SAMPLING AND STRUCTURAL PROPERTIES OF DISCRETIZED LINEAR MODELS

BY

N. TAMVAKLIS

THESIS SUBMITTED FOR THE
DEGREE OF
DOCTOR OF PHILOSOPHY

IN

CONTROL THEORY

CONTROL ENGINEERING CENTRE,
DEPARTMENT OF ELECTRICAL, ELECTRONIC AND INFORMATION ENGINEERING,
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DECLARATION

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MATHEMATICAL NOTATIONS

- $\mathbb{R}, \mathbb{C}$: fields of real, complex numbers.

- $\mathbb{R}(s)$: field of rational functions in the variable s with real coefficients.

- $\mathbb{R}[s]$: ring of polynomials in s with real coefficients.

- $\mathcal{F}$: denotes a general field, or ring.

- $\mathcal{F}^{p\times m}$: set of matrices with $p \times m$ dimensions and elements over $\mathcal{F}$.

- $\mathbb{R}^{p\times m}(s)$, $\mathbb{R}^{p\times m}[s]$: denote set of matrices with elements over $\mathbb{R}^{p\times m}$, $\mathbb{R}^{p\times m}$.

- $\mathcal{V}$: denotes a finite dimensional vector space over some field $\mathcal{F}$ (usual cases the real vector spaces ($\mathcal{R}$-vector spaces), rational vector spaces $\mathbb{R}(s)$-vector spaces).

- $\mathcal{F}^n$: set of all n-dimensional vectors (n-tuples) of elements of $\mathcal{F}$.Usual cases $\mathbb{R}^n$ (or $\mathcal{R}$), $\mathbb{C}^n, \mathbb{R}^n(s), ...$: n-dimensional vector spaces over $\mathcal{F}$.

- If $\mathcal{V}$ is a subspace of $\mathcal{R}$ (or $\mathbb{R}^n$), the $\underline{v} \in \mathcal{V}$ denotes a vector of $\mathcal{R}$ that belongs to $\mathcal{V}$. If $\dim\mathcal{V} = d$ and $\{\underline{v}_1, \underline{v}_2, ..., \underline{v}_d\}$ is a basis of $\mathcal{V}$, then $V = [\underline{v}_1, \underline{v}_2, ..., \underline{v}_d] \in \mathbb{R}^{n\times d}$ denotes a basis matrix of $\mathcal{V}$.

- If $P \in \mathcal{F}^{p\times m}, \mathcal{F}$ a field, then $\text{rank}P$ denotes the rank of P over $\mathcal{F}$, $\mathcal{N}_r\{P\}$ the right null space and $\mathcal{N}_l\{P\}$ the left null space of P.

- Z denotes the set of integers, Z^+ the positive integers, Z_0^+ the nonnegative integers $(Z^+ \cup \{0\})$and $Z_{\neq 0}$ the set of nonzero integers $(Z - \{0\})$.

- If $n \in Z^+$, then $\mathbf{n} = \{1, 2, ..., n\}$ and if a property holds for $i \in \mathbf{n}$, that implies that it is true for all $i = 1, 2, ..., n$.

- If $A \in \mathcal{F}^{n\times p}$:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ . & . & \cdots & . \\ a_{n1} & a_{n2} & \cdots & a_{np} \end{bmatrix}$$

we denote as column vectors of A $\underline{u}_1$, $\underline{u}_2$, ...,$\underline{u}_p$:

$$\underline{u}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ . \\ a_{n1} \end{bmatrix}, \underline{u}_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ . \\ a_{n2} \end{bmatrix}, ..., \underline{u}_p = \begin{bmatrix} a_{1p} \\ a_{2p} \\ . \\ a_{np} \end{bmatrix}$$

and as row vectors of A $\underline{v}_1^\top, \underline{v}_2^\top, ..., \underline{v}_n^\top$:

$$\underline{v}_1^\top = \begin{bmatrix} a_{11} & a_{12} & ... & a_{1p} \end{bmatrix}, \underline{v}_2^\top = \begin{bmatrix} a_{21} & a_{22} & ... & a_{2p} \end{bmatrix}, ..., \underline{v}_n^\top = \begin{bmatrix} a_{n1} & a_{n2} & ... & a_{np} \end{bmatrix}$$

- If $A \in \mathcal{F}^{n \times n}$, $|A|$ denotes the determinant of A, $\sigma(A)$ be the set of roots of the characteristic polynomial of A, $\Phi(A) = \det [sI - A]$.

- J denotes the similar to A Jordan matrix $A = UJU^{-1} = UJV$. where $U = V^{-1}$ is the matrix defined by the chains of eigenvectors of A and where :

$$J = \text{diag}\{J(\lambda_1), J(\lambda_2), ..., J(\lambda_i), ..., J(\lambda_f)\}$$

and $J(\lambda_i)$ is the diagonal matrix block formed by all the ν_i Jordan blocks associated with the distinct eigenvalue λ_i :

$$J(\lambda_i) = \text{diag}\{J_{i1}, ..., J_{ik}, ...J_{i\nu_i}\}$$

and where J_{ik} is the $\tau_{ik} \times \tau_{ik}$ Jordan diagonal block corresponding to the generalized eigenvectors chain of length τ_{ik}, associated with λ_i :

$$J_{ik} = \begin{bmatrix} \lambda_i & 1 & ... & 0 & 0 \\ 0 & \lambda_i & ... & 0 & 0 \\ . & . & ... & . & . \\ 0 & 0 & ... & \lambda_i & 1 \\ 0 & 0 & ... & 0 & \lambda_i \end{bmatrix}, \; J_{ik} \in \mathbb{C}^{\tau_{ik}}$$

- If $\sigma(A) = \{\lambda_1, \lambda_2, ..., \lambda_f\}$ are all the distinct eigenvalues of A, $\pi_1, \pi_2, ..., \pi_f$ are their algebraic multiplicities and $\nu_1, \nu_2, ..., \nu_f$ are defined as the corresponding geometric multiplicity.

- $\wp_{\lambda_i}(A)$: denotes the Segré Characteristic of A at λ_i,

$$\wp_{\lambda_i}(A) = \{\tau_{ik},\ k = 1, 2, ..., \nu_i,\ \tau_{i\nu_i} \geq ... \geq \tau_{ik} \geq ... \geq \tau_{i1} > 0\}$$

or the shorted notation $\wp_{\lambda_i}(A) = \{\tau_{i\nu_i} \geq ... \geq \tau_{ik} \geq ... \geq \tau_{i1} > 0\}$.

- State space description in time domain :

$$S(A, B, C, D) : \begin{cases} \dot{\underline{x}}(t) = A\underline{x}(t) + B\underline{u}(t) \\ \underline{x}(t) = C\underline{y}(t) + D\underline{u}(t) \end{cases}$$

Where $A \in R^{nxn}, B \in R^{nxl}, C \in R^{mxn}, D \in^{lxm}$ and $u(t)$ is the $l \times 1$ input vector, $\underline{y}(t)$ is the $m \times 1$ output vector and $\underline{x}(t)$ is the $n \times 1$ state variable vector.

- The Jordan canonical description of the system $S(A, B, C, D)$is:

$$S_J(J, \mathcal{B}, \Gamma, \Delta) : \begin{cases} \dot{\underline{z}}(t) = J\underline{z}(t) + \mathcal{B}\underline{u}(t) \\ \underline{y}(t) = \Gamma\underline{z}(t) + \Delta\underline{u}(t) \end{cases}$$

where: $\underline{z}(t) = U\underline{x}(t)$, $J = U^{-1}AU = VAU$, $\mathcal{B} = U^{-1}B$, $\Gamma = CU$, $\Delta = D$.

- ZOH denotes a Digital to Analogue Converter with zero order hold.

- FOH denotes a Digital to Analogue Converter with first order hold.

- The discretized model of $S(A, B, C, D)$ in a configuration involving ZOH is:

$$\hat{S}(\hat{A}, \hat{B}, \hat{C}, \hat{D}) : \begin{cases} \underline{x}[(k+1)T] = \hat{A}\underline{x}(kT) + \hat{B}\underline{u}(kT) \\ \underline{y}(kT) = \hat{C}\underline{x}(kT) + \hat{D}\underline{u}(kT) \end{cases}$$

where $\hat{A} = e^{AT}$, $\hat{B} = \left(\int\limits_0^T e^{A\sigma} d\sigma\right) B$, $\hat{C} = C$, $\hat{D} = D$.

- The Jordan canonical description of the system $\hat{S}(\hat{A},\hat{B},\hat{C},\hat{D})$:

$$\hat{S}_{\hat{J}}(\hat{J},\widehat{\mathcal{B}},\hat{\Gamma},\hat{\Delta}) : \begin{cases} \underline{z}\left[(k+1)T\right] = \hat{J}\underline{z}(kT) + \widehat{\mathcal{B}}\underline{u}(kT) \\ \underline{y}(kT) = \hat{\Gamma}\underline{z}(kT) + \hat{\Delta}\underline{u}(kT) \end{cases}$$

where :

$$\underline{z}(kT) = \hat{U}\underline{x}(kT),\ \hat{J} = \hat{U}^{-1}\hat{A}\hat{U} = \hat{V}\hat{A}\hat{U},\ \widehat{\mathcal{B}} = \hat{U}^{-1}\hat{B} = \hat{V}(\int_0^T e^{J\sigma}d\sigma)B = \tilde{V}\Xi B$$

$$\Xi \triangleq \int_0^T e^{J\sigma}d\sigma,\ \hat{\Gamma} = \hat{C}\hat{U} = C\hat{U} = \Gamma\tilde{U},\ \hat{\Delta} = \hat{D} = D$$

- The pencil $P(s) \in \mathbb{R}^{(n+m)\times(n+l)}[s]$ is defined as the Rosenbrock's system matrix pencil :

$$P(s) \triangleq \begin{bmatrix} sI-A & -B \\ -C & -D \end{bmatrix}$$

- The transfer function matrix $G(s) \in \mathbb{R}^{m\times l}(s)$ is given as $G(s) = C(sI-A)^{-1}B + D$

- e.d. denotes the elementary divisor.

- i.e.d. denotes infinite elementary divisor.

- i.d.z. denotes the input decoupling zeros of the system S (the roots of the e.d. of the pencil $[sI-A, B]$).

- o.d.z. denotes the output decoupling zeros of the system S (the roots of the e.d. of the pencil $\begin{bmatrix} sI-A \\ C \end{bmatrix}$).

- r.c.i. of the system S denotes the elements of the set of the *i-th* spectrum row controllability indices (r.c.i.) of A, B : $\Theta(A,B)_{\lambda_i} = \{\theta_{i1} \geq \theta_{i2} \geq \ldots \geq \theta_{i\nu_i} \geq 0\}$.

- c.o.i. of the system S denotes the elements of the set of the *i-th* spectrum column observability indices (c.o.i.) of A, C : $Z(A,C)_{\lambda_i} = \{\zeta_{i1} \geq \zeta_{i2} \geq \ldots \geq \zeta_{i\nu_i} \geq 0\}$.

ABSTRACT

The implementation of digital control schemes, involves issues such as fixed-point arithmetic, computer quantization, round off error effects and the selection of sampling scheme. The selection of sampling is crucial in the design of digital controllers and may affect drastically the quality of the discretized model on which design is based. The selection of sampling is so far dominated by the rules of signal processing theory and practical heuristics. The development of a theory and methodology for selection of sampling based on the overall quality of the discretized model, which is complementary to that provided by signal processing theory, is a long term objective of this research area and this thesis aspires to contribute to its development.

The thesis is mainly concerned with the study of the effect of sampling on the fundamental structural properties of the resulting discretized model. As such, this study is part of the more general area of investigating the transformation-preservation of qualitative and quantitative properties of continuous time models to discrete time models under sampling. Throughout the thesis we assume linear systems and constant sampling rate. The emphasis is studying the effect of sampling on fundamental model characteristics such as Jordan forms, eigenspaces, controllability, observability properties and finite-infinite zeros. Central to the approach developed here is the study of implications of a phenomenon referred to as "*eigenvalue collapsing*" that corresponds to the case where distinct eigenvalues of the continuous model become repeated eigenvalues of the discretized model. This phenomenon provides a classification of sampling rates into regular and irregular. A thorough investigation of the "*eigenvalue collapsing*" phenomena is given and their implication on the structural properties of the discretized model is given. In particular we examine the effect of such phenomena on the Segré characteristics, structure of eigenspaces, Jordan forms, controllability, observability, dimensions of controllability, observability properties, degrees of decoupling zeros and finite-infinite zeros of the discretized model.

The developments in the above directions have required some additional work in the study of certain structural properties of continuous time models, such as a detailed study of spectral properties of controllability, observability, which lead to a new characterization of decoupling zeros and their computation.

The result presented here provide a basis for the development of a model based theory of

sampling, which is significant for the development of a general implementation methodology of digital systems.

Chapter 1

INTRODUCTION

Digital computers are used increasingly as tools for analysis and design of control systems. Because of the revolutionary development of microelectronics in the last decade, advanced regulators can be implemented in many control applications areas. Sampling is a fundamental element of computer-controlled systems because of the discrete-time nature of the digital computer. So far, discretisation has been considered mainly from the signal viewpoint and rules for selection of sampling are signal based (Shannon's theorem etc.). The main objective in this thesis is the development of an alternative approach to selection of sampling that is based on preservation of structural properties of the continuous model. We shall refer to this as the development of the model based theory of sampling selection.

With the advent of microprocessor in 1969, the area of digital control systems applications has increased rapidly and this has also motivated a corresponding growth of digital control theory [Rag. & Fra., 1], [Ast. & Wit., 1], [Lew., 1]. Because of these developments the analysis, design, and implementation of control systems is changing rapidly and a number of issues which have been previously overlooked, have now started to emerge as important research topics. It is now realized that there is much to be gained by exploiting the full potential of the new technology, rather than simply "*translating*" the earlier analog designs into the new technology. The area of theory and design of digital control systems has attracted renewed interest recently and efforts have been made to put it on a more solid foundation. We distinguish three main areas of activity.

(i) Study of implementation issues, such as fixed point arithmetic, finite word-length, quantization effects and round-off errors.

(ii) Conversion of an already designed continuous-time controller to a discrete controller.

(iii) Direct digital design of control schemes.

The first area is involved in the implementation of schemes designed either by analog, or directly by digital methods. For continuous, or discrete systems it is assumed that the process model has real coefficients and thus the system is defined over an infinite field. The assumption that the computer has infinite precision in representing numbers is not true. In fact, microprocessors have limited accuracy and they use fixed-point arithmetic, or floating point arithmetic, and thus quantization is involved and both signals and model coefficients are represented by finite computer word-length. As a result, the discrete system model is not any more defined over the real numbers, but over a finite field; the latter implies the emergence of round-off errors, with significant effects on the overall performance of the digital implementation. Furthermore, it should be noted that the presence of a quantizer in the loop makes the overall system nonlinear, even when the plant is represented by a linear model and thus exhibits the features of nonlinear systems. The study of these issues is the topic of the first area described above and has been considered for a number of years [Wil., 1]. Issues considered so far are minimization of round-off errors [Mul. & Rob., 1], Optimal Finite Word Length Selection and choice of digital realization [And., Li & Gev., 1], etc.

In the design of digital control schemes we distinguish two general approaches; The first will be referred to as the Continuous Controller Design Approach (CCDA) and deals with the conversion of an already designed continuous-time controller to a discrete-time controller. The second is called the Direct Discrete Design Approach (DDDA) and deals with the design of discrete controllers on a discretised plant. These two areas have attracted interest recently as areas of potential applications of the H_∞ Optimization and related techniques [Che. & Fra., 1], [Dul. & Fra., 1], [Kel. & And., 1]. The advantage of the CCD approach is that all tools from continuous design can be deployed and the sample period T does not have to be selected until after the continuous time controller has been designed. However, all these controller discretized schemes are approximations and so far the whole area has been based on heuristics on how to modify the continuous design so that a more suitable controller is obtained. All discretisation methods found in texts [Rag. & Fra., 1], [Ast. Wit., 1], [Lew., 1] etc. suffer from the disadvantage that recovery of analog performance can be guaranteed only in the limit as the sampling period goes to zero; however, small sampling periods can be problematic. Another disadvantage of CCDA is that it gives little insight into the properties of the sampling process, such as the appearance of non minimum-phase zeros, or the properties of the discrete systems.

The general problem of the traditional discretisation of controller methodologies found in the literature is that these techniques ignore the plant, whereas the closed-loop properties clearly depend on the plant, as well as the controller. A methodology that leads to discretisation of the controller which overcomes the traditional deficiencies of CCDA, and which also preserves closed-loop properties, such as stability, has been recently developed in [Kel. & And., 1] using tools from H_∞ optimization. This approach represents the modern trend in controller discretisation and it is still in its early development stages. The deficiencies of the CCDA have motivated the emergence of a strong trend which deals with direct digital design of the sample-data controllers. These techniques are exact and usually allow significantly larger sample periods than those of the CCDA type. An additional advantage of the DDDA methods is that they provide additional insight and guarantee performance at the sample points. Within this area of work, two main tendencies have emerged. In the first H_∞, or related optimization techniques are used to accommodate intersample effects and the main feature is analysis and design of sampled-data compensators using induced norms as the performance measure [Che. & Fra., 1], [Dul. & Fra., 1]. In the second, attention is focused on the effect of sampling on the structural properties of the discretised model prior to any design of discrete controllers and will be referred to here as, Sampling and Plant Model Quality (SPMQ). The present thesis is within the latter area of work and it is focused on the structural properties of the discretised model.

The selection of sampling is crucial in the design of computer controlled systems. This problem has two main aspects; The first is of a signal nature and deals with the question under which conditions a signal can be recovered from its values in discrete points only; a solution to this problem was given by the Nyquist-Shannon theory [Ast. & Wit., 1]. The second approach is based on the quality of the discretised model as a function of the sampling period was initiated by the work of Kalman [Kal. & Ber., 1], [Kal., Ho & Nar., 1], on the role of sampling on the controllability and observability properties of discretised models. The latter work was of a preliminary nature and it was restricted only to the derivation of sufficient conditions for preservation of controllability and observability rather than providing a detailed study of these properties for all values of the sampling rate. Since the 1960s no developments have taken place in this area until 1980, when the mapping of zeros of discretised single input single output systems (SISO) was examined in [Ast. & Wit., 1], where some results on the asymptotic properties of zeros have been derived. The effect of sampling on the location of the resulting finite zeros has been an issue that has attracted attention, [Pas. & Ant.], [Fu & Dum.], [Har., Kon. & Kat., 1], [Ish., 1] ; most of the work in this area has been focused on determining conditions for the stability of the discretised zeros and has been restricted to SISO systems. The overall area of studying properties of the discretised model as a function of sampling is in its early stages of development. This thesis aspires to contribute in the development of an overall integrated approach by examining the effect of sampling on a number of structural characteristics and associated properties. The overall philosophy that is adopted is that the selection of sampling must satisfy the signal recovery criteria and also preserve structural properties, as well as degree of their presence in the discretised model.

The overall aim of the thesis is to provide a unifying approach to the study of mapping properties of the continuous model to equivalent properties of the discretised model. It is realized that the overall study involves structural and non structural properties. We focus our attention here on the structural properties. Issues related to the study of design indicators are considered as topics of further research. The basic philosophy is that structural properties are central in shaping the values of design indicators and the study of structural features precedes those which are more directly linked to design. The main objective here is to study properties such as stability, controllability, observability and structural characteristics such as Jordan forms, eigenspaces, controllable unobservable spaces, decoupling zeros, finite and infinite zeros of the discretised model as a function of the sampling rate T. The behavior of the eigenvalues of the discretised model as a function of sampling is central to our approach and provides the means for investigating further structural properties. An interesting phenomenon, referred to us *eigenvalue collapsing* is studied and criteria for different types of *collapsing* to occur are given. This leads to a complete characterization of phenomena such as merging of Segré characteristics and Jordan forms and collapsing of eigenspaces. A by-product of this analysis is the classification of sampling rates to regular and irregular. The study of the controllability and observability properties under both regular and irregular sampling then follows. The results here for irregular sampling, provide a complete treatment of phenomena such as loss of controllability, observability, emergence of new decoupling zeros and characterization of their order, as well migration of zeros at infinity. Furthermore, it also provides tests for controllability, observability which go beyond the structural collapsing and are model parameter dependent. The development of the structural properties of the discretised models depend mostly on existing theory for continuous time linear models. We have developed some additional results on the spectral characterization of controllability, observability, as well as characterization of degrees of decoupling zeros for continuous time models. The latter results are then integrated with the discrete system studies and provide criteria for dimension of resulting controllable, unobservable subspaces and orders of resulting decoupling zeros. The work on zeros of discretised models is mostly concerned with the study of phenomena where zeros migrate to infinity as a result of the selected sampling rate.

The thesis is structured as follow:

In Chapter 2 examine the general problem of discretisation and we introduce some general issues related to the discretisation process. This includes quantization, time delay, mathematical idealization and the main issues derived from the process of discretisation of continuous signals, as well of the process of reconstruction of a continuous signal from a discrete one.

In Chapter 3 a comprehensive introduction to the fundamental mathematical tools and systems theory is given: these are relevant in the study of the structural properties of the system under discretisation. The specific objective of this Chapter is to provide a short review of descriptions, basic concepts and tools from mathematics and control theory, which will be used as background material for the following investigations.

Chapter 4 deals with the state space description of a discretised model and the basic structural properties of such a model for the different types of the control signal reconstruction. In this Chapter we investigate the eigenstructure of the discretised state matrix $\tilde{A}$ and the Jordan equivalent description of the discretized model. The study of properties of the eigenvalues of the discretised model as a function of sampling is considered here and this leads to a development of the theory for eigenvalue collapsing. The importance of the relationships between A and $\tilde{A}$ is that although we have preservation of the cyclic and invariant subspaces, distinct eigenvalues of A may transformed to coinciding eigenvalues of $\tilde{A}$. This is called collapsing of eigenvalues and as a result we have phenomena associated with the merging of the corresponding generalized null-spaces and Segré characteristics. The collapsing of eigenvalues, the conditions of collapsing and the merging of generalized null-spaces and Segré characteristics are subjects examined in this Chapter. The results here classify the sampling process into two cases : the regular sampling where no collapsing phenomena occur and the irregular sampling where collapsing occurs between the eigenvalues of the discretised system. The significance of irregular sampling is investigated in the following Chapters.

Controllability and observability matrices provide one type of criteria for testing the corresponding properties. Controllability and observability matrices of a model in Jordan canonical form lead to the use of the spectral controllability and observability matrices. In the case of a discretised model the use of such tests enables the investigation of the effect of collapsing on the above structural properties. A detailed account of the effect of collapsing on the changes in the controllability and observability properties is given in Chapter 5. The work here generalizes the results derived by Kalman [Kal. & Ber., 1] by providing a complete study of the effects of irregular sampling.

An extension of the classical results on the spectral characterization of the structural properties of controllability and observability, is developed in Chapter 6. New sets of invariant indices, that is the set of *i-th* spectrum row controllability indices (r.c.i.) $\Theta_{\lambda_i}(A, B)$ and the dual set of *i-th* spectrum column observability indices (c.o.i.) $Z_{\lambda_i}(A, C)$are introduced. The role of the system parameters of the Jordan canonical description in the determination of the dimension of the controllable (unobservable) subspace $\mathcal{R}$ ($\mathcal{P}$) of linear systems is also examined here. These new derivations enables the investigation of the relation between the dimension of the controllable (unobservable) subspace of the discretised system and the corresponding of continuous system under the different types of sampling (regular or irregular).

Chapter 7 examines the role of the system parameters of the Jordan canonical description in the determination of the structure of i.d.z. (o.d.z.). A new left (right) sequence of λ_i-

Characteristic Toeplitz matrices is used to determine the set $\Sigma(A,B)_{\lambda_i}$ ($\Psi(A,B)_{\lambda_i}$) of degrees of elementary divisors of the input (output) pencil of the system at $s = \lambda_i$ or what is equivalent the degrees of input (output) decoupling zeros. These results extend the modal characterization of controllability, observability (classical results of Gilbert [Gil., 1] by providing a characterization of the degrees of elementary divisors associated with the input and output decoupling zeros. The results for continuous system provide new relationships between the Segré Characteristic of A at λ_i, $\wp_{\lambda_i}(A)$ the set of r.c.i.(c.o.i.) $\Theta(A,B)_{\lambda i}$ ($Z(A,B)_{\lambda i}$) and the set of degrees of i.d.z.(o.d.z.) $\Sigma(A,B)_{\lambda_i}$ ($\Psi(A,B)_{\lambda_i}$). This relation enables the investigation of the changes in the set of i.d.z.(o.d.z.) under irregular sampling and thus completes the study of collapsing of controllability, observability properties under irregular sampling.

In Chapter 8 the expressions derived in Chapter 3 for the zero polynomial of the continuous system are applied to the case of discretised model for the calculation of the discretised zero polynomial coefficients. The existence of a set of eigenvalues located on the imaginary axis and the collapsing of such eigenvalues to 0 is a precondition for a further migration of finite zeros to infinity under irregular sampling.

Finally, Chapter 9 provides a summary of the overall contribution of the thesis and specifies a list of open issues, which form the subject for future research.

Chapter 2

SAMPLING THEORY AND SYSTEM PROPERTIES : BACKGROUND RESULTS

2.1 Introduction

The purpose of this Chapter is to introduce some of the fundamental notions associated with the theory of computer control systems and provide the motivation for the work that follows in the thesis. This is intended as a brief introduction rather than a proper treatment of the general issues, which are properly treated in textbooks such as [Ast. & Wit., 1], [Lew., 1], [Feu. & Goo., 1], [Wil., 1] etc.

2.2 The Computer Control Configuration

In a modern feedback control system the information processing device used in generating the required controller action is almost invariably a digital computer. This is connected to the physical system being controlled through an interface as shown in the Figure 2-1.

The configuration contains essentially six parts :

1. The dynamic system (or the process) to be controlled. The vector of system's outputs

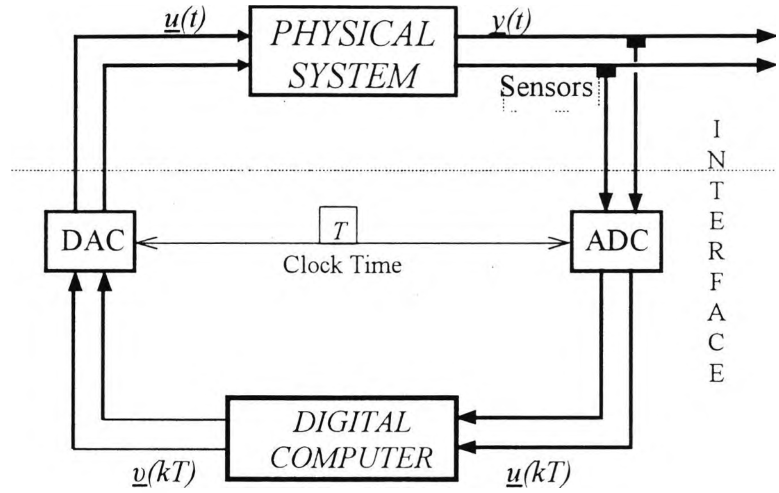

Figure 2-1: Typical Scheme of a Computer Controlled System

$\underline{y}(t)$ consists of such physical time continuous signals as position, velocity, pressure etc.

2. The sensors that produce voltage (or current) proportional to the system outputs $\underline{y}(t)$ signals.

3. The Analog to Digital Converter (ADC) that transforms the sensors continuous time signals into digital number sequences (digital signals) $\underline{\psi}(kT)$ to be fed to the computer.

4. The digital computer providing the desired control action by the resident in his memory control algorithm. The control algorithm acts on the digital signals $\underline{\psi}(kT)$ to provides further a vector of digital signals $\underline{v}(kT)$.

5. The Digital to Analog Converter (DAC) converts the sequences $\underline{v}(kT)$ back into a vector of continuous time signals $\underline{u}(t)$. This is known as reconstruction process. The continuous time control signals $\underline{u}(t)$ are then fed to the dynamic system as inputs.

6. The Clock Time determines the sampling period T.

More complicated sampling schemes can also be used. For instance, different sampling periods can be used for different control loops. This is called multirate sampling and can be considered to be the superposition of several periodic sampling schemes.

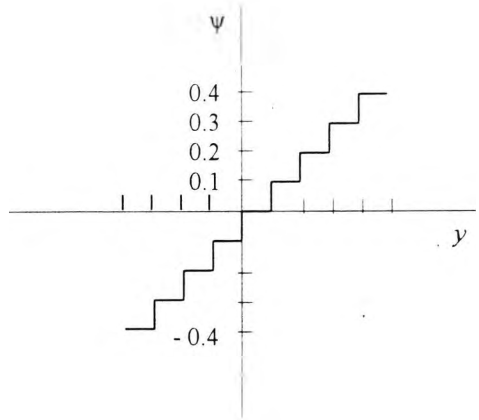

Figure 2-2: Output versus input characteristic of the ADC

A digital controller is implemented as a computer program using the above configuration. The basic functions of a digital controller are thus the following: The controller samples and quantizes a continuous time signal to produce a digital signal; it processes this digital signal using a digital computer and then it converts the resulting signal back into a continuous-time signal. Such a control system thus involves both continuous time and discrete time signals, in a continuous-time framework.

A digital signal is a discrete-time signal with a quantized amplitude.

2.3 Quantized Signals

The output of the ADC must be stored in digital logic composed of a finite number of digits. Most commonly, the logic is based on binary digits (bits) composed of $0's$ and $1's$, but the essential feature is that the representation has a finite number of digits [Fra., Pow. & Wor., 1]. A common situation is that the conversion of the analogue to digital signal is done so that the digital can be thought of as a number with fixed number of places of accuracy. If we plot the values of the analogue signal $y(t)$ versus the quantized $\psi(t)$ we can obtain a plot like that shown in the Figure 2-2.We would say that $\psi(t)$ has been truncated to one decimal place, or that $y(t)$ is quantized with a q of 0.1, since $\psi(t)$ changes only in fixed quanta of (in this case) 0.1 units. Note that quantization is a nonlinear function.

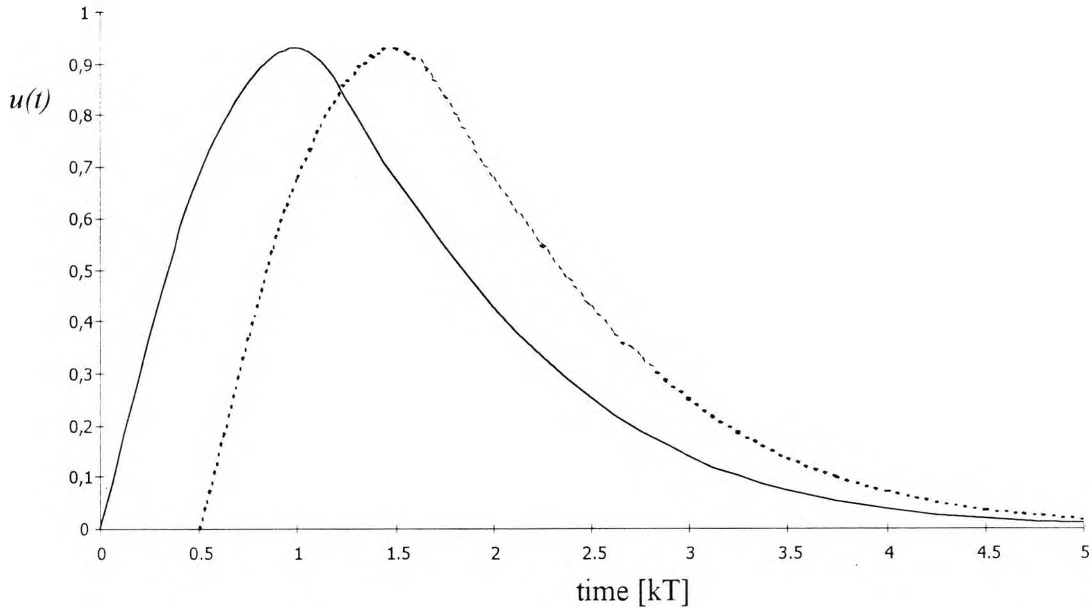

Figure 2-3: Time delay due to the DAC operation

2.4 Time Delay

The function of DAC is associated with time delay. Each value of $u(kT)$ is typically held constant until the next value is available from the computer [Fra., Pow. & Wor., 1]. Thus the continuous values of $u(t)$ consists of steps that on the average lag $u(kT)$ by $T/2$, as shown by the dashed line in the Figure 2-3.

2.5 Mathematical idealization

For purposes of analysis and design, the standard digital control system is idealized [Che. & Fra., 1], [Wil., 1]. In this idealization the three components implemented as shown in Figure 2-4, the ADC, the Digital Computer and the DAC are considered as follows :

1. The ADC became a ideal sampler S. It periodically samples the continuous signal $y(t)$ (Figure 2-5) to yield the discrete-time (and not quantized in amplitude) signal $y(kT) =$

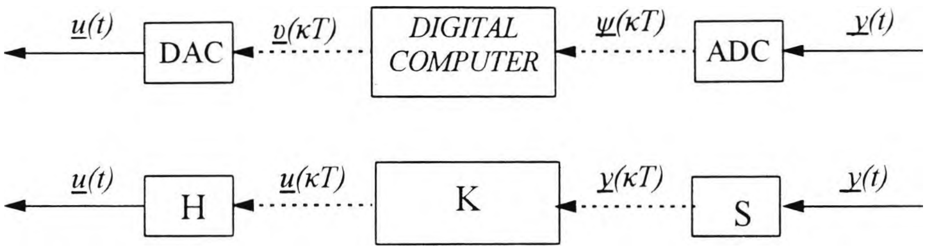

Figure 2-4: Mathematical Idealization

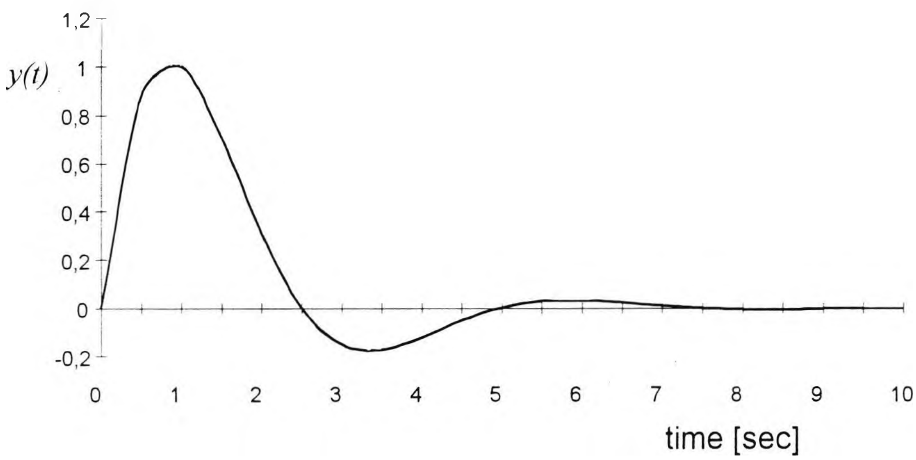

Figure 2-5: Continuous Signal

$y(kT)$ (Figure 2-6). In the general multi-output case $\underline{y}(t)$ and $\underline{\hat{y}}(kT)$ are both vectors of the same dimension.

2. The digital computer is described as a finite dimensional, linear time-invariant, causal, discrete-time system K. Its input and output at time kT are $\underline{\hat{y}}(kT)$ and $\underline{\hat{u}}(kT) = \underline{v}(kT)$ (Figure 2-7).

3. The DAC is a hold operator H. It converts the discrete-time signals $\underline{\hat{u}}(kT)$ into the continuous time signals $\underline{u}(t)$. A common and typically valid assumption is that of a H with zero-order hold ZOH. Each value of $u(kT)$ is held constant until the next value is available from the computer (Figure 2-8). If H is implemented as a first order hold (FOH) oper-

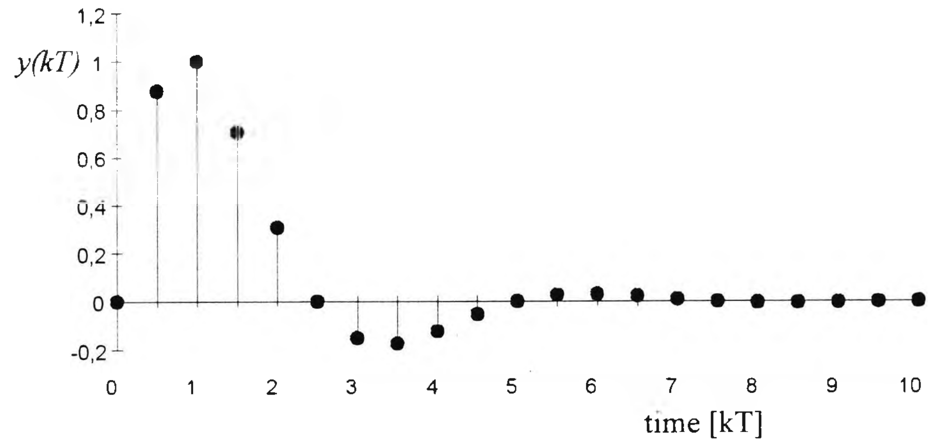

Figure 2-6: Discretized Signal

ator, then $u(t)$ is held as a straight line determined by the two last numbers $u(kT)$ and $u(kT-T)$ until a new number $u(kT+T)$ is available from the computer. It is possible to consider a H with a hold of upper than one order, say m. That means, in the time interval from kT to $kT+T$, the signal $u(T)$ is held as the extrapolation of a curve determined by the last m numbers of the sequence.

Note that S and H are synchronized, physically by a clock. Using the idealizations of S and H, we obtain the idealized model of the standard control system. This is called the standard sampled-data (SD) system. The sampled data system has both continuous-time and discrete time signals, whereas *"digital"* refers to a system having digital signals

2.6 Shannon Reconstruction

It is of course, essential to know when a continuous-time signal is uniquely given by its sampled version [Feu., 1]. The following theorem [Ast & Wit., 1] gives the conditions for the case of periodic sampling.

Theorem 1 *A continuous-time signal with a Fourier transform that is zero outside the interval* $(-\omega_0, \omega_0)$ *is given uniquely by its values in equidistant points if the sampling angular frequency*

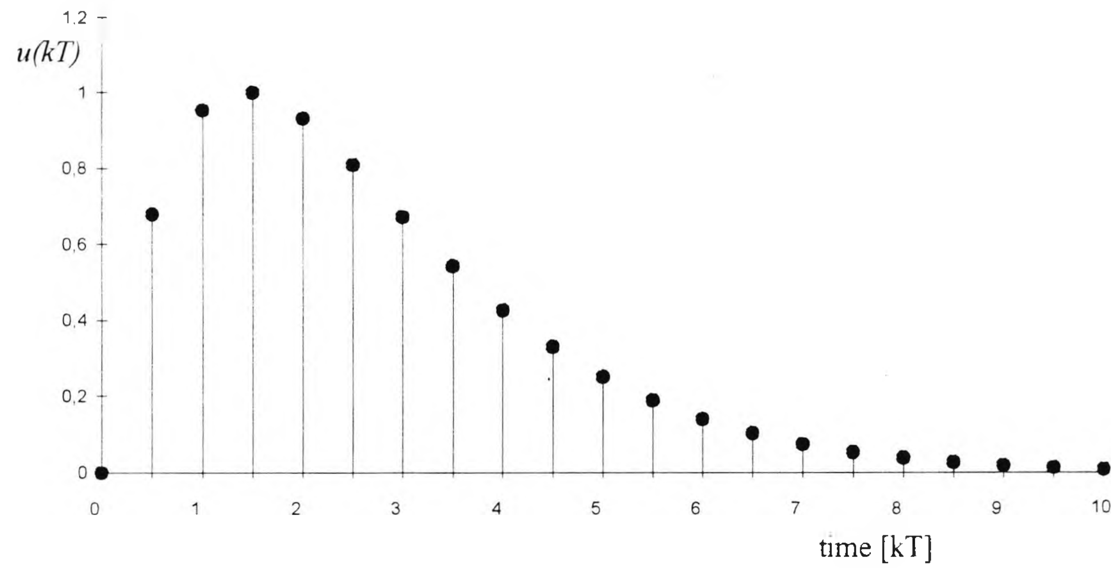

Figure 2-7: Digital Signal from Computer

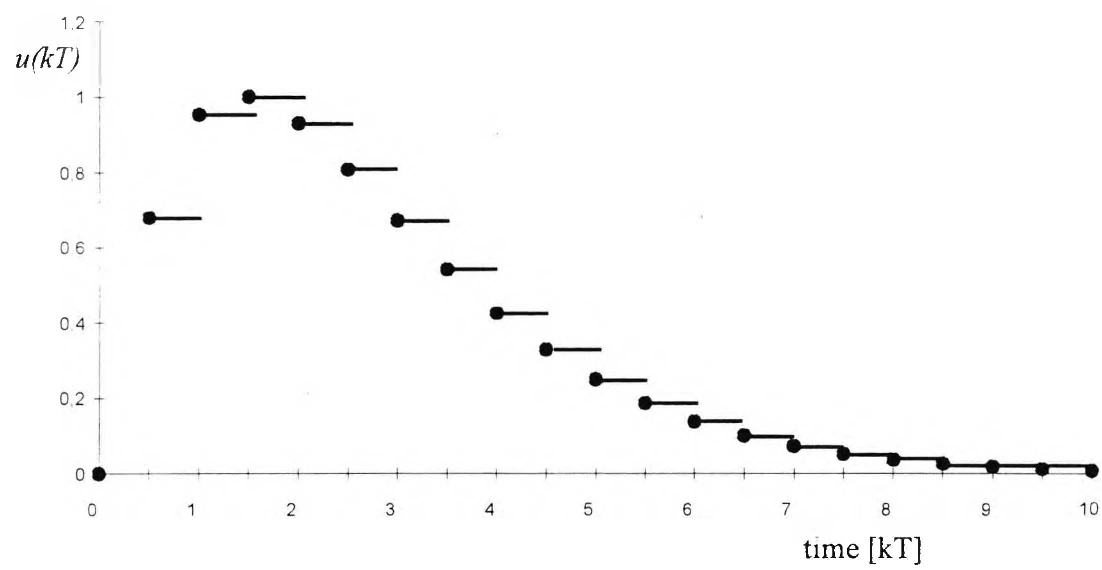

Figure 2-8: Signal Reconstruction from DAC with ZOH

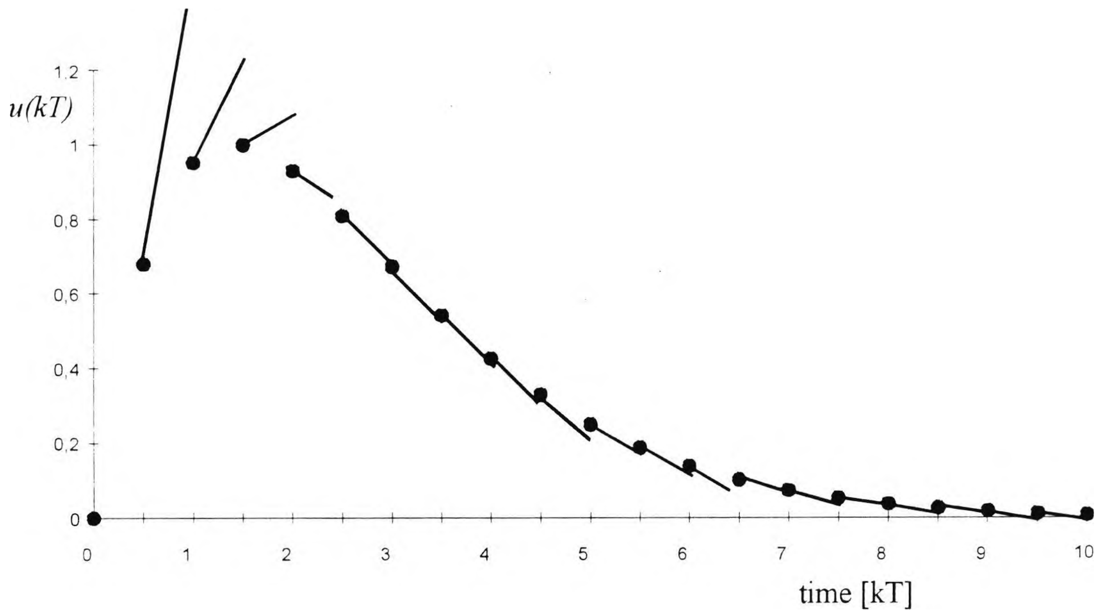

Figure 2-9: Signal Reconstruction from DAC with FOH

$\omega_s = 2\pi T$ is higher than $2\omega_0$. The continuous-time signal can be computed from the sampled signal by the interpolation formula:

$$y(t) = \sum_{k=-\infty}^{\infty} y(kT) \frac{\sin \omega_s (t - kT)/2}{\omega_s (t - kT)/2} \quad . \tag{2.1}$$

Remark 1 *The frequency $\omega_N = \omega_s/2$ is called the Nyquist frequency.*

Remark 2 *Equation (2.1) defines the reconstruction of signals whose Fourier transforms vanish for frequencies larger than the Nyquist frequency ω_N*

The inversion of the sampling operation, i.e., the conversion of a sequence of numbers to a continuous-time function is called reconstruction. In computer-controlled systems, it is necessary to convert the control actions produced by the computer as a sequence of numbers into a form of a continuous-time function using a hold operator.

2.7 Aliasing [Ast. & Wit., 1]

Stable linear time-invariant systems have the property that the steady-state response to sinusoidal excitations is sinusoidal with the frequency of the excitation signal. Computer control systems behave however, in a much move complicated way because sampling will create signals with new frequencies and this can drastically affect performance unless precautions are taken. The phenomenon that sampling process creates new frequency components is called aliasing. Whenever the signal contains frequencies that are larger than half the sampling frequencies, there will be new low frequency components which are created.

In fact, if a continuous-time signal $f(t)$ that has the Fourier transform,

$$F(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} f(t) dt$$

is sampled periodically, it has been proved that the sampled signal $f(kT)$ can be interpreted as the coefficients of the Fourier series of a periodic function $F_s(\omega)$ defined as [Ast. & Wit., 1].

$$F_s(\omega) = \frac{1}{T} \sum F(\omega + k\omega_s) \tag{2.2}$$

The period of $F_s(\omega)$ is equal to ω_s and it is

$$f(kT) = \frac{1}{\omega_s} \int_{0}^{\omega_s} e^{ikT\omega} F_s(\omega) d\omega$$

then the function $F_s(\omega)$ can be interpreted as the Fourier transform of the sampled signal $f(kT)$. If the continuous time signal has no frequency components higher than the Nyquist frequency, the Fourier transform $F_s(\omega)$ is simply a periodic repetition of the Fourier transform of the continuous-time signal. It follows from (2.2) that the value of the Fourier transform of the sampled signal at ω is the sum of the values of the Fourier transform of the continuous-time signal at the frequencies $\omega + n\omega_s$.

An illustration of the aliasing effect in computer systems is illustrated by the following diagram [Ast. & Wit., 1] in Figure 2-11 representing the response of a computer system under certain conditions on the value of sampling.

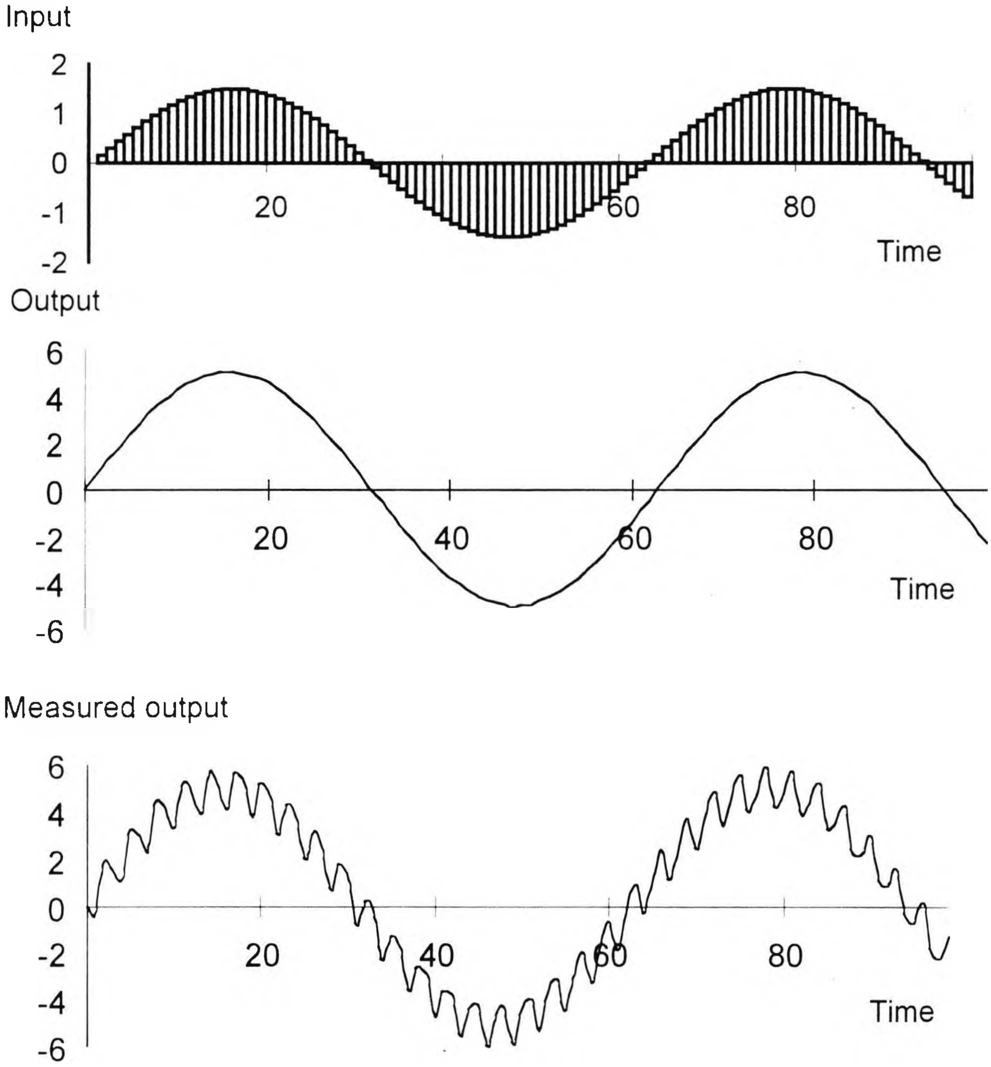

Figure 2-10: Simulation of a sampled data system exhibiting aliasing phenomena

To avoid the difficulties associated with aliasing, it is essential that all signal components with frequencies higher than the Nyquist frequencies are removed before the signal is sampled. This involves the use of an antialiasing filter in the overall configuration. The proper selection of sampling periods and antialiasing filters are important aspects of the design of computer-controlled systems.

2.8 The z-Transform

The z-transform maps a semi-infinite time sequence into a function of a complex variable. A summary of the basics of the transform theory is given below.

Definition 1 *The z-transform of the discrete-time signal $\{f(kT) : k = 0, 1, ...\}$, is defined as.*

$$F(z) = \sum_{k=0}^{\infty} f(kT) z^{-k}$$

where z is a complex variable. The inverse of the z-transform of $f(kT)$ is given by.

$$f(kT) = \frac{1}{2\pi i} \oint F(z) z^{k-1} dz$$

where the contour of integration encloses all singularities of $F(z)$. The z-transform of $f(kT)$ is denoted by $\mathcal{Z} f$ or F.

The basic properties of the z-transform are,

1. Linearity:

$$\mathcal{Z}(\alpha f + \beta g) = \alpha \mathcal{Z} f + \beta \mathcal{Z} g$$

2. Time shift:

$$\mathcal{Z} q^{-n} f = z^{-n} F$$

$$\mathcal{Z} \{q^n f\} = z^n (F - F_1)$$

$$\text{where, } F_1(z) = \sum_{j=0}^{n-1} f(jT) z^{-j}$$

where by q is denoted the forward-shift operator and by q^{-1} the backward-shift operator [Ast. & Wit., 1].

3. Initial value theorem:

$$f(0) = \lim_{z \to \infty} F(z)$$

4. Final value theorem: If $(1 - z^{-1})F(z)$ does not have any poles on or outside the unit circle, then,

$$\lim_{k \to \infty} f(kT) = \lim_{z \to 1} (1 - z^{-1})F(z)$$

5. Convolution:

$$\mathcal{Z}(f * g) = \mathcal{Z} \sum_{n=0}^{k} f(n)g(k-n) = (Zf)\,(Zg)$$

2.9 Conclusions, Emerging Issues

The selection of sampling is an important issue in the design of computer control schemes. So far, the methodology for selection of the appropriate sampling has been based on signal type criteria. The need to develop a model based theory, that exploits also the quality of the resulting discretised model has been recognized, but the area is in its early stages of development. In practice, it is essential that sampling is selected using both signal recovery criteria and preservation of continuous model features criteria. The model based criteria express the general aim that the open loop discretised model has characteristics and properties, as close as possible to those of the open loop continuous time model. To achieve the above general objectives research is needed in many areas related to the mapping of continuous model properties characteristics, to the corresponding ones of the discretised model as a function of the sampling rate. Amongst the issues worth examining are those related to the mapping of structural invariants and associated properties, as well as the transformation of features of design indicators. This thesis is concerned with the study of properties of a number of system invariants and their associated properties as a function of the sampling rate.

Chapter 3

SYSTEMS AND MATHEMATICS BACKGROUND

3.1 Introduction

Modern control theory and design uses concepts and tools from almost every single branch of mathematics. The aim of this chapter is to introduce some terminology and define the basic mathematical concepts and tools, which are essential for the presentation of the system concepts in the following sections. The following topics are considered as essential:

1. Basic concepts and tools from Polynomial and Rational Matrix theory.
2. Basic concepts and definitions from the structure of a linear operator in an n-dimensional space.
3. Basic concepts and tools from matrix pencil theory.
4. Review of relevant topics from systems theory.

It should be stressed that this section serves as basic terminology and does not aspire to be an introduction to mathematics for control theory. Details may be found in the references. Certain nonstandard topics, which have an impact on the subsequent chapters are treated in some more detail.

3.2 Polynomial and Rational Matrices

Let $\mathbb{R}, \mathbb{C}$ denote the fields of real and complex numbers respectively, Z^+ is the set of positive integers and $Z_0^+ = Z^+ \cup \{0\}$, $\mathbb{R}[s]$ be the ring of polynomials with coefficients in $\mathbb{R}$ and $\mathbb{R}(s)$ be the field of rational functions :

$$\mathbb{R}(s) \triangleq \{t(s) : t(s) = \frac{n(s)}{d(s)}, n(s), d(s) \in \mathbb{R}[s], d(s) \not\equiv 0\} \tag{3.1}$$

Let $\mathbb{R}(s)^p, p \in Z^+$ be the set of ordered p-tuples of rational functions considered as column vectors:

$$\mathbb{R}(s)^p \triangleq \{t(s) : t(s) = [t_1(s), t_2(s), ..., t_p(s)]^T, t_i(s) \in \mathbb{R}(s), i \in \mathbf{p}\} \tag{3.2}$$

then $\mathbb{R}(s)^p$ has the structure of a linear vector space which we call a *rational vector space*. Let $\mathbb{R}(s)^{p\times m}, p, m \in Z^+$ denote the set of $p \times m$ matrices with elements in $\mathbb{R}(s)$. A matrix $T(s) \in \mathbb{R}(s)^{p\times m}$ is called a rational matrix.

The ring of polynomials $\mathbb{R}[s]$ is a Euclidean ring i.e. there is a map $\partial : \mathbb{R}[s] \longrightarrow Z_0^+$ such that for every $a(s) \in \mathbb{R}[s], a(s) \not\equiv 0$ we denote $\partial a(s) \triangleq \deg a(s) \in Z_0^+$. The units $u(s)$ of $\mathbb{R}[s]$ are the non-zero elements of $\mathbb{R}$.

A rational matrix $T(s)$ whose elements are polynomials is called a polynomial matrix. The set of polynomial matrices is denoted by $\mathbb{R}[s]^{p\times m}$. A polynomial matrix $T(s) \in \mathbb{R}[s]^{p\times p}$ is called unimodular if there exist a $\hat{T}(s) \in \mathbb{R}[s]^{p\times p}$ such that $T(s)\hat{T}(s) = I_p$, equivalently if $\det T(s) = c \in \mathbb{R}, \ c \neq 0$.

Definition 2 *The degree of a polynomial matrix $T(s) \in \mathbb{R}[s]^{p\times m}$, denoted by $\deg T(s)$, is defined as the maximum degree among the degrees of all its maximum order (non-zero) minors.* □

3.2.1 Smith-McMillan Form [Var., Lim. & Kar., 1]

"*Elementary row and column operations*" on a $T(s) \in \mathbb{R}^{p\times m}(s)$ are defined in the following usual way :

1. interchange any two rows (columns) of $T(s)$

2. multiply row (column) i of $T(s)$ by a unit $u(s) \in \mathbb{R}[s]$ and

3. add to row (column) i a multiple by a non zero element $t(s) \in \mathbb{R}[s]$ of row (column) j

These elementary operations can be accomplished by multiplying the given $T(s)$ on the left (right) by "elementary" unimodular matrices, namely matrices obtained by performing the above elementary operations on the identity matrix $I_p(m)$.

Definition 3 *$T_1(s) \in \mathbb{R}^{p\times m}(s), T_2(s) \in \mathbb{R}^{p\times m}(s)$ are called equivalent in $\mathbb{C}$ if there exist unimodular matrices $T_L(s) \in \mathbb{R}^{p\times p}(s), T_R(s) \in \mathbb{R}^{m\times m}(s)$ such that*

$$T_L(s)T_1(s)T_R(s) = T_2(s) \tag{3.3}$$

The above equation defines an equivalence relation of $T_1(s), T_2(s)$ on $\mathbb{R}^{p\times m}(s)$ which we denote by $E^{\mathbb{C}}$. The set of all the equivalent matrices of a fixed $T(s) \in \mathbb{R}^{p\times m}(s)$ defines $E^{\mathbb{C}}$-equivalence class or the orbit of the $T(s)$. □

Theorem 2 *Let $T(s) \in \mathbb{R}^{p\times m}(s)$ with $\mathrm{rank}_{\mathbb{R}(s)}T(s) = r$. Then $T(s)$ is equivalent in $\mathbb{C}$ to a diagonal matrix $S^{\mathbb{C}}_{T(s)}$ having the following form:*

$$S^{\mathbb{C}}_{T(s)} = diag\{\frac{\varepsilon_1(s)}{\psi_1(s)}, \frac{\varepsilon_2(s)}{\psi_2(s)}, ..., \frac{\varepsilon_r(s)}{\psi_r(s)}, 0_{m-r,p-r}\} \tag{3.4}$$

where $\varepsilon_i(s), \psi_i(s) \in \mathbb{R}[s]$ are monic and coprime such that $\varepsilon_i(s)$ divides $\varepsilon_{i+1}(s)$, while $\psi_{i+1}(s)$ divides $\psi_i(s), i = 1, 2, ...r-1$. □

Definition 4 *The rational functions*

$$f_i \triangleq \varepsilon_i(s)/\psi_i(s) \in \mathbb{R}(s), i \in \mathbf{r},$$

constitute a complete set of invariants of $E^{\mathbb{C}}$ and are called the invariant rational functions of $T(s)$. □

Definition 5 *The zeros of $T(s) \in \mathbb{R}^{p\times m}(s)$ in $\mathbb{C}$ are defined as the zeros of the polynomials $\varepsilon_i(s), i \in \mathbf{r}$. The poles of $T(s)$ in $\mathbb{C}$ are defined as the zeros of the polynomials $\psi_i(s), i \in \mathbf{r}$.* □

Remark 3 (Smith form) *If $T(s) \in \mathbb{R}^{p\times m}[s]$ then $\psi_i(s) = 1, i \in \mathbf{r}$, that is, $S^{\mathbb{C}}_{T(s)}$ is also a polynomial matrix and it is called the Smith form of $T(s)$ in $\mathbb{C}$. Otherwise if $T(s)$ is non-polynomial, for some $i = 1, 2, ..., j; j \in \mathbf{r}$, the $\psi_i(s)$ are non constant, that is called the McMillan form of $T(s)$ in $\mathbb{C}$.* □

Let $T(s) \in \mathbb{R}^{p\times m}[s]$ and

$$S^{\mathbb{C}}_{T(s)} = \mathrm{diag}\{\varepsilon_1(s), \varepsilon_2(s), ..., \varepsilon_r(s), 0_{p-r,m-r}\}$$

Then we have.

Definition 6 *The polynomials $\varepsilon_i(s) \in \mathbb{R}[s], i \in \mathbf{r}$, constitute a complete set of invariants of E and are called the invariant polynomials of $T(s)$.* □

The invariant polynomials $\varepsilon_i(s)$ can also be obtained by.

$$\varepsilon_i(s) = \frac{D_i(s)}{D_{i-1}(s)}, i \in \mathbf{r}$$

where $D_0(s) \triangleq 1$ and $D_i(s)$ is the greatest common divisor of minors of order i in $T(s)$.

Let the invariant polynomials $\varepsilon_i(s)$ be factorized into their monic irreducible factors $\varphi_j(s)$ over the field of $\mathbb{R}$ and let the power of $\varphi_j(s)$ occurring in $\varepsilon_i(s)$ be k_{ij}. Then those of $\varphi_j^{k_{ij}}(s)$ with $k_{ij} \neq 0$ are called the elementary divisors of $T(s)$.

3.2.2 Smith-McMillan form at $s = \infty$ [Var., Lim. & Kar., 1]

Define the map $\delta_\infty : \mathbb{R}(s) \longrightarrow Z \cup (+\infty)$ via

$$\delta_\infty(t(s)) = \begin{cases} \deg d(s) - \deg n(s), & t(s) \not\equiv 0 \\ +\infty. & t(s) \equiv 0 \end{cases} \tag{3.5}$$

The map $\delta_\infty(.)$ is a discrete valuation on $\mathbb{R}(s)$ and every $t(s) \in \mathbb{R}(s)$ can be factored as,

$$t(s) = \left(\frac{1}{s}\right)^{q_\infty} \frac{n_1(s)}{d_1(s)} \tag{3.6}$$

where $q_\infty \triangleq \delta_\infty(t(s))$ and $\deg n_1(s) = \deg d_1(s)$.

Definition 7 *If $q_\infty > 0$ we say that $t(s)$ has a zero at $s = \infty$ of order q_∞ and if $q_\infty < 0$, then we say that $t(s)$ has a pole of order $|q_\infty|$ at $s = \infty$.* $\square$

If $t(s) \in \mathbb{R}(s)$ and $\delta_\infty(t(s)) \geq 0$, then $t(s)$ is called a proper rational function. Thus, proper rational functions have no poles at $s = \infty$. It is easily verified that the set of all proper rational functions, which we denote by $\mathbb{R}_{pr}(s)$, is an integral domain. The units $u(s) \in \mathbb{R}_{pr}(s)$ are those proper rational functions for which there exist a $\acute{u}(s) \in \mathbb{R}_{pr}(s)$ such that $u(s)\acute{u}(s) = 1$. Such functions have no zeros at $s = \infty$ and thus, if $u(s) = n(s)/d(s) \in \mathbb{R}_{pr}(s)$ is a unit, $\delta_\infty(u(s)) = 0$, i.e. $\deg n(s) = \deg d(s)$.

Denote by $\mathbb{R}_{pr}^{p\times m}(s)$ the set of $p \times m$ matrices with elements in $\mathbb{R}_{pr}(s)$. Such matrices are called *proper rational matrices*. Let $T(s) \in \mathbb{R}_{pr}^{p\times p}(s)$, then $T(s)$ is called $\mathbb{R}_{pr}(s)$*-unimodular* or *biproper* if there exists a $\bar{T}(s) \in \mathbb{R}_{pr}^{p\times p}(s)$ such that $T(s)\bar{T}(s) = I_p$.

"*Elementary row and column operations*" on a $T(s) \in \mathbb{R}^{p\times p}(s)$ are defined in the following usual way :

1. interchange any two rows (columns) of $T(s)$
2. multiply row (column) of $T(s)$ by a unit $u(s) \in \mathbb{R}_{pr}(s)$ and
3. add to row (column) i a multiple by a $t(s) \in \mathbb{R}_{pr}(s)$ of row (column) j

These elementary operations can be accomplished by multiplying the given $T(s)$ on the left (right) by "*elementary*" biproper matrices obtained by performing the above elementary operations on the identity matrix $I_p(m)$.

Definition 8 *$T_1(s) \in \mathbb{R}_{pr}^{p\times m}(s), T_2(s) \in \mathbb{R}_{pr}^{p\times m}(s)$ are called equivalent at $s = \infty$ if there exist biproper rational matrices $T_L(s) \in \mathbb{R}_{pr}^{p\times p}(s), T_{\mathbb{R}}(s) \in \mathbb{R}_{pr}^{m\times m}(s)$ such that*

$$T_L(s)T_1(s)T_{\mathbb{R}}(s) = T_2(s) \tag{3.7}$$

$\square$

We have the following .

Theorem 3 *(Smith-McMillan form of a rational matrix at $s = \infty$) Let $T(s) \in \mathbb{R}^{p\times m}(s)$ with $rank_{\mathbb{R}(s)}T(s) = r$. Then $T(s)$ is equivalent at $s = \infty$ to a diagonal matrix $S^\infty_{T(s)}$ having the* *following form:*

$$S^{\infty}_{T(s)} = diag\{s^{q^1_\infty}, s^{q^2_\infty}, ..., s^{q^j_\infty}, \frac{1}{s^{\hat{q}^{j+1}_\infty}}, \frac{1}{s^{\hat{q}^{j+2}_\infty}}, ..., \frac{1}{s^{\hat{q}^{r}_\infty}}, 0_{p-r,m-r}\} \tag{3.8}$$

where

$$q^1_\infty \geq q^2_\infty \geq ... \geq q^j_\infty \geq 0 \tag{3.9}$$

$$\hat{q}^r_\infty \geq \hat{q}^{r-1}_\infty \geq ... \geq \hat{q}^{j+1}_\infty \geq 0 \tag{3.10}$$

□

Remark 4 *If $T(s) \in \mathbb{R}^{p\times p}_{pr}(s)$then $q^i_\infty = 0$, $i = 1, 2, ..., j$, i.e. $S^{\infty}_{T(s)}$ is a proper rational matrix ($S^{\infty}_{T(s)} \in \mathbb{R}^{p\times p}_{pr}(s)$and it is called the Smith form of $T(s)$ at $s = \infty$. Otherwise, i.e., if $T(s)$ is nonproper, then $S^{\infty}_{T(s)}$is also nonproper and it is called the McMillan form of $T(s)$ at $s = \infty$. If p_∞ is the number of q^i_∞'s in (3.8) with $q^i_\infty > 0$, $i \in j$, then we say that $T(s)$ has p_∞ poles at infinity, each one of order $q^i_\infty > 0$. Also if z_∞ is the number of $\hat{q}^i_\infty$'s in* (5) *with $\hat{q}^i_\infty > 0$, $i = j+1, ..., r$, then we say that $T(s)$ has z_∞ zeros at infinity, each one of order $\hat{q}^i_\infty > 0$.* □

An alternative, equivalent characterization of the poles, zeros at infinity is given below.

Definition 9 (Ver., 1) *,[Pug. & Rat., 1]The rational matrix $G(s)$ is said to have a pole (zero) at infinity, if the matrix $G(\frac{1}{w})$ has a pole (zero) at $w = 0$.* □

3.3 The Structure of a Linear Operator in an n-Dimensional Space

The existence of a matrix of normal form in a class of similar matrices is closely connected with important and deep properties of a linear operator in a n-dimensional space.

3.3.1 Geometric Theory of Elementary Divisors [Gan., 1]

Consider an n-dimensional vector space $\mathcal{R}$ over the field $\mathcal{F}$ and a linear operator A in this space and a vector $\underline{u} \in \mathcal{R}$. The following definitions and propositions are valid :

Definition 10 *The monic annihilating polynomial $\phi(s)$ of least degree for which $\phi(A)\underline{x} = \underline{0}$ will be called the minimal polynomial of $\underline{x}$.* □

Proposition 1 *Every vector $\underline{x}$ has only one minimal polynomial $\phi(s)$ which divides every annihilating polynomial of $\underline{x}$.* □

If the vectors $\underline{x}_1, \underline{x}_2, ..., \underline{x}_n \in \mathcal{R}$, define a basis in $\mathcal{R}$ and the corresponding minimal polynomials $\phi_1(s), \phi_2(s), ..., \phi_n(s)$ then if $\psi(s)$ is the least common multiple of $\phi_1(s), \phi_2(s), ..., \phi_n(s)$ it is an annihilating polynomial for every vector $\underline{x} \in \mathcal{R}$. It can be proved that $\psi(s)$ has the least degree and divides all the annihilating polynomials for the whole space $\mathcal{R}$ and it is called the minimal polynomial of the space $\mathcal{R}$. Then the minimal polynomial of the space $\mathcal{R}$ does not depend of the choice of the basis. This polynomial is divisible by the minimal polynomial of every $\underline{x} \in \mathcal{R}$ and annihilates every $\underline{x} \in \mathcal{R}$.

The space $\mathcal{R}$ is decomposed into two subspaces $\mathcal{R}_1$ and $\mathcal{R}_2$: $\mathcal{R} = \mathcal{R}_1 \oplus \mathcal{R}_2$ if,

1. $\mathcal{R}_1 \cap \mathcal{R}_2 = \{\underline{0}\}$

2. $\forall \underline{x} \in \mathcal{R} \Rightarrow \underline{x} = \underline{x}_1 + \underline{x}_2,\ \underline{x}_1 \in \mathcal{R}_1,\ \underline{x}_2 \in \mathcal{R}_2$

A subspace $\mathcal{R}' \subset \mathcal{R}$ is called invariant with respect to the operator A if $A\mathcal{R}' \subset \mathcal{R}'$ or $\forall \underline{x} \in \mathcal{R}' \Rightarrow A\underline{x} \in \mathcal{R}'$.

