cost should decrease with the valuation rate, as a higher proportion of the final benefit would be funded by the higher (expected) returns.

Figure 6.13 displays the cost of the funding strategy, for different contribution and investment strategies [20] In particular, on the x-axis there is the parameter p (the higher p, the lower L_{AF}), and on the y-axis there is the continuous force of valuation δ_{β_S}. Isocost lines have the same colour.

Stripes of the same colour can be seen from one end of the surface to the other. Hence, at the same cost, the fund manager can implement two different and (to some extent) opposite strategies: either requiring late contributions and then investing them into higher expected return assets; or, combining early contributions and lower expected return investments.

Indifference on the plane of cost simply means that from the point of view of implementation cost there is equivalence between the two strategies. However, we have seen in previous chapters that cost is not the only feature considered when

[20]The financial economic model is used to generate the investment returns and the fund can be allocated either in bonds or inequities. The number of new entrants is deterministic and constant.

Figure 6.13: Cost of funding strategies for increasing L_{AF} and force of valuation δ.

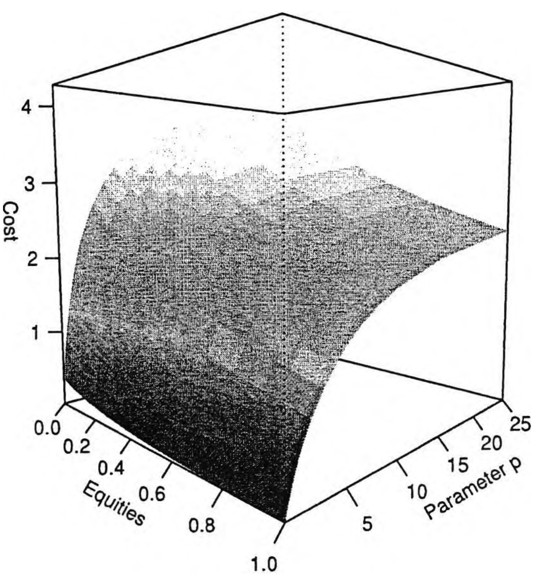
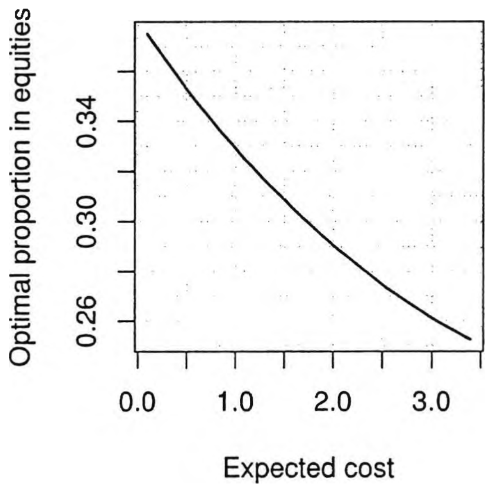


Figure 6.14: Optimal funding strategies.



choosing how to fund a retirement benefit, as the risk of meeting the liabilities plays a crucial role as well. In fact, strategies sharing the same cost are not equally risky - at least not necessarily - and thus the question to be answered is whether it is possible to find efficient combination of contribution and investment strategies which simultaneously minimise the cost and the risk.

Optimal funding.

If we fix the cost that the scheme is willing to face in order to fund the retirement benefit, it is possible to identify the combinations of p and β_S which leads to the same cost.

Figure 6.14 displays the efficient frontier of optimal allocation equities for different (expected) costs of pension provision. In agreement with the results relative to classical normal cost methods, the optimal proportion in equities (which minimises the variance) decreases as the expected cost of implementation increases.

Looking at the numerical figures in Table 6.3, we can see that, for instance, having set the expected cost equal to 1, a contribution strategy identified by $p = 0.86$ (so with large L_{AF}) combined with an allocation of roughly 33% of the fund value in equities, minimises the variance of ul [21]. Any other combination of β_S and p which generates a contribution strategy of the same expected cost, will also generates a higher risk.

Table 6.3: Table of optimal funding strategies.

Expected Cost	p	*Optimal equity alloc.*
0.102	0.001	0.375
0.449	0.287	0.356
0.735	0.572	0.341
0.977	0.858	0.331
1.084	1.000	0.325
1.661	2.000	0.302
3.048	9.000	0.260

So for instance, a proportion of 30% requires a (rough) value of $p = 0.2$ in order to minimise the variance of ul, however, such a combinations produces a set of normal contribution having an expected cost equal to 1.67. Hence, if there is the need to reduce the expected cost up to 1, while leaving an investment of 30% in equities, this can be achieved by increasing the L_{AF} , i.e. decreasing p. However, the resulting combination would be sub-optimal, since the variance of ul will not be at its minimum.

It is of interest to note that the range of variation of the optimal allocation in equities is rather narrow, whereas the expected cost varies from being almost null to a fairly high cost.

[21] From the graph, it is not possible to extract the value of p. However, this is uniquely determined once the expected cost and the valuation rate (inferred from the asset allocation) are fixed.

6.7 Optimal strategies under a financial economic approach.

As part of the pension funding literature suggests, the characteristic structure of the pension debt, together with the theoretical support of the Modigliani-Miller theorem, invites us to consider the possibility of funding the retirement benefit by exclusively investing in bonds.

Although this approach has been widely criticised by the more classical actuarial literature, this method provides a measure of certain interest. In fact, if the scheme were to wind up at the time of the valuation, the current fund would be invested in a low risk, and low return, asset in order to ensure its value throughout time with the highest degree of confidence. From this perspective, the valuation of the liability, at a valuation rate computed from bonds, measures what would be the necessary capital to cover future liabilities in case the scheme were to stop its activity.

In light of this, we have carried out an analysis of optimal allocation in equities, where the liabilities are valuated in two different ways.

A first approach consists of valuing the liabilities at a rate equal to the expected return from investing in bonds. Thus, the valuation rate is constant and the financial realisations do not affect the actuarial liability. However, a systematic mismatch between the rate of return from investments and the valuation rate arises with consequences which will be analysed in the forthcoming section.

A pure market based approach is then implemented, whereby the actual annual rate of return from bonds is used to value the liability. Haberman *et al* (2003) employ a similar approach, whereby the liabilities are valued using the current market conditions (the real yield index-linked bond, as generated by the Wilkie model).

Using such an approach, the resulting liabilities are volatile due to the variations in the financial markets, which affect the valuation rate.

A (third) hybrid approach, whereby the valuation rate is determined as a smoothed value of a number of previously realised returns, could be implemented in order to provide more stable estimates of the actuarial liability. However, the aim of this section is to show the effects of a financial economic approach on optimal asset allocation, thus we leave out the development of extensions for future research [22].

6.7.1 Expected return bond based valuation.

Before illustrating the results, it is of interest to explain what kind of results are expected. Using the expected rate of return from bonds as the valuation rate used for discounting future liabilities leads to two main problems.

First, throughout the analysis the spreading surpluses/deficits method is used in order to deal with the *ul* arising from the mismatch between the assumptions made and the actual realisations. Such a method of supplementary funding has been devised in order to deal with the situation whereby the valuation rate coincides with the expected rate of return. However, in this case the valuation rate is no longer a function of the asset allocation, as it is fixed and equal to the expected rate of return from bonds. Under this circumstances, the framework in §6.2.5 is implemented in order to guarantee that *ul* is finite in the long term. The alternative method of amortising gains/losses, as described in Dufresne (1988), is affected by the same problem. A more general approach is proposed in Owadally (2003), where the method of spreading surplus/deficits is modified in order to overcome this specific problem.

Second, the assumption that the valuation rate is a function of the asset allocation explains the results obtained in the previous sections. In fact, we have seen that a higher valuation rate on one hand reduces the magnitude of AL; while on the other it increases the variability of the financial performances. Conversely, a lower valuation rate increases the value of the future liabilities, whereas it reduces the investment volatility.

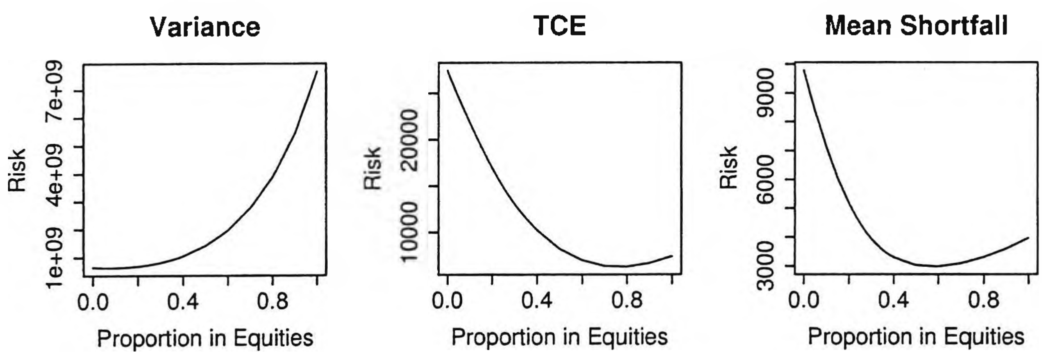
In this specific case, where the valuation rate is independent of the asset allocation, a change in the proportion of fund invested in equities is not going to affect the valuation of future liabilities. Hence, only the financial part of the variability of *ul* will be sensitive to the investment strategy.

In the light of this, Figure 6.15 illustrates three different risk measures in order

[22] A possible avenue is identified in Haberman and Vigna (2002), who investigate such an approach in the case of defined contribution pension schemes.

to capture different aspects of the way in which asset allocation is going to affect the overall risk of a DB pension scheme [23].

Figure 6.15: Optimal allocation in equities.



The variance of *ul* suggests that the strategy which determines the lowest variance of the return is optimal. With the current values of the parameters, this means that the an *all-bonds* (or a very low holding of equities) is the optimal strategy. This result is not surprising in the light of what has been discussed above.

By taking into account the tail of the loss, the other two risk measures are more sensitive to the allocation strategy. In particular, both **TCE** and **MS** show that a high allocation into equities could be beneficial in terms of risk. In detail, the **MS** suggests that investing around 50% in equities is optimal, whereas the **TCE** requires an even higher proportion.

