Once upon a time there was a magic formula ...

Laura Ballotta[a]

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Few results have struck me as elegant and meaningful as the Black-Scholes formula for option contracts (Black & Scholes, 1973). As someone trained in (micro)economics, the pricing formula spoke to me as a clear example of general equilibrium at work: risk preferences do not matter when the market prices securities.

The mathematical formalization of this intuitive idea though is an illuminating example of cross-pollination, which has lead to a new field of study, mathematical finance, and a deeper understanding of the pricing mechanisms for any security in the market. The mathematical tools which led to the Black-Scholes formula had been known, but the point was to see how these tools could be used to create a formal infrastructure for the economic theory.

The driving force of this process was the 'simple' concept of no-arbitrage, which in this context could be expressed as follows: two securities or portfolios offering the same payoff at a specific date must have the same value at any other point in time. Indeed this led to the breakthrough of the Black-Scholes model. As a consequence, contracts can be hedged by buying and selling the underlying asset in just the right amount in order to eliminate risk - a procedure nowadays referred to as delta hedging. The economic concept of no-arbitrage was later translated into a mathematical concept, namely that of a martingale and the corresponding martingale measure by Harrison & Kreps (1979) and Harrison & Pliska (1981, 1983). Mathematical finance had found its natural setting. In the years following this insight, academic researchers pushed the equivalence between the two concepts to models of increasing sophistication.

The end result of this research programme is what we call risk neutral valuation, a revised version of the expected present value technique according to which the price of a derivative security can be obtained as the expected value of the terminal payoff discounted, crucially, at the risk free rate. The expectation is taken with respect to the so called risk neutral measure: the measure under which discounted asset prices are martingales. A significant point to clarify about this statement is that it is not equivalent to say that investors are risk neutral. They are not, of course. Rather, it means that in the process of changing the probability measure from the historical one (i.e. the physical one), investors receive their compensation for risk in the form of the market price of risk. An application of the Girsanov theorem in the Black-Scholes setting indeed shows that the 'shift' in the probability measure is given by the Sharpe ratio.

50 years later and the panorama in the financial industry has profoundly changed: new markets, new products, new challenges. All of these arose in part thanks to the availability of this

[a]Laura Ballotta is a Professor of Mathematical Finance at the Faculty of Finance, Bayes Business School (formerly Cass), City, University of London, UK.
email: L.Ballotta@city.ac.uk.

general and robust pricing theory. Also our understanding has changed. Risk neutral valuation is a general pricing principle, which we apply to all derivative securities, regardless of any exotic feature they might possess. Log-returns are clearly non Gaussian, and markets are far from complete. Nevertheless, risk neutral valuation can still be applied: the risk neutral measure is no longer unique but we can 'let the market (the data) choose the pricing measure' (Eberlein et al., 1998) through calibration. Given a model described by a parameter set Θ under some risk neutral measure, we use market quotes C^{mkt} of prespecified derivatives to determine an appropriate value $\vartheta \in \Theta$ - and implicitly the pricing measure - by solving

$$\min_{\vartheta} \sum_{i=1}^{N} \Big(C_i(\vartheta) - C_i^{mkt} \Big)^2, \tag{1}$$

where N is the size of the dataset and $C_i(\vartheta)$ the model price of the derivative securities under consideration.

Although it is clear that $C_i(\vartheta)$ is an expected value of the discounted contract's payoff, obtaining the corresponding analytical expression which allows us to solve (1) is not straightforward. Models for the underlying assets which cater for more realistic distributions are in general not as tractable as the geometric Brownian motion underpinning the Black-Scholes formula.

Tractability is indeed one of the fundamental traits of the Black-Scholes formula together with interpretability, and both represent reasons why researchers always tend to gravitate around it. As a natural consequence, researchers have attempted to keep both features when developing more realistic models for the underlying asset, as they help us to better understand these alternative models, including what they can do for us towards capturing and reproducing important empirical stylised facts.

To illustrate this point, let us consider the large class of models generated by pure jump Lévy processes L_t which can be represented in the form of a subordinated Brownian motion

$$L_t = \mu t + \beta \tau(t) + W_{\tau(t)}, \tag{2}$$

where W_t is a standard Brownian motion, $\tau(t)$ is a subordinator with distributions $P^{\tau(t)}(\cdot)$ independent of W_t, and μ, β are the location and skewness parameters. The risk driver in equation (2) is a Brownian motion with drift evolving along a time axis which is not controlled by the calendar clock, but by the stochastic clock $\tau(t)$ capturing the level of activity in the market. In other words, the subordinator $\tau(t)$ can be effectively regarded as a clock recording business time. We note that representation (2) applies in particular to generalized hyperbolic (GH) Lévy motions for which the subordinator is in the class of generalized inverse Gaussian (GIG) processes, see Eberlein (2001). Special cases of GH Lévy processes are the hyperbolic Lévy motion (Eberlein & Keller, 1995), the normal inverse Gaussian (NIG) process (Barndorff-Nielsen, 1997) and the Variance Gamma process (Madan et al., 1998).

The corresponding model for the stock price under the risk neutral measure is

$$S_t = S_0 \; exp\Big(\left(r - \varphi(-\mathrm{i})\right) t + L_t \Big)$$

with r representing the risk free rate of interest, and $\varphi(u)$ denoting the characteristic exponent of L_t, i.e. $\mathbb{E}\left[exp\left(\mathrm{i}uL_t\right)\right] = exp\left(\varphi(u)t\right)$. Let $X_t = \ln\left(S_t/S_0\right)$, then representation (2) implies

$$X_t = \left(r - \varphi(-\mathrm{i}) + \mu\right)t + \beta\tau(t) + W_{\tau(t)}.$$

By conditioning on $\tau(t) = \theta$, this becomes

$$X_t = \left(r - \varphi(-\mathrm{i}) + \mu + \beta\frac{\theta}{t}\right)t + W_\theta.$$

In distribution W_θ is equal to $\sqrt{\theta/t}\; W_t$.