Theorem 4 (First Decomposition Theorem) *If for a given operator A the minimal polynomial $\psi(s)$ of the space is represented over $\mathcal{F}$ in the form of a product of two co-prime polynomials $\psi_1(s)$ and $\psi_2(s)$ (with highest coefficients 1).*

$$\psi(s) = \psi_1(s)\psi_2(s)$$

then the whole space $\mathcal{R}$ splits into two invariant subspaces I_1 and I_2,

$$\mathcal{R} = I_1 \oplus I_2$$

whose minimal polynomials are $\psi_1(s)$ and $\psi_2(s)$ respectively. □

Theorem 5 *In a vector space there always exists a vector whose minimal polynomial coincides with the minimal polynomial of the whole space.* □

Lemma 1 *If the minimal polynomials of the vectors $\underline{x}_1$ and $\underline{x}_2$ are co-prime, then the minimal polynomial of the sum vector $\underline{x}_1 + \underline{x}_2$ is equal to the product of the minimal polynomials of the constituent vectors.* □

Let the minimal polynomial of the vector $\underline{e}$ be

$$\phi(s) = s^p + a_1 s^{p-1} + ... + a_{p-1}s + a_p$$

then the vectors $\underline{e}, A\underline{e}, ..., A^{p-1}\underline{e}$ are linearly independent, form a basis for a p-dimensional A-invariant and cyclic subspace $\mathcal{V}$.

Every vector $\underline{x} \in \mathcal{V}$ is representable in the form of a linear combination of the basis vectors, i.e. in the form $\underline{x} = \chi(A)\underline{e}$ where $\chi(s) \in \mathbb{R}[s]$ of degree $\leq p-1$ with coefficients in $\mathcal{F}$.

Remark 5 *1. Vector $\underline{e}$ is defined as the generating vector of the subspace $\mathcal{V}$ and*

2. the minimal polynomial of $\underline{e}$ is also the minimal polynomial of the whole subspace $\mathcal{V}$. □

Theorem 6 (Second Decomposition Theorem) *Relative to a given linear operator A the space can always be split into cyclic subspaces $\mathcal{V}_1$, $\mathcal{V}_2$, ..., $\mathcal{V}_t$ with minimal polynomials $\psi_1(s)$, $\psi_2(s)$, ..., $\psi_t(s)$,*

$$R = \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus ... \oplus \mathcal{V}_t$$

such that $\psi_1(s)$ coincides with the minimal polynomial $\psi(s)$ of the whole space and that each $\psi_i(s)$ is divisible by $\psi_{i+1}(s)(i = 1, 2, ..., t-1)$. □

Theorem 7 *A space is cyclic if and only if its dimension is equal to the degree of its minimal polynomial.* □

Theorem 8 *A space does not split into invariant subspaces if and only if*

1. it is cyclic and

2. its minimal polynomial is a power of an irreducible polynomial over $\mathcal{F}$. □

Theorem 9 (Third Decomposition Theorem) *A space can always split into cyclic invariant subspaces*

$$R = \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus ... \oplus \mathcal{V}_t$$

such that the minimal polynomial of each of these cyclic subspaces is a power of an irreducible polynomial. □

3.3.2 Jordan Form of a Matrix [Gan., 1],[Kar., 2]

Let the characteristic polynomial of a matrix $A \in \mathbb{R}^{n\times n}$ be:

$$\phi(s) = \det(A - sI) = (s-\lambda_1)^{\pi_1}(s-\lambda_2)^{\pi_2}...(s-\lambda_f)^{\pi_f}$$

Where $\lambda_1, \lambda_2, ..., \lambda_f \in \mathbb{C}$ are all the distinct eigenvalues of A and $\pi_1, \pi_2, ..., \pi_f$ are their corresponding algebraic multiplicities, with $\pi_1 + \pi_2 + ... + \pi_f = n$. Also let that the invariant polynomials of $A - sI$ are decomposed into elementary divisors.

$$\begin{aligned}
f_1(s) &= (s-\lambda_1)^{\tau_{1\nu_1}}(s-\lambda_2)^{\tau_{2\nu_2}}...(s-\lambda_i)^{\tau_{i\nu_i}}...(s-\lambda_f)^{\tau_{f\nu_f}} \\
f_2(s) &= (s-\lambda_1)^{\tau_{1\nu_1}-1}(s-\lambda_2)^{\tau_{2\nu_2}-1}...(s-\lambda_i)^{\tau_{i\nu_i}-1}...(s-\lambda_f)^{\tau_{f\nu_f}-1} \\
& \quad \cdots\cdots\cdots\cdots \\
f_k(s) &= (s-\lambda_1)^{\tau_{1k}}(s-\lambda_2)^{\tau_{2k}}...(s-\lambda_i)^{\tau_{ik}}...(s-\lambda_f)^{\tau_{fk}} \\
& \quad \cdots\cdots\cdots\cdots \\
f_{\nu_1}(s) &= (s-\lambda_1)^{\tau_{11}}(s-\lambda_2)^{\tau_{21}}...(s-\lambda_i)^{\tau_{i1}}...(s-\lambda_f)^{\tau_{f1}}
\end{aligned}$$

where, $\nu_1 \geq \nu_2 \geq ... \geq \nu_i \geq ... \geq \nu_f \geq 0$ and for $i = 1, 2, ..., f$ it is, $\tau_{i\nu_i} \geq \tau_{i\nu_i-1} \geq ... \geq \tau_{i1} > 0$ if $k \leq \nu_i \Rightarrow \tau_{ik} = \tau_{ik}$ and if $k > \nu_i \Rightarrow \tau_{ik} = 0$. Also it is:

$$\tau_{i1} + \tau_{i2} + ... + \tau_{i\nu_i} = \pi_i$$

and where ν_i is the rank deficiency of matrix $A - sI$ at $s = \lambda_i$ i.e.,

$$\nu_i = n - \text{rank}\,(A - \lambda_i I) \tag{3.11}$$

To each one of the above elementary divisors, say $(s-\lambda_i)^{\tau_{ik}}$ there corresponds a definite cyclic subspace $\mathcal{V}_{ik}$ generated by a vector $\underline{e}$. For this vector the elementary divisor $(s-\lambda_i)^{\tau_{ik}}$ is the minimal polynomial:

$$(A-\lambda_i I)^{\tau_{ik}}\underline{e}=\underline{0}$$

and the vectors,

$$\underline{u}_{ik_1}=(A-\lambda_i I)^{\tau_{ik}-1}\underline{e},\ \underline{u}_{ik_2}=(A-\lambda_i I)^{\tau_{ik}-2}\underline{e},\ \ldots,\ \underline{u}_{ik_{\tau_{ik}}}\triangleq\underline{e}$$

are linearly independent and consist a basis for the cyclic subspace $\mathcal{V}_{ik}$. Vector $\underline{u}_{ik_1}$ is defined as an eigenvector of A associated with the eigenvalue λ_i. It is,

$$(A-\lambda_i I)\underline{u}_{ik_1}=\underline{0}$$

The maximum number of linearly independent eigenvectors associated with each one of the distinct eigenvalues λ_i, is given by the rank deficiency ν_i. Number ν_i is defined also as the geometric multiplicity of λ_i.

To each one of the ν_i real eigenvectors associated with the eigenvalue λ_i, corresponds one chain of generalized eigenvectors. Let to the eigenvector $\underline{u}_{ik}$ $(k=1,2,...,\nu_i)$ associated with λ_i corresponds a chain of τ_{ik} generalized eigenvectors $\underline{u}_{ik_1},\underline{u}_{ik_2},...,\underline{u}_{ik_{\tau_{ik}}}$, given by the equations :

$$\begin{array}{ccc} (A-\lambda_i I)\,\underline{u}_{ik_1}=\underline{0} & \text{and} & \underline{u}_{ik_1}\triangleq\underline{u}_{ik} \\ (A-\lambda_i I)\,\underline{u}_{ik_2}=\underline{u}_{ik_1} & \Leftrightarrow & (A-\lambda_i I)^2\,\underline{u}_{ik_2}=\underline{0} \\ \ldots\ldots & \ldots & \ldots\ldots \\ (A-\lambda_i I)\,\underline{u}_{ik_{\tau_{ik}}}=\underline{u}_{ik_{\tau_{ik}-1}} & \Leftrightarrow & (A-\lambda_i I)^{\tau_{ik}}\,\underline{u}_{ik_{\tau_{ik}}}=\underline{0} \end{array} \tag{3.12}$$

where τ_{ik} is now defined as the eigenvectors chain length. The maximum possible value for an eigenvector chain length is equal to the minimum power $\tau_{i\nu_i}$ of matrix $A-\lambda_i I$, for which :

$$\text{rank}(A-\lambda_i I)^{\tau_{i\nu_i}}=\text{rank}(A-\lambda_i I)^{\tau_{i\nu_i}+1} \tag{3.13}$$

and we have :

Definition 11 *We define as the index of annihilation of matrix A at λ_i, the minimum power $\tau_{i\nu_i}$ of matrix $A - \lambda_i I$ for which the relation (3.13) is valid. The annihilation index is equal to the length of the longest chain of generalized eigenvectors, associated with the eigenvalue λ_i.* □

From (3.12) it follows that every generalized eigenvector of a chain $\underline{u}_{ik_j}$, belongs to the null space $\mathcal{N}_{ij}$ of the matrix $(A-\lambda_i I)^j$, as well as to the null space $\mathcal{N}_{i(j+1)}$ of the matrix $(A-\lambda_i I)^{j+1}$, but it does not belong to the null space $\mathcal{N}_{i(j-1)}$ of the matrix $(A - \lambda_i I)^{j-1}$. So we have :

$$\mathcal{N}_{i0} \subset \mathcal{N}_{i1} \subset \ldots \subset \mathcal{N}_{i\tau_{i\nu_i}} \triangleq \mathcal{N}_i$$

The dimension of $\mathcal{N}_{i0}$ is defined as 0. The dimension of $\mathcal{N}_{i1}$ is equal to ν_i, the geometric multiplicity of λ_i given by (3.11) and the dimension of $\mathcal{N}_i$ is equal to π_i i.e. the algebraic multiplicity of λ_i.

Definition 12 *We define as the generalized null-space $\mathcal{N}_i$ corresponding to the distinct eigenvalue λ_i, the null space of the matrix $(A - \lambda_i I)^{\tau_{i\nu_i}}$, where $\tau_{i\nu_i}$ is the index of annihilation of A at λ_i.* □

From the above we conclude that the set of the π_i generalized eigenvectors of all the ν_i chains associated with λ_i, belongs to the generalized null space $\mathcal{N}_i$. From the linear independence of the set of the π_i generalized eigenvectors, it can been shown, that it defines a basis for the generalized null-space $\mathcal{N}_i$, and π_i (the algebraic multiplicity of λ_i), denotes the dimension of the generalized null-space $\mathcal{N}_i$. So it is :

$$\operatorname{rank}(A - \lambda_i I)^{\tau_{i\nu_i}} = n - \pi_i$$

Also from equations (3.12) it can been shown that the generalized null space $\mathcal{N}_i$ is A-invariant. So we have :

Proposition 2 *To each one of the ν_i real eigenvectors $\underline{u}_{i_1}, \underline{u}_{i_2}, \ldots, \underline{u}_{i_{\nu_i}}$ associated with the distinct eigenvalue λ_i (of algebraic multiplicity π_i), corresponds one chain of generalized eigenvectors. The set of the π_i generalized eigenvectors of all the ν_i chains associated with λ_i forms a basis for the A-invariant generalized null space $\mathcal{N}_i$.* □

As each chain of τ_{ik} generalized eigenvectors forms a basis for the corresponding A-invariant and cyclic subspace $\mathcal{V}_{ik}$ it is,

Proposition 3 *The generalized A-invariant null-space $\mathcal{N}_i$ corresponding to the distinct eigenvalue λ_i may be written as a direct sum of ν_i, A-invariant and cyclic subspaces, each one of which is defined by a generalized eigenvectors chain .*

$$\mathcal{N}_i = \mathcal{V}_{i1} \oplus \mathcal{V}_{i2} \oplus ... \oplus \mathcal{V}_{ik} \oplus ... \oplus \mathcal{V}_{i\nu_i}$$

□

Proposition 4 *A generalized null-space $\mathcal{N}_i$ is cyclic relative to A, if and only if, is composed by only one subspace V_{ik}, or (what is the same) to the distinct eigenvalue λ_i corresponds only one real eigenvector and consequently only one chain of generalized eigenvectors.* □

Proposition 5 *The whole space $\mathbb{R}^n$ is cyclic relative to A, if and only if, all the generalized null spaces $\mathcal{N}_i$ corresponding to the distinct eigenvalues are cyclic.* □

Also the dual eigenvectors and dual generalized eigenvectors are defined as following.

To each real eigenvector $\underline{u}_{ik}$ $(k = 1, 2, ..., \nu_i)$ associated with λ_i $(i = 1, 2, ...f)$ corresponds a dual eigenvector $\underline{v}_{ik}$ such that, $\underline{v}_{ik}(A - \lambda_i I) = \underline{0}$, and $\underline{v}_{ik}^T \underline{u}_{ik} = 1$. The chain of τ_{ik} generalized dual eigenvectors $\underline{v}_{ik_1}, \underline{v}_{ik_2}, ..., \underline{v}_{ik_{\tau_{ik}}}$, is given by the equations :

$$\begin{array}{ccc}
\underline{v}_{i1}\,(A - \lambda_i I)^{\tau_{ik}} = \underline{0} & \Leftrightarrow & \underline{v}_{i1}\,(A - \lambda_i I) = \underline{v}_{i2} \\
..... & ... & ..... \\
\underline{v}_{ik_{\tau_{ik}-1}}\,(A - \lambda_i I)^2 = \underline{0} & \Leftrightarrow & \underline{v}_{ik_{\tau_{ik}-1}}\,(A - \lambda_i I) = \underline{v}_{ik_{\tau_{ik}}} \\
\underline{v}_{ik_{\tau_{ik}}} \triangleq \underline{v}_{ik} & \text{and} & \underline{v}_{ik_{\tau_{ik}}}\,(A - \lambda_i I) = \underline{0}
\end{array}$$

Let the $n \times n$ transforming matrix U of A defined by the chains of generalized eigenvectors associated with the distinct eigenvalues $\lambda_1, \lambda_2, ..., \lambda_f$ (eigenbasis) as following :

$$U = [U(\lambda_1), ..., U(\lambda_i), ..., U(\lambda_f)] \tag{3.14}$$

Where the columns of $U(\lambda_i) \in \mathbb{C}^{n \times \pi_i}$ define a basis for the A-invariant generalized null space

$\mathcal{N}_i$. Matrix $U(\lambda_i)$ is formed by the ν_i matrix blocks corresponding to the chains of the distinct eigenvalue λ_i :

$$U(\lambda_i) = [U_{i1}, ..., U_{ik}, ..., U_{i\nu_i}] \tag{3.15}$$

The columns of $U_{ik} \in \mathbb{C}^{n \times \tau_{ik}}$ are formed by the corresponding chain of the τ_{ik} generalized eigenvectors which define a basis for the A-invariant and cyclic subspace $\mathcal{V}_{ik}$:

$$U_{ik} = \left[\underline{u}_{i\tau_1}, ..., \underline{u}_{i\tau_{ik}}\right] \tag{3.16}$$

Matrix $V = U^{-1}$ is defined in the same as above line from the dual eigenvectors chains of A.

Then matrix A is similar to the Jordan matrix J :

$$A = UJU^{-1} = UJV \tag{3.17}$$

where,

$$J = \mathrm{diag}\{J(\lambda_1), J(\lambda_2), ..., J(\lambda_i), ..., J(\lambda_f)\} \tag{3.18}$$

and $J(\lambda_i)$ is the diagonal matrix block formed by all the ν_i Jordan blocks associated with the distinct eigenvalue λ_i :

$$J(\lambda_i) = \mathrm{diag}\{J_{i1}, ..., J_{ik}, ...J_{i\nu_i}\} \tag{3.19}$$

and where J_{ik} is the $\tau_{ik} \times \tau_{ik}$ Jordan diagonal block corresponding to the generalized eigenvectors chain of length τ_{ik}, associated with λ_i :

$$J_{ik} = \begin{bmatrix} \lambda_i & 1 & ... & 0 & 0 \\ 0 & \lambda_i & ... & 0 & 0 \\ . & . & ... & . & . \\ 0 & 0 & ... & \lambda_i & 1 \\ 0 & 0 & ... & 0 & \lambda_i \end{bmatrix}, \; J_{ik} \in \mathbb{C}^{\tau_{ik}} \tag{3.20}$$

From the above is concluded the following :

Proposition 6 *Under the partition (3.14) of the transforming matrix U, the whole space $\mathbb{R}^n$*

is decomposed into f, A-invariant generalized null-spaces,

$$\mathbb{R}^n = \mathcal{N}_1 \oplus \mathcal{N}_2 \oplus \ldots \oplus \mathcal{N}_i \oplus \ldots \oplus \mathcal{N}_f \tag{3.21}$$

The subspaces $\mathcal{N}_1, \mathcal{N}_2, \ldots, \mathcal{N}_f$ are uniquely defined as they coincide with the null-spaces of the corresponding matrices $(A - \lambda_1 I)^{\tau_{1\nu_1}}$, $(A - \lambda_2 I)^{\tau_{2\nu_2}}$, ..., $(A - \lambda_f I)^{\tau_{f\nu_f}}$. □

Proposition 7 *Under the partition (3.15) of the matrix block $U(\lambda_i)$, the A-invariant subspace $\mathcal{N}_i$ is decomposed into ν_i, A-invariant and cyclic subspaces, where ν_i is the geometric multiplicity of λ_i :*

$$\mathcal{N}_i = \mathcal{V}_{i1} \oplus \mathcal{V}_{i2} \oplus \ldots \oplus \mathcal{V}_{ik} \oplus \ldots \oplus \mathcal{V}_{i\nu_i}$$

and the whole space is decomposed into a total number η of cyclic and A-invariant subspaces.

$$\eta = \nu_1 + \nu_2 + \ldots + \nu_i + \ldots + \nu_f$$

□

The subspaces $\mathcal{V}_{i1}, \mathcal{V}_{i2}, \ldots, \mathcal{V}_{i\nu_i}$ are not uniquely defined, as we can have different ordering of the corresponding basis column vectors of $U_1^i, \ldots, U_k^i, \ldots, U_{\nu_i}^i$. On the contrary, the number ν_i of the subspaces, as well the dimensions of each one, are uniquely defined , as so are the number of the ν_i real eigenvectors associated with the distinct eigenvalue λ_i and the lengths of the corresponding generalized eigenvector chains. Otherwise, to each one of the elementary divisors $(s - \lambda_i)^{\tau_{ik}}$ of the continuous system matrix A, corresponds the Jordan block J_{ik}, as in (3.20). Also, each one of the elementary divisors, is the minimal polynomial of the corresponding Jordan block as well the minimal annihilating polynomial of the corresponding A-invariant and cyclic subspace $\mathcal{V}_{ik}$. So the elementary divisors of A can be arranged as following:

$$(s-\lambda_1)^{\tau_{11}}, (s-\lambda_1)^{\tau_{12}}, \ldots, (s-\lambda_1)^{\tau_{1k}}, \ldots, (s-\lambda_1)^{\tau_{1\nu_1}}$$

$$(s-\lambda_2)^{\tau_{21}}, (s-\lambda_2)^{\tau_{22}}, \ldots, (s-\lambda_2)^{\tau_{2k}}, \ldots, (s-\lambda_2)^{\tau_{2\nu_2}}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$(s-\lambda_i)^{\tau_{i1}}, (s-\lambda_i)^{\tau_{i2}}, \ldots, (s-\lambda_i)^{\tau_{ik}}, \ldots, (s-\lambda_i)^{\tau_{i\nu_i}}$$

$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

$$(s-\lambda_f)^{\tau_{f1}}, (s-\lambda_f)^{\tau_{f2}}, ..., (s-\lambda_f)^{\tau_{fk}}, ..., (s-\lambda_f)^{\tau_{f\nu_f}}$$

where for $i = 1, 2, ..., f$:

$$\tau_{i\nu_i} \geq ... \geq \tau_{ik} \geq ... \geq \tau_{i1} > 0, \ \tau_{i\nu_i} + ... + \tau_{ik} + ... + \tau_{i1} = \pi_i$$

Definition 13 *We define as the Segré Characteristic of A at λ_i, the set of the degrees of the elementary divisors of A at λ_i:*

$$\wp_{\lambda_i}(A) = \{\tau_{i\nu_i} \geq ... \geq \tau_{ik} \geq ... \geq \tau_{i1} > 0\} \tag{3.22}$$

□

Remark 6 *From the above we conclude that :*

1. *The number ν_i of the elements of $\wp_{\lambda_i}(A)$ is equal to the number of Jordan blocks associated with the distinct eigenvalue λ_i.*
2. *The sum π_i of the elements of $\wp_{\lambda_i}(A)$ is equal to the dimension of the generalized null space $\mathcal{N}_i$.*
3. *Each one of the elements of $\wp_{\lambda_i}(A)$, let the τ_{ik} is equal to the dimension of the corresponding Jordan block $J_{ik}(\lambda_i)$, as well to the dimension of the corresponding cyclic and A-invariant subspace $\mathcal{V}_{ik}$.*
4. *The first element $\tau_{i\nu_i}$ of $\wp_{\lambda_i}(A)$ is equal to the annihilation index of A at λ_i.* □

The minimal polynomial of A is determined as :

$$\Psi(A) \triangleq f_1(s) = (s-\lambda_1)^{\tau_{1\nu_1}}(s-\lambda_2)^{\tau_{2\nu_2}}...(s-\lambda_i)^{\tau_{i\nu_i}}..., (s-\lambda_f)^{\tau_{f\nu_f}}$$

Also we conclude that Proposition 5 can be stated as following:

Proposition 8 *The whole space* $\mathbb{R}^n$ *is cyclic relative to* A, *if and only if the* $\wp_{\lambda_i}(A)$ *includes only one element:*

$$\wp_{\lambda_i}(A) = \{\tau_{i\nu_i}\} \text{ and } \tau_{i\nu_i} = \pi_i, \text{ for } i = 1, 2, ..., f$$

or what is the same. to each one distinct eigenvalue λ_i *of* A *corresponds only one Jordan block.*□

Definition 14 *We define as index of cyclicity* ν *of a matrix* A, *the maximum of* ν_i *for* $i = 1, 2, ..., f$. *where* ν_i *is the geometric multiplicity of* A *at* λ_i *or (what is the same) the dimension of the null-space of* $(A - \lambda_i I)$ *i.e.*

$$\nu = \max\{\nu_i\} \text{ for } i = 1, 2, ..., f$$

□

From the above we conclude that :

Proposition 9 *The whole space* $\mathbb{R}^n$ *is cyclic relative to* A, *if and only if* $\nu = 1$. □

3.4 Definition of Finite and Infinite e.d. Structure of Right (Left) Regular Pencils [Kar. & Kal., 1]

3.4.1 Definitions

If $\mathcal{F}$ is a field or a ring $\mathcal{F}^{p\times m}$ denotes the set of $p \times m$ matrices with elements from $\mathcal{F}$. The right (left) null space of a map (matrix) W is denoted respectively by $\mathcal{N}_{\mathrm{r}}(W)$,$(\mathcal{N}_{\mathrm{l}}(W))$.

Let the set of matrix pencils be defined as :

$$\begin{aligned} \mathcal{L}_{p,m} &\triangleq \{W = (F, G) : F, G \in \mathbb{R}^{p\times m}\} \\ \mathcal{L}_{p,m}(s, w) &\triangleq \{W(s, w) = sF - wG, W = (F, G) \in \mathcal{L}_{p,m}\} \end{aligned}$$

where (s, w) is an ordered pair of indeterminate. The pair $W = (F, G)$ is called right regular, if $\mathcal{N}_{\mathrm{r}\mathbb{R}(s,w)}(sF - wG) = \{0\}$. The subset of $\mathcal{L}_{p,m}$ which is made up from all right regular pairs will be denoted by $\mathcal{L}^{\mathrm{rr}}_{p,m}$ and the corresponding set of pencils will be denoted by $\mathcal{L}^{\mathrm{rr}}_{p,m}(s, w)$. The set $\mathcal{L}^{\mathrm{lr}}_{p,m}$ of all left regular pairs is defined in a similar manner. It is clear that a necessary and sufficient condition for $W \in \mathcal{L}^{\mathrm{rr}}_{p,m}$ is that $W_{p,m}(s,w)$ has full rank over $\mathbb{R}(s,w)$ and $p \geq m$. A special case of right (left) regular pairs are those which $p = m$. This set is denoted by $\mathcal{L}^{\mathrm{r}}_{p,p}$ and will be referred to as the set of *entirely regular* or *simply regular* pairs. Clearly if $W = (F,G) \in \mathcal{L}^{\mathrm{r}}_{p,p}$ then $|sF - wG| \in \mathbb{R}[s,w] - \{0\}$. The set of matrix pencils that correspond to $\mathcal{L}^{\mathrm{r}}_{p,p}$ is denoted by $\mathcal{L}^{\mathrm{r}}_{p,p}(s,w)$.

Consider the following set of ordered pairs,

$$\mathcal{H} \triangleq \left\{h : h = (R,T), R \in \mathbb{R}^{p\times p}, T = \mathrm{diag}\{Q,Q\}, Q \in \mathbb{R}^{m\times m}, |R|, |Q| \neq \{0\}\right\}$$

and a composition rule $(*)$ defined on $\mathcal{H}$ as follows

$$\begin{aligned} \mathcal{H} \times \mathcal{H} \quad &\longrightarrow \quad \mathcal{H} : \forall h_1 = (R_1,T_1), h_2 = (R_2,T_2) \in \mathcal{H}, \text{then} \\ h_1 * h_2 \quad &\triangleq \quad (R_1,T_1) * (R_2,T_2) = (R_1R_2, T_2T_1) \end{aligned}$$

It may be readily verified that $(\mathcal{H},*)$ is a group with identity element (I_p, I_{2m}) and it is known as the *strict-equivalence group* (SEG). The action of $(\mathcal{H},*)$ on $W = (F,G) \in \mathcal{L}_{p,m}$ is defined by:

$$\begin{aligned} \mathcal{H} \times \mathcal{L}_{p,m} \quad &\longrightarrow \quad \mathcal{L}_{p,m} : \forall h = (R, \mathrm{diag}(Q,Q)) \in \mathcal{H}, \text{then} \\ h * W \quad &\triangleq \quad (R,T) \circ (F,G) = L' = (F',G') = (RFQ, RGQ) \end{aligned}$$

The above action defines an equivalence relation $\mathcal{E}_{\mathcal{H}}$ on $\mathcal{L}_{p,m}$ which is known as *strict-equivalence* (SE). Two pencils $W_1(s,w) = sF_1 - wG_1$, $W_2(s,w) = sF_2 - wG_2$ are said to be strict equivalent, $W_1(s,w)\mathcal{E}_{\mathcal{H}}W_2(s,w)$, if there exists $h \in \mathcal{H} : W_2 = h \circ W_1$. By $\mathcal{E}_{\mathcal{H}}(F,G)$ it is denoted the SE class, or orbit, of $W = (F,G)$ or equivalently of $W(s,w) = sF - wG$.

The above definitions, clearly apply to the $\mathcal{L}^{\mathrm{rr}}_{p,m}, \mathcal{L}^{\mathrm{lr}}_{p,m}, \mathcal{L}^{\mathrm{r}}_{p,m}$ cases. In the following, we concentrate on $\mathcal{L}^{\mathrm{rr}}_{p,m}$ while the treatment for $\mathcal{L}^{\mathrm{lr}}_{p,m}$ is dual. For all $L = (F,G) \in \mathcal{L}_{p,m}$, the SE class, $\mathcal{E}_{\mathcal{H}}(F,G)$, is characterized by a complete set of invariants, known as strict equivalence invariants (SEI).

Let $W = (F,G), G, F \in \mathbb{R}^{p\times m}$, $\rho = \mathrm{rank}_{\mathbb{R}(s,w)}(sF - wG) \leq \min(p,m)$ and let $\mathcal{D}(G,F)$ be the set of homogeneous elementary divisors (e.d.) of $sF - wG$. These are of the following three types :

$$(s-aw)^{r_i},\ s^{\sigma_i},\ w^{\mu_i}$$

Where $a \in \mathbb{C}-\{0\}$. The subsets of $\mathcal{D}(G,F)$ which corresponds to the same point of $\mathbb{C}\cup\{\infty\}$ will be denoted by

$$\begin{aligned}
\mathcal{D}_{1,a} &\triangleq \left\{ \begin{array}{c} (s-aw)^{r_i},\ \text{or}\ (\tilde{a}s-w)^{r_i},\ a\in\mathbb{C}\cup\{\infty\},\ \tilde{a}=a^{-1} \\ i\in\boldsymbol{\nu}_a,\ 0<r_1\leq ...\leq r_{\nu_a} \end{array} \right\} \\
\mathcal{D}_{1,0} &\triangleq \{s^{\sigma_i},\ i\in\boldsymbol{\nu}_0,\ 0<\sigma_1\leq ...\leq\sigma_{\nu_0}\} \\
\mathcal{D}_{0,1} &\triangleq \{w^{\mu_i},\ i\in\boldsymbol{\nu}_\infty,\ 0<\mu_1\leq ...\leq\mu_{\nu_\infty}\}
\end{aligned} \tag{3.23}$$

For the single variable pencils $sF-G$, $F-wG$ derived from $sF-wG$, the above sets of e.d. may be interpreted as [Kar. & Kal., 1]: for $sF-G$, $\mathcal{D}_{1,a}$, $\mathcal{D}_{1,0}$, $\mathcal{D}_{0,1}$, represent the sets of a-e.d., $a\neq 0$. 0-e.d., ∞-e.d. respectively and thus they will be denoted in short by $\mathcal{D}_a$. $\mathcal{D}_0$. $\mathcal{D}_\infty$. For the "*dual pencil*", $F-wG$, the sets $\mathcal{D}_{1,a}$. $\mathcal{D}_{1,0}$. $\mathcal{D}_{0,1}$, represent the sets of $\tilde{a}$–e.d., $\tilde{a}=a^{-1}$, ∞–e.d., 0–e.d., respectively and thus will be denoted by $\widetilde{\mathcal{D}}_a$. $\widetilde{\mathcal{D}}_0$. $\widetilde{\mathcal{D}}_\infty$ correspondingly. In the following, the case $sF-G$ will be considered and thus the notation $\mathcal{D}_a$. $\mathcal{D}_0$. $\mathcal{D}_\infty$ will be adopted, the results concerning $F-wG$ are dual.

Remark 7 *The set $\mathcal{D}(F,G)$ is self conjugate and thus if $\mathcal{D}_a\in\mathcal{D}(F,G)$, $a\in\mathbb{C}-\mathbb{R}$. then $\mathcal{D}_{a*}\in\mathcal{D}(F.G)$ (where $a*$ is the complex conjugate of a).* □

Definition 15 *The set*

$$\Phi(F.G)\triangleq\{a_i : a_i\in\mathbb{C}\cup\{\infty\},\ a_i\neq a_j,\ i\in\boldsymbol{\nu} : \text{rank}(a_iF-G)<\rho\}$$

will be called the root range of $(F.G)$. □

Also following the definitions given in the previous section for the $sI-A$ case (3.22), the sets of integers,

$$\begin{aligned}
\wp_a(F,G) &\triangleq \{r_{\nu_a}\geq ...\geq r_1>0\},\quad \wp_0(F,G)\triangleq\{\sigma_{\nu_0}\geq ...\geq\sigma_1>0\} \\
\wp_\infty(F,G) &\triangleq \{\mu_{\nu\infty}\geq ...\geq\mu_1>0\}
\end{aligned} \tag{3.24}$$

characterizing the degrees of the e.d. in the sets $\mathcal{D}_a, \mathcal{D}_0, \mathcal{D}_\infty$, will be defined as the $a-, 0-, \infty-$Segré characteristic of the pair (F, G) respectively [Kar. & Kal., 1].

A pair $W = (F, G) \in \mathcal{L}_{p,m}$ such that $W \notin \mathcal{L}^{\mathrm{r}}_{p,m}$ will be called singular and the set of all singular pairs will be denoted by $\mathcal{L}^{\mathrm{s}}_{p,m}$; clearly $\mathcal{L}^{\mathrm{rr}}_{p,m}, \mathcal{L}^{\mathrm{lr}}_{p,m} \in \mathcal{L}^{\mathrm{s}}_{p,m}$.

3.4.2 The a-Toeplitz Matrices

The following results indicate the procedure for the computation of the Segré characteristics $\wp_a, \wp_0, \wp_\infty$ without resorting to their algebraic definitions.

Theorem 10 *Let $W = (G, F) \in \mathcal{L}^{\mathrm{rr}}_{p,m}(p \geq m)$. The pencil $W_{p,m}(s) = sF - G$ has an e.d. $(s-a)^{r_i}$, $a \in \mathbb{C}$, if and only if there exists a maximal chain of linearly independent vectors $\{\underline{x}_1, \underline{x}_2, ..., \underline{x}_{r_i}\} \in \mathbb{C}^p$ such that*

$$\begin{bmatrix} G-aF & 0 & \cdots & 0 & 0 \\ -F & G-aF & \cdots & 0 & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \cdots & -F & G-aF \end{bmatrix} \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \underline{x}_j \end{bmatrix} = \underline{0}, \ \text{for } j = 1, 2, ..., r_i \tag{3.25}$$

□

Theorem 11 *Let $W = \{G, F\} \in \mathcal{L}^{\mathrm{rr}}_{p,m}$ $(p \geq m)$. The pencil $W_{p,m}(s) = sF - G$ has an i.e.d. w^{μ_i}, if and only if there exist a maximal chain of linearly independent vectors $\{\underline{x}_1, \underline{x}_2, ..., \underline{x}_{\mu_i}\} \in R^p$ such that*

$$\begin{bmatrix} F & 0 & \cdots & 0 & 0 \\ -G & F & \cdots & 0 & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \cdots & -G & F \end{bmatrix} \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \underline{x}_j \end{bmatrix} = \underline{0}, \ \text{for } j = 1, 2, ..., \mu_i \tag{3.26}$$

$\square$

For $\forall a \in \mathbb{C}$ is defined the sequence of matrices :

$$
\begin{aligned}
T_a^1 &\triangleq G - aF \\
T_a^2 &\triangleq \begin{bmatrix} G - aF & 0 \\ F & G - aF \end{bmatrix} \\
&\cdots\cdots \\
T_a^i &\triangleq \begin{bmatrix} G-aF & 0 & \ldots & 0 & 0 \\ F & G-aF & \ldots & 0 & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \ldots & F & G-aF \end{bmatrix} \in \mathbb{C}^{ip \times im}, \text{ for all } i = 1, 2, \ldots
\end{aligned} \tag{3.27}
$$

For $a = \infty$, we may also define the sequence of matrices :

$$
\begin{aligned}
T_\infty^1 &\triangleq F \\
T_\infty^2 &\triangleq \begin{bmatrix} F & 0 \\ -G & F \end{bmatrix} \\
T_\infty^i(F, -G) &\triangleq \begin{bmatrix} F & 0 & \ldots & 0 & 0 \\ -G & F & \ldots & 0 & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \ldots & -G & F \end{bmatrix} \in R^{ip \times im}, \text{ for all } i = 1, 2, \ldots
\end{aligned}
$$

Matrices of the type T_a^i, T_∞^i will be referred to as the *i-th* order a-,∞-Toeplitz matrices of the pair $(G, F,)$. Let us denote by :

$$
N_a^k \triangleq \mathcal{N}_r\left\{T_a^k\right\}, \quad \tilde{N}_a^k \triangleq \mathcal{N}_l\{T_a^k\}, \ \forall a \in \mathbb{C}, \ k = 1, 2, \ldots \tag{3.28}
$$

$$
N_\infty^k \triangleq \mathcal{N}_r\{T_\infty^k\}, \quad \tilde{N}_\infty^k \triangleq \mathcal{N}_l\{T_\infty^k\}, \ k = 1, 2, \ldots \tag{3.29}
$$

For all the pairs $W = (G, F)$ and $b \in \mathbb{C} \cup \{\infty\}$ we define the sequences,

$$J_b^{\mathrm{r}}(G,F) \triangleq \left\{\eta_k^b : \eta_0^b = 0,\ \eta_k^b = \dim N_b^k;\ k \geq 1\right\} \tag{3.30}$$

$$J_b^{\mathrm{l}}(G,F) \triangleq \left\{\vartheta_k^b : \vartheta_0^b = 0,\ \vartheta_k^b = \dim \tilde{N}_b^k;\ k \geq 1\right\} \tag{3.31}$$

$J_b^{\mathrm{r}}(G.F)$, $J_b^{\mathrm{l}}(G.F)$ will be referred to as the right b-(G,F), left b-(G,F)-sequence of the pair (G,F). A sequence $J_b^{\mathrm{r}}(G,F)$, $J_b^{\mathrm{l}}(G,F)$ will be called neutral, if its elements are zero for all k : $k = 1, 2, ...$

Theorem 12 *The differences $\eta_{k+1} - \eta_k$ provide the following information about the e.d. structure of $sF - G$ at $s = b$:*

1. *η_1 is the number of e.d. at $s = b$.*

2. *The smallest index k for which $\eta_{k+1} - \eta_k = 0$ gives the maximal of the degrees of e.d. at $s = b$.*

3. *The difference $\eta_{k+1} - \eta_k$ defines the number of e.d. with degrees higher than k.* □

Definition 16 *The set of the first non-zero successive differences in $J_b^r(G,F)$ is defined as the Weyr characteristic of (G,F) at b and it is denoted by $\mathcal{W}_b$. Clearly is given by :*

$$\mathcal{W}_b \triangleq \{\Gamma_1 = \eta_1 - \eta_0,\ \Gamma_2 = \eta_2 - \eta_1,\ ...,\ \Gamma_k = \eta_k - \eta_{k-1}\}$$

□

Proposition 10 *Let $W = \{G, F\} \in \mathcal{L}_{p,m}^{\mathrm{r}}$. Then,*

1. *$\Gamma_j \geq \Gamma_{j+1}$ for all $j = 1, ..., k$ and $\Gamma_{j+1} = 0$ for all $j = k, k+1, ...$*

2. *The strict inequality $\Gamma_j > \Gamma_{j+1}$ holds true if and only if $j = q_i$, where q_i is the degree of a e.d. The multiplicity of the e.d. is then defined by $\Gamma_{q_i} - \Gamma_{q_i+1}$.* □

3.5 Structure at Infinity of Matrix Pencils [Var. & Kar., 1],[Eli. & Kar., 1]

3.5.1 Regular Pencils

Let $W(s) = sF - G \in \mathcal{L}^r_{p,p}(s)$, $G \in \mathbb{R}^{p \times p}$, $F \in \mathbb{R}^{p \times p}$, and let $\deg |W(s)| \triangleq n \geq 0$. It is known that $W(s)$ is strictly equivalent to its Weierstrass normal form $W_W(s)$, uniquely characterized by the set of homogeneous elementary divisors 3.23 and consequently by the $a-, 0-, \infty-$Segré characteristic of the pair (F, G) ?? :

$$W_W(s) \triangleq sF_W - G_W = s \begin{bmatrix} I_0 & 0 & 0 \\ 0 & I_a & 0 \\ 0 & 0 & H_\infty \end{bmatrix} - \begin{bmatrix} H_0 & 0 & 0 \\ 0 & J_a & 0 \\ 0 & 0 & I_\infty \end{bmatrix} \tag{3.32}$$

where :

1. $H_\infty \in \mathbb{R}^{(p-n)\times(p-n)}$, defined by $\mathcal{D}_\infty$ and nilpotent :

$$H_\infty = \text{block diag}\,\{H_{\mu_1}, ..., H_{\mu_{\nu_\infty}}\} \tag{3.33}$$

$$H_{\mu_i} = \begin{bmatrix} 0 & 1 & 0 & . & . & . & 0 \\ 0 & 0 & 1 & . & . & . & 0 \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & 1 \\ 0 & 0 & 0 & . & . & . & 0 \end{bmatrix} \in \mathbb{R}^{\mu_i \times \mu_i}, \quad i \in \boldsymbol{\nu}_\infty \tag{3.34}$$

$$\nu_\infty = \text{rank defect of } F = p - \text{rank} F \geq 0 \tag{3.35}$$

2. H_0 defined by $\mathcal{D}_0$ and nilpotent :

$$H_0 = \text{block diag}\,\{H_{\sigma_1}, ..., H_{\sigma_{\nu_0}}\} \tag{3.36}$$

$$H_{\sigma_i} = \begin{bmatrix} 0 & 1 & 0 & . & . & . & 0 \\ 0 & 0 & 1 & . & . & . & 0 \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & 1 \\ 0 & 0 & 0 & . & . & . & 0 \end{bmatrix} \in \mathbb{R}^{\sigma_i \times \sigma_i}, \quad i \in \boldsymbol{\nu}_0 \tag{3.37}$$

3. J_a defined by $\mathcal{D}_a$, in Jordan form :

$$J_a = \text{block diag}\,\{J(a_1), ..., J(a_m)\} \tag{3.38}$$

$$\text{where } a_i \in \Phi(F,G) - \{0\},\ i = 1, ..., m \tag{3.39}$$

and if the a_i–Segré characteristic of the pair (F,G) is :

$$\wp_{a_i}(F,G) \triangleq \{r_{i\nu_a} \geq ... \geq r_{i1} > 0\}\,,\ i \in \mathbf{m}$$

then also it is :

$$J(a_i) = \text{block diag}\,\{J_{i1}, ..., J_{i\nu_{a_i}}\},\ i \in \mathbf{m}$$

$$J_{ij} = \begin{bmatrix} a_i & 1 & 0 & . & . & . & 0 \\ 0 & a_i & 1 & . & . & . & 0 \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & 1 \\ 0 & 0 & 0 & . & . & . & a_i \end{bmatrix} \in \mathbb{R}^{r_{ij} \times r_{ij}}\ r_{ij} > 0\ j \in \boldsymbol{\nu}_{a_i},\ i \in \mathbf{m} \tag{3.40}$$

The infinite elementary divisors (i.e.d.) of $W(s)$ are given by,

$$w^{\mu_1}, w^{\mu_2}, ..., w^{\mu_{\nu_\infty}} \tag{3.41}$$

where μ_i, $i \in \nu_\infty$ are the sizes of the blocks H_{μ_i}, $i \in \nu_\infty$.

Let now that,

$$S^{\infty}_{W(s)} = diag\{s^{q^1_\infty}, s^{q^2_\infty}, ..., s^{q^j_\infty}, \frac{1}{s^{q^{j+1}_\infty}}, \frac{1}{s^{q^{j+2}_\infty}}, ..., \frac{1}{s^{q^p_\infty}}\} \tag{3.42}$$

is the McMillan form of $W(s)$ at $s = \infty$ where $q_\infty^1 \geq q_\infty^2 \geq ... \geq q_\infty^j > 0$ are the orders of its infinite poles and $\hat{q}_\infty^r \geq \hat{q}_\infty^{r-1} \geq ... \geq \hat{q}_\infty^{j+1} \geq 0$ are the orders of its infinite zeros. Then we have the following.

Proposition 11 *The McMillan form at $s = \infty$, of a regular pencil $W(s) = sF - G \in \mathcal{L}_{p,p}^r(s)$, is given by 3.42 where.*

1. *The number j of its poles at $s = \infty$ is given by $j = rankF$, and their orders satisfy $q_\infty^i = 1,\ i \in \mathbf{j}$.*

2. *The degrees μ_i of its i.e.d.'s w^{μ_i} $(\mu_i > 0)$, $i \in \boldsymbol{\nu}_\infty$, $\nu_\infty = p - rankF$, satisfy $\mu_i = \hat{q}_\infty^i + 1$, $i \in \boldsymbol{\nu}_\infty$, where $\hat{q}_\infty^i$ are the orders of the zeros at $s = \infty$ of $W(s)$.* □

3.5.2 Singular Pencils

Let $W(s) = sF - G \in \mathcal{L}_{p,m}^s(s)$ and let $W_k(s) = sF_k - G_k$ be its Kronecker form. Then:

$$W_k(s) = \text{block diag}\{0_{h,g}, L\varepsilon(s), L\eta(s), s\tilde{F} - \tilde{G}\} \tag{3.43}$$

where $s\tilde{F} - \tilde{G}$ is a regular pencil in its Weierstrass form,

$$L\varepsilon(s) = \text{block diag}\{L_{\varepsilon_{g+1}}(s), ..., L_{\varepsilon_l}(s)\} \tag{3.44}$$

$$L_{\varepsilon_i}(s) = \begin{bmatrix} s & 1 & . & . & 0 & 0 \\ o & s & . & . & 0 & 0 \\ . & . & . & . & . & . \\ 0 & 0 & . & . & 1 & 0 \\ 0 & 0 & . & . & s & 1 \end{bmatrix} \in \mathbb{R}^{\varepsilon_i \times (\varepsilon_i + 1)}[s] \tag{3.45}$$

$$L\eta(s) = \text{block diag}\{L_{\eta_{h+1}}(s), ..., L_{\eta_l}(s)\} \tag{3.46}$$

$$L_{\eta_i}(s) = \begin{bmatrix} s & 0 & . & 0 & 0 \\ 1 & s & . & 0 & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & . & 1 & s \\ 0 & 0 & . & 0 & 1 \end{bmatrix} \in \mathbb{R}^{(\eta_i+1)\times\eta_i}[s] \tag{3.47}$$

and $\varepsilon_1 = ... = \varepsilon_g = 0 < \varepsilon_{g+1} \leq ... \leq \varepsilon_l$ are the column minimal indexes (c.m.i.) and $\eta_1 = ... = \eta_h = 0 < \eta_{h+1} \leq ... \leq \eta_t$ are the row minimal indices (r.m.i.). Then we have:

Proposition 12 *For a singular pencil* $W(s) = sF - G \in \mathcal{L}^s_{p,m}(s)$*:*

1. *the number* j *of its poles at* $s = \infty$ *is given by* $j = n\varepsilon + n\eta + \eta_w = \text{rank}F$, *where* $n\varepsilon = \sum \varepsilon_i$, $n\eta = \sum \eta_i$, $\eta_w = \text{rank}\tilde{F}$ *and their orders satisfy* $q^i_\infty = 1, i \in \mathbf{j}$,

   - *the number* k *of its i.e.d.'s* w^{μ_i} $(\mu_i > 0), i \in \mathbf{k}$, *is given by* $k = rank_{\mathbb{R}(s)}(sF - G) - rank_{\mathbb{R}}F$ *and*

   - *we have* $\mu_i = \hat{q}^i_\infty + 1$, $i \in \mathbf{k}$, *where* $\hat{q}^i_\infty$ *are the orders of its zeros at* $s = \infty$, *and*

2. *The McMillan form at* $s = \infty$, *of* $W(s)$ *is :*

$$\begin{aligned} S^\infty_{W(s)} &= block\ diag\{sI_{n\varepsilon+n\eta}, S_{\hat{G}+s\hat{F}}, 0_{t,l}\} = \\ &= block\ diag\{sI_j, \frac{1}{s^{\hat{q}^{j+1}_\infty}}, ..., \frac{1}{s^{\hat{q}^{j+k}_\infty}}, 0_{t,l}\} \end{aligned} \tag{3.48}$$

□

3.6 Exterior Algebra-The Grassman Products [Mar. & Min., 1]

In this section we first introduce some useful notations and definitions on the sequences of integer numbers and on the submatrices of a given matrix.

If $\mathcal{F}$ is a field or a ring, $\mathcal{F}^{p\times m}$ denotes the set of $p \times m$ matrices with elements from $\mathcal{F}$. For $1 \leq k \leq n$, let $Q_{k,n}$ denote the totality of strictly increasing sequences of k integers chosen from $1, \ldots, n$. In general $Q_{k,n}$ has $\binom{n}{k}$ sequences in it. If $a, b \in Q_{k,n}$, then a precedes b (written $a < b$), if there exists an integer $t, (1 \leq t \leq k)$ for which $a_1 = b_1, \ldots, a_{t-1} = b_{t-1}, a_t < b_t$.

Suppose $A = (a_{ij}) \in \mathcal{F}^{m \times n}$; k and r are positive integers satisfying $1 \leq k \leq m$, $1 \leq r \leq n$ and $a = (i_1, i_2, \ldots, i_k) \in Q_{k,m}$, $b = (j_1, j_2, \ldots, j_r) \in Q_{r,n}$. Then the matrix $B \in \mathcal{F}^{k \times r}$ is called the submatrix of A lying in rows a and columns b and may be designated by

$$B = A^{a]}_{b]}$$

We use the notation $A^{a)}_{b]}$ to designate the submatrix of A whose rows are precisely those complementary to a and whose columns are designated by b. Similarly $A^{a]}_{b)}$ includes rows a and excludes columns b, whereas $A^{a)}_{b)}$ excludes rows a and columns b.

Theorem 13 (Binet-Cauchy) *Suppose $A \in \mathcal{F}^{n \times p}$, $B \in \mathcal{F}^{p \times m}$ and $C = AB \in \mathcal{F}^{n \times m}$. If $1 \leq r \leq \min(n, m, p)$, $a \in Q_{r,n}$, $b \in Q_{r,m}$, then*

$$\det\{C^{a]}_{b]}\} = C^{a}_{b} = \sum_{\omega \in Q_{r,p}} \det\{A^{a]}_{\omega]}\} \det\{B^{\omega]}_{b]}\} \tag{3.49}$$

□

3.6.1 Compound Matrices

If $A \in \mathcal{F}^{m \times n}$ and $1 \leq r \leq \min(m, n)$, then the *r-th compound matrix* of A is the $\binom{m}{r} \times \binom{n}{r}$ matrix whose entries are $\det A^{\alpha]}_{\beta]}$, $\alpha \in Q_{r,m}$, $\beta \in Q_{r,n}$ arranged lexicographically in α and β and this matrix will be designated by $\mathfrak{C}_r(A)$.

If $A \in \mathcal{F}^{n \times p}$, $B \in \mathcal{F}^{p \times m}$, $1 \leq r \leq \min(n, m, p)$, another way of writing down the Binet-Cauchy theorem is:

$$\mathfrak{C}_r(AB) = \mathfrak{C}_r(A)\mathfrak{C}_r(B)$$

If $A \in \mathcal{F}^{r \times n}$ and the r rows of A are denoted by $\underline{v}_1^T, \ldots, \underline{v}_r^T$ in succession, $(1 \leq r \leq n)$ then $\mathfrak{C}_r(A)$ is an $\binom{n}{r}$-tuple and is sometimes called the Grassman product or skew symmetric product of the vectors $\underline{v}_1^T, \ldots, \underline{v}_r^T$. The Grassman product of the columns of an $A \in \mathcal{F}^{n \times r}$ matrix may be defined in a similar manner; the product in this case however will be a column vector in contrast to the row vector product obtained from the previous case. The usual notation for this $\binom{n}{r}$-tuple of subdeterminants of A is $\underline{v}_1^\top, ..., \underline{v}_r^\top$ or $\underline{v}_1 \wedge ... \wedge \underline{v}_r$.

Example 1 *Let A and B be two matrices given by,*

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & a_{2,5} \end{bmatrix} = \begin{bmatrix} \underline{a}_1^\top \\ \underline{a}_2^\top \end{bmatrix}$$

$$B = \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \\ b_{3,1} & b_{3,2} \\ b_{4,1} & b_{4,2} \\ b_{5,1} & b_{5,2} \end{bmatrix} = \begin{bmatrix} \underline{b}_1 & \underline{b}_2 \end{bmatrix}$$

Then the previously defined Grassman vectors can be denoted by the bold letters,

$$\mathbf{a} \triangleq \underline{a}_1^\top \wedge \underline{a}_2^\top = \begin{bmatrix} A_{1,2}^{1,2} & A_{1,3}^{1,2} & A_{1,4}^{1,2} & A_{2,3}^{1,2} & A_{2,4}^{1,2} & A_{3,4}^{1,2} \end{bmatrix}$$

$$\mathbf{b} \triangleq \underline{b}_1 \wedge \underline{b}_2 = \begin{bmatrix} B_{1,2}^{1,2} & B_{1,2}^{1,3} & B_{1,2}^{1,4} & B_{1,2}^{2,3} & B_{1,2}^{2,4} & B_{1,2}^{3,4} \end{bmatrix}^\top$$

□

3.6.2 Multi-orthogonality [Kar., 3]

Lemma 2 (Kar., 3) *Given the matrices $C \in \mathcal{F}^{m\times n}$ and $B \in \mathcal{F}^{n\times m}$ we form the matrices $\Gamma = CU$, $\mathcal{B} = VB$, where $UV = I_m$, as well the matrices $C\varphi(\Gamma\varphi)$, $B\varphi(\mathcal{B}\varphi)$, obtained by interchanging the φ set of columns of $C(\Gamma)$ and the φ set of rows of $B(\mathcal{B})$ in the same way. Then the rank tests on CB, $\Gamma\mathcal{B}$, $C\varphi B\varphi$, $\Gamma\varphi\mathcal{B}\varphi$ are equivalent; this may be also expressed by the following conditions :*

$$\mathfrak{C}_r(CB) = \mathfrak{C}_r(\Gamma\mathcal{B}) = \mathfrak{C}_r(C\varphi B\varphi) = \mathfrak{C}_r(\Gamma\varphi\mathcal{B}\varphi) \tag{3.50}$$

where $1 \leq r \leq m$. □

Let the non singular square matrices Q and R such that,

$$C^* = QC, \; B^* = BR$$

We consider the submatrices $C_{a)}(C^*_{a)})$, $B^{a)}(B^{*a)})$ and their corresponding Grassman products, where a is the set of omitted columns and rows of C, B respectively. Then it is readily seen that $C^*_{a)} = QC_{a)}$, $B^{*a)} = B^{a)}R$ and the determinant of the product of these two matrices is now expressed as,

$$\left|C^*_{a)}B^{*a)}\right| = |Q| \left|C_{a)}B^{a)}\right| |R|$$

Let the bold letters $\mathbf{c}, \mathbf{b}$ denote the Grassman product of rows of C, columns of B and using the notation previously introduced, the above expression may be written as,

$$\left\langle \mathbf{c}^*_{a)}, \mathbf{b}^{*a)} \right\rangle = |Q| \left\langle \mathbf{c}_{a)}, \mathbf{b}^{a)} \right\rangle |R|$$

where by $\langle . \rangle$ is designated the inner product operator. The above condition implies that the orthogonality properties of any pair of Grassman vectors $(\mathbf{c}_{a)}, \mathbf{b}^{a)})$ is invariant under any set of non singular coordinate transformations or equivalently, the rank of CB does not change under the Q, R transformations .

Theorem 14 *Let the matrices $C \in \mathcal{F}^{m \times n}$ and $B \in \mathcal{F}^{n \times m}$ and let their product CB be rank deficient of deficiency d. Let a be a set of indices such that $a \in Q_{\mu,n}$ where $\mu \leq d$. Then, in connection with the Grassman products of different order $\mathbf{c}_{a)}, \mathbf{b}^{a)}$ of the C, B matrices respectively, we have the following conditions:*

$$\left.\begin{array}{rl}
\langle \mathbf{c}, \mathbf{b} \rangle = 0 & \\
\left\langle \mathbf{c}_{a_1)}, \mathbf{b}^{a_1)} \right\rangle = 0, & a_1 \in Q_{1,n} \\
\cdots & \\
\left\langle \mathbf{c}_{a_k)}, \mathbf{b}^{a_k)} \right\rangle = 0, & 1 \leq k < d, \; a_k \in Q_{k,n} \\
\cdots & \\
\left\langle \mathbf{c}_{a_{d-1})}, \mathbf{b}^{a_{d-1})} \right\rangle = 0, & a_{d-1} \in Q_{d-1,n} \\
\left\langle \mathbf{c}_{a_d)}, \mathbf{b}^{a_d)} \right\rangle \neq 0, & \text{for at least some } a_d \in Q_{d,n}
\end{array}\right\} \tag{3.51}$$

□

From the above theorem is clear that if the rank defect of CB is $d = 1$, then the vectors $\mathbf{c}$ and $\mathbf{b}$ become orthogonal, but such an orthogonality does not hold for every other pair of Grassman subvectors $\mathbf{c}_{a)}, \mathbf{b}^{a)}$ $a \in Q_{k,n}$. As the rank defect of CB increases from $d = 1$ to $d = 2$, then not only are $\mathbf{c}$, $\mathbf{b}$ orthogonal, but any pair $\mathbf{c}_{a_1)}, \mathbf{b}^{a_1)}$, $a_1 \in Q_{1,n}$ is orthogonal too. The orthogonality property then extends to the class of $\mathbf{c}_{a_1)}, \mathbf{b}^{a_1)}$ subvectors, but not for to any higher class $\mathbf{c}_{a_k)}, \mathbf{b}^{a_k)}$ with $a_k \in Q_{k,n}$ and $k \geq 2$. Generally speaking for every additional degree of rank defect of CB, the orthogonality property of the Grassman vectors $\mathbf{c}$, $\mathbf{b}$ extends to a new class of Grassman subvectors of $\mathbf{c}$ and $\mathbf{b}$. The number of classes of Grassman subvectors to which the orthogonality property extends is called the multi-orthogonality degree of the vectors $\mathbf{c}$, $\mathbf{b}$ and it is equal to the rank deficiency of the product CB.

3.7 Continuous Time Linear Systems and Structural Properties [Che., 1],[Rug., 1]

We assume that a plant is described by a continuous time linear state space model $S(A, B, C, D)$:

$$\dot{\underline{x}}(t) = A\underline{x}(t) + B\underline{u}(t) \tag{3.52}$$

$$\underline{x}(t) = C\underline{y}(t) + D\underline{u}(t) \tag{3.53}$$

Where $A \in R^{nxn}, B \in R^{nxl}, C \in R^{mxn}, D \in^{lxm}$ and $u(t)$ is the $l \times 1$ input vector, $\underline{y}(t)$ is the $m \times 1$ output vector and $\underline{x}(t)$ is the $n \times 1$ state variable vector.

The solution of the equation 3.52 takes the following form in the time interval from t_1 to t_2 :

$$\underline{x}(t_2) = e^{A(t_2-t_1)}\underline{x}(t_1) + \int_{t_1}^{t_2} e^{A(t_2-\tau)}B\underline{u}(\tau)d\tau \tag{3.54}$$

3.7.1 Controllability

Definition 17 *A system $S(A, B, C, D)$ is said to be controllable at time t_0 if for any initial state $\underline{x}_0$ in the space $\mathbb{R}^n$ and any state $\underline{x}_1$ there exists a finite time $t_1 > t_0$ and an input $\underline{u}[t_0, t_1]$ that will transfer the state $\underline{x}_0$ to $\underline{x}_1$ in time $t_1 - t_0$. Otherwise the system is said to be* *uncontrollable.* □

It can be shown that the system $S(A, B, C, D)$ described by the equations (3.52) and (3.53) is controllable if one of the following equivalent condition are valid [Che., 1]:

- The rows of the matrix $e^{At}B$ are linearly independent over the field of complex numbers.
- The rows of $(sI - A)^{-1}B$ are linearly independent over the field of complex numbers.
- The pencil $[sI - A, B]$ has no e.d.
- The rows of controllability matrix $Q \in \mathbb{R}^{n \times nl}$,

$$Q = [B, AB, A^2B, ..., A^{n-1}B] \tag{3.55}$$

are linearly independent.

If $\underline{b}_1, \underline{b}_2, ..., \underline{b}_l$ are the columns of B,

$$Q = [\underline{b}_1, ..., \underline{b}_l, A\underline{b}_1, ..., A\underline{b}_l, ..., A^{n-1}\underline{b}_1, ..., A^{n-1}\underline{b}_l]$$

The linear A-invariant vector subspace of $\mathbb{R}^n$ consisting of all the states $\underline{x}(t)$ that can be reached from any initial state $\underline{x}_0$ within a finite time,

$$\mathcal{R} = \text{span}[B, AB, A^2B, ..., A^{n-1}B]$$

is defined as the controllable space of the system. The dimension r of the controllable space is

$$r = \dim\mathcal{R} = \text{rank}Q$$

3.7.2 Observability

For the property of observability of a linear system $S(A, B, C, D)$ we have the following dual to controllability definition :

Definition 18 *A system $S(A, B, C, D)$ is said to be observable at time t_0 if for any initial state $\underline{x}_0$ in the space $\mathbb{R}^n$ there exists a finite time $t_1 > t_0$ such that knowledge of the output $\underline{y}(t)$ over* *the interval* $[0, t_1]$ *suffices to determine (to observe) the initial state* $\underline{x}_0$. *Otherwise the system is said to be unobservable.* □

It can be shown [3],[1] that the system $S(A, B, C, D)$ described by the equations (3.52) and (3.53) is observable if one of the following equivalent condition is valid:

- the columns of the matrix $Ce^{A(t-t_0)}$ are linearly independent over the field of complex numbers.
- the columns of $C(sI - A)^{-1}$ are linearly independent over the field of complex numbers.
- The pencil $\begin{bmatrix} sI - A \\ C \end{bmatrix}$ has no e.d.
- the columns of observability matrix $M \in \mathbb{R}^{mn \times n}$, are linearly independent.

$$M = \begin{bmatrix} C \\ CA \\ \cdots \\ CA^{n-1} \end{bmatrix} \tag{3.56}$$

If $c_1, c_2, \ldots, c_m$ are the rows of C,

$$M = \begin{bmatrix} c_1 \\ \cdots \\ c_m \\ c_1 A \\ \cdots \\ c_m A \\ \cdots \\ c_1 A^{n-1} \\ \cdots \\ c_m A^{n-1} \end{bmatrix}$$

The linear A-invariant vector subspace of $\mathbb{R}^n$ consisting of all the states $\underline{x}(t)$ that cannot be observed,

$$\mathcal{P} = \mathcal{N}_{\text{right}} M$$

is defined as the unobservable space of the system. The dimension p of the unobservable space is,

$$p = \dim\mathcal{P} = n - \text{rank}M$$

3.8 Poles and Zeros of Continuous Time Linear Systems [Kar., 2][Kar. & Mac., 1]

3.8.1 Definitions

Consider the system $S(A, B, C, D)$, described in the time domain by:

$$\begin{aligned} \dot{\underline{x}}(t) &= A\underline{x}(t) + B\underline{u}(t), \ \underline{x}(0) = \underline{x}_0 \\ \underline{y}(t) &= C\underline{x}(t) + D\underline{u}(t) \end{aligned} \tag{3.57}$$

Where $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times l}, C \in \mathbb{R}^{m\times n}, D \in \mathbb{R}^{m\times l}$ and $\underline{u}(t)$ is the $l \times 1$ input vector, $\underline{y}(t)$ is the $m \times 1$ output vector, $\underline{x}(t)$ is the $n \times 1$ state variable vector and $\underline{x}_0$ is the vector of initial conditions and let the same system $S(A, B, C, D)$, described in the s domain by the equation:

$$\begin{bmatrix} sI - A & -B \\ -C & -D \end{bmatrix} \begin{bmatrix} \underline{x}(s) \\ \underline{u}(s) \end{bmatrix} = \begin{bmatrix} \underline{x}_0 \\ -\underline{y}(s) \end{bmatrix} \tag{3.58}$$

or by the transfer function matrix $G(s) \in \mathbb{R}^{m\times l}(s)$:

$$G(s) = C(sI - A)^{-1}B + D \tag{3.59}$$

In the case of a proper system it is $D = 0$.

The pencil $P(s)$ is defined as the Rosenbrock's system matrix pencil :

$$P(s) \triangleq \begin{bmatrix} sI - A & -B \\ -C & -D \end{bmatrix} = \begin{bmatrix} -A & -B \\ -C & -D \end{bmatrix} + s \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{(n+m)\times(n+l)}[s] \tag{3.60}$$

According to the definitions of section 3.4 the system the system described by (3.60) is right (left) regular if $\mathcal{N}_r(P(s)) = 0$ ($\mathcal{N}_l(P(s)) = 0$). A system is said to be regular if and $m = l$ and $\det P(s) \not\equiv 0$.

Let Q be $n \times n$ nonsingular constant matrix. If :

$$P(s) = \begin{bmatrix} sI - A & -B \\ -C & -D \end{bmatrix} \text{ and } P_1(s) = \begin{bmatrix} sI - A_1 & -B_1 \\ -C_1 & -D_1 \end{bmatrix}$$

are related by the transformation

$$\begin{bmatrix} Q^{-1} & 0 \\ 0 & I_m \end{bmatrix} \begin{bmatrix} sI - A & -B \\ -C & -D \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & I_l \end{bmatrix} = \begin{bmatrix} sI - A_1 & -B_1 \\ -C_1 & -D_1 \end{bmatrix}$$

then we shall say that $P(s)$ and $P_1(s)$ are *system similar.*

Theorem 15 *Two similar system pencils have the same order and give rise to the same transfer function matrix.*

If $Q = U = V^{-1}$ is the matrix defined by the chains of eigenvectors of A of a system $S(A, B, C, D)$, then the Jordan canonical description $S_J(J, \mathcal{B}, \Gamma, \Delta)$is determined as follows:

$$J = U^{-1}AU = VAU, \quad \mathcal{B} = U^{-1}B = V(\int_0^T e^{J\sigma} d\sigma)B = V\Xi B \tag{3.61}$$

$$\Gamma = CU, \quad \Delta = D \tag{3.62}$$

Let $\sigma(A)$ be the set of roots of the characteristic polynomial of A, $\Phi(A) = \det[sI - A]$ and $\underline{u}_i$ a eigenvector corresponding to the eigenvalue λ_i. □

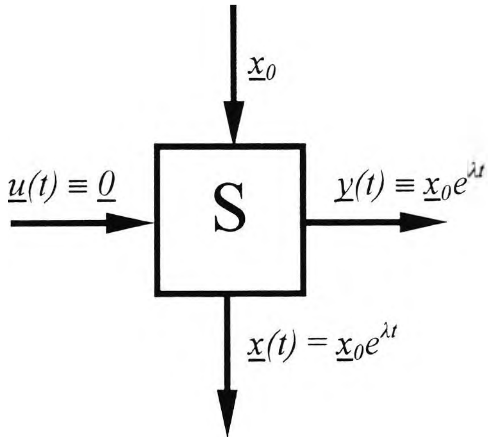

Figure 3-1: Zero Input Problem

3.8.2 System Poles

Definition 19 *An $s \in \mathbb{C}$ is a pole of $S(A, B, C, D)$ if and only if there exists an initial state $\underline{x}_0$ such that the zero-input ($\underline{u}(t) \equiv 0$) response at the output of the system is equal to $\underline{y}(t) = \underline{y}_0 e^{st}$ for some nonzero vector $\underline{y}_0$.*

It can be proved that s is a pole of S if and only if is an eigenvalue of A. □

- *Zero Input Problem-Free Response: Find the system output $\underline{y}(t)$ under zero system input ($\underline{u}(t) \equiv 0$) and $\underline{x}_0$ vector of initial state conditions.*

The solutions of state equations for $\underline{u}(t) \equiv 0$ are given as,

$$\underline{x}(t) = e^{At}\underline{x}_0 = Ue^{Jt}V\underline{x}_0 =$$

$$= \sum_{i=1}^{f} \left\{ \sum_{k=1}^{\nu_i} \left\{ e^{\lambda_i t} \left[\underline{u}_{ik_1}, \dots, \underline{u}_{ik_{\tau_{ik}}} \right] \left[I_{ik} + H_{ik}t + \dots + \frac{(H_{ik}t)^{\tau_{ik}-1}}{(\tau_{ik}-1)!} \right] \begin{bmatrix} \underline{v}_{ik_1}^{\top} \\ \dots \\ \underline{v}_{ik_{\tau_{ik}}}^{\top} \end{bmatrix} \right\} \right\} \underline{x}_0$$

$$\underline{y}(t) = C\underline{x}(t), \quad \underline{y}_0 = C\underline{x}_0$$

where $\underline{u}_{i\tau_1}, \dots, \underline{u}_{i\tau_{ik}}$ and $\underline{v}_{i\tau_1}^{\top}, \dots, \underline{v}_{i\tau_{ik}}^{\top}$ $(k = 1, 2, \dots, \nu_i)$ are the sets of eigenvectors and dual eigenvectors chains associated with the distinct eigenvalue λ_i.

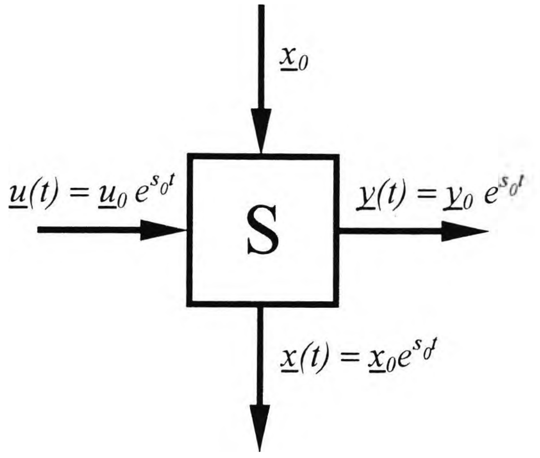

Figure 3-2: Forced Rectilinear Problem

- *Forced rectilinear motion problem : Find* $\underline{x}_0$ *and* $\underline{u}(t)$ *such that* $\underline{x}(t) = \underline{x}_0 e^{s_0 t}$, $\forall t \geq 0$, *for some* $s_0 \in \mathbb{C}$.