The discrepancy between the two downside risk measures and the variance arises because a return higher than expected, reduces the probability of a positive *ul* (i.e. when the liabilities are larger than the assets), with the ultimate effect of reducing the risk measures. In contrast, since the variance measures the dispersion around the mean, it does not capture the fact that the expected *ul* decreases because of a systematic mismatch between the average (rate of) return from investments and the

[23]Returns are generated with the financial economic models. By carrying out the same analysis with the other financial models, the results seem to be conceptually consistent with those here illustrated and they are presented in Appendix C.3, Figure C.8

valuation rate.

6.7.2 Market bond based approach.

As already mentioned, a market value approach is also included in this section. In particular, the valuation rate is set equal to the last experienced rate of return from bonds. So if valuation is at time t, we are interested in the simulated return for the year $(t-1,\ t)$.

On the one hand, this assumption allows us to investigate the issue of asset allocation when the liabilities are measured at a real cost, in the sense that expected but not yet realised returns are not accounted for. Thus, this measure provides a realistic valuation of the cost which the pension scheme would incur if it were to wind up at the time of valuation.

On the other hand, this assumption increases the variability of the estimates of AL, because of the volatility of the valuation rate. This happens because bonds are not risk-free products and hence the returns provided are not deterministic but random. For this reason, the market rewards the scheme for taking on this risk with an extra premium on top of the risk-free rate provided by cash.

Figure 6.16 displays the three considered measures of risk when the proportion of equities varies, and remaining portfolio is invested in bonds [24].

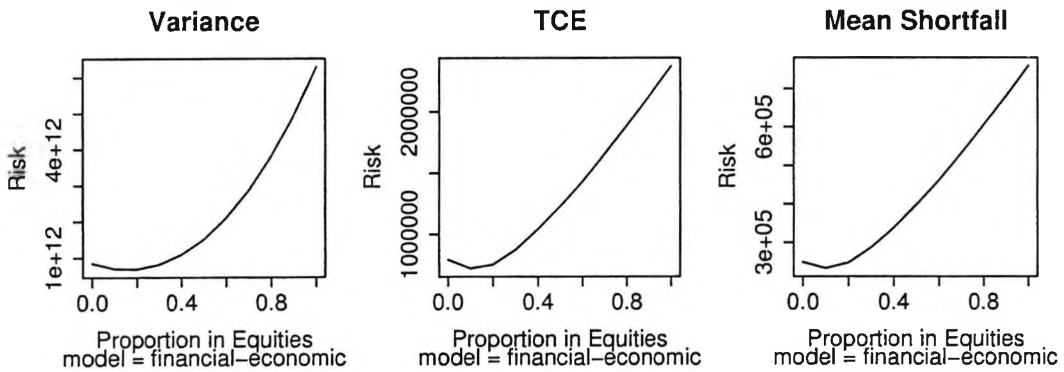
These graphs suggest that it is optimal to invest a relatively small proportion of the fund value in equities. Such results are also consistent with those obtained in Haberman *et al* (2003).

As we have previously seen, increasing the proportion invested in equities has a two fold effect: it increases the variance of ul by increasing the variability of the function ϕ, which summarises the returns from the market. In addition, it reduces the probability of positive ul, which in turn reduces the downside risk measures.

However, when AL is valued at a constant valuation rate (as in section 6.7.1), the reduction in the probability of positive ul compensates for the increase in the expected ul (conditional on being positive) and so the resulting downside risk measure is lower

[24] The returns are simulated according to the financial economic model. Similar figures concerning the other financial models are presented in Appendix C.3, Figure C.9

Figure 6.16: Optimal allocation in equities.



with a higher proportion in equities. In contrast, this reduction of downside risk measures does not happen when the valuation rate is equal to the random rate of returns from bonds.

6.8 Summary.

In this chapter, the results relative to optimal contribution strategies are extended to the more general case of optimal funding strategies. Specifically, the level of advance funding is analysed in terms of the scheduling of contributions (as in chapter 4 and 5) and of the investment strategy.

Following a traditional actuarial approach to pension funding, whereby the discount rate is related to the asset allocation, we have shown that a considerable proportion of assets should be invested in equities in order to minimise a set of selected risk measures.

The optimal proportion of assets to be invested in equities varies according to (a) the financial model implemented to forecast future investment returns; and (b) the calibration of the parameters. Nonetheless, a remarkably narrow interval can be identified, once the parameters are estimated given the data on the specific financial market where the considered DB pension scheme operates.

Adding a risk-less asset (cash) to the set of possible choices, we find that such an asset is efficient in terms of minimising the risk. Taking into account the unfeasibility of the *all-cash* strategy, we find the optimal allocation in equities when a fixed fraction of the fund is invested in cash for liquidity purposes.

Optimal equity proportions are relatively insensitive to the random variation of the pension plan membership population; although, a high variability in the number of new entrants can eventually make risky strategies much riskier. Conversely, under the considered specifications, the size of the scheme does not seem to affect the investment strategy.

Analysing combinations of contribution and investment strategies, we find evidences that the investment strategy should support the contribution strategy. In other words, the more the normal cost method employed relies on investment returns in order to fund the retirement benefit, the more it should be invested in risky assets, in order to reduce the computed risk measures. Evidence for this is found using classical normal cost methods, as well as implementing the more general family of the power function cost methods.

In a concluding section, the same questions are addressed when a financial economic approach to pension funding is implemented. Specifically, liabilities are evaluated with two different discount rates: (a) the expected return from bond investments; (b) the returns from bond investments as read from the market at the valuation date.

In case (a), the variance suggests that an *all-bond* strategy is optimal, thus supporting the financial-economic view that assets should be invested exclusively in bonds. However, the variance - measuring the variability of ul around the mean - fails to identify the advantage of a systematic positive mismatch between the expected rate of return and the discount rate. By contrast, the two downside risk measures **TCE** and **MS**, suggest the optimality of investing a proportion of the fund in equities, in order to reduce the probability of positive ul.

In case (b), the variability of AL, induced by a randomly variable discount rate, overwhelms the reduction in the variability of ul due to an increase in the proportion of equities. As a consequence of this, numerical figures suggest the optimality of investing a minimal proportion of the fund in equities.

Chapter 7

Conclusions

In this Chapter we briefly summarise the main achievements of the thesis, as well as proposing possible areas of extension for future research.

Chapter 3 - Population plan theory Chapter 3 deals with the evolution of the membership population of a DB pension scheme.

In the classical theory, strong assumptions are usually made with regard to the structure and to the evolution of the membership population. These assumptions are relaxed in this chapter, where we allow the population to evolve randomly; i.e. when the number of new entrants is random and the eliminations are deterministic.

Since a direct link exists between the population and the structure of the liabilities, such an extension to the stochastic case represents a step forward in understanding the effect of the demographic risk on the pension fund liabilities.

In detail, we decompose the demographic evolution into two streams of forces: positive factors, which increase the number of members; and negative factors, which instead decrease it. Using this distinction, we demonstrate that a set of generalised results exists when the positive factors evolve randomly.

Specifically, for a stochastic stable population we associate the corresponding liability growth process, in analogy to the Bowers' liability growth equation, which holds for deterministic stable population. Similarly, the stochastic counterpart of a stationary population leads to Trowbridge's equation of maturity - or maturity equilibrium

- on average.

Attention is subsequently directed to the effects of a randomly evolving population on the unfunded liability of a pension scheme, and in particular on its variance.

In addition to the stochastic demographic evolution, we also allow for random realisations in the financial market. Thus, two sources of uncertainty are included while analysing the variance of *ul*.

Different from the classic literature in the field, we also investigate the case in which these two sources of uncertainty are not independent.

By doing so, we are able to provide closed expressions of the variance of *ul* in a number of specific cases. Moreover, we show how it is possible to identify and separate the variables describing the dependence structure *within* the two risks, from those describing the dependence *between* the two of them.

Under certain conditions, we also find that the independence assumption leads to an upper-bound of the risk under dependence.

Future research. A more general theory allowing for stochastic negative factors in the determination of the membership population is certainly an area of great interest.

With respect to the implications on the variance of *ul*, we have not proposed a model for linking the number of new entrant to the financial realisations, other than presenting the dependence in terms of covariances. Although it may be difficult to gather accurate data, it is possible that a significant link between the two phenomena exists, perhaps with some time lag. However, it is likely that, whatever the nature of an existing relationship, the effect on the liabilities would be negligible. This is because, the two events of joining the scheme and receiving the benefit happen at very distant points in time, by which time the correlation effect are likely to have damped.

Chapter 4 - Contribution strategy. The aim of this chapter is to develop a classification of normal cost methods, which takes into account (a) the risk that, for each member of the scheme, the fund accrued at retirement differs from the expected value of the benefit; and (b) the cost of pension provision.

We use the accrual density function, as introduced in Cooper and Hickman (1967), Bowers *et al* (1976, 1979, 1982) and Economou (2003), in order to define the level of advance funding of a contribution strategy; i.e., the proportion of the retirement benefit which is funded by the investment income. Risk and cost are linked to this quantity.

Financial risk and the risk of the sponsor defaulting in its payments are separately modelled, and in addition we show the way to include the demographic risk as a third source of uncertainty.

Using such a decomposition, it is possible to illustrate the contrasting effects of the two sources of uncertainty on the level of risk of a given contribution strategy.

The accrual density function is also used in the definition of the cost of implementation of any given contribution strategy. Introducing the cost has the effect of creating a force opposite to the financial risk. In other words, the impact of financial risk can be reduced only by implementing a more expensive contribution strategy. Conversely, in order to reduce the cost of pension provision, a higher risk of mismatch between accrued and expected values has to be faced. Hence a tradeoff between the two exists.

By contrast, we illustrate that, when the risk of default is the only source of uncertainty, the cheapest strategy is also the safest one and therefore the above tradeoff does not arise.

Future research. Although explicitly expressed in the equations, the demographic risk is not included in the risk-based classification of contribution strategies. Considering the probability of surviving (or the force of mortality) as stochastic processes would extend this line of research.

Chapter 5 - Optimal contribution strategies. Starting from the tradeoff between risk and cost of contribution strategies, the aim of this chapter is to devise a methodology in order to identify optimal strategies which minimise a certain measure of risk.

First, we set the constrained quadratic problem of minimising the variance of a general loss function. Then, we find an analytical solution under some fairly general conditions.