In order to determine the price of a European call option with strike K and maturity date T, we have to evaluate

$$C(K,T) = \mathbb{E}\left[e^{-rT}\left(S_T - K\right)^+\right] = \mathbb{E}\left[\mathbb{E}\left[e^{-rT}\left(S_T - K\right)^+ |\tau(T)\right]\right].$$

Let us define

$$\begin{aligned}
q(\theta) &= \varphi(-\mathrm{i}) - \mu - \frac{\theta}{T}\left(\beta + \frac{1}{2}\right), \\
v(\theta) &= \sqrt{\frac{\theta}{T}},
\end{aligned}$$

then, conditioned on $\tau(T) = \theta$, S_T is in distribution equal to

$$S_0\; exp\left(\left(r - q(\theta) - \frac{1}{2}v(\theta)^2\right)T + v(\theta)W_T\right),$$

which is the form appropriate for the application of the Black-Scholes formula to a price process with dividend yield $q(\theta)$ and volatility $v(\theta)$. The price of the call option is then

$$C(K,T) = \int_0^\infty C^{BS}\left(S_0, K, T, r, q(\theta), v(\theta)\right) dP^{\tau(T)}(\theta).$$

Thus, in this setting we can interpret the option price as an average of Black-Scholes prices $C^{BS}(\cdot)$ across all possible values of the business time $\tau(T)$. In particular, we note the two inputs of the Black-Scholes formula $q(\theta)$ and $v(\theta)$. The first one, $q(\theta)$, which could be seen as an artificial 'dividend' yield, corrects for the additional drift induced by the jump process. The second - more important - input is the volatility $v(\theta)$ which is now stochastic as it depends on the distribution of the subordinator. Therefore, this can be interpreted as a model for stochastic volatility, which is powerful enough to generate a volatility smirk, as shown in Figure 1.

A similar approach had already been used by Hull & White (1987) to price options in a general stochastic volatility model. In this case, the price is given once again by the Black-Scholes formula now averaged across all possible values of the 'mean variance'

$$\frac{1}{T-t}\int_t^T \sigma_s^2 ds.$$

From a practical point of view, though, such representations are not particularly useful in order to actually obtain numbers out of a model. This was already evident in Hull & White (1987), as the density of the mean variance is in general not known. The switch to more realistic

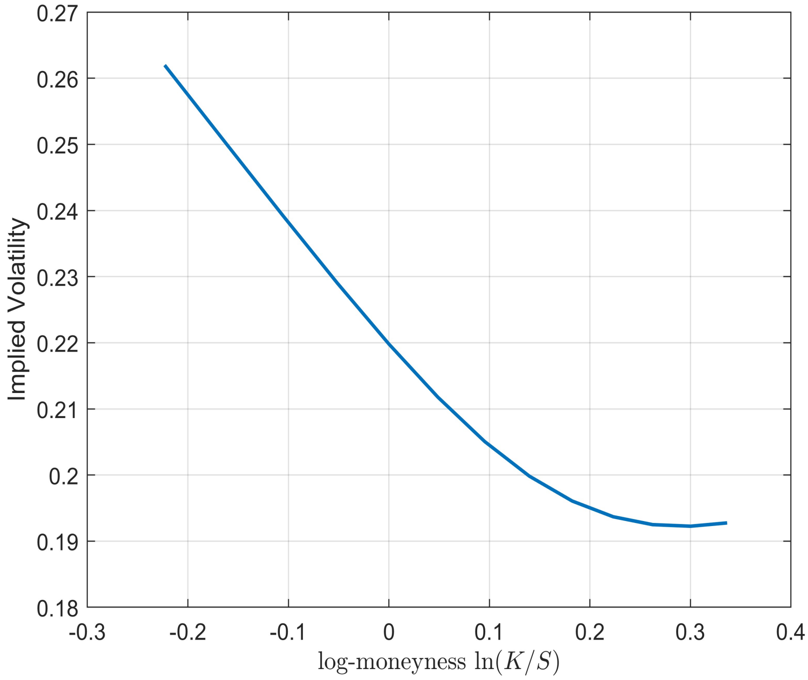

Figure 1: Implied volatility smirk from a GH model with $\lambda = -0.5$ (NIG). Parameter set: $T = 6$ months, $r = 5\%$ p.a.

models does pose the problem of overcoming computational barriers. This in turn has contributed to the development of what nowadays can be considered a field in its own right: computational finance.

From Monte Carlo simulation (Boyle, 1977) and its many refinements (for comprehensive treatments see Glasserman, 2003, Kienitz & Wetterau, 2012), to valuation methods based on Fourier transforms (see Carr & Madan, 1999, Eberlein et al., 2010, Fang & Oosterlee, 2008 to mention the key methods - see also Ballotta, 2022 for a short review), to deep learning methods (Horvath et al., 2020), computational finance has developed and keeps developing to meet the financial industry's needs originated by the proliferation of more sophisticated products and models.

Today we can look back at an impressive development of mathematical finance and financial engineering, which also gave origin to a whole industry devoted to the implementation of sophisticated financial models. This quantum leap started with a magic formula published in 1973.

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