The solution to this problem is described below

Remark 8 *Necessary and sufficient condition for the existence of a rectilinear motion* $e^{s_0 t}\underline{x}_0$, *along* $\underline{x}_0$, *is that* $\underline{u}(t)$ *is rectilinear* $\underline{u}_0 e^{s_0 t}, \forall t \geq 0$ *and that* $(s_0, \underline{x}_0, \underline{u}_0)$, *satisfy condition (3.63).*

$$[s_0 I - A, -B] \begin{bmatrix} \underline{x}_0 \\ \underline{u}_0 \end{bmatrix} = \underline{0} \tag{3.63}$$

$$\underline{u}(t) = \underline{u}_0 e^{s_0 t}, \forall t \geq 0 \tag{3.64}$$

In the case of a proper system the above conditions implies,

$$\underline{y}(t) = C\underline{x}(t) = e^{s_0 t} C\underline{x}_0 \triangleq e^{s_0 t}\underline{y}_0$$

$$\begin{bmatrix} s_0 I - A & -B \\ -C & 0 \end{bmatrix} \begin{bmatrix} \underline{x}_0 \\ \underline{u}_0 \end{bmatrix} = \begin{bmatrix} 0 \\ -\underline{y}_0 \end{bmatrix} \tag{3.65}$$

□

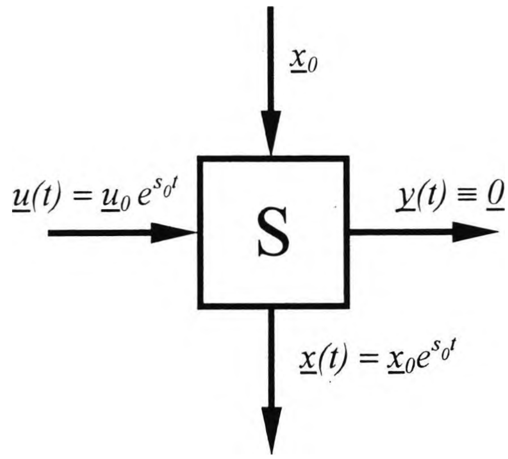

Figure 3-3: Zero Output Problem

3.8.3 Zeros-Output Zeroing

Definition 20 *A zero of the dynamic system is a value of frequency $s = s_0 \in \mathbb{C}$, such that if the input is exponential of the type $\underline{u}(t) = \underline{u}_0 e^{s_0 t}$ then there exist an initial condition $\underline{x}_0$ such that the output is identically zero.* □

There will, however, be rectilinear motion in the state-space, of the form $\underline{x}(t) = e^{s_0 t}\underline{x}_0$, so we have :

$$P(s)\begin{bmatrix} \underline{x}_0 \\ \underline{u}_0 \end{bmatrix} = 0 \tag{3.66}$$

The vectors $\underline{x}_0$ and $\underline{u}_0$ are termed as the zero directions in the state-space and input space respectively.

A zero of the state-space model is thus a value of frequency s for which the above equation has a non-trivial solution.

Definition 21 *A number $s_0 \in \mathbb{C}$ is a state-space zero or system invariant zero if*

$$\mathrm{rank}P(s_0) < rankP(s) = n + min(p, m), \ \forall s \in \mathbb{C}$$

□

Remark 9 *Another equivalent derivation of the system invariant zeros can be from the Smith form* $S^{\mathbb{C}}_{P(s)}$ *of* $P(s)$ *as the roots of finite e.d. of* $P(s)$.

□

Definition 22 *The set of zeros described by the Smith form* $S^{\mathbb{C}}_{G(s)}$ *as the roots of finite e.d. of* $G(s)$ *are defined as the transmission zeros.* □

In general the set of *system invariant zeros* is larger and always contains the set of *transmission zeros.*

Theorem 16 *If a proper system* $S(A, B, C)$ *is both controllable and observable then the set of* system invariant zeros *and the set of* transmission zeros *are the same.* □

3.8.4 Modal Controllability, Observability

For a proper system

$$G(s) = C(sI - A)^{-1}B = \Gamma(sI - J)^{-1}\mathcal{B}$$

and from the diagonal structure of J :

$$J = \text{diag}\{J(\lambda_1), J(\lambda_2), ..., J(\lambda_i), ..., J(\lambda_f)\}$$

and

$$J(\lambda_i) = \text{diag}\{J_{i1}, ..., J_{ik}, ..., J_{i\nu_i}\}$$

also it is.

$$\mathcal{B} = \begin{bmatrix} V(\lambda_1) \\ \cdots \\ V(\lambda_i) \\ \cdots \\ V(\lambda_f) \end{bmatrix} B = \begin{bmatrix} \mathcal{B}_1 \\ \cdots \\ \mathcal{B}_i \\ \cdots \\ \mathcal{B}_f \end{bmatrix} \tag{3.67}$$

$$\Gamma = C\left[U(\lambda_1), ..., U(\lambda_i), ..., U(\lambda_f)\right] = \begin{bmatrix} \Gamma_1 & ... & \Gamma_i & ... & \Gamma_f \end{bmatrix} \tag{3.68}$$

where :

$$\mathcal{B}_i = \begin{bmatrix} V_{i1} \\ \dots \\ V_{ik} \\ \dots \\ V_{i\nu_i} \end{bmatrix} B = \begin{bmatrix} \mathcal{B}_{i1} \\ \dots \\ \mathcal{B}_{ik} \\ \dots \\ \mathcal{B}_{i\nu_i} \end{bmatrix} \tag{3.69}$$

$$\Gamma_i = C \begin{bmatrix} U_{i1} & \dots & U_{ik} & \dots & U_{i\nu_i} \end{bmatrix} = \begin{bmatrix} \Gamma_{i1} & \dots & \Gamma_{ik} & \dots & \Gamma_{i\nu_i} \end{bmatrix} \tag{3.70}$$

and

$$\mathcal{B}_{ik} = \begin{bmatrix} \underline{v}_{ik_1}^{\top} \\ \underline{v}_{ik_2}^{\top} \\ \dots \\ \underline{v}_{ik_{\tau_{ik}}}^{\top} \end{bmatrix} B = \begin{bmatrix} \underline{\beta}_{ik_1}^{\top} \\ \underline{\beta}_{ik_2}^{\top} \\ \dots \\ \underline{\beta}_{ik_{\tau_{ik}}}^{\top} \end{bmatrix} \tag{3.71}$$

$$\Gamma_{ik} = C \left[\underline{u}_{ik_1}, \underline{u}_{ik_2}, \dots, \underline{u}_{ik_{\tau_{ik}}}\right] = \left[\underline{\gamma}_{ik_1}, \underline{\gamma}_{ik_2}, \dots, \underline{\gamma}_{ik_{\tau_{ik}}}\right] \tag{3.72}$$

it is :

$$\begin{aligned} G(s) &= \begin{bmatrix} \Gamma_1 & \dots & \Gamma_i & \dots & \Gamma_f \end{bmatrix} \operatorname{diag}\left\{(sI_i - J_i)^{-1}\right\} \begin{bmatrix} \mathcal{B}_1 \\ \dots \\ \mathcal{B}_i \\ \dots \\ \mathcal{B}_f \end{bmatrix} = \\ &= \sum_{i=1}^{f} \Gamma_i (sI_i - J_i)^{-1} \mathcal{B}_i = \sum_{i=1}^{f} G_i, \ \left(G_i \triangleq \Gamma_i (sI_i - J_i)^{-1} \mathcal{B}_i\right) \end{aligned}$$

where.

$$\begin{aligned} G_i &= \sum_{k=1}^{\nu_i} \Gamma_{ik} (sI_{ik} - J_{ik})^{-1} \mathcal{B}_{ik} = \sum_{k=1}^{\nu_i} G_{ik} \\ &\quad \left(G_{ik} \triangleq \Gamma_{ik} (sI_{ik} - J_{ik})^{-1} \mathcal{B}_{ik}\right) \end{aligned}$$

and where

$$G_{ik} = \left[\underline{\gamma}_{ik_1}, \ldots, \underline{\gamma}_{ik_{\tau_{ik}}}\right] \begin{bmatrix} \frac{1}{s-\lambda_i} & \frac{1}{-(s-\lambda_i)^2} & \cdots & \frac{1}{(-1)^{\tau_i-1}(s-\lambda_i)^{\tau_i}} \\ 0 & \frac{1}{s-\lambda_i} & \cdots & \frac{1}{(-1)^{\tau_i-2}(s-\lambda_i)^{\tau_i-1}} \\ . & . & \cdots & . \\ 0 & 0 & \cdots & \frac{1}{s-\lambda_i} \end{bmatrix} \begin{bmatrix} \underline{\beta}^{\top}_{ik_1} \\ \cdots \\ \underline{\beta}^{\top}_{ik_{\tau_{ik}}} \end{bmatrix}$$

Definition 23 *The i-th spectrum controllability matrix $\mathcal{B}_i^S$ is the $l \times \nu_i$ matrix formed by the ν_i rows of $\mathcal{B}_i$ corresponding to the last rows of the Jordan blocks associated with the eigenvalue λ_i:*

$$\mathcal{B}_i^S = \begin{bmatrix} \underline{\beta}^{\top}_{i1_{\tau_{i1}}} \\ \cdots \\ \underline{\beta}^{\top}_{ik_{\tau_{ik}}} \\ \cdots \\ \underline{\beta}^{\top}_{i\nu_i,\tau_{i\nu_i}} \end{bmatrix} \tag{3.73}$$

□

Definition 24 *The i-th spectrum observability matrix Γ_i^F is the $\nu_i \times l$ matrix formed by the ν_i rows of $\Gamma_i^{\top}$ corresponding to the first columns of the Jordan blocks associated with the eigenvalue λ_i:*

$$\Gamma_i^F \triangleq \left[\underline{\gamma}_{i1_1}, \ldots, \underline{\gamma}_{ik_1}, \ldots, \underline{\gamma}_{i\nu_{i,1}}\right] \tag{3.74}$$

□

Theorem 17 *The mode $(\lambda_i, U(\lambda_i), V(\lambda_i))$ is controllable if and only if the rows of the i-th spectrum controllability matrix $\mathcal{B}_i^S$ are linearly independent over the field of complex numbers.*□

The above result implies that the subsystem associated with the λ_i mode is controllable. This also implies that all λ_i eigenvalues may be change under feedback.

Theorem 18 *If the mode $(\lambda_i, U(\lambda_i), V(\lambda_i))$ for $i = 1, 2, ..., f$ is controllable, then the rows of the pencil $[sI - A, B]$ are linearly independent over the field of complex numbers.* □

Remark 10 *If the mode* $(\lambda_i, U(\lambda_i), V(\lambda_i))$ *is uncontrollable then it is,*

$$\operatorname{rank}[\lambda_i I - A, B] < \operatorname{rank}[sI - A, B]$$

and λ_i *is an input decoupling zero of* S. $\square$

Definition 25 *The pencil* $[sI - A, B]$ *is defined as the input state pencil.* $\square$

Definition 26 *The roots of the e.d. of the pencil* $[sI - A, B]$ *are defined as the input decoupling zeros (i.d.z.) of the system* S. $\square$

Theorem 19 *The mode* $(\lambda_i, U(\lambda_i), V(\lambda_i))$ *is observable if and only if the columns of the* i-th *spectrum observability matrix* Γ_i^F *are linearly independent over the field of complex numbers.* $\square$

The above result implies that all initial conditions associated with the space span$U(\lambda_i)$ may be reconstructed.

Theorem 20 *If the mode* $(\lambda_i, U(\lambda_i), V(\lambda_i))$ *for* $i = 1, 2, ..., f$ *is observable, then the rows of the pencil* $\begin{bmatrix} sI - A \\ C \end{bmatrix}$ *are linearly independent over the field of complex numbers.* $\square$

Remark 11 *If the mode* $(\lambda_i, U(\lambda_i), V(\lambda_i))$ *is unobservable then*

$$\operatorname{rank}\begin{bmatrix} \lambda_i I - A \\ C \end{bmatrix} < \operatorname{rank}\begin{bmatrix} sI - A \\ C \end{bmatrix}$$

and λ_i *is an output decoupling zero of* S. $\square$

Definition 27 *The pencil* $\begin{bmatrix} sI - A \\ C \end{bmatrix}$ *is defined as the output state pencil.* $\square$

Definition 28 *The roots of the e.d. of the pencil* $\begin{bmatrix} sI - A \\ C \end{bmatrix}$ *are defined as the output decoupling zeros (o.d.z.) of the system* S. $\square$

Theorem 21 *The mode* $(\lambda_i, U(\lambda_i), V(\lambda_i))$ *is observable if and only if the columns of the* i-th *spectrum observability matrix* Γ_i^F *are linearly independent over the field of complex numbers.* $\square$

Theorem 22 *The set of controllable and observable eigenvalues of state space model define the poles of $S^{\mathbb{C}}_{G(s)}$.* □

3.8.5 Kalman Decomposition

Let $\mathcal{R}$,$\overline{\mathcal{R}}$,$\mathcal{N}$ the controllable, the uncontrollable and the unobservable respectively subspaces of $\mathbb{R}^n$.

If $\mathcal{V}_{c\bar{o}}$, $\mathcal{V}_{co}$, $\mathcal{V}_{\bar{c}\bar{o}}$, $\mathcal{V}_{\bar{c}o}$, are the controllable-unobservable, the controllable-observable, the uncontrollable unobservable and the uncontrollable-observable respectively subspaces of $\mathbb{R}^n$, then it is :

$$
\begin{aligned}
\mathbb{R}^n &= \mathcal{R} \oplus \overline{\mathcal{R}} \\
\mathcal{R} &= \mathcal{R} \cap \mathcal{N} \oplus \mathcal{V}_{co} = \mathcal{V}_{c\bar{o}} \cap \mathcal{V}_{co} \\
\overline{\mathcal{R}} &= \mathcal{V}_{\bar{c}\bar{o}} \oplus \mathcal{V}_{co} \\
\mathcal{N} &= \mathcal{V}_{c\bar{o}} \oplus \mathcal{V}_{\bar{c}\bar{o}} \\
\mathbb{R}^n &= \mathcal{V}_{c\bar{o}} \cap \mathcal{V}_{co} \oplus \mathcal{V}_{\bar{c}\bar{o}} \oplus \mathcal{V}_{co}
\end{aligned}
$$

The subspaces $\mathcal{R}.\mathcal{N}$ and $\mathcal{V}_{c\bar{o}}$ are A-invariant. Consider a base P of $\mathbb{R}^n$:

$$P = [V_{c\bar{o}}, V_{co}, V_{\bar{c}\bar{o}}, V_{\bar{c}o}]$$

where $V_{c\bar{o}}, V_{co}. V_{\bar{c}\bar{o}}, V_{\bar{c}o}$ are respectively bases of the subspaces $\mathcal{V}_{c\bar{o}}, \mathcal{V}_{co}; \mathcal{V}_{\bar{c}\bar{o}}, \mathcal{V}_{\bar{c}o}$ and the coordinate transformation of the model $S(A, B, C) \sim \tilde{S}(\tilde{A}, \tilde{B}, \tilde{C})$:

$$\tilde{A} = P^{-1}AP.\ \tilde{B} = P^{-1}B,\ \tilde{C} = CP$$

Then.

1. As $[V_{c\bar{o}}, V_{\bar{c}\bar{o}}] \in \ker C$, it is :

$$\tilde{C} = C\,[V_{c\bar{o}}, V_{co}, V_{\bar{c}\bar{o}}, V_{\bar{c}o}] = \left[0, \tilde{C}_{co}, 0, \tilde{C}_{\bar{c}o}\right]$$

2. As $\left(\bar{P} \triangleq P^{-1}\right)$,

$$\bar{P}P = \begin{bmatrix} \bar{V}_{c\bar{o}}^{\top} \\ \bar{V}_{co}^{\top} \\ \bar{V}_{\bar{c}\bar{o}}^{\top} \\ \bar{V}_{\bar{c}o}^{\top} \end{bmatrix} [V_{c\bar{o}}, V_{co}, V_{\bar{c}\bar{o}}, V_{\bar{c}o}] = I_n \Rightarrow$$

$$\Rightarrow \begin{bmatrix} \bar{V}_{\bar{c}\bar{o}}^{\top} \\ \bar{V}_{\bar{c}o}^{\top} \end{bmatrix} [V_{c\bar{o}}, V_{co}] = 0$$

and as $[V_{c\bar{o}}, V_{co}]$ defines a basis for the controllable subspace $\mathcal{R}$,

$$\begin{bmatrix} \bar{V}_{\bar{c}\bar{o}} \\ \bar{V}_{\bar{c}o} \end{bmatrix} B = 0$$

it is,

$$\tilde{B} = P^{-1}B = \begin{bmatrix} \tilde{B}_{c\bar{o}} \\ \tilde{B}_{co} \\ 0 \\ 0 \end{bmatrix}$$

3. As,

$$\begin{aligned}
AV_{c\bar{o}} &= V_{co}\tilde{A}_{c\bar{o}} \\
AV_{co} &= V_{co}\tilde{A}_{co} + V_{c\bar{o}}\tilde{A}_{12} \\
AV_{\bar{c}\bar{o}} &= V_{c\bar{o}}\tilde{A}_{13} + V_{\bar{c}\bar{o}}A_{\bar{c}\bar{o}} \\
AV_{\bar{c}o} &= V_{c\bar{o}}\tilde{A}_{14} + V_{co}\tilde{A}_{24} + V_{\bar{c}\bar{o}}\tilde{A}_{34} + V_{\bar{c}o}A_{\bar{c}o}
\end{aligned}$$

it is :

$$\tilde{A} = \begin{bmatrix} \tilde{A}_{c\bar{o}} & \tilde{A}_{12} & \tilde{A}_{13} & \tilde{A}_{14} \\ 0 & \tilde{A}_{co} & 0 & \tilde{A}_{24} \\ 0 & 0 & \tilde{A}_{\bar{c}\bar{o}} & \tilde{A}_{34} \\ 0 & 0 & 0 & \tilde{A}_{\bar{c}o} \end{bmatrix}$$

From the above coordinate transformation of the Rosenbrock's system matrix pencil we have the Kalman decomposition of the system :

$$P(s) = \begin{bmatrix} \bar{P} & 0 \\ 0 & I_m \end{bmatrix} \begin{bmatrix} sI - A & -B \\ -C & 0 \end{bmatrix} \begin{bmatrix} P & 0 \\ 0 & I_l \end{bmatrix} =$$

$$= \begin{bmatrix} sI - \tilde{A}_{c\bar{o}} & -\tilde{A}_{12} & -\tilde{A}_{13} & -\tilde{A}_{14} & -\tilde{B}_{c\bar{o}} \\ 0 & sI - \tilde{A}_{co} & 0 & -\tilde{A}_{24} & -\tilde{B}_{co} \\ 0 & 0 & sI - \tilde{A}_{\bar{c}\bar{o}} & -\tilde{A}_{34} & 0 \\ 0 & 0 & 0 & sI - \tilde{A}_{\bar{c}o} & 0 \\ 0 & -\tilde{C}_{co} & 0 & -\tilde{C}_{\bar{c}o} & 0 \end{bmatrix}$$

Remark 12 *From the above decomposition we conclude :*

1. *Matrix* $\begin{bmatrix} sI - \tilde{A}_{\bar{c}\bar{o}} & -\tilde{A}_{34} \\ 0 & sI - \tilde{A}_{\bar{c}o} \end{bmatrix}$ *defines the uncontrollable modes of the system and so defines the set of e.d. of* $[sI - A, B]$ *or the set of i.d.z.*

2. *Matrix* $\begin{bmatrix} sI - \tilde{A}_{c\bar{o}} & -\tilde{A}_{13} \\ 0 & sI - \tilde{A}_{\bar{c}\bar{o}} \end{bmatrix}$ *defines the unobservable modes of the system and so defines the set of e.d. of* $\begin{bmatrix} sI - A \\ C \end{bmatrix}$ *or the set of o.d.z.*

3. *Matrix* $sI - \tilde{A}_{\bar{c}\bar{o}}$ *defines the uncontrollable and unobservable modes of the system and so defines the set of i.o.d.z.* □

3.8.6 Infinite Zeros

It is important to note that the definitions we have used apply only to poles and zeros at finite points in the complex s plane, because the unimodular matrices used to get the Smith-McMillan form destroy information about the behavior at $s = \infty$.

Definition 29 *The infinite zeros are defined as the zeros of the Smith form at* $s = \infty$, $S^{\infty}_{G(s)}$ *of the transfer function* $G(s)$. □

Theorem 23 *Let* $r = rank_{\mathbb{R}[s]}P(s), \rho = rank_{\mathbb{R}(s)}G(s), \vartheta = rankD$, *then :*

1. *If* k *is the number of i.e.d. of* $P(s)$ *it is* $k = \rho$.

2. *The number of linear i.e.d. of $P(s)$ is equal to ϑ*

3. *If $w^{\mu_i}, i \in \mathbf{k}$ is the set of i.e.d. of $P(s)$ then the Smith-McMillan form of $G(s)$at $s = \infty$ is:*

$$S^{\infty}_{G(s)} = diag\{\frac{1}{s^{f_1}}, \frac{1}{s^{f_2}}, ..., \frac{1}{s^{f_k}}, 0_{m-\rho,p-\rho}\} \tag{3.75}$$

where $f_i = \mu_i - 1, i \in \mathbf{k}$. □

Remark 13 *If $\delta = \vartheta$ is the number of linear i.e.d. of $P(s)$ then it is:*

$$S^{\infty}_{G(s)} = diag\{\underbrace{1, ..., 1}_{\delta}, \frac{1}{s^{f_{\delta+1}}}, \frac{1}{s^{f_{\delta+2}}}, ..., \frac{1}{s^{f_{\delta+k}}}, 0_{m-\rho,p-\rho}\} \tag{3.76}$$

□

The above characterization is that based on properties over the ring of proper rational functions. Alternatively the structure at infinity may be defined as the structure at $w = 0$ of $G(\frac{1}{w})$, [Pug. & Rat., 1].

3.9 The Zero Polynomial of a Linear Square System [Kar., 3]

The zeros of a system are characteristic of the coupling between the energetic mechanism of the system and its environment, where the coupling is represented by the input and output operators B and C. It is shown that exist an expression for the invariant zero polynomial in terms of the A, B, C parameters of the system. The coefficients of the polynomial are functions of the eigenvalues of A and the Grassman products of the matrices C and B. The investigation is restricted to proper and square systems. The operator A is assumed to have a simple structure although the results may be easily generalized to the case of a non-simple structure A but the derived expressions for the zero polynomial are no more in a simple form. However as the zeros of the system are invariant under state feedback, a simple method of avoiding a non simple structure is to apply an arbitrary state feedback which can change the structure of A from non-simple to a simple one.

The Rosenbrock's system matrix pencil (3.60) in the case of a square ($m = l$) and proper system is given by:

$$P(s) \triangleq \begin{bmatrix} sI - A & -B \\ -C & 0 \end{bmatrix} \in \mathbb{R}^{(n+m)\times(n+m)}[s] \tag{3.77}$$

In order to have a non-trivial solution in equation (3.66) then :

$$\det P(s) = 0$$

where $\det P(s)$ is termed as the zero polynomial. It has been proved that the degree of the zero polynomial of such a system is $n - m$:

$$z(s) = \alpha_{n-m}s^{n-m} + a_{n-m-1}s^{n-m-1} + \dots + a_1 s + a_0 \tag{3.78}$$

The case $\alpha_{n-m} = 0$ means that a finite zero moves to infinity (finite zeros are transformed to zeros at infinity [Kar., 3]). A further reduction in the degree of $z(s)$ implies the migration of another zero to infinity and so on.

3.9.1 Calculation of Coefficients

Theorem 24 *Given the square, proper linear system $S(A, B, C)$ strict equivalent to the $S_J(J, \mathcal{B}, \Gamma)$, where J is in simple structure, Jordan canonical form of A, then the zero polynomial $z(s)$ may be expressed as,*

$$z(s) = \boldsymbol{\gamma}^{\top}\boldsymbol{\beta}s^{n-m} + \left\{ \sum_{\omega_1} \lambda_{\omega_1}\boldsymbol{\gamma}^{\top}_{\omega_1)}\boldsymbol{\beta}^{\omega_1)} \right\} s^{n-m-1} + \dots + \left\{ \sum_{\omega_k} \lambda_{\omega_k}\boldsymbol{\gamma}^{\top}_{\omega_k)}\boldsymbol{\beta}^{\omega_k)} \right\} s^{n-m-k} + \\ + \dots + \left\{ \sum_{\omega_{n-m}} \lambda_{\omega_{n-m}}\boldsymbol{\gamma}^{\top}_{\omega_{n-m})}\boldsymbol{\beta}^{\omega_{n-m})} \right\}$$

where the bold letters, $\boldsymbol{\gamma}^{\top}, \boldsymbol{\beta}$ denote the Grassman products of the rows of Γ, columns of $\mathcal{B}$ respectively. If,

$$\Gamma = \begin{bmatrix} \underline{\gamma}_1^{\top} \\ \underline{\gamma}_2^{\top} \\ \dots \\ \underline{\gamma}_n^{\top} \end{bmatrix}, \; \mathcal{B} = \begin{bmatrix} \underline{\beta}_1 & \underline{\beta}_2 & \dots & \underline{\beta}_n \end{bmatrix} \tag{3.79}$$

then,

$$\gamma^{\top} = \underline{\gamma}_1^{\top} \wedge \underline{\gamma}_2^{\top} \wedge \ldots \wedge \underline{\gamma}_n^{\top} \tag{3.80}$$
$$\beta = \underline{\beta}_1 \wedge \underline{\beta}_2 \wedge \ldots \wedge \underline{\beta}_n$$

$\omega_k \in Q_{k,n}$, $\lambda_i \in \wp_{\lambda_i}(A)$. □

It is clear that the number of finite invariant zeros is given by the degree of the zero polynomial. The location of the finite zeros depends on the values which the coefficients of $z(s)$ assume. The expressions for the coefficients of $z(s)$ are :

$$\begin{aligned}
\alpha_{n-m} &= \gamma^{\top}\beta \\
\alpha_{n-m-1} &= (-1)\left\{\sum_{\omega_1} \lambda_{\omega_1} \gamma^{\top}_{\omega_1)} \beta^{\omega_1)}\right\} \\
&\ldots \\
\alpha_{n-m-k} &= (-1)^{k-1}\left\{\sum_{\omega_k} \lambda_{\omega_k} \gamma^{\top}_{\omega_k)} \beta^{\omega_k)}\right\} \\
&\ldots \\
\alpha_0 &= (-1)^{m-n-1}\left\{\sum_{\omega_{m-n}} \lambda_{\omega_{m-n}} \gamma^{\top}_{\omega_{m-n})} \beta^{\omega_{m-n})}\right\}
\end{aligned} \tag{3.81}$$

Thus for systems with CB full rank, the number of finite finite invariant zeros of the system is equal to $n - m$.

If CB has rank defect 1, then the coefficient of s^{n-m} becomes zero and the maximum number of finite invariant zeros gets less than 1. Generally, when the rank defect of CB becomes d, the inner products of the Grassman vectors $\left\langle \gamma^{\top}_{\omega)} \beta^{\omega)} \right\rangle$, $\omega \in Q_{\rho,n}$ $\rho = 0, 1, d-1$ become identically zero due to the multi-orthogonality property (*Theorem 14*) of the vectors γ, β; therefore, the first d terms in $z(s)$ become zero, irrespective of the eigenvalues of A and hence the maximum number of finite invariant zeros is reduced from $n-m$ to at least $n-m-d$. This is summarized below :

3.9.2 Migration of Zeros to Infinity [Kar., 1], [Kar. & Kou., 1]

Theorem 25 *Let the square, proper linear system $S(A, B, C)$and let CB have rank defect d.*

The maximum number of system invariant zeros is $n - m - d$. □

In physical terms as γ becomes orthogonal to β at least one finite zero migrates to infinity. If the multi-orthogonality degree as this has been defined before, increases from 1 to 2, at least one zero migrates to infinity and so on. Generally, as the multi-orthogonality of the vectors γ, β gains d degrees, at least d zeros more vanish at infinity, thus reducing the maximum number of finite zeros from $n - m$ to $n - m - d$. The set of zeros which are lost at infinity because of the multi-orthogonality of the vectors γ, β will be called *class 1 of zeros at infinity* .

A further loss of zeros at infinity may take place, if the coefficient of s^{n-m-d} term becomes zero; however, such a further loss of zeros no longer depends on the properties of γ, β alone, but involves the eigenvalues of the system, too; in such a case it is believed that some more general forms of multi-orthogonality are involved which need further investigation. Finally, we note that if the coefficients $a_{n-m}, ..., a_0$ are all zero, then the zero polynomial is identically zero and the system becomes *degenerate*.

Example 2 *Consider the continuous system $S(A, B, C)$:*

$$A = \begin{bmatrix} -6.0 & 0 & 0 & 0 & 0 \\ 0 & -2.0 & 3.0 & -3.0 & 1.0 \\ 0 & -1.5 & -2.0 & -1.0 & 1.5 \\ 0 & 1.5 & 1.0 & -2.0 & 1.5 \\ 0 & -1.0 & -3.0 & -3.0 & -2.0 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 2 & 0 \\ 0 & -1 \\ 1 & 3 \\ 2 & 0 \end{bmatrix}, C = \begin{bmatrix} -7 & 0 & -2 & 0 & 0 \\ 3 & -6 & -3 & -1 & -2 \end{bmatrix}$$

with the following simple structure, Jordan form of A :

$$J = \begin{bmatrix} -6.0 & 0 & 0 & 0 & 0 \\ 0 & -2.0 - 2.0i & 0 & 0 & 0 \\ 0 & 0 & -2.0 + 2.0i & 0 & 0 \\ 0 & 0 & 0 & -2.0 + 4.0i & 0 \\ 0 & 0 & 0 & 0 & -2.0 - 4.0i \end{bmatrix} = VAU$$

where,

$$V = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1+i & -1-i & -i \\ 0 & 1 & -1-i & -1+i & i \\ 0 & 1 & 1-i & 1+i & -i \\ 0 & 1 & 1+i & 1-i & i \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & -\frac{1}{8}-\frac{1}{8}i & -\frac{1}{8}+\frac{1}{8}i & \frac{1}{8}+\frac{1}{8}i & \frac{1}{8}-\frac{1}{8}i \\ 0 & -\frac{1}{8}+\frac{1}{8}i & -\frac{1}{8}-\frac{1}{8}i & \frac{1}{8}-\frac{1}{8}i & \frac{1}{8}+\frac{1}{8}i \\ 0 & \frac{1}{4}i & -\frac{1}{4}i & \frac{1}{4}i & -\frac{1}{4}i \end{bmatrix}$$

and consequently the parameters $\mathcal{B}$ and Γ of the Jordan equivalent system are,

$$\mathcal{B} = VB = \begin{bmatrix} 0 & 1 \\ 1-3i & -2-4i \\ 1+3i & -2+4i \\ 3-i & 2+4i \\ 3+i & 2-4i \end{bmatrix}, \quad \Gamma = CU = \begin{bmatrix} -7 & \frac{1}{4}+\frac{1}{4}i & \frac{1}{4}-\frac{1}{4}i & -\frac{1}{4}-\frac{1}{4}i & -\frac{1}{4}+\frac{1}{4}i \\ 3 & -1-\frac{1}{4}i & -1+\frac{1}{4}i & -2-\frac{3}{4}i & -2+\frac{3}{4}i \end{bmatrix}$$

The zero polynomial of the above system, is of the form,

$$z(s) = a_3 z^3 + a_2 z^2 + a_1 z + a_0$$

We proceed to the calculation of the Grassman products of $\mathcal{B}$ and Γ and also to the coefficients of the zero polynomial of the system,

$$\begin{aligned} \beta &= \mathfrak{C}_r(B) = \begin{bmatrix} B_{1,2}^{1,2} & B_{1,2}^{1,3} & B_{1,2}^{1,4} & B_{1,2}^{2,3} & B_{1,2}^{2,4} & B_{1,2}^{3,4} \end{bmatrix}^{\top} \\ &= \begin{bmatrix} -1+3i & -1-3i & -3+i & -3-i & 20i & 24+8i & -8+4i & -8-4i & 24-8i & -28i \end{bmatrix}^{\top} \end{aligned}$$

$$\begin{aligned} \gamma &= \mathfrak{C}_r(C) = \begin{bmatrix} C_{1,2}^{1,2} & C_{1,3}^{1,2} & C_{1,4}^{1,2} & C_{2,3}^{1,2} & C_{2,4}^{1,2} & C_{3,4}^{1,2} \end{bmatrix} \\ &= \begin{bmatrix} \frac{25}{4}+i & \frac{25}{4}-i & \frac{59}{4}+6i & \frac{59}{4}-6i & -\frac{3}{8}i & -\frac{1}{2}-i & -1-\frac{1}{8}i & -1+\frac{1}{8}i & -\frac{1}{2}+i & \frac{5}{8}i \end{bmatrix} \end{aligned}$$

it is,

$$a_3 = \langle \gamma^{\top} \beta \rangle = -85$$

also we have,

$$\beta^{1)} = \begin{bmatrix} 20i \\ 24+8i \\ -8+4i \\ -8-4i \\ 24-8i \\ -28i \end{bmatrix}, \beta^{2)} = \begin{bmatrix} -1-3i \\ -3+i \\ -3-i \\ -8-4i \\ 24-8i \\ -28i \end{bmatrix}, \beta^{3)} = \begin{bmatrix} -1+3i \\ -3+i \\ -3-i \\ 24+8i \\ -8+4i \\ -28i \end{bmatrix}, \beta^{4)} = \begin{bmatrix} -1+3i \\ -1-3i \\ -3-i \\ 20i \\ -8+4i \\ 24-8i \end{bmatrix}.$$

$$\beta^{5)} = \begin{bmatrix} -1+3i \\ -1-3i \\ -3+i \\ 20i \\ 24+8i \\ -8-4i \end{bmatrix}$$

$$\begin{aligned}
\gamma_{1)} &= \begin{bmatrix} -\frac{3}{8}i & -\frac{1}{2}-i & -1-\frac{1}{8}i & -1+\frac{1}{8}i & -\frac{1}{2}+i & \frac{5}{8}i \end{bmatrix} \\
\gamma_{2)} &= \begin{bmatrix} \frac{25}{4}-i & \frac{59}{4}+6i & \frac{59}{4}-6i & -1+\frac{1}{8}i & -\frac{1}{2}+i & \frac{5}{8}i \end{bmatrix} \\
\gamma_{3)} &= \begin{bmatrix} \frac{25}{4}+i & \frac{59}{4}+6i & \frac{59}{4}-6i & -\frac{1}{2}-i & -1-\frac{1}{8}i & \frac{5}{8}i \end{bmatrix} \\
\gamma_{4)} &= \begin{bmatrix} \frac{25}{4}+i & \frac{25}{4}-i & \frac{59}{4}-6i & -\frac{3}{8}i & -1-\frac{1}{8}i & -\frac{1}{2}+i \end{bmatrix} \\
\gamma_{5)} &= \begin{bmatrix} \frac{25}{4}+i & \frac{25}{4}-i & \frac{59}{4}+6i & -\frac{3}{8}i & -\frac{1}{2}-i & -1+\frac{1}{8}i \end{bmatrix}
\end{aligned}$$

and the coefficient a_2 of the zero polynomial is defined as,

$$\left.\begin{aligned}
(-6.0)\left\langle \gamma_{1)}^{\top}\beta^{1)} \right\rangle &= -204.0 \\
(-2.0-2.0i)\left\langle \gamma_{2)}^{\top}\beta^{2)} \right\rangle &= 202.0+149.0i \\
(-2.0+2.0i)\left\langle \gamma_{3)}^{\top}\beta^{3)} \right\rangle &= 202.0-149.0i \\
(-2.0+2.0i)\left\langle \gamma_{3)}^{\top}\beta^{3)} \right\rangle &= .5-283.5i \\
(-2.0-4.0i)\left\langle \gamma_{5)}^{\top}\beta^{5)} \right\rangle &= .5+283.5i
\end{aligned}\right\} \Longrightarrow$$

$$\Longrightarrow a_2 = -\left(-204.0 + 202.0 + 149.0i + 202.0 - 149.0i + .5 - 283.5i + .5 + 283.5i\right) = -201.0$$

in order to define the coefficient a_1 we have to calculate the column vectors,

$$\beta^{12)} = \begin{bmatrix} -8-4i \\ 24-8i \\ -28i \end{bmatrix}, \ \beta^{13)} = \begin{bmatrix} 24+8i \\ -8+4i \\ -28i \end{bmatrix}, \ \beta^{14)} = \begin{bmatrix} 20i \\ -8+4i \\ 24-8i \end{bmatrix}, \ \beta^{15)} = \begin{bmatrix} 20i \\ 24+8i \\ -8-4i \end{bmatrix},$$

$$\beta^{23)} = \begin{bmatrix} -3+i \\ -3-i \\ -28i \end{bmatrix}, \ \beta^{24)} = \begin{bmatrix} -1-3i \\ -3-i \\ 24-8i \end{bmatrix}, \ \beta^{25)} = \begin{bmatrix} -1-3i \\ -3+i \\ -8-4i \end{bmatrix}, \ \beta^{34)} = \begin{bmatrix} -1+3i \\ -3-i \\ -8+4i \end{bmatrix}.$$

$$\beta^{35)} = \begin{bmatrix} -1+3i \\ -3+i \\ 24+8i \end{bmatrix}, \ \beta^{45)} = \begin{bmatrix} -1+3i \\ -1-3i \\ 20i \end{bmatrix}$$

and correspondingly the row vectors,

$$\begin{array}{ll}
\gamma_{12)} = \begin{bmatrix} -1+\frac{1}{8}i & -\frac{1}{2}+i & \frac{5}{8}i \end{bmatrix} & \gamma_{13)} = \begin{bmatrix} -\frac{1}{2}-i & -1-\frac{1}{8}i & \frac{5}{8}i \end{bmatrix} \\
\gamma_{14)} = \begin{bmatrix} -\frac{3}{8}i & -1-\frac{1}{8}i & -\frac{1}{2}+i \end{bmatrix} & \gamma_{15)} = \begin{bmatrix} -\frac{3}{8}i & -\frac{1}{2}-i & -1+\frac{1}{8}i \end{bmatrix} \\
\gamma_{23)} = \begin{bmatrix} \frac{59}{4}+6i & \frac{59}{4}-6i & \frac{5}{8}i \end{bmatrix} & \gamma_{24)} = \begin{bmatrix} \frac{25}{4}-i & \frac{59}{4}-6i & -\frac{1}{2}+i \end{bmatrix} \\
\gamma_{25)} = \begin{bmatrix} \frac{25}{4}-i & \frac{59}{4}+6i & -1+\frac{1}{8}i \end{bmatrix} & \gamma_{34)} = \begin{bmatrix} \frac{25}{4}+i & \frac{59}{4}-6i & -1-\frac{1}{8}i \end{bmatrix} \\
\gamma_{35)} = \begin{bmatrix} \frac{25}{4}+i & \frac{59}{4}+6i & -\frac{1}{2}-i \end{bmatrix} & \gamma_{45)} = \begin{bmatrix} \frac{25}{4}+i & \frac{25}{4}-i & -\frac{3}{8}i \end{bmatrix}
\end{array}$$

so we have,

$$\left.\begin{array}{ll}
\lambda_1\lambda_2\left\langle\gamma_{12)}^{\top}\beta^{12)}\right\rangle = -108.0+636.0i & \lambda_1\lambda_3\left\langle\gamma_{13)}^{\top}\beta^{13)}\right\rangle = -108.0-636.0i \\
\lambda_1\lambda_4\left\langle\gamma_{14)}^{\top}\beta^{14)}\right\rangle = 744.0+12.0i & \lambda_1\lambda_5\left\langle\gamma_{15)}^{\top}\beta^{15)}\right\rangle = 744.0-12.0i \\
\lambda_2\lambda_3\left\langle\gamma_{23)}^{\top}\beta^{23)}\right\rangle = -664.0 & \lambda_2\lambda_4\left\langle\gamma_{24)}^{\top}\beta^{24)}\right\rangle = -708.0+416.0i \\
\lambda_2\lambda_5\left\langle\gamma_{25)}^{\top}\beta^{25)}\right\rangle = 420.0-540.0i & \lambda_3\lambda_4\left\langle\gamma_{34)}^{\top}\beta^{34)}\right\rangle = 420.0+540.0i \\
\lambda_3\lambda_5\left\langle\gamma_{35)}^{\top}\beta^{35)}\right\rangle = -708.0-416.0i & \lambda_4\lambda_5\left\langle\gamma_{45)}^{\top}\beta^{45)}\right\rangle = -220.0
\end{array}\right\} \Longrightarrow$$

$$\Longrightarrow a_1 = -108.0 + 636.0i - 108.0 - 636.0i + 744.0 + 12.0i + 744.0 - 12.0i - 664.0 - 708.0 +$$

$$+416.0i + 420.0 - 540.0i + 420.0 + 540.0i - 708.0 - 416.0i - 220.0 = -188.0$$

and finally for the calculation of a_0

$$\begin{aligned}\beta^{123)} &= -28i,\ \beta^{124)} = 24-8i,\ \beta^{125)} = -8-4i,\ \beta^{134)} = -8+4i,\ \beta^{135)} = 24+8i,\\ \beta^{145)} &= 20i,\ \beta^{234)} = -3-i,\ \beta^{235)} = -3+i,\ \beta^{245)} = -1-3i,\ \beta^{345)} = -1+3i\end{aligned}$$

and

$$\begin{aligned}\gamma^{123)} &= \frac{5}{8}i,\ \gamma^{124)} = -\frac{1}{2}+i,\ \gamma^{125)} = -1+\frac{1}{8}i,\ \gamma^{134)} = -1-\frac{1}{8}i,\ \gamma^{135)} = -\frac{1}{2}-i,\\ \gamma^{145)} &= -\frac{3}{8}i,\ \gamma^{234)} = \frac{59}{4}-6i,\ \gamma^{235)} = \frac{59}{4}+6i,\ \gamma^{245)} = \frac{25}{4}-i,\ \gamma^{345)} = \frac{25}{4}+i\end{aligned}$$

we have

$$\left.\begin{array}{ll}\lambda_1\lambda_2\lambda_3\left\langle\beta^{123)}\gamma^{123)}\right\rangle = -840.0 & \lambda_1\lambda_2\lambda_4\left\langle\beta^{124)}\gamma^{124)}\right\rangle = -384.0-2112.0i\\ \lambda_1\lambda_2\lambda_5\left\langle\beta^{125)}\gamma^{125)}\right\rangle = 420.0-540.0i & \lambda_1\lambda_3\lambda_4\left\langle\beta^{134)}\gamma^{134)}\right\rangle = 420.0+540.0i\\ \lambda_1\lambda_3\lambda_5\left\langle\beta^{135)}\gamma^{135)}\right\rangle = -384.0+2112.0i & \lambda_1\lambda_4\lambda_5\left\langle\beta^{145)}\gamma^{145)}\right\rangle = -900.0\\ \lambda_2\lambda_3\lambda_4\left\langle\beta^{234)}\gamma^{234)}\right\rangle = 700.0-1660.0i & \lambda_2\lambda_3\lambda_5\left\langle\beta^{235)}\gamma^{235)}\right\rangle = 700.0+1660.0i\\ \lambda_2\lambda_4\lambda_5\left\langle\beta^{245)}\gamma^{245)}\right\rangle = -340.0+1080.0i & \lambda_3\lambda_4\lambda_5\left\langle\beta^{345)}\gamma^{345)}\right\rangle = -340.0-1080.0i\end{array}\right\}\Longrightarrow$$

$$\Longrightarrow a_0 = -(-840.0+420.0-540.0i-384.0+2112.0i+700.0-1660.0i-340.0+1080.0i+$$

$$+1080.0i-384.0-2112.0i+420.0+540.0i-900.0+700.0+1660.0i-340.0-1080.0i) = 948.0$$

the zero polynomial is given as :

$$z(s) = -85.0s^3 - 201.0s^2 - 188.0s + 948.0$$

and the system invariant zeros are defined as,

$$z_1 = 1.4451,\ z_2 = -1.9049+2.0221i,\ z_3 = -1.9049-2.0221i$$

$\square$

3.10 Conclusions

An extensive review of the fundamentals of relevant mathematics and systems theory has been given, which provide the basis for the investigations undertaken in the following chapters. In the subsequent Chapters we investigate the effect of sampling on the structural characteristics of the discretised models.

Chapter 4

SAMPLING THEORY AND BASIC DYNAMICS OF DISCRETISED MODELS

4.1 Introduction

State space description of the linear, time invariant, continuous systems $S(A, B, C, D)$ and of the corresponding discretised models $S(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ provide the framework for the study of basic structural properties such as controllability, observability, decoupling zeros etc. Moreover Jordan canonical description of a linear, time invariant continuous system enables the use of spectral criteria for study of the above properties. The Jordan canonical description is used as a natural tool that demonstrates the structure of the internal dynamics and it is crucial in the investigation of the mapping of structural properties from the continuous to the discrete model.

In this Chapter we introduce,

- The Jordan canonical description of discretised models equipped with ZOH, or FOH we define the eigenbasis matrix $\hat{U}$ of the linear operator $\hat{A} = e^{AT}$ and the relation between U and $\hat{U}$.

- The problem of mapping of the set of eigenvalues of the continuous model matrix A to the set of the corresponding eigenvalues of the discretised matrix $\hat{A}$.

- The consequences of sampling on the eigenspaces, the Segré characteristic and the cyclicity properties.

- The classification of the sampling period values between regular and irregular.

This chapter introduces the key problem studied here and provides some introductory result on the properties of mapping between continuous and discrete model properties.

4.2 State Space Description of discretised Models [Che., 1], [Fra., Pow. & Wor., 1], [Kar., 2]

We assume the general configuration of Fig 2-1. We consider the solution of the first state space equation over one sample period T, to obtain the difference equation :

$$\underline{x}(kT+T) = e^{AT}\underline{x}(kT) + \int_{kT}^{kT+T} e^{A(kT+T-\tau)}B\underline{u}(\tau)d\tau \tag{4.1}$$

Here, we have to distinguish the two cases based on the implementation of hold device (H), one with ZOH and one with FOH. Each one of these leads to a corresponding discretised model of the physical system. Those two cases are considered next.

4.2.1 Case of a system with ZOH

With the assumption of a ZOH with no delay we have :

$$\underline{u}(\tau) = \underline{u}(kT) \text{ for } kT \leq \tau < kT + T$$

and if we change variables in the integral from τ to σ, i.e. $\sigma = kT + T - \tau$, the difference equation (4.1) becomes:

$$x(kT+T) = e^{AT}\underline{x}(kT) + \left(\int_0^T e^{A\sigma}d\sigma\right)B\underline{u}(kT) \tag{4.2}$$

If we define as,

$$\hat{A} = e^{AT}, \ \hat{B} = \left(\int_0^T e^{A\sigma} d\sigma \right) B \tag{4.3}$$

we have:

$$\underline{x}[(k+1)T] = \hat{A}\underline{x}(kT) + \hat{B}\underline{u}(kT) \tag{4.4}$$

Also sampling of equation (3.53) gives :

$$\underline{y}(kT) = \hat{C}\underline{x}(kT) + \hat{D}\underline{u}(kT) \tag{4.5}$$

Where:

$$\hat{C} = C, \ \hat{D} = D \tag{4.6}$$

The above analysis leads to the following result:

Proposition 13 *The discretised model of the system $S(A, B, C, D)$ in a configuration involving a ZOH and for a sampling period T is defined by $\hat{S}(\hat{A}, \hat{B}, \hat{C}, \hat{D})$, where the state parameters are defined as above by equations (4.3) and (4.6).* □

4.2.2 Case of a system with FOH

With the assumption of a H which is of the FOH type and assuming no delay we have,

$$u(\tau) = \frac{\tau - kT}{T}\left(\underline{u}(kT) - \underline{u}(kT - T)\right) + \underline{u}(kT), \ \text{for } kT \leq \ \tau < kT + T$$

and if we change variables in the integral from τ to σ, i.e. $\sigma = kT + T - \tau$, the difference equation (4.1) becomes:

$$\underline{x}(kT + T) = e^{AT}\underline{x}(kT) + \int_0^T \left(2 - \frac{\sigma}{T}\right) e^{A\sigma} d\sigma B\underline{u}(kT) - \frac{1}{T}\int_0^T \sigma e^{A\sigma} d\sigma B\underline{u}(kT - T)$$

If we define as,

$$\hat{A} = e^{AT},\ \hat{E} = \int_0^T \left(2 - \frac{\sigma}{T}\right) e^{A\sigma} d\sigma B,\ \hat{Z} = -\frac{1}{T}\int_0^T \sigma e^{A\sigma} d\sigma B \tag{4.7}$$

we have

$$\underline{x}(kT + T) = \hat{A}\underline{x}(kT) + \hat{E}\underline{u}(kT) + \hat{Z}\underline{u}(kT - T) \tag{4.8}$$

Also sampling of equation (3.53) yields,

$$\underline{y}(kT) = \hat{C}\underline{x}(kT) + \hat{D}\underline{u}(kT) \tag{4.9}$$

Where,

$$\hat{C} = C,\ \hat{D} = D \tag{4.10}$$

The above analysis leads to the following result:

Proposition 14 *The discretised model of the system $S(A, B, C, D)$ in a configuration involving a FOH and for a sampling period T is defined by $\hat{S}(\hat{A}, \hat{E}, \hat{Z}, \hat{C}, \hat{D})$, where the state parameters are defined as above by equations (4.7) and (4.10).* □

From the above we can conclude that for any order of the H hold the state parameters of the resulting discretised system remain functions of the sampling period T we select. In the following we concentrate on study of the structural properties for the discretised models where discretisation involves ZOH and FOH.

4.3 Structural Properties of Discretised Models

From the introductory analysis in the previous section it is clear that the parameters of a discretised model are functions of the sampling period T we select. The investigation of the effect of sampling on the structural properties of the resulting discretised model, such as controllability and observability is the aim of this section. So we recall the definition of these two properties from the theory of linear, time invariant, continuous systems.

4.3.1 Controllability, Observability of Continuous Systems [Che., 1], [Rug., 1], [Kar., 2]

Definition 30 *A system $S(A,B,C,D)$ is said to be controllable at time t_0 if for any initial state $\underline{x}_0$ in the space $\mathbb{R}^n$ and any state $\underline{x}_1$ there exists a finite time $t_1 > t_0$ and an input $\underline{u}[t_0,t_1]$ that will transfer the state $\underline{x}_0$ to $\underline{x}_1$ in time $t_1 - t_0$. Otherwise the system is said to be uncontrollable.* □

It can be shown [Che., 1] that the system $S(A,B,C,D)$ described by the equations (3.52) and (3.53) is controllable if the rows of the matrix $e^{At}B$ are linearly independent over the field of complex numbers. By using Laplace transforms we have that :

$$\mathcal{L}\{e^{At}B\} = (sI - A)^{-1}B$$

and this leads to an equivalent test.

Proposition 15 *The system $S(A,B,C,D)$ is controllable , if the rows of $(sI - A)^{-1}B$ are linearly independent over the field of complex numbers.* □

For the property of observability of a linear system $S(A,B,C,D)$ we have the following dual to controllability definition :

Definition 31 *A system $S(A,B,C,D)$ is said to be observable at time t_0 if for any initial state $\underline{x}_0$ in the space $\mathbb{R}^n$ there exists a finite time $t_1 > t_0$ such that knowledge of the output $\underline{y}(t)$, over the interval $[t_0,t_1]$ suffices to determine the initial state $\underline{x}_0$. Otherwise the system is said to be unobservable.* □

It can be shown [3],[1] that the system $S(A,B,C,D)$ described by the equations (3.52) and (3.53) is observable if the columns of the matrix $Ce^{A(t-t_0)}$ are linearly independent over the field of complex numbers. This is equivalently expressed using Laplace transforms as

$$L\{Ce^{A(t-t_0)}\} = C(sI - A)^{-1}$$

So we have the following proposition for observability (dual to Proposition 15) :

Proposition 16 *The system $S(A, B, C, D)$ is observable , if the columns of $C(sI - A)^{-1}$ are linearly independent over the field of complex numbers.* □

4.3.2 Controllability of Discretised Models

After the definition of these structural properties of the linear continuous system, we can proceed defining controllability and observability of the discretised model of the linear system. In order to define the controllability and observability matrices of such a model we have to distinguish two cases of hold implementation, one with ZOH and one with FOH. For the definition of controllability of the discretised model we have :

Definition 32 *A discretised model $\hat{S}$ is said to be controllable if for any initial state $\underline{x}(0) = \underline{x}_1$ and any state $\underline{x}_2$ there exists a finite time $nT > 0$ and a sequence of inputs $\underline{u}(0)$, $\underline{u}(T)$, $\underline{u}(2T)$, ..., $\underline{u}[(n-1)T]$ that will transfer the state $\underline{x}(0) = \underline{x}_1$, to $\underline{x}(nT) = \underline{x}_2$. Otherwise the system is said to be uncontrollable.* □

Case of system with ZOH [Kar., 2]

For the case of ZOH the controllability test becomes :

Proposition 17 *A discretised model $\hat{S}(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ of a system with ZOH is controllable if and only if,*

$$rank \left[\hat{B}, \hat{A}\hat{B}, \hat{A}^2\hat{B}, ..., \hat{A}^{n-1}\hat{B}\right] = n$$

Proof. (For convenience we drop T from the difference equations) :

$$\begin{aligned} \underline{x}(k+1) &= \hat{A}\underline{x}(k) + \hat{B}\underline{u}(k) \\ \underline{y}(k) &= \hat{C}\underline{x}(k) + \hat{D}\underline{u}(k) \end{aligned}$$

for $k = 0, 1, 2, ..., n$ we have :

$$\begin{aligned} \underline{x}(1) &= \hat{A}\underline{x}(0) + \hat{B}\underline{u}(0) \\ \underline{x}(2) &= \hat{A}\underline{x}(1) + \hat{B}\underline{u}(1) = \hat{A}^2\underline{x}(0) + \hat{A}\hat{B}\underline{u}(0) + \hat{B}\underline{u}(1) \\ & \quad \cdots\cdots \end{aligned}$$

$$\underline{x}(n) = \hat{A}^n\underline{x}(0) + \hat{A}^{n-1}\hat{B}\underline{u}(0) + \hat{A}^{n-2}\hat{B}\underline{u}(1) + ... + \hat{A}\hat{B}\underline{u}(n-2) + \hat{B}\underline{u}(n-1) \Leftrightarrow$$

$$\Leftrightarrow \underline{x}(n) = \hat{A}^n\underline{x}(0) + \sum_{i=0}^{n-1} \hat{A}^{n-1-i}\hat{B}\underline{u}(i) \Leftrightarrow$$

$$\Leftrightarrow \underline{x}(n) - \hat{A}^n\underline{x}(0) = \sum_{i=0}^{n-1} \hat{A}^{n-1-i}\hat{B}\underline{u}(i) = \left[\hat{B}, \hat{A}\hat{B}, ..., \hat{A}^{n-1}\hat{B}\right] \begin{bmatrix} \underline{u}(n-1) \\ \underline{u}(n-2) \\ \cdots \\ \underline{u}(0) \end{bmatrix}$$

For any $\underline{x}(0)$ and any $\underline{x}(n)$ there exists a solution of the system of equations if and only if $rank[\hat{B}, \hat{A}\hat{B}, (\hat{A})2\hat{B}, ..., (\hat{A})^{n-1}\hat{B}] = n.$ □

An equivalent criterion is :

Proposition 18 *The discretised model $\hat{S}(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ with ZOH is controllable if and only if the rows of $(zI - \hat{A})^{-1}\hat{B}$ are linearly independent over the field of complex numbers.*

Proof. Consider the equality,

$$(zI - \hat{A})^{-1}\hat{B} = z^{-1}[I - z^{-1}\hat{A}]^{-1}\hat{B}$$

and the binomial expansion,

$$\left(I - z^{-1}\hat{A}\right)^{-1} = I + z^{-1}\hat{A} + z^{-2}\hat{A}^2 + z^{-3}\hat{A}^3 + ... \tag{4.11}$$

then.

$$(zI - \hat{A})^{-1}\hat{B} = z^{-1}\hat{B} + z^{-2}\hat{A}\hat{B} + z^{-3}\hat{A}^2\hat{B} + ...$$

the n rows of the above matrix are linearly independent over the field of complex numbers, if and only if

$$rank\left[\hat{B}, \hat{A}\hat{B}, \hat{A}^2\hat{B}, ...\right] = n$$

From the Gayley-Hamilton theorem, we know that $\hat{A}^m$ with $m \geq n$ can be written as a linear combination of $I, \hat{A}, \hat{A}^2, ..., \hat{A}^{n-1}$. Hence the columns of $\hat{A}^m\hat{B}$ with $m \geq n$ are linearly dependent on the columns of $\hat{B}$, $\hat{A}\hat{B}$, $\hat{A}^2\hat{B}$, ..., $\hat{A}^{n-1}\hat{B}$. Consequently,

$$rank\left[\hat{B}, \hat{A}\hat{B}, \hat{A}^2\hat{B}, ...\right] = rank\left[\hat{B}, \hat{A}\hat{B}, \hat{A}^2\hat{B}, ..., \hat{A}^{n-1}\hat{B}\right]$$

and the rows of $(zI - \hat{A})^{-1}\hat{B}$ are linearly independent over the field of complex numbers if and only if :

$$rank\left[\hat{B}, \hat{A}\hat{B}, \hat{A}^2\hat{B}, ..., \hat{A}^{n-1}\hat{B}\right] = n$$

and the discretised system according to Proposition 17 is said to be controllable. □

Definition 33 *We define $(zI - \hat{A})^{-1}\hat{B}$ as the discretised controllability matrix of a system with ZOH.* □

Case of a system with FOH

A similar analysis is now given for the case of FOH implementation

Proposition 19 *A discretised model $\hat{S}(\hat{A}, \hat{E}, \hat{Z}, \hat{C}, \hat{D})$ of a system with FOH is said to be controllable if and only if,*

$$rank\left[\hat{E},\ \hat{A}\hat{E} + \hat{Z},\ \hat{A}^2\hat{E} + \hat{A}\hat{Z},\ ...,\ \hat{A}^{n-1}\hat{E} + \hat{A}^{n-2}\hat{Z}\right] = n$$

Proof. The difference equations for a discretised model with FOH, are :

$$\begin{aligned} \underline{x}(k+1) &= \hat{A}\underline{x}(k) + \hat{E}\underline{u}(k) + \hat{Z}\underline{u}(k-1) \\ \underline{y}(k) &= \hat{C}\underline{x}(k) + \hat{D}\underline{u}(k) \end{aligned}$$

for $k = 0, 1, 2, ..., n$ correspondingly we have $(\underline{u}(-T) = 0)$:

$$\begin{aligned} \underline{x}(1) &= \hat{A}\underline{x}(0) + \hat{E}\underline{u}(0) \\ \underline{x}(2) &= \hat{A}\underline{x}(1) + \hat{E}\underline{u}(1) + \hat{Z}\underline{u}(0) = \hat{A}^2\underline{x}(0) + \hat{A}\hat{E}\underline{u}(0) + \hat{E}\underline{u}(1) + \hat{Z}\underline{u}(0) \\ & \quad \cdots\cdots\cdots\cdots \\ \underline{x}(n) &= \hat{A}^n\underline{x}(0) + \hat{A}^{n-1}\hat{E}\underline{u}(0) + \hat{A}^{n-2}\hat{E}\underline{u}(1) + ... + \hat{A}\hat{E}\underline{u}(n-2) + \hat{E}\underline{u}(n-1) + \end{aligned}$$

$$+\hat{A}^{n-2}\hat{Z}\underline{u}(0)+\hat{A}^{n-3}\hat{Z}\underline{u}(1)+...+\hat{A}\hat{Z}\underline{u}(n-3)+\hat{Z}\underline{u}(n-2)\Leftrightarrow$$

$$\Leftrightarrow \underline{x}(n)=\hat{A}^n\underline{x}(0)+\sum_{i=0}^{n-1}\hat{A}^{n-1-i}\hat{E}\underline{u}(i)+\sum_{i=0}^{n-2}\hat{A}^{n-2-i}\hat{Z}\underline{u}(i)\Leftrightarrow$$

$$\Leftrightarrow \underline{x}(n)-\hat{A}^n\underline{x}(0)=\sum_{i=0}^{n-2}\left[\hat{A}^{n-1-i}\hat{E}+\hat{A}^{n-2-i}\hat{Z}\right]\underline{u}(i)+\hat{E}\underline{u}(n-1)\Leftrightarrow$$

$$\Leftrightarrow \underline{x}(n)-\hat{A}^n\underline{x}(0)=\left[\hat{E},\ \hat{A}\hat{E}+\hat{Z},\ \hat{A}^2\hat{E}+\hat{A}\hat{Z},\ ...,\ \hat{A}^{n-1}\hat{E}+\hat{A}^{n-2}\hat{Z}\right]\begin{bmatrix}\underline{u}(n-1)\\ \underline{u}(n-2)\\ \cdots\\ \underline{u}(0)\end{bmatrix}$$

For any $\underline{x}(0)$ and any $\underline{x}(n)$ there exists a solution for the system of equations if and only if.

$$rank\left[\hat{E},\ \hat{A}\hat{E}+\hat{Z},\ \hat{A}^2\hat{E}+\hat{A}\hat{Z},\ ...,\ \hat{A}^{n-1}\hat{E}+\hat{A}^{n-2}\hat{Z}\right]=n$$

and Proposition is proved. □

Proposition 20 *The discretised model with FOH is controllable if and only if the* n *rows of* $(zI-\hat{A})^{-1}(z\hat{E}+\hat{Z})$ *are linearly independent over the field of complex numbers.*

Proof. From the binomial expansions in (4.11) we have that

$$\left.\begin{array}{l}(zI-\hat{A})^{-1}(z\hat{E})=\hat{E}+z^{-1}\hat{A}\hat{E}+z^{-2}\hat{A}^2\hat{E}+...\\ (zI-\hat{A})^{-1}\hat{Z}=z^{-1}\hat{Z}+z^{-2}\hat{A}\hat{Z}+z^{-3}\hat{A}^2\hat{Z}+...\end{array}\right\}\Rightarrow$$

$$\Rightarrow (zI-\hat{A})^{-1}(z\hat{E}+\hat{Z})=\hat{E}+z^{-1}\left(\hat{A}\hat{E}+\hat{Z}\right)+z^{-2}\left(\hat{A}^2\hat{E}+\hat{A}\hat{Z}\right)+...$$

and from the Cayley-Hamilton theorem , as in *Proposition 19*, the rows of $(zI-\hat{A})^{-1}(z\hat{E}+\hat{Z})$ are linearly independent over the field of complex numbers if and only if

$$rank\left[\hat{E},\ \hat{A}\hat{E}+\hat{Z},\ \hat{A}^2\hat{E}+\hat{A}\hat{Z},\ ...,\ \hat{A}^{n-1}\hat{E}+\hat{A}^{n-2}\hat{Z}\right]=n$$

and the discretised system according to *Proposition 19* is said to be controllable. □

Definition 34 *We define* $(zI - \hat{A})^{-1}(z\hat{E} + \hat{Z})$ *as the discretised controllability matrix of a system with FOH.* □

4.3.3 Observability of Discretised Models

For the observability property of discretised models we have the following dual to controllability definitions and propositions :

Definition 35 *A discretised system is said to be observable at time* t_0 *if for any initial state* $\underline{x}_0$ *in the space* $\mathbb{R}^n$*,there exists a finite time* $nT > t_0$ *such that knowledge of the sequence of outputs* $\underline{y}(0)$, $\underline{y}(T)$, $\underline{y}(2T)$, ..., $\underline{y}[(n-1)T]$, *as well as inputs* $\underline{u}(0)$, $\underline{u}(T)$,...,$\underline{u}[(n-1)T]$ *over the interval* $[0, nT]$ *suffices to determine the initial state* $\underline{x}_0$. *Otherwise the system is said to be unobservable.* □

In the case of a system with ZOH or FOH we have the following tests:

Proposition 21 *The discretised model* $\hat{S}(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ *of a system with ZOH or the discretised model* $\hat{S}(\hat{A}, \hat{E}, \hat{Z}, \hat{C}, \hat{D})$*of a system with FOH, is observable if and only if :*

$$rank \begin{bmatrix} \hat{C} \\ \hat{C}\hat{A} \\ \hat{C}\hat{A}^2 \\ \cdots \\ \hat{C}\hat{A}^{n-1} \end{bmatrix} = n$$

□

Proposition 22 *The discretised model with ZOH or FOH is observable if and only if the* n *columns of* $\hat{C}(zI - \hat{A})^{-1}$ *are linearly independent over the field of complex numbers.* □

Definition 36 *We define* $\hat{C}(zI - \hat{A})^{-1}$ *as the discretised observability matrix of a system with ZOH or FOH.* □

Remark 14 *From the above we conclude that for the controllability test of a discretised model we have two different types of controllability matrices for ZOH and FOH respectively. This can* *be extended to models with higher than one order H holds i.e. to each order hold corresponds a different type of controllability matrix. On the contrary, for the observability test, the same type of observability matrix is valid for all the orders of DAC holds as both the state matrix $\hat{A}$ and matrix $\hat{C}$ of the discretised model remain unaffected by the order of the H hold.* □

4.4 Jordan form of the discretised matrix $\hat{\mathbf{A}}$

The matrix $\hat{A}$ of the discretised system, after the transformation of the continuous system matrix A to Jordan form (3.17) and (3.18) becomes :

$$\hat{A} = e^{AT} = e^{UJTV} = Ue^{JT}V = U(\text{block diag}\{e^{J(\lambda_1)T}, ..., e^{J(\lambda_i)T}, ..., e^{J(\lambda_f)T}\})V$$

where

$$e^{J(\lambda_i)T} = \text{diag}\{e^{J_{i1}T}, ..., e^{J_{ik}T}, ..., e^{J_{i\nu_i}T}\}$$

and from (3.20) we have that $e^{J_{ik}T}$ is the upper triangular matrix block of the type :

$$e^{J_{ik}T} = \begin{bmatrix} e^{\lambda_i T} & Te^{\lambda_i T} & \dots & \frac{T^{\tau_{ik}-2}e^{\lambda_i T}}{(\tau_{ik}-2)!} & \frac{T^{\tau_{ik}-1}e^{\lambda_i T}}{(\tau_{ik}-1)!} \\ 0 & e^{\lambda_i T} & \dots & \frac{T^{\tau_{ik}-3}e^{\lambda_i T}}{(\tau_{ik}-3)!} & \frac{T^{\tau_{ik}-2}e^{\lambda_i T}}{(\tau_{ik}-2)!} \\ . & . & \dots & . & . \\ 0 & 0 & \dots & e^{\lambda_i T} & Te^{\lambda_i T} \\ 0 & 0 & \dots & 0 & e^{\lambda_i T} \end{bmatrix} \in \mathbb{C}^{\tau_{ik}\times\tau_{ik}} \tag{4.12}$$

The characteristic polynomial of the above upper triangular type matrix is given by

$$\det\left(zI_{ik} - e^{J_{ik}T}\right) = \left(z - e^{\lambda_i T}\right)^{\tau_{ik}}$$

and so $e^{\lambda_i T}$ is the only distinct eigenvalue of matrix $e^{J_{ik}T}$, with algebraic multiplicity τ_{ik}.

Then, if we define,

$$\hat{\lambda}_i \triangleq e^{\lambda_i T} \tag{4.13}$$

we have :

$$e^{J_{ik}T} - \hat{\lambda}_i I_{ik} = \begin{bmatrix} 0 & Te^{\lambda_i T} & \frac{T^2 e^{\lambda_i T}}{2!} & \dots & \frac{T^{\tau_{ik}-1} e^{\lambda_i T}}{(\tau_{ik}-1)!} \\ 0 & 0 & Te^{\lambda_i T} & \dots & \frac{T^{\tau_{ik}-2} e^{\lambda_i T}}{(\tau_{ik}-2)!} \\ . & . & . & \dots & . \\ 0 & 0 & 0 & \dots & Te^{\lambda_i T} \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \in \mathbb{C}^{\tau_{ik} \times \tau_{ik}}$$

Given that $Te^{\lambda_i T} \neq 0$ for every value of the time period $T > 0$, the rank of the above upper triangular matrix is :

$$rank[e^{J_{ik}T} - \hat{\lambda}_i I] = \tau_{ik} - 1$$

or the rank deficiency (geometric multiplicity of $\hat{\lambda}_i$) of the matrix block $e^{J_{ik}T}$ at $\hat{\lambda}_i$ is 1.