Furthermore, numerical routines are used to find optimal contribution strategies in two specific situations: when the objective is to minimise (a) the variance of a loss function relative to a single member of the scheme (disaggregated approach); (b) the variance of the unfunded liability (aggregated approach).

For the disaggregated case, we include two sources of uncertainty: namely, the rates of investment return and the risk of sponsor's default. Thus, numerical optimisation identifies the efficient frontier in the risk/cost plane. Moreover, we show that the normal cost methods generated by the power accrual density function are inefficient in this plane.

With respect to case (b), financial and demographic risks are included as sources of uncertainty in the analytical derivation of the optimal solution.

A numerical application focuses on the effects of stochastic new entrants when a deterministic mismatch between the expected and actual returns from investments exists. Specifically, we use numerical algorithms in order to find optimal strategies, when new entrants at a fixed age are modelled as a sequence of *iid* random variables. Moreover, once the cost is included in the investigation, we identify the efficient frontier in the cost/risk plane.

In this framework, optimal strategies are compared to normal cost methods, showing the inefficiency of these in terms both of cost and risk.

Future research. As far as the mathematical development is concerned, analytical conditions assuring the non negativity of the optimal solution *a priori* may exist.

On the numerical implementation side, multiple objective functions optimisation could be used to minimise simultaneously the cost and the risk. This approach is not used here, but it could provide us with a valuable tool for comparing the validity of the results.

More realistic financial models, as proposed in Chapter 6, could be taken into account when looking for optimal contribution strategy from the individual member's point of view. Similarly, including financial risk and the risk of sponsor's default in the aggregated problem may be interesting and worth pursuing.

Moreover, part of the results may be explained in terms of the chosen risk measure. From this perspective, allowing for different measures, such as those satisfying the requirements of coherence, would be a natural extension to this research.

Chapter 6 - Optimal funding strategies. In this chapter, we investigate the asset allocation as a tool for controlling the investment income. This issue is part of a broader discussion involving different approaches to the funding of pension scheme. These approaches arise from different schools of thinking, which here we label as *classic actuarial* and *financial-economic.*

Following a traditional actuarial approach to pension funding, whereby the discount rate is related to the asset allocation, we find that a considerable proportion of assets should be invested in equities in order to minimise a selection of risk measures.

This result is tested with different financial models, as well as with different parameters, and it seems that the optimal allocation in equities lies in a remarkably narrow interval.

Furthermore, we extend the results from previous chapters by looking for optimal funding strategies, i.e optimal ways of combining contribution and investment strategies. We find evidence that the investment strategy should support the contribution strategy. In other words, the more the normal cost method relies on investment returns in order to fund the retirement benefit, the more should be invested in risky assets. In this way we reduce the computed risk measures. Evidence for this is found using classical normal cost methods, as well as implementing the more general family of the power function cost methods.

In order to compare the two approaches, the actuarial and the financial-economic ones, and in order to understand to which extent they lead to different optimal solutions, we perform the same analysis following an approach consistent with the financial-economic school.

Specifically, liabilities are evaluated with two different discount rates: (a) the expected return from bond investments; and (b) the returns from bond investments as read from the market at the valuation date.

In the case (a), we find that a two-sided risk measure suggests the optimality of an *all-bond* strategy, thus corroborating the view that assets should not be invested in equities. In contrast, downside risk measures account for the systematic positive mismatch between the expected rate of return and the discount rate, and because of this the optimal strategy calls for investments in equities. Thus, different risk measures suggest the optimality of considerably different strategies.

In the case (b), the variability of the liabilities, due to the randomness of the discount rate, overwhelms the advantages of investing in equities, and hence, numerical figures suggest the optimality of investing a minimal proportion of the fund in equities.

Future research The literature illustrates the importance of taxation and solvency protection policies in determining the efficiency of asset allocation.

Specifically, these two factors have a contrasting effect on the proportion to be invested in equities: taxation calls for a reduction of equities, whereas solvency insurance for an increase.

In this thesis, these factors are excluded, and it would be of interest to investigate their effects in this framework, where investment and contribution incomes are efficiently combined.

Furthermore, this analysis is limited to the case of a static optimal problem, where we minimise a risk measure at a specific point in time. Indeed, further research may be developed in order to extend this analysis to a dynamic case through the use of dynamic programming.

Appendix A

Analytical proofs

A.1 Proof of Liability Growth process

Under the assumption of deterministic decrements and independently of the fact that the number of members aged x at time t is deterministic or not, the following relation holds: $n(x,t) \cdot \ {}_{y-x}p_x = n(y, t+y-x)$

Hence, the **GLG** equation (2.2) has the following expression:

$$
\begin{aligned}
AL(t) &= \sum_{x=0}^{\infty} AL_x n(x,t) = \sum_{x=0}^{\infty} n(x,t) \sum_{y=x}^{\infty} \Big(B_y - NC_y\Big) v^{y-x} \ {}_{y-x}p_x \\
&= \sum_{x=0}^{\infty} \sum_{y=x}^{\infty} \Big(B_y - NC_y\Big) v^{y-x} \ n(y, t+y-x) \\
&= \sum_{x=0}^{\infty} \sum_{k=0}^{\infty} \Big(B_{x+k} - NC_{x+k}\Big) v^{k} \ n(x+k, k+t) \\
&= \sum_{k=0}^{\infty} \sum_{x=0}^{\infty} \Big(B_{x+k} - NC_{x+k}\Big) v^{k} \ n(x+k, k+t) \\
&= \sum_{k=0}^{\infty} \sum_{y=k}^{\infty} \Big(B_y - NC_y\Big) v^{k} \ n(y, t+k)
\end{aligned}
$$

Thus, we can split the two cases of $k = 0$ and $k \geq 1$, hence obtaining the following:

$$
\begin{aligned}
AL(t) &= \sum_{y=0}^{\infty}\Big(B_y - NC_y\Big)\; n(y,t) + \sum_{k=1}^{\infty}\sum_{y=k}^{\infty}\Big(B_y - NC_y\Big)v^k\; n(y,t+k) \\
&= B(t) - NC(t) + \sum_{h=0}^{\infty}\sum_{y=h+1}^{\infty}\Big(B_y - NC_y\Big)v^{h+1}\; n(y,t+h+1) \\
&= B(t) - NC(t) + v\Bigg[\sum_{h=0}^{\infty}\sum_{y=h}^{\infty}\Big(B_y - NC_y\Big)v^h\; n(y,t+h+1) \\
&\quad - \sum_{h=0}^{\infty}\Big(B_h - NC_h\Big)v^h\; n(h,t+1+h)\Bigg] \\
&= B(t) - NC(t) + vAL(t+1) - v\; n(0,t+1)\sum_{h=0}^{\infty}\Big(B_h - NC_h\Big)v^h\; {}_kp_0 \\
&= B(t) - NC(t) + vAL(t+1) - v\; n(0,t+1)AL_0 \\
&= B(t) - NC(t) + vAL(t+1)
\end{aligned}
$$

A.2 Derivation of the recursive formula of *ul*.

In section 3.5.2 a recursive formulation of the variance of *ul* is illustrated, for the case of deterministic eliminations from the populations and of random rates of investment returns. Under the assumption of deterministic elimination the **LGP** equation (3.6) holds, and hence, equation (3.27) is derived in the following way:

$$
\begin{aligned}
ul(t+1) &= AL(t+1) - f(t+1) = \\
&= \big[AL(t) + NC(t) - B(t)\big](1+i) - \big[f(t) + c(t) - B(t)\big]\big(1 + r(t)\big) \\
&= \big[AL(t) + NC(t) - B(t) - f(t) - c(t) + B(t)\big]\big(1 + r(t)\big) \\
&\quad + \big[AL(t) + NC(t) - B(t)\big]\big(i - r(t)\big) \\
&= \big[AL(t) - f(t) - kul(t)\big]\big(1 + r(t)\big) + AL(t+1)\frac{i - r(t)}{1+i} \\
&= ul(t)\big(1 + r(t)\big)(1-k) + AL(t+1)\frac{i - r(t)}{1+i} \\
&= -\sum_{h=1}^{t+1} AL(h)\ \phi(h, t+1)
\end{aligned}
$$

where $\phi(h, t+1)$ is as defined in equation (3.28).

A.3 Proof of covariance decomposition.

The covariance between two random variable is defined as the first mixed moment less the product of the two first moments:

$$\text{Cov}(X, Y) = E[XY] - E[X]E[Y] \tag{A.1}$$

The covariance is equal to 0 when X and Y are independent, because in such a case $E[XY] = E[X]E[Y]$.

By the definition of covariance, the following relations hold.

A.3.1 Covariance between products of two independent variables.

Let X, Y, ξ and υ be 4 random variables, such that X and ξ are independent, and so are Y and υ then:

$$
\begin{aligned}
\mathrm{Cov}_{\perp}\Big(X\xi, Yv\Big) &= E[X\xi Yv] - E[X\xi]E[Yv] \\
&= E[XY]E[\xi v] - E[X]E[Y]E[\xi]E[v] \\
&= \mathrm{Cov}(X,Y)E[\xi v] + E[X]E[Y]E[\xi v] - E[X]E[Y]E[\xi]E[v] \\
&= \mathrm{Cov}(X,Y)\mathrm{Cov}(\xi,v) + \mathrm{Cov}(X,Y)E[\xi]E[v] + \mathrm{Cov}(\xi,v)E[X]E[Y]
\end{aligned}
$$

A.3.2 Covariance between products of two variables.

Let X, Y, ξ and v be 4 random variables, then:

$$
\begin{aligned}
\mathrm{Cov}\Big(X\xi, Yv\Big) &= E[X\xi Yv] - E[X\xi]E[Yv] \\
&= E[X\xi Yv] - E[X\xi]\mathrm{Cov}\big(Y,v\big) - E[X\xi]E[Y]E[v] \\
&= E[X\xi Yv] - \mathrm{Cov}\big(X,\xi\big)\mathrm{Cov}\big(Y,v\big) - \mathrm{Cov}\big(Y,v\big)E[X]E[\xi] \\
&\quad -\mathrm{Cov}\big(X,\xi\big)E[Y]E[v] - E[X]E[\xi]E[Y]E[v] \\
&= \mathrm{Cov}_{\perp}\Big(X\xi, Yv\Big) - \mathrm{Cov}\big(X,\xi\big)\mathrm{Cov}\big(Y,v\big) \\
&\quad -\mathrm{Cov}\big(Y,v\big)E[X]E[\xi] - \mathrm{Cov}\big(X,\xi\big)E[Y]E[v]
\end{aligned}
$$

A.4 Example of same cost and different risk.