Thus we conclude that for every $T > 0$, the matrix block $e^{J_{ik}T}$ has only one distinct eigenvalue $\hat{\lambda}_i = e^{\lambda_i T}$ and only one real eigenvector. If this eigenvector is $\underline{\tilde{u}}_{ik}$, then a chain of generalized eigenvectors is defined by the equations :

$$\left[e^{J_{ik}T} - \hat{\lambda}_i I\right] \underline{\tilde{u}}_{ik} = \underline{0} \quad \Leftrightarrow \quad \underline{\tilde{u}}_{ik1} \triangleq \underline{\tilde{u}}_{ik}$$

$$\left[e^{J_{ik}T} - \hat{\lambda}_i I\right] \underline{\tilde{u}}_{ik2} = \underline{\tilde{u}}_{ik1} \quad \Leftrightarrow \quad \left[e^{J_{ik}T} - \hat{\lambda}_i I\right]^2 \underline{\tilde{u}}_{ik2} = \underline{0}$$

$$\dots \quad \dots \quad \dots$$

$$\left[e^{J_{ik}T} - \hat{\lambda}_i I\right] \underline{\tilde{u}}_{ik\tau_{ik}} = \underline{\tilde{u}}_{ik\tau_{ik}-1} \quad \Leftrightarrow \quad \left[e^{J_{ik}T} - \hat{\lambda}_i I\right]^{\tau_{ik}} \underline{\tilde{u}}_{ik\tau_{ik}} = \underline{0}$$

and the matrix block $e^{J_{ik}T}$ can be transformed to a Jordan matrix $\hat{J}_{ik}$ as shown below,

$$e^{J_{ik}T} = \tilde{U}_{ik} \hat{J}_{ik} \tilde{V}_{ik} \tag{4.14}$$

where :

$$\hat{J}_{ik} = \begin{bmatrix} e^{\lambda_i T} & 1 & \dots & 0 & 0 \\ 0 & e^{\lambda_i T} & \dots & 0 & 0 \\ . & . & \dots & . & . \\ 0 & 0 & \dots & e^{\lambda_i T} & 1 \\ 0 & 0 & \dots & 0 & e^{\lambda_i T} \end{bmatrix} \in \mathbb{C}^{\tau_{ik} \times \tau_{ik}} \tag{4.15}$$

and $\tilde{U}_{ik}$ is a $\tau_{ik} \times \tau_{ik}$ matrix defined by the eigenvector $\tilde{\underline{u}}_{ik}$ and the corresponding chain of generalized eigenvectors:

$$\tilde{U}_{ik} = \left[\tilde{\underline{u}}_{ik} \triangleq \tilde{\underline{u}}_{ik1}, \tilde{\underline{u}}_{ik2}, ..., \tilde{\underline{u}}_{ik\tau_{ik}} \right] \tag{4.16}$$

so we have :

Proposition 23 *For every value of the sampling period T, the matrix block $e^{J_{ik}T}$ given by (4.14) has only one distinct eigenvalue $\hat{\lambda}_i = e^{\lambda_i T}$, of geometric multiplicity 1. Hence, a matrix block $e^{J_{ik}T}$ can be transformed to a Jordan matrix $\hat{J}_{ik}$ formed by only one Jordan block of dimensions $\tau_{ik} \times \tau_{ik}$.* □

We examine next the nature of the eigenchains of $e^{J_{ik}T}$. By inspection of matrix $e^{J_{ik}T} - \hat{\lambda}_i I$ we conclude that the solution of the matrix equation,

$$\left[e^{J_{ik}T} - \hat{\lambda}_i I \right] \tilde{\underline{u}}_{ik1} = \underline{0}$$

for every value of the sampling period $T > 0$ is the $\tau_{ik} \times 1$ vector:

$$\tilde{\underline{u}}_{ik} \triangleq \tilde{\underline{u}}_{ik1} = [1, 0, ..., 0]^{\top}$$

Then we determine the $\tau_{ik} \times 1$ vector $\tilde{\underline{u}}_{ik2}$ from the equation:

$$\left[e^{J_{ik}T} - \hat{\lambda}_i I \right] \tilde{\underline{u}}_{ik2} = \tilde{\underline{u}}_{ik1} \Leftrightarrow \tilde{\underline{u}}_{ik2} = \left[0, \frac{e^{-\lambda_i T}}{T}, 0, ..., 0 \right]^{\top}$$

and also from the equations:

$$\left[e^{J_{ik}T} - \hat{\lambda}_i I \right] \tilde{\underline{u}}_{ik3} = \tilde{\underline{u}}_{ik2} \Leftrightarrow \tilde{\underline{u}}_{ik3} = \left[0, -\frac{e^{-2\lambda_i T}}{2T}, \frac{e^{-2\lambda_i T}}{T^2}, 0, ..., 0 \right]^{\top}$$

$$\left[e^{J_{ik}T} - \hat{\lambda}_i I \right] \tilde{\underline{u}}_{ik4} = \tilde{\underline{u}}_{ik3} \Leftrightarrow \tilde{\underline{u}}_{ik4} = \left[0, \frac{e^{-3\lambda_i T}}{3T}, -\frac{e^{-3\lambda_i T}}{T^2}, \frac{e^{-3\lambda_i T}}{T^3}, 0, ..., 0 \right]^{\top}$$

It is thus clear that we can determine any number of generalized eigenvectors following the above procedure. In general the generalized eigenvector $\tilde{\underline{u}}_{ikj}$ has its first, $(j-1)$-*th* and j-*th* to τ_{ik}-*th* entries as follows:

$$\underline{\tilde{u}}_{ikj} = \left[0, \#, ..., \#, \frac{-(j-2)e^{-(j-1)\lambda_i T}}{2T^{j-2}}, \frac{e^{-(j-1)\lambda_i T}}{T^{j-1}}, 0, ..., 0\right]^T$$

where by # denotes a nonzero entry. The generalized eigenvector $\underline{\tilde{u}}_{ik(j+1)}$ is defined by the equation :

$$\left[e^{J_{ik}T} - \hat{\lambda}_i I\right] \underline{\tilde{u}}_{ik(j+1)} = \underline{\tilde{u}}_{ikj} \Leftrightarrow$$

$$\Leftrightarrow \underline{\tilde{u}}_{ik(j+1)} = \left[0, \#, ..., \#, \frac{-(j-1)e^{-j\lambda_i T}}{2T^{j-1}}, \frac{e^{-j\lambda_i T}}{T^j}, 0, ..., 0\right]^T$$

We have thus proved that the $\tau_{ik} \times \tau_{ik}$ transformation matrix $\tilde{U}_{ik}$ of the matrix block $e^{J_{ik}T}$, has the following upper triangular form :

$$\tilde{U}_{ik} = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & \frac{e^{-\lambda_i T}}{T} & -\frac{e^{-2\lambda_i T}}{2T} & \frac{e^{-3\lambda_i T}}{3T} & \dots & \# & \# \\ 0 & 0 & \frac{e^{-2\lambda_i T}}{T^2} & -\frac{e^{-3\lambda_i T}}{T^2} & \dots & \# & \# \\ 0 & 0 & 0 & \frac{e^{-3\lambda_i T}}{T^3} & \dots & \# & \# \\ . & . & . & . & \dots & . & . \\ 0 & 0 & 0 & 0 & \dots & \frac{e^{-(\tau_{ik}-2)\lambda_i T}}{T^{\tau_{ik}-2}} & \frac{-(\tau_{ik}-2)e^{-(\tau_{ik}-1)\lambda_i T}}{2T^{\tau_{ik}-2}} \\ 0 & 0 & 0 & 0 & \dots & 0 & \frac{e^{-(\tau_{ik}-1)\lambda_i T}}{T^{\tau_{ik}-1}} \end{bmatrix} \tag{4.17}$$

Proposition 24 *The matrix $e^{J_{ik}T}$ has only one set of generalized eigenvectors associated with the eigenvalue $\hat{\lambda}_i = e^{\lambda_i T}$, which form the columns of the triangular matrix $\tilde{U}_{ik}$ given by (4.17).*□

From the previous analysis it follows that the diagonal matrix $e^{J(\lambda_i)T}$ can be transformed to the Jordan form matrix $\hat{J}(\hat{\lambda}_i)$ by the transformation :

$$e^{J(\lambda_i)T} = \tilde{U}(\lambda_i)\hat{J}(\hat{\lambda}_i)\tilde{V}(\lambda_i)$$

where the transforming matrix $\tilde{U}(\lambda_i)$ is also an upper diagonal matrix:

$$\tilde{U}(\lambda_i) = \text{block diag}\{\tilde{U}_{i1}, ..., \tilde{U}_{ik}, ..., \tilde{U}_{i\nu_i}\} \tag{4.18}$$

and where $\hat{J}(\hat{\lambda}_i)$ is a Jordan matrix formed by the Jordan blocks associated with the distinct eigenvalue $\hat{\lambda}_i$:

$$\hat{J}(\hat{\lambda}_i) = \text{block diag}\{\hat{J}_{i1}, ..., \hat{J}_{ik}, ..., \hat{J}_{i\nu_i}\}$$

Then the diagonal matrix e^{JT}, can also be transformed to the Jordan form matrix $\hat{J}$, by the transformation :

$$e^{JT} = \tilde{U}\hat{J}\tilde{V}$$

where :

$$\tilde{U} = \text{block diag}\{\tilde{U}(\lambda_1), ..., \tilde{U}(\lambda_i), ..., \tilde{U}(\lambda_f)\}$$

and where :

$$\hat{J} = \{\hat{J}(\lambda_1), ..., \hat{J}(\lambda_i), ..., \hat{J}(\lambda_f)\}$$

From the above we have :

$$\hat{A} = e^{AT} = e^{UJTV} = Ue^{JT}V = U\tilde{U}\hat{J}\tilde{V}V$$

and if we define as $\hat{U} \triangleq U\tilde{U}$, $\hat{V} \triangleq \tilde{V}V$ then:

$$\hat{A} = \hat{U}\hat{J}\hat{V}$$

So, we have proved that there exists a transformation matrix $\hat{U}$ which transforms the discretised matrix $\hat{A}$ to a Jordan form matrix $\hat{J}$. The transformation matrix $\hat{U}$ is :

$$\begin{aligned}\hat{U} &= U\tilde{U} = [U(\lambda_1), ..., U(\lambda_i), ..., U(\lambda_f)] \text{ block diag}\left\{\tilde{U}(\lambda_1), ..., \tilde{U}(\lambda_i), ..., \tilde{U}(\lambda_f)\right\} = \\ &= \left[U(\lambda_1)\tilde{U}(\lambda_1), ..., U(\lambda_i)\tilde{U}(\lambda_i), ..., U(\lambda_f)\tilde{U}(\lambda_f)\right] = \left[\hat{U}(\lambda_1), ..., \hat{U}(\lambda_i), ..., \hat{U}(\lambda_f)\right]\end{aligned}$$

where $\hat{U}_i(\lambda_i) \triangleq U(\lambda_i)\tilde{U}(\lambda_i)$, is a $\pi_i \times n$ matrix as the product of multiplication, of the $\pi_i \times n$ matrix $U(\lambda_i)$ and the $\pi_i \times n$ block diagonal matrix $\tilde{U}(\lambda_i)$. Furthermore from (3.15) and (4.18) we have :

$$\hat{U}_i = U(\lambda_i)\tilde{U}(\lambda_i) = [U_{i1}, ..., U_{ik}, ..., U_{i\nu_i}] \text{ block diag}\left\{\tilde{U}_{i1}, ..., \tilde{U}_{ik}, ..., \tilde{U}_{i\nu_i}\right\} =$$

$$= \left\{U_{i1}\tilde{U}_{i1}, ..., U_{ik}\tilde{U}_{ik}, ..., U_{i\nu_i}\tilde{U}_{i\nu_i}\right\}$$

and from (3.16) and (4.16) we have:

$$\begin{aligned} \hat{U}_{ik} &= U_{ik}\tilde{U}_{ik} = \left[\underline{u}_{ik1}, \underline{u}_{ik2}, ..., \underline{u}_{ik\tau_{ik}}\right]\left[\tilde{\underline{u}}_{ik1}, \tilde{\underline{u}}_{ik2}, ..., \tilde{\underline{u}}_{ik\tau_{ik}}\right] = \\ &= \left[\hat{\underline{u}}_{ik1}, \hat{\underline{u}}_{ik2}, ..., \hat{\underline{u}}_{ik\tau_{ik}}\right] \end{aligned}$$

where $\hat{\underline{u}}_{ik1} \triangleq \hat{\underline{u}}_{ik}$ is an eigenvector of $\hat{A}$ associated with the eigenvalue $\hat{\lambda}_i$ and $\hat{\underline{u}}_{ik1}$, $\hat{\underline{u}}_{ik2}$, ..., $\hat{\underline{u}}_{ik\tau_{ik}}$ is a chain of generalized eigenvectors, satisfying the equations:

$$(\hat{A} - \hat{\lambda}_i I)\hat{\underline{u}}_{ik1} = \underline{0}, \quad (\hat{A} - \hat{\lambda}_i I)\hat{\underline{u}}_{ik2} = \hat{\underline{u}}_{ik1}, \quad ..., \quad (\hat{A} - \hat{\lambda}_i I)\hat{\underline{u}}_{ik\tau_{ik}} = \hat{\underline{u}}_{ik\tau_{ik}-1}$$

From (4.17) and the above definition of matrix $\tilde{U}_{ik}$ we have:

$$\begin{aligned} \hat{\underline{u}}_{ik1} &= U_{ik}\tilde{\underline{u}}_{ik1} = \underline{u}_{ik1} \\ \hat{\underline{u}}_{ik2} &= U_{ik}\tilde{\underline{u}}_{ik2} = \frac{e^{-\lambda_i T}}{T}\underline{u}_{ik2} \\ \hat{\underline{u}}_{ik3} &= U_{ik}\tilde{\underline{u}}_{ik3} = \frac{e^{-2\lambda_i T}}{2T}\underline{u}_{ik2} + \frac{e^{-2\lambda_i T}}{T^2}\underline{u}_{ik3} \\ \hat{\underline{u}}_{ik4} &= U_{ik}\tilde{\underline{u}}_{ik4} = \frac{e^{-3\lambda_i T}}{3T}\underline{u}_{ik2} - \frac{e^{-3\lambda_i T}}{T^2}\underline{u}_{ik3} + \frac{e^{-3\lambda_i T}}{T^3}\underline{u}_{ik4} \end{aligned}$$

and it is possible to determine any number of generalized eigenvectors following this procedure. From the above and from the linear independence of the generalized eigenvectors we can conclude that

$$\text{span}\left[\underline{u}_{ik1}, \underline{u}_{ik2}, ..., \underline{u}_{ik\tau_{ik}}\right] = \text{span}\left[\hat{\underline{u}}_{ik1}, \hat{\underline{u}}_{ik2}, ..., \hat{\underline{u}}_{ik\tau_{ik}}\right]$$

Every chain of the above generalized eigenvectors forms a basis for the invariant and cyclic relative to A and $\hat{A}$ subspaces V_{ik} and $\widehat{\mathcal{V}}_{\tau_{ik}}$ So we conclude that $\mathcal{V}_{ik} \equiv \widehat{\mathcal{V}}_{ik}$. From the above we have the following result :

Theorem 26 *For every value of the sampling period T, the following properties hold true:*

1. *To every distinct eigenvalue λ_i of A corresponds the eigenvalue $\hat{\lambda}_i = e^{\lambda_i T}$ of $\hat{A}$.*

2. *A and $\hat{A}$ have the same eigenvectors.*

3. *To every chain of generalized eigenvectors of A associated with the distinct eigenvalue λ_i, there corresponds a chain of the same length of generalized eigenvectors of $\hat{A}$ associated with the not necessarily distinct eigenvalue $\hat{\lambda}_i = e^{\lambda_i T}$.*

4. *Each A-invariant and cyclic relative to A subspace V_{ik} is also $\hat{A}$-invariant and cyclic relative to $\hat{A}$.*

5. *To each Jordan block, of the Jordan form J of A, associated with the eigenvalue λ_i there corresponds a Jordan block of the same dimension of the Jordan form matrix $\hat{J}$ of $\hat{A}$. associated with the eigenvalue $\hat{\lambda}_i$.*

6. *To every elementary divisor $(s-\lambda_i)^{\tau_{ik}}$ of A, there corresponds the elementary divisor $\left(z-\hat{\lambda}_i\right)^{\tau_{ik}}$ of $\hat{A}$.* □

Because the eigenvalues $\hat{\lambda}_1, \hat{\lambda}_2, ... \hat{\lambda}_f$ of $\hat{A}$ may be not distinct, for reasons referred to as eigenvalue collapsing and which will be examined in the following Chapter 5, we cannot say that for every value of the sampling period T, to the generalized null-space $\mathcal{N}_{ij}$ of A corresponds a generalized null-space $\widehat{\mathcal{N}}_{ij}$ of the same dimension of $\hat{A}$, since merging for such spaces may occur. For the same reason, the Segré characteristic and the index of cyclicity of $\hat{A}$ for every value of the sampling period T cannot be defined for any value of sampling, but needs special attention.

We return now to the previous relations in order to define the inverse transformation matrix $\hat{V}(\lambda_i)$. In fact,

$$\tilde{V}_{ik} = \left(\tilde{U}_{ik}\right)^{-1} = \begin{bmatrix} \underline{\tilde{v}}_{ik1}^{\top} \\ \underline{\tilde{v}}_{ik2}^{\top} \\ \cdots \\ \underline{\tilde{v}}_{ik\tau_{ik}}^{\top} \end{bmatrix}$$

where $\underline{\tilde{v}}_{ik\tau_{ik}}^{\top}$ is a dual eigenvector and $\underline{\tilde{v}}_{ik1}^{\top}, \underline{\tilde{v}}_{ik2}^{\top}, ..., \underline{\tilde{v}}_{ik\tau_{ik}-1}^{\top}$ are dual generalized eigenvectors, defined by the equations:

$$\underline{\tilde{v}}_{ik1}^{\top}(e^{J_iT}-\hat{\lambda}_i I) = \underline{\tilde{v}}_{ik2}^{\top},\ \underline{\tilde{v}}_{ik2}^{\top}(e^{J_iT}-\hat{\lambda}_i I) = \underline{\tilde{v}}_{ik3}^{\top},\ ...,\ \underline{\tilde{v}}_{ik\tau_{ik}}^{\top}(e^{J_iT}-\hat{\lambda}_i I) = \underline{0}$$

Also we have:

$$\tilde{V}_{ik}\tilde{U}_{ik} = \begin{bmatrix} \underline{\tilde{v}}_{ik1}^{\top} \\ \underline{\tilde{v}}_{ik2}^{\top} \\ \dots \\ \underline{\tilde{v}}_{ik\tau_{ik}}^{\top} \end{bmatrix} \left[\underline{\tilde{u}}_{ik1}, \underline{\tilde{u}}_{ik2}, \dots, \underline{\tilde{u}}_{ik\tau_{ik}}\right] = I_{\tau_{ik}}$$

or :

$$\underline{\tilde{v}}_{ikh}^{\top}\underline{\tilde{u}}_{ikj} = \delta_{hj} \text{ where, } \begin{cases} \delta_{hj} = 0 \text{ for } h \neq j \\ \delta_{hj} = 1 \text{ for } h = j \end{cases}$$

From the above we conclude that :

$$\underline{\tilde{v}}_{ik1}^{\top} = [1, 0, \dots, 0] \text{ and } \underline{\tilde{v}}_{ik\tau_{ik}}^{\top} = \left[0, \dots, 0, T^{\tau_{ik}-1}e^{(\tau_{ik}-1)\lambda_i T}\right]$$

and thus in general

$$\underline{\tilde{v}}_{ikj}^{\top} = \left[0, \dots, 0, T^{j-1}e^{(j-1)\lambda_i T}, \frac{(j-1)T^j e^{(j-1)\lambda_i T}}{2}, \#, \dots \#\right]$$

From the above we conclude that matrix $\tilde{V}_{ik} = (\tilde{U}_{ik})^{-1}$ has the following upper triangular form:

$$\tilde{V}_{ik} = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & Te^{\lambda_i T} & \frac{T^2 e^{\lambda_i T}}{2} & \# & \dots & \# & \# \\ 0 & 0 & T^2 e^{2\lambda_i T} & T^3 e^{2\lambda_i T} & \dots & \# & \# \\ 0 & 0 & 0 & T^3 e^{3\lambda_i T} & \dots & \# & \# \\ . & . & . & . & \dots & . & . \\ 0 & 0 & 0 & 0 & \dots & T^{(\tau_{ik}-2)}e^{(\tau_{ik}-2)\lambda_i T} & \frac{(\tau_{ik}-2)T^{(\tau_{ik}-1)}e^{(\tau_{ik}-2)\lambda_i T}}{2} \\ 0 & 0 & 0 & 0 & \dots & 0 & T^{(\tau_{ik}-1)}e^{(\tau_{ik}-1)\lambda_i T} \end{bmatrix} \tag{4.19}$$

The matrix $e^{J_{ik}T}$ has only one set of generalized dual eigenvectors, associated with the eigenvalue $\hat{\lambda}_i = e^{\lambda_i T}$, which form the rows of the above triangular matrix $\tilde{V}_{ik}$.

4.5 Jordan equivalent equations of a discretised system

4.5.1 Case of a system with ZOH

Consider the system $S(A, B, C, D)$, described in the time domain by the equations (3.52) and (3.53). If $U = V^{-1}$ is the matrix defined by the chains of eigenvectors of A, the Jordan canonical description of the system $S_J(J, \mathcal{B}, \Gamma, \Delta)$ is given by the equations

$$\dot{\underline{z}}(t) = J\underline{z}(t) + \mathcal{B}\underline{u}(t) \tag{4.20}$$

$$\underline{y}(t) = \Gamma\underline{z}(t) + \Delta\underline{u}(t) \tag{4.21}$$

where :

$$\underline{z}(t) = U\underline{x}(t),\ J = U^{-1}AU = VAU,\ \mathcal{B} = U^{-1}B,\ \Gamma = CU,\ \Delta = D$$

We have seen before that the state-space description of a discretised model $\hat{S}(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ of a system $S(A, B, C, D)$, with ZOH and sampling period T, is given by the equations:

$$\underline{x}[(k+1)T] = \hat{A}\underline{x}(kT) + \hat{B}\underline{u}(kT)$$

$$\underline{y}(kT) = \hat{C}\underline{x}(kT) + \hat{D}\underline{u}(kT)$$

$$\text{where:} \quad \hat{A} = e^{AT}, \quad \hat{B} = \left(\int_0^T e^{A\sigma}d\sigma\right)B, \quad \hat{C} = C, \quad \hat{D} = D$$

Defining a new basis for $\hat{A}$ as the eigenbasis matrix $\hat{U}$, then the state space description becomes:

$$\underline{z}\left[(k+1)\,T\right] = \hat{J}\underline{z}(kT) + \widehat{\mathcal{B}}\underline{u}(kT) \tag{4.22}$$

$$\underline{y}(kT) = \hat{\Gamma}\underline{z}(kT) + \hat{\Delta}\underline{u}(kT) \tag{4.23}$$

where :

$$\underline{z}(kT) = \hat{U}\underline{x}(kT),\ \hat{J} = \hat{U}^{-1}\hat{A}\hat{U} = \hat{V}\hat{A}\hat{U} \tag{4.24}$$

$$\widehat{\mathcal{B}} = \hat{U}^{-1}\hat{B} = \hat{V}(\int_0^T e^{J\sigma}d\sigma)B = \hat{V}\Xi\mathcal{B} \tag{4.25}$$

$$\Xi \triangleq \int_0^T e^{J\sigma} d\sigma \tag{4.26}$$

$$\hat{\Gamma} = \hat{C}\hat{U} = C\hat{U} = \Gamma\tilde{U}, \ \hat{\Delta} = \hat{D} = D \tag{4.27}$$

From (3.16) we have :

$$\Xi = \text{block diag}\left\{\int_0^T e^{J(\lambda_1)\sigma} d\sigma, ..., \int_0^T e^{J(\lambda_i)\sigma} d\sigma, ..., \int_0^T e^{J(\lambda_f)\sigma} d\sigma\right\}$$

where,

$$\int_0^T e^{J_i\sigma} d\sigma = \text{block diag}\left\{\int_0^T e^{J_{\tau_{i1}}\sigma} d\sigma, ..., \int_0^T e^{J_{\tau_{ik}}\sigma} d\sigma, ..., \int_0^T e^{J_{\tau_{i\nu_i}}\sigma} d\sigma\right\}$$

For the above integration we have to distinguish two cases, one for $\lambda_i = 0$ and one for $\lambda_i \neq 0$. The following results are readily established :

Lemma 3 *Let $\tau_{11}, ..., \tau_{1k}, ..., \tau_{1\nu_1}$ the dimensions of the Jordan blocks of J associated with the eigenvalue $\lambda_1 = 0$. Then we have,*

$$\int_0^T e^{J(0)\sigma} d\sigma = \text{block diag}\left\{\int_0^T e^{J_{\tau_{11}}\sigma} d\sigma, ..., \int_0^T e^{J_{\tau_{1k}}\sigma} d\sigma, ..., \int_0^T e^{J_{\tau_{1\nu_1}}\sigma} d\sigma\right\}$$

where,

$$P_{\tau_{1k}}(0, T) \triangleq \int_0^T e^{J_{\tau_{1k}}\sigma} d\sigma = \begin{bmatrix} T & \frac{T^2}{2!} & \frac{T^3}{3!} & \cdots & \frac{T^{\tau_{1k}}}{\tau_{1\nu_1}!} \\ 0 & T & \frac{T^2}{2!} & \cdots & \frac{T^{\tau_{1k}-1}}{(\tau_{1k}-1)!} \\ . & . & . & \cdots & . \\ 0 & 0 & 0 & \cdots & \frac{T^2}{2!} \\ 0 & 0 & 0 & \cdots & T \end{bmatrix}$$

□

Lemma 4 *Let $\tau_{i1}, ..., \tau_{ik}, ..., \tau_{i\nu_i}$ be the dimensions of the Jordan blocks of J associated with the* *non zero eigenvalue* λ_i, *then*

$$\int_0^T e^{J(\lambda_i)\sigma} d\sigma = \text{block diag}\left\{ \int_0^T e^{J_{\tau_{i1}}\sigma} d\sigma, ..., \int_0^T e^{J_{\tau_{ik}}\sigma} d\sigma, ..., \int_0^T e^{J_{\tau_{i\nu_i}}\sigma} d\sigma \right\}$$

where :

$$P_{\tau_{ik}}(\lambda_i, T) \triangleq \int_0^T e^{J_{\tau_{ik}}\sigma} d\sigma = \left(e^{J_{\tau_{ik}}T} - I_{\tau_{ik}}\right)\left(J_{\tau_{ik}}\right)^{-1}$$

□

From the above two Lemmas 3 and 4 the following Proposition is directly concluded :

Proposition 25 *With the notation previously introduced we have that*

$$\Xi = \text{block diag}\left\{P_1, ..., P_i, ..., P_f\right\}$$

where :

$$\begin{aligned} P_1 &= \text{block diag}\left\{P_{\tau_{11}}(0,T), ..., P_{\tau_{ik}}(0,T), ..., P_{\tau_{1\nu_1}}(0,T)\right\} \\ P_i &= \text{block diag}\left\{P_{\tau_{i1}}(\lambda_i,T), ..., P_{\tau_{ik}}(\lambda_i,T), ..., P_{\tau_{i\nu_i}}(\lambda_i,T)\right\} \ (i = 2, ..., f) \end{aligned}$$

□

From the above two Lemmas 3 and 4 and Proposition 25 the following Theorem is directly concluded :

Theorem 27 *For every value of the sampling period* $T > 0$, *the matrix* Ξ *is a block diagonal with the same structure of diagonal blocks as the* J *matrix. In particular:*

1. *For* $\lambda_i = 0$ *the elements of the main diagonal are equal to* T.
2. *For* $\lambda_i \neq 0$ *the elements of the main diagonal are equal to* $(e^{\lambda_i T} - 1)\lambda_i^{-1}$.
3. Ξ *is a non singular matrix.* □

4.5.2 Case of a system with FOH

We have seen in the previous section that the state-space description of a discretised model of a system $S(A, B, C, D)$, with FOH and sampling period T is given by the equations :

$$\begin{aligned} \underline{x}(kT+T) &= \hat{A}\underline{x}(kT) + \hat{E}\underline{u}(kT) + \hat{Z}\underline{u}(kT-T) \\ \underline{y}(kT) &= \hat{C}\underline{x}(kT) + \hat{D}\underline{u}(kT) \end{aligned}$$

where

$$\hat{A} = e^{AT},\ \hat{E} = \int_0^T \left(2 - \frac{\sigma}{T}\right) e^{A\sigma} d\sigma B,\ \hat{Z} = -\frac{1}{T}\int_0^T \sigma e^{A\sigma} d\sigma B,\ \hat{C} = C,\ \hat{D} = D$$

Defining a new basis for $\hat{A}$ as the eigenbasis matrix $\hat{U}$, introduced by (3.43), the state space description becomes :

$$z[(k+1)T] = \hat{J}z(kT) + \hat{\mathcal{E}}u(kT) + \hat{\mathcal{Z}}u(kT-T) \tag{4.28}$$

$$y(kT) = \hat{\Gamma}z(kT) + \hat{\Delta}u(kT) \tag{4.29}$$

where : $\underline{z}(kT) = \hat{U}\underline{x}(kT),\ \hat{J} = \hat{U}^{-1}\hat{A}\hat{U} = \hat{V}\hat{A}\hat{U}$

$$\hat{\mathcal{E}} = \hat{U}^{-1}\hat{E} = \hat{V}\int_0^T \left(2 - \frac{\sigma}{T}\right) e^{A\sigma} d\sigma B \tag{4.30}$$

$$\hat{\mathcal{Z}} = \hat{U}^{-1}\hat{Z} = -\hat{V}\frac{1}{T}\int_0^T \sigma e^{A\sigma} d\sigma B \tag{4.31}$$

$$\hat{\Gamma} = \hat{C}\hat{U} = C\hat{U} = \Gamma\tilde{U}, \qquad \hat{\Delta} = \hat{D} = D \tag{4.32}$$

If, we set

$$\Xi \triangleq \int_0^T e^{J\sigma} d\sigma, \qquad \Sigma \triangleq \frac{1}{T}\int_0^T \sigma e^{A\sigma} d\sigma \tag{4.33}$$

then,

$$\hat{\mathcal{E}} = \hat{V}(2\Xi - \Sigma)\mathcal{B}, \qquad \hat{\mathcal{Z}} = \hat{V}\Sigma\mathcal{B}$$

Analytical expressions for Ξ have been derived in the previous section for the ZOH. So we have the corresponding expressions for Σ :

$$\Sigma = \frac{1}{T}\text{block diag}\left\{\int_0^T \sigma e^{J(\lambda_1)\sigma}d\sigma, ..., \int_0^T \sigma e^{J(\lambda_i)\sigma}d\sigma, ..., \int_0^T \sigma e^{J(\lambda_f)\sigma}d\sigma\right\}$$

$$\int_0^T \sigma e^{J_i\sigma}d\sigma = \text{block diag}\left\{\int_0^T e^{J_{\tau_{i1}}\sigma}d\sigma, ..., \int_0^T e^{J_{\tau_{ik}}\sigma}d\sigma, ..., \int_0^T e^{J_{\tau_{i\nu_i}}\sigma}d\sigma\right\}$$

For the integration of Σ, as for the integration of Ξ in the previous paragraph, we have to distinguish two cases, one for $\lambda_i = 0$ and one for $\lambda_i \neq 0$ .The following results are readily established :

Lemma 5 *Let $\tau_{11}, ..., \tau_{1k}, ..., \tau_{1\nu_1}$ be the dimensions of the Jordan blocks of J associated with the eigenvalue $\lambda_1 = 0$, then we have :*

$$\int_0^T \sigma e^{J(\lambda_1)\sigma}d\sigma = \text{block diag}\left\{Q_{\tau_{11}}(0,T), ..., Q_{\tau_{1k}}(0,T), ..., Q_{\tau_{1\nu_1}}(0,T)\right\}$$

$$Q_{\tau_{1k}}(0,T) \triangleq \begin{bmatrix} \frac{T^2}{2} & \frac{T^3}{1!3} & \cdots & \frac{T^{(\tau_{1k}+1)}}{(\tau_{1k}-1)!(\tau_{1k}+1)} \\ 0 & \frac{T^2}{2} & \cdots & \frac{T^{\tau_{1k}}}{(\tau_{1k}-2)!\tau_{1k}} \\ . & . & \cdots & . \\ 0 & 0 & \cdots & \frac{T^2}{2} \end{bmatrix}$$

□

Lemma 6 *Let $\tau_{i1}, ..., \tau_{ik}, ..., \tau_{i\nu_i}$ the dimensions of the Jordan blocks of J associated with the eigenvalue $\lambda_i \neq 0$, then*

$$\int_0^T \sigma e^{J(\lambda_i)\sigma}d\sigma = \text{block diag}\left\{Q_{\tau_{ii}}(\lambda_i,T), ..., Q_{\tau_{ik}}(\lambda_i,T), ..., Q_{\tau_{i\nu_i}}(\lambda_i,T)\right\}$$

$$Q_{\tau_{ik}}(\lambda_i,T) \triangleq \left(e^{J_{ik}T}\left(J_{ik} - I_{\tau_{ik}}\right) + I_{\tau_{ik}}\right)\left(J_{ik}\right)^{-2}$$

□

From the above two Lemmas 5 and 6 we have :

Proposition 26 *With the notation previously introduced we have :*

$$\Sigma = \frac{1}{T}\text{block diag}\left\{Q_1, ..., Q_i, ..., Q_f\right\}$$

where :

$$Q_1 = \text{block diag}\left\{Q_{\tau_{11}}(0,T), ..., Q_{\tau_{ik}}(0,T), ..., Q_{\tau_{1\nu_1}}(0,T)\right\}$$

$$Q_i = \text{block diag}\left\{Q_{\tau_{i1}}(\lambda_i,T), ..., Q_{\tau_{ik}}(\lambda_i,T), ..., Q_{\tau_{i\nu_i}}(\lambda_i,T)\right\}, \ (i = 2, ..., f)$$

□

From the above two Lemmas 5 and 6 and Proposition 26 we have :

Theorem 28 *For every value of the sampling period $T > 0$, matrix Σ is block diagonal matrix with the same structure as the structure of the diagonal matrix J. In particular :*

1. *For $\lambda_i = 0$ the elements of the main diagonal are equal to $\frac{T^2}{2}$.*
2. *For $\lambda_i \neq 0$ the elements of the main diagonal are equal to $\left(e^{\lambda_i T}(\lambda_i - 1) + 1\right)\lambda_i^{-2}$.*
3. *Σ is a non singular matrix.* □

4.6 Eigenvalue Collapsing

4.6.1 Introduction

From the derivation of the discretised parameters it is evident that for any $\lambda_i = \sigma + j\omega \in \Phi(A)$, then $\hat{\lambda}_i = e^{\lambda_i T} = e^{\sigma T + j\omega T}$ is an eigenvalue of $\hat{A} = e^{AT}$ i.e. $\hat{\lambda}_i \in \Phi(\hat{A})$. However, for two distinct eigenvalues $\lambda_1, \lambda_2 \in \Phi(A)$ there may be values of T such that $e^{\lambda_1 T} = e^{\lambda_2 T} = \hat{\lambda}_c \in \Phi(\hat{A})$. This phenomenon, where distinct elements of $\Phi(A)$ are mapped to one element of $\Phi(\hat{A})$ is called *collapsing of eigenvalues* and any value of T for which such phenomena occur will be referred to as *irregular sampling.* All values of T for which any $\lambda_1, \lambda_2 \in \Phi(A), \lambda_1 \neq \lambda_2$ is mapped to $\hat{\lambda}_1 \neq \hat{\lambda}_2$ will be *regular sampling.* The property of eigenvalue collapsing may occur not only on a pair of (λ_1, λ_2), but on a subset of $\Phi(A)$: $\mathcal{L}(A) = \{\lambda_i \in \Phi(A), \lambda_i \neq \lambda_j\}$, such sets for which there exists T such that $\hat{\lambda}_i = e^{\lambda_i T} = \hat{\lambda}_c \in \Phi(\hat{A})$, for all $\lambda_i \in \mathcal{L}(A)$, will be called collapsing sets and depending on whether $\hat{\lambda}_c \in \mathbb{C}$ or $\in \mathbb{R}$ this collapsing will be called *complex* or *real.* Clearly, this property depends on how we select T, as well as the nature of the set. The presence of collapsing sets in $\Phi(A)$, as well as the characterization of values of associated irregular sampling, is of great importance in the development of model based theory for sampling, since it affects the basic structure of $\hat{A}$, as well as related properties and it is subject of this section.

4.6.2 Collapsing Sets

As we have seen, for two complex eigenvalues λ_1, λ_2 of A with different real parts σ_1, σ_2 : $\lambda_1 = \sigma_1 + j\omega_1$, $\lambda_2 = \sigma_2 + j\omega_2$ correspond the two eigenvalues of $\hat{A}$: $\hat{\lambda}_1 = e^{\lambda_1 T} = e^{\sigma_1 T} e^{j\omega_1 T}$, $\hat{\lambda}_2 = e^{\lambda_2 T} = e^{\sigma_2 T} e^{j\omega_2 T}$. Because $\hat{\lambda}_1$, $\hat{\lambda}_2$ are distinct, independently of the values of ω_1, ω_2, we have:

Proposition 27 *For any two distinct eigenvalues λ_1, λ_2 of the continuous system with different real parts ($\sigma_1 \neq \sigma_2$), there correspond for $\forall T > 0$ two distinct eigenvalues $\hat{\lambda}_1$, $\hat{\lambda}_2$ of the discretised model.* □

Remark 15 *If A has all its eigenvalues real and distinct for $\forall T > 0$, then there is no collapsing set.* □

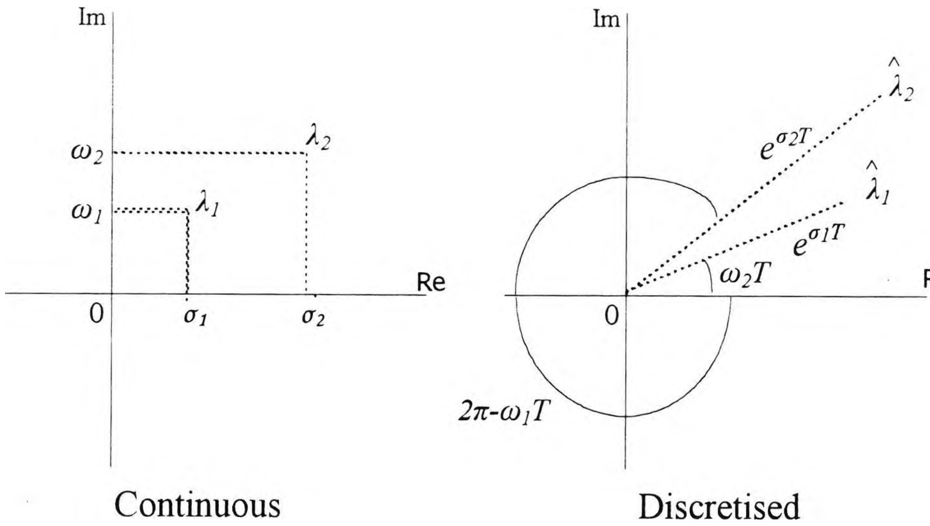

Figure 4-1: Mapping of two eigenvalues with $\sigma_1 \neq \sigma_2$

Remark 16 *If $\lambda_1,\ \lambda_2 \in \Phi(A)$, then a necessary condition for the existence of T such that collapsing occurs is that* $\mathrm{Re}(\lambda_1) = \mathrm{Re}(\lambda_2)$. □

Definition 37 *Let $\Phi(A) = \{\lambda_i \in \mathbb{C},\ \lambda_i = \sigma_i + j\omega_i,\ i = 1, 2..., f\}$, and let*

$$\Lambda(A) \triangleq \{\sigma_p, p = 1, ..., \mu, \sigma_p \in \mathbb{R}\}$$

be the set of distinct values of the real parts of $\Phi(A)$, and $\Phi_{\sigma_p}(A) = \{\forall \lambda_j \in \Phi(A) : \mathrm{Re}(\lambda_j) = \sigma_p\}$. Then $\Lambda(A)$ will be the real trace of $\Phi(A)$ and $\Phi_{\sigma_p}(A)$ is its σ_p-root range. □

It is evident that collapsing occurs for subsets of $\Phi_{\sigma_p}(A)$.This is defined below:

Theorem 29 *Let $\lambda_1 = r + j\omega_1,\ \lambda_2 = r + j\omega_2 \in \Phi_r(A)$ and $\hat{\lambda}_1 = e^{-rT}e^{j\omega_1 T},\ \hat{\lambda}_2 = e^{-rT}e^{j\omega_2 T}$, then the following properties hold true :*

1. *For any $\lambda_1, \lambda_2 \in \Phi_r(A)$, we have:*

   - $\hat{\lambda}_1 = \hat{\lambda}_2 \in \mathbb{C}$, *if and only if,*

$$T = \frac{2k\pi}{|\omega_1 - \omega_2|},\ k \in Z^+ \tag{4.34}$$

- *there exists T such that $\hat{\lambda}_1 = \hat{\lambda}_2 \in \mathbb{R}$ if and only if ω_2/ω_1 is rational i.e.*

$$\frac{\omega_2}{\omega_1} = \frac{\mu}{\nu}, \ \mu, \nu \in Z_{\neq 0} \tag{4.35}$$

Furthermore, if the latter condition is satisfied, then the corresponding T is :

$$T = \frac{2\,|\nu - \mu|\, k\pi}{|\omega_1 - \omega_2|}, \ k \in Z^+ \tag{4.36}$$

2. *If $\lambda_1 = r + j\omega$, $\lambda_2 = r - j\omega = \lambda_1^*$, then $\hat{\lambda}_1 = \hat{\lambda}_2$ if and only if,*

$$T = \frac{k\pi}{\omega}, \ k \in Z^+ \tag{4.37}$$

Furthermore, for all such T we have $\hat{\lambda}_1 = \hat{\lambda}_2 = e^{rT}$, if $k = 2, 4, 6, \ldots$ and $\hat{\lambda}_1 = \hat{\lambda}_2 = -e^{rT}$, if $k = 1, 3, 5, \ldots$

3. *If $\lambda_1 = r + j\omega_1$, $\lambda_1^* = r - j\omega_1$, $\lambda_2 = r + j\omega_2$, $\lambda_2^* = r - j\omega_2 \in \Phi_r(A)$ then for any T such that $\hat{\lambda}_1 = \hat{\lambda}_2$ then also $\hat{\lambda}_1^* \left(= e^{\lambda_1^* T}\right) = \hat{\lambda}_2^* \left(= e^{\lambda_2^* T}\right)$. Furthermore, if T is selected as in (4.37), then $\hat{\lambda}_1 = \hat{\lambda}_2 = \hat{\lambda}_1^* = \hat{\lambda}_2^* \in \mathbb{R}$.*

Proof. :

1. If $\omega_1 > \omega_2 > 0$, then we have the following representation of Figure (4.2) below and:

   - Also we have,

$$\hat{\lambda}_1 = e^{\lambda_1 T} = e^{rT} e^{j\frac{2k\pi\omega_1}{\omega_1 - \omega_2}} = e^{rT} e^{j\left(2k\pi + \frac{2k\pi\omega_2}{\omega_1 - \omega_2}\right)} = e^{rT} e^{j\frac{2k\pi\omega_2}{\omega_1 - \omega_2}}$$

$$\hat{\lambda}_2 = e^{\lambda_2 T} = e^{rT} e^{j\frac{2k\pi\omega_2}{\omega_1 - \omega_2}} = \hat{\lambda}_1$$

   - Let $\omega_1 > \omega_2 > 0$, $\nu > \mu$,then

$$\hat{\lambda}_1 = e^{\lambda_1 T} = e^{rT} e^{j\frac{2k(\nu - \mu)\pi\omega_1}{\omega_1 - \omega_2}} = e^{rT} e^{j\frac{2k(\nu - \mu)\pi\nu\omega_1}{\nu\omega_1 - \mu\omega_1}} = e^{rT} e^{j2k\nu\pi} = \hat{\lambda}_2$$

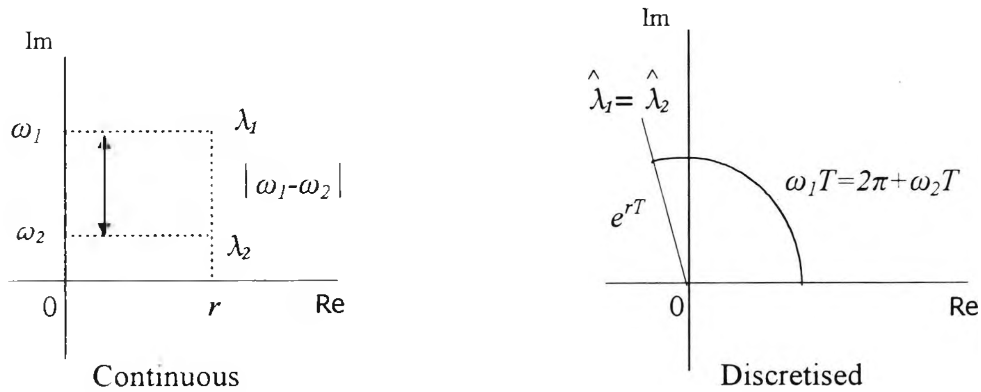

Figure 4-2: Collapsing of a pair of eigenvalues

2. For this case

$$\hat{\lambda}_1 = \hat{\lambda}_2 = e^{rT}e^{jk\pi} \Rightarrow \begin{cases} k = 2, 4, 6, ..., \Rightarrow e^{jk\pi} = 1 \\ k = 2, 4, 6, ..., \Rightarrow e^{jk\pi} = -1 \end{cases}$$

and proof is completed. □

The above result clearly establishes the existence of irregular sampling for complex and real collapsing on simple subsets of a general $\Phi_r(A)$ set. More specifically:

Corollary 1 *Given any set of $\Phi_r(A)$. then the following properties hold true for simple subsets of $\Phi_r(A)$:*

1. *For the set $\Phi_r^a(A) = \{r, r \pm j\omega\}$, there always exists T such that there is total real collapsing to e^{rT}.*

2. *For the set $\Phi_r^b(A) = \{r \pm j\omega_1, r \pm j\omega_2\}$, there is always a T such that we have complex collapsing to $\hat{\lambda}_c, \hat{\lambda}_c^* \in \mathbb{C}$. Furthermore, if $\frac{\omega_1}{\omega_2}$ is rational, there exists T such that we have total real collapsing to e^{rT}.*

3. *For the set $\Phi_r^c(A) = \{r, r \pm j\omega_1, r \pm j\omega_2\}$ there exists total real collapsing to e^{rT} for some T if and only if $\frac{\omega_1}{\omega_2}$ is rational.* □

For sets with more elements the problem of total complex, or real collapsing depends on the structure of the set and this is established below:

Remark 17 *Let*

$$\Phi_r(A) = \{\lambda_i, \lambda_i^* = r \pm j\omega_i, i = 1, 2, ..., \mu, 0 < \omega_1 < ... < \omega\mu\}$$

The following properties hold true :

1. *If all the μ^2 differences $|\omega_i - \omega_j|$ for the total number of the 2μ eigenvalues of $\Phi_r(A)$ are distinct then there exist μ^2 corresponding sequences of T for which collapsing occurs between two pairs of eigenvalues λ_i, λ_j and λ_i^*, λ_j^*.*

2. *There exists a T such that for all $\hat{\lambda}_i, \hat{\lambda}_i^*$ we have $\hat{\lambda}_1 = \hat{\lambda}_2 = ... = \hat{\lambda}\mu = \hat{\lambda}_c$, $\hat{\lambda}_1^* = \hat{\lambda}_2^* = ... = \hat{\lambda}\mu^* = \hat{\lambda}_c^*$, $\hat{\lambda}_c, \hat{\lambda}_c^* \in \mathbb{C}$, if and only if,*

$$\omega\mu - \omega_{\mu-1} = ... = \omega_2 - \omega_1 = \delta\omega \tag{4.38}$$

   If the above holds true, then the appropriate sampling is

$$T = \frac{2k\pi}{\delta\omega}, \ k \in Z^+ \tag{4.39}$$

3. *If the above condition holds true and for some ω_i*

$$\frac{\omega_i}{\delta\omega} = \frac{\mu}{\nu}, \ \mu, \nu \in Z^+ \tag{4.40}$$

   then there exists T such that $\hat{\lambda}_i = \hat{\lambda}_i^* = e^{rT}$ for all $i = 1, 2, ..., \mu$. □

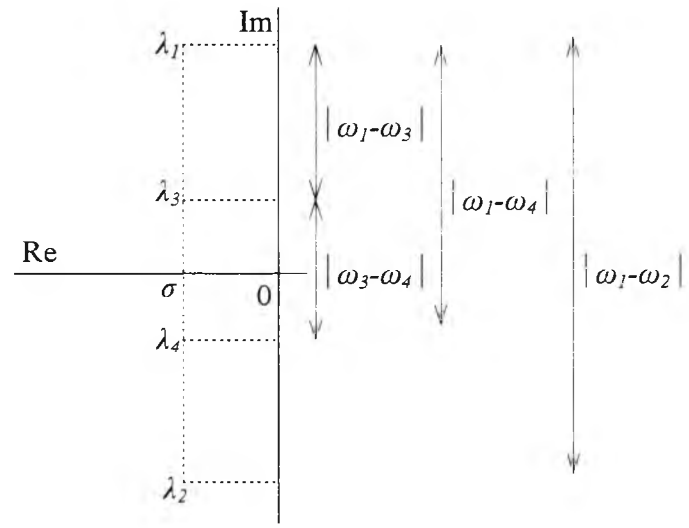

Figure 4-3: Collapsing Example

Example 3 *Consider a continuous system with the root range at $r = -5$ which is defined as*

$$\Phi_{-5}(A) = \{\lambda_1 = -5 + 12i,\ \lambda_2 = -5 - 12i,\ \lambda_3 = -5 + 3i,\ \lambda_4 = -5 - 3i\}$$

According to Theorem 29 there exist the following differences and the corresponding sequences of irregular T,

$$\begin{aligned} |\omega_1 - \omega_2| &= 24 \Rightarrow T_{1,2} = \frac{2k\pi}{24},\ |\omega_1 - \omega_3| = 9 \Rightarrow T_{1.4} = \frac{2k\pi}{15} \\ |\omega_1 - \omega_3| &= 9 \Rightarrow T_{1,3} = \frac{2k\pi}{9},\ |\omega_3 - \omega_4| = 6 \Rightarrow T_{3,4} = \frac{2k\pi}{6} \end{aligned}$$

a) The values of the sequence $T_{3.4}$ are included in $T_{1,2}$ (for $k = 4, 8, 12...$) and so for the values of $T_{1.2}$ the following collapsing occurs

$T_{1,2} = \frac{2k\pi}{24}$	*a.1*) for $k = 1, 2, 3, \ldots$	$\hat{\lambda}_1 = \hat{\lambda}_2 \in \mathbb{R}$
	a.2) for $k = 4, 8, 12, \ldots$	$\hat{\lambda}_1 = \hat{\lambda}_2,\ \hat{\lambda}_3 = \hat{\lambda}_4 \in \mathbb{R}$
	a.3) for $k = 8, 16, \ldots$	$\hat{\lambda}_1 = \hat{\lambda}_2 = \hat{\lambda}_3 = \hat{\lambda}_4 \in \mathbb{R}$

the above are verified by the following table,

k	$T_{1.2}$	$\hat{\lambda}_1 = \hat{\lambda}_2$	$\hat{\lambda}_3$	$\hat{\lambda}_4$
1	$\frac{\pi}{12}$	$-e^{-\frac{5}{12}\pi}$	$\frac{\sqrt{2}}{2}e^{-\frac{5}{12}\pi} + \frac{\sqrt{2}}{2}ie^{-\frac{5}{12}\pi}$	$\frac{\sqrt{2}}{2}e^{-\frac{5}{12}\pi} - \frac{\sqrt{2}}{2}ie^{-\frac{5}{12}\pi}$
2	$\frac{\pi}{6}$	$e^{-\frac{5}{6}\pi}$	$ie^{-\frac{5}{6}\pi}$	$-ie^{-\frac{5}{6}\pi}$
3	$\frac{\pi}{4}$	$-e^{-\frac{5}{4}\pi}$	$-\frac{\sqrt{2}}{2}e^{-\frac{5}{4}\pi} + \frac{\sqrt{2}}{2}ie^{-\frac{5}{4}\pi}$	$-\frac{\sqrt{2}}{2}e^{-\frac{5}{4}\pi} - \frac{\sqrt{2}}{2}ie^{-\frac{5}{4}\pi}$
4	$\frac{\pi}{3}$	$e^{-\frac{5}{3}\pi}$	$-e^{-\frac{5}{3}\pi}$	$-e^{-\frac{5}{3}\pi}$
5	$\frac{5\pi}{12}$	$-e^{-\frac{25}{12}\pi}$	$-\frac{\sqrt{2}}{2}e^{-\frac{25}{12}\pi} - \frac{\sqrt{2}}{2}ie^{-\frac{25}{12}\pi}$	$-\frac{\sqrt{2}}{2}e^{-\frac{25}{12}\pi} + \frac{\sqrt{2}}{2}ie^{-\frac{25}{12}\pi}$
6	$\frac{\pi}{2}$	$e^{-\frac{5}{2}\pi}$	$-ie^{-\frac{5}{2}\pi}$	$ie^{-\frac{5}{2}\pi}$
7	$\frac{7\pi}{12}$	$-e^{-\frac{35}{12}\pi}$	$\frac{\sqrt{2}}{2}e^{-\frac{35}{12}\pi} - \frac{\sqrt{2}}{2}ie^{-\frac{35}{12}\pi}$	$\frac{\sqrt{2}}{2}e^{-\frac{35}{12}\pi} + \frac{\sqrt{2}}{2}ie^{-\frac{35}{12}\pi}$
8	$\frac{2\pi}{3}$	$e^{-\frac{10}{3}\pi}$	$e^{-\frac{10}{3}\pi}$	$e^{-\frac{10}{3}\pi}$
9	$\frac{2\pi}{3}$	$-e^{-\frac{15}{4}\pi}$	$\frac{\sqrt{2}}{2}e^{-\frac{15}{4}\pi} + \frac{\sqrt{2}}{2}ie^{-\frac{15}{4}\pi}$	$\frac{\sqrt{2}}{2}e^{-\frac{15}{4}\pi} - \frac{\sqrt{2}}{2}ie^{-\frac{15}{4}\pi}$
10	$\frac{5\pi}{6}$	$e^{-\frac{25}{6}\pi}$	$ie^{-\frac{25}{6}\pi}$	$-ie^{-\frac{25}{6}\pi}$
11	$\frac{11\pi}{12}$	$-e^{-\frac{55}{12}\pi}$	$-\frac{\sqrt{2}}{2}e^{-\frac{55}{12}\pi} + \frac{\sqrt{2}}{2}ie^{-\frac{55}{12}\pi}$	$-\frac{\sqrt{2}}{2}e^{-\frac{55}{12}\pi} - \frac{\sqrt{2}}{2}ie^{-\frac{55}{12}\pi}$
12	π	$e^{-5\pi}$	$-e^{-5\pi}$	$-e^{-5\pi}$
13	$\frac{13\pi}{12}$	$-e^{-\frac{65}{12}\pi}$	$-\frac{\sqrt{2}}{2}e^{-\frac{65}{12}\pi} - \frac{\sqrt{2}}{2}ie^{-\frac{65}{12}\pi}$	$-\frac{\sqrt{2}}{2}e^{-\frac{65}{12}\pi} + \frac{\sqrt{2}}{2}ie^{-\frac{65}{12}\pi}$
14	$\frac{7\pi}{6}$	$e^{-\frac{35}{6}\pi}$	$-ie^{-\frac{35}{6}\pi}$	$ie^{-\frac{35}{6}\pi}$
15	$\frac{15\pi}{12}$	$-e^{-\frac{25}{4}\pi}$	$\frac{\sqrt{2}}{2}e^{-\frac{25}{4}\pi} - \frac{\sqrt{2}}{2}ie^{-\frac{25}{4}\pi}$	$\frac{\sqrt{2}}{2}e^{-\frac{25}{4}\pi} + \frac{\sqrt{2}}{2}ie^{-\frac{25}{4}\pi}$
16	$\frac{4\pi}{3}$	$e^{-\frac{20}{3}\pi}$	$e^{-\frac{20}{3}\pi}$	$e^{-\frac{20}{3}\pi}$

b) As $\frac{\omega_1}{\omega_4} = \frac{4}{-1}$, according to Theorem 29 for the values of the sequence $T_{1,4}$ the following collapsing occurs

$$T_{1,4} = \frac{2k\pi}{15} \quad b.1)\ k = 1, 2, 3, \ldots \quad \hat{\lambda}_1 = \hat{\lambda}_4,\ \hat{\lambda}_2 = \hat{\lambda}_3 \in \mathbb{C},$$

$$b.2)\ k = 5, 10, 15, \ldots \quad \hat{\lambda}_1 = \hat{\lambda}_2 = \hat{\lambda}_3 = \hat{\lambda}_4 \in \mathbb{R}$$

the above are verified by the following table,

k	$T_{1,4}$	$\hat{\lambda}_1 = \hat{\lambda}_4$	$\hat{\lambda}_2 = \hat{\lambda}_3$
1	$\frac{2\pi}{15}$	$\frac{\sqrt{5}-1}{4}e^{-\frac{2}{3}\pi} - \frac{\sqrt{2(5+\sqrt{5})}}{4}ie^{-\frac{2}{3}\pi}$	$\frac{\sqrt{5}-1}{4}e^{-\frac{2}{3}\pi} + \frac{\sqrt{2(5+\sqrt{5})}}{4}ie^{-\frac{2}{3}\pi}$
2	$\frac{4\pi}{15}$	$-\frac{\sqrt{5}+1}{4}e^{-\frac{4}{3}\pi} - \frac{\sqrt{2(5-\sqrt{5})}}{4}ie^{-\frac{4}{3}\pi}$	$-\frac{\sqrt{5}+1}{4}e^{-\frac{4}{3}\pi} + \frac{\sqrt{2(5-\sqrt{5})}}{4}ie^{-\frac{4}{3}\pi}$
3	$\frac{6\pi}{15}$	$-\frac{\sqrt{5}+1}{4}e^{-2\pi} + \frac{\sqrt{2(5-\sqrt{5})}}{4}ie^{-2\pi}$	$-\frac{\sqrt{5}+1}{4}e^{-2\pi} - \frac{\sqrt{2(5-\sqrt{5})}}{4}ie^{-2\pi}$
4	$\frac{8\pi}{15}$	$\frac{\sqrt{5}-1}{4}e^{-\frac{8}{3}\pi} + \frac{\sqrt{2(5+\sqrt{5})}}{4}ie^{-\frac{8}{3}\pi}$	$\frac{\sqrt{5}-1}{4}e^{-\frac{8}{3}\pi} - \frac{\sqrt{2(5+\sqrt{5})}}{4}ie^{-\frac{8}{3}\pi}$
5	$\frac{10\pi}{15}$	$e^{-\frac{10}{3}\pi}$	$e^{-\frac{10}{3}\pi}$
6	$\frac{12\pi}{15}$	$\frac{\sqrt{5}-1}{4}e^{-4\pi} - \frac{\sqrt{2(5+\sqrt{5})}}{4}ie^{-4\pi}$	$\frac{\sqrt{5}-1}{4}e^{-4\pi} + \frac{\sqrt{2(5+\sqrt{5})}}{4}ie^{-4\pi}$
7	$\frac{14\pi}{15}$	$-\frac{\sqrt{5}+1}{4}e^{-\frac{14}{3}\pi} - \frac{\sqrt{2(5-\sqrt{5})}}{4}ie^{-\frac{14}{3}\pi}$	$-\frac{\sqrt{5}+1}{4}e^{-\frac{14}{3}\pi} + \frac{\sqrt{2(5-\sqrt{5})}}{4}ie^{-\frac{14}{3}\pi}$
8	$\frac{16\pi}{15}$	$-\frac{\sqrt{5}+1}{4}e^{-\frac{16}{3}\pi} + \frac{\sqrt{2(5-\sqrt{5})}}{4}ie^{-\frac{16}{3}\pi}$	$-\frac{\sqrt{5}+1}{4}e^{-\frac{16}{3}\pi} - \frac{\sqrt{2(5-\sqrt{5})}}{4}ie^{-\frac{16}{3}\pi}$
9	$\frac{18\pi}{15}$	$\frac{\sqrt{5}-1}{4}e^{-6\pi} + \frac{\sqrt{2(5+\sqrt{5})}}{4}ie^{-6\pi}$	$\frac{\sqrt{5}-1}{4}e^{-6\pi} - \frac{\sqrt{2(5+\sqrt{5})}}{4}ie^{-6\pi}$
10	$\frac{20\pi}{15}$	$e^{-\frac{20}{3}\pi}$	$e^{-\frac{20}{3}\pi}$

c) Finally for the values of the sequence $T_{1,3}$ ($\frac{\omega_1}{\omega_3} = \frac{4}{1}$), the following collapsing occurs,

$$T_{1,3} = \frac{2k\pi}{9} \quad c.1)\ k = 1, 2, 3, \ldots \quad \hat{\lambda}_1 = \hat{\lambda}_3,\ \hat{\lambda}_2 = \hat{\lambda}_4 \in \mathbb{C},$$
$$c.2)\ k = 3, 6, 9, \ldots \quad \hat{\lambda}_1 = \hat{\lambda}_2 = \hat{\lambda}_3 = \hat{\lambda}_4 \in \mathbb{R}$$

the above are verified by the following table,

k	$T_{1,3}$	$\hat{\lambda}_1 = \hat{\lambda}_3$	$\hat{\lambda}_2 = \hat{\lambda}_4$
1	$\frac{2\pi}{9}$	$-\frac{1}{2}e^{-\frac{10}{9}\pi} + \frac{1}{2}ie^{-\frac{10}{9}\pi}\sqrt{3}$	$-\frac{1}{2}e^{-\frac{10}{9}\pi} - \frac{1}{2}ie^{-\frac{10}{9}\pi}\sqrt{3}$
2	$\frac{4\pi}{9}$	$-\frac{1}{2}e^{-\frac{20}{9}\pi} - \frac{1}{2}ie^{-\frac{20}{9}\pi}\sqrt{3},$	$-\frac{1}{2}e^{-\frac{20}{9}\pi} + \frac{1}{2}ie^{-\frac{20}{9}\pi}\sqrt{3}$
3	$\frac{6\pi}{9}$	$e^{-\frac{10}{3}\pi}$	$e^{-\frac{10}{3}\pi}$
4	$\frac{8\pi}{9}$	$-\frac{1}{2}e^{-\frac{40}{9}\pi} + \frac{1}{2}ie^{-\frac{40}{9}\pi}\sqrt{3}$	$-\frac{1}{2}e^{-\frac{40}{9}\pi} - \frac{1}{2}ie^{-\frac{40}{9}\pi}\sqrt{3}$
5	$\frac{10\pi}{9}$	$-\frac{1}{2}e^{-\frac{50}{9}\pi} - \frac{1}{2}ie^{-\frac{50}{9}\pi}\sqrt{3}$	$-\frac{1}{2}e^{-\frac{50}{9}\pi} + \frac{1}{2}ie^{-\frac{50}{9}\pi}\sqrt{3}$
6	$\frac{12\pi}{9}$	$e^{-\frac{20}{3}\pi}$	$e^{-\frac{20}{3}\pi}$

$\square$

4.6.3 Structural Consequences of Eigenvalue Collapsing

The collapsing of eigenvalues has been examined so far for distinct eigenvalue sets. We examine next the consequences of such phenomena on the Segrè characteristics and the geometry of eigenspaces of $\hat{A}$.

Let the characteristic polynomial of the continuous system matrix $A \in \mathbb{R}^{n \times n}$ be :

$$\Phi(s) = \det(A - sI) = (s - \lambda_1)^{\pi_1}(s - \lambda_2)^{\pi_2} \ldots (s - \lambda_f)^{\pi_f}$$

where $\lambda_1, \lambda_2, ..., \lambda_f$ are all the distinct eigenvalues of A and $\pi_1, \pi_2, ..., \pi_f$ be their corresponding algebraic multiplicities, with $\pi_1 + \pi_2 + ... + \pi_f = n$. For regular sampling , to each one of the distinct eigenvalues λ_i of A there corresponds one distinct eigenvalue $\hat{\lambda}_i$ of $\hat{A}$.

From Proposition 26 we know that for every value of T the eigenvectors of $\hat{A}$ associated with the eigenvalue $\hat{\lambda}_i$ as well as the corresponding eigenvectors chain lengths, are equal to the eigenvectors and eigenvectors chain lengths of A associated with λ_i. So, under regular sampling the algebraic multiplicity of $\hat{\lambda}_i$ is :

$$\hat{\pi}_i = \tau_{i\nu_i} + ... + \tau_{ik} + ... + \tau_{i1} = \pi_i$$

and the characteristic polynomial of $\hat{A}$ is :

$$\hat{\Phi}(z) = \det(\hat{A} - zI) = (z - \hat{\lambda}_1)^{\pi_1}(z - \hat{\lambda}_2)^{\pi_2} \ldots (z - \hat{\lambda}_f)^{\pi_f}$$

The generalized null-space $\widehat{\mathcal{N}_i}$ of $\hat{A}$ associated with the eigenvalue $\hat{\lambda}_i$ may be written according to Proposition 3 as a direct sum of the cyclic and $\hat{A}$-invariant subspaces $\widehat{\mathcal{V}}_{\tau_{i1}}, \widehat{\mathcal{V}}_{\tau_{i2}}, ..., \widehat{\mathcal{V}}_{\tau_{i\nu_i}}$. Also from Proposition 26 for every value of T, the above subspaces are equal to the corresponding cyclic and A-invariant subspaces of the continuous system $\mathcal{V}_{\tau_{i1}}, \mathcal{V}_{\tau_{i2}}, ..., \mathcal{V}_{\tau_{i\nu_i}}$ associated with λ_i, the direct sum of which forms the generalized null-space $\mathcal{N}_i$ of A . So we conclude that under regular sampling the generalized null-space $\widehat{\mathcal{N}_i}$ of $\hat{A}$, associated with $\hat{\lambda}_i$ is equal to the generalized null-space $\mathcal{N}_i$ of A, associated with λ_i. From the above we can readily conclude :

Theorem 30 *For the values of the sampling period T for which no collapsing occurs between the eigenvalues of $\hat{A}$ (i.e. under regular sampling), we have the properties:*

1. *To each pair of distinct eigenvalue* $\lambda_i, \lambda_j \in \Phi(A)$ *there corresponds one pair of distinct eigenvalues* $\hat{\lambda}_i, \hat{\lambda}_j \in \Phi(\hat{A})$.

2. *The generalized* A*-invariant null-space* $\mathcal{N}_i$ *of* A *associated with* λ_i, *is also a generalized* $\hat{A}$*-invariant null-space of* $\hat{A}$, *associated with* $\hat{\lambda}_i$.

3. *The Segré characteristic of* A *at* λ_i *is equal to the Segré characteristic of* $\hat{A}$ *at* $\hat{\lambda}_i$:

$$\wp_{\lambda_i}(A) = \wp_{\hat{\lambda}_i}\left(\hat{A}\right) = \{\tau_{i\nu_i} \geq ... \geq \tau_{ik} \geq ... \geq \tau_{i1} > 0\}$$

4. *The index of cyclicity of* $\hat{A}$ *is equal to the index of cyclicity of* A.

5. *The minimal polynomial of* $\hat{A}$ *is given as:*

$$\hat{\Psi}(z) = (z - \hat{\lambda}_1)^{\tau_{1\nu_1}}(z - \hat{\lambda}_2)^{\tau_{2\nu_2}}...(z - \hat{\lambda}_f)^{\tau_{f\nu_f}}$$

where $\tau_{1\nu_1}, \tau_{2\nu_2}, ..., \tau_{f\nu_f}$ *are the annihilation indices of* A. □

Consider two complex eigenvalues of A , λ_p and λ_q with equal real parts, geometric multiplicities π_p and π_q and with Segré characteristic of A at λ_p and λ_q, $\wp_{\lambda_p}(A)$, $\wp_{\lambda_q}(A)$ respectively, where:

$$\begin{aligned} \wp_{\lambda_p}(A) &= \left\{\tau_{p\nu_p} \geq ... \geq \tau_{pk} \geq ... \geq \tau_{p1} > 0\right\} \\ \wp_{\lambda_q}(A) &= \left\{\tau_{q\nu_q} \geq ... \geq \tau_{qk} \geq ... \geq \tau_{q1} > 0\right\} \end{aligned}$$

Let us now assume that for some value of the sampling period T collapsing occurs between $\hat{\lambda}_p$ and $\hat{\lambda}_q$ to the value $\hat{\lambda}_c$ (i.e. irregular sampling).

According to Theorem 26, for every value of the sampling period T, to the pair of eigenvalues $\hat{\lambda}_p$, $\hat{\lambda}_q$, there are associated ν_p,ν_q eigenvectors of $\hat{A}$ (or chains of generalized eigenvectors, or $\hat{A}$-invariant and cyclic subspaces, or Jordan blocks, or elementary divisors). So, under irregular sampling, to the collapsed eigenvalue $\hat{\lambda}_c$ there correspond $\nu_c = \nu_p + \nu_q$ eigenvectors of $\hat{A}$ (or chains of generalized eigenvectors, or $\hat{A}$-invariant and cyclic subspaces, or Jordan blocks, or elementary divisors). The generalized null-space $\hat{\mathcal{N}}_c$ of $\hat{A}$ associated with $\hat{\lambda}_c$ may be written according to Proposition 3 as a direct sum of the $\nu_c = \nu_p + \nu_q$, $\hat{A}$-invariant and cyclic subspaces associated with the eigenvalues $\hat{\lambda}_p$ and $\hat{\lambda}_q$ (which are equal to the direct sum of the corresponding A-invariant and cyclic subspaces associated with λ_p and λ_q), that is

$$\widehat{\mathcal{N}_c} = \left\{\mathcal{V}_{p1} \oplus \mathcal{V}_{p2} \oplus ... \oplus \mathcal{V}_{p\nu_p}\right\} \oplus \left\{\mathcal{V}_{q1} \oplus \mathcal{V}_{q2} \oplus ... \oplus \mathcal{V}_{q\nu_q}\right\} = \mathcal{N}_p \oplus \mathcal{N}_q$$

If we rearrange all the above subspaces according to increasing order, then we have :

$$\widehat{\mathcal{N}_c} = \mathcal{V}_{c1} \oplus \mathcal{V}_{c2} \oplus ... \oplus \mathcal{V}_{c\nu_c}$$

where:

$$\nu_c = \nu_p + \nu_q, \;\; \mathcal{V}_{c\nu_c} = \max\left(\mathcal{V}_{p\nu_p}, \mathcal{V}_{q\nu_q}\right), \;\; \mathcal{V}_{c1} = \min\left(\mathcal{V}_{p1}, \mathcal{V}_{q1}\right)$$

As $\widehat{\mathcal{N}_c}$ is formed as the direct sum of the generalized null-spaces $\mathcal{N}_p$, $\mathcal{N}_q$ associated with the eigenvalues λ_p and λ_q it follows that $\pi_c = \pi_p + \pi_q$. From the above we conclude the following result.

Theorem 31 *For the irregular values of the sampling period T for which a collapsing occurs between the pair of eigenvalues (λ_p, λ_q) of A, then the following properties hold true.*

1. *To the pair of distinct eigenvalues λ_p, λ_q of A there corresponds one eigenvalue $\hat{\lambda}_c$ of $\hat{A}$.*

2. *To the generalized null-spaces $\mathcal{N}_p$ and $\mathcal{N}_q$ associated with λ_p and λ_q there corresponds the generalized null-space $\widehat{\mathcal{N}_c}$ associated with $\hat{\lambda}_c$, defined as the direct sum, $\widehat{\mathcal{N}_c} = \mathcal{N}_p \oplus \mathcal{N}_q$.*

3. *The algebraic multiplicity of $\hat{\lambda}_c$ is given as $\pi_c = \pi_p + \pi_q$.*

4. *The Segré characteristic of $\hat{A}$ at $\hat{\lambda}_c$ is defined as a union of Segré characteristics i.e.*

$$\begin{aligned} \wp_{\hat{\lambda}_c}\left(\hat{A}\right) &= \wp_{\lambda_p}(A) \cup \wp_{\lambda_q}(A) = \\ = \left\{\tau_{p\nu_p} \geq ... \geq \tau_{pk} \geq ... \geq \tau_{p1} > 0\right\} &\cup \left\{\tau_{q\nu_q} \geq ... \geq \tau_{qk} \geq ... \geq \tau_{q1} > 0\right\} = \\ &= \left\{\tau_{c\nu_c} \geq ... \geq \tau_{ck} \geq ... \geq \tau_{c1} > 0\right\} \end{aligned}$$

Where $\nu_c = \nu_p + \nu_q$, $\tau_{c\nu_c} = \max\left(\tau_{p\nu_p}, \tau_{q\nu_q}\right)$, $\tau_{c1} = \min\left(\tau_{p1}, \tau_{q1}\right)$.