Let us consider two contribution strategies, whereby the first one requires the payment of a lump sum at a general age $y \in (\alpha,\ R)$. Hence, the corresponding accrual function $m_1(x)$ is equal to 1 for $x = y$ and null elsewhere. In contrast, the second strategy requires the payment of two contributions at two different ages y_1 and y_2, such that $y_1 \leq y \leq y_2$. Then the resulting accrual function is as the following:

$$m(x) = \begin{cases} p & \text{if } x = y_1 \\ 1-p & \text{if } x = y_2 \\ 0 & \text{otherwise} \end{cases}$$

Without loss of generality we can make a valuation assumption and consider the case of a constant force of interest $\delta_u = \delta$. So, recalling the equation of cost (4.22), the two strategies have the following expected costs, respectively:

$$\begin{aligned} E[\text{Cost}|m_1]/PVFB_\alpha &= e^{\delta(y-\alpha)} \\ E[\text{Cost}|m_2]/PVFB_\alpha &= p\ e^{\delta(y_1-\alpha)} + (1-p)\ e^{\delta(y_2-\alpha)} \end{aligned}$$

Hence, we can see that the two strategies have the same cost if the first contribution in the second strategy is equal to the following:

$$p = \frac{e^{\delta y_2} - e^{\delta y}}{e^{\delta y_2} - e^{\delta y_1}}$$

According to equation (4.12), these two strategies have a risk which is given by

$$\begin{aligned} \Phi[m_1] &= \Phi\left[\gamma_y\ e^{\int_y^R \left(r(u)-\delta\right) du}\right] = \Phi\left[\xi(y)\right] \\ \Phi[m_2] &= \Phi\left[\gamma_{y_1}\ e^{\int_{y_1}^R \left(r(u)-\delta\right) du} + \gamma_{y_2}\ e^{\int_{y_2}^R \left(r(u)-\delta\right) du}\right] = \Phi\left[\xi(y_1) + \xi(y_2)\right] \end{aligned}$$

where, Φ is a general risk measure, and $\xi(y) = \gamma_y \exp\{\int_y^R r(u)du\}$ is a random variable summarising the combination of the two random variables financial realisation and sponsor's default.

In a similar way, it can be shown that the two strategies have the same risk if the first contribution in the second strategy is equal to the following proportion:

$$p' = \frac{\xi(y_2) - \xi(y)}{\xi(y_2) - \xi(y_1)}$$

Trivially, $p' = p$ if either of the two following cases are considered: (1) if $\xi(y) = e^{\delta y}$, $\forall y$, i.e. if the random variable ξ is to a constant, so losing its randomness. Therefore, this means that there is no risk involved, and thus, this case degenerates to a case of no interest.

Alternatively, (2) $p' = p$ if $y_1 = y = y_2$. In such a situation, the two strategies have the same cost and same risk, only if the payments are scheduled in the same way, i.e. if these two strategies coincide. Hence, two different strategies cannot have same cost and same risk (unless the risk is null).

A.5 Equivalent quadratic problems.

Let Σ be a square matrix of dimensions $n \times n$ and let $\mathbf{x}$ and $\mathbf{e}$ be two vectors of dimension n. Consider now, the following problem of quadratic programming:

$$\begin{aligned} \min_{x} \quad & \mathrm{x}'\Sigma\mathrm{x} \\ \text{s.t.} \quad & \mathrm{x}'\mathrm{e} = K \end{aligned} \tag{A.2}$$

If (A.2) has a solution $\mathbf{x}^*$, then $\exists\, \nu > 0$ such that $\mathbf{x}^*$ is also solution of the following problem

$$\min_{x} \mathrm{x}'\Sigma\mathrm{x} + \nu\mathrm{x}'\mathrm{e} \tag{A.3}$$

and

$$\nu = K\left(\mathrm{e}'\Sigma^{-1}\mathrm{e}\right)^{-1}$$

Proof: According to the Lagrangian method, problem (A.2) is equivalent to

$$\min_{x,\lambda} \mathrm{x}'\Sigma\mathrm{x} - \lambda\left(\mathrm{x}'\mathrm{e} - K\right)$$

which has solution and Lagrangian multiplier

$$\mathrm{x}' = \frac{\mathrm{e}'\Sigma^{-1}K}{\mathrm{e}'\Sigma^{-1}\mathrm{e}}; \qquad \tilde{\lambda} = \frac{2K}{\mathrm{e}'\Sigma^{-1}\mathrm{e}}$$

In a similar fashion, problem (A.3) has solution

$$\mathrm{x}' = \mathrm{e}'\Sigma^{-1}\nu \tag{A.4}$$

Trivially, solution (A.4) is equal to solution (A.4) if $\nu = \tilde{\lambda}$ as stated in the hypothesis.

A.6 Derivation of $\Sigma_{\phi,\gamma}[x,y]$.

In section 5.2.2, the general (x^{th}, y^{th}) element of the variance/covariance matrix $\Sigma_{\phi,\gamma}$ is defined in equation (5.33); where ϕ_x is the product of $R - x + 1$ log-normally distributed random variables.

Derivation Without losing generality, let us consider the case $x \leq y$.

$$\begin{aligned}\Sigma_{\phi,\gamma}[x,y] &= \text{Cov}\Big(\phi_x\gamma_x;\ \phi_y\gamma_y\Big)\\ &= \text{Cov}\Big(\phi_x,\phi_y\Big)\text{E}\Big[\gamma_x\gamma_y\Big] + \text{Cov}\Big(\gamma_x,\gamma_y\Big)\text{E}\Big[\phi_x\Big]\text{E}\Big[\phi_y\Big]\end{aligned}$$

as it is proved in Appendix A.3.1. Looking separately at each component, we can show that

$$\begin{aligned}\text{Cov}\Big(\phi_x,\phi_y\Big) &= \text{Cov}\Big(e^{\sum_{z=x}^{R}\left(r(z)-\delta_z\right)};\ e^{\sum_{w=y}^{R}\left(r(w)-\delta_w\right)}\Big)\\ &= \text{Cov}\Big(e^{\sum_{z=x}^{y-1}\left(r(z)-\delta_z\right)}e^{\sum_{z=y}^{R}\left(r(z)-\delta_z\right)};\ e^{\sum_{w=y}^{R}\left(r(w)-\delta_w\right)}\Big)\\ &= \text{E}\Big[e^{\sum_{z=x}^{y-1}\left(r(z)-\delta_z\right)}\phi_y^2\Big] - \text{E}\Big[e^{\sum_{z=x}^{y-1}}\phi_y\Big]\text{E}\Big[\phi_y\Big]\end{aligned}$$

The deviations from the expected returns are assumed to be independent, and hence, we can rearrange the moments, such that the last member is also equal to the following:

$$\begin{aligned}\text{Cov}\Big(\phi_x,\phi_y\Big) &= \text{E}\Big[e^{\sum_{z=x}^{y-1}\left(r(z)-\delta_z\right)}\Big]\text{E}\Big[\phi_y^2\Big] - \text{E}\Big[e^{\sum_{z=x}^{y-1}}\Big]\text{E}\Big[\phi_y\Big]^2\\ &= \text{E}\Big[e^{\sum_{z=x}^{y-1}}\Big]\text{Var}\Big(\phi_y\Big) \qquad \text{(A.5)}\end{aligned}$$

Finally, using equations (5.31) and (5.32), the covariance sought is given by

$$\mathrm{Cov}\Big(\phi_x, \phi_y\Big) = e^{\left(2R-y-x+2\right)\left(\bar{r}+\frac{\sigma_{\bar{r}}^2}{2}\right)}\left(e^{\left(R-y+1\right)\sigma_{\bar{r}}^2} - 1\right) \tag{A.6}$$

With regard to the covariance of the default indicator variable, we have that

$$\begin{aligned}
\mathrm{Cov}\Big(\gamma_x;\ \gamma_y\Big) &= \mathrm{E}\Big[\gamma_x\gamma_y\Big] - \mathrm{E}\Big[\gamma_x\Big]\mathrm{E}\Big[\gamma_y\Big] \\
&= \Pr\Big\{\gamma_y = 1\Big\} - \Pr\Big\{\gamma_y = 1\Big\}\Pr\Big\{\gamma_x = 1\Big\} \\
&= \Pr\Big\{\gamma_y = 1\Big\}\Big(1 - \Pr\Big\{\gamma_x = 1\Big\}\Big) \\
&= \Pr\Big\{\gamma_y = 1\Big\}\Pr\Big\{\gamma_x = 0\Big\} \qquad \text{(A.7)}
\end{aligned}$$

since $\mathrm{E}\Big[\gamma_x\Big] = \Pr\Big\{\gamma_x = 1\Big\}$ and $\Pr\Big\{\gamma_x = 1,\ \gamma_y = 1\Big\} = \Pr\Big\{\gamma_y = 1\Big\}$ for $x \leq y$.

Combining the two results equation (5.33) can easily be derived.

A.7 Derivation of $\mathrm{Var}\big(ul(t)\big)$.

$$
\begin{aligned}
\mathrm{Var}\big[ul(t)\big] &= \mathrm{Var}\Big\{\sum_{h=1}^{t} AL(h)\phi(h,t)\Big\} = \mathrm{Var}\Big\{\sum_{h=1}^{t}\phi(h,t)\sum_{k=0}^{h-1}\big[NC(k)-B(k)\big]u^{h-k}\Big\} \\
&= \mathrm{Var}\Big\{\sum_{h=1}^{t}\phi(h,t)\sum_{j=0}^{h-1}u^{h-j}\sum_{x=0}^{\tau}NC_{\alpha+x}\sum_{\kappa=0}^{\tau-\alpha}n(\alpha+x,j,\kappa)-\widehat{B}(t)\Big\} \\
&= \mathrm{Var}\Big\{\sum_{x=0}^{\tau}NC_{\alpha+x}\,G(x,t)-\widehat{B}(t)\Big\} \\
&= \sum_{x=0}^{\tau}\sum_{y=0}^{\tau}NC_{\alpha+x}NC_{\alpha+y}\mathrm{Cov}\Big\{G(x,t),\ G(y,t)\Big\}+\mathrm{Var}\big\{\widehat{B}(t)\big\} \\
&\quad -2\sum_{x=0}^{\tau}NC_{\alpha+x}\mathrm{Cov}\Big\{G(x,t),\widehat{B}(t)\Big\}
\end{aligned}
$$

A.8 Closed covariances for *iid* new entrants.

Under the assumption that a sequence of *iid* random variable describes the number of new entrants joining the scheme at a fixed age α, it is possible to provide closed expressions for the elements in the matrix $\Sigma_{G(t)}$ and in the vector $\underline{C}(t)$:

$$
\begin{aligned}
\sum_{G(t)}[x,y] &= {}_{x}p_{\alpha}\ {}_{y}p_{\alpha}\Bigg\{\frac{(u^{t-x}-w^{t-x})(u^{t-y}-w^{t-y})}{(u-w)^2}-\frac{u^{|x-y|+1}}{u-w}\cdot\frac{(uw)^{t-\max(x,y)}-1}{uw-1} \\
&- \frac{uw^{t-\min(x,y)}}{uw-1}\cdot\frac{u^{t-\max(x,y)}-w^{t-\max(x,y)}}{u-w} \\
&+ w^{t-\min(x,y)}\left[\frac{1}{uw-1}+\frac{u}{u-w}\right]\cdot\frac{w^{2(t-\max(x,y))}-1}{w^2-1}\Bigg\}
\end{aligned}
\tag{A.8}
$$

$$
\begin{aligned}
\underline{C}_x(t) = {}& {}_x p_\alpha w^{t-x} B \sum_{y=0}^{M_2(t-1)} \left\{ \frac{u^{t-\tau-y+1} - w^{t-\tau-y+1}}{u-w} \left[\frac{u^{t-x} - w^{t-x}}{(u-w)w^{t-\tau-y}} - \frac{uw}{uw-1} \right] \right. \\
& - \frac{u^{\tau+y-x+1}\left[(uw)^{t-\tau-y+1} - 1\right]}{w^{t-x-1}(u-w)(uw-1)} + \left[\frac{1}{uw-1} + \frac{u}{u-w} \right] \cdot \frac{w^{2(t-\tau-y+1)} - 1}{(w^2-1)w^{t-\tau-y}} \\
& \left. + \frac{u^{t-\tau-y+1} - w^{t-\tau-y+1}}{u-w} \right\} {}_{\tau+y} p_\alpha \qquad \text{(A.9)}
\end{aligned}
$$

Appendix B

Assumptions and parameters

Unless differently stated, the following assumptions have been used when carrying out computer simulations of the DB pension scheme.

Model assumptions.

- Actuarial cost method: Entry age.
- Supplemental cost method: surplus/deficit spreading over a fixed term.
- Asset valuation: market value.

Model parameters.

- Valuation rate: $i = 3\%$; force of discount: $\delta = \log(1 + i)$.
- Investment rate of return: $r = 3\%$; fore of return: $\bar{r} = \log(1 + r)$.
- Amortisation period m has been initially set equal to 4.
- Fixed entry age $\alpha = 20$, retirement age $R = 65$, extreme age $\omega = 105$.
- Data from Watson&Wyatt, have been used to derive the service table (probabilities of remaining in the scheme) and salary scale.

Simulation assumptions and parameters.

- New entrants process$\{g_t\}$: indipendent and normally distributed with mean $\bar{g} = 1000$ and standard deviation $\sigma_g = 250$; negative values are artificially set equal to zero.
- Number of simulations: 10,000.
- Multinormal financial model (from Blake *et al* (2000)):
  - Vector of expected returns: $\hat{\mu} = (3\%, 6\%, 13\%)'$.
  - Correlation matrix:

$$\mathbf{V} = \begin{pmatrix} 2.045\% & - & - \\ 0.256\% & 7.953\% & - \\ -0.016\% & 0.544\% & 19.113\% \end{pmatrix}$$

- Regime switching multinormal model (from Blake *et al* (2000)):
  - Vector of expected returns: $\hat{\mu} = c(1.28\%, 5.14\%, 9.21\%)'$.
  - Correlation matrix:

$$\mathbf{V} = \begin{pmatrix} 4.045\% & - & - \\ 0.256\% & 14.57\% & - \\ -0.061\% & 0.544\% & 18.24\% \end{pmatrix}$$

  - Parameters in the high volatility state

    Bond expected return: $\mu_B^{(H)} = -6.92$

    Bond variance: $\text{Var}_B^{(H)} = 6.21$

    Equity expected return: $\mu_S^{(H)} = 21.92$

    Equity variance: $\text{Var}_S^{(H)} = 66.62$

– Transition probabilities:

$$\Pr\Big\{\text{Bond in state}H\Big\} = 29.82\%$$
$$\Pr\Big\{\text{Equity in state}H\Big\} = 9.13\%$$

- Wilkie model: the parameters are taken from Wilkie (1995)
- Financial-economic model (parameters are taken from Boullier *et al* (2001) and partly modified, in order to obtain expectations and volatilities comparable to those from the other models):

  – Short term rate
  $a = 0.2$
  $\bar{\gamma} = 0.03$
  $\sigma_r = 0.015$
  $r_0 = \bar{\gamma}$

  – Bond
  $K = 10$
  $\sigma_K = \sigma_r \cdot \frac{1-e^{-a \cdot K}}{a}$
  $\lambda_r = \bar{\gamma}/\sigma_K$

  – Equity
  $\lambda_{S|r} = 0.46$
  $\sigma = 0.02$
  $\sigma_{S|r} = 0.199$

  These parameters lead to following values for the equity:
  expected mean $\mu_S = 13\%$
  volatility $\sigma_S = 20\%$

correlation $\rho_{r,S} = 10\%$

- Sponsor default model:
  - γ varies such that that the expected number of years before the default lies in the range from 1 to 500 years.
  - β is equal to 0, when independence between financial markets and sponsor's default is assumed. Otherwise, following Ngwira and Wright (2004), we set $\beta = 4.5\%$.

Appendix C

Additional graphs

C.1 Adding a third option to the allocation problem - §6.4.3

The following figures C.1, C.2 and C.3 display the optimal allocation of cash versus bonds and equities (separately), when multinormal, regime-switching and Wilkie models are implemented, respectively. The results are in line with those illustrated in section 6.4.3.

Figure C.4 suggests that the results achieved in section 6.4.3 also hold when the rates of return are generated by the other models. In particular graph (a) refers to the multinormal model, graph (b) refers to the regime-switching model, and graph (c) to the Wilkie model.

Figure C.1: Optimal allocation in bonds and equities versus cash. Multinormal model

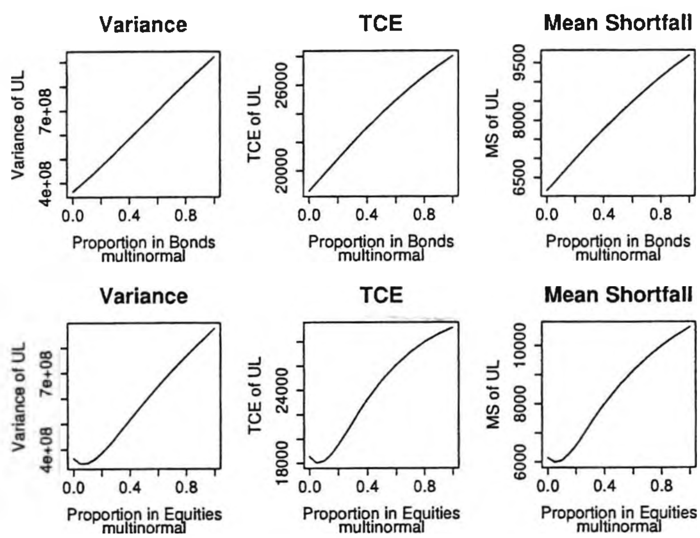


Figure C.2: Optimal allocation in bonds and equities versus cash. Regime-switching model