5. *If the minimal polynomial of A is :*

$$\Psi(s) = (s-\lambda_1)^{\tau_{1\nu_1}} \ldots (s-\lambda_p)^{\tau_{p\nu_p}} \ldots (s-\lambda_q)^{\tau_{q\nu_q}} \ldots (s-\lambda_f)^{\tau_{f\nu_f}}$$

then the minimal polynomial of $\hat{A}$ is :

$$\hat{\Psi}(z) = (z-\hat{\lambda}_1)^{\tau_{1\nu_1}} \ldots (z-\hat{\lambda}_c)^{\tau_{c\nu_c}} \ldots (z-\hat{\lambda}_f)^{\tau_{f\nu_f}}$$

where the elementary divisor $(z-\hat{\lambda}_c)^{\tau_{c\nu_c}}$ corresponds to the elementary divisors $(s-\lambda_p)^{\tau_{p\nu_p}}$ and $(s-\lambda_q)^{\tau_{q\nu_q}}$ of A. □

The above conclusions can readily be extended to the case of collapsing of a set of eigenvalues.

Theorem 32 *For the irregular values of the sampling period T for which a collapsing occurs between the subset of μ eigenvalues $\lambda_1, \lambda_2, ..., \lambda\mu$ of A, the following properties hold:*

1. *To the set of distinct eigenvalues $\lambda_1, \lambda_2, ...\lambda\mu$, of A corresponds one distinct eigenvalue $\hat{\lambda}_c$ of $\hat{A}$.*

2. *To the set of generalized null-spaces $\mathcal{N}_1, \mathcal{N}_2, ..., \mathcal{N}\mu$ associated with the eigenvalues λ_1, λ_2, ..., $\lambda\mu$ corresponds one generalized null-space $\widehat{\mathcal{N}_c}$ associated with $\hat{\lambda}_c$ equal to the direct sum*

$$\widehat{\mathcal{N}_c} = \mathcal{N}_1 \oplus \mathcal{N}_2 \oplus \ldots \oplus \mathcal{N}\mu$$

and the algebraic multiplicity of $\hat{\lambda}_c$ is given as $\pi_c = \pi_1 + \pi_2 + ... + \pi\mu$.

3. *If the Segré characteristics of A at $\lambda_1, \lambda_2, ...\lambda\mu$, are $\wp_{\lambda_1}(A), \wp_{\lambda_2}(A), ..., \wp_{\lambda\mu}(A)$, then the Segré characteristics of $\hat{A}$ at $\hat{\lambda}_c$ is:*

$$\wp_{\hat{\lambda}_c}\left(\hat{A}\right) = \wp_{\lambda_1}(A) \cup \wp_{\lambda_2}(A) \cup \ldots \cup \wp_{\lambda\mu}(A) = \{\tau_{c\nu_c} \geq \ldots \geq \tau_{ck} \geq \ldots \geq \tau_{c1} > 0\}$$

Where :

$$\nu_c = \nu_1 + \nu_2 + \ldots + \nu\mu, \;\; \tau_{c\nu_c} = \max\left(\tau_{1\nu_1}, \tau_{2\nu_2}, \ldots, \tau_{\mu\nu\mu}\right), \;\; \tau_{c1} = \min\left(\tau_{11}, \tau_{21}, \ldots, \tau_{\mu 1}\right)$$

4. If the minimal polynomial of A is :

$$\Psi(s) = (s - \lambda_1)^{\tau_{1\nu_1}} ... (s - \lambda_p)^{\tau_{\mu\nu_\mu}} . . . (s - \lambda_f)^{\tau_{f\nu_f}}$$

then, the minimal polynomial of $\hat{A}$ is :

$$\hat{\Psi}(z) = (z - \hat{\lambda}_c)^{\tau_{c\nu_c}} . . . (z - \hat{\lambda}_f)^{\tau_{f\nu_f}}$$

where the elementary divisor $(z - \hat{\lambda}_c)^{\tau_{c\nu_c}}$ of $\hat{A}$, corresponds to the elementary divisors $(s - \lambda_1)^{\tau_{1\nu_1}} ... (s - \lambda_p)^{\tau_{\mu\nu_\mu}}$ of A. □

From the above Theorems we have :

Remark 18 *1. If the space $\mathbb{R}^n$ is cyclic relative to A. for the regular values of the sampling period T, then the space $\mathbb{R}^n$ is also cyclic relative to $\hat{A}$.*

2. For the irregular values of the sampling period T, the whole space $\mathbb{R}^n$ is not cyclic relative to $\hat{A}$.

3. For the irregular values of the sampling period T for which a collapsing occurs between a pair of eigenvalues, the degree of the minimal polynomial $\hat{\Psi}(z)$ is decreased by a number. This number is equal to the minimal of annihilation indices associated with the two collapsing eigenvalues.

4. For the irregular values of the sampling period T for which a collapsing occurs between the subset of μ eigenvalues $\lambda_1, \lambda_2, ... \lambda_\mu$, with corresponding annihilation indices $\tau_{1\nu_1}, \tau_{2\nu_2}, ..., \tau_{\mu\nu_\mu}$ the degree of the minimal polynomial $\hat{\Psi}(z)$ is decreased by a number. This number is equal to

$$\tau_{1\nu_1} + \tau_{2\nu_2} + ... + \tau_{\mu\nu_\mu} - \tau_{c\nu_c}$$

where $\tau_{c\nu_c} = \max(\tau_{1\nu_1}, \tau_{2\nu_2}, ..., \tau_{\mu\nu_\mu})$. □

Example 4 *Let that the Segré characteristics of the eigenvalues of the set $\Phi_{-5}(A)$ in the Example 3 be*

$$\wp_{\lambda_1}(A) = \wp_{\lambda_2}(A) = \{6,3,1\} \ \ and\ \wp_{\lambda_3}(A) = \wp_{\lambda_4}(A) = \{3,1\}$$

and $\Psi(s) = (s-\lambda_1)^8(s-\lambda_2)^8(s-\lambda_3)^5(s-\lambda_4)^5$. *Then for the irregular sampling we have,*

a) for $T = \frac{2k\pi}{24}$

Segré Characteristic and minimal polynomial

a.1) $k = 1,2,3,5,6,7,9,...$ $\quad \wp_{\hat{\lambda}_{1,2}}\left(\hat{A}\right) = \wp_{\lambda_1}(A) \cup \wp_{\lambda_2}(A) = \{6,6,3,3,1,1\}$
$$\Rightarrow \hat{\Psi}(z) = (z-\hat{\lambda}_{1,2})^6(z-\hat{\lambda}_3)^3(z-\hat{\lambda}_4)^3$$

a.2) $k = 4,8,12,....$ $\quad \wp_{\hat{\lambda}_{1,2}}\left(\hat{A}\right) = \{6,6,3,3,1,1\},\ \wp_{\hat{\lambda}_{3,4}}\left(\hat{A}\right) = \{3,3,1,1\}$
$$\Rightarrow \hat{\Psi}(z) = (z-\hat{\lambda}_{1,2})^6(z-\hat{\lambda}_{3,4})^3$$

a.3) $k = 8,16,24,...$ $\quad \wp_{\hat{\lambda}_{1,2,3,4}}\left(\hat{A}\right) = \{6,6,3,3,3,3,1,1,1,1\}$
$$\Rightarrow \hat{\Psi}(z) = (z-\hat{\lambda}_{1,2,3,4})^8$$

b) for $T = \frac{2k\pi}{15}$

Segré Characteristic and minimal polynomial

b.1) $k = 1,2,3,4,6,7,8,9,11,...$ $\quad \wp_{\hat{\lambda}_{1,4}}\left(\hat{A}\right) = \wp_{\hat{\lambda}_{2,3}}\left(\hat{A}\right) = \{6,3,3,1,1\}$
$$\Rightarrow \hat{\Psi}(z) = (z-\hat{\lambda}_{1,4})^6(z-\hat{\lambda}_{2,3})^6$$

b.2) $k = 5,10,15,20,....$ $\quad \wp_{\hat{\lambda}_{1,2,3,4}}\left(\hat{A}\right) = \{6,6,3,3,3,3,1,1,1,1\}$
$$\Rightarrow \hat{\Psi}(z) = (z-\hat{\lambda}_{1,2,3,4})^6$$

c) for $T = \frac{2k\pi}{9}$

Segré Characteristic and minimal polynomial

c.1) $k = 1,2,4,5,7,8,10,...$ $\quad \wp_{\hat{\lambda}_{1,3}}\left(\hat{A}\right) = \wp_{\hat{\lambda}_{2,4}}\left(\hat{A}\right) = \{6,3,3,1,1\}$
$$\Rightarrow \hat{\Psi}(z) = (z-\hat{\lambda}_{1,3})^6(z-\hat{\lambda}_{2,4})^6$$

c.2) $k = 5,10,15,20,....$ $\quad \wp_{\hat{\lambda}_{1,2,3,4}}\left(\hat{A}\right) = \{6,6,3,3,3,3,1,1,1,1\}$
$$\Rightarrow \hat{\Psi}(z) = (z-\hat{\lambda}_{1,2,3,4})^6$$

□

4.7 Conclusion

In this Chapter we have introduced the basics of the nature of the discretised models as functions of the sampling period T. The effect of sampling on properties such as eigenstructure, Segré characteristics and minimal polynomial has been thoroughly investigated. The work here provides the background for the work that will follow.

The consequences of collapsing in the structural properties of $\hat{A}$ has been examined and so we have to investigate in the next Chapter the consequences in the basic characteristics of the discretised model.

Chapter 5

COLLAPSING OF EIGENVALUES AND CONTROLLABILITY OBSERVABILITY PROPERTIES

5.1 Introduction

The Jordan canonical description of a linear continuous time system, enables the testing of controllability and observability of the system by the Spectral controllability and observability matrices $\mathcal{B}_i^S$ and Γ_i^F. Controllability and observability properties of a discretised model have been defined in CHAPTER 4. The Jordan canonical description for the discretised models which were introduced in CHAPTER 4, as well the investigation of the effect of collapsing of eigenvalues and of the merging of Segré Characteristics, leads to the study in this CHAPTER 5 of the following issues:

- The discrete Spectral controllability and observability matrices $\widehat{\mathcal{B}}_i^S$ and $\widehat{\Gamma}_i^F$ for the discretised model under regular sampling.
- The composite discrete Spectral controllability and observability matrices respectively under irregular sampling.

The results here provide a description of the effects of collapsing on the controllability, observability properties of the discretised models and thus enhance our understanding on the selection of sampling based on system properties.

5.2 Structural Properties of a Linear Continuous System in Jordan Form

5.2.1 Controllability

Some preliminary results on the spectral controllability Properties are considered first.

Proposition 28 *The n-dimensional, linear ,time invariant system $S(A, B, C, D)$ described by the Jordan equivalent equations, is controllable if and only if for each $i = 1, 2, ..f$, the rows of the $\mathcal{B}_i^S$ matrix (defined in 3.73) are linearly independent over the field of complex numbers.*

Remark 19 *The linear independence of the rows of $\mathcal{B}_i^S$ are tested individually for each i.*

Proof. [Kar., 1]If the continuous system is in Jordan form then according to Proposition 18 it is controllable, if and only if the rows of matrix $(sI - A)^{-1} B$, or the equivalent in Jordan form $(sI - J)^{-1} \mathcal{B}$ are linearly independent over the field of complex numbers. That is:

$$(sI - J)^{-1} \mathcal{B} = \text{block diag} \left\{ (sI_k^i - J_{ik})^{-1} \mathcal{B}_{ik} \right\}, i = 1, 2, ..f, \ k = 1, 2, ..., \nu_i$$

where,

$$(sI_k^i - J_{ik})^{-1} \mathcal{B}_{ik} = \begin{bmatrix} \frac{1}{s-\lambda_i} & \frac{1}{(s-\lambda_i)^2} & \cdots & \frac{1}{(\tau_{ik}-1)!(s-\lambda_i)^{\tau_i}} \\ 0 & \frac{1}{s-\lambda_i} & \cdots & \frac{1}{(\tau_{ik}-2)!(s-\lambda_i)^{\tau_i - 1}} \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & \frac{1}{s-\lambda_i} \end{bmatrix} \begin{bmatrix} \underline{\beta}_{ik_1}^{\top} \\ \underline{\beta}_{ik_2}^{\top} \\ \cdots \\ \underline{\beta}_{ik\tau_{ik}}^{\top} \end{bmatrix}$$

so there are ν_i rows of matrix $(sI - J)^{-1} \mathcal{B}$ of the form

$$(s - \lambda_i)^{-1} \underline{\beta}_{i1_{\tau_{i1}}}^{\top}, \ (s - \lambda_i)^{-1} \underline{\beta}_{i2_{\tau_{i2}}}^{\top}, \ ..., \ (s - \lambda_i)^{-1} \underline{\beta}_{(i\nu_i)_{\tau_{i\nu_i}}}^{\top}$$

Hence if the rows $\underline{\beta}^{\top}_{i1_{\tau_{i1}}}$, $\underline{\beta}^{\top}_{i2_{\tau_{i2}}}$, ..., $\underline{\beta}^{\top}_{(i\nu_i)_{\tau_{i\nu_i}}}$ (i.e. the rows of the $\mathcal{B}_i^S$) are linearly independent for $i = 1, 2, ..f$ then all the rows of $(sI - J)^{-1}\mathcal{B}$ are linearly independent over the field of complex numbers and the system $S(A, B, C, D)$ is controllable. □

Proposition 29 *If the rows of $(sI_k^i - J_{ik})^{-1}\mathcal{B}_{ik}$ are to be linearly independent, then the row $\underline{\beta}^{\top}_{ik_{\tau_{ik}}}$ must be non zero.*

Proof. The first row of $(sI_k^i - J_{ik})^{-1}\mathcal{B}_{ik}$ has a factor of,

$$\frac{\underline{\beta}^{\top}_{ik_{\tau_{ik}}}}{(\tau_{ik} - 1)!\,(s - \lambda_i)^{\tau_i}}$$

, the second row has a factor of

$$\frac{\underline{\beta}^{\top}_{ik_{\tau_{ik}}}}{(\tau_{ik} - 2)!\,(s - \lambda_i)^{\tau_i - 1}}$$

and so on. Hence if $\underline{\beta}^{\top}_{ik_{\tau_{ik}}}$ is a non zero tow of $\mathcal{B}$, then all the rows of $(sI_k^i - J_{ik})^{-1}\mathcal{B}_{ik}$ are linearly independent and the system $S(A, B, C, D)$ can be controllable. □

From the above and from the Definition of the cyclicity index ν of A it can be readily concluded:

Proposition 30 *Necessary condition for the n-dimensional, continuous, linear, time invariant system $S(A, B, C, D)$, to be controllable is that $\nu \leq l$, where l is the number of system inputs.* □

Remark 20 *Necessary condition for a single input system to be controllable is that, all the eigenvalues are distinct, each have only one associated Jordan block (i.e. $\nu = 1$, the system must be cyclic), and the rows of $\mathcal{B}$ corresponding to the last row of each Jordan block is non-zero.* □

5.2.2 Observability

For the observability property of a linear continuous time system $S(A, B, C, D)$ described by Jordan equivalent equations, we have the following dual statements which are to the controllability definitions and employ similar notation:

Proposition 31 *The n-dimensional, linear, time invariant system $S(A, B, C, D)$ described by the Jordan equivalent equations, is observable if and only if for each $i = 1, 2, ..f$, the columns of the Γ_i^F matrix (defined in 3.74) are linearly independent over the field of complex numbers.* $\square$

Remark 21 *The linear independence of the columns of Γ_i^F is tested individually for each i.* $\square$

Remark 22 *If the columns of $\Gamma_{ik}(sI_k^i - J_{ik})^{-1}$ are to be linearly independent, then the column $\underline{\gamma}_{ik_1}$ must be non zero.* $\square$

Proposition 32 *Necessary condition for the n-dimensional, continuous, linear, time invariant system $S(A, B, C, D)$, to be observable is that $\nu \leq m$, where m is the number of system outputs.* $\square$

Remark 23 *Necessary condition for a single output system to be observable is that, all the eigenvalues are distinct, i.e. each have only one associated Jordan block (i.e. $\nu = 1$. the system must be cyclic), and the rows of $\mathcal{B}$ corresponding to the first row of each Jordan block is non-zero.* $\square$

5.3 Controllability of a Discretised Model

In the Chapter 4 we have proved that the form of the controllability matrix of a discretised model depends on the type of H used in the implementation of the control scheme. Thus we have to examine the cases of ZOH and FOH separately.

5.3.1 Case of a system with ZOH

The discretised controllability matrix of a discretised model with ZOH (Proposition 18) described by the Jordan equivalent equations (4.22 and 4.25) is given as,

$$(zI - \hat{A})^{-1}\hat{B} \sim (zI - \hat{J})^{-1}\widehat{\mathcal{B}} = (zI - \hat{J})^{-1}\tilde{V}\Xi\mathcal{B}$$

Hence if the rows of $(zI - \hat{J})^{-1}\tilde{V}\Xi\mathcal{B}$ are shown to be independent over the field of complex numbers then the discretised model will be controllable.

The matrix $\tilde{V}\Xi$ is a block diagonal matrix with the same structure as J of the diagonal block type and with each block of $\tilde{V}\Xi$ of an upper triangular form. If the rows of the spectral controllability matrix of the continuous model (3.73) corresponding to the eigenvalue $\lambda_1 = 0$, are given as,

$$\mathcal{B}_1^S = \begin{bmatrix} \underline{\beta}^{\mathsf{T}}_{11_{\tau_{11}}} \\ \cdots \\ \underline{\beta}^{\mathsf{T}}_{1k_{\tau_{1k}}} \\ \cdots \\ \underline{\beta}^{\mathsf{T}}_{1\nu_1,\tau_{1\nu_1}} \end{bmatrix} \tag{5.1}$$

and those of the spectral controllability matrix of the continuous model corresponding to the eigenvalue $\lambda_i \neq 0$, are given as,

$$\mathcal{B}_i^S = \begin{bmatrix} \underline{\beta}^{\mathsf{T}}_{i1_{\tau_{i1}}} \\ \cdots \\ \underline{\beta}^{\mathsf{T}}_{ik_{\tau_{ik}}} \\ \cdots \\ \underline{\beta}^{\mathsf{T}}_{i\nu_i,\tau_{i\nu_i}} \end{bmatrix} \tag{5.2}$$

thus, the last elements of the main diagonal of $\tilde{V}\Xi$ corresponding to the last rows of the Jordan blocks are :

1. for $\lambda_1 = 0$: the last elements of the main diagonal of $\tilde{V}\Xi$ corresponding to the last rows of the Jordan blocks are $T \times T^{\tau_{1k}-1} = T^{\tau_{1k}}$ and the corresponding rows of the discretised model are,

$$\begin{bmatrix} T^{\tau_{11}}\underline{\beta}^{\top}_{11_{\tau_{11}}} \\ \dots \\ T^{\tau_{1k}}\underline{\beta}^{\top}_{1k_{\tau_{1k}}} \\ \dots \\ T^{\tau_{1\nu_1}}\underline{\beta}^{\top}_{1\nu_1,\tau_{1\nu_1}} \end{bmatrix} = \begin{bmatrix} T^{\tau_{11}} & 0 & \dots & 0 \\ 0 & T^{\tau_{12}} & \dots & 0 \\ . & . & \dots & . \\ 0 & 0 & \dots & T^{\tau_{1\nu_1}} \end{bmatrix} \mathcal{B}_1^S \tag{5.3}$$

2. for $\lambda_i \neq 0$: the last elements of the main diagonal of $\tilde{V}\Xi$ corresponding to the last rows of the Jordan blocks are $\frac{(e^{\lambda_i T}-1)(Te^{\lambda_i T})^{\tau_{ik}-1}}{\lambda_i}$ and the corresponding rows of the discretised model are,

$$\frac{(e^{\lambda_i T}-1)}{\lambda_i}\begin{bmatrix} (Te^{\lambda_i T})^{\tau_{i1}-1}\,\underline{\beta}^{\top}_{i1_{\tau_{i1}}} \\ (Te^{\lambda_i T})^{\tau_{i2}-1}\,\underline{\beta}^{\top}_{i2_{\tau_{i2}}} \\ \dots \\ (Te^{\lambda_i T})^{\tau_{i\nu_i}-1}\,\underline{\beta}^{\top}_{i\nu_i,\tau_{i\nu_i}} \end{bmatrix} =$$

$$= \frac{(e^{\lambda_i T}-1)}{\lambda_i}\begin{bmatrix} (Te^{\lambda_i T})^{\tau_{i1}-1} & 0 & \dots & 0 \\ 0 & (Te^{\lambda_i T})^{\tau_{i2}-1} & \dots & 0 \\ . & . & \dots & . \\ 0 & 0 & \dots & (Te^{\lambda_i T})^{\tau_{i\nu_i}-1} \end{bmatrix} \mathcal{B}_i^S \tag{5.4}$$

Definition 38 *The i-th discrete Spectral controllability matrix $\hat{\mathcal{B}}_i^S$, is the matrix formed by the rows of $\hat{\mathcal{B}}$ corresponding to the last rows of the Jordan blocks associated with the eigenvalue λ_i.*$\square$

With the above introduced notation, we have the following propositions, similar to the corresponding propositions of the continuous model.

Proposition 33 *The discretised model $\hat{S}(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ of a linear, time invariant system with ZOH is controllable if and only if for each $i = 1, 2, ..., f$ the rows of the $\hat{\mathcal{B}}_i^S$ matrix are linearly* *independent over the field of complex numbers.* □

Remark 24 *The linear independence of the rows of $\widehat{\mathcal{B}}_i^S$ are tested individually for each i.* □

In order to determine the relation between the *i-th* discrete Spectral controllability matrix $\widehat{\mathcal{B}}_i^S$ and the Spectral controllability matrix $\mathcal{B}_i^S$ we have to distinguish the two cases of sampling.

Regular Sampling

From the above analysis we conclude the following:

Proposition 34 *Under regular sampling the* i-th *discrete Spectral controllability matrix $\widehat{\mathcal{B}}_i^S$, is related to $\mathcal{B}_i^S$ as is shown below:*

1. *for* $\lambda_1 = 0$:

$$\widehat{\mathcal{B}}_1^S = \begin{bmatrix} T^{\tau_{11}} & 0 & \dots & 0 \\ 0 & T^{\tau_{12}} & \dots & 0 \\ . & . & \dots & . \\ 0 & 0 & \dots & T^{\tau_{1\nu_1}} \end{bmatrix} \mathcal{B}_1^S = \hat{D}(0)\mathcal{B}_1^S, \text{ where } \hat{D}(0) \triangleq \begin{bmatrix} T^{\tau_{11}} & 0 & \dots & 0 \\ 0 & T^{\tau_{12}} & \dots & 0 \\ . & . & \dots & . \\ 0 & 0 & \dots & T^{\tau_{1\nu_1}} \end{bmatrix} \tag{5.5}$$

2. *for* $\lambda_i \neq 0$:

$$\widehat{\mathcal{B}}_i^S = \frac{\hat{\lambda}_i - 1}{\lambda_i} \begin{bmatrix} \left(T\hat{\lambda}_i\right)^{\tau_{i1}-1} & 0 & \dots & 0 \\ 0 & \left(T\hat{\lambda}_i\right)^{\tau_{i2}-1} & \dots & 0 \\ . & . & \dots & . \\ 0 & 0 & \dots & \left(T\hat{\lambda}_i\right)^{\tau_{i\nu_i}-1} \end{bmatrix} \mathcal{B}_i^S = \frac{\hat{\lambda}_i - 1}{\lambda_i} \hat{D}(\lambda_i)\mathcal{B}_1^S$$

where,

$$\hat{D}(\lambda_i) \triangleq \begin{bmatrix} \left(T\hat{\lambda}_i\right)^{\tau_{i1}-1} & 0 & \dots & 0 \\ 0 & \left(T\hat{\lambda}_i\right)^{\tau_{i2}-1} & \dots & 0 \\ . & . & \dots & . \\ 0 & 0 & \dots & \left(T\hat{\lambda}_i\right)^{\tau_{i\nu_i}-1} \end{bmatrix} \tag{5.6}$$

□

So, under regular sampling we have for the controllability of the discretised model the following proposition:

Proposition 35 *The discretised model $\hat{S}(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ of a linear, time invariant system with ZOH, under regular sampling, is controllable, if and only if the corresponding continuous time linear system $S(A, B, C, D)$ is controllable.* □

Irregular Sampling

The effect of irregular sampling on the controllability properties is examined next.

Proposition 36 *Under irregular sampling for which collapsing occurs of a subset of μ eigenvalues $\lambda_1, \lambda_2, ..., \lambda\mu \in \Phi_\tau(A)$ to the distinct eigenvalue $\hat{\lambda}_c \in \Phi\left(\hat{A}\right)$, the c-th discrete Spectral controllability matrix $\widehat{\mathcal{B}}_c^S$, is related to the corresponding matrices $\mathcal{B}_1^S$, $\mathcal{B}_2^S$, ..., $\mathcal{B}\mu^S$, as follows:*

1. *If $\lambda_1, \lambda_2, ..., \lambda\mu \neq 0$ then:*

$$\widehat{\mathcal{B}}_c^S = \left(\hat{\lambda}_c - 1\right) \begin{bmatrix} \frac{1}{\lambda_1}\hat{D}(\lambda_1) & 0 & \cdots & 0 \\ 0 & \frac{1}{\lambda_2}\hat{D}(\lambda_2) & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & \frac{1}{\lambda\mu}\hat{D}(\lambda\mu) \end{bmatrix} \begin{bmatrix} \mathcal{B}_1^S \\ \mathcal{B}_2^S \\ \cdots \\ \mathcal{B}\mu^S \end{bmatrix}$$

where $\hat{D}(\lambda_1), \hat{D}(\lambda_2), ..., \hat{D}(\lambda\mu)$ are defined as previously for $\lambda_i \neq 0$ (5.6).

2. *If $\lambda_1 = 0, (\lambda_2, ..., \lambda\mu \neq 0)$:*

$$\widehat{\mathcal{B}}_c^S = \begin{bmatrix} \hat{D}(0) & 0 & \cdots & 0 \\ 0 & \frac{\hat{\lambda}_c-1}{\lambda_2}\hat{D}(\lambda_2) & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & \frac{\hat{\lambda}_c-1}{\lambda\mu}\hat{D}(\lambda\mu) \end{bmatrix} \begin{bmatrix} \mathcal{B}_1^S \\ \mathcal{B}_2^S \\ \cdots \\ \mathcal{B}\mu^S \end{bmatrix}$$

where $\hat{D}(0)$ is defined as previously for $\lambda_1 = 0$ (5.5) and similarly $\hat{D}(\lambda_2), ..., \hat{D}(\lambda\mu)$ are defined as previously for $\lambda_i \neq 0$ (5.6).

Proof. From the definition 38 relations (5.3), (5.4) the above Proposition is directly concluded.□

Definition 39 *Under irregular sampling where a collapsing occurs between a subset of μ eigenvalues $\lambda_1, \lambda_2, ..., \lambda\mu \in \Phi_r(A)$ to the distinct eigenvalue $\hat{\lambda}_c \in \Phi\left(\hat{A}\right)$, the* c-th *composite spectral Controllability matrix $\mathcal{B}_c^S$ is defined as the matrix consisting of the rows of all the Spectral controllability matrices corresponding to the eigenvalues $\lambda_1, \lambda_2, ..., \lambda\mu$:*

$$\mathcal{B}_c^S \triangleq \begin{bmatrix} \mathcal{B}_1^S \\ \mathcal{B}_2^S \\ \dots \\ \mathcal{B}\mu^S \end{bmatrix}$$

The relation between the c-th *discrete Spectral controllability matrix $\widehat{\mathcal{B}}_c^S$ and the* c-th *composite spectral controllability matrix $\mathcal{B}_c^S$ is defined as:*

1. *If $\lambda_1, \lambda_2, ..., \lambda\mu \neq 0$:*

$$\widehat{\mathcal{B}}_c^S = \left(\hat{\lambda}_c - 1\right) \begin{bmatrix} \frac{1}{\lambda_1}\hat{D}(\lambda_1) & 0 & \dots & 0 \\ 0 & \frac{1}{\lambda_2}\hat{D}(\lambda_2) & \dots & 0 \\ . & . & \dots & . \\ 0 & 0 & \dots & \frac{1}{\lambda\mu}\hat{D}(\lambda\mu) \end{bmatrix} \mathcal{B}_c^S$$

2. *If $\lambda_1 = 0, (\lambda_2, ..., \lambda\mu \neq 0)$:*

$$\widehat{\mathcal{B}}_c^S = \begin{bmatrix} \hat{D}(0) & 0 & \dots & 0 \\ 0 & \frac{\hat{\lambda}_c - 1}{\lambda_2}\hat{D}(\lambda_2) & \dots & 0 \\ . & . & \dots & . \\ 0 & 0 & \dots & \frac{\hat{\lambda}_c - 1}{\lambda\mu}\hat{D}(\lambda\mu) \end{bmatrix} \mathcal{B}_c^S$$

□

The above leads to:

Proposition 37 *The discretised model $\hat{S}(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ of a linear, time invariant controllable system, with ZOH, under irregular sampling, for which collapsing occurs between a subset of μ eigenvalues $\lambda_1, \lambda_2, ..., \lambda\mu \in \Phi_r(A)$ to the distinct eigenvalue $\hat{\lambda}_c \in \Phi\left(\hat{A}\right)$, becomes uncon-*

trollable, if and only if the ν_c *rows of the* c-th *composite spectral controllability matrix* $\mathcal{B}_c^S$ *are linearly dependent, where,* $\nu_c = \nu_1 + \nu_2 + ... + \nu\mu$. □

Definition 40 *We define as structural loss of controllability of the discretised model* $\hat{S}(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ *the case where* $\nu_c > l$ *(where* l *is the number of system inputs).* □

Definition 41 *We define as numerical loss of controllability of the discretised model* $\hat{S}(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ *the case where* $\nu_c \leq l$ *and the rows of the* c-th *composite spectral controllability matrix* $\mathcal{B}_c^S$ *are linearly dependent.* □

It is clear that structural loss of controllability implies loss of controllability independent from the numerical values of A, B matrices. □

Remark 25 *The discretised model* $\hat{S}(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ *of a linear, time invariant, single input, controllable system, with ZOH, under any irregular sampling, becomes uncontrollable.* □

5.3.2 Case of a system with FOH

The discretised controllability matrix of a discretised model with FOH (*Proposition* 20) described by the Jordan equivalent equations (4.28) is given as,

$$(zI - \hat{A})^{-1}(z\hat{E} + \hat{Z}) \sim (zI - \hat{J})^{-1}(z\hat{\mathcal{E}} + \hat{\mathcal{Z}}) = (zI - \hat{J})^{-1}\tilde{V}\left[z\,(2\Xi - \Sigma) + \Sigma\right]\mathcal{B}$$

Hence if the rows of $(zI - \hat{J})^{-1}\tilde{V}\left[z\,(2\Xi - \Sigma) + \Sigma\right]\mathcal{B}$ are shown to be independent over the field of complex numbers then the discretised model will be controllable.

Matrix $\tilde{V}\left[z\,(2\Xi - \Sigma) + \Sigma\right]$, as in the case of systems with ZOH, is a block diagonal matrix with the same structure as J of the diagonal block type and with each block of an upper triangular form. If the rows of the spectral controllability matrix of the continuous model corresponding to the eigenvalue $\lambda_1 = 0$, are given as in relation (5.1) and if the rows of the spectral controllability matrix of the continuous model corresponding to the eigenvalue $\lambda_i \neq 0$, are given as in relation (5.2) the last elements of the main diagonal of $\tilde{V}\left[z\,(2\Xi - \Sigma) + \Sigma\right]$ corresponding to the last rows of the Jordan blocks are :

Proposition 38 *For every value of the sampling period* $T > 0$, *then the matrix* $\tilde{V}\left[z\,(2\Xi - \Sigma) + \Sigma\right]$ *is block diagonal with the same structure of diagonal blocks as the* J *matrix. In particular: The* *last elements of the main diagonal of* $\tilde{V}\left[z\left(2\Xi-\Sigma\right)+\Sigma\right]$ *corresponding to the last rows of the Jordan blocks are:*

1. *for* $\lambda_1 = 0$ *:*

$$T^{\tau_{1k}-1}\left[z\left(2T-\frac{T^2}{2}\right)+\frac{T^2}{2}\right]=T^{\tau_{1k}}\left[z\left(2-\frac{T}{2}\right)+\frac{T}{2}\right]$$

and the corresponding rows of the discretised model are,

$$\left[z\left(2-\frac{T}{2}\right)+\frac{T}{2}\right]\hat{D}(0)\mathcal{B}_1^S$$

where $\hat{D}(0)$ *is defined as in the case of ZOH.*

2. *for* $\lambda_i \neq 0$ *:*

$$\left(Te^{\lambda_i T}\right)^{\tau_{ik}-1}\left[z\frac{e^{\lambda_i T}(\lambda_i+1)-2\lambda_i-1}{\lambda_i^2}+\frac{e^{\lambda_i T}(\lambda_i-1)+1}{\lambda_i^2}\right]$$

and the corresponding rows of the discretised model are,

$$\frac{\left[e^{\lambda_i T}(\lambda_i+1)-2\lambda_i-1\right]z+e^{\lambda_i T}(\lambda_i-1)+1}{\lambda_i^2}\hat{D}(\lambda_i)\mathcal{B}_i^S$$

where $\hat{D}(\lambda_i)$ *is also as defined in the case of ZOH.*

Proof. From the definition of the diagonal block $\tilde{V}_{ik}$ of matrix $\tilde{V}$ (in 4.19), Theorem 27 on the structure of matrix Ξ, Theorem 28 on the structure of matrix Σ Proposition is directly concluded. □

From the above we conclude that all the Definitions, Propositions and Remarks for the controllability property of a discretised model with ZOH under regular or irregular sampling previously exposed, can be directly applied to the case of a discretised model with FOH.

5.4 Observability of a Discretised Model

In Chapter 4 we have also proved that the form of the observability matrix of a discretised model is independent from the type of H used in the implementation of the control scheme. Thus we do not have to examine the cases of ZOH and FOH separately

The form of the discretised observability matrix of a discretised model is independent of the H implementing ZOH or FOH (Proposition 22) and it is described by the Jordan equivalent equations (4.21) or (4.29) given below

$$\hat{C}(zI - \hat{A})^{-1} \sim \hat{\Gamma}(zI - \hat{J})^{-1} = \Gamma\tilde{U}(zI - \hat{J})^{-1}$$

Hence if the columns of $\Gamma\tilde{U}(zI - \hat{J})^{-1}$ are shown to be independent over the field of complex numbers, then the discretised model is said to be observable.

As $\tilde{U}$ is a block diagonal matrix with the same structure as J and as each block of $\tilde{U}$ is an upper triangular matrix, then the columns of the spectral observability matrix of the continuous model corresponding to the eigenvalue λ_i, are given as,

$$\Gamma_i^F = \left[\underline{\gamma}_{i1_1}, ..., \underline{\gamma}_{ik_1}, ..., \underline{\gamma}_{i\nu_{i,1}}\right]$$

and the first elements of the main diagonal of $\tilde{U}$ corresponding to the first column of the Jordan blocks are 1 the corresponding columns of the discretised model are the same to those of the continuous system.

Definition 42 *The i-th discrete Spectral observability matrix $\hat{\Gamma}_i^F$, is the matrix formed by the columns of $\hat{\Gamma}$ corresponding to the first columns of the Jordan blocks associated with the eigenvalue λ_i.* □

With the above notation, we have the following propositions, which are similar to these corresponding for the continuous model.

Proposition 39 *The discretised model $\hat{S}(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ of a linear, time invariant system with ZOH or FOH is observable if and only if for each $i = 1, 2, ..., f$ the columns of the $\hat{\Gamma}_i^F$ matrix are linearly independent over the field of complex numbers.* □

Remark 26 *The linear independence of the columns of* $\widehat{\Gamma}_i^F$ *are tested individually for each* i.□

In order to determine the relation between the *i-th* discrete Spectral observability matrix $\widehat{\Gamma}_i^F$ and the Spectral observability matrix Γ_i^F we have to distinguish the two cases of sampling

5.4.1 Regular Sampling

For the case of regular sampling the above analysis leads to:

Proposition 40 *Under regular sampling the* i-th *discrete Spectral observability matrix* $\widehat{\Gamma}_i^F$, *remains the same to* Γ_i^F. □

Thus for the regular sampling case we have the following results:

Proposition 41 *The discretised model* $\hat{S}(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ *of a linear, time invariant system with ZOH or FOH, under regular sampling. is observable, if and only if the corresponding linear system* $S(A, B, C, D)$ *is observable.* □

5.4.2 Irregular Sampling

We examine now the case of irregular sampling.

Proposition 42 *Under irregular sampling when collapsing occurs between a subset of* μ *eigenvalues* $\lambda_1, \lambda_2, ..., \lambda\mu \in \Phi_r(A)$ *to the distinct eigenvalue* $\hat{\lambda}_c \in \Phi\left(\hat{A}\right)$, *the* c-th *discrete Spectral observability matrix* $\widehat{\Gamma}_c^F$ *is defined as the matrix:*

$$\widehat{\Gamma}_c^F = \left[\begin{array}{cccc} \Gamma_1^F, & \Gamma_2^F, & ..., & \Gamma\mu^F \end{array}\right]$$

□

Definition 43 *Under irregular sampling for which collapsing occurs for a subset of* μ *eigenvalues* $\lambda_1, \lambda_2, ..., \lambda\mu \in \Phi_r(A)$ *to the distinct eigenvalue* $\hat{\lambda}_c \in \Phi\left(\hat{A}\right)$, *the* c-th *composite spectral observability matrix* Γ_c^F *is defined as the matrix consisting of the columns of all the Spectral observability matrices corresponding to the eigenvalues* $\lambda_1, \lambda_2, ..., \lambda\mu$:

$$\Gamma_c^F = \left[\begin{array}{cccc} \Gamma_1^F, & \Gamma_2^F, & ..., & \Gamma\mu^F \end{array}\right] = \widehat{\Gamma}_c^F$$

$\square$

From the above we have:

Proposition 43 *The discretised model* $\hat{S}(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ *corresponding to a linear, time invariant observable system, with ZOH or FOH, under irregular sampling for which collapsing occurs for a subset of* μ *eigenvalues* $\lambda_1, \lambda_2, ..., \lambda\mu \in \Phi_r(A)$ *to the distinct eigenvalue* $\hat{\lambda}_c \in \Phi\left(\hat{A}\right)$, *becomes unobservable, if and only if the* ν_c *columns of the* c-th *composite spectral observability matrix* Γ_c^F *are linearly dependent, where,* $\nu_c = \nu_1 + \nu_2 + ... + \nu\mu$. $\square$

Definition 44 *We define as structural loss of observability of the discretised model* $\hat{S}(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ *the case where* $\nu_c > m$ *(where* m *is the number of system outputs).* $\square$

Definition 45 *We define as numerical loss of observability of the discretised model* $\hat{S}(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ *the case where* $\nu_c \leq m$ *and the columns of the* c-th *composite spectral observability matrix* Γ_c^F *are linearly dependent.* $\square$

Remark 27 *The discretised model* $\hat{S}(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ *of a linear, time invariant, single output, observable system, with ZOH, under any irregular sampling, becomes unobservable.* $\square$

Example 5 *Let the spectrum controllability matrices for the eigenvalues* λ_1,λ_2,λ_3,λ_4 *of the set* $\Phi_{-5}(A)$ *in Example 3 are (number of system inputs* $l = 6$*):*

$$\mathcal{B}_1^S = \begin{bmatrix} 2-4i & 0 & 0 & 0 & 1+i & 3-4i \\ 0 & 1+5i & 0 & -1+3i & 0 & 0 \\ 1-3i & 0 & -2-7i & 1+5i & 0 & -4i \end{bmatrix}$$

$$\mathcal{B}_2^S = \begin{bmatrix} 2+4i & 0 & 0 & 0 & 1-i & 3+4i \\ 0 & 1-5i & 0 & -1-3i & 0 & 0 \\ 1+3i & 0 & -2+7i & 1-5i & 0 & 4i \end{bmatrix}$$

$$\mathcal{B}_3^S = \begin{bmatrix} 0 & 1+5i & 0 & -1+3i & 0 & 0 \\ -2+3i & 0 & -1 & 0 & 3-i & 9+4i \end{bmatrix}$$

$$\mathcal{B}_4^S = \begin{bmatrix} 0 & 1-5i & 0 & -1-3i & 0 & 0 \\ -2-3i & 0 & -1 & 0 & 3+i & 9-4i \end{bmatrix}$$

where $\text{rank}\mathcal{B}_1^S = 3$, $\text{rank}\mathcal{B}_2^S = 3$, $\text{rank}\mathcal{B}_3^S = 2$, $\text{rank}\mathcal{B}_4^S = 2$ *and the continuous system is said to be modal controllable for* λ_1, λ_2, λ_3, λ_4. *Then for the different cases of collapsing we have*

a) $T = \frac{2k\pi}{24}$,

a.1) for $k = 1, 2, 3, 5, 6, 7, 9, 10, 11, ...$

$$\mathcal{B}_{1,2}^S = \begin{bmatrix} 2-4i & 0 & 0 & 0 & 1+i & 3-4i \\ 0 & 1+5i & 0 & -1+3i & 0 & 0 \\ 1-3i & 0 & -2-7i & 1+5i & 0 & -4i \\ 2+4i & 0 & 0 & 0 & 1-i & 3+4i \\ 0 & 1-5i & 0 & -1-3i & 0 & 0 \\ 1+3i & 0 & -2+7i & 1-5i & 0 & 4i \end{bmatrix}, \ \text{rank}\mathcal{B}_{1,2}^S = 6 \Rightarrow$$

$\Rightarrow$ mode$\hat{\lambda}_{1,2}$ controllable a.2) for $k = 4, 12, 20, 28, ...$

$$\text{rank}\mathcal{B}^S_{1,2} = 6 \Rightarrow \text{mode}\hat{\lambda}_{1,2} \text{ controllable}$$

$$\mathcal{B}^S_{3,4} = \begin{bmatrix} 0 & 1+5i & 0 & -1+3i & 0 & 0 \\ -2+3i & 0 & -1 & 0 & 3-i & 9+4i \\ 0 & 1-5i & 0 & -1-3i & 0 & 0 \\ -2-3i & 0 & -1 & 0 & 3+i & 9-4i \end{bmatrix}, \text{ rank}\mathcal{B}^S_{3,4} = 4$$

$$\Rightarrow \text{mode}\hat{\lambda}_{3,4} \text{ controllable}$$

a.3) for $k = 8, 16, 24, ...$

$$\mathcal{B}^S_{1,2,3,4} = \begin{bmatrix} 2-4i & 0 & 0 & 0 & 1+i & 3-4i \\ 0 & 1+5i & 0 & -1+3i & 0 & 0 \\ 1-3i & 0 & -2-7i & 1+5i & 0 & -4i \\ 2+4i & 0 & 0 & 0 & 1-i & 3+4i \\ 0 & 1-5i & 0 & -1-3i & 0 & 0 \\ 1+3i & 0 & -2+7i & 1-5i & 0 & 4i \\ 0 & 1+5i & 0 & -1+3i & 0 & 0 \\ -2+3i & 0 & -1 & 0 & 3-i & 9+4i \\ 0 & 1-5i & 0 & -1-3i & 0 & 0 \\ -2-3i & 0 & -1 & 0 & 3+i & 9-4i \end{bmatrix}, \text{ rank}\mathcal{B}^S_{1,2,3,4} = 6$$

$$\Rightarrow \text{mode}\hat{\lambda}_{1,2,3,4} \text{ uncontrollable (structural loss)}$$

b) $T = \frac{2k\pi}{15}$,

b.1) for $k = 1, 2, 3, 4, 6, 7, 8, 9, 11, ...$

$$\mathcal{B}^S_{1,4} = \begin{bmatrix} 2-4i & 0 & 0 & 0 & 1+i & 3-4i \\ 0 & 1+5i & 0 & -1+3i & 0 & 0 \\ 1-3i & 0 & -2-7i & 1+5i & 0 & -4i \\ 0 & 1-5i & 0 & -1-3i & 0 & 0 \\ -2-3i & 0 & -1 & 0 & 3+i & 9-4i \end{bmatrix}, \text{ rank}\mathcal{B}^S_{1,4} = 5$$

$\Rightarrow$ mode$\hat{\lambda}_{1,4}$ controllable

$$\mathcal{B}^S_{2,3} = \begin{bmatrix} 2+4i & 0 & 0 & 0 & 1-i & 3+4i \\ 0 & 1-5i & 0 & -1-3i & 0 & 0 \\ 1+3i & 0 & -2+7i & 1-5i & 0 & 4i \\ 0 & 1+5i & 0 & -1+3i & 0 & 0 \\ -2+3i & 0 & -1 & 0 & 3-i & 9+4i \end{bmatrix}, \ \mathrm{rank}\mathcal{B}^S_{2,3} = 5$$

$\Rightarrow$ mode$\hat{\lambda}_{2,3}$ controllable

b.2) for $k = 5, 10, 15, \ldots$ It is, rank$\mathcal{B}^S_{1,2,3,4} = 6 \Rightarrow$mode$\hat{\lambda}_{1,2,3,4}$ uncontrollable (structural loss)

c) $T = \frac{2k\pi}{9}$,

c.1) for $k = 1, 2, 4, 5, 7, 8, \ldots$

$$\mathcal{B}^S_{1,3} = \begin{bmatrix} 2-4i & 0 & 0 & 0 & 1+i & 3-4i \\ 0 & 1+5i & 0 & -1+3i & 0 & 0 \\ 1-3i & 0 & -2-7i & 1+5i & 0 & -4i \\ 0 & 1+5i & 0 & -1+3i & 0 & 0 \\ -2+3i & 0 & -1 & 0 & 3-i & 9+4i \end{bmatrix}, \ \mathrm{rank}\mathcal{B}^S_{1,3} = 4$$

$\Rightarrow$ mode$\hat{\lambda}_{1,3}$ uncontrollable (numerical loss)

$$\mathcal{B}^S_{2,4} = \begin{bmatrix} 2+4i & 0 & 0 & 0 & 1-i & 3+4i \\ 0 & 1-5i & 0 & -1-3i & 0 & 0 \\ 1+3i & 0 & -2+7i & 1-5i & 0 & 4i \\ 0 & 1-5i & 0 & -1-3i & 0 & 0 \\ -2-3i & 0 & -1 & 0 & 3+i & 9-4i \end{bmatrix}, \ \mathrm{rank}\mathcal{B}^S_{2,4} = 4$$

$\Rightarrow$ mode$\hat{\lambda}_{2,4}$ uncontrollable (numerical loss)

c.2) for $k = 3, 6, 9, \ldots$ It is rank$\mathcal{B}^S_{1,2,3,4} = 6 \Rightarrow$mode$\hat{\lambda}_{1,2,3,4}$ uncontrollable. □

5.5 Conclusions

In this Chapter we have defined the effect of sampling on the controllability and observability properties and we have determined spectral criteria for the above properties of discretised models under regular and irregular sampling. In the next Chapter we define the effect of sampling on the dimension of the controllable and unobservable space, under the use of irregular sampling.

Chapter 6

SPECTRAL CHARACTERIZATION OF THE CONTROLLABLE (UNOBSERVABLE) SPACE AND COLLAPSING PHENOMENA

6.1 Introduction

This chapter examines the role of the system parameters of the Jordan canonical description in the determination of the dimension of the controllable (unobservable) subspace $\mathcal{R}$ $(\mathcal{P})$ of linear systems. A new test based on the properties of rows of matrix $\mathcal{B}$ and the set of *i-th* spectrum row controllability indices (r.c.i.) $\Theta_{\lambda_i}(A, B)$ is derived, using properties of cyclicity and associated minimal polynomials. Also a relation is presented between the controllability index of the system and the set of $\Theta_{\lambda_i}(A, B)$.

The above results provide an extension of the classical results on the spectral characterization of controllability (observability) and enables the study of the corresponding characteristics of the discretised model, under the different types of irregular sampling.

6.2 Spectral Characterization of the Controllable Space

We explore here the properties of Jordan decomposition in determining the dimension of the controllable subspace of a continuous system.

Under the partition (3.67) of the matrix $\mathcal{B}$, the columns $\underline{\beta}_1, \underline{\beta}_2, ..., \underline{\beta}_j, ..., \underline{\beta}_l$ are also partitioned as follows,

$$\mathcal{B} = \begin{bmatrix} \mathcal{B}_1 \\ \mathcal{B}_2 \\ \cdots \\ \mathcal{B}_i \\ \cdots \\ \mathcal{B}_f \end{bmatrix} = \left[\underline{\beta}_1, \underline{\beta}_2, ..., \underline{\beta}_j, ..., \underline{\beta}_l\right] = \begin{bmatrix} \underline{\beta}_{11} & \underline{\beta}_{12} & \cdots & \underline{\beta}_{1j} & \cdots & \underline{\beta}_{1l} \\ \underline{\beta}_{21} & \underline{\beta}_{22} & \cdots & \underline{\beta}_{2j} & \cdots & \underline{\beta}_{2l} \\ . & . & \cdots & . & \cdots & . \\ \underline{\beta}_{i1} & \underline{\beta}_{i2} & \cdots & \underline{\beta}_{ij} & \cdots & \underline{\beta}_{il} \\ . & . & \cdots & . & \cdots & . \\ \underline{\beta}_{f1} & \underline{\beta}_{f2} & \cdots & \underline{\beta}_{fj} & \cdots & \underline{\beta}_{fl} \end{bmatrix} \tag{6.1}$$

where

$$\underline{\beta}_{i1}, \underline{\beta}_{i2}, ..., \underline{\beta}_{ij}, ..., \underline{\beta}_{il} \in \mathcal{N}_i (i = 1, 2, ..., f)$$

and

$$\mathcal{B}_i \triangleq \begin{bmatrix} \underline{\beta}_{i1} & \underline{\beta}_{i2} & \cdots & \underline{\beta}_{ij} & \cdots & \underline{\beta}_{il} \end{bmatrix}, (i = 1, 2, ..., f) \tag{6.2}$$

Also, under the partition (3.69), each one of the vectors $\underline{\beta}_{ij}$ is also partitioned as :

$$\underline{\beta}_{ij} = \begin{bmatrix} \underline{\beta}_{1ij} \\ \underline{\beta}_{2ij} \\ \cdots \\ \underline{\beta}_{kij} \\ \cdots \\ \underline{\beta}_{\nu_i ij} \end{bmatrix}$$

Where

$$\underline{\beta}_{kij} \in \mathcal{V}_{ik} (j = 1, 2, ..., l), (k = 1, 2, ..., \nu_i)$$

6.2.1 The Minimal Polynomial and Spectral Controllability Properties

First we develop new criteria for determining the dimension of the controllable space based on properties of the spectral form and the concept of minimal polynomial associated with invariant spaces. We first state :

Theorem 33 *1. The minimal polynomial of the vector $\underline{\beta}_{kij} \in \mathcal{V}_{ik}$ is of the form $(s-\lambda_i)^{\delta_{kij}}$, where $\delta_{kij} \leq \tau_{ik}$.*

2. The minimal polynomial of the vector $\underline{\beta}_{ij} \in \mathcal{N}_i$ is of the form $(s-\lambda_i)^{\delta_{ij}}$, where,

$$\delta_{ij} = \max(\delta_{1ij}, \delta_{2ij}, ..., \delta_{kij}, ..., \delta_{\nu_i ij}) \text{ and } \delta_{ij} \leq \tau_{i1}$$

Proof.

1. The minimal polynomial of the elementary subspace $\mathcal{V}_{ik}$, is $(s-\lambda_i)^{\tau_{ik}}$ and the minimal polynomial of the vector $\underline{\beta}_{kij} \in \mathcal{V}_{ik}$ $(j = 1, 2, ..., t)$ is of the form $(s-\lambda_i)^{\delta_{kij}}$, where $\delta_{kij} \leq \tau_{ik}$.

2. The minimal polynomial of the generalized null-space $\mathcal{N}_i$ is $(s-\lambda_i)^{\tau_{i1}}$. The minimal polynomial of the vector $\underline{\beta}_{ij} \in \mathcal{N}_i$ $(i = 1, 2, ..., f)$ is of the form $(s-\lambda_i)^{\delta_{ij}}$, where $\delta_{ij} \leq \tau_{i1}$. Also the minimal polynomial of the vector $\underline{\beta}_{ij}$ is equal to the least common multiple of the minimal polynomials of the constituent vectors $\underline{\beta}_{1ij}, \underline{\beta}_{2ij}, ..., \underline{\beta}_{kij}, ..., \underline{\beta}_{\nu_i ij}$ i.e. to the least common multiple of $(s-\lambda_1)^{\delta_{1ij}}$, $(s-\lambda_i)^{\delta_{2ij}}$, ..., $(s-\lambda_i)^{\delta_{kij}}$, ..., $(s-\lambda_i)^{\delta_{\nu_i ij}}$ and so $\delta_{ij} = \max(\delta_{1ij}, \delta_{2ij}, ..., \delta_{kij}, ..., \delta_{\nu_i ij})$. □

Lemma 7 *The degree δ_{kij} of the minimal polynomial of the vector $\underline{\beta}_{kij} \in \mathcal{V}_{ik}$ $(j = 1, 2, ..., l)$ is given by the order of its last non zero element.*

Proof. According to the definition of the minimal polynomial of a vector, the vector $\underline{\beta}_{kij}$ is annihilated by the matrix,

$$(J_{ik} - \lambda_i I_{ik})^{\delta_{kij}-1} = (H_{ik})^{\delta_{kij}-1}$$

but not by the matrix,

$$(J_{ik} - \lambda_i I_{ik})^{\delta_{kij}} = (H_{ik})^{\delta_{kij}}$$

where $I_{ik}, H_{ik} \in \mathbb{C}^{\tau_{ik} \times \tau_{ik}}$ are correspondingly, the unit and nilpotent matrices of order τ_{ik}. So we have $(H_{ik})^{\delta_{kij}} \underline{\beta}_{kij} \neq \underline{0}$, and $(H_{ik})^{\delta_{kij}-1} \underline{\beta}_{kij} = \underline{0}$. Thus, if:

$$\underline{\beta}_{kij} = \left[x_1, x_2, ..., x_{\delta_{kij}-1}, x_{\delta_{kij}}, x_{\delta_{kij}+1}, ..., x_{\tau_{ik}}\right]^{\top}$$

then from the above two relations we conclude $x_{\delta_{kij}+1} = ... = x_{\tau_{ik}} = 0$ and $x_{\delta_{kij}} \neq 0$ and the lemma is proved. □

Lemma 8 *The degree δ_{ij} of the minimal polynomial of the vector $\beta_{ij} \in N_i$ $(j = 1, 2, ..., l)$ is given by the order, of the last non zero element, of its constituent vector with minimal polynomial of maximum degree.* □

The controllability matrix $Q \in \mathbb{R}^{n \times nl}$ of a continuous system $S(A, B, C, D)$, is defined in (3.55). Under the transformation of the system $S(A, B, C, D)$ to the Jordan equivalent $S_J(J, \mathcal{B}, \Gamma. \Delta)$ the controllability matrix Q is also equivalent to the matrix Q_J i.e.:

$$Q \sim Q_J \triangleq \left[\mathcal{B}, J\mathcal{B}, ..., J^{n-1}\mathcal{B}\right]$$

Let matrix Q_J be also partitioned according to (3.18) :

$$Q_J = \begin{bmatrix} Q_{J_1} \\ Q_{J_2} \\ \cdots \\ Q_{J_i} \\ \cdots \\ Q_{J_f} \end{bmatrix}$$

then from (6.1) it follows that,

$$Q_{J_i} = \begin{bmatrix} \mathcal{B}_i & J(\lambda_i)\mathcal{B}_i & (J(\lambda_i))^2 \mathcal{B}_i & ... & (J(\lambda_i))^{n-1} \mathcal{B}_i \end{bmatrix} \tag{6.3}$$

and from (6.2) we have,

$$\begin{aligned} Q_{J_i} &= \Big[\underline{\beta}_{i1},...,\underline{\beta}_{ij},...,\underline{\beta}_{il}, J(\lambda_i)\underline{\beta}_{i1},...,J(\lambda_i)\underline{\beta}_{ij},...,J(\lambda_i)\underline{\beta}_{il},... \\ &\quad ...,(J(\lambda_i))^{n-1}\underline{\beta}_{i1},...,(J(\lambda_i))^{n-1}\underline{\beta}_{ij},...,(J(\lambda_i))^{n-1}\underline{\beta}_{il}\Big] \end{aligned}$$

All the column vectors of Q_{J_i} span the A-invariant subspace $\mathcal{R}_i$ of the generalized A-invariant null-space $\mathcal{N}_i$ ($\mathcal{R}_i \subseteq \mathcal{N}_i$). Let r_i be the number of linearly independent column vectors of Q_{J_i} defining a basis for the subspace $\mathcal{R}_i$, then

$$\dim \mathcal{R}_i = r_i = \mathrm{rank} Q_{J_i} \leq \dim \mathcal{N}_i = \tau_{i1} + ... + \tau_{ik} + ... + \tau_{i\nu_i} = \pi_i \tag{6.4}$$

Theorem 34 *The dimension r of the controllable space $\mathcal{R}$ is given by the sum of the dimensions r_i of the controllable subspaces $\mathcal{R}_i$,*

$$r = r_1 + r_2 + ... + r_i + ... + r_f$$

Proof. Let $\delta_{i1},....\delta_{ij},...,\delta_{il}$ be the degrees of the minimal polynomials corresponding to the vectors $\underline{\beta}_{i1},...,\underline{\beta}_{ij},...,\underline{\beta}_{il}$ and let $\underline{\beta}_{i\mu}$ be the vector with the minimal polynomial $(s-\lambda_i)^{\delta_{i\mu}}$ of maximum degree. Then according to Theorem 1 we have, $\delta_{i\mu} = \max(\delta_{\mu i1},...,\delta_{\mu ij},...,\delta_{\mu il})$.

Since all the vectors of $\mathcal{R}_i$ are generated by chains of the vectors $\underline{\beta}_{i1},...,\underline{\beta}_{ij},...,\underline{\beta}_{il}$ and the minimal polynomial of $\mathcal{R}_i$ is equal to the least common multiple of the basis vectors, the minimal polynomial of $\mathcal{R}_i$ is also equal to $(s-\lambda_i)^{\delta_{i\mu}}$ and $\delta_{i\mu} = \delta_i$ $(i = 1, 2, ..., f)$. Then, the minimal polynomial of $\mathcal{R}$ is the product of the co-prime polynomials,

$$(s-\lambda_1)^{\delta_1},...,(s-\lambda_i)^{\delta_i},...,(s-\lambda_f)^{\delta_f}$$

and according to the Theorem 4 we have,

$$\mathcal{R} = \mathcal{R}_1 \oplus ... \oplus \mathcal{R}_i \oplus ... \oplus \mathcal{R}_f \subseteq \mathcal{R}^n$$

where, $r = r_1 + ... + r_i + ... + r_f \leq n$ and Theorem 34 is proved. □

6.2.2 The set of *i-th* Spectrum Row Controllability Indices

By Theorem 34 it is concluded that the dimension r of the controllable space $\mathcal{R}$ is defined from the dimensions r_i of $\mathcal{R}_i$ for $i = 1, 2, ..., f$. Thus, in the following we concentrate on the definition of the dimension r_i of the controllable subspace $\mathcal{R}_i$ corresponding to only one eigenvalue λ_i or equivalently, to the rank of matrix Q_{J_i}.

Since $J(\lambda_i) = \lambda_i I + H_i$ where H_i is a nilpotent block diagonal matrix of the same block structure as $J(\lambda_i)$, i.e.

$$H_i = \text{diag}\{H_{i1}, ..., H_{ik}, ..., H_{i\nu_i}\}, \text{ where, } H_{ik} = \begin{bmatrix} 0 & 1 & ... & 0 & 0 \\ 0 & 0 & ... & 0 & 0 \\ . & . & ... & . & . \\ 0 & 0 & ... & 0 & 1 \\ 0 & 0 & ... & 0 & 0 \end{bmatrix} \in \mathbb{R}^{\tau_{ik} \times \tau_{ik}} \tag{6.5}$$

then from (6.3) we have,

$$Q_{J_i} = \begin{bmatrix} \mathcal{B}_i & (\lambda_i I + H_i)\mathcal{B}_i & ... & (\lambda_i I + H_i)^{n-1}\mathcal{B}_i \end{bmatrix}$$

Using only column operations on the above matrix we have,

$$Q_{J_i} \sim Q_{H_i} \triangleq \begin{bmatrix} \mathcal{B}_i & H_i\mathcal{B}_i & ... & (H_i)^{n-1}\mathcal{B}_i \end{bmatrix} \tag{6.6}$$

Let now the rows matrix $\mathcal{B}_i$ be partitioned according to the Jordan structure, corresponding to $\wp_{\lambda_i}(A)$ as in (3.69) and (3.71). Then the matrix Q_{H_i} can be described as follows,

$$Q_{H_i} = \begin{bmatrix} Q_{H_{i1}} \\ ..... \\ Q_{H_{ik}} \\ ..... \\ Q_{H_{i\nu_i}} \end{bmatrix}$$

where

$$Q_{H_{i1}} = \begin{bmatrix} \underbrace{\begin{matrix} \underline{\beta}^{\top}_{i1_1} & \underline{\beta}^{\top}_{i1_2} & \dots & \underline{\beta}^{\top}_{i1_{\tau_{i1}-1}} & \underline{\beta}^{\top}_{i1_{\tau_{i1}}} \\ \underline{\beta}^{\top}_{i1_2} & \underline{\beta}^{\top}_{i1_3} & \dots & \underline{\beta}^{\top}_{i1_{\tau_{i1}}} & 0 \\ . & . & \dots & . & . \\ \underline{\beta}^{\top}_{i1_{\tau_{i1}-1}} & \underline{\beta}^{\top}_{i1_{\tau_{i1}}} & \dots & 0 & 0 \\ \underline{\beta}^{\top}_{i1_{\tau_{i1}}} & 0 & \dots & 0 & 0 \end{matrix}}_{\tau_{i1} \text{ columns}} \end{bmatrix}$$

$$\cdots\cdots\cdots$$

$$Q_{H_{ik}} = \begin{bmatrix} \underbrace{\begin{matrix} \underline{\beta}^{\top}_{ik_1} & \underline{\beta}^{\top}_{ik_2} & \dots & \underline{\beta}^{\top}_{ik_{\tau_{ik}-1}} & \underline{\beta}^{\top}_{ik_{\tau_{ik}}} \\ \underline{\beta}^{\top}_{ik_2} & \underline{\beta}^{\top}_{ik_3} & \dots & \underline{\beta}^{\top}_{ik_{\tau_{ik}}} & 0 \\ . & . & \dots & . & . \\ \underline{\beta}^{\top}_{ik_{\tau_{ik}-1}} & \underline{\beta}^{\top}_{ik_{\tau_{ik}}} & \dots & 0 & 0 \\ \underline{\beta}^{\top}_{ik_{\tau_{ik}}} & 0 & \dots & 0 & 0 \end{matrix}}_{\tau_{ik} \text{ columns}} \end{bmatrix}$$

$$\cdots\cdots\cdots$$

$$Q_{H_{i\nu_i}} = \begin{bmatrix} \underbrace{\begin{matrix} \underline{\beta}^{\top}_{i\nu_{i1}} & \underline{\beta}^{\top}_{i\nu_{i2}} & \dots & \underline{\beta}^{\top}_{i\nu_{i\tau_{i\nu_i}-1}} & \underline{\beta}^{\top}_{i\nu_{i\tau_{i\nu_i}}} \\ \underline{\beta}^{\top}_{i\nu_{i2}} & \underline{\beta}^{\top}_{i\nu_{i3}} & \dots & \underline{\beta}^{\top}_{i\nu_{i\tau_{i\nu_i}}} & 0 \\ . & . & \dots & . & . \\ \underline{\beta}^{\top}_{i\nu_{i\tau_{i\nu_i}-1}} & \underline{\beta}^{\top}_{i\nu_{i\tau_{i\nu_i}}} & \dots & 0 & 0 \\ \underline{\beta}^{\top}_{i\nu_{i\tau_{i\nu_i}}} & 0 & \dots & 0 & 0 \end{matrix}}_{\tau_{i\nu_i} \text{ columns}} \end{bmatrix}$$

Searching for the linear independent rows on Q_{H_i} from top to bottom, we have the following remarks :

Remark 28 *For the matrix Q_{H_i} we have the properties:*

1. *If the last row of $\mathcal{B}_{i1}$ is nonzero: $\underline{\beta}^{\top}_{i1_{\tau_{i1}}} \neq \underline{0}^{\top}$ then all the τ_{i1} rows of the block $Q_{H_{i1}}$ are linearly independent.*

2. *If the* ξ_1 *last rows of* $\mathcal{B}_{i1}$ *are zero:* $\underline{\beta}^{\top}_{i1_{\tau_{i1}}} = \underline{0}^{\top}$, $\underline{\beta}^{\top}_{i1_{\tau_{i1}-1}} = \underline{0}^{\top}$, ..., $\underline{\beta}^{\top}_{i1_{\tau_{i1}-\xi_1+1}} = \underline{0}^{\top}$ *and* $\underline{\beta}^{\top}_{i1_{\tau_{i1}-\xi_1}} \neq \underline{0}^{\top}$ *then all the first* $\tau_{i1} - \xi_1$ *rows of the block* $Q_{H_{i1}}$ *are linearly independent.*

3. *If the last nonzero row of* $\mathcal{B}_{i2}$: $\underline{\beta}^{\top}_{i2_{\tau_{i2}-\xi_2}} \neq \underline{0}^{\top}$ *is linearly independent of the last nonzero row of* $\mathcal{B}_{i1}$: $\underline{\beta}^{\top}_{i1_{\tau_{i1}-\xi_1}}$, *then all the first* $\tau_{i2} - \xi_2$ *rows of the block* $Q_{H_{i2}}$ *are linearly independent.*

4. *If the last nonzero row of* $\mathcal{B}_{i2}$: $\underline{\beta}^{\top}_{i2_{\tau_{i2}-\xi_2}} \neq \underline{0}^{\top}$ *is linearly dependent on the last nonzero row of* $\mathcal{B}_{i1}$: $\underline{\beta}^{\top}_{i1_{\tau_{i1}-\xi_1}}$, *but the row vector consisting of the last two nonzero rows of* $\mathcal{B}_{i2}$: $\left[\underline{\beta}^{\top}_{i2_{\tau_{i2}-\xi_2-1}}, \underline{\beta}^{\top}_{i2_{\tau_{i2}-\xi_2}}\right]$, *is linearly independent of the row vector formed by the two last nonzero rows of* $\mathcal{B}_{i1}$: $\left[\underline{\beta}^{\top}_{i1_{\tau_{i1}-\xi_1-1}}, \underline{\beta}^{\top}_{i1_{\tau_{i1}-\xi_1}}\right]$, *then all the first* $\tau_{i2} - \xi_2 - 1$ *rows of the block* $Q_{H_{i2}}$ *are linearly independent.*

5. *If the above row vector consisting of the last two nonzero rows of* $\mathcal{B}_{i2}$: $\left[\underline{\beta}^{\top}_{i2_{\tau_{i2}-\xi_2-1}}, \underline{\beta}^{\top}_{i2_{\tau_{i2}-\xi_2}}\right]$, *is linearly dependent on the above row vector formed by the last nonzero rows of* $\mathcal{B}_{i1}$: $\left[\underline{\beta}^{\top}_{i1_{\tau_{i1}-\xi_1-1}}, \underline{\beta}^{\top}_{i1_{\tau_{i1}-\xi_1}}\right]$, *but the row vector consisting of the last three nonzero rows of* $\mathcal{B}_{i2}$: $\left[\underline{\beta}^{\top}_{i2_{\tau_{i2}-\xi_2-2}}, \underline{\beta}^{\top}_{i2_{\tau_{i2}-\xi_2-1}}, \underline{\beta}^{\top}_{i2_{\tau_{i2}-\xi_2}}\right]$, *is linearly independent of the row vector formed by the last three nonzero rows of* $\mathcal{B}_{i1}$: $\left[\underline{\beta}^{\top}_{i1_{\tau_{i1}-\xi_1-2}}, \underline{\beta}^{\top}_{i1_{\tau_{i1}-\xi_1-1}}, \underline{\beta}^{\top}_{i1_{\tau_{i1}-\xi_1}}\right]$, *then all the first* $\tau_{i2} - \xi_2 - 2$ *rows of the block* $Q_{H_{i2}}$ *are linearly independent.*

6. *If the last nonzero row of* $\mathcal{B}_{i3}$: $\underline{\beta}^{\top}_{i3_{\tau_{i2}-\xi_3}} \neq \underline{0}^{\top}$ *is linearly independent of the above last nonzero row of* $\mathcal{B}_{i1}$ *and the possibly linear independent last nonzero row of* $\mathcal{B}_{i2}$ *then all the first* $\tau_{i3} - \xi_3$ *rows of the block* $Q_{H_{i3}}$ *are linearly independent.*

7. *Following the same procedure we determine the number of linearly independent rows from top to bottom for each block of* Q_{H_i}. □

Let $\theta_{i1}, \theta_{i2}, ..., \theta_{i\nu_i}$ be the number denoting the orders of the above defined rows into each one block of Q_{H_i} and let the blocks be rearranged from top to bottom in a way such that :

$$\theta_{i1} \geq \theta_{i2} \geq ... \geq \theta_{i\nu_i} \geq 0$$

The above remark summarizes the conditions for characterization of controllability in spectral form. This result can also be used to provide a characterization of i.d.z. as it will be shown below.