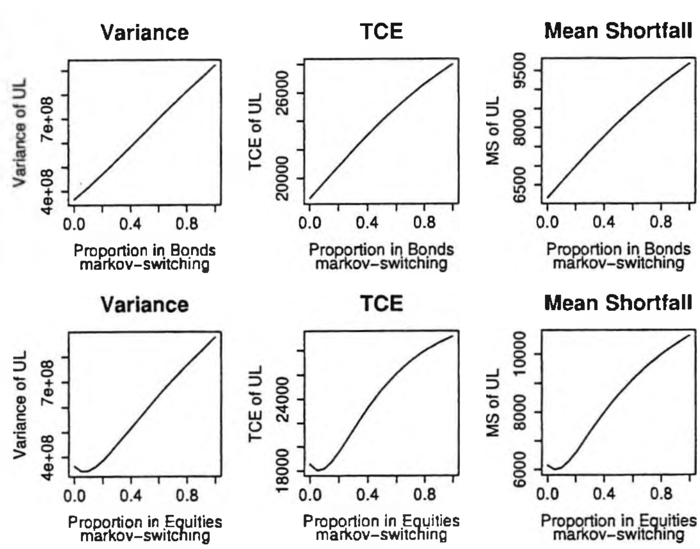


Figure C.3: Optimal allocation in bonds and equities versus cash. Wilkie model

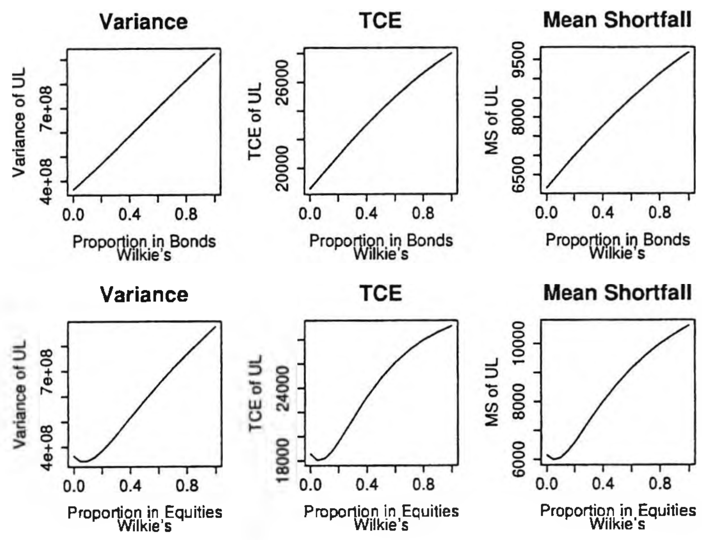
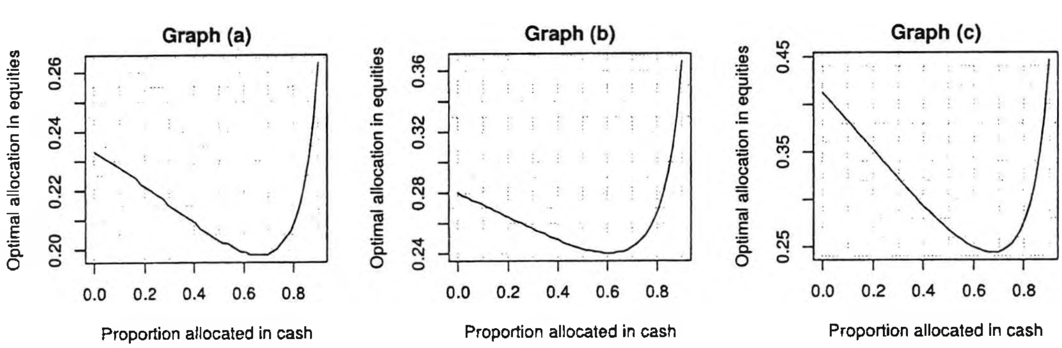


Figure C.4: Optimal allocation with fixed cash.



C.2 Adding a random demographic component - §6.5.

We have seen in section 6.5 that the optimal allocation does not particularly vary, when a random demographic component is added. Specifically, here we model the number of new entrants with a sequence of independent and normally distributed random variables. The following Figures C.5 display the optimal allocation equities (the remaining is invested in bonds), when the rates of return are simulated with the multinormal, regime-switching and Wilkie models, respectively. Graph (a) illustrates the surface of the variance (y-axis) when the asset allocation (x-axis) and the variability of new entrants (z-axis) change, whereas graph (b) display the surface of the MS.

The results from this other investment models are consistent with those illustrated in section 6.5, obtained using the financial-economic model. The optimal allocation is not particularly affected by the variability of the new entrants.

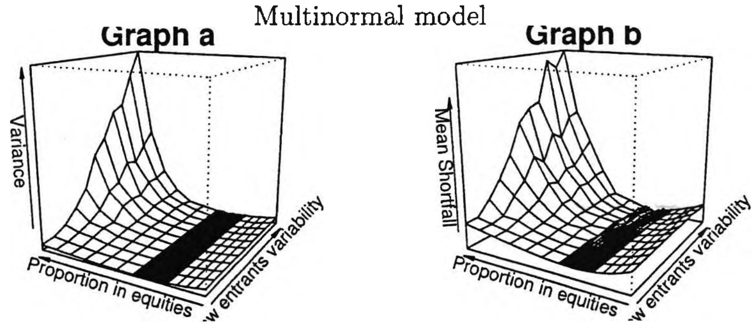
C.2.1 Small and large scheme

The following Figures C.6 and C.7 show the differences in asset allocation for small and large schemes, when models of the rates of return different from the financial-economic one are used. Specifically, Figures C.6 display the case of deterministic new entrants; whereas Figures C.7 illustrate the case of random new entrants at two points in time: year 45 and year 90.

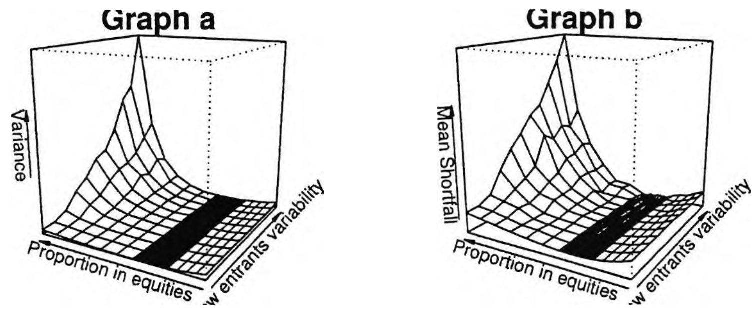
It is of interest to highlight that the Wilkie model does not always suggest the optimality of investing at least a minimal amount in equities.

Figure C.5: Optimal allocation in equities with increasingly variable random new entrants.

Multinormal model



Regime-switching model



Wilkie model

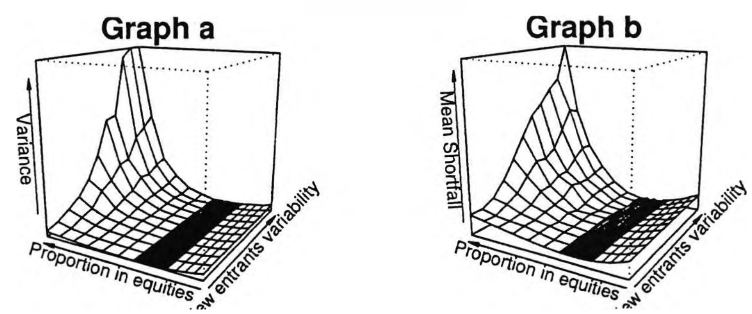
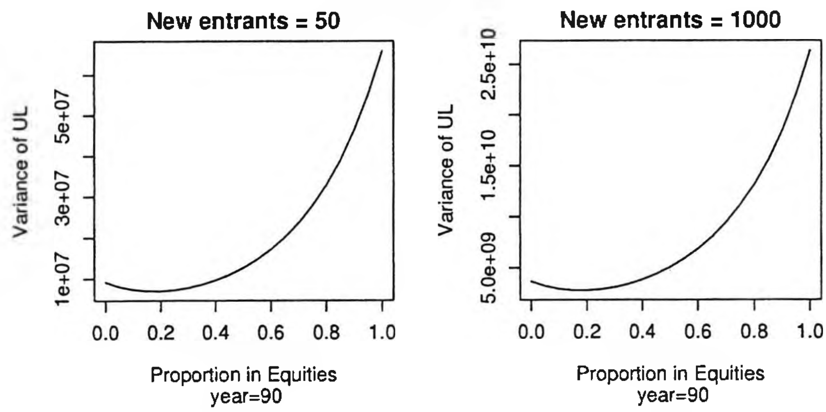
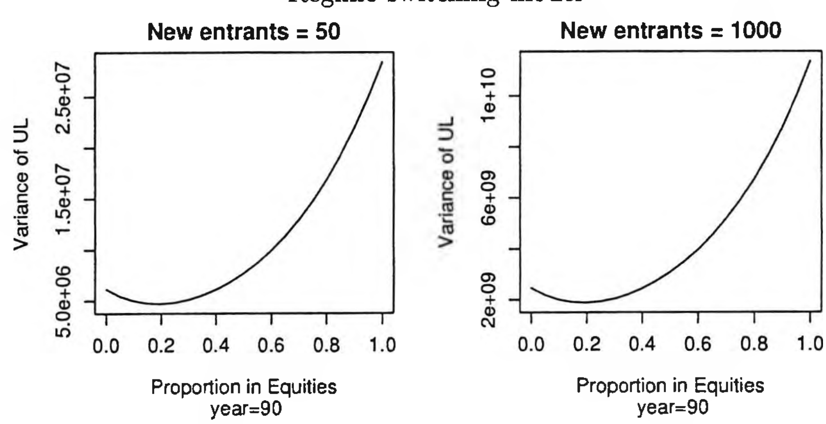


Figure C.6: Optimal allocation in equities in small and large schemes, with *deterministic* new entrants.