Definition 46 *The set of the above numbers is defined as the set of the* i-th *spectrum row controllability indices (r.c.i.) of* A, B :

$$\Theta(A,B)_{\lambda_i} = \{\theta_{i1} \geq \theta_{i2} \geq \ldots \geq \theta_{i\nu_i} \geq 0\} \tag{6.7}$$

From the above the following propositions are directly concluded. □

Theorem 35 *The dimension* r_i *of the controllable subspace* $\mathcal{R}_i$ *is given as,*

$$r_i = \theta_{i1} + \theta_{i2} + \ldots + \theta_{i\nu_i}$$

Proof. From the construction of the set $\Theta(A,B)_{\lambda_i}$ (Remark 28) it is :

$$\theta_{i1} + \theta_{i2} + \ldots + \theta_{i\nu_i} = \text{rank} Q_{H_i}$$

from (6.6) it is : $\text{rank}Q_{H_i} = \text{rank}Q_{J_i}$ and from (6.4) the Theorem is proved. □

Proposition 44 *The mode* $(\lambda_i, U(\lambda_i), V(\lambda_i))$ *of the system* $S(A,B)$ *is controllable, if and only if the set* $\Theta(A,B)_{\lambda_i}$ *coincides with the set* $\wp_{\lambda_i}(A)$:

$$\Theta(A,B)_{\lambda_i} \equiv \wp_{\lambda_i}(A)$$

Proof. According to Theorem 17 the mode $(\lambda_i, U(\lambda_i), V(\lambda_i))$ is controllable if and only if the rows of the *i-th* spectrum controllability matrix $\mathcal{B}_i^S$ are linearly independent over the field of complex numbers. From the definition of the set $\Theta(A,B)_{\lambda_i}$ we conclude that this is equivalent to the statement of the Proposition. □

Corollary 2 *The first of the* i-th *spectrum r.c.i.* θ_{i1} *is equal to the degree of the minimal polynomial of the controllable subspace* R_i. □

Corollary 3 *The minimal polynomial of the whole controllable space* $\mathcal{R}$ *is given by*

$$(s-\lambda_1)^{\theta_{11}}(s-\lambda_2)^{\theta_{21}} \ldots (s-\lambda_i)^{\theta_{i1}} \ldots (s-\lambda_i)^{\theta_{f1}} \tag{6.8}$$

and the degree d of the minimal polynomial of the whole controllable space $\mathcal{R}$ is given as,

$$d = \theta_{11} + \theta_{21} + ... + \theta_{i1} + ... + \theta_{f1} \tag{6.9}$$

□

Definition 47 *We define as normal structure of Q_{H_t}, the matrix Q_{H_t} for which all the nonzero r.c.i. are determined by the orders of the two last nonzero rows of the row blocks.* □

6.2.3 Spectral Restriction of the Controllability Index

Some further results related to the controllability index are given below,

Proposition 45 *The degree d of the minimal polynomial of the whole controllable space $\mathcal{R}$ is equal to the degree of the minimal polynomial of the subspace determined by the columns of matrix B.*

Proof. The minimal polynomial of the subspace determined by the columns of matrix B is the least common multiple of the minimal polynomial of the basis vectors. As all the basis vectors of $\mathcal{R}$ are derived from chains of the column vectors of B , $\underline{\beta}_1, \underline{\beta}_2, ..., \underline{\beta}_l$ the common multiple of their minimal polynomials coincides with the common multiple of the minimal polynomials of $\underline{\beta}_1, \underline{\beta}_2, ..., \underline{\beta}_l$. □

We define the partial controllability matrix [Che., 1],

$$Q_k \triangleq \left[B, AB, ..., A^k B\right] \quad k = 0, 1, 2, ...$$

where matrix $Q \triangleq Q_{n-1}$ is the controllability matrix. Searching for the linear independent columns of Q_k from left to right, let r_i be the number of linearly dependent columns in $A^i B$ for $i = 1, 2, ..k$ and r_0 be the number of linearly dependent columns of B. Then it is,

$$0 \leq r_0 \leq r_1 \leq ... \leq r_k$$

and,

$$\text{rank}Q_0 < \text{rank}Q_1 < ... < \text{rank}Q_{\mu-1} = \text{rank}Q\mu = ... = \text{rank}Q_{n-1}$$

Hence the property of controllability of $S(A,B)$ can be checked from $Q_{\mu-1}$ and μ is defined as the *controllability index* of the system $S(A,B)$.

It is clear that the following equivalence relations hold true,

$$\begin{aligned} Q_{\mu-1} &\sim \left[\mathcal{B}, J\mathcal{B}, ..., J^{\mu-1}\mathcal{B}\right] = \\ &= \left[\underline{\beta}_1, \underline{\beta}_2, ..., \underline{\beta}_l, J\underline{\beta}_1, J\underline{\beta}_2, ..., J\underline{\beta}_l, ..., J^{\mu-1}\underline{\beta}_1, J^{\mu-1}\underline{\beta}_2, ..., J^{\mu-1}\underline{\beta}_l\right] \end{aligned} \tag{6.10}$$

and the linear independent columns of the above matrix define a basis for the controllable space $\mathcal{R}$.

Proposition 46 *For the controllability index μ of the system $S(A,B)$ we have:*

$$\mu \leq \theta_{11} + \theta_{21} + ... + \theta_{i1} + ... + \theta_{f1} = d$$

Proof. As all the column vectors in (6.10) belong to chains generated by the column vectors of B and given that the maximum possible chain length of linearly independent column vectors is given by the degree of the minimal polynomial of $\mathcal{R}$ then the result follows from (6.9). □

Example 6 *The sets of r.c.i. corresponding to the controllable continuous system eigenvalues* λ_1,λ_2.λ_3,λ_4 *of the set* $\Phi_{-5}(A)$ *in the Example 5 are given as follows:*

1. *mode* $\lambda_1 : \Theta(A,B)_{\lambda_1} = \wp_{\lambda_1}(A) = \{6,3,1\}$, *the dimension of the controllable subspace* $\mathcal{R}_1$ *is* $r_1 = 6+3+1 = 10$
2. *mode* $\lambda_2 : \Theta(A,B)_{\lambda_2} = \wp_{\lambda_2}(A) = \{6,3,1\}$, $r_2 = 6+3+1 = 10$
3. *mode* $\lambda_3 : \Theta(A,B)_{\lambda_3} = \wp_{\lambda_3}(A) = \{3,2\}$, $r_2 = 3+2 = 5$
4. *mode* $\lambda_4 : \Theta(A,B)_{\lambda_4} = \wp_{\lambda_4}(A) = \{3,2\}$, $r_4 = 3+2 = 5$

The dimension of the whole controllable space for the set $\Phi_{-5}(A)$ *is :* $d = 10+10+5+5 = 30$.
Spectral restriction of the corresponding controllability index : $\mu \leq 6+6+3+3 = 18$. □

Example 7 *Let us now consider an uncontrollable system, of the same eigenstructure as that of the above set* $\Phi_{-5}(A)$ *and of the following row blocks of* $\mathcal{B}$ *and corresponding r.c.i. :*

1. *For the mode* λ_1 *we have:*

$$\mathcal{B}_1 = \left\{ \begin{array}{l} \mathcal{B}_{11} = \begin{bmatrix} -3-i & 5+4i & 0 & 0 & 7 & 2i \\ -9-5i & 0 & -6 & 0 & 0 & 2+3i \\ 2+6i & 0 & 0 & -3+2i & -9-5i & -7+2i \\ 0 & -2i & 0 & 0 & 3i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \leftarrow \theta_{11} = 4 \\ \mathcal{B}_{12} = \begin{bmatrix} 0 & 0 & 2-i & -5+3i & 0 & 0 \\ 0 & -3+3i & 0 & 3-2i & 7i & -9i \\ -4-2i & 0 & 0 & 0 & -4-i & 0 \end{bmatrix} \leftarrow \theta_{12} = 3 \\ \mathcal{B}_{13} = \begin{bmatrix} 2+6i & 0 & 0 & 5-9i & 0 & 0 \end{bmatrix} \leftarrow \theta_{13} = 1 \end{array} \right\}$$

$\Rightarrow \Theta(A,B)_{\lambda_1} = \{4,3,1\}$, $r_1 = 4+3+1 = 8$

2. For the mode λ_2 we have:

$$\mathcal{B}_2 = \left\{ \begin{array}{l} \mathcal{B}_{21} = \begin{bmatrix} -3+i & 5-4i & 0 & 0 & 7 & -2i \\ -9+5i & 0 & -6 & 0 & 0 & 2-3i \\ 2-6i & 0 & 0 & -3-2i & -9+5i & -7-2i \\ 0 & 2i & 0 & 0 & -3i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \leftarrow \theta_{21} = 4 \\ \mathcal{B}_{22} = \begin{bmatrix} 0 & 0 & 2+i & -5-3i & 0 & 0 \\ 0 & -3-3i & 0 & 3+2i & -7i & 9i \\ -4+2i & 0 & 0 & 0 & -4+i & 0 \end{bmatrix} \leftarrow \theta_{22} = 3 \\ \mathcal{B}_{23} = \begin{bmatrix} 2+6i & 0 & 0 & 5-9i & 0 & 0 \end{bmatrix} \leftarrow \theta_{23} = 1 \end{array} \right\}$$

$$\Rightarrow \Theta(A,B)_{\lambda_2} = \{4,3,1\}\,,\ r_2 = 4+3+1 = 8$$

3. For the mode λ_3 we have:

$$\mathcal{B}_3 = \left\{ \begin{array}{l} \mathcal{B}_{31} = \begin{bmatrix} 3-8i & 3 & 0 & 0 & -2+3i & 0 \\ 0 & 0 & -1+i & 0 & 8i & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \leftarrow \theta_{31} = 2 \\ \mathcal{B}_{32} = \begin{bmatrix} 0 & -8i & 0 & 0 & 12i & 0 \end{bmatrix} \leftarrow \theta_{32} = 1 \end{array} \right\}$$

$$\Rightarrow \Theta(A,B)_{\lambda_3} = \{2,1\}\,,\ r_3 = 2+1 = 3$$

4. For the mode λ_4 we have:

$$\mathcal{B}_4 = \left\{ \begin{array}{l} \mathcal{B}_{41} = \begin{bmatrix} 3+8i & 3 & 0 & 0 & -2-3i & 0 \\ 0 & 0 & -1-i & 0 & -8i & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \leftarrow \theta_{41} = 2 \\ \mathcal{B}_{42} = \begin{bmatrix} 0 & 8i & 0 & 0 & -12i & 0 \end{bmatrix} \leftarrow \theta_{42} = 1 \end{array} \right\}$$

$$\Rightarrow \Theta(A,B)_{\lambda_4} = \{2,1\}\,,\ r_4 = 2+1 = 3$$

The dimension of the whole controllable space for the set $\Phi_{-5}(A)$ is $r = r_1 + r_2 + r_3 + r_4 = 22$. The spectral restriction of the corresponding controllability index is $\mu = 4 + 4 + 2 + 2 = 12$. □

6.3 Spectral Characterization of the Unobservable Space

A similar analysis may now be applied for the study of unobservability. Under the partition (3.68) of matrix Γ, the row vectors $\underline{\gamma}_1^{\top}, \underline{\gamma}_2^{\top}, ..., \underline{\gamma}_j^{\top}, ..., \underline{\gamma}_m^{\top}$ are also partitioned as follows,

$$\Gamma = \begin{bmatrix} \Gamma_1 & ... & \Gamma_i & ... & \Gamma_f \end{bmatrix} = \begin{bmatrix} \underline{\gamma}_1^{\top} \\ \underline{\gamma}_2^{\top} \\ \cdots \\ \underline{\gamma}_j^{\top} \\ \cdots \\ \underline{\gamma}_m^{\top} \end{bmatrix} = \begin{bmatrix} \underline{\gamma}_{11}^{\top} & \underline{\gamma}_{21}^{\top} & \cdots & \underline{\gamma}_{i1}^{\top} & \cdots & \underline{\gamma}_{f1}^{\top} \\ \underline{\gamma}_{12}^{\top} & \underline{\gamma}_{22}^{\top} & \cdots & \underline{\gamma}_{i2}^{\top} & \cdots & \underline{\gamma}_{f2}^{\top} \\ \cdot & \cdot & \cdots & \cdot & \cdots & \cdot \\ \underline{\gamma}_{1j}^{\top} & \underline{\gamma}_{2j}^{\top} & \cdots & \underline{\gamma}_{ij}^{\top} & \cdots & \underline{\gamma}_{fj}^{\top} \\ \cdot & \cdot & \cdots & \cdot & \cdots & \cdot \\ \underline{\gamma}_{1m}^{\top} & \underline{\gamma}_{2m}^{\top} & \cdots & \underline{\gamma}_{im}^{\top} & \cdots & \underline{\gamma}_{fm}^{\top} \end{bmatrix} \tag{6.11}$$

where $\underline{\gamma}_{i1}^{\top}, \underline{\gamma}_{i2}^{\top}, ..., \underline{\gamma}_{ij}^{\top}, ..., \underline{\gamma}_{im}^{\top} \in \mathcal{O}_i$ $(i = 1, 2, ..., f)$ and

$$\Gamma_i \triangleq \begin{bmatrix} \underline{\gamma}_{i1}^{\top} \\ \underline{\gamma}_{i2}^{\top} \\ \cdot \\ \underline{\gamma}_{ij}^{\top} \\ \cdot \\ \underline{\gamma}_{im}^{\top} \end{bmatrix}, (i = 1, 2, ..., f) \tag{6.12}$$

Also, under (3.19) each one of the A-invariant generalized row null-spaces $\mathcal{O}_i$ is decomposed to the A-invariant and cyclic row elementary subspaces corresponding to the Segré Characteristic $\wp_{\lambda_i}(A)$,

$$\mathcal{O}_i = \mathcal{F}_{i1} \oplus \mathcal{F}_{i2} \oplus ... \oplus \mathcal{F}_{ik} \oplus ... \oplus \mathcal{F}_{i\nu_i}$$

and under this decomposition, each one of the row vectors $\underline{\gamma}_{ij}^{\top}$ is partitioned as in (3.70) i.e.

$$\underline{\gamma}_{ij}^{\top} = \begin{bmatrix} \underline{\gamma}_{1ij}^{\top} & \underline{\gamma}_{2ij}^{\top} & \cdots & \underline{\gamma}_{kij}^{\top} & \cdots & \underline{\gamma}_{\nu_i ij}^{\top} \end{bmatrix}$$

Where $\underline{\gamma}_{kij}^{\top} \in \mathcal{F}_{ik}$ $(j = 1, 2, ..., m)$, $(k = 1, 2, ..., \nu_i)$.

6.3.1 The Minimal Polynomial and Unobservability Properties

The corresponding dual to controllability criteria for determining the dimension of the unobservable space based on properties of the spectral form and the concept of minimal polynomial associated with invariant spaces, are given below:

Theorem 36 *1. The minimal polynomial of the row vector $\underline{\gamma}_{kij}^{\top} \in \mathcal{F}_{ik}$ is of the form $(s - \lambda_i)^{\varepsilon_{kij}}$, where $\varepsilon_{kij} \leq \tau_{ik}$.*

2. The minimal polynomial of the row vector $\underline{\gamma}_{ij}^{\top} \in \mathcal{O}_i$ is of the form $(s - \lambda_i)^{\varepsilon_{ij}}$, where $\varepsilon_{ij} = \max(\varepsilon_{1ij}, \varepsilon_{2ij}, ..., \varepsilon_{kij}, ..., \varepsilon_{\nu_i ij})$ and $\varepsilon_{ij} \leq \tau_{i1}$.

Proof.

1. The minimal polynomial of the elementary subspace $\mathcal{F}_{ik}$, is $(s - \lambda_i)^{\tau_{ik}}$ and the minimal polynomial of the row vector $\underline{\gamma}_{kij}^{\top} \in \mathcal{F}_{ik}$ $(j = 1, 2, ..., t)$ is of the form $(s - \lambda_i)^{\varepsilon_{kij}}$, where $\varepsilon_{kij} \leq \tau_{ik}$.

2. The minimal polynomial of the generalized null-space $\mathcal{O}_i$ is $(s - \lambda_i)^{\tau_{i1}}$. The minimal polynomial of the row vector $\underline{\gamma}_{ij}^{\top} \in \mathcal{O}_i$ $(i = 1, 2, ..., f)$ is of the form $(s - \lambda_i)^{\varepsilon_{ij}}$, where $\varepsilon_{ij} \leq \tau_{i1}$. Also the minimal polynomial of the row vector $\underline{\gamma}_{ij}^{\top}$ is equal to the least common multiple of the minimal polynomials of the constituent vectors $\underline{\gamma}_{1ij}^{\top}, \underline{\gamma}_{2ij}^{\top}, ..., \underline{\gamma}_{kij}^{\top}, ..., \underline{\gamma}_{\nu_i ij}^{\top}$ i.e. to the least common multiple of

$$(s - \lambda_1)^{\varepsilon_{1ij}}, (s - \lambda_i)^{\varepsilon_{2ij}}, ..., (s - \lambda_i)^{\varepsilon_{kij}}, ...(s - \lambda_i)^{\varepsilon_{\nu_i ij}}$$

   and so $\varepsilon_{ij} = \max(\varepsilon_{1ij}, \varepsilon_{2ij}, ..., \varepsilon_{kij}, ..., \varepsilon_{\nu_i ij})$. □

Lemma 9 *The degree ε_{kij} of the minimal polynomial of the row vector $\underline{\gamma}_{kij}^{\top} \in \mathcal{F}_{ik}$ $(j = 1, 2, ..., m)$ is given by the order (measured from right to left) of its last non zero element.* □

Lemma 10 *The degree ε_{ij} of the minimal polynomial of the row vector $\underline{\gamma}_{ij}^{\top} \in \mathcal{O}_i$ $(j = 1, 2, ..., m)$ is given by the order (measured from right to left) of the last non zero element, of its constituent vector with minimal polynomial of maximum degree.* □

The observability matrix $M \in \mathbb{R}^{mn \times n}$ of a continuous system $S(A, B, C, D)$, is defined in (3.56). Under the transformation of the system $S(A, B, C, D)$ to the Jordan equivalent $S_J(J, \mathcal{B}, \Gamma, \Delta)$ the observability matrix M is also equivalent to the matrix M_J i.e.:

$$M \sim M_J \triangleq \begin{bmatrix} \Gamma \\ \Gamma J \\ \dots \\ \Gamma J^{n-1} \end{bmatrix}$$

Let the matrix M_J be also partitioned according to (3.18) as:

$$M_J = \begin{bmatrix} M_{J_1} & M_{J_2} & \dots & M_{J_i} & \dots & M_{J_f} \end{bmatrix}$$

Then from (6.11) follows that :

$$M_{J_i} = \begin{bmatrix} \Gamma_i \\ \Gamma_i J(\lambda_i) \\ \Gamma_i \left(J(\lambda_i)\right)^2 \\ \dots \\ \Gamma_i \left(J(\lambda_i)\right)^{n-1} \end{bmatrix} \tag{6.13}$$

and from (6.12) we have:

$$M_{J_i} = \begin{bmatrix} \underline{\gamma}_{i1}^{\top} \\ \dots \\ \underline{\gamma}_{im}^{\top} \\ \dots \\ \dots \\ \underline{\gamma}_{i1}^{\top} J(\lambda_i)^{n-1} \\ \dots \\ \underline{\gamma}_{im}^{\top} J(\lambda_i)^{n-1} \end{bmatrix}$$

As in the case of controllability, the q_i linearly independent rows of M_{J_i} define a row vector space $\mathcal{Q}_i \subseteq \mathcal{O}_i$ with minimal polynomial $(s - \lambda_i)^{\varepsilon_{i\mu}}$ where $\varepsilon_{i\mu}$ is the degree of the minimal polynomial of the row vector with the minimal polynomial of maximum degree.

The right null-space of M_{J_i} determines the A-invariant subspace $\mathcal{P}_i$ of the generalized A-invariant null-space $\mathcal{N}_i$ ($\mathcal{P}_i \subseteq \mathcal{N}_i$).Then it is,

$$\dim \mathcal{P}_i \triangleq p_i = \pi_i - q_i = \pi_i - \operatorname{rank} M_{J_i} \leq \dim \mathcal{N}_i = \pi_i \tag{6.14}$$

Theorem 37 *The dimension p of the unobservable space $\mathcal{P}$ is given by the sum of the dimensions p_i of the unobservable subspaces $\mathcal{P}_i$, that is:*

$$p = p_1 + p_2 + \dots + p_i + \dots + p_f$$

Proof. Let $\mathcal{Q}$ be the row vector space spanned by the rows of M_J. It can be proved, as for the case controllability that

$$\mathcal{Q} = \mathcal{Q}_1 \oplus \mathcal{Q}_2 \oplus \dots \oplus \mathcal{Q}_f$$

and

$$q = q_1 + q_2 + \dots + q_f$$

and from (6.14) the Proposition is proved. □

6.3.2 The set of *i-th* Spectrum Column Observability Indices

Following along similar lives as for the case of controllability it follows that the dimension p of the unobservable space $\mathcal{P}$ is defined if the dimensions p_i of $\mathcal{P}_i$ for $i = 1, 2, \dots, f$ are also defined. After that, the object of the following work is to define the dimension p_i of the unobservable subspace $\mathcal{P}_i$ corresponding to only one eigenvalue λ_i or equivalently, to the rank q_i of matrix M_{J_i}. Note that,

$$M_{J_i} = \begin{bmatrix} \Gamma_i \\ \Gamma_i (\lambda_i I + H_i) \\ \dots \\ \Gamma_i (\lambda_i I + H_i)^{n-1} \end{bmatrix}$$

Using only row operations on the above matrix, we have that

$$M_{J_i} \sim M_{H_i} \triangleq \begin{bmatrix} \Gamma_i \\ \Gamma_i H_i \\ \dots \\ \Gamma_i H_i^{n-1} \end{bmatrix} \tag{6.15}$$

and if the columns matrix Γ_i is partitioned as in (3.70) and (3.72) then the matrix M_{H_i} can be described as follows:

$$\underbrace{\begin{matrix} \underline{\gamma}_{i1_1} & \underline{\gamma}_{i1_2} & \cdots & \underline{\gamma}_{i1_{\tau_{i1}}} \\ 0 & \underline{\gamma}_{i1_1} & \cdots & \underline{\gamma}_{i1_{\tau_{i1}-1}} \\ . & . & \cdots & . \\ 0 & 0 & \cdots & \underline{\gamma}_{i1_1} \end{matrix}}_{M_{H_{i1}} \in \mathbb{C}^{\tau_{i1}\times\tau_{i1}}} \quad \cdots \quad \underbrace{\begin{matrix} \underline{\gamma}_{ik_1} & \underline{\gamma}_{ik_2} & \cdots & \underline{\gamma}_{ik_{\tau_{ik}}} \\ 0 & \underline{\gamma}_{ik_1} & \cdots & \underline{\gamma}_{ik_{\tau_{ik}-1}} \\ . & . & \cdots & . \\ 0 & 0 & \cdots & \underline{\gamma}_{ik_1} \end{matrix}}_{M_{H_{ik}} \in \mathbb{C}^{\tau_{ik}\times\tau_{ik}}} \quad \cdots \quad \underbrace{\begin{matrix} \underline{\gamma}_{i\nu_{i1}} & \underline{\gamma}_{i\nu_{i2}} & \cdots & \underline{\gamma}_{i\nu_{i\tau_{i\nu_i}}} \\ 0 & \underline{\gamma}_{i\nu_{i1}} & \cdots & \underline{\gamma}_{i\nu_{i\tau_{i\nu_i}-1}} \\ . & . & \cdots & . \\ 0 & 0 & \cdots & \underline{\gamma}_{i\nu_{i1}} \end{matrix}}_{M_{H_{i\nu_i}} \in \mathbb{C}^{\tau_{i\nu_i}\times\tau_{i\nu_i}}}$$

Searching for the linearly independent columns on M_{H_i}, from left to right, we have the following remarks :

Remark 29 *For the M_{H_i} we have the Properties:*

1. *If the first column of Γ_{i1} is nonzero: $\underline{\gamma}_{i1_1} \neq \underline{0}$ then all the τ_{i1} columns of the block $M_{H_{i1}}$ are linearly independent.*

2. *If the ξ_1 first columns of Γ_{i1} are zero: $\underline{\gamma}_{i1_1} = \underline{0}$, $\underline{\gamma}_{i1_2} = \underline{0}$, ..., $\underline{\gamma}_{i1_{\xi_1}} = \underline{0}$ and $\underline{\gamma}_{i1_{\xi_1+1}} \neq \underline{0}$ then all the remaining $\tau_{i1} - \xi_1$ columns of the block $M_{H_{i1}}$ are linearly independent.*

3. *If the first nonzero column of Γ_{i2} : $\underline{\gamma}_{i2_{\xi_2}} \neq \underline{0}$ is linearly independent of the above first nonzero column of Γ_{i1} : $\underline{\gamma}_{i1_{\xi_1+1}}$ then all the remaining $\tau_{i2} - \xi_2$ columns of the block $M_{H_{i2}}$ are linearly independent.*

4. *If the first nonzero column of Γ_{i2} : $\underline{\gamma}_{i2_{\xi_2}} \neq \underline{0}$ is linearly dependent on the above first nonzero column of Γ_{i1} : $\underline{\gamma}_{i1_{\xi_1+1}}$ but the column vector consisting of the first two nonzero columns of Γ_{i2} : $\left[\underline{\gamma}_{i2_{\xi_2+1}}, \underline{\gamma}_{i2_{\xi_2}}\right]^{\top}$, is linearly independent of the column vector, formed*

by the first two nonzero columns of Γ_{i1} : $\left[\underline{\gamma}_{i1_{\xi_1+1}}, \underline{\gamma}_{i1_{\xi_1}}\right]^{\top}$, *then all the last* $\tau_{i2} - \xi_2 - 1$ *columns of the block* $M_{H_{i2}}$ *are linearly independent.*

5. *If the above column vector consisting of the first two nonzero columns of* Γ_{i2} : $\left[\underline{\gamma}_{i2_{\xi_2+1}}, \underline{\gamma}_{i2_{\xi_2}}\right]^{\top}$, *is linearly dependent to the above column vector formed by the columns of* Γ_{i1} : $\left[\underline{\gamma}_{i1_{\xi_1+1}}, \underline{\gamma}_{i1_{\xi_1}}\right]^{\top}$, *but the column vector consisting of the three nonzero columns of* Γ_{i2} : $\left[\underline{\gamma}_{i2_{\xi_2+2}}, \underline{\gamma}_{i2_{\xi_2+1}}, \underline{\gamma}_{i2_{\xi_2}}\right]^{\top}$, *is linearly independent from the column vector formed by the first three nonzero columns of* Γ_{i1} : $\left[\underline{\gamma}_{i1_{\xi_1+2}}, \underline{\gamma}_{i1_{\xi_1+1}}, \underline{\gamma}_{i1_{\xi_1}}\right]^{\top}$, *then all the last* $\tau_{i2} - \xi_2 - 2$ *columns of the block* $M_{H_{i2}}$ *are linearly independent.*

6. *If the first nonzero column of* Γ_{i3} : $\underline{\gamma}_{i3_{\xi_3}} \neq \underline{0}$ *is linearly independent of the above first nonzero column of* Γ_{i1} *and the possibly linear independent first nonzero column of* Γ_{i2} *then all the last* $\tau_{i3} - \xi_3$ *columns of the block* $M_{H_{i3}}$ *are linearly independent.*

7. *Following the same procedure we determine the number of linearly independent columns from left to right into each one block of* M_{H_i}. □

Let $\zeta_{i1}, \zeta_{i2}, \ldots, \zeta_{i\nu_i}$ be the number of linearly independent columns into each one block of M_{H_i} and let the blocks be rearranged from left to right such that :

$$\zeta_{i1} \geq \zeta_{i2} \geq \ldots \geq \zeta_{i\nu_i} \geq 0$$

The above remark summarizes the conditions for characterization of observability in spectral form. This result can also be used to provide a characterization of o.d.z. as it will be shown below. Then, we may define:

Definition 48 *The set of the above numbers is defined as the set of the* i-th *spectrum column observability indices (c.o.i.) of* A, C :

$$Z(A,C)_{\lambda_i} = \{\zeta_{i1} \geq \zeta_{i2} \geq \ldots \geq \zeta_{i\nu_i} \geq 0\} \tag{6.16}$$

From the above the following propositions are directly concluded. □

Theorem 38 *The dimension p_i of the unobservable subspace $\mathcal{P}_i$ is given as,*

$$p_i = \pi_i - q_i = \pi_i - (\zeta_{i1} + \zeta_{i2} + \ldots + \zeta_{i\nu_i})$$

Proof. From the construction of the set $Z(A, C)_{\lambda_i}$ (Remark 29) it is :

$$\zeta_{i1} + \zeta_{i2} + \ldots + \zeta_{i\nu_i} = \mathrm{rank} M_{H_i}$$

from (6.15) it is : $\mathrm{rank} M_{H_i} = \mathrm{rank} M_{J_i}$ and from (6.14) the Theorem is proved. □

Proposition 47 *The mode $(\lambda_i, U(\lambda_i), V(\lambda_i))$ of the system $S(A, B)$ is observable, if and only if the set $Z(A, C)_{\lambda_i}$ coincide with the set $\wp_{\lambda_i}(A)$ i.e.*

$$Z(A, C)_{\lambda_i} \equiv \wp_{\lambda_i}(A)$$

Proof. According to Theorem 19 the mode $(\lambda_i, U(\lambda_i), V(\lambda_i))$ is observable if and only if the rows of the *i-th* spectrum observability matrix Γ_i^F are linearly independent over the field of complex numbers. From the definition of the set $Z(A, C)_{\lambda_i}$ we conclude that this is equivalent to the statement of the Proposition. □

Corollary 4 *The first of the i-th spectrum c.o.i. ζ_{i1} is equal to the degree of the minimal polynomial of the row subspace $\mathcal{Q}_i$.* □

Corollary 5 *The minimal polynomial of the whole row space $\mathcal{Q}$ is given as,*

$$(s - \lambda_1)^{\zeta_{11}} (s - \lambda_2)^{\zeta_{21}} \ldots (s - \lambda_i)^{\zeta_{i1}} \ldots (s - \lambda_i)^{\zeta_{f1}} \tag{6.17}$$

and the degree g of the minimal polynomial of the whole row space $\mathcal{Q}$ is given as,

$$g = \zeta_{11} + \zeta_{21} + \ldots + \zeta_{i1} + \ldots + \zeta_{f1} \tag{6.18}$$

□

Definition 49 *We define as normal structure of M_{H_i}, the matrix M_{H_i} for which all the nonzero c.o.i. are determined by the orders of the two first nonzero columns of the row blocks.* □

6.3.3 Spectral Restriction of the Observability Index

Some further properties related to the observability index are given below:

Proposition 48 *The degree g of the minimal polynomial of the whole row space $\mathcal{Q}$ is equal to the degree of the minimal polynomial of the subspace determined by the rows of matrix C .*

Proof. The minimal polynomial of the subspace spanned by the rows of matrix C is the least common multiple of the minimal polynomial of the basis row vectors $\underline{c}_1^T, \underline{c}_2^T, ..., \underline{c}_m^T$. As all the basis vectors of $\mathcal{Q}$ are derived from chains of the row vectors of C, the common multiple of their minimal polynomials coincides with the common multiple of the minimal polynomials of $\underline{c}_1^T, \underline{c}_2^T, ...\underline{c}_m^T$, the result follows. □

We define the following matrix [Che., 1]:

$$M_k \triangleq \begin{bmatrix} C \\ CA \\ CA^2 \\ \cdots \\ CA^k \end{bmatrix} \quad k = 0, 1, 2, ...$$

as the k-th partial observability matrix and the matrix $M \triangleq M_{n-1}$ is the observability matrix. Searching for the linear independent rows of M_k from the top to the bottom, let r_i be the number of linearly dependent rows in CA^i for $i = 1, 2, ..k$ and r_0 be the number of linearly dependent rows in C. Then, we have:

$$0 \leq r_0 \leq r_1 \leq ... \leq r_k$$

and

$$\text{rank} M_0 < \text{rank} M_1 < ... < \text{rank} M_{\xi-1} = \text{rank} M\xi = ... = \text{rank} M_{\xi-1}$$

Hence the property of observability of $S(A, C)$ can be checked from $M_{\xi-1}$ and ξ is defined as the *Observability index* of the system $S(A, C)$.

In fact,

$$Q_{\mu-1} \sim \begin{bmatrix} \Gamma \\ \Gamma J \\ \cdots \\ \Gamma J^{\xi-1} \end{bmatrix} = \begin{bmatrix} \underline{\gamma}_1^{\mathsf{T}} \\ \cdots \\ \underline{\gamma}_m^{\mathsf{T}} \\ \underline{\gamma}_1^{\mathsf{T}} J(\lambda_i) \\ \cdots \\ \underline{\gamma}_1^{\mathsf{T}} J(\lambda_i) \\ \cdots \\ \cdots \\ \underline{\gamma}_1^{\mathsf{T}} J(\lambda_i)^{\xi-1} \\ \cdots \\ \underline{\gamma}_m^{\mathsf{T}} J(\lambda_i)^{\xi-1} \end{bmatrix} \tag{6.19}$$

and the linear independent rows of the above matrix define a basis for the row space $\mathcal{Q}$.

Proposition 49 *For the observability index ξ of the system $S(A, C)$ we have the property:*

$$\xi \leq \zeta_{11} + \zeta_{21} + \ldots + \zeta_{i1} + \ldots + \zeta_{f1} = g$$

□

6.4 Spectral Properties of a discretised Model with ZOH

The state space description of a discretised model $\hat{S}(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ of a continuous system $S(A, B, C, D)$ with ZOH and sampling period T is given by the equations (4.4) and (4.5) and the Jordan description of the same model is presented in section 4.5.1. Here we examine the effect of sampling on the controllable subspace using the spectral properties defined earlier

6.4.1 Spectral Characterization of the discretised Controllable Space

Following the same procedure as for the continuous system, the controllable subspace $\widehat{\mathcal{R}}$ of the discretised model is defined as,

$$\widehat{\mathcal{R}} = \text{span}\left[\hat{B}, \hat{A}\hat{B}, \hat{A}^2 B, \ldots, \hat{A}^{n-1}\hat{B}\right]$$

Following exactly the same steps as in the case of the continuous system, it can been shown directly that the same conclusions, theorems, propositions, lemmas, remarks etc., proved in the previous sections for the Spectral Characterization of the Controllable Space of the continuous system are also valid for the Spectral Characterization of the Controllable Space of the discretised system. So, it can be also proved for the discretized model, that $\widehat{\mathcal{R}}$ can be expressed as the direct sum :

$$\widehat{\mathcal{R}} = \widehat{\mathcal{R}}_1 \oplus ... \oplus \widehat{\mathcal{R}}_i \oplus ... \oplus \widehat{\mathcal{R}}_f \subseteq \mathbb{R}^n$$

where the controllable subspaces $\widehat{\mathcal{R}}_1, .., \widehat{\mathcal{R}}_f$ are defined as in the continuous system case ($\widehat{\mathcal{R}}_i \subseteq \mathcal{N}_i$). Following the continuous system analysis it can be proved that :

Theorem 39 *For every value of the sampling period T, the dimension $\hat{r}$ of the controllable subspace $\widehat{\mathcal{R}}$, is given by the sum of the dimensions $\hat{r}_i$ of the controllable subspaces $\widehat{\mathcal{R}}_i$,*

$$\hat{r} = \hat{r}_1 + \hat{r}_2 + ... + \hat{r}_i + ... + \hat{r}_f$$

□

As it has already been proved in the previous sections for a system with ZOH and for every value of the sampling period T, Propositions 26 and 27 are valid. Under the same conditions the following Theorem also holds true :

Theorem 40 *For every value of the sampling period T, the minimal polynomial of the vector $\hat{\underline{\beta}}_{kij} \in \mathcal{V}_{ik}$ is $(z-\lambda_i)^{\delta_{kij}}$, where δ_{kij} is the degree of the minimal polynomial of the corresponding vector $\underline{\beta}_{kij}$ of the continuous system.*

Proof. Let $\widehat{\mathcal{B}} = [\hat{\underline{\beta}}_1, \hat{\underline{\beta}}_2, ..., \hat{\underline{\beta}}_j, ..., \hat{\underline{\beta}}_l]$. From (4.25) we have, $\hat{\underline{\beta}}_j = \tilde{V}\Xi\underline{\beta}_j$ $(j = 1, 2, ..., l)$. The matrices $\tilde{V}$ and Ξ (and consequently their product) are non singular, in block diagonal form and of the same structure as J. Also each diagonal block is an upper triangular matrix. So, if

$\underline{\beta}_j$ is partitioned according to the eigenstructure of A and if,

$$\underline{\beta}_{kji} = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_{\delta_{kij}} \\ 0 \\ \cdots \\ 0 \end{bmatrix}, x_{\delta_{kij}} \neq 0$$

is its constituent vector in $\mathcal{V}_{ik}$, with minimal polynomial $(s-\lambda_i)^{\delta_{kij}}$ then,

$$\hat{\underline{\beta}}_{kji} = \tilde{V}_{ik} \Xi_{ik} \underline{\beta}_{kji}$$

where $\tilde{V}_{ik}$ and Ξ_{ik} are the diagonal blocks of dimensions $\tau_{ik} \times \tau_{ik}$ corresponding to J_{ik}. From (4.19) and Theorem 27 the main diagonal of matrix $\tilde{V}_{ik}\Xi_{ik}$ is of the type:

(a) if $\lambda_i = 0$:

$$1,\ T,\ T^2,\ \ldots,\ T^{\tau_{ik}-1} \tag{6.20}$$

(b) if $\lambda_i \neq 0$:

$$\frac{e^{\lambda_i T}-1}{\lambda_i},\ \frac{Te^{\lambda_i T}\left(e^{\lambda_i T}-1\right)}{\lambda_i},\ \frac{T^2 e^{2\lambda_i T}\left(e^{\lambda_i T}-1\right)}{\lambda_i},\ \ldots,\ \frac{T^{\tau_{ik}-1}e^{(\tau_{ik}-1)\lambda_i T}\left(e^{\lambda_i T}-1\right)}{\lambda_i} \tag{6.21}$$

and so it is

$$\hat{\underline{\beta}}_{kji} = \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \cdots \\ \hat{x}_{\delta_{kij}} \\ 0 \\ \cdots \\ 0 \end{bmatrix}, \hat{x}_{\delta_{kij}} \neq 0$$

where :

(a) for $\lambda_i = 0 \Rightarrow \hat{x}_{\delta_{kij}} = T^{\delta_{kij}-1} x_{\delta_{kij}} \neq 0$ and

(b) for $\lambda_i \neq 0 \Rightarrow \hat{x}_{\delta_{kij}} = \dfrac{T^{\delta_{kij}-1} e^{\left(\delta_{kij}-1\right)\lambda_i T} \left(e^{\lambda_i T}-1\right)}{\lambda_i} x_{\delta_{kij}} \neq 0$

From the above and from Lemma 7 it follows that the minimal polynomial of the vector $\hat{\underline{\beta}}_{kij} \in \mathcal{V}_{ik}$ is $(z - \hat{\lambda}_i)^{\delta_{kij}}$ and the Theorem is proved. □

Corollary 6 *The last non zero rows of the vectors* $\underline{\beta}_{kij}$, $\hat{\underline{\beta}}_{kij}$ *have the same order.* □

For reasons of eigenvalue collapsing , we cannot claim that for every value of the sampling period T, the degree of the minimal polynomial of the vector $\hat{\underline{\beta}}_{ij} \in \widehat{\mathcal{N}}_i$ as well the dimensions $\hat{r}_i$ of the controllable subspace $\widehat{\mathcal{R}}_i$ are automatically defined from the continuous system. Therefore we have to distinguish the two cases of sampling.

Regular Sampling

It is already known from Theorem 30 that under regular sampling,

1. The generalized A-invariant subspace $\mathcal{N}_i$, is also an $\hat{A}$-invariant and generalized null-space of $\hat{A}$.
2. The Segré Characteristic of A at λ_i is equal to the Segré characteristic of $\hat{A}$ at $\hat{\lambda}_i$: $\wp_{\lambda_i}(A) = \wp_{\hat{\lambda}_i}\left(\hat{A}\right)$

Theorem 41 *Under regular sampling the minimal polynomial of the vector* $\hat{\underline{\beta}}_{ij} \in \mathcal{N}_i$ *is* $(z - \hat{\lambda}_i)^{\delta_{ij}}$*, where* δ_{ij} *is the degree of the minimal polynomial of the corresponding vector* $\underline{\beta}_{ij}$ *of the continuous system.*

Proof. From Proposition 1 the degree of the minimal polynomial of the vector $\underline{\beta}_{ij} \in \mathcal{N}_i$ is

$$\delta_{ij} = \max(\delta_{1ij}, \delta_{2ij}, ..., \delta_{kij}, ..., \delta_{\nu_i ij})$$

From Proposition 40 and from the above remark for the Segré Characteristic under regular sampling, it is concluded that δ_{ij} is also the degree of the minimal polynomial of the corresponding vector $\hat{\underline{\beta}}_{ij}$. □

Theorem 42 *Under regular sampling, the discretised model $\hat{S}(\hat{A},\hat{B})$ of a continuous system having Q_{H_i} as normal structure (according to Definition 47) it has the properties :*

(a) The set of r.c.i. of the discretised model is equal to the set of r.c.i. of the continuous system, i.e.

$$\Theta(\hat{A},\hat{B})_{\hat{\lambda}_i} \equiv \Theta(A,B)_{\lambda_i}$$

(b) The dimension $\hat{r}_i$ of the controllable subspace $\widehat{\mathcal{R}}_i$ is equal to the corresponding dimension r_i of the controllable subspace $\mathcal{R}_i$ of the continuous system.

(c) The dimension $\hat{r}$ of the controllable subspace $\widehat{\mathcal{R}}$ of the discretised model $\hat{S}(\hat{A},\hat{B})$ is the same with the corresponding dimension r of the continuous system, i.e. $\hat{r} \equiv r$.

(d) The degree $\hat{d}$ of the minimal polynomial of the controllable space $\widehat{\mathcal{R}}$ (as well the controllability index restriction) is the same with the corresponding degree d (and the controllability index restriction) of the continuous system, i.e. $\hat{d} \equiv d$.

Proof.

(a) As in the continuous system case, it can be proved that,

$$\hat{Q}_{\hat{J}_i} \sim \hat{Q}_{H_i} = \left[\begin{array}{ccccc} \widehat{\mathcal{B}}_i & H_i\widehat{\mathcal{B}}_i & H_i^2\widehat{\mathcal{B}}_i & \dots & H_i^{n-1}\widehat{\mathcal{B}}_i \end{array}\right]$$

From Lemma 8 it is directly concluded that the first element of the set $\Theta(A,B)_{\lambda_i} : \theta_{i1}$ is equal to the first element of the set $\Theta(\hat{A},\hat{B})_{\hat{\lambda}_i} : \hat{\theta}_{i1} = \theta_{i1}$. From (4.25) and also from the structure of the matrices $\tilde{V}, \Xi$ (of the same structure as J) it follows that,

$$\widehat{\mathcal{B}}_i = \tilde{V}(\lambda_i)\Xi_i\mathcal{B}_i$$

The matrices $\tilde{V}(\lambda_i), \Xi_i$ (and thus their product) are non singular and they are in block upper diagonal form and of the same block structure as $J(\lambda_i)$. Consequently for each block we have,

$$\widehat{\mathcal{B}}_{ik} = \tilde{V}_{ik}\Xi_{ik}\mathcal{B}_{ik} = \Phi_{ik}\mathcal{B}_{ik}, \;\; \Phi_{ik} \triangleq \tilde{V}_{ik}\Xi_{ik}$$

and as Φ_{ik} is in upper diagonal form, the last non zero row of $\widehat{\mathcal{B}}_{ik}$ is the scalar multiple of the last non zero row of $\mathcal{B}_{ik}$ and the corresponding diagonal element of Φ_{ik}. Thus in the case where the r.c.i. are defined by the last non zero rows of $\mathcal{B}_{ik}$ $(k = 1, 2, .., \nu_i)$ the set $\Theta(A, B)_{\lambda_i}$ remains the same under regular sampling. Let now the last non zero row of $\mathcal{B}_{i2}$: $\underline{\beta}^{\top}_{i2_{\tau_{i2}-\xi_2}}$ is linearly dependent on the last non zero row $\underline{\beta}^{\top}_{i1_{\tau_{i1}-\xi_1}}$ of the above block i.e. $\underline{\beta}^{\top}_{i1_{\tau_{i1}-\xi_1}} = \mu\underline{\beta}^{\top}_{i2_{\tau_{i2}-\xi_2}}$, $\mu \in \{\mathbb{R} - 0\}$ but the row vector consisting of the last two nonzero rows of $\mathcal{B}_{i2}$: $\left[\underline{\beta}^{\top}_{i2_{\tau_{i2}-\xi_2-1}}, \underline{\beta}^{\top}_{i2_{\tau_{i2}-\xi_2}}\right]$, is linearly independent of the row vector $\left[\underline{\beta}^{\top}_{i1_{\tau_{i1}-\xi_1-1}}, \underline{\beta}^{\top}_{i1_{\tau_{i1}-\xi_1}}\right]$. In order to simplify the notation we consider the rows of $\mathcal{B}$ corresponding to the Jordan blocks of dimension 4 and 5 in a system with 3 inputs:

$$\widehat{\mathcal{B}}_{i1} = \begin{bmatrix} \hat{\underline{\beta}}^{\top}_{11} \\ \hat{\underline{\beta}}^{\top}_{12} \\ \hat{\underline{\beta}}^{\top}_{13} \\ \hat{\underline{\beta}}^{\top}_{14} \\ \hat{\underline{\beta}}^{\top}_{15} \end{bmatrix} = \Psi_{i1}\mathcal{B}_{i1} = \begin{bmatrix} \psi_{11} & \psi_{12} & \psi_{13} & \psi_{14} & \psi_{15} \\ 0 & \psi_{22} & \psi_{23} & \psi_{24} & \psi_{25} \\ 0 & 0 & \psi_{33} & \psi_{34} & \psi_{35} \\ 0 & 0 & 0 & \psi_{44} & \psi_{45} \\ 0 & 0 & 0 & 0 & \psi_{55} \end{bmatrix} \begin{bmatrix} \underline{\beta}^{\top}_{11} \\ \underline{\beta}^{\top}_{12} \\ \underline{\beta}^{\top}_{13} \\ \underline{\beta}^{\top}_{14} \\ \underline{\beta}^{\top}_{15} \end{bmatrix} \tag{6.22}$$

$$\widehat{\mathcal{B}}_{i2} = \begin{bmatrix} \hat{\underline{\beta}}^{\top}_{21} \\ \hat{\underline{\beta}}^{\top}_{22} \\ \hat{\underline{\beta}}^{\top}_{23} \\ \hat{\underline{\beta}}^{\top}_{24} \end{bmatrix} = \Psi_{i2}\mathcal{B}_{i2} = \begin{bmatrix} \psi_{11} & \psi_{12} & \psi_{13} & \psi_{14} \\ 0 & \psi_{22} & \psi_{23} & \psi_{24} \\ 0 & 0 & \psi_{33} & \psi_{34} \\ 0 & 0 & 0 & \psi_{44} \end{bmatrix} \begin{bmatrix} \underline{\beta}^{\top}_{21} \\ \underline{\beta}^{\top}_{22} \\ \underline{\beta}^{\top}_{23} \\ \underline{\beta}^{\top}_{24} \end{bmatrix} \tag{6.23}$$

and let

a.1 for the continuous system assume that:

$$\underline{\beta}^{\top}_{15} \neq \underline{0}^{\top} \Rightarrow \theta_{i1} = 5$$

and

$$\left\{ \begin{array}{c} \underline{\beta}^{\top}_{15} \\ \underline{\beta}^{\top}_{24} \end{array} \right\} \text{ linearly dependent} \Rightarrow \exists \mu \in \{\mathbb{R} - 0\} : \underline{\beta}^{\top}_{24} = \mu \underline{\beta}^{\top}_{15}$$

a.2 assume also for the continuous system that:

$$\left\{ \begin{array}{c} \left[\underline{\beta}_{14}^{\top}, \underline{\beta}_{15}^{\top}\right] \\ \left[\underline{\beta}_{23}^{\top}, \underline{\beta}_{24}^{\top}\right] \end{array} \right\} = \left\{ \begin{array}{c} \left[\underline{\beta}_{14}^{\top}, \underline{\beta}_{15}^{\top}\right] \\ \left[\underline{\beta}_{23}^{\top}, \mu\underline{\beta}_{15}^{\top}\right] \end{array} \right\} \text{ linearly independent} \Leftrightarrow$$

$$\Leftrightarrow \underline{\beta}_{23}^{\top} \neq \mu \underline{\beta}_{14}^{\top} \Leftrightarrow \theta_{i2} = 3$$

the corresponding vectors of the discretised system are related to the continuous as it is shown below,

$$\text{from } (6.22) \Rightarrow \left\{ \begin{array}{l} \hat{\underline{\beta}}_{14}^{\top} = \psi_{44}\underline{\beta}_{14}^{\top} + \psi_{45}\underline{\beta}_{15}^{\top} \\ \hat{\underline{\beta}}_{15}^{\top} = \psi_{55}\underline{\beta}_{15}^{\top} \Rightarrow \hat{\underline{\beta}}_{15}^{\top} \neq \underline{0}^{\top} \Rightarrow \hat{\theta}_{i1} = \theta_{i1} = 5 \end{array} \right.$$

$$\text{and from } (6.23) \Rightarrow \left\{ \begin{array}{l} \hat{\underline{\beta}}_{23}^{\top} = \psi_{33}\underline{\beta}_{23}^{\top} + \psi_{34}\underline{\beta}_{24}^{\top} \\ \hat{\underline{\beta}}_{24}^{\top} = \psi_{44}\underline{\beta}_{24}^{\top} \end{array} \right.$$

and we have to prove that the following two vectors of the discretised system are linearly independent,

$$\left\{ \begin{array}{c} \left[\hat{\underline{\beta}}_{14}^{\top}, \hat{\underline{\beta}}_{15}^{\top}\right] \\ \left[\hat{\underline{\beta}}_{23}^{\top}, \hat{\underline{\beta}}_{24}^{\top}\right] \end{array} \right\} = \left\{ \begin{array}{c} \left[\psi_{44}\underline{\beta}_{14}^{\top} + \psi_{45}\underline{\beta}_{15}^{\top}, \psi_{55}\underline{\beta}_{15}^{\top}\right] \\ \left[\psi_{33}\underline{\beta}_{23}^{\top} + \psi_{34}\underline{\beta}_{24}^{\top}, \psi_{44}\underline{\beta}_{24}^{\top}\right] \end{array} \right\} = \left\{ \begin{array}{c} \left[\psi_{44}\underline{\beta}_{14}^{\top} + \psi_{45}\underline{\beta}_{15}^{\top}, \psi_{55}\underline{\beta}_{15}^{\top}\right] \\ \left[\psi_{33}\underline{\beta}_{23}^{\top} + \psi_{34}\mu\underline{\beta}_{15}^{\top}, \psi_{44}\mu\underline{\beta}_{15}^{\top}\right] \end{array} \right\}$$

From (6.20) and (6.21) it is :

$$\frac{\psi_{55}}{\psi_{44}} = \frac{\psi_{44}}{\psi_{33}} = Te^{\lambda_i T} \triangleq \varphi$$

$$\frac{\psi_{45}}{\psi_{44}} = \frac{3T}{2} \frac{\lambda_i T e^{\lambda_i T} - e^{\lambda_i T} + 1}{\lambda_i (e^{\lambda_i T} - 1)} = 3\omega, \ \omega \triangleq \frac{T}{2} \frac{\lambda_i T e^{\lambda_i T} - e^{\lambda_i T} + 1}{\lambda_i (e^{\lambda_i T} - 1)}$$

$$\frac{\psi_{34}}{\psi_{33}} = T \frac{\lambda_i T e^{\lambda_i T} - e^{\lambda_i T} + 1}{\lambda_i (e^{\lambda_i T} - 1)} = 2\omega$$

and so the above two vectors of the discretised model are linearly independent if and only if the following two vectors are linearly independent :

$$\left\{ \begin{array}{c} \left[\underline{\beta}_{14}^{\top} + 3\omega\underline{\beta}_{15}^{\top}, \varphi\underline{\beta}_{15}^{\top}\right] \\ \left[\underline{\beta}_{23}^{\top} + 2\omega\mu\underline{\beta}_{15}^{\top}, \varphi\mu\underline{\beta}_{15}^{\top}\right] \end{array} \right\} \text{linearly independent} \Leftrightarrow$$

$$\Leftrightarrow \left[\underline{\beta}_{23}^{\top} + 2\omega\mu\underline{\beta}_{15}^{\top}\right] \neq \mu \left[\underline{\beta}_{14}^{\top} + 3\omega\underline{\beta}_{15}^{\top}\right] \Leftrightarrow \underline{\beta}_{23}^{\top} - \mu\underline{\beta}_{14}^{\top} \neq \omega\mu\underline{\beta}_{15}^{\top}$$

The above relation between the row vectors $\underline{\beta}_{23}^{\top}, \underline{\beta}_{14}^{\top}, \underline{\beta}_{15}^{\top}$ is impossible for every value of the sampling period T. Thus the vectors $\left[\hat{\underline{\beta}}_{14}^{\top}, \hat{\underline{\beta}}_{15}^{\top}\right], \left[\hat{\underline{\beta}}_{23}^{\top}, \hat{\underline{\beta}}_{24}^{\top}\right]$ are linearly independent and it is $\theta_{i2} = \hat{\theta}_{i2} = 3$.

The above can be applied directly to the general case and so we have under regular sampling for a system with Q_{H_i} of normal structure : $\Theta(\hat{A}, \hat{B})_{\hat{\lambda}_i} \equiv \Theta(A, B)_{\lambda_i}$.

(b) Then for regular sampling it is established that: $r_i = \theta_{i1} + ... + \theta_{i\nu_i} = \hat{r}_i$

(c) From the above and from Theorem 34 the result follows.

(d) From the above and from Propositions 45 and 46 the result follows. □

Proposition 50 *In the case of a continuous system with Q_{H_i} having abnormal structure, then the process of a regular sampling, acts for the "normalization" of Q_{H_i} and then we have a "restoration" of the controllable space.*

Proof. Let for the continuous system in the proof of the above Theorem 42 be valid the following :

$$\left\{ \begin{array}{c} \left[\underline{\beta}_{14}^{\top}, \underline{\beta}_{15}^{\top}\right] \\ \left[\underline{\beta}_{23}^{\top}, \underline{\beta}_{24}^{\top}\right] \end{array} \right\} \text{ linearly dependent} \Leftrightarrow \underline{\beta}_{23}^{\top} = \mu\underline{\beta}_{14}^{\top},\ \underline{\beta}_{24}^{\top} = \mu\underline{\beta}_{15}^{\top} \Leftrightarrow \theta_{i2} = 2$$

Then for the discretised system we have

$$\left\{ \begin{array}{c} \left[\hat{\underline{\beta}}_{14}^{\top}, \hat{\underline{\beta}}_{15}^{\top}\right] \\ \left[\hat{\underline{\beta}}_{23}^{\top}, \hat{\underline{\beta}}_{24}^{\top}\right] \end{array} \right\} = \left\{ \begin{array}{c} \left[\psi_{44}\underline{\beta}_{14}^{\top} + \psi_{45}\underline{\beta}_{15}^{\top}, \psi_{55}\underline{\beta}_{15}^{\top}\right] \\ \left[\psi_{33}\underline{\beta}_{23}^{\top} + \psi_{34}\underline{\beta}_{24}^{\top}, \psi_{44}\underline{\beta}_{24}^{\top}\right] \end{array} \right\} =$$

$$= \left\{ \begin{array}{c} \left[\psi_{44}\underline{\beta}_{14}^{\top} + \psi_{45}\underline{\beta}_{15}^{\top}, \psi_{55}\underline{\beta}_{15}^{\top}\right] \\ \mu\left[\psi_{33}\underline{\beta}_{14}^{\top} + \psi_{34}\underline{\beta}_{15}^{\top}, \psi_{44}\underline{\beta}_{15}^{\top}\right] \end{array} \right\} \sim \left\{ \begin{array}{c} \left[\underline{\beta}_{14}^{\top} + 3\omega\underline{\beta}_{15}^{\top}, \varphi\underline{\beta}_{15}^{\top}\right] \\ \left[\underline{\beta}_{14}^{\top} + 2\omega\underline{\beta}_{15}^{\top}, \varphi\underline{\beta}_{15}^{\top}\right] \end{array} \right\} \text{ linearly independent}$$

and thus $\hat{\theta}_{i2} = 3 > \theta_{i2} = 2$ which proves the result. □

Irregular sampling

For notational simplicity the case of two eigenvalues λ_u and λ_y collapsing is examined here. The obtained results can be extended directly to any other case of a partial or a total collapsing, for each r-root range of A, $\Phi_r(A)$. We assume that under an irregular sampling, for which the distinct eigenvalues λ_u and λ_y of the continuous system correspond to the eigenvalue $\hat{\lambda}_c$ of the discrete model. It is already known (Theorem 31) that,

- The generalized $\hat{A}$-invariant null-space $\mathcal{N}_c$, is the direct sum of $\mathcal{N}_u$ and $\mathcal{N}_y$: $\mathcal{N}_c = \mathcal{N}_u \oplus \mathcal{N}_y$
- The Segré Characteristic $\wp_{\hat{\lambda}_c}\left(\hat{A}\right)$ is formed by the merging of $\wp_{\lambda_u}(A)$ and $\wp_{\lambda_y}(A)$.

Then, for the discretised model $\hat{S}(\hat{A}, \hat{B})$ it follows:

Proposition 51 *Under the irregular sampling, for which a collapsing occurs between the eigenvalues λ_u and λ_y to the eigenvalue $\hat{\lambda}_c$, for the discretised model $\hat{S}(\hat{A}, \hat{B})$, the minimal polynomial of the vector $\underline{\hat{\beta}}_{cj} \in \mathcal{N}_c$ is $(z - \hat{\lambda}_c)^{\delta_{cj}}$ $(j = 1, 2, ..., l)$, where, $\delta_{cj} = \max(\delta_{uj}, \delta_{yj})$*

Proof. For the above irregular sampling each one of the vectors $\underline{\hat{\beta}}_{cj} \in \mathcal{N}_c$ $(j = 1, 2, ..., l)$ is created from the component vectors of the corresponding $\underline{\hat{\beta}}_{uj} \in \mathcal{N}_u$ and $\underline{\hat{\beta}}_{yj} \in \mathcal{N}_y$ $(j = 1, 2, ..., l)$. Then $\delta_{cj} = max(\delta_{uj}, \delta_{yj})$. □

Theorem 43 *Under the irregular sampling, for which a collapsing occurs between the eigenvalues λ_u and λ_y to the eigenvalue $\hat{\lambda}_c$, for the discretised model $\hat{S}(\hat{A}, \hat{B})$ with Q_{H_u}, Q_{H_y} as normal structure, the following holds true:*

(a) If $r_u = \dim\mathcal{R}_u$, $\mathcal{R}_u \subseteq \mathcal{N}_u$ and $r_y = \dim\mathcal{R}_y$, $\mathcal{R}_y \subseteq \mathcal{N}_y$, and $\hat{r}_c$ the dimension of the controllable subspace $\widehat{\mathcal{R}}_c \subseteq \mathcal{N}_c$, then $\hat{r}_c \leq r_u + r_y$.

(b) If $\hat{r}$ is the dimension of the whole controllable space $\widehat{\mathcal{R}}$ of the discretised model $\hat{S}(\hat{A}, \hat{B})$ and r is the corresponding dimension of the continuous system, then, $\hat{r} \leq r$.

(c) If $\hat{d}$ is the degree of the minimal polynomial of the controllable space $\widehat{\mathcal{R}}$ (as well the controllability index restriction) and d is the corresponding degree (and the controllability index restriction) of the continuous system, then, $\hat{d} = d - \min(\theta_{u1}, \theta_{y1})$.

Proof.

(a) The matrix $\widehat{\mathcal{B}_c}$ is composed by the matrix blocks of $\widehat{\mathcal{B}_u}$ and $\widehat{\mathcal{B}_y}$: $\widehat{\mathcal{B}_c} = \begin{bmatrix} \widehat{\mathcal{B}_u} \\ \widehat{\mathcal{B}_y} \end{bmatrix}$. Let ,

$$\Theta(A,B)_{\lambda_u} = \{\theta_{u1} \geq \theta_{u2} \geq ... \geq \theta_{u\nu_u} \geq 0\} \Rightarrow r_u = \theta_{u1} + \theta_{u2} + ... + \theta_{u\nu_u}$$

and

$$\Theta(A,B)_{\lambda_y} = \{\theta_{y1} \geq \theta_{y2} \geq ... \geq \theta_{y\nu_y} \geq 0\} \Rightarrow r_y = \theta_{y1} + \theta_{y2} + ... + \theta_{y\nu_y}$$

From the construction procedure of $\Theta(\hat{A},\hat{B})_{\hat{\lambda}_c}$ it follows that, $\theta_{c1} = \max(\theta_{u1}, \theta_{y1})$ and each one of the following indices $\theta_{c2}, ..., \theta_{c\nu_c}$ is equal or smaller to the corresponding index of the same matrix block of $\mathcal{B}_u$ and $\mathcal{B}_y$ of the continuous system. Thus,

$$\hat{r}_c = \theta_{c1} + \theta_{c2} + ... + \theta_{c\nu_c} \leq (\theta_{u1} + \theta_{u2} + ... + \theta_{u\nu_u}) + (\theta_{y1} + \theta_{y2} + ... + \theta_{y\nu_y})$$

(b) From the above and from Theorem 34 the result follows.

(c) Also from the above and from Propositions 45 and 46 the result follows. □

Example 8 *Consider the uncontrollable continuous system the Example 7 under the following irregular values of the sampling period.*

a) $T = \frac{2k\pi}{24}$,

a.1) For $k = 1, 2, 3, 5, 6, 7, 9, 10, 11, \ldots$ *we have for the mode* $\hat{\lambda}_{12}$ *:*

$$\mathcal{B}_{12,1} = \begin{bmatrix} -3-i & 5+4i & 0 & 0 & 7 & 2i \\ -9-5i & 0 & -6 & 0 & 0 & 2+3i \\ 2+6i & 0 & 0 & -3+2i & -9-5i & -7+2i \\ 0 & -2i & 0 & 0 & 3i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} . \\ . \\ . \\ \leftarrow \hat{\theta}_{12,1} = 4 \\ . \\ . \end{array}$$

$$\mathcal{B}_{12,2} = \begin{bmatrix} -3+i & 5-4i & 0 & 0 & 7 & -2i \\ -9+5i & 0 & -6 & 0 & 0 & 2-3i \\ 2-6i & 0 & 0 & -3-2i & -9+5i & -7-2i \\ 0 & 2i & 0 & 0 & -3i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} . \\ . \\ \leftarrow \hat{\theta}_{12,2} = 3 \\ . \\ . \\ . \end{array}$$

$$\mathcal{B}_{12,3} = \begin{bmatrix} 0 & 0 & 2-i & -5+3i & 0 & 0 \\ 0 & -3+3i & 0 & 3-2i & 7i & 5-9i \\ -4-2i & 0 & 0 & 0 & -4-i & 0 \end{bmatrix} \begin{array}{l} . \\ . \\ \leftarrow \hat{\theta}_{12,3} = 3 \end{array}$$

$$\mathcal{B}_{12,5} = \begin{bmatrix} 2+6i & 0 & 0 & 5-9i & 0 & 0 \end{bmatrix} \leftarrow \hat{\theta}_{12,5} = 1$$

$$\mathcal{B}_{12,6} = \begin{bmatrix} 2-6i & 0 & 0 & 5+9i & 0 & 0 \end{bmatrix} \leftarrow \hat{\theta}_{12,6} = 1$$

$$\Rightarrow \left\{ \begin{array}{l} \Theta(\hat{A},\hat{B})_{\hat{\lambda}_{12}} = \{4,3,3,3,1,1\}\,,\ \hat{r}_{12} = 15\,(< r_1 + r_2 = 16) \\ \Theta(\hat{A},\hat{B})_{\hat{\lambda}_3} = \Theta(A,B)_{\lambda_3} = \{2,1\}\,,\ \hat{r}_3 = 3\,(= r_3) \\ \Theta(\hat{A},\hat{B})_{\hat{\lambda}_4} = \Theta(A,B)_{\lambda_4} = \{2,1\}\,,\ \hat{r}_4 = 3\,(= r_4) \end{array} \right\}$$

The dimension of the controllable space of the discretised system corresponding to the set $\Phi_{-5}(A)$, *is* $\hat{r} = \hat{r}_{12} + \hat{r}_3 + \hat{r}_4 = 21\,(< r = 22)$. *The spectral restriction of the* *corresponding controllability index is* $\mu = 4 + 2 + 2 = 8\,(< 12)$.

a.2) For $k = 4, 8, 12, \ldots.$ *we have for the mode* $\hat{\lambda}_{12}$ *as above and for the mode* $\hat{\lambda}_{34}$:

$$\mathcal{B}_{34,1} = \begin{bmatrix} 3-8i & 3 & 0 & 0 & -2+3i & 0 \\ 0 & 0 & -1+i & 0 & 8i & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \leftarrow \hat{\theta}_{34,1} = 2$$

$$\mathcal{B}_{34,2} = \begin{bmatrix} 3+8i & 3 & 0 & 0 & -2-3i & 0 \\ 0 & 0 & -1-i & 0 & -8i & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \leftarrow \hat{\theta}_{34,2} = 2$$

$$\mathcal{B}_{34,3} = \begin{bmatrix} 0 & -8i & 0 & 0 & 12i & 0 \end{bmatrix} \leftarrow \hat{\theta}_{34,3} = 1$$

$$\mathcal{B}_{34,4} = \begin{bmatrix} 0 & 8i & 0 & 0 & -12i & 0 \end{bmatrix} \leftarrow \hat{\theta}_{34,4} = 0$$

$$\Rightarrow \left\{ \begin{array}{l} \Theta(\hat{A}, \hat{B})_{\hat{\lambda}_{12}} = \{4, 3, 3, 3, 1, 1\}, \; \hat{r}_{12} = 15\,(< r_1 + r_2 = 16) \\ \Theta(\hat{A}, \hat{B})_{\hat{\lambda}_{34}} = \{2, 2, 1, 0\}, \; \hat{r}_{34} = 5\,(< r_3 + r_4 = 6) \end{array} \right\}$$

It is $\hat{r} = \hat{r}_{12} + \hat{r}_{34} = 20\,(< r = 22)$, $\mu = 4 + 2 = 6\,(< 12)$.

a.3) For $k = 8, 16, 24, \ldots$ *we have for the mode* $\hat{\lambda}_{1234}$:

$$\mathcal{B}_{1234,1} = \begin{bmatrix} -3-i & 5+4i & 0 & 0 & 7 & 2i \\ -9-5i & 0 & -6 & 0 & 0 & 2+3i \\ 2+6i & 0 & 0 & -3+2i & -9-5i & -7+2i \\ 0 & -2i & 0 & 0 & 3i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \leftarrow \hat{\theta}_{1234,1} = 4$$

$$\mathcal{B}_{1234,2} = \begin{bmatrix} -3+i & 5-4i & 0 & 0 & 7 & -2i \\ -9+5i & 0 & -6 & 0 & 0 & 2-3i \\ 2-6i & 0 & 0 & -3-2i & -9+5i & -7-2i \\ 0 & 2i & 0 & 0 & -3i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \leftarrow \hat{\theta}_{1234,2} = 3$$

$$\mathcal{B}_{1234,3} = \begin{bmatrix} 0 & 0 & 2-i & -5+3i & 0 & 0 \\ 0 & -3+3i & 0 & 3-2i & 7i & 5-9i \\ -4-2i & 0 & 0 & 0 & -4-i & 0 \end{bmatrix} \begin{matrix} . \\ . \\ \leftarrow \hat{\theta}_{1234,3} = 3 \end{matrix}$$

$$\mathcal{B}_{1234,4} = \begin{bmatrix} 0 & 0 & 2+i & -5-3i & 0 & 0 \\ 0 & -3-3i & 0 & 3+2i & -7i & 5+9i \\ -4+2i & 0 & 0 & 0 & -4+i & 0 \end{bmatrix} \begin{matrix} . \\ . \\ \leftarrow \hat{\theta}_{1234,4} = 3 \end{matrix}$$

$$\mathcal{B}_{1234,5} = \begin{bmatrix} 3+8i & 3 & 0 & 0 & -2-3i & 0 \\ 0 & 0 & -1-i & 0 & -8i & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} . \\ \leftarrow \hat{\theta}_{1234,5} = 2 \\ . \end{matrix}$$

$$\mathcal{B}_{1234,6} = \begin{bmatrix} 3-8i & 3 & 0 & 0 & -2+3i & 0 \\ 0 & 0 & -1+i & 0 & 8i & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} . \\ \leftarrow \hat{\theta}_{1234,6} = 2 \\ . \end{matrix}$$

$$\mathcal{B}_{1234,7} = \begin{bmatrix} 2+6i & 0 & 0 & 5-9i & 0 & 0 \end{bmatrix} \leftarrow \hat{\theta}_{1234,7} = 0$$

$$\mathcal{B}_{1234,8} = \begin{bmatrix} 2-6i & 0 & 0 & 5+9i & 0 & 0 \end{bmatrix} \leftarrow \hat{\theta}_{1234,8} = 0$$

$$\mathcal{B}_{1234,9} = \begin{bmatrix} 0 & -8i & 0 & 0 & 12i & 0 \end{bmatrix} \leftarrow \hat{\theta}_{1234,9} = 0$$

$$\mathcal{B}_{1234,10} = \begin{bmatrix} 0 & 8i & 0 & 0 & -12i & 0 \end{bmatrix} \leftarrow \hat{\theta}_{1234,10} = 0$$

$$\Rightarrow \Theta(\hat{A}, \hat{B})_{\hat{\lambda}_{1234}} = \{4, 3, 3, 3, 2, 2, 0, 0, 0, 0\}$$

It is $\hat{r} = 17\,(< r = 22)$, $\mu \leq 4\,(< 12)$.

b) $T = \frac{2k\pi}{15}$

b.1) For $k = 1, 2, 3, 4, 6, 7, 8, 9, 11, \ldots$ *we have* *b.1.1) for the mode $\hat{\lambda}_{14}$:*

$$\mathcal{B}_{14,1} = \begin{bmatrix} -3-i & 5+4i & 0 & 0 & 7 & 2i \\ -9-5i & 0 & -6 & 0 & 0 & 2+3i \\ 2+6i & 0 & 0 & -3+2i & -9-5i & -7+2i \\ 0 & -2i & 0 & 0 & 3i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \leftarrow \hat{\theta}_{14,1} = 4$$

$$\mathcal{B}_{14,2} = \begin{bmatrix} 0 & 0 & 2-i & -5+3i & 0 & 0 \\ 0 & -3+3i & 0 & 3-2i & 7i & 5-9i \\ -4-2i & 0 & 0 & 0 & -4-i & 0 \end{bmatrix} \leftarrow \hat{\theta}_{14,2} = 3$$

$$\mathcal{B}_{14,3} = \begin{bmatrix} 3+8i & 3 & 0 & 0 & -2-3i & 0 \\ 0 & 0 & -1-i & 0 & -8i & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \leftarrow \hat{\theta}_{14,3} = 2$$

$$\mathcal{B}_{14,4} = \begin{bmatrix} 2-6i & 0 & 0 & 5+9i & 0 & 0 \end{bmatrix} \leftarrow \hat{\theta}_{14,4} = 1$$

$$\mathcal{B}_{14,5} = \begin{bmatrix} 0 & -8i & 0 & 0 & 12i & 0 \end{bmatrix} \leftarrow \hat{\theta}_{14,5} = 0$$

$$\Rightarrow \Theta(\hat{A}, \hat{B})_{\hat{\lambda}_{14}} = \{4, 3, 2, 1, 0\}$$

b.1.2) and for the mode $\hat{\lambda}_{23}$:

$$\mathcal{B}_{23,1} = \begin{bmatrix} -3+i & 5-4i & 0 & 0 & 7 & -2i \\ -9+5i & 0 & -6 & 0 & 0 & 2-3i \\ 2-6i & 0 & 0 & -3-2i & -9+5i & -7-2i \\ 0 & 2i & 0 & 0 & -3i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \leftarrow \hat{\theta}_{23,1} = 4$$

$$\mathcal{B}_{23,2} = \begin{bmatrix} 0 & 0 & 2+i & -5-3i & 0 & 0 \\ 0 & -3-3i & 0 & 3+2i & -7i & 5+9i \\ -4+2i & 0 & 0 & 0 & -4+i & 0 \end{bmatrix} \begin{array}{l} . \\ . \\ \leftarrow \hat{\theta}_{23,2} = 3 \end{array}$$

$$\mathcal{B}_{23,3} = \begin{bmatrix} 3-8i & 3 & 0 & 0 & -2+3i & 0 \\ 0 & 0 & -1+i & 0 & 8i & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} . \\ \leftarrow \hat{\theta}_{23,3} = 2 \\ . \end{array}$$

$$\mathcal{B}_{23,4} = \begin{bmatrix} 2-6i & 0 & 0 & 5+9i & 0 & 0 \end{bmatrix} \leftarrow \hat{\theta}_{23,4} = 1$$

$$\mathcal{B}_{23,5} = \begin{bmatrix} 0 & -8i & 0 & 0 & 12i & 0 \end{bmatrix} \leftarrow \hat{\theta}_{23,5} = 0$$

$\Rightarrow \Theta(\hat{A},\hat{B})_{\hat{\lambda}_{23}} = \{4,3,2,1,0\}$ *It is* $\Phi_{-5}(A)$, *is* $\hat{r} = \hat{r}_{14} + \hat{r}_{23} = 20\,(< r = 22)$, $\mu = 4+4 = 8\,(< 12)$.

b.2) For $k = 5, 10, 15, 20, \ldots \Rightarrow \Theta(\hat{A},\hat{B})_{\hat{\lambda}_{1234}} = \{4,3,3,3,2,2,0,0,0,0\} \Rightarrow \hat{r} = 17$, $\mu = 4$, *as above.*

c) $T = \frac{2k\pi}{9}$

c.1) For $k = 1, 2, 4, 5, 7, 8, \ldots$ *we have*

c.1.1) for the mode $\hat{\lambda}_{13}$*:*

$$\mathcal{B}_{13,1} = \begin{bmatrix} -3-i & 5+4i & 0 & 0 & 7 & 2i \\ -9-5i & 0 & -6 & 0 & 0 & 2+3i \\ 2+6i & 0 & 0 & -3+2i & -9-5i & -7+2i \\ 0 & -2i & 0 & 0 & 3i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} . \\ . \\ . \\ \leftarrow \hat{\theta}_{13,1} = 4 \\ . \\ . \end{array}$$

$$\mathcal{B}_{13,2} = \begin{bmatrix} 0 & 0 & 2-i & -5+3i & 0 & 0 \\ 0 & -3+3i & 0 & 3-2i & 7i & 5-9i \\ -4-2i & 0 & 0 & 0 & -4-i & 0 \end{bmatrix} \begin{array}{l} . \\ . \\ \leftarrow \hat{\theta}_{13,2} = 3 \end{array}$$

$$\mathcal{B}_{13,3} = \begin{bmatrix} 3-8i & 3 & 0 & 0 & -2+3i & 0 \\ 0 & 0 & -1+i & 0 & 8i & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} . \\ \leftarrow \hat{\theta}_{13,3} = 2 \\ . \end{matrix}$$

$$\mathcal{B}_{13,4} = \begin{bmatrix} 2+6i & 0 & 0 & 5-9i & 0 & 0 \end{bmatrix} \leftarrow \hat{\theta}_{13,4} = 1$$

$$\mathcal{B}_{13,5} = \begin{bmatrix} 0 & -8i & 0 & 0 & 12i & 0 \end{bmatrix} \leftarrow \hat{\theta}_{13,5} = 0$$

$\Rightarrow \Theta(\hat{A}, \hat{B})_{\hat{\lambda}_{13}} = \{4, 3, 2, 1, 0\}$

c.1.2) for the mode $\hat{\lambda}_{24}$*:*

$$\mathcal{B}_{24,1} = \begin{bmatrix} -3+i & 5-4i & 0 & 0 & 7 & -2i \\ -9+5i & 0 & -6 & 0 & 0 & 2-3i \\ 2-6i & 0 & 0 & -3-2i & -9+5i & -7-2i \\ 0 & 2i & 0 & 0 & -3i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} . \\ . \\ . \\ \leftarrow \hat{\theta}_{13,1} = 4 \\ . \\ . \end{matrix}$$

$$\mathcal{B}_{24,2} = \begin{bmatrix} 0 & 0 & 2+i & -5-3i & 0 & 0 \\ 0 & -3-3i & 0 & 3+2i & -7i & 5+9i \\ -4+2i & 0 & 0 & 0 & -4+i & 0 \end{bmatrix} \begin{matrix} . \\ . \\ \leftarrow \hat{\theta}_{13,2} = 3 \end{matrix}$$

$$\mathcal{B}_{24,3} = \begin{bmatrix} 3+8i & 3 & 0 & 0 & -2-3i & 0 \\ 0 & 0 & -1-i & 0 & -8i & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} . \\ \leftarrow \hat{\theta}_{13,3} = 2 \\ . \end{matrix}$$

$$\mathcal{B}_{24,4} = \begin{bmatrix} 2-6i & 0 & 0 & 5+9i & 0 & 0 \end{bmatrix} \leftarrow \hat{\theta}_{13,4} = 1$$

$$\mathcal{B}_{24,5} = \begin{bmatrix} 0 & 8i & 0 & 0 & -12i & 0 \end{bmatrix} \leftarrow \hat{\theta}_{13,5} = 0$$

$\Rightarrow \Theta(\hat{A}, \hat{B})_{\hat{\lambda}_{24}} = \{4, 3, 2, 1, 0\}$. *It is* $\hat{r} = \hat{r}_{13} + \hat{r}_{24} = 20\,(< r = 22)$, $\mu = 4 + 4 = 8\,(< 12)$.

c.2) For $k = 5, 10, 15, 20, \ldots. \Rightarrow \Theta(\hat{A}, \hat{B})_{\hat{\lambda}_{1234}} = \{4, 3, 3, 3, 2, 2, 0, 0, 0, 0\} \Rightarrow \hat{r} = 17$, $\mu = 4$, *as above.* □

6.4.2 Spectral Characterization of the discretised Unobservable Space

Following the same procedure as for the continuous system, the unobservable subspace $\widehat{\mathcal{P}}$ of the discretised model is defined as the right null-space,

$$\widehat{\mathcal{P}} = \mathcal{N}_{\text{riht}} \begin{bmatrix} \hat{C} \\ \hat{C}\hat{A} \\ \hat{C}\hat{A}^2 \\ \dots \\ \hat{C}\hat{A}^{n-1} \end{bmatrix}$$

It can be proved, as for the case of continuous system, that $\widehat{\mathcal{P}}$ can be expressed as the direct sum,

$$\widehat{\mathcal{P}} = \widehat{\mathcal{P}}_1 \oplus \dots \oplus \widehat{\mathcal{P}}_i \oplus \dots \oplus \widehat{\mathcal{P}}_f \subseteq \mathbb{R}^n$$

where the unobservable subspaces $\widehat{\mathcal{P}}_1, \dots, \widehat{\mathcal{P}}_f$ are defined as for continuous system ($\widehat{\mathcal{P}}_i \subseteq \mathcal{N}_i$). Similarly to the continuous system case it can be proved that :

Theorem 44 *For every value of the sampling period T, the dimension $\hat{p}$ of the controllable subspace $\widehat{\mathcal{P}}$, is given by the sum of the dimensions $\hat{p}_i$ of the controllable subspaces $\widehat{\mathcal{P}}_i$,*

$$\hat{p} = \hat{p}_1 + \hat{p}_2 + \dots + \hat{p}_i + \dots + \hat{p}_f$$

□

Similarly as for the case of the controllable space we have:

Theorem 45 *For every value of the sampling period T, the minimal polynomial of the row vector*

$$\underline{\gamma}_{kij}^{\top} \in \mathcal{F}_{ik}, \ (i = 1, 2, .., f), \ (k = 1, 2, ..\nu_i), \ (j = 1, 2, \dots, m)$$

is $(z - \hat{\lambda}_i)^{\varepsilon_{kij}}$, where ε_{kij} is the degree of the minimal polynomial of the corresponding row vector $\underline{\gamma}_{kij}^{\top}$ of the continuous system.

Proof. Let.

$$\widehat{\Gamma} = \begin{bmatrix} \hat{\underline{\gamma}}_1^\top \\ \dots \\ \hat{\underline{\gamma}}_j^\top \\ \dots \\ \hat{\underline{\gamma}}_m^\top \end{bmatrix}$$

From (4.27) we have, $\hat{\underline{\gamma}}_j^\top = \underline{\gamma}_j^\top \tilde{U}$ $(j = 1, 2, ..., m)$. The matrix $\tilde{U}$ is non singular, it is in block diagonal form and of the same structure as J; also each diagonal block is an upper triangular matrix. So, if $\underline{\gamma}_j^\top$ is partitioned according to the eigenstructure of A then,

$$\underline{\gamma}_{kji}^\top = \begin{bmatrix} 0 & \dots & 0 & x_{\varepsilon_{kij}} & \dots & x_2 & x_1 \end{bmatrix}, x_{\varepsilon_{kij}} \neq 0$$

is its constituent vector in $\mathcal{F}_{ik}$, with minimal polynomial $(s - \lambda_i)^{\varepsilon_{kij}}$ and $\hat{\underline{\gamma}}_{kji}^\top = \underline{\gamma}_{kji}^\top \tilde{U}_{ik}$. Furthermore $\tilde{U}_{ik}$ is the diagonal block of dimensions $\tau_{ik} \times \tau_{ik}$ corresponding to J_{ik}. From the computation of $\tilde{U}_{ik}$ in (4.17) the main diagonal of matrix $\tilde{U}_{ik}$ is,

$$1, \ \frac{e^{-\lambda_i T}}{2T}, \ \dots, \ \frac{e^{-(\tau_{ik}-1)\lambda_i T}}{2T^{\tau_{ik}-1}}$$

and so it is

$$\hat{\underline{\gamma}}_{kji}^\top = \begin{bmatrix} 0 & \dots & 0 & \hat{x}_{\varepsilon_{kij}} & \dots & \hat{x}_2 & \hat{x}_1 \end{bmatrix}, \ \hat{x}_{\varepsilon_{kij}} \neq 0$$

where :

$$\hat{x}_{\varepsilon_{kij}} = x_{\varepsilon_{kij}} \frac{e^{-(\varepsilon_{kij}-1)\lambda_i T}}{2T^{\varepsilon_{kij}-1}}$$

From the above it is concluded that the minimal polynomial of the row vector $\hat{\underline{\gamma}}_{kji}^\top \in \mathcal{F}_{ik}$ is $(z - \lambda_i)^{\varepsilon_{kij}}$. □

Corollary 7 *The last non zero rows of the vectors $\underline{\gamma}_{kji}^\top$, $\hat{\underline{\gamma}}_{kji}^\top$ have the same order.* □

Due to eigenvalue collapsing phenomena, we cannot say that for every value of the sampling period T, the degree of the minimal polynomials of the row vectors $\hat{\underline{\gamma}}_{ij}^\top \in \widehat{\mathcal{O}}_i$ $(j = 1, 2, ..., m)$, and the dimensions $\hat{p}_i$ of the unobservable subspace $\widehat{\mathcal{P}}_i$ are automatically defined from the continuous system. Therefore we have to distinguish the two cases of sampling.

Regular Sampling

As in the case of controllable space, for the discretised model $\hat{S}(\hat{A}, \hat{C})$ we have,

Theorem 46 *Under regular sampling of a continuous system having M_{H_i} as normal structure (according to Definition 49) the following properties hold true:*

(a) The minimal polynomial of the row vector $\hat{\underline{\gamma}}_{ij}^{\top} \in \widehat{\mathcal{O}}_i$ is $(z - \hat{\lambda}_i)^{\varepsilon_{ij}}$, where ε_{ij} is the degree of the minimal polynomial of the corresponding vector $\underline{\gamma}_{ij}^{\top} \in \widehat{\mathcal{O}}_i$ of the continuous system.

(b) The set of c.o.i. of the discretised model is equal to the corresponding set for the continuous system i.e.

$$\hat{Z}(\hat{A}, \hat{C})_{\hat{\lambda}_i} = Z(A, C)_{\lambda_i}$$

(c) The dimension $\hat{p}_i$ of the unobservable subspace $\widehat{\mathcal{P}}_i$ is equal to the corresponding dimension p_i of the unobservable subspace $\mathcal{P}_i$ of the continuous system.

(d) The dimension $\hat{p}$ of the unobservable subspace $\widehat{\mathcal{P}}$ of the discretised model $\hat{S}(\hat{A}, \hat{C})$ is the same with the corresponding dimension p of the continuous system, that is

$$\hat{p} = p$$

(e) The degree $\hat{g}$ of the minimal polynomial of the row space $\widehat{\mathcal{Q}}$ (or the observability index restriction) is the same with the corresponding degree g (or the observability index restriction) of the continuous system i.e.

$$\hat{g} = g$$

Proof.

(a) From Theorem 1 the degree of the minimal polynomial of the vector $\underline{\gamma}_{ij}^{\top} \in \widehat{\mathcal{O}}_i$ is

$$\varepsilon_{ij} = \max(\varepsilon_{1ij}, \varepsilon_{2ij}, ..., \varepsilon_{kij}, ..., \varepsilon_{\nu_i ij})$$

From Proposition 1 as for the case of controllable space, it is concluded that ε_{ij} is also the degree of the minimal polynomial of the corresponding vector $\hat{\underline{\gamma}}_{ij}^{\top}$.

(b) As for the continuous system case, it can be proved that,

$$\hat{M}_{\hat{J}_i} \sim \hat{M}_{H_i} = \begin{bmatrix} \hat{\Gamma}_i \\ \hat{\Gamma}_i H_i \\ \hat{\Gamma}_i H_i^2 \\ \dots \\ \hat{\Gamma}_i H_i^{n-1} \end{bmatrix}$$

From Lemma 10 it is directly concluded that the first element of the set $Z(A, C)_{\lambda_i} : \zeta_{i1}$ is equal to the first element of the set $\hat{Z}(\hat{A}, \hat{C})_{\hat{\lambda}_i} : \hat{\zeta}_{i1} = \zeta_{i1}$. From (4.27) and also from the structure of the matrix $\tilde{U}$ (of the same structure as J) it follows that, $\widehat{\Gamma}_i = \Gamma_i \tilde{U}(\lambda_i)$. The matrix $\tilde{U}(\lambda_i)$ is non singular, it is in block diagonal form and of the same structure as $J(\lambda_i)$. Consequently the remaining c.o.i. $\zeta_{i2}, ..., \zeta_{i\nu_i}$, of the discretised system are equal to the corresponding c.o.i. of the continuous system; otherwise the set $Z(A, C)_{\lambda_i}$ remains the same under regular sampling.

(c) Then for regular sampling it follows that: $p_i = \pi_i - (\zeta_{i1} + ... + \zeta_{i\nu_i}) = \hat{p}_i$ and the Proposition for the regular sampling is proved.

(d) From the above and from Theorem 37 the result follows.

(e) Also from the above and from Propositions 48 and 49 the result follows. □

Irregular sampling

As for the case of the controllable space the collapsing of two eigenvalues λ_u and λ_y to $\hat{\lambda}_c$ is examined now for the discretised model $\hat{S}(\hat{A}, \hat{C})$.

Theorem 47 *Under the irregular sampling for which a collapsing occurs between the eigenvalues λ_u and λ_y to the eigenvalue $\hat{\lambda}_c$, with M_{H_u}, M_{H_y} as normal structure, the following properties hold true:*

(a) The minimal polynomial of the row vector $\underline{\gamma}_{cj}^T \in \mathcal{N}_c$ is $(z - \hat{\lambda}_c)^{\varepsilon_{cj}}$ $(j = 1, 2, ..., m)$, where, $\varepsilon_{cj} = \max(\varepsilon_{uj}, \varepsilon_{yj})$.

(b) If $p_u = \dim\mathcal{P}_u$, $\mathcal{P}_u \subseteq \mathcal{N}_u$ and $p_y = \dim\mathcal{P}_y$, $\mathcal{P}_y \subseteq \mathcal{N}_y$, the dimension $\hat{p}_c$ of the observable subspace $\widehat{\mathcal{P}}_c \in \mathcal{N}_c$ is , $\hat{p}_c \geq p_u + p_y$.

(c) If $\hat{p}$ is the dimension of the observable subspace $\widehat{\mathcal{P}}$ of the discretised model $\hat{S}(\hat{A}, \hat{C})$ and p is the corresponding dimension of the continuous system, then $\hat{p} \geq p$.

(d) If $\hat{p}$ is the degree of the minimal polynomial of the observable space $\widehat{\mathcal{P}}$ (or the observability index restriction) and p is the corresponding degree (or the observability index restriction) of the continuous system, then we have, $\hat{p} = p + \min(\theta_{u1}, \theta_{y1})$.

Proof.

(a) For the above irregular sampling, each one of the row vectors $\underline{\hat{\gamma}}_{cj}^{\top} \in \mathcal{N}_c$ $(j = 1, 2, \ldots, m)$ is created from the component row vectors of the corresponding $\underline{\hat{\gamma}}_{uj}^{\top} \in \mathcal{N}_u$ and $\underline{\hat{\gamma}}_{yj}^{\top} \in \mathcal{N}_y$ $(j = 1, 2, \ldots, m)$. Then $\varepsilon_{cj} = max(\varepsilon_{uj}, \varepsilon_{yj})$.

(b) The matrix $\widehat{\Gamma}_c$ is composed by the matrix blocks of Γ_u and Γ_y i.e.: $\widehat{\Gamma}_c = \begin{bmatrix} \Gamma_u & \Gamma_y \end{bmatrix}$. Let us assume,

$$Z(A, C)_{\lambda_u} = \{\zeta_{u1} \geq \ldots \geq \zeta_{u\nu_u} \geq 0\} \Rightarrow p_u = \pi_u - (\zeta_{u1} + \ldots + \zeta_{u\nu_u})$$

and

$$Z(A, C)_{\lambda_y} = \{\zeta_{y1} \geq \ldots \geq \zeta_{y\nu_y} \geq 0\} \Rightarrow p_y = \pi_y - (\zeta_{y1} + \ldots + \zeta_{y\nu_y})$$

Then from the construction procedure of $\hat{Z}(\hat{A}, \hat{C})_{\hat{\lambda}_c}$ we have, $\hat{\zeta}_{c1} = \max(\zeta_{u1}, \zeta_{y1})$ and each one of the next indices $\hat{\zeta}_{c1}, \ldots, \hat{\zeta}_{c\nu_c}$ is equal or smaller to the corresponding index of the same matrix block of Γ_u and Γ_y of the continuous system. Thus since $\hat{\pi}_c = \pi_u + \pi_y$ it follows that,

$$(\hat{\zeta}_{c1} + \ldots + \hat{\zeta}_{c\nu_c}) \leq (\zeta_{u1} + \ldots + \zeta_{u\nu_u}) + (\zeta_{y1} + \ldots + \zeta_{y\nu_y}) \Rightarrow$$

$$\Rightarrow \hat{\pi}_c - (\zeta_{c1} + \ldots + \zeta_{c\nu_c}) \geq \hat{\pi}_c - \left((\zeta_{u1} + \ldots + \zeta_{u\nu_u}) + (\zeta_{y1} + \ldots + \zeta_{y\nu_y})\right) \Rightarrow \hat{p}_c \geq p_u + p_y$$

(c) From the above and from Theorem 37 the results follows.

(d) Also from the above and from Propositions 48 and 49 the result follows. □

6.5 Conclusions

A new approach for the characterization of spectral properties of the controllable and observable space has been established based on the properties of minimal polynomials of vectors. New sets of invariants indices are introduced which enables:

a) The determination of the dimension of the controllable (unobservable) space from the Jordan canonical description.

b) The investigation of the relation between the dimension of the controllable (unobservable) space of the discretised model and the corresponding of continuous system under the different types of sampling.

Such effects are also examined in the next chapter, where the study of the degrees of decoupling zeros, formed under irregular sampling is considered.

Chapter 7

SPECTRAL DETERMINATION OF THE STRUCTURE OF DECOUPLING ZEROS

7.1 Introduction

This chapter examines the role of the system parameters of the Jordan canonical description in the determination of the structure of i.d.z. (o.d.z.). A new left (right) sequence of λ_i-Characteristic Toeplitz matrices is used to determine the set $\Sigma(A,B)_{\lambda_i}$ $(\Psi(A,B)_{\lambda_i})$ of degrees of elementary divisors of the input (output) pencil of the system at $s=\lambda_i$ or what is equivalent the degrees of input (output) decoupling zeros. The result has been proved for continuous system and provide new relationships between the Segré Characteristic of A at λ_i, $\wp_{\lambda_i}(A)$ the set of r.c.i.(c.o.i.) $\Theta(A,B)_{\lambda i}$ $(Z(A,B)_{\lambda i})$ and the set of degrees of i.d.z.(o.d.z.) $\Sigma(A.B)_{\lambda_i}$ $(\Psi(A,B)_{\lambda_i})$. This relation enables the investigation of the changes in the set of i.d.z.(c.o.i.) under regular and irregular sampling for the discrete models. The work here generalizes some classical results on the spectral characterization of controllability to the spectral characterization of degrees of decoupling zeros.

7.2 Spectral Determination of the Structure of i.d.z. of a Continuous System $S(A, B)$.

The set of i.d.z. of a system $S(A, B)$ is defined in Chapter 3 (Definition 26) as the set of roots of e.d. of the input state pencil. Let the input state pencil of the equivalent system in Jordan form $S(J, \mathcal{B})$ be, $[sI - J, \mathcal{B}] = s[I, 0] - [J, -\mathcal{B}] \in \mathcal{L}^{\mathrm{lr}}_{n,n+l}$. Consequently the structure of i.d.z. of the system $S(A, B)$ is determined equivalently by the root range of the input state pencil (Definition 15). As a first step in this direction the following sequence of left a-characteristic Toeplitz matrices is defined as in (3.27) :

$$T_a^1 = \begin{bmatrix} J - aI & \mathcal{B} \end{bmatrix} \in \mathbb{C}^{n \times (n+l)}, \qquad T_a^2 = \begin{bmatrix} J - aI & \mathcal{B} & I & 0 \\ 0 & 0 & J - aI & \mathcal{B} \end{bmatrix} \in \mathbb{C}^{2n \times 2(n+l)}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$T_a^j = \begin{bmatrix} J - aI & \mathcal{B} & I & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & J - aI & \mathcal{B} & I & 0 & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & J - aI & \mathcal{B} \end{bmatrix} \in \mathbb{C}^{jn \times j(n+l)}$$

The properties of the above sequence will be considered next.

7.2.1 Basic Properties of the Rank of the a-Characteristic TOEPLITZ Matrices

We start the investigation of the properties of the above Toeplitz sequence by considering their rank properties.

Proposition 52 *For $\forall a \in \mathbb{C} : a \notin \Phi(A)$ the matrix T_a^j has full rank.*

Proof.

$$\text{Let, } \begin{bmatrix} \underline{y}_1^\top & \underline{y}_2^\top & \cdots & \underline{y}_j^\top \end{bmatrix} T_a^j = \underline{0} \Leftrightarrow \begin{cases} \underline{y}_1^\top [J - aI, \mathcal{B}] = 0 \\ \underline{y}_1^\top [I, 0] = -\underline{y}_2^\top [J - aI, \mathcal{B}] \\ \cdots\cdots\cdots \\ \underline{y}_{j-1}^\top [I, 0] = -\underline{y}_j^\top [J - aI, \mathcal{B}] \end{cases}$$

If $a \notin \Phi(A) \Rightarrow \operatorname{rank}[J - aI, \mathcal{B}] = n \Leftrightarrow \underline{y}_1^\top = \underline{0}$ and recursively $\underline{y}_2^\top = \underline{0}, \ldots, \underline{y}_j^\top = \underline{0}$ □

Proposition 53 *Let* $a = \lambda_i \in \Phi(A)$ *and express,* $T^1_{\lambda_i} = \begin{bmatrix} H_i & 0 & \mathcal{B}_i \\ 0 & T' & \mathcal{B}' \end{bmatrix}$ *where* $H_i = J(\lambda_i) - \lambda_i I \in \mathbb{R}^{\pi_i \times \pi_i}$ *is nilpotent,* $T' \in \mathbb{C}^{(n-\pi_i)\times(n-\pi_i)}$ *is full rank,* $\mathcal{B}_i \in \mathbb{C}^{\pi_i \times l}$ *is (as defined in 3.67) the matrix block of* $\mathcal{B}$ *corresponding to* $J(\lambda_i)$*. Then the left nullity of the matrix* $T^j_{\lambda_i}$ *is defined by the left nullity of the matrix* $\check{T}^j_{\lambda_i}$*, where,*

$$\check{T}^j_{\lambda_i} \triangleq \left.\begin{bmatrix} H_i & \mathcal{B}_i & I & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & H_i & \mathcal{B}_i & I & 0 & \dots & 0 & 0 \\ . & . & . & . & . & . & \dots & . & . \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & H_i & \mathcal{B}_i \end{bmatrix}\right\} j\text{-blocks} \tag{7.1}$$

Proof. for $j = 1$:

$$\begin{bmatrix} \underline{y}_1^\top & \underline{y}_2^\top \end{bmatrix} \begin{bmatrix} H_i & 0 & \mathcal{B}_i \\ 0 & T' & \mathcal{B}' \end{bmatrix} = \underline{0} \Leftrightarrow \begin{cases} \underline{y}_1^\top H_i = \underline{0} \\ \underline{y}_2^\top T' = \underline{0} \\ \underline{y}_1^\top \mathcal{B}_i + \underline{y}_2^\top \mathcal{B}' = \underline{0} \end{cases} \Leftrightarrow \begin{cases} \underline{y}_2^\top = \underline{0} \\ \underline{y}_1^\top [H_i, \mathcal{B}_i] = \underline{0} \Rightarrow \check{T}^1_{\lambda_i} = [H_i, \mathcal{B}_i] \end{cases}$$

for $j = 2$:

$$\begin{bmatrix} \underline{y}_1^\top & \underline{y}_2^\top & \underline{y}_3^\top & \underline{y}_4^\top \end{bmatrix} \begin{bmatrix} H_i & 0 & \mathcal{B}_i & I & 0 & 0 \\ 0 & T' & \mathcal{B}' & 0 & I & 0 \\ 0 & 0 & 0 & H_i & 0 & \mathcal{B}_i \\ 0 & 0 & 0 & 0 & T' & \mathcal{B}' \end{bmatrix} = \underline{0} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \underline{y}_2^\top = \underline{0} \\ \underline{y}_4^\top = \underline{0} \\ \begin{bmatrix} \underline{y}_1^\top & \underline{y}_3^\top \end{bmatrix} \begin{bmatrix} H_i & \mathcal{B}_i & I & 0 \\ 0 & 0 & H_i & \mathcal{B}_i \end{bmatrix} = \underline{0} \Rightarrow \check{T}^2_{\lambda_i} = \begin{bmatrix} H_i & \mathcal{B}_i & I & 0 \\ 0 & 0 & H_i & \mathcal{B}_i \end{bmatrix} \end{cases}$$

The general step follows along similar lines. □

Remark 30 *From the above we conclude that only the numbers* $\lambda_i \in \Phi(A)$ *are candidate for i.d.z.* □

Remark 31 *We can study the sequence of left nullities and the corresponding tests, for determining the degree of i.d.z. by considering the case of each eigenvalue λ_i for which $\tilde{T}^j_{\lambda_i}$ is rank deficient.* □

7.2.2 Left Nullity of the j-*th* Single Block Matrix $\tilde{T}^j_{\lambda_i}$

From the above we conclude that any $\lambda_i \in \Phi(A)$ is a candidate decoupling zero. If the A-Segré Characteristic at λ_i is $\wp_{\lambda_i}(A)$ (given by 3.22), then the matrix $\mathcal{B}_i$ can be partitioned according to $\wp_{\lambda_i}(A)$ as in 3.69 and the nilpotent matrix H_i can also be represented according to the eigenstructure of A at λ_i as in 6.5.

Definition 50 *The t-th reduced matrix of $\mathcal{B}^{t)}_{ik}$ is defined as the matrix derived from $\mathcal{B}_{ik}$ (where $\mathcal{B}_{ik}$ is defined in 3.71) as indicated below :*

$$\mathcal{B}^{t)}_{ik} = \begin{bmatrix} 0 \\ \dots \\ 0 \\ \underline{\beta}^{\top}_{ik_t} \\ \dots \\ \underline{\beta}^{\top}_{ik_{\tau_{ik}}} \end{bmatrix} \in \mathbb{C}^{\tau_{ik} \times l}, \ \ t = 1, 2, ..., \tau_{ik}$$

and where $\mathcal{B}^{t)}_{ik} \triangleq \mathcal{B}_{ik}$ for $\forall t = 0, -1, -2, ...$ □

The same notation can be applied to any other matrix. Thus if I_{ik} is the $\tau_{ik} \times \tau_{ik}$ identity matrix, then $I^{t)}_{ik}$ is also a $\tau_{ik} \times \tau_{ik}$ matrix,

$$I^{t)}_{ik} = \begin{array}{c} \begin{bmatrix} 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ . & \dots & . & . & . & \dots & . \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & 1 & \dots & 0 \\ . & \dots & . & . & . & \dots & . \\ 0 & \dots & 0 & 0 & 0 & \dots & 1 \end{bmatrix} \end{array} \begin{array}{l} \\ \\ \\ \longleftarrow t\text{-}th \text{ row} \\ \\ \\ \\ \end{array}$$

For any $p \geq 1,\ p \in \mathbb{Z}$ we define the $*$ operation on $\mathcal{B}_{ik} \in \mathbb{C}^{\tau_{ik} \times l}$ by :

$$p * \mathcal{B}_{ik} \triangleq \mathcal{B}_{ik}^{*p} \triangleq \mathcal{B}_{ik}^{\tau_{ik}+1-p)},\ p = 1, 2, ... \tag{7.2}$$

Let $\mathcal{B}_i$ be partitioned into blocks as in 3.69, then we define the $*$ operation on $\mathcal{B}_i$ by some $p \in \mathbb{Z}$ as :

$$p * \mathcal{B}_i \triangleq \mathcal{B}_i^{*p} \triangleq \begin{bmatrix} p * \mathcal{B}_{i1} \\ \cdots \\ p * \mathcal{B}_{ik} \\ \cdots \\ p * \mathcal{B}_{i\nu_i} \end{bmatrix} = \begin{bmatrix} \mathcal{B}_{i1}^{*p} \\ \cdots \\ \mathcal{B}_{ik}^{*p} \\ \cdots \\ \mathcal{B}_{i\nu_i}^{*p} \end{bmatrix} = \begin{bmatrix} \mathcal{B}_{i1}^{\tau_{i1}+1-p)} \\ \cdots \\ \mathcal{B}_{ik}^{\tau_{ik}+1-p)} \\ \cdots \\ \mathcal{B}_{i\nu_i}^{\tau_{i\nu_i}+1-p)} \end{bmatrix}$$

7.2.3 Normal Description of the j-*th* Left Toeplitz Matrix $\tilde{T}_{\lambda_i}^j$

Using the above notation we may simplify the computation of nullities of $\tilde{T}_{\lambda_i}^j$ using simpler matrices.

Proposition 54 *The above defined* j-th *left Toeplitz matrix* $\tilde{T}_{\lambda_i}^j$ *is equivalent over* $\mathbb{C}$ *by elementary column operations to the following form :*

$$\tilde{T}_{\lambda_i}^j \triangleq \left.\begin{bmatrix} H_i & \mathcal{B}_i^{*1} & I_i^{*1} & 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & H_i & \mathcal{B}_i^{*2} & I_i^{*2} & 0 & \ldots & 0 & 0 \\ . & . & . & . & . & . & \ldots & . & . \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots & I_i^{*(j-1)} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots & H_i & \mathcal{B}_i^{*j} \end{bmatrix}\right\} j - blocks$$

Proof. Assume for the sake of simplicity, $\wp_{\lambda_i}(A) \triangleq \{5, 4, 2\}$. We shall establish the Proposition for this case, whereas the general case follows along similar lines.

1. for $j = 1$:

$$\tilde{T}_{\lambda_i}^1 = \begin{bmatrix} H_2 & 0 & 0 & \mathcal{B}_2 \\ 0 & H_4 & 0 & \mathcal{B}_4 \\ 0 & 0 & H_5 & \mathcal{B}_5 \end{bmatrix}$$

for a typical block, we have by column transformations and using the notation introduced above, we have :

$$\left[\begin{array}{cc} H_{ik} & \mathcal{B}_{ik} \end{array}\right] = \left[\begin{array}{ccc} 0 & I' & \mathcal{B}'_{ik} \\ 0 & 0 & \underline{\beta}^{\mathsf{T}}_{ik_{\tau_{ik}}} \end{array}\right] \sim \left[\begin{array}{ccc} 0 & I' & 0 \\ 0 & 0 & \underline{\beta}^{\mathsf{T}}_{ik_{\tau_{ik}}} \end{array}\right] \sim \left[\begin{array}{cc} H_{ik} & \mathcal{B}^{\tau_{ik})}_{ik} \end{array}\right]$$

and thus by column transformations we have,

$$\tilde{T}^1_{\lambda_i} \sim \left[\begin{array}{cccc} H_2 & 0 & 0 & \mathcal{B}_2^{2)} \\ 0 & H_4 & 0 & \mathcal{B}_4^{4)} \\ 0 & 0 & H_5 & \mathcal{B}_5^{5)} \end{array}\right] = \left[\begin{array}{cc} H_i & \mathcal{B}_i^{*1} \end{array}\right] = \tilde{T}^1_{\lambda_i}$$

2. for $j = 2$: by elementary column operations we have that

$$\tilde{T}^2_{\lambda_i} = \left[\begin{array}{cccccccc} H_2 & 0 & 0 & \mathcal{B}_2 & I_2 & 0 & 0 & 0 \\ 0 & H_4 & 0 & \mathcal{B}_4 & 0 & I_4 & 0 & 0 \\ 0 & 0 & H_5 & \mathcal{B}_5 & 0 & 0 & I_5 & 0 \\ 0 & 0 & 0 & 0 & H_2 & 0 & 0 & \mathcal{B}_2 \\ 0 & 0 & 0 & 0 & 0 & H_4 & 0 & \mathcal{B}_4 \\ 0 & 0 & 0 & 0 & 0 & 0 & H_5 & \mathcal{B}_5 \end{array}\right] \sim \left[\begin{array}{cccccccc} H_2 & 0 & 0 & \mathcal{B}_2^{2)} & I_2^{2)} & 0 & 0 & 0 \\ 0 & H_4 & 0 & \mathcal{B}_4^{4)} & 0 & I_4^{4)} & 0 & 0 \\ 0 & 0 & H_5 & \mathcal{B}_5^{5)} & 0 & 0 & I_5^{5)} & 0 \\ 0 & 0 & 0 & 0 & H_2 & 0 & 0 & \mathcal{B}_2 \\ 0 & 0 & 0 & 0 & 0 & H_4 & 0 & \mathcal{B}_4 \\ 0 & 0 & 0 & 0 & 0 & 0 & H_5 & \mathcal{B}_5 \end{array}\right]$$

we use column transformation from each of the $\left[\begin{array}{c} \cdots \\ I_j^{j)} \\ \cdots \\ H_j \\ \cdots \end{array}\right]$ to the corresponding $\left[\begin{array}{c} \cdots \\ 0 \\ \cdots \\ \mathcal{B}_j \\ \cdots \end{array}\right]$,

and thus we have :

$$\left[\begin{array}{ccc} \cdots & \cdots & \cdots \\ I_j^{j)} & \cdots & 0 \\ \cdots & \cdots & \cdots \\ H_j & \cdots & \mathcal{B}_j \\ \cdots & \cdots & \cdots \end{array}\right] \sim \left[\begin{array}{ccc} \cdots & \cdots & \cdots \\ I_j^{j)} & \cdots & 0 \\ \cdots & \cdots & \cdots \\ H_j & \cdots & \mathcal{B}_j^{j-1)} \\ \cdots & \cdots & \cdots \end{array}\right]$$

which also leads to :

$$\tilde{T}^2_{\lambda_i} \sim \begin{bmatrix} H_2 & 0 & 0 & \mathcal{B}_2^{2)} & I_2^{2)} & 0 & 0 & 0 \\ 0 & H_4 & 0 & \mathcal{B}_4^{4)} & 0 & I_4^{4)} & 0 & 0 \\ 0 & 0 & H_5 & \mathcal{B}_5^{5)} & 0 & 0 & I_5^{5)} & 0 \\ 0 & 0 & 0 & 0 & H_2 & 0 & 0 & \mathcal{B}_2^{1)} \\ 0 & 0 & 0 & 0 & 0 & H_4 & 0 & \mathcal{B}_4^{3)} \\ 0 & 0 & 0 & 0 & 0 & 0 & H_5 & \mathcal{B}_5^{4)} \end{bmatrix} = \begin{bmatrix} H_i & \mathcal{B}_i^{*1} & I_i^{*1} & 0 \\ 0 & 0 & H_i & \mathcal{B}_i^{*2} \end{bmatrix} = \tilde{T}^2_{\lambda_i}$$

3. $j = 3$:

$$\tilde{T}^3_{\lambda_i} = \begin{bmatrix} H_2 & 0 & 0 & \mathcal{B}_2 & I_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & H_4 & 0 & \mathcal{B}_4 & 0 & I_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & H_5 & \mathcal{B}_5 & 0 & 0 & I_5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & H_2 & 0 & 0 & \mathcal{B}_2 & I_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & H_4 & 0 & \mathcal{B}_4 & 0 & I_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & H_5 & \mathcal{B}_5 & 0 & 0 & I_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & H_2 & 0 & 0 & \mathcal{B}_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & H_4 & 0 & \mathcal{B}_4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & H_5 & \mathcal{B}_5 \end{bmatrix}$$

Using the transformations of step $j = 2$ we have that

$$\tilde{T}^3_{\lambda_i} \sim \begin{bmatrix} H_2 & 0 & 0 & \mathcal{B}_2^{2)} & I_2^{2)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & H_4 & 0 & \mathcal{B}_4^{4)} & 0 & I_4^{4)} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & H_5 & \mathcal{B}_5^{5)} & 0 & 0 & I_5^{5)} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & H_2 & 0 & 0 & \mathcal{B}_2^{1)} & I_2^{1)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & H_4 & 0 & \mathcal{B}_4^{3)} & 0 & I_4^{3)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & H_5 & \mathcal{B}_5^{4)} & 0 & 0 & I_5^{4)} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & H_2 & 0 & 0 & \mathcal{B}_2^{0)} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & H_4 & 0 & \mathcal{B}_4^{2)} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & H_5 & \mathcal{B}_5^{3)} \end{bmatrix} =$$

$$= \begin{bmatrix} H_i & B_i^{*1} & I_i^{*1} & 0 & 0 & 0 \\ 0 & 0 & H_i & B_i^{*2} & I_i^{*2} & 0 \\ 0 & 0 & 0 & 0 & H_i & B_i^{*3} \end{bmatrix} = \tilde{T}^3_{\lambda_i}$$

The above analysis readily implies the general step and this completes the proof. □

Remark 32 *The form* $\tilde{T}^j_{\lambda_i}$ *is column equivalent to* $\check{T}^j_{\lambda_i}$ *and* $\tilde{T}^j_{\lambda_i}$ *will be called the normal description of the* j-th *left normal Toeplitz matrix. Clearly,*

$$\text{rank}\check{T}^j_{\lambda_i} = \text{rank}\tilde{T}^j_{\lambda_i}$$

The left null-space of $\check{T}^j_{\lambda_i}$ *may be studied by using* $\tilde{T}^j_{\lambda_i}$ *since the two are column space equivalent.* □

Proposition 55 *Let* $\underline{y} \in \mathbb{C}^{jn}$ *and be partitioned as,*

$$\underline{y}^\top = \left[\underline{y}_1^\top, \underline{y}_2^\top, ..., \underline{y}_{j-1}^\top, \underline{y}_j^\top\right]$$

then $\underline{y}^\top \in \mathcal{N}_l\{\check{T}^j_{\lambda_i}\}$ *where* $\check{T}^j_{\lambda_i} \in \mathbb{C}^{jn \times j(n+l)}$ *if and only if the following conditions are satisfied,*

$$\begin{cases} \underline{y}_1^\top H_i = \underline{0} \\ \underline{y}_2^\top H_i = -\underline{y}_1^\top I_i^{*1} \\ \underline{y}_3^\top H_i = -\underline{y}_2^\top I_i^{*2} \\ \quad \cdots \\ \underline{y}_j^\top H_i = -\underline{y}_{j-1}^\top I_i^{*(j-1)} \end{cases} \quad and \quad \begin{cases} \underline{y}_1^\top B_i^{*1} = 0 \\ \underline{y}_2^\top B_i^{*2} = 0 \\ \underline{y}_3^\top B_i^{*3} = 0 \\ \quad \cdots \\ \underline{y}_j^\top B_i^{*j} = 0 \end{cases} \tag{7.3}$$

Proof. Since $\mathcal{N}_l\{\check{T}^j_{\lambda_i}\} = \mathcal{N}_l\{\tilde{T}^j_{\lambda_i}\}$, by writing the condition $\underline{y}^\top \tilde{T}^j_{\lambda_i} = 0$ and considering the description of $\tilde{T}^j_{\lambda_i}$ and the natural partitioning the result follows. □

The set of equations 7.3 comprises from two subsets i.e. the equations of the first column are referred to as the *left recurrent equations* of the set, and the equations of the second column, called the *left Kernel equations*. We consider first the recurrent equations.

Remark 33 *Let* $\underline{y}_i^\top$ *be partitioned according to Segré characteristic defined in (3.22) as,*

$$\underline{y}_i^\top = \left[\underline{y}_{\tau_{i1}}^\top, ..., \underline{y}_{\tau_{ik}}^\top, ..., \underline{y}_{\tau_{i\nu_i}}^\top\right], \ i = 1, 2, ..., f$$

and from the block diagonal structure of H_i and I_i^{*j} we have that the set of the recurrent equations is equivalent to,

$$\left\{\begin{array}{ll} \underline{y}_{\tau_{ik}}^{\top 1} H_{ik} = \underline{0}^{\top} & (1) \\ \underline{y}_{\tau_{ik}}^{\top 2} H_{ik} = -\underline{y}_{\tau_{ik}}^{\top 1} I_{ik}^{*1} & (2) \\ \underline{y}_{\tau_{ik}}^{\top 3} H_{ik} = -\underline{y}_{\tau_{ik}}^{\top 2} I_{ik}^{*2} & (3) \\ \quad \cdots\cdots\cdots & \cdots \\ \underline{y}_{\tau_{ik}}^{\top j} H_{ik} = -\underline{y}_{\tau_{ik}}^{\top j-1} I_{ik}^{*j-1} & (j) \end{array}\right. \tag{7.4}$$

where τ_{ik} takes values from the set of $\wp_{\lambda_i}(A)$. Equations 7.4 will be called the basic recurrent equations. □

Lemma 11 *For any $\tau_{ik} \geq 1$ the solution of the basic recurrent equation (7.4) is given by :*

1. *for $j \leq \tau_{ik}$:*

$$\underline{y}_{\tau_{ik}}^{\top j} = \left[\underbrace{0, \ \ldots, \ 0}_{\tau_{ik}-j}, \ (-1)^{j-1} c_{\tau_{ik}}^{1}, \ (-1)^{j-2} c_{\tau_{ik}}^{2}, \ldots, \ -c_{\tau_{ik}}^{j-1}, \ c_{\tau_{ik}}^{j} \right] \tag{7.5}$$

where $c_{\tau_{ik}}^{1}, c_{\tau_{ik}}^{2}, \ldots, c_{\tau_{ik}}^{j}$ arbitrary,

2. *for $j > \tau_{ik}$:*

$$\left.\begin{array}{l} \underline{y}_{\tau_{ik}}^{\top 1} = \underline{y}_{\tau_{ik}}^{\top 2} = \ldots = \underline{y}_{\tau_{ik}}^{\top j-\tau_{ik}} = \underline{0}^{\top} \\ \underline{y}_{\tau_{ik}}^{\top j-\tau_{ik}+1} = [0, \ldots 0, c_{\tau_{ik}}^{j-\tau_{ik}+1}] \\ \cdots\cdots\cdots\cdots \\ \underline{y}_{\tau_{ik}}^{\top j-1} = [0, (-1)^{\tau_{ik}-2} c_{\tau_{ik}}^{j-\tau_{ik}+1}, \ldots, -c_{\tau_{ik}}^{j-2}, c_{\tau_{ik}}^{j-1}] \\ \underline{y}_{\tau_{ik}}^{\top j} = [(-1)^{\tau_{ik}-1} c_{\tau_{ik}}^{j-\tau_{ik}+1}, \ldots, -c_{\tau_{ik}}^{j-1}, c_{\tau_{ik}}^{j}] \end{array}\right\} \tag{7.6}$$

where $c_{\tau_{ik}}^{j-\tau_{ik}+1}, \ldots, c_{\tau_{ik}}^{j}$ arbitrary.

Proof. The result is established by induction. Thus,

1. $j = 1$: Let

$$\underline{y}_{\tau_{ik}}^{\top 1} \triangleq [x_1^1, \ldots, x_{\tau_{ik}}^1]$$

from the first of the recurrent equations (7.4) it follows :

$$(1) \Leftrightarrow \left[0, x_1^1, ..., x_{\tau_{ik}-1}^1\right] = \underline{0} \Leftrightarrow x_1^1 = ... = x_{\tau_{ik}-1}^1 = 0,\ x_{\tau_{ik}}^1 = c_{\tau_{ik}}^1 \text{ arbitrary}$$

$$\Leftrightarrow \underline{y}_{\tau_{ik}}^{\top 1} = \left[0, ..., 0, c_{\tau_{ik}}^1\right]$$

2. $j = 2$: Let

$$\underline{y}_{\tau_{ik}}^{\top 1} \triangleq \left[x_1^1, ..., x_{\tau_{ik}}^1\right], \quad \underline{y}_{\tau_{ik}}^{\top 2} \triangleq \left[x_1^2, ..., x_{\tau_{ik}}^2\right]$$

from the (1) and (2) of the recurrent equations (7.4) we have :

$$\left.\begin{array}{l} \underline{y}_{\tau_{ik}}^{\top 1} H_{ik} = \underline{0} \\ \underline{y}_{\tau_{ik}}^{\top 2} H_{ik} = -\underline{y}_{\tau_{ik}}^{\top 1} I_{\tau_{ik}}^{*1} \end{array}\right\} (a)$$

where,

$$I_{\tau_{ik}}^{*1} = I_{\tau_{ik}}^{\tau_{ik})} = \begin{bmatrix} 0 & ... & 0 & 0 \\ . & ... & . & . \\ 0 & ... & 0 & 0 \\ 0 & ... & 0 & 1 \end{bmatrix}$$

Thus,

$$(a) \Leftrightarrow \left\{ \begin{array}{l} y_{\tau_{ik}}^1 = \left[0, ..., 0, c_{\tau_{ik}}^1\right] \\ \left[0, x_1^2, ..., x_{\tau_{ik}-1}^2\right] = \left[0, ..., 0, c_{\tau_{ik}}^1\right] \Leftrightarrow \end{array}\right.$$

$$\Leftrightarrow x_1^2 = ... = x_{\tau_{ik}-2}^2 = 0,\ x_{\tau_{ik}-1}^2 = c_{\tau_{ik}}^1,\ \ x_{\tau_{ik}}^2 = c_{\tau_{ik}}^2 \text{ arbitrary}$$

$$\Leftrightarrow y_{\tau_{ik}}^2 = \left[0, ..., 0, c_{\tau_{ik}}^1, c_{\tau_{ik}}^2\right]$$

. . . .

3. for the general $j \leq \tau_{ik}$ step, let us assume :

$$\underline{y}_{\tau_{ik}}^{\top 1} \triangleq \left[x_1^1, ..., x_{\tau_{ik}}^1\right], \ \underline{y}_{\tau_{ik}}^{\top 2} \triangleq \left[x_1^2, ..., x_{\tau_{ik}}^2\right], ..., \ y_{\tau_{ik}}^j \triangleq \left[x_1^j, ..., x_{\tau_{ik}}^j\right]$$

then, from the j first of the recurrent equations (7.4) it follows :

$$\left.\begin{array}{l} \underline{y}^{\top 1}_{\tau_{ik}} H_{ik} = \underline{0} \\ \underline{y}^{\top 2}_{\tau_{ik}} H_{ik} = -\underline{y}^{\top 1}_{\tau_{ik}} I^{*1}_{\tau_{ik}} \\ \underline{y}^{\top 3}_{\tau_{ik}} H_{ik} = -\underline{y}^{\top 2}_{\tau_{ik}} I^{*2}_{\tau_{ik}} \\ \cdots\cdots\cdots \\ \underline{y}^{\top j}_{\tau_{ik}} H_{ik} = -\underline{y}^{\top j-1}_{\tau_{ik}} I^{*j-1}_{\tau_{ik}} \end{array}\right\} (b)$$

where :

$$\underline{y}^{\top j-1}_{\tau_{ik}} = [\underbrace{0, ..., 0}_{\tau_{ik}-j+1}, (-1)^{j-2} c^1_{\tau_{ik}}, (-1)^{j-3} c^2_{\tau_{ik}}, ..., -c^{j-2}_{\tau_{ik}}, c^{j-1}_{\tau_{ik}}]$$

and

$$\begin{array}{cc} I^{*(j-1)}_{\tau_{ik}} = I^{(\tau_{ik}+2-j)}_{\tau_{ik}} = \begin{bmatrix} 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ . & . & . & . & . & \ldots & . \\ 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \\ 0 & \ldots & 0 & 0 & 1 & \ldots & 0 \\ . & \ldots & . & . & . & \ldots & . \\ 0 & \ldots & 0 & 0 & 0 & \ldots & 1 \end{bmatrix} & \begin{array}{l} \\ \\ \\ \longleftarrow \tau_{ik}+2-j \\ \\ \\ \\ \end{array} \end{array}$$

Then it follows that :

$$(b) \Leftrightarrow \left\{\begin{array}{l} y^1_{\tau_{ik}} = \left[0, ..., 0, c^1_{\tau_{ik}}\right] \\ y^2_{\tau_{ik}} = \left[0, ..., 0, -c^1_{\tau_{ik}}, c^2_{\tau_{ik}}\right] \\ y^3_{\tau_{ik}} = \left[0, ..., 0, c^1_{\tau_{ik}}, -c^2_{\tau_{ik}}, c^3_{\tau_{ik}}\right] \\ \cdots\cdots\cdots \\ \left[0, x^j_1, ..., x^j_{\tau_{ik}-1}\right] = \left[\underbrace{0, ..., 0}_{\tau_{ik}-j+1}, (-1)^{j-2} c^1_{\tau_{ik}}, ..., c^{j-1}_{\tau_{ik}}\right] \end{array}\right.$$

and relation (7.5) is proved.

4. for $j > \tau_{ik}$, let again,

$$\underline{y}_{\tau_{ik}}^{\top 1} \triangleq \left[x_1^1, ..., x_{\tau_{ik}}^1\right], \ \underline{y}_{\tau_{ik}}^{\top 2} \triangleq \left[x_1^2, ..., x_{\tau_{ik}}^2\right], ..., \ y_{\tau_{ik}}^{\top j} \triangleq \left[x_1^j, ..., x_{\tau_{ik}}^j\right]$$

It is

$$I_{\tau_{ik}}^{*\tau_{ik}} = I_{\tau_{ik}}$$

and

$$\underline{y}_{\tau_{ik}}^{\top j} = [(-1)^{\tau_{ik}-1} c_{\tau_{ik}}^{j-\tau_{ik}+1}, ..., -c_{\tau_{ik}}^{j-1}, c_{\tau_{ik}}^{j}] \tag{7.7}$$

from the j-*th* of the recurrent equations (7.4) it follows that :

$$[(-1)^{\tau_{ik}-1} c_{\tau_{ik}}^{j-\tau_{ik}+1}, ..., -c_{\tau_{ik}}^{j-1}, c_{\tau_{ik}}^{j}]H_{ik} = -\left[x_1^{j-1}, ..., x_{\tau_{ik}}^{j-1}\right] I_{\tau_{ik}}^{*\tau_{ik}} \Leftrightarrow$$

$$\Leftrightarrow x_1^{j-1} = 0, x_2^{j-1} = (-1)^{\tau_{ik}-2} c_{\tau_{ik}}^{j-\tau_{ik}+1}, ..., x_{\tau_{ik}}^{j-1} = c_{\tau_{ik}}^{j-1} \Leftrightarrow$$

$$\underline{y}_{\tau_{ik}}^{\top j-1} = [0, (-1)^{\tau_{ik}-2} c_{\tau_{ik}}^{j-\tau_{ik}+1}, ..., -c_{\tau_{ik}}^{j-2}, c_{\tau_{ik}}^{j-1}] \tag{7.8}$$

and from the $(j-1)$-*th* of the recurrent equations we have :

$$[0, (-1)^{\tau_{ik}-2} c_{\tau_{ik}}^{j-\tau_{ik}+1}, ..., -c_{\tau_{ik}}^{j-2}, c_{\tau_{ik}}^{j-1}]H_{ik} = -\left[x_1^{j-2}, ..., x_{\tau_{ik}}^{j-2}\right] I_{\tau_{ik}} \Leftrightarrow$$

$$\Leftrightarrow x_1^{j-1} = 0, x_1^{j-2} = 0, x_2^{j-3} = (-1)^{\tau_{ik}-3} c_{\tau_{ik}}^{j-\tau_{ik}+1}, ..., x_{\tau_{ik}}^{j-2} = c_{\tau_{ik}}^{j-2} \Leftrightarrow$$

$$\underline{y}_{\tau_{ik}}^{\top j-2} = [0, 0, (-1)^{\tau_{ik}-3} c_{\tau_{ik}}^{j-\tau_{ik}+1}, ..., --c_{\tau_{ik}}^{j-3}, c_{\tau_{ik}}^{j-2}] \tag{7.9}$$

So the solution of the basic recurrent equations is as described by relations (7.5) and (7.6). □

Example 9 *Consider the system characterized by* $\wp_{\lambda_i}(A) \triangleq \{5,4,2\}$ *where,*

$$H_i = \begin{bmatrix} H_2 & 0 & 0 \\ 0 & H_4 & 0 \\ 0 & 0 & H_5 \end{bmatrix}, \quad \mathcal{B}_i = \begin{bmatrix} \mathcal{B}_2 \\ \mathcal{B}_4 \\ \mathcal{B}_5 \end{bmatrix}, \quad I_i = \begin{bmatrix} I_2 & 0 & 0 \\ 0 & I_4 & 0 \\ 0 & 0 & I_5 \end{bmatrix}$$

we will first examine the solution of equations (7.3) :

1. for $j = 1$

$$\underline{y}_1^\top H_i = \underline{0}, \quad \underline{y}_1^\top \mathcal{B}_i^{*1} = \underline{0}$$

and if

$$\underline{y}_1^\top = \begin{bmatrix} \underline{y}_2^{\top 1} & \underline{y}_4^{\top 1} & \underline{y}_5^{\top 1} \end{bmatrix}$$

then

$$\underline{y}_2^{\top 1} H_2 = \underline{0}, \quad \underline{y}_4^{\top 1} H_4 = \underline{0}, \quad \underline{y}_5^{\top 1} H_5 = \underline{0}, \quad \begin{bmatrix} \underline{y}_2^{\top 1} & \underline{y}_4^{\top 1} & \underline{y}_5^{\top 1} \end{bmatrix} \begin{bmatrix} \mathcal{B}_2^{2)} \\ \mathcal{B}_4^{4)} \\ \mathcal{B}_5^{5)} \end{bmatrix} = 0$$

By Lemma 11 we have :

$$\underline{y}_2^{\top 1} = \begin{bmatrix} 0 & c_2^1 \end{bmatrix}, \quad \underline{y}_2^{\top 1} = \begin{bmatrix} 0 & c_2^1 \end{bmatrix}, \quad \underline{y}_5^{\top 1} = \begin{bmatrix} 0 & 0 & 0 & 0 & c_5^1 \end{bmatrix}$$

and from the Kernel condition :

$$\underline{y}_2^{\top 1} \mathcal{B}_2^{2)} + \underline{y}_4^{\top 1} \mathcal{B}_4^{4)} + \underline{y}_5^{\top 1} \mathcal{B}_5^{5)} = 0$$

the latter is reduced to

$$\begin{bmatrix} 0 & c_2^1 \end{bmatrix} \begin{bmatrix} 0 \\ \underline{\beta}_{2,2}^\top \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & c_4^1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \underline{\beta}_{4,4}^\top \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & c_5^1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \underline{\beta}_{5,5}^\top \end{bmatrix} = 0$$

$$\Leftrightarrow \begin{bmatrix} c_2^1 & c_4^1 & c_5^1 \end{bmatrix} \begin{bmatrix} \underline{\beta}_{2,2}^\top \\ \underline{\beta}_{4,4}^\top \\ \underline{\beta}_{5,5}^\top \end{bmatrix} = 0$$

If we define

$$\underline{c}_1^\top \triangleq \begin{bmatrix} c_2^1 & c_4^1 & c_5^1 \end{bmatrix}$$

then

$$\underline{c}_1^\top \begin{bmatrix} \underline{\beta}_{2,2}^\top \\ \underline{\beta}_{4,4}^\top \\ \underline{\beta}_{5,5}^\top \end{bmatrix} = 0$$

2. for $j = 2$:

$$\begin{array}{ll} \underline{y}_1^\top H_i = \underline{0} & \underline{y}_1^\top \mathcal{B}_i^{*1} = \underline{0} \\ \underline{y}_2^\top H_i = -\underline{y}_1^\top I_i^{*1} & \underline{y}_2^\top \mathcal{B}_i^{*2} = \underline{0} \end{array}$$

and if

$$\underline{y}_1^\top = \begin{bmatrix} \underline{y}_2^{\top 1} & \underline{y}_4^{\top 1} & \underline{y}_5^{\top 1} \end{bmatrix} \quad \underline{y}_2^\top = \begin{bmatrix} \underline{y}_2^{\top 2} & \underline{y}_4^{\top 2} & \underline{y}_5^{\top 2} \end{bmatrix}$$

then by *Lemma 11* we have :

$$\begin{array}{lll} \underline{y}_2^{\top 1} = \begin{bmatrix} 0 & c_2^1 \end{bmatrix}, & \underline{y}_2^{\top 1} = \begin{bmatrix} 0 & c_2^1 \end{bmatrix}, & \underline{y}_5^{\top 1} = \begin{bmatrix} 0 & 0 & 0 & 0 & c_5^1 \end{bmatrix} \\ \underline{y}_2^{\top 2} = \begin{bmatrix} -c_2^1 & c_2^2 \end{bmatrix}, & \underline{y}_4^{\top 2} = \begin{bmatrix} 0 & 0 & -c_4^1 & c_4^2 \end{bmatrix}, & \underline{y}_5^{\top 2} = \begin{bmatrix} 0 & 0 & 0 & -c_5^1 & c_5^2 \end{bmatrix} \end{array}$$

If we now define

$$\underline{c}_2^\top \triangleq \begin{bmatrix} -c_2^1 & c_2^2 & -c_4^1 & c_4^2 & -c_5^1 & c_5^2 \end{bmatrix}$$

then we have :

$$\underline{c}_2^\top \begin{bmatrix} \underline{\beta}_{1,2}^\top & \underline{\beta}_{2,2}^\top \\ \underline{\beta}_{2,2}^\top & 0 \\ \underline{\beta}_{3,4}^\top & \underline{\beta}_{4,4}^\top \\ \underline{\beta}_{4,4}^\top & 0 \\ \underline{\beta}_{4,5}^\top & \underline{\beta}_{5,5}^\top \\ \underline{\beta}_{5,5}^\top & 0 \end{bmatrix} = \underline{0}$$

3. for $j = 3$ and by following the same as above procedure we have:

$$
\begin{array}{lll}
\underline{y}_2^{\top 1} = \begin{bmatrix} 0 & 0 \end{bmatrix}, & \underline{y}_4^{\top 2} = \begin{bmatrix} 0 & 0 & 0 & c_4^1 \end{bmatrix}, & \underline{y}_5^{\top 2} = \begin{bmatrix} 0 & 0 & 0 & 0 & c_5^1 \end{bmatrix} \\
\underline{y}_2^{\top 2} = \begin{bmatrix} 0 & c_2^2 \end{bmatrix}, & \underline{y}_4^{\top 2} = \begin{bmatrix} 0 & 0 & -c_4^1 & c_4^2 \end{bmatrix}, & \underline{y}_5^{\top 2} = \begin{bmatrix} 0 & 0 & 0 & -c_5^1 & c_5^2 \end{bmatrix} \\
\underline{y}_2^{\top 3} = \begin{bmatrix} -c_2^2 & c_2^3 \end{bmatrix}, & \underline{y}_4^{\top 3} = \begin{bmatrix} 0 & c_4^1 & -c_4^2 & c_4^3 \end{bmatrix}, & \underline{y}_5^{\top 3} = \begin{bmatrix} 0 & 0 & c_5^1 & -c_5^2 & c_5^3 \end{bmatrix}
\end{array}
$$

If we set

$$
\underline{c}_3^{\top} \triangleq \begin{bmatrix} -c_2^2 & c_2^3 & c_4^1 & -c_4^2 & c_4^3 & c_5^1 & -c_5^2 & c_4^3 \end{bmatrix}
$$

then we have :

$$
\underline{c}_3^{\top} \begin{bmatrix}
\underline{\beta}_{1,2}^{\top} & \underline{\beta}_{2,2}^{\top} & 0 \\
\underline{\beta}_{2,2}^{\top} & 0 & 0 \\
\underline{\beta}_{2,4}^{\top} & \underline{\beta}_{3,4}^{\top} & \underline{\beta}_{4,4}^{\top} \\
\underline{\beta}_{3,4}^{\top} & \underline{\beta}_{4,4}^{\top} & 0 \\
\underline{\beta}_{4,4}^{\top} & 0 & 0 \\
\underline{\beta}_{3,5}^{\top} & \underline{\beta}_{4,5}^{\top} & \underline{\beta}_{5,5}^{\top} \\
\underline{\beta}_{4,5}^{\top} & \underline{\beta}_{5,5}^{\top} & 0 \\
\underline{\beta}_{5,5}^{\top} & 0 & 0
\end{bmatrix} = \underline{0}
$$

4. Similarly for $j = 4$ we have :

$$
\begin{array}{lll}
\underline{y}_2^{\top 1} = \begin{bmatrix} 0 & 0 \end{bmatrix}, & \underline{y}_4^{\top 4} = \begin{bmatrix} 0 & 0 & 0 & c_4^1 \end{bmatrix}, & \underline{y}_5^{\top 4} = \begin{bmatrix} 0 & 0 & 0 & 0 & c_5^1 \end{bmatrix} \\
\underline{y}_2^{\top 2} = \begin{bmatrix} 0 & 0 \end{bmatrix}, & \underline{y}_4^{\top 4} = \begin{bmatrix} 0 & 0 & -c_4^1 & c_4^2 \end{bmatrix}, & \underline{y}_5^{\top 4} = \begin{bmatrix} 0 & 0 & 0 & -c_5^1 & c_5^2 \end{bmatrix} \\
\underline{y}_2^{\top 3} = \begin{bmatrix} 0 & c_2^3 \end{bmatrix}, & \underline{y}_4^{\top 3} = \begin{bmatrix} 0 & c_4^1 & -c_4^2 & c_4^3 \end{bmatrix}, & \underline{y}_5^{\top 4} = \begin{bmatrix} 0 & 0 & c_5^1 & -c_5^2 & c_5^3 \end{bmatrix} \\
\underline{y}_2^{\top 4} = \begin{bmatrix} -c_2^3 & c_2^4 \end{bmatrix}, & \underline{y}_4^{\top 4} = \begin{bmatrix} -c_4^1 & c_4^2 & -c_4^3 & c_4^4 \end{bmatrix}, & \underline{y}_5^{\top 4} = \begin{bmatrix} 0 & -c_5^1 & c_5^2 & -c_5^3 & c_5^4 \end{bmatrix}
\end{array}
$$

and if we set

$$
\underline{c}_4^{\top} \triangleq \begin{bmatrix} -c_2^3 & c_2^4 & -c_4^1 & c_4^2 & -c_4^3 & c_4^4 & -c_5^1 & c_5^2 & -c_5^3 & c_5^4 \end{bmatrix}
$$

then we have:

$$\underline{c}_4^{\top}\begin{bmatrix} \underline{\beta}_{1,2}^{\top} & \underline{\beta}_{2,2}^{\top} & 0 & 0 \\ \underline{\beta}_{2,2}^{\top} & 0 & 0 & 0 \\ \underline{\beta}_{1,4}^{\top} & \underline{\beta}_{2,4}^{\top} & \underline{\beta}_{3,4}^{\top} & \underline{\beta}_{4,4}^{\top} \\ \underline{\beta}_{2,4}^{\top} & \underline{\beta}_{3,4}^{\top} & \underline{\beta}_{4,4}^{\top} & 0 \\ \underline{\beta}_{3,4}^{\top} & \underline{\beta}_{4,4}^{\top} & 0 & 0 \\ \underline{\beta}_{4,4}^{\top} & 0 & 0 & 0 \\ \underline{\beta}_{2,5}^{\top} & \underline{\beta}_{3,5}^{\top} & \underline{\beta}_{4,5}^{\top} & \underline{\beta}_{5,5}^{\top} \\ \underline{\beta}_{3,5}^{\top} & \underline{\beta}_{4,5}^{\top} & \underline{\beta}_{5,5}^{\top} & 0 \\ \underline{\beta}_{4,5}^{\top} & \underline{\beta}_{5,5}^{\top} & 0 & 0 \\ \underline{\beta}_{5,5}^{\top} & 0 & 0 & 0 \end{bmatrix} = \underline{0}$$

5. Similarly for $j = 5$ and $j = 6$ we have

$$\underline{c}_5^{\top}\begin{bmatrix} \underline{\beta}_{1,2}^{\top} & \underline{\beta}_{2,2}^{\top} & 0 & 0 & 0 \\ \underline{\beta}_{2,2}^{\top} & 0 & 0 & 0 & 0 \\ \underline{\beta}_{1,4}^{\top} & \underline{\beta}_{2,4}^{\top} & \underline{\beta}_{3,4}^{\top} & \underline{\beta}_{4,4}^{\top} & 0 \\ \underline{\beta}_{2,4}^{\top} & \underline{\beta}_{3,4}^{\top} & \underline{\beta}_{4,4}^{\top} & 0 & 0 \\ \underline{\beta}_{3,4}^{\top} & \underline{\beta}_{4,4}^{\top} & 0 & 0 & 0 \\ \underline{\beta}_{4,4}^{\top} & 0 & 0 & 0 & 0 \\ \underline{\beta}_{1,5}^{\top} & \underline{\beta}_{2,5}^{\top} & \underline{\beta}_{3,5}^{\top} & \underline{\beta}_{4,5}^{\top} & \underline{\beta}_{5,5}^{\top} \\ \underline{\beta}_{2,5}^{\top} & \underline{\beta}_{3,5}^{\top} & \underline{\beta}_{4,5}^{\top} & \underline{\beta}_{5,5}^{\top} & 0 \\ \underline{\beta}_{3,5}^{\top} & \underline{\beta}_{4,5}^{\top} & \underline{\beta}_{5,5}^{\top} & 0 & 0 \\ \underline{\beta}_{4,5}^{\top} & \underline{\beta}_{5,5}^{\top} & 0 & 0 & 0 \\ \underline{\beta}_{5,5}^{\top} & 0 & 0 & 0 & 0 \end{bmatrix} = \underline{0} \quad .$$

and

$$\underline{c}_6^\top \begin{bmatrix} \underline{\beta}_{1,2}^\top & \underline{\beta}_{2,2}^\top & 0 & 0 & 0 \\ \underline{\beta}_{2,2}^\top & 0 & 0 & 0 & 0 \\ \underline{\beta}_{1,4}^\top & \underline{\beta}_{2,4}^\top & \underline{\beta}_{3,4}^\top & \underline{\beta}_{4,4}^\top & 0 \\ \underline{\beta}_{2,4}^\top & \underline{\beta}_{3,4}^\top & \underline{\beta}_{4,4}^\top & 0 & 0 \\ \underline{\beta}_{3,4}^\top & \underline{\beta}_{4,4}^\top & 0 & 0 & 0 \\ \underline{\beta}_{4,4}^\top & 0 & 0 & 0 & 0 \\ \underline{\beta}_{1,5}^\top & \underline{\beta}_{2,5}^\top & \underline{\beta}_{3,5}^\top & \underline{\beta}_{4,5}^\top & \underline{\beta}_{5,5}^\top \\ \underline{\beta}_{2,5}^\top & \underline{\beta}_{3,5}^\top & \underline{\beta}_{4,5}^\top & \underline{\beta}_{5,5}^\top & 0 \\ \underline{\beta}_{3,5}^\top & \underline{\beta}_{4,5}^\top & \underline{\beta}_{5,5}^\top & 0 & 0 \\ \underline{\beta}_{4,5}^\top & \underline{\beta}_{5,5}^\top & 0 & 0 & 0 \\ \underline{\beta}_{5,5}^\top & 0 & 0 & 0 & 0 \end{bmatrix} = \underline{0}$$

So for the general example we have that the dimension of the left Kernel of $\tilde{T}^j_{\lambda_i}$ is defined by the dimensions of the left Kernel of the following Toeplitz type matrices :

$$Q^1_{\lambda_i} = \begin{bmatrix} \underline{\beta}_{2,2}^\top \\ \underline{\beta}_{4,4}^\top \\ \underline{\beta}_{5,5}^\top \end{bmatrix} \quad Q^2_{\lambda_i} = \begin{bmatrix} \underline{\beta}_{1,2}^\top & \underline{\beta}_{2,2}^\top \\ \underline{\beta}_{2,2}^\top & 0 \\ \underline{\beta}_{3,4}^\top & \underline{\beta}_{4,4}^\top \\ \underline{\beta}_{4,4}^\top & 0 \\ \underline{\beta}_{4,5}^\top & \underline{\beta}_{5,5}^\top \\ \underline{\beta}_{5,5}^\top & 0 \end{bmatrix} \quad Q^3_{\lambda_i} = \begin{bmatrix} \underline{\beta}_{1,2}^\top & \underline{\beta}_{2,2}^\top & 0 \\ \underline{\beta}_{2,2}^\top & 0 & 0 \\ \underline{\beta}_{2,4}^\top & \underline{\beta}_{3,4}^\top & \underline{\beta}_{4,4}^\top \\ \underline{\beta}_{3,4}^\top & \underline{\beta}_{4,4}^\top & 0 \\ \underline{\beta}_{4,4}^\top & 0 & 0 \\ \underline{\beta}_{3,5}^\top & \underline{\beta}_{4,5}^\top & \underline{\beta}_{5,5}^\top \\ \underline{\beta}_{4,5}^\top & \underline{\beta}_{5,5}^\top & 0 \\ \underline{\beta}_{5,5}^\top & 0 & 0 \end{bmatrix}$$

$$
Q_{\lambda_i}^4 = \begin{bmatrix}
\underline{\beta}_{1,2}^\top & \underline{\beta}_{2,2}^\top & 0 & 0 \\
\underline{\beta}_{2,2}^\top & 0 & 0 & 0 \\
\underline{\beta}_{1,4}^\top & \underline{\beta}_{2,4}^\top & \underline{\beta}_{3,4}^\top & \underline{\beta}_{4,4}^\top \\
\underline{\beta}_{2,4}^\top & \underline{\beta}_{3,4}^\top & \underline{\beta}_{4,4}^\top & 0 \\
\underline{\beta}_{3,4}^\top & \underline{\beta}_{4,4}^\top & 0 & 0 \\
\underline{\beta}_{4,4}^\top & 0 & 0 & 0 \\
\underline{\beta}_{2,5}^\top & \underline{\beta}_{3,5}^\top & \underline{\beta}_{4,5}^\top & \underline{\beta}_{5,5}^\top \\
\underline{\beta}_{3,5}^\top & \underline{\beta}_{4,5}^\top & \underline{\beta}_{5,5}^\top & 0 \\
\underline{\beta}_{4,5}^\top & \underline{\beta}_{5,5}^\top & 0 & 0 \\
\underline{\beta}_{5,5}^\top & 0 & 0 & 0
\end{bmatrix}
\quad
Q_{\lambda_i}^5 = \begin{bmatrix}
\underline{\beta}_{1,2}^\top & \underline{\beta}_{2,2}^\top & 0 & 0 & 0 \\
\underline{\beta}_{2,2}^\top & 0 & 0 & 0 & 0 \\
\underline{\beta}_{1,4}^\top & \underline{\beta}_{2,4}^\top & \underline{\beta}_{3,4}^\top & \underline{\beta}_{4,4}^\top & 0 \\
\underline{\beta}_{2,4}^\top & \underline{\beta}_{3,4}^\top & \underline{\beta}_{4,4}^\top & 0 & 0 \\
\underline{\beta}_{3,4}^\top & \underline{\beta}_{4,4}^\top & 0 & 0 & 0 \\
\underline{\beta}_{4,4}^\top & 0 & 0 & 0 & 0 \\
\underline{\beta}_{1,5}^\top & \underline{\beta}_{2,5}^\top & \underline{\beta}_{3,5}^\top & \underline{\beta}_{4,5}^\top & \underline{\beta}_{5,5}^\top \\
\underline{\beta}_{2,5}^\top & \underline{\beta}_{3,5}^\top & \underline{\beta}_{4,5}^\top & \underline{\beta}_{5,5}^\top & 0 \\
\underline{\beta}_{3,5}^\top & \underline{\beta}_{4,5}^\top & \underline{\beta}_{5,5}^\top & 0 & 0 \\
\underline{\beta}_{4,5}^\top & \underline{\beta}_{5,5}^\top & 0 & 0 & 0 \\
\underline{\beta}_{5,5}^\top & 0 & 0 & 0 & 0
\end{bmatrix} \tag{7.10}
$$

whereas for $j > 5$ we have that $Q_{\lambda_i}^j = Q_{\lambda_i}^5$. $\square$

The above sequence of matrices are defined from the spectral decomposition of the system description and are of simpler nature than the original Toeplitz matrices. Their significance is described below.