Multinormal model



Regime-switching model



Wilkie model

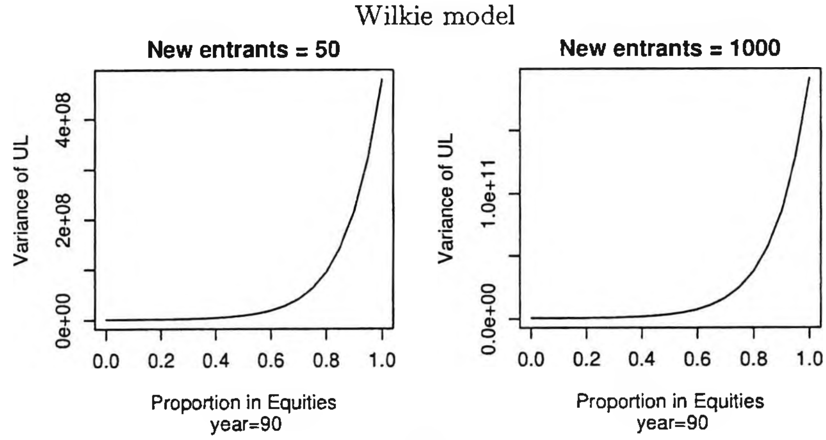
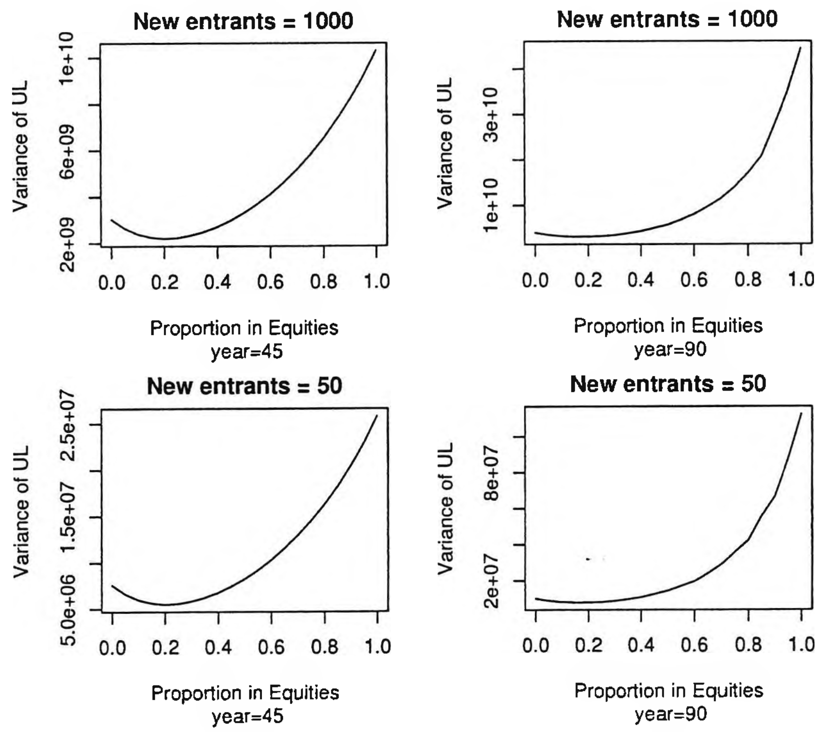
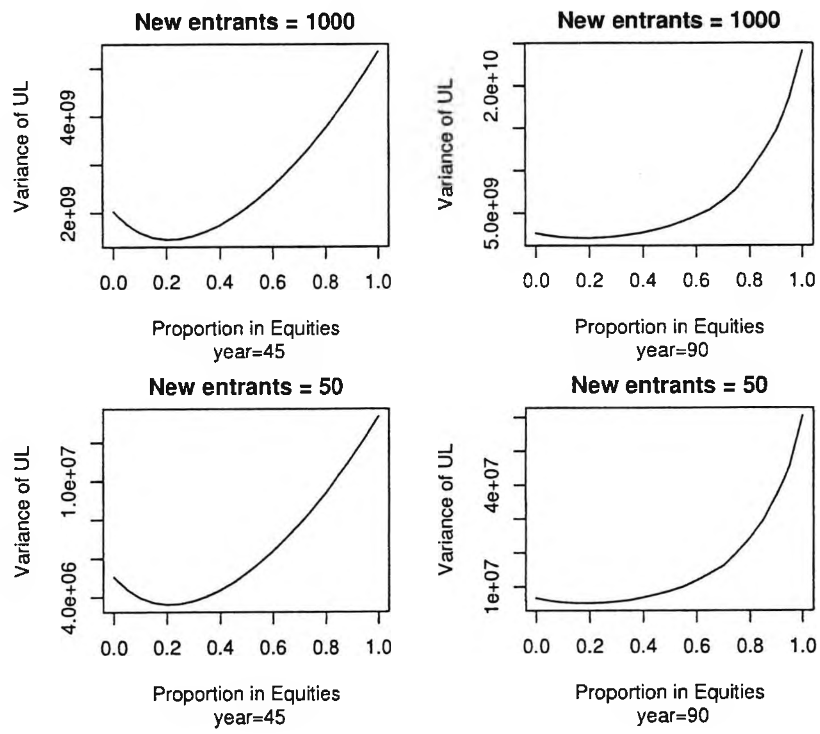


Figure C.7: Optimal allocation in equities in small and large schemes, with *random* new entrants.

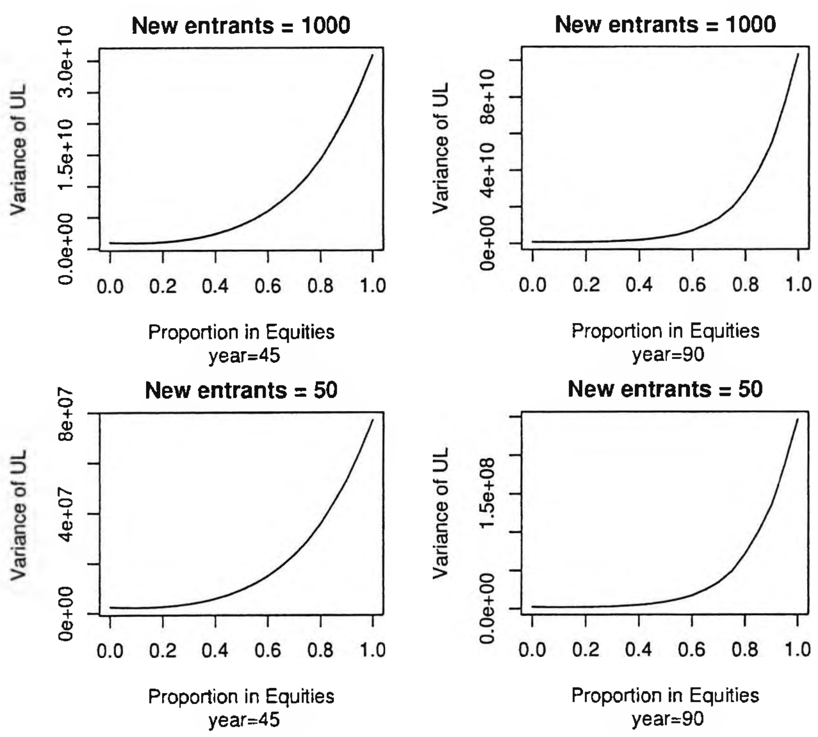
C.7.a : Multinormal model



C.7.b : Regime-switching model

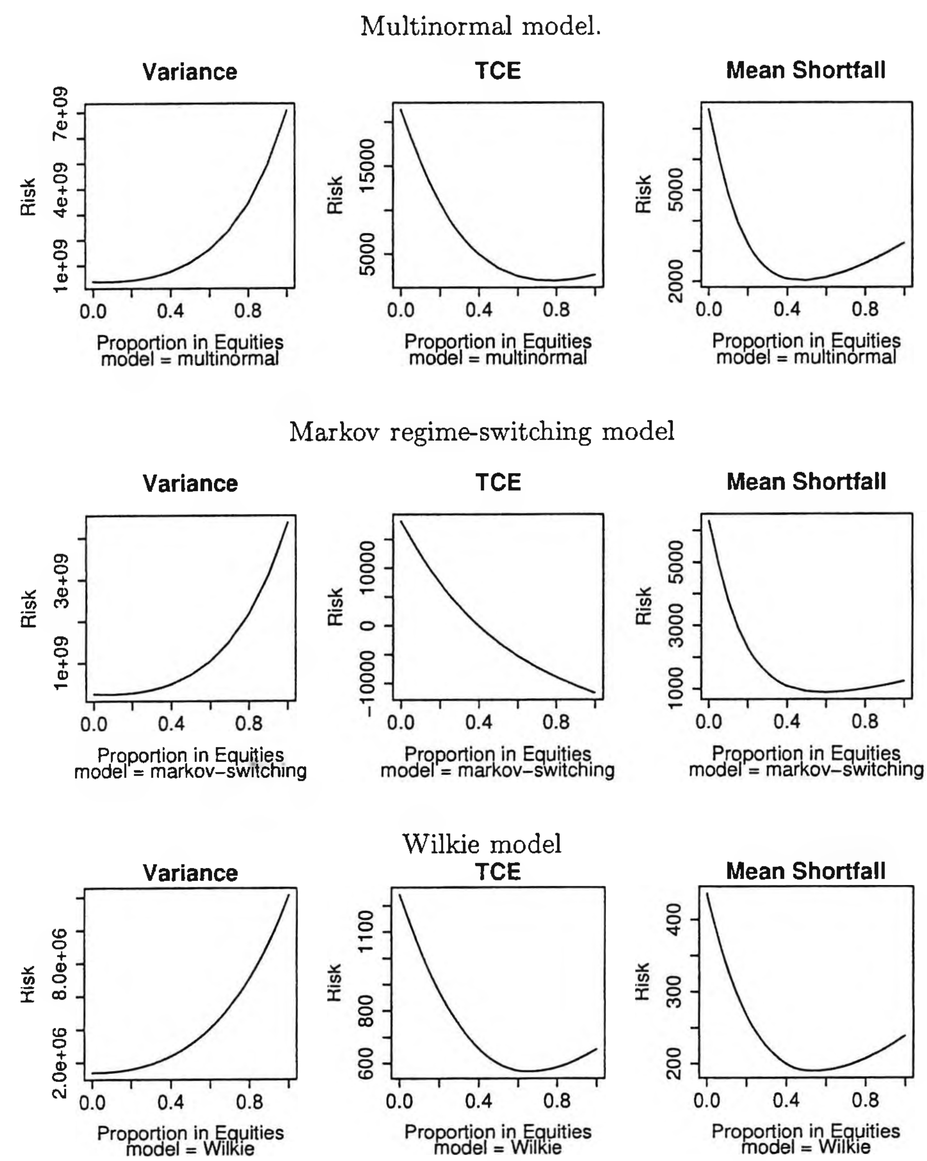


C.7.c : Wilkie model



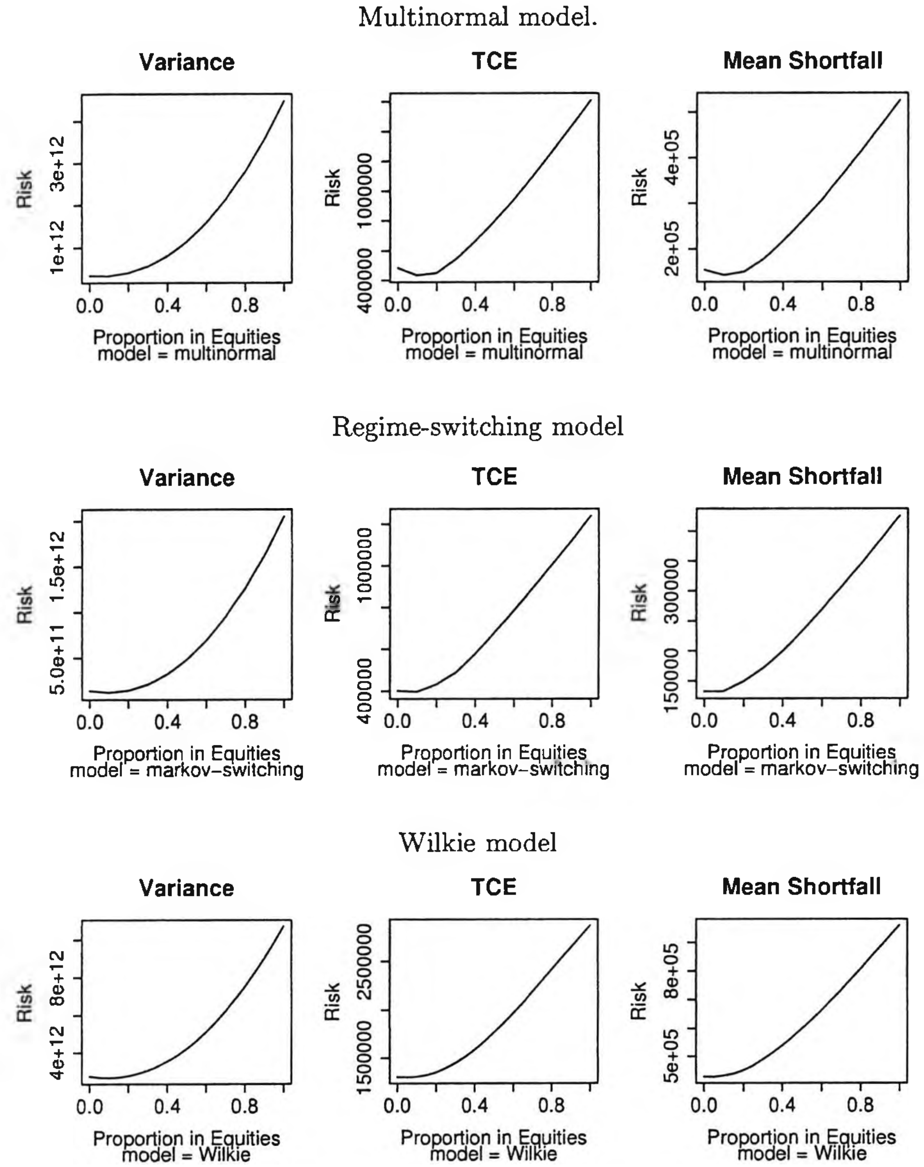
C.3 Financial economic approach - §6.7

Figure C.8: Optimal allocation with constant valuation rate.



In figure C.8, the valuation rate is set equal to the expected rate of returns from bonds. The results are conceptually similar to those presented in section 6.7.1, although the TCE in the regime-switching model would suggest that all-equities would be the optimal strategy.

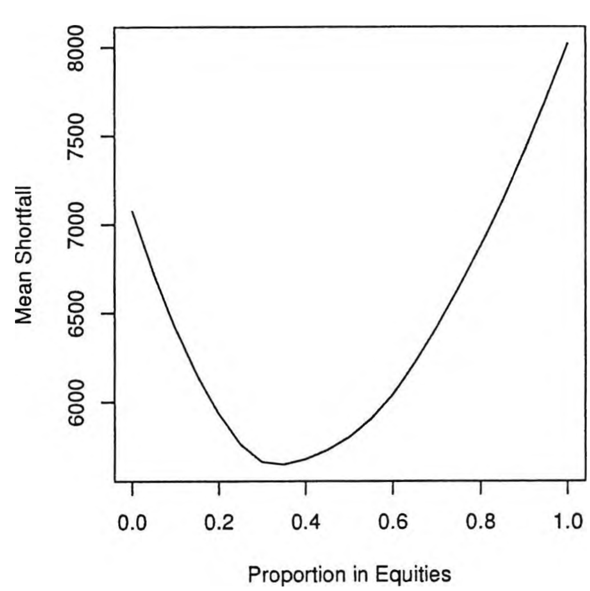
Figure C.9: Optimal allocation with market based valuation rate.



In the calculation producing figure C.9, the bond rate of return experienced during the year before the valuation rate is used for evaluating the liability. These other models seem to produce results very similar to those presented in section 6.7.2.

The results the Wilkie model seems to agree with those presented in Haberman *et al* (2003). Figure C.10 displays the optimal allocation in equities after a period of 3 years (compare), using the **MS** as a risk measure:

Figure C.10: Optimal allocation with market based valuation rate. Wilkie model, year 3 and **MS**



Appendix D

Sensitivity analysis

D.1 Spreading period.

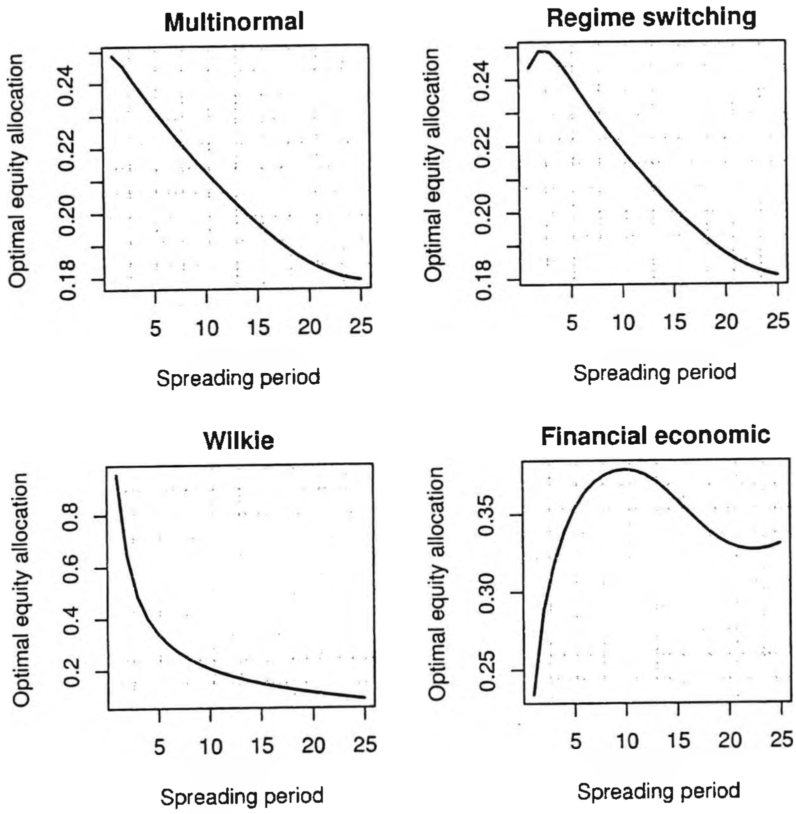
Figure 6.1 suggests the optimality of investing roughly 32% of the fund in equities. It is of interest to show how this optimal value changes, when the amortisation period changes. Figure D.1 displays the results of this investigation.

Focusing on the financial-economic model, we can see that, as the spreading period increases from 1 up to 9 years, the line identifying the optimal proportion in equities rises up to hitting a maximum. From that point on, optimality requires to reduce the allocation in the risky asset.

The other models for the rates of return generate results, which are partly consistent with the financial-economic model. Specifically, as the other charts in Figure D.1 suggest, all the models call for a reduction of the proportion in equities, as the spreading period increases. However, only in the regime-switching model the proportion increases reaching a maximum, before eventually dropping.

It is also of interest to remark, that the Wilkie model produces a completely different shape. In fact, the optimal allocation in equities decreases quickly as a longer spreading period is used. However, this phenomenon is particularly pronounced when m is short. Moreover, the scale on the y-axis suggests that for short periods of m the optimal allocation can be considerably different.

Figure D.1: Optimal allocation in equities with increasing spreading period.



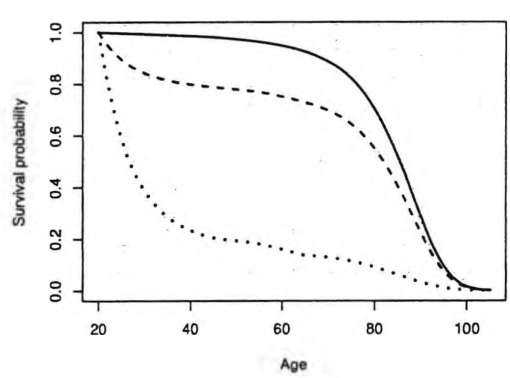
D.2 Service table.

In this section, it is of interest to test the robustness of the optimal allocation, when different assumptions regarding the eliminations from the scheme are used.

Specifically, we compare the optimal proportion to be invested in equities using the following probabilities of elimination. The first table, to which we refer as the standard service table, is the basic assumption used throughout the thesis. As an alternative, we take into account a mortality table (called $RG48$) derived from data of the general population of Italy, in order to describe the mortality of Italian life annuitants. This table is subsequently adjusted to reflect both the higher mortality of a population of pensioners, compared to a population of life annuitants; and the improvements in mortality during the past 10 years. [1] This table is chosen because it is characterised by low mortality. We also include an artificial table, built by averaging out the two previous tables.

Figures D.2 displays the probability functions derived from the 3 tables: specifically, the thick line is the table $RG48$, the dotted line is the "standard" table and the dashed lines is the "average" table. On the x-axis there is the age, whereas on the y-axis there is the probability of surviving one year.

Figure D.2: Comparison among the three mortality tables.

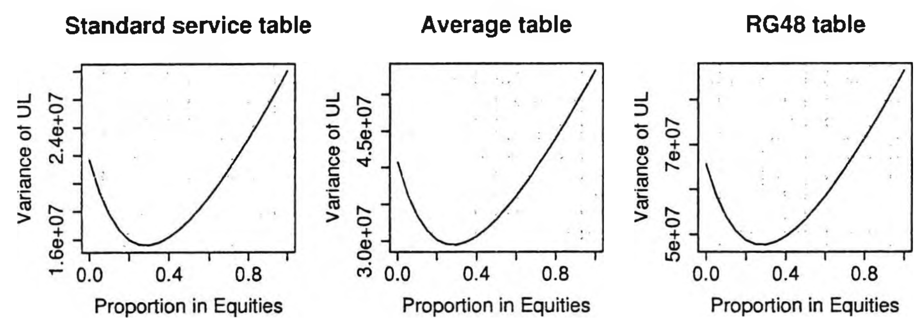


Figures D.3 display the optimal allocation in equities, under standard assumptions

[1]Details on the projection techniques are in Sithole *et al* (2000) and Colombo and Esposito (2003).

(i.e. financial-economic model for rates of return, spreading period equal to 4 years and entry age cost method), but varying the demographic assumptions as described above.

Figure D.3: Optimal allocation in equities, under different demographic assumptions.



It is of interest to highlight that the optimal allocation does not change, or at least not significantly. It can be seen that the scale of variability of ul under different demographic assumptions is massively different (the higher th probability of surviving the higher the values on the y-axis). Notwithstanding this, the optimal allocation is the same.

Running the same investigation under different assumptions, such as different models for the rates of return, spreading period and contribution strategy, leads to the same result: the optimal allocation is robust with respect to the model for the survival probability that is used.

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