7.2.4 Input Spectral TOEPLITZ Matrices

Assume for the system $S(A, B)$ the corresponding partition of the Jordan description $\mathcal{B}$ of B is given as in (3.67). If the Segré Characteristic of A at $s = \lambda_i$, $\lambda_i \in \Phi(A)$ is given by (3.22) and the corresponding partition of $\mathcal{B}_i$ by (3.69) and if assume that the typical spectral block $\mathcal{B}_{ik}$ is described by (3.71). We may define:

Definition 51 *The j-th input spectral Toeplitz matrix is defined in a row block partition form as shown below,*

$$Q^j_{\lambda_i} \triangleq \begin{bmatrix} Q^j_{\tau_1} \\ \cdots \\ Q^j_{\tau_{ik}} \\ \cdots \\ Q^j_{\tau_{i\nu_i}} \end{bmatrix} \tag{7.11}$$

where.

1. *For $\forall j \leq \tau_{i\nu_i}$:*

   (a) *if $j \leq \tau_{ik}$,*

$$Q^j_{\tau_{ik}} \triangleq \begin{bmatrix} \underline{\beta}^{\top}_{\tau_{ik}+1-j,\tau_{ik}} & \underline{\beta}^{\top}_{\tau_{ik}-j,\tau_{ik}} & \cdots & \underline{\beta}^{\top}_{\tau_{ik}-1,\tau_{ik}} & \underline{\beta}^{\top}_{\tau_{ik},\tau_{ik}} \\ \underline{\beta}^{\top}_{\tau_{ik}-j,\tau_{ik}} & \underline{\beta}^{\top}_{\tau_{ik}-j-1,\tau_{ik}} & \cdots & \underline{\beta}^{\top}_{\tau_{ik},\tau_{ik}} & 0 \\ . & . & \cdots & . & . \\ \underline{\beta}^{\top}_{\tau_{ik}-1,\tau_{ik}} & \underline{\beta}^{\top}_{\tau_{ik},\tau_{ik}} & \cdots & 0 & 0 \\ \underline{\beta}^{\top}_{\tau_{ik},\tau_{ik}} & 0 & \cdots & 0 & 0 \end{bmatrix} \in \mathbb{C}^{j \times lj} \tag{7.12}$$

   (b) *if $j > \tau_{ik}$,*

$$Q^j_{\tau_{ik}} \triangleq \begin{bmatrix} \underline{\beta}^{\top}_{1,\tau_{ik}} & \underline{\beta}^{\top}_{2,\tau_{ik}} & \cdots & \underline{\beta}^{\top}_{\tau_{ik}-1,\tau_{ik}} & \underline{\beta}^{\top}_{\tau_{ik},\tau_{ik}} & 0 & \cdots & 0 \\ \underline{\beta}^{\top}_{2,\tau_{ik}} & \underline{\beta}^{\top}_{3,\tau_{ik}} & \cdots & \underline{\beta}^{\top}_{\tau_{ik},\tau_{ik}} & 0 & 0 & \cdots & 0 \\ . & . & \cdots & . & . & . & \cdots & . \\ \underline{\beta}^{\top}_{\tau_{ik}-1,\tau_{ik}} & \underline{\beta}^{\top}_{\tau_{ik},\tau_{ik}} & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \underline{\beta}^{\top}_{\tau_{ik},\tau_{ik}} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{C}^{\tau_{ik} \times lj} \tag{7.13}$$

2. *For* $\forall k > \tau_{i\nu_i} : Q^j_{\tau_{ik}} = Q^{\tau_{i\nu_i}}_{\tau_{ik}}$ □

From the above definition, Lemma 11 and using induction following the steps of the rather general Example 9, the following *Proposition* is readily established.

Proposition 56 *For any set of indices* $\{\tau_{i\nu_i} \geq ... \geq \tau_{ik} \geq ... \geq \tau_{i1} > 0\}$ *the solution of the set of equations (7.3) (recurrent and Kernel equations) is determined by the vectors* $\underline{c}_j^\top$, *where,*

$$\underline{c}_j^\top Q_i^j = \underline{0}$$

□

Remark 34 *The degrees of freedom of the set of equations (7.3 recurrent and Kernel equations) are determined by* $\dim \mathcal{N}_1 \left\{ Q_i^j \right\}$. *Furthermore, for all* $j > \tau_{i\nu_i} : Q^j_{\tau_{ik}} = Q^{\tau_{i\nu_i}}_{\tau_{ik}}$ *there are no more degrees of freedom to the solution of equations (7.3).* □

Proposition 57 *Consider the system* $S(A, B)$ *with* $\wp_{\lambda_i}(A) = \{\tau_{i\nu_i} \geq ... \geq \tau_{ik} \geq ... \geq \tau_{i1} > 0\}$ *and let* $S(J, \mathcal{B})$ *be the corresponding Jordan normal description. If* $T^j_{\lambda_i}$ *is the* j-th, λ_i*-characteristic Toeplitz matrix of* $S(J, \mathcal{B})$ *and* $Q^j_{\lambda_i}$ *is the* j-th *input spectral matrix of the system, then,*

$$\mathcal{N}_1 \left\{ T^j_{\lambda_i} \right\} = \mathcal{N}_1 \left\{ Q^j_{\lambda_i} \right\}$$

Proof. By *Proposition 53* it follows that $\tilde{T}^j_{\lambda_i}$ determines the left null-space properties of $T^j_{\lambda_i}$ ($\mathcal{N}_1 \left\{ T^j_{\lambda_i} \right\} = \mathcal{N}_1 \left\{ \tilde{T}^j_{\lambda_i} \right\}$), whereas by *Proposition 54* the study of the $\mathcal{N}_1 \left\{ \tilde{T}^j_{\lambda_i} \right\}$ is further reduced to the study of $\mathcal{N}_1 \left\{ \tilde{T}^j_{\lambda_i} \right\}$. Finally by *Proposition 56* follows that the solution of equations (7.3) is given by the $\mathcal{N}_1\{Q^j_{\lambda_i}\}$ and this completes the proof. □

Definition 52 *Using the* $Q^j_{\lambda_i}$, $j = 1, 2, ...$ *input spectral matrices we define the* λ_i*-input spectral sequence as in (3.31) i.e.*

$$J^1_{\lambda_i} \triangleq \left\{ n_j^{\lambda_i} : n_0^{\lambda_i} = 0,\ n_j^{\lambda_i} = \dim \mathcal{N}_l(Q^j_{\lambda_i}); j \geq 1 \right\}$$

□

Theorem 48 *The sequence $J^1_{\lambda_i}$ is piecewise arithmetic progression satisfying, the condition*

$$n_j^{\lambda_i} \geq \frac{n_{j-1}^{\lambda_i} + n_{j+1}^{\lambda_i}}{2}, \ j = 1, 2, ...$$

In particular we have that strict inequality holds if $j = \mu$ is the degree of an input decoupling zero of the $S(A, B)$ pair. In this case the multiplicity of the degree $j = \mu$ is,

$$\sigma = 2n_j^{\lambda_i} - n_{j-1}^{\lambda_i} - n_{j+1}^{\lambda_i}$$

Proof. The set of i.d.z. of the system $S(A, B)$ is defined as the set of roots of e.d. of the input state pencil : $s[I, 0] - [J, -\mathcal{B}] \in \mathcal{L}^{\mathrm{lr}}_{n,n+l}$. From Theorem 12, Definition 16 and Proposition 10 we conclude that the Weyr characteristic of $([J, -\mathcal{B}], [I, 0])$ determines the structure of e.d. of the input state pencil. Then from the above Proposition 57 the Theorem is proved. □

7.2.5 Calculation of the i.d.z. at λ_i from the Set of r.c.i.

Some further results on the characterization of i.d.z. are given below.

Remark 35 *From the Definition 51 it is directly concluded that :*

1. *The matrix $Q^1_{\lambda_i}$ coincides with the i-th spectrum controllability matrix $\mathcal{B}_i^S$ defined by (3.73).*

$$Q^1_{\lambda_i} \equiv \mathcal{B}_i^S$$

2. *The matrix $Q_i^{\tau_{i\nu_i}}$ coincides with the matrix Q_{H_i} defined by (6.6)*

$$Q_i^{\tau_{i\nu_i}} \equiv Q_{H_i}$$

□

Consider the set of r.c.i. $\Theta(A, B)_{\lambda_i}$ given by (6.7). Let the set of $\Theta(A, B)_{\lambda_i}$ be rearranged such that the index $\acute{\theta}_{ik}$, $k = 1, 2, ..., \nu_i$ be the r.c.i. which corresponds to the block $\mathcal{B}_{ik}$ of $\mathcal{B}$. Then this is denoted as,

$$\Theta'(A, B)_{\lambda_i} = \{\acute{\theta}_{i1}, \acute{\theta}_{i2}, ..., \acute{\theta}_{ik}, ..., \acute{\theta}_{i\nu_i}\}, \ \acute{\theta}_{ik} \geq 0, \ k = 1, 2, ..., \nu_i \tag{7.14}$$

Consider now the set of differences

$$\Sigma'(A,B)_{\lambda_i} \triangleq \left\{ q'_{i_1}, q'_{i_2}, ..., q'_{ik}, ..., q'_{i\nu_i} \right\}$$

between the corresponding elements of the two sets, $\Theta'(A,B)_{\lambda_i}$ and the set of the Segré characteristic of A at λ_i, $\wp_{\lambda_i}(A)$:

$$\tau_{i1} - \acute{\theta}_{i1} \triangleq q'_{i1},\ \tau_{i2} - \acute{\theta}_{i2} \triangleq q'_{i2}, ...,\ \tau_{ik} - \acute{\theta}_{ik} \triangleq q'_{ik},\ \tau_{i\nu_i} - \acute{\theta}_{i\nu_i} \triangleq q'_{i\nu_i} \tag{7.15}$$

and let $\Sigma(A,B)_{\lambda_i}$ be the set of the non zero values of the above differences, described by the ordered set of integers,

$$\Sigma(A.B)_{\lambda_i} \triangleq \left\{ (\mu_i^j, \sigma_i^j); \mu_i^{s_i} \geq ... \geq \mu_i^2 \geq \mu_i^1 > 0 \right\}; i = 1,2,...,f \tag{7.16}$$

Where σ_i^j is the multiplicity of μ_i^j $(j = 1,2...,s_i)$. Then we have the following result :

Theorem 49 *The degrees of the input decoupling zeros of a system $S(A,B)$ at $s = \lambda_i$ are defined by the above defined set of indices $\Sigma(A.B)_{\lambda_i}$ (or the $\Sigma'(A,B)_{\lambda_i}$).*

Proof. Let rank$Q^1_{\lambda_i}$ $(=$ rank$\mathcal{B}_i^S) = \rho$ .Then, the number of linearly dependent rows of $Q^1_{\lambda_i}$ is $\nu_i - \rho$ and from the definition (3.31) of $J^1_{\lambda_i}$ we have $n_1^{\lambda_i} - n_0^{\lambda_i} = \nu_i - \rho$.

1. For $j = 1,2,... \leq \mu_i^1$: The total number of linearly dependent rows of the matrices $Q^1_{\lambda_i}$, $Q^2_{\lambda_i}$, ..., $Q^{\mu_i^1}_{\lambda_i}$, is correspondingly $\nu_i - \rho$, $2(\nu_i - \rho)$, ..., $\mu_i^1(\nu_i - \rho)$. Then for the successive differences of $J^1_{\lambda_i}$ we have:

$$n_j^{\lambda_i} - n_{j-1}^{\lambda_i} = \nu_i - \rho,\ j = 1,2,...,\mu_i^s$$

2. For $\mu_i^1 < j \leq \mu_i^2$: In the matrix $Q^{\mu_i^1+1}_{\lambda_i}$ there are σ_i^1 new linearly independent rows of $\mathcal{B}_i$. Then,

$$\text{rank}Q^{\mu_i^1+1}_{\lambda_i} = \left(\mu_i^1 + 1\right)\rho + \sigma_i^1$$

and the total number of linearly dependent rows of $Q_{\lambda_i}^{\mu_i^1+1}$, is

$$\left(\mu_i^1+1\right)\left(\nu_i-\rho\right)-\sigma_i^1$$

Thus, the corresponding total number of linearly dependent rows of $Q_{\lambda_i}^{\mu_i^1+2}, ..., Q_{\lambda_i}^{\mu_i^2}$ is

$$\left(\mu_i^1+2\right)\left(\nu_i-\rho\right)-2\sigma_i^1,\ \left(\mu_i^1+3\right)\left(\nu_i-\rho\right)-3\sigma_i^1, ...,\ \mu_i^2\left(\nu_i-\rho\right)-\mu_i^2\sigma_i^1$$

and the successive differences of $J_{\lambda_i}^1$ are :

$$n_j^{\lambda_i}-n_{j-1}^{\lambda_i}=\nu_i-\rho-\sigma_i^1$$

From the above we conclude that for $j=\mu_i^1$, it is

$$n_{\mu_i^1}^{\lambda_i}>\frac{n_{\mu_i^1-1}^{\lambda_i}+n_{\mu_i^1+1}^{\lambda_i}}{2}\Leftrightarrow$$

$$\Leftrightarrow \mu_i^1\left(\nu_i-\rho\right)>\frac{\left(\mu_i^1-1\right)\left(\nu_i-\rho\right)+\left(\mu_i^1+1\right)\left(\nu_i-\rho\right)-\sigma_i^1}{2}\Leftrightarrow$$

$$\Leftrightarrow 0>-\sigma_i^1/2$$

and μ_i^1 is a degree of i.d.z. at $s=\lambda_i$ with multiplicity,

$$2n_{\mu_i^1}^{\lambda_i}-\left(n_{\mu_i^1-1}^{\lambda_i}+n_{\mu_i^1+1}^{\lambda_i}\right)=\sigma_i^1$$

The above result can be extended by induction for $j=\mu_i^2, ..., \mu_i^s$. □

Proposition 58 *The sum of the degrees of i.d.z. at $s=\lambda_i$ is given as,*

$$q_{i_1}'+q_{i_2}'+...+q_{ik}'+...+q_{i\nu_i}'=\pi_i-r_i$$

where π_i is the algebraic multiplicity of λ_i and r_i is the dimension of the controllable subspace $\mathcal{R}_i$.

Proof. From the above relation (7.15) and Theorem 35 the relation is directly deduced. □

Example 10 *Let now the uncontrollable continuous system given in Example 7. We proceed to the calculation of the elements of the set* $\Sigma'(A,B)_{\lambda_1}$ *by subtraction of the elements of* $\Theta(A,B)_{\lambda_1}$ *from the corresponding of* $\wp_{\lambda_1}(A)$. *Then the i.d.z. are determined directly:*

(a) mode λ_1:

$\wp_{\lambda_1}(A)$	$\Theta(A,B)_{\lambda_1}$	$\Sigma'(A,B)_{\lambda_1}$	idz
6	4	2	$(s-\lambda_1)^2$
3	3	0	1
1	1	0	1

(b) mode λ_2:

$\wp_{\lambda_2}(A)$	$\Theta(A,B)_{\lambda_2}$	$\Sigma'(A,B)_{\lambda_2}$	idz
6	4	2	$(s-\lambda_2)^2$
3	3	0	1
1	1	0	1

(c) mode λ_3:

$\wp_{\lambda_3}(A)$	$\Theta(A,B)_{\lambda_3}$	$\Sigma'(A,B)_{\lambda_3}$	idz
3	2	1	$(s-\lambda_3)$
1	1	0	1

(d) mode λ_4:

$\wp_{\lambda_4}(A)$	$\Theta(A,B)_{\lambda_4}$	$\Sigma'(A,B)_{\lambda_4}$	idz
3	2	1	$(s-\lambda_4)$
1	1	0	1

or we have: $\Sigma(A.B)_{\lambda_1}=\{2\}$, $\Sigma(A,B)_{\lambda_2}=\{2\}$, $\Sigma(A,B)_{\lambda_3}=\{1\}$, $\Sigma(A,B)_{\lambda_4}=\{1\}$. □

7.3 Spectral Determination of the Structure of o.d.z. of a Continuous System $S(A, C)$.

The set of o.d.z. of a system $S(A, C)$ is defined in *Chapter 3* (*Definition 28*) as the set of roots of e.d. of the output state pencil. Let the output state pencil of the equivalent system in Jordan form $S(J, \Gamma)$, be as $\begin{bmatrix} sI - J \\ \Gamma \end{bmatrix} = s \begin{bmatrix} I \\ 0 \end{bmatrix} - \begin{bmatrix} J \\ -\Gamma \end{bmatrix} \in \mathcal{L}^{\Gamma\Gamma}_{n+m,n}$.

Consequently the structure of o.d.z. of the system $S(A, C)$ is determined equivalently by the root range of the output state pencil (Definition 15). As a first step in this direction the following sequence of right a-characteristic Toeplitz matrices as in (3.27) may be constructed :

$$T_b^1 = \begin{bmatrix} J - bI \\ \Gamma \end{bmatrix} \in \mathbb{C}^{(n+m)\times n},\ T_b^2 = \begin{bmatrix} J - bI & 0 \\ \Gamma & 0 \\ I & J - bI \\ 0 & \Gamma \end{bmatrix} \in \mathbb{C}^{2(n+m)\times 2n}$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$T_b^j = \begin{bmatrix} J - bI & 0 & \dots & 0 & 0 \\ \Gamma & 0 & \dots & 0 & 0 \\ I & J - bI & \dots & 0 & 0 \\ 0 & \Gamma & \dots & 0 & 0 \\ \cdot & \cdot & \dots & \cdot & \cdot \\ 0 & 0 & \dots & I & J - bI \\ 0 & 0 & \dots & 0 & \Gamma \end{bmatrix} \in \mathbb{C}^{j(n+m)\times jn}$$

The analysis that follows is similar to that of the previous case and the results mainly follow by duality.

7.3.1 Basic Properties of the Rank of the b-Characteristic TOEPLITZ Matrices

The characterization of candidate values for o.d.z. is defined by the following result:

Proposition 59 *For $\forall b \in \mathbb{C} : b \notin \Phi(A)$ the matrix T_b^j has full rank.*

Proof. Let,

$$T_b^j \begin{bmatrix} \underline{y}_1 \\ \underline{y}_2 \\ \cdots \\ \underline{y}_j \end{bmatrix} = \underline{0} \Leftrightarrow \begin{cases} \begin{bmatrix} J-bI \\ \Gamma \end{bmatrix} \underline{y}_1 = 0 \\ \begin{bmatrix} I \\ 0 \end{bmatrix} \underline{y}_1 = - \begin{bmatrix} J-bI \\ \Gamma \end{bmatrix} \underline{y}_2 \\ \dots\dots\dots\dots \\ \begin{bmatrix} I \\ 0 \end{bmatrix} \underline{y}_{j-1} = - \begin{bmatrix} J-bI \\ \Gamma \end{bmatrix} \underline{y}_j \end{cases}$$

If $b \notin \Phi(A) \Rightarrow \text{rank} \begin{bmatrix} J-bI \\ \Gamma \end{bmatrix} = n \Leftrightarrow \underline{y}_1 = \underline{0}$ and recursively $\underline{y}_2 = \underline{0}, \ \ldots, \underline{y}_j = \underline{0}$. □

Proposition 60 *Let $b = \lambda_i \in \Phi(A)$ and as in the case of i.d.z. express,* $T_{\lambda_i}^1 = \begin{bmatrix} H_i & 0 \\ 0 & T' \\ \Gamma_i & \Gamma' \end{bmatrix}$ *where $H_i = J(\lambda_i) - \lambda_i I \in \mathbb{R}^{\pi_i \times \pi_i}$ is nilpotent, $T' \in \mathbb{C}^{(n-\pi_i)\times(n-\pi_i)}$ is full rank, $\Gamma_i \in \mathbb{C}^{\pi_i \times l}$ is (as defined in 3.68) the matrix block of Γ corresponding to $J(\lambda_i)$. Then the nullity of the matrix $T_{\lambda_i}^j$ is defined by the nullity of the matrix $\check{T}_{\lambda_i}^j$, where,*

$$\check{T}_{\lambda_i}^j = \underbrace{\begin{bmatrix} H_i & 0 & \ldots & 0 & 0 \\ \Gamma_i & 0 & \ldots & 0 & 0 \\ I & H_i & \ldots & 0 & 0 \\ . & . & \ldots & . & . \\ 0 & 0 & \ldots & \Gamma_i & 0 \\ 0 & 0 & \ldots & I & H_i \\ 0 & 0 & \ldots & 0 & \Gamma_i \end{bmatrix}}_{j\text{-blocks}}$$

□

Remark 36 *We conclude that only the $\lambda_i \in \Phi(A)$ are candidate values for o.d.z.* □

Remark 37 *We can study the sequence of nullity and the corresponding tests, for determining the degree of o.d.z. by considering the case that corresponds to each one of the eigenvalues.* □

7.3.2 Right Nullity of the j-*th* Single Block Matrix $\tilde{T}^{j}_{\lambda_i}$

From the above we conclude that any $\lambda_i \in \Phi(A)$ is also a candidate o.d.z. The matrix Γ_i can also be partitioned according to $\wp_{\lambda_i}(A)$ as in (3.70) .

Definition 53 *The* t-th *reduced matrix of* $\Gamma_{ik}^{(t}$ *is defined as the matrix derived from* Γ_{ik} *(where* Γ_{ik} *is defined in (3.72)) as it is indicated below:*

$$\Gamma_{ik}^{(t} = \left[\underline{\gamma}_{ik_1}, ..., \underline{\gamma}_{ik_t}, ..., 0, ..., 0\right] \in \mathbb{C}^{m \times \tau_{ik}},\ t = 1, 2, ..., \tau_{ik} \text{ and } \Gamma_{ik}^{(t} \triangleq \Gamma_{ik} \text{ for } \forall t = 0, -1, -2, ... \square$$

For any $p \geq 1,\ p \in \mathbb{Z}$ we define the $\ast$ operation on $\Gamma_{ik} \in \mathbb{C}^{\tau_{ik} \times l}$ by :

$$\Gamma_{ik} \ast p \triangleq \Gamma_{ik}^{\ast p} \triangleq \Gamma_{ik}^{(\tau_{ik}+1-p},\ p = 1, 2, ...$$

Let Γ_i be partitioned as in (3.70), then we define the $\ast$ operation on Γ_i by some $p \in \mathbb{Z}$ as:

$$p \ast \Gamma_i \triangleq \Gamma_i^{\ast p} \triangleq [p \ast \Gamma_{i1}, ..., p \ast \Gamma_{ik}, ..., p \ast \Gamma_{i\nu_i}] = \left[\Gamma_{i1}^{(\tau_{i1}+1-p}, ..., \Gamma_{ik}^{(\tau_{ik}+1-p}, ..., \Gamma_{i\nu_i}^{(\tau_{i\nu_i}+1-p}\right]$$

7.3.3 Normal Description of the j-*th* Right Toeplitz Matrix $\tilde{T}^{j}_{\lambda_i}$

The properties of the corresponding Toeplitz matrices are defined below.

Proposition 61 *The above defined* j-th *right Toeplitz matrix* $\tilde{T}^{j}_{\lambda_i}$ *is equivalent over* $\mathbb{C}$ *by elementary column operations to the following form :*

$$\tilde{T}^{j}_{\lambda_i} = \underbrace{\begin{bmatrix} H_i & 0 & \dots & 0 & 0 \\ \Gamma_i^{\ast 1} & 0 & \dots & 0 & 0 \\ I_i^{\ast 1} & H_i & \dots & 0 & 0 \\ . & . & \dots & . & . \\ 0 & 0 & \dots & \Gamma_i^{\ast (j-1)} & 0 \\ 0 & 0 & \dots & I_i^{\ast (j-1)} & H_i \\ 0 & 0 & \dots & 0 & \Gamma_i^{\ast j} \end{bmatrix}}_{j-blocks}$$

$\square$

Remark 38 *The form* $\tilde{T}^j_{\lambda_i}$ *is row equivalent to* $\hat{T}^j_{\lambda_i}$ *and* $\tilde{T}^j_{\lambda_i}$ *will be called the normal description of the* j-th *right normal Toeplitz matrix. Clearly,* $\mathrm{rank}\tilde{T}^j_{\lambda_i} = \mathrm{rank}\hat{T}^j_{\lambda_i}$. *The right null-space of* $\hat{T}^j_{\lambda_i}$ *may be studied by using* $\tilde{T}^j_{\lambda_i}$ *since the two are row space equivalent.* □

Proposition 62 *Let* $\underline{y} \in \mathbb{C}^{jn}$ *and be partitioned as,* $\underline{y} = \begin{bmatrix} \underline{y}_1 & \underline{y}_2 & \cdots & \underline{y}_{j-1} & \underline{y}_j \end{bmatrix}^\top$. *Then* $\underline{y} \in \mathcal{N}_r\{\tilde{T}^j_{\lambda_i}\}$, *where* $\tilde{T}^j_{\lambda_i} \in \mathbb{C}^{j(n+m)\times jn}$, *if and only if the following conditions are satisfied,*

$$\begin{cases} H_i\underline{y}_1 = \underline{0} \\ H_i\underline{y}_2 = -I_i^{*1}\underline{y}_1 \\ H_i\underline{y}_3 = -I_i^{*2}\underline{y}_2 \\ \quad\cdots \\ H_i\underline{y}_j = -I_i^{*(j-1)}\underline{y}_{j-1} \end{cases} \quad \text{and} \quad \begin{cases} B_i^{*1}\underline{y}_1 = 0 \\ B_i^{*2}\underline{y}_2 = 0 \\ B_i^{*3}\underline{y}_3 = 0 \\ \quad\cdots \\ B_i^{*j}\underline{y}_j = 0 \end{cases} \tag{7.17}$$

□

The set of equations (7.17) comprises from two subsets i.e. the equations of the first column are referred as the *right recurrent equations* of the set and the equations of the second column as the *right Kernel equations.* We consider first the recurrent equations.

Remark 39 *Let* $\underline{y}_i$ *be partitioned according to Segré characteristic defined in (3.22) as,*

$$\underline{y}_i = \begin{bmatrix} \underline{y}_{\tau_{i1}} & \cdots & \underline{y}_{\tau_{ik}} & \cdots & \underline{y}_{\tau_{i\nu_i}} \end{bmatrix}^\top, \; i = 1, 2, ..., f$$

and from the block diagonal structure of H_i *and* I_i^{*j} *we have that the set of the recurrent equations is equivalent to,*

$$\begin{cases} H_{ik}\underline{y}_{\tau_{ik}} &= 0 \\ H_{ik}\underline{y}_{\tau_{ik}} &= -I_{ik}^{*1}\underline{y}_{\tau_{ik}} \\ H_{ik}\underline{y}_{\tau_{ik}} &= -I_{ik}^{*2}\underline{y}_{\tau_{ik}} \\ \quad\cdots\cdots\cdots \\ H_{ik}\underline{y}_{\tau_{ik}} &= -I_{ik}^{*j-1}\underline{y}_{\tau_{ik}} \end{cases} \tag{7.18}$$

where τ_{ik} *takes values from the set of* $\wp_{\lambda_i}(A)$. *Equations (7.18) will be called the basic right recurrent equations.* □

Lemma 12 *For any $\tau_{ik} \geq 1$ the solution of the basic right recurrent equations (7.18) is given by :*

1. *for $j \leq \tau_{ik}$:*

$$\underline{y}^j_{\tau_{ik}} = \left[d^1_{\tau_{ik}},\ -d^2_{\tau_{ik}},\ ...,\ (-1)^{j-1}\, d^j_{\tau_{ik}},\ \underbrace{0,\ ...,\ 0}_{\tau_{ik}-j} \right]^{\top} \tag{7.19}$$

where $d^1_{\tau_{ik}}, d^2_{\tau_{ik}}, ..., d^j_{\tau_{ik}}$ arbitrary,

2. *for $j > \tau_{ik}$:*

$$\left. \begin{array}{l} \underline{y}^1_{\tau_{ik}} = \underline{y}^2_{\tau_{ik}} = ... = \underline{y}^{j-\tau_{ik}}_{\tau_{ik}} = 0 \\ \underline{y}^{j-\tau_{ik}+1}_{\tau_{ik}} = \left[d^{j-\tau_{ik}+1}_{\tau_{ik}}, 0, ...0 \right]^{\top} \\ \ldots\ldots\ldots\ldots \\ \underline{y}^{j-1}_{\tau_{ik}} = \left[d^{j-1}_{\tau_{ik}}, -d^{j-2}_{\tau_{ik}}, ..., (-1)^{\tau_{ik}-2}\, d^{j-\tau_{ik}+1}_{\tau_{ik}}, 0 \right]^{\top} \\ \underline{y}^{j}_{\tau_{ik}} = \left[d^{j}_{\tau_{ik}}, -d^{j-2}_{\tau_{ik}}, ..., (-1)^{\tau_{ik}-1}\, d^{j-\tau_{ik}+1}_{\tau_{ik}} \right]^{\top} \end{array} \right\} \tag{7.20}$$

where $d^{j-\tau_{ik}+1}_{\tau_{ik}}, ..., d^j_{\tau_{ik}}$ arbitrary. □

7.3.4 Output Spectral TOEPLITZ Matrices

The above results lead to the definition of the output spectral matrix the properties of which define a characterization of the o.d.z. Assume for the system $S(A, C)$ the corresponding partition of the Jordan description Γ of C is given by (3.68). Let the Segré Characteristic of A at $s = \lambda_i$, $\lambda_i \in \Phi(A)$ be given by (3.22) and the corresponding partition of Γ_i as in (3.70) and that the typical spectral block Γ_{ik} is described as in (3.72).

Definition 54 *The j-th output spectral Toeplitz matrix is defined in a row block partitioned form as shown below.*

$$M^j_{\lambda_i} \triangleq \begin{bmatrix} M^j_{\tau_1} \\ \cdots \\ M^j_{\tau_{ik}} \\ \cdots \\ M^j_{\tau_{i\nu_i}} \end{bmatrix} \tag{7.21}$$

where.

1. *For* $\forall j \leq \tau_{i\nu_i}$:

   (a) *if* $j \leq \tau_{ik}$, *then*

$$M^j_{\tau_{ik}} \triangleq \begin{bmatrix} \underline{\gamma}_{1,\tau_{ik}} & \underline{\gamma}_{2,\tau_{ik}} & \cdots & \underline{\gamma}_{j-1,\tau_{ik}} & \underline{\gamma}_{j,\tau_{ik}} \\ 0 & \underline{\gamma}_{1,\tau_{ik}} & \cdots & \underline{\gamma}_{j-2,\tau_{ik}} & \underline{\gamma}_{j-1,\tau_{ik}} \\ . & . & \cdots & . & . \\ 0 & 0 & \cdots & \underline{\gamma}_{1,\tau_{ik}} & \underline{\gamma}_{2,\tau_{ik}} \\ 0 & 0 & \cdots & 0 & \underline{\gamma}_{1,\tau_{ik}} \end{bmatrix} \in \mathbb{C}^{j\times j}$$

   (b) *if* $j > \tau_{ik}$, *then*

$$M^j_{\tau_{ik}} \triangleq \begin{bmatrix} \underline{\gamma}_{1,\tau_{ik}} & \underline{\gamma}_{2,\tau_{ik}} & \cdots & \underline{\gamma}_{\tau_{ik}-1,\tau_{ik}} & \underline{\gamma}_{\tau_{ik},\tau_{ik}} \\ 0 & \underline{\gamma}_{1,\tau_{ik}} & \cdots & \underline{\gamma}_{\tau_{ik}-2,\tau_{ik}} & \underline{\gamma}_{\tau_{ik}-1,\tau_{ik}} \\ . & . & \cdots & . & . \\ 0 & 0 & \cdots & \underline{\gamma}_{1,\tau_{ik}} & \underline{\gamma}_{2,\tau_{ik}} \\ 0 & 0 & \cdots & 0 & \underline{\gamma}_{1,\tau_{ik}} \\ 0 & 0 & \cdots & 0 & 0 \\ . & . & \cdots & . & . \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \in \mathbb{C}^{j\times \tau_{ik}}$$

2. *For* $\forall k > \tau_{i\nu_i} : M^j_{\tau_{ik}} = M^{\tau_{i\nu_i}}_{\tau_{ik}}$. □

Proposition 63 *For any set of indices* $\{\tau_{i\nu_i} \geq ... \geq \tau_{ik} \geq ... \geq \tau_{i1} > 0\}$ *the solution of the set of equations (7.3) (right recurrent and Kernel equations) is determined by the vectors* $\underline{d}_j$. *where,*

$$M^j_i \underline{d}_j = \underline{0}$$

□

Remark 40 *The degrees of freedom of the set of equations (7.17 right recurrent and Kernel equations) are determined by* $\dim\mathcal{N}_1\left\{Q^j_i\right\}$. *Furthermore, as for all* $j > \tau_{i\nu_i} : Q^j_{\tau_{ik}} = Q^{\tau_{i\nu_i}}_{\tau_{ik}}$ *there are no more degrees of freedom to the solution of equations (7.17).* □

Corollary 8 *Consider the system $S(A,C)$ with $\wp_{\lambda_i}(A) = \{\tau_{iv_i} \geq ... \geq \tau_{ik} \geq ... \geq \tau_{i1} > 0\}$ and let $S(J,\Gamma)$ be the corresponding Jordan normal description. If $T^j_{\lambda_i}$ is the j-th, λ_i-characteristic Toeplitz matrix of $S(J,\Gamma)$ and $M^j_{\lambda_i}$ is the j-th output spectral matrix of the system, then,*

$$n_{\mathrm{r}}\left\{T^j_{\lambda_i}\right\} = n_{\mathrm{r}}\left\{M^j_{\lambda_i}\right\}$$

□

Definition 55 *Using the $M^j_{\lambda_i}$, $j = 1,2,...$ output spectral matrices we define the λ_i-input spectral sequence as in (3.30), i.e.*

$$J^{\mathrm{r}}_{\lambda_i}(G,F) \triangleq \left\{\eta^{\lambda_i}_k : \eta^{\lambda_i}_0 = 0,\ \eta^{\lambda_i}_k = \dim N^k_{\lambda_i};\ k \geq 1\right\} \tag{7.22}$$

□

Theorem 50 *The sequence $J^{\mathrm{r}}_{\lambda_i}$ is piecewise arithmetic progression satisfying, the condition*

$$n^{\lambda_i}_j \geq \frac{n^{\lambda_i}_{j-1} + n^{\lambda_i}_{j+1}}{2},\ j = 1,2,... \tag{7.23}$$

In particular we have that strict inequality holds if $j = \xi$ is the degree of an o.d.z. of the $S(A,C)$ pair. In this case the multiplicity of the degree $j = \xi$ is,

$$\varphi = 2n^{\lambda_i}_j - n^{\lambda_i}_{j-1} - n^{\lambda_i}_{j+1} \tag{7.24}$$

□

The above result provides the means for computing the degrees of o.d.z. with matrix based tests.

7.3.5 Calculation of the o.d.z. from the Set of c.o.i.

The computation of o.d.z. using alternative means provided by the c.o.i. is considered below and the analysis is similar to that given for i.d.z.

Remark 41 *From the Definition 54 it is directly concluded that:*

1. *The matrix* $M^1_{\lambda_i}$ *coincides with the i-th spectral observability matrix* Γ^F_i *defined by (3.74), i.e.*

$$M^1_{\lambda_i} \equiv \Gamma^F_i$$

2. *The matrix* $M_i^{\tau_{i\nu_i}}$ *coincides with matrix* M_{H_i} *defined by (6.15), i.e.*

$$M_i^{\tau_{i\nu_i}} \equiv M_{H_i}$$

□

Consider the set of c.o.i. $Z(A,C)_{\lambda_i}$ given by (6.16). Let the set of $Z'(A,C)_{\lambda_i}$ be rearranged such that the index $\acute{\zeta}_{ik}$, $k = 1,2,...,\nu_i$ be the c.o.i. which corresponds to the block Γ_{ik} of Γ . Then we have

$$Z'(A,C)_{\lambda_i} = \{\acute{\zeta}_{i1}, \acute{\zeta}_{i2}, ..., \acute{\zeta}_{ik}, ..., \zeta_{i\nu_i}\}, \acute{\zeta}_{ik} \geq 0, k = 1,2,...,\nu_i \tag{7.25}$$

Consider now the differences

$$\Psi'(A,C)_{\lambda_i} \triangleq \left\{p'_{i_1}, p'_{i_2}, ..., p'_{ik}, ..., p'_{i_{\nu_i}}\right\}$$

between the corresponding elements of the two sets, the $\wp_{\lambda_i}(A)$ and the $Z'(A,C)_{\lambda_i}$, that is

$$\tau_{i1} - \acute{\zeta}_{i1} \triangleq p'_{i1},\ \tau_{i2} - \acute{\zeta}_{i2} \triangleq p'_{i2}, ...,\ \tau_{ik} - \acute{\zeta}_{ik} \triangleq p'_{ik},\ \tau_{i\nu_i} - \acute{\zeta}_{i\nu_i} \triangleq p'_{i\nu_i} \tag{7.26}$$

and let $\Psi(A,C)_{\lambda_i}$ be the set of the non zero of the above differences, described by the ordered set of integers,

$$\Psi(A,C)_{\lambda_i} \triangleq \{(\xi_i^j, \varphi_i^j); \xi_i^{t_i} > ... > \xi_i^1 > 0\}; i = 1,2,...,f \tag{7.27}$$

Where φ_i^j denotes the multiplicity of ξ_i^j $(j = 1,2,...t_i)$. Then we have the following result :

Theorem 51 *The degrees of the o.d.z. of a system* $S(A,C)$ *at* $s = \lambda_i$ *are given by the set of indices* $\Psi(A,C)_{\lambda_i}$. □

Proposition 64 *The sum of the degrees of o.d.z. at $s = \lambda_i$ is given as,*

$$p'_{i_1} + p'_{i_2} + \ldots + p'_{ik} + \ldots + p'_{i\nu_i} = p_i \tag{7.28}$$

where p_i is the dimension of the unobservable subspace $\mathcal{P}_i$.

Proof. From the above relation (7.26) and theorem (38) the relation is directly concluded. □

7.4 Spectral Determination of the Structure of Input (Output) Decoupling Zeros of a discretised System

The set of i.d.z. of a discretised system $S(\hat{A},\hat{B})$ is defined as the set of roots of e.d. of the input state pencil $\left[zI-\hat{A},\hat{B}\right]$. Let the input state pencil of the equivalent discretised system in Jordan form be $S(\hat{J},\hat{\mathcal{B}})$: $\left[zI-\hat{J},\hat{\mathcal{B}}\right]=z[I,0]-[\hat{J},-\mathcal{B}]\in\mathcal{L}^{\text{lr}}_{n,n+l}$.

The set of o.d.z. of the $S(\hat{A},\hat{C})$ is also defined as the set of roots of e.d. of the output state pencil $\begin{bmatrix} zI-\hat{A} \\ \hat{C}\end{bmatrix}$ or of the equivalent discretised system in Jordan form $S(\hat{J},\hat{\Gamma})$:

$$\begin{bmatrix} sI-\hat{J} \\ \hat{\Gamma}\end{bmatrix}=s\begin{bmatrix} I \\ 0\end{bmatrix}-\begin{bmatrix} \hat{J} \\ -\hat{\Gamma}\end{bmatrix}\in\mathcal{L}^{\text{rr}}_{n+m,n}$$

Consequently the structure of i.d.z. (o.d.z.) of the discretised system $S(\hat{A},\hat{B},\hat{\Gamma})$ is determined equivalently by the root range of the discretised input (output) state pencil. Following exactly the same steps as in the case of the continuous system, it can been shown directly that the same conclusions, theorems, propositions, lemmas, remarks etc., proved in the previous sections for the determination of the i.d.z.(o.d.z.) structure of the continuous system are also valid for the structure of i.d.z.(o.d.z.) of the discretised system.

In this section, we have to investigate the mapping of the structure of i.d.z.(o.d.z.) of a continuous system $S(A,B,C)$ to the corresponding structure of i.d.z. (o.d.z.) of the discretised model $S(\hat{A},\hat{B},\hat{C})$ under the two types of sampling.

7.4.1 Regular Sampling

For the case of regular sampling we have the following result:

Theorem 52 *Under regular sampling, the structure of the i.d.z.(o.d.z.) of the discretised model $S(\hat{A},\hat{B},\hat{C})$ remains the same as the corresponding structure of the continuous system $S(A,B,C)$. Then it is,*

(a) To each one of the i.d.z. $(s-\lambda_i)^{\mu_i^j}$ $(j=1,2,...,s_i)$ of the continuous system corresponds the i.d.z. $\left(z-\hat{\lambda}_i\right)^{\mu_i^j}$ of the discretised model.

(b) To each one of the o.d.z. $(z-\lambda_i)^{\xi_i^j}$ $(j=1,2,...,t_i)$*of the continuous system corresponds the o.d.z.* $\left(z-\hat{\lambda}_i\right)^{\xi_i^j}$ *of the discretised model.*

Proof. : It is already known from Theorem 30 that under regular sampling, the Segré Characteristic of A at λ_i is equal to the Segré characteristic of $\hat{A}$ at $\hat{\lambda}_i$: $\wp_{\lambda_i}(A) \equiv \wp_{\hat{\lambda}_i}\left(\hat{A}\right)$

(a) It have been shown (Proposition 53, Remark 30) that candidate i.d.z. of $S(\hat{A},\hat{B})$ are exist at $z=\hat{\lambda}_i \in \Phi(\hat{A})$ and from Theorem 41 it follows that the set of *i-th* spectrum r.c.i. of the discretised model is equal to the set of *i-th* spectrum r.c.i. of the continuous system: $\Theta(A,B)_{\lambda_i} \equiv \Theta(\hat{A},\hat{B})_{\hat{\lambda}_i}$. Then the set of differences between the corresponding elements of the sets $\wp_{\lambda_i}(A)\,,\Theta(A,B)_{\lambda_i}$ and $\wp_{\hat{\lambda}_i}\left(\hat{A}\right),\Theta(\hat{A},\hat{B})_{\hat{\lambda}_i}$, is determined as in (7.15) and remain also the same. From (7.16) we conclude that for regular sampling it is, $\Sigma(A,B)_{\lambda_i} \equiv \Sigma(\hat{A},\hat{B})_{\hat{\lambda}_i}$.

(b) Also candidate o.d.z. of $S(\hat{A},\hat{C})$ exist at $z=\hat{\lambda}_i \in \Phi(\hat{A})$ and from Proposition 46 it is known that the set of *i-th* spectrum c.o.i. of the discretised model is equal to the set of the *i-th* spectrum c.o.i. of the continuous system, $Z(\hat{A},\hat{C})_{\hat{\lambda}_i} = Z(A,C)_{\lambda_i}$. Then the set of differences between the corresponding elements of the sets $\wp_{\lambda_i}(A)\,,Z(A,C)_{\lambda_i}$ and $\wp_{\hat{\lambda}_i}\left(\hat{A}\right),Z(\hat{A},\hat{C})_{\hat{\lambda}_i}$, is determined as in (7.26) and remain also the same. From (7.27) we conclude that for regular sampling,

$$\Psi(A,B)_{\lambda_i} \equiv \Psi(\hat{A},\hat{B})_{\hat{\lambda}_i}$$

and the theorem is proved. □

7.4.2 Irregular Sampling

For notational simplicity again the case of two eigenvalues λ_u and λ_y collapsing is examined. The obtained results can directly be extended to any case of a partial or a total collapsing, to each σ-root range of A, $\Phi\sigma(A)$. We assume that under an irregular sampling, the distinct eigenvalues λ_u and λ_y of the continuous system correspond to the eigenvalue $\hat{\lambda}_c$ of the discretised system.

Theorem 53 *Under irregular sampling, the structure of the i.d.z.(o.d.z.) of the discretised model $S(\hat{A},\hat{B},\hat{C})$ is transformed from the corresponding structure of the continuous system $S(A,B,C)$.*

(a) The two sets,

$$\Sigma'(A,B)_{\lambda_u} = \left\{q'_{u1}, q'_{u2}, ..., q'_{u\nu_u}; q'_{uk} \geq 0,\ k = 1,2,...,\nu_u\right\}$$
$$\Sigma'(A,B)_{\lambda_y} = \left\{q'_{y1}, q'_{y2}, ..., q'_{y\nu_y}; q'_{yk} \geq 0,\ k = 1,2,...,\nu_y\right\}$$

determining according to (7.15) the structure of i.d.z. of the continuous system at $s = \lambda_u$ and $s = \lambda_y$, corresponds the set,

$$\Sigma'(\hat{A},\hat{B})_{\hat{\lambda}_c} = \left\{q'_{c1}, q'_{c2}, ..., q'_{c\nu_c}; q'_{ck} \geq 0,\ k = 1,2,...,\nu_c\right\}$$

determining the structure of i.d.z. at $s = \hat{\lambda}_c$ of the discretised model; each one of the $q'_{ck}(k = 1,2,...,\nu_c)$ is greater or equal to the corresponding number (one of the q'_{uk} $(k = 1,2,...,\nu_u)$or q'_{yk} $(k = 1,2,...,\nu_y)$) derived from the same matrix block of $\mathcal{B}_u$ or $\mathcal{B}_y$ of the continuous system

(b) The two sets,

$$Z'(A,C)_{\lambda_u} = \left\{p'_{u1}, p'_{u2}, ..., p'_{u\nu_u}; p'_{uk} \geq 0,\ k = 1,2,...,\nu_u\right\}$$
$$Z'(A,C)_{\lambda_y} = \left\{p'_{y1}, p'_{y2}, ..., p'_{y\nu_y}; p'_{yk} \geq 0,\ k = 1,2,...,\nu_y\right\}$$

determining according to (7.26) the structure of o.d.z. of the continuous system at $s = \lambda_u$ and $s = \lambda_y$, corresponds the set,

$$Z'(\hat{A},\hat{C})_{\hat{\lambda}_c} = \left\{p'_{c1}, p'_{c2}, ..., p'_{c\nu_c}; p'_{ck} \geq 0,\ k = 1,2,...,\nu_c\right\}$$

determining the structure of o.d.z. at $s = \hat{\lambda}_c$ of the discretised model; each one of the $p'_{ck}(k = 1,2,...,\nu_c)$ is greater or equal to the corresponding number (one of the p'_{uk} $(k = 1,2,...,\nu_u)$or p'_{yk} $(k = 1,2,...,\nu_y)$) derived from the same matrix block of $\mathcal{B}_u$ or $\mathcal{B}_y$ of the continuous system.

Proof. It is already known (Theorem 31) that, the Segré Characteristic $\wp_{\hat{\lambda}_c}\left(\hat{A}\right)$ is formed by the merging of $\wp_{\lambda_u}(A)$ and $\wp_{\lambda_y}(A)$.

$$\wp_{\hat{\lambda}_c}\left(\hat{A}\right) = \wp_{\lambda_u}(A) \cup \wp_{\lambda_y}(A) \Leftrightarrow \{\tau_{c1}, \tau_{c2}, ..., \tau_{c\nu_c}\} = \{\tau_{u1}, \tau_{u2}, ..., \tau_{u\nu_u}\} \cup \{\tau_{y1}, \tau_{y2}, ..., \tau_{y\nu_y}\}$$

(a) The set of *c-th* spectrum r.c.i. rearranged as in (7.14) is,

$$\Theta'(\hat{A}, \hat{B})_{\hat{\lambda}_c} = \{\acute{\theta}_{c1}, \acute{\theta}_{c2}, ..., \acute{\theta}_{ck}, ..., \acute{\theta}_{c\nu_c}\}, \acute{\theta}_{ck} \geq 0, k = 1, 2, ..., \nu_c \tag{7.29}$$

and each one of the above indices of $\Theta'(\hat{A}, \hat{B})_{\hat{\lambda}_c}$ is, according to the proof of Theorem 43, equal or smaller to the corresponding index of the same matrix block of $\mathcal{B}_u$ and $\mathcal{B}_y$ of the continuous system. Then from (7.15) the result is proved.

(b) The set of *c-th spectrum* c.o.i. rearranged as in (7.25) is,

$$Z'(\hat{A}, \hat{C})_{\hat{\lambda}_c} = \{\acute{\zeta}_{c1}, \acute{\zeta}_{c2}, ..., \acute{\zeta}_{ck}, ..., \acute{\zeta}_{c\nu_c}\}, \acute{\zeta}_{ck} \geq 0, k = 1, 2, ..., \nu_c \tag{7.30}$$

and each one of the above indices of $Z'(\hat{A}, \hat{C})_{\hat{\lambda}_c}$ is, according to the proof of Theorem 47, equal or smaller to the corresponding index of the same matrix block of Γ_u and Γ_y of the continuous system. Then from (7.26) the result is proved. □

The above Proposition suggests that under irregular sampling it is possible to have i.d.z. (o.d.z.) of the discretized model with greater degrees than the corresponding i.d.z. (o.d.z.) of the continuous system. Also under the conditions described in the following Remark, it is possible to have the generation of new i.d.z. (o.d.z.).

Remark 42 *If into some block of $\mathcal{B}_u$ or $\mathcal{B}_y$ (Γ_u or Γ_y) of the continuous system (with Q_{H_u}, Q_{H_y} (M_{H_u}, M_{H_y}) as normal structure according to Definition 47 (49)), let into the block $\mathcal{B}_{uk}$ (Γ_{uk}) we have $\theta_{uk} = \tau_{uk}$ ($\zeta_{uk} = \tau_{uk}$), while the corresponding index of the discretised system is $\theta_{uk} < \tau_{uk}$ ($\zeta_{uk} < \tau_{uk}$) then we have the generation of a new i.d.z. under irregular sampling.* □

As an illustration of the above consider a continuous system $S_J(J, \mathcal{B}, \Gamma)$, the Jordan block $J_{uk} \in \mathbb{C}^{\tau_{ik} \times \tau_{ik}}$ of J and the corresponding blocks $\mathcal{B}_{uk}$ of $\mathcal{B}$ and Γ_{uk} of $\Gamma$and $\theta_{uk} = \tau_{uk} - 1$.

$\zeta_{uk} = \tau_{uk}$:

$$
\begin{array}{ccccccc}
\lambda_u & 1 & . & 0 & 0 & \underline{\beta}^\top_{uk_1} & 1 \\
0 & \lambda_u & . & 0 & 0 & \underline{\beta}^\top_{uk_2} & 2 \\
. & . & . & . & . & . & . \\
0 & 0 & . & 1 & 0 & \underline{\beta}^\top_{uk_{\tau_{uk}-2}} & \tau_{uk}-2 \\
0 & 0 & . & \lambda_u & 1 & \underline{\beta}^\top_{uk_{\tau_{uk}-1}} & \leftarrow \grave{\theta}_{uk} = \tau_{uk}-1 \\
0 & 0 & . & 0 & \lambda_u & \underline{\beta}^\top_{uk_{\tau_{uk}}} & \\
\underline{\gamma}_{uk_1} & \underline{\gamma}_{uk_2} & . & \underline{\gamma}_{uk_{\tau_{uk}-1}} & \underline{\gamma}_{uk_{\tau_{uk}}} & & \\
\uparrow & & & & & & \\
\grave{\zeta}_{uk} = \tau_{uk} & \tau_{uk}-1 & . & 2 & 1 & &
\end{array}
$$

So for the continuous system we have the i.d.z. $(s - \lambda_u)$ and no one o.d.z. Let under irregular sampling for the corresponding discretized model $S(\hat{J}, \widehat{\mathcal{B}}, \widehat{\Gamma})$ be :

$$
\begin{array}{ccccccc}
\hat{\lambda}_c & 1 & . & 0 & 0 & \hat{\underline{\beta}}^\top_{ck_1} & 1 \\
0 & \hat{\lambda}_c & . & 0 & 0 & \hat{\underline{\beta}}^\top_{ck_2} & 2 \\
. & . & . & . & . & . & . \\
0 & 0 & . & 1 & 0 & \hat{\underline{\beta}}^\top_{ck_{\tau_{ck}-2}} & \leftarrow \hat{\theta}_{ck} = \tau_{ck}-2 \\
0 & 0 & . & \hat{\lambda}_c & 1 & \hat{\underline{\beta}}^\top_{ck_{\tau_{ck}-1}} & \\
0 & 0 & . & 0 & \hat{\lambda}_c & \hat{\underline{\beta}}^\top_{ck_{\tau_{ck}}} & \\
\hat{\underline{\gamma}}_{ck_1} & \hat{\underline{\gamma}}_{ck_2} & . & \hat{\underline{\gamma}}_{ck_{\tau_{ck}-1}} & \hat{\underline{\gamma}}_{ck_{\tau_{ck}}} & & \\
 & \uparrow & & & & & \\
 & \hat{\zeta}_{ck} = \tau_{ck}-1 & . & 2 & 1 & &
\end{array}
$$

where $\tau_{ck} = \tau_{uk}$, $\hat{\theta}_{ck} = \tau_{ck} - 2 < \theta_{uk}$, $\hat{\zeta}_{ck} = \tau_{ck} - 1 < \zeta_{uk}$. Then we have the i.d.z. $(z - \lambda_c)^2$ and the generation of the o.d.z. $(z - \lambda_c)$.

Example 11 *Consider the uncontrollable continuous system given in Examples 7 and 10 be under the irregular values of the sampling period as already have been examined in Example 8.*

a) $T = \frac{2k\pi}{24}$.

a.1) For $k = 1, 2, 3, 5, 6, 7, 9, 10, 11, ...$

a.1.1) mode $\hat{\lambda}_{12}$:

$\wp_{\hat{\lambda}_{12}}\left(\hat{A}\right)$	$\Theta(\hat{A},\hat{B})_{\hat{\lambda}_{12}}$	$\Sigma'(\hat{A},\hat{B})_{\hat{\lambda}_{12}}$	*i.d.z.*
6	4	2	$(z-\hat{\lambda}_{12})^2$
6	3	3	$(z-\hat{\lambda}_{12})^3$
3	3	0	1
3	3	0	1
1	1	0	1
1	1	0	1

to the above i.d.z. correspond the i.d.z. $(s-\lambda_1)^2$ *and* $(s-\lambda_2)^2$ *of the system*

a.1.2) mode $\hat{\lambda}_3$:

$\wp_{\hat{\lambda}_{3}}\left(\hat{A}\right)$	$\Theta(\hat{A},\hat{B})_{\hat{\lambda}_{3}}$	$\Sigma'(\hat{A},\hat{B})_{\hat{\lambda}_{3}}$	*i.d.z.*
3	2	1	$(z-\hat{\lambda}_3)$
1	1	0	1

to the above i.d.z. correspond the i.d.z. $(s-\lambda_3)$ *of the system*

a.1.3) mode $\hat{\lambda}_4$:

$\wp_{\hat{\lambda}_{4}}\left(\hat{A}\right)$	$\Theta(\hat{A},\hat{B})_{\hat{\lambda}_{4}}$	$\Sigma'(\hat{A},\hat{B})_{\hat{\lambda}_{4}}$	*i.d.z.*
3	2	1	$(z-\hat{\lambda}_4)$
1	1	0	1

to the above i.d.z. correspond the i.d.z. $(s-\lambda_4)$ *of the system.*

a.2) For $k = 4, 8, 12, ...$

a.2.1) mode $\hat{\lambda}_{12}$ *: as above.*

a.2.2) mode $\hat{\lambda}_{34}$:

$\wp_{\hat{\lambda}_{34}}\left(\hat{A}\right)$	$\Theta(\hat{A},\hat{B})_{\hat{\lambda}_{34}}$	$\Sigma'(\hat{A},\hat{B})_{\hat{\lambda}_{34}}$	*i.d.z.*
3	2	1	$(z-\hat{\lambda}_{34})$
3	2	1	1
1	1	0	1
1	0	1	$(z-\hat{\lambda}_{34})$

to the above i.d.z. correspond the i.d.z. $(s-\lambda_3)$ *and* $(s-\lambda_4)$ *of the system.*

a.3) For $k = 8, 16, 24, ...$

a.3.1) mode $\hat{\lambda}_{1234}$:

$\wp_{\hat{\lambda}_{1234}}\left(\hat{A}\right)$	$\Theta(\hat{A},\hat{B})_{\hat{\lambda}_{1234}}$	$\Sigma'(\hat{A},\hat{B})_{\hat{\lambda}_{1234}}$	*i.d.z.*
6	4	2	$(z-\hat{\lambda}_{1234})^2$
6	3	3	$(z-\hat{\lambda}_{1234})^3$
3	3	0	1
3	3	0	1
3	2	1	$(z-\hat{\lambda}_{1234})$
3	2	1	$(z-\hat{\lambda}_{1234})$
1	0	1	$(z-\hat{\lambda}_{1234})$
1	0	1	$(z-\hat{\lambda}_{1234})$
1	0	1	$(z-\hat{\lambda}_{1234})$
1	0	1	$(z-\hat{\lambda}_{1234})$

to the above i.d.z. correspond the i.d.z. $(s-\lambda_1)^2, (s-\lambda_2)^2, (s-\lambda_3)$ *and* $(s-\lambda_4)$ *of the system.*

b) $T = \frac{2k\pi}{15}$

b.1) For $k = 1, 2, 3, 4, 6, 7, 8, 9, 11, ...$

b.1.1) mode $\hat{\lambda}_{14}$*:*

$\wp_{\hat{\lambda}_{14}}\left(\hat{A}\right)$	$\Theta(\hat{A},\hat{B})_{\hat{\lambda}_{14}}$	$\Sigma'(\hat{A},\hat{B})_{\hat{\lambda}_{14}}$	*i.d.z.*
6	4	2	$(z-\hat{\lambda}_{14})^2$
3	3	0	1
3	2	1	$(z-\hat{\lambda}_{14})$
1	1	0	1
1	0	1	$(z-\hat{\lambda}_{14})$

to the above i.d.z. correspond the i.d.z. $(s-\lambda_1)^2$ *and* $(s-\lambda_4)$ *of the system*

b.1.2) mode $\hat{\lambda}_{23}$*:*

$\wp_{\hat{\lambda}_{23}}\left(\hat{A}\right)$	$\Theta(\hat{A},\hat{B})_{\hat{\lambda}_{23}}$	$\Sigma'(\hat{A},\hat{B})_{\hat{\lambda}_{23}}$	*i.d.z.*
6	4	2	$(z-\hat{\lambda}_{23})^2$
3	3	0	1
3	2	1	$(z-\hat{\lambda}_{23})$
2	1	0	1
1	0	1	$(z-\hat{\lambda}_{23})$

to the above i.d.z. correspond the i.d.z. $(s-\lambda_2)^2$ *and* $(s-\lambda_3)$ *of the system.*

b.2) For $k = 5, 10, 15, 20, ...$

b.2.1. mode $\hat{\lambda}_{1234}$ *as above.* □

c) $T = \frac{2k\pi}{9}$

c.1) For $k = 1, 2, , 4, 5, 7, 8, 10, ...$

c.1.1) mode $\hat{\lambda}_{13}$*:*

$\wp_{\hat{\lambda}_{13}}\left(\hat{A}\right)$	$\Theta(\hat{A},\hat{B})_{\hat{\lambda}_{13}}$	$\Sigma'(\hat{A},\hat{B})_{\hat{\lambda}_{13}}$	*idz*
6	4	2	$(z-\hat{\lambda}_{13})^2$
3	3	0	1
3	2	1	$(z-\hat{\lambda}_{13})$
1	1	0	1
1	0	1	$(z-\hat{\lambda}_{13})$

to the above i.d.z. correspond the i.d.z. $(s-\lambda_1)^2$ *and* $(s-\lambda_3)$ *of the system*

c.1.2) mode $\hat{\lambda}_{24}$*:*

$\wp_{\hat{\lambda}_{24}}\left(\hat{A}\right)$	$\Theta(\hat{A},\hat{B})_{\hat{\lambda}_{24}}$	$\Sigma'(\hat{A},\hat{B})_{\hat{\lambda}_{24}}$	*i.d.z.*
6	4	2	$(z-\hat{\lambda}_{24})^2$
3	3	0	1
3	2	1	$(z-\hat{\lambda}_{24})$
1	1	0	1
1	0	1	$(z-\hat{\lambda}_{24})$

to the above i.d.z. correspond the i.d.z. $(s-\lambda_2)^2$ *and* $(s-\lambda_4)$ *of the system.*

c.2) For $k = 5, 10, 15, 20, \ldots$

c.2.1) mode $\hat{\lambda}_{1234}$ *as above.* □

7.5 Conclusions

The results of this section provide an extension of the classical spectral analysis for the characterization of controllability and observability results (Gilbert results [Gil., 1]) to the characterization of degrees of divisors associated with the input and output decoupling zeros of a continuous system. The new framework provided here is a natural one for characterizing the corresponding degrees of input and output decoupling zeros to the case of irregular sampling. In fact, the merging of spectral matrices and the spectral analysis provide a simple method for characterizing i.d.z., o.d.z. without resorting to algebraic tests.

Chapter 8

SAMPLING PROPERTIES AND FINITE AND INFINITE ZEROS

8.1 Introduction

A number of general results on the asymptotic properties of zeros of discretised SISO systems under special forms of sampling have been defined in [Ast., Hag. & Ste.]. The overall problem of the mapping of zeros under sampling is an open issue however. Here we examine for the case of multivariable systems with the same number of inputs and outputs a particular aspect of the zero mapping problem which has to do with the migration of finite zeros under special conditions affecting geometric aspects of the system to migrate at infinity. An integral part of the work here is the computation of the discretised zero polynomial and some of its properties under special types of sampling.

The described expressions in Chapter 3 for the zero polynomial of the continuous system [Kar., 3] may also be applied in the case of discretised model for the calculation of the discretised zero polynomial coefficients. The existence of a set of eigenvalues located on the imaginary axis and the collapsing of such eigenvalues to 0 is a precondition for a further migration of finite zeros to infinity under irregular sampling. This case is investigated here and this highlights another aspect of the effects of irregular sampling on discretised systems.

8.2 The Zero Polynomial of the Discretised Model

The Rosenbrock's matrix pencil of a square and strictly proper system $S(A,B,C)$ has been given in Chapter 3 by relation (3.77). The corresponding Rosenbrock's matrix pencil of the discretised model $\hat{S}(\hat{A},\hat{B},\hat{C})$ of a square and strictly proper system $S(A,B,C)$ is also defined by :

$$\hat{P}(z) \triangleq \begin{bmatrix} zI-\hat{A} & -\hat{B} \\ -\hat{C} & 0 \end{bmatrix} \in \mathbb{R}^{(n+m)\times(n+m)}[z] \tag{8.1}$$

Where according to the previous analysis for a system with ZOH we have,

$$\hat{A}=e^{AT},\ \hat{B}=(\int_0^T e^{A\sigma}d\sigma)B,\ \hat{C}=C \tag{8.2}$$

or in a Jordan description,

$$\hat{J}=e^{JT},\ \widehat{\mathcal{B}}=\Xi\mathcal{B},\ \hat{\Gamma}=\Gamma \tag{8.3}$$

Let $\hat{S}_{\hat{J}}(\hat{J},\widehat{\mathcal{B}},\Gamma)$ denote the Jordan description of the discretised system $\hat{S}(\hat{A},\hat{B},\hat{C})$. From Theorem 26 it is directly concluded that if the continuous system matrix J is in simple form, then for every value of the sampling period T the corresponding matrix $\hat{J}$ of the discretised system is also simple. From Chapter 4 the relations between the discretised and continuous system parameters are also known. From this description we have:

Proposition 65 *For every value of the sampling period T, the matrix Ξ of the discretised model $\hat{S}(\hat{A},\hat{B},\hat{C})$ of a square, strictly proper system $S(A,B,C)$ with simple matrix A and with none of its eigenvalues on the imaginary axis, except possibly 0 $(\lambda_i \notin (\mathbb{I}-0),\ i=1,2,...,n)$ is a diagonal and non singular matrix.*

Proof. From the simple structure of $\hat{J}$, the relations 4.19, 4.17 and Theorem 27, we have,

$$\tilde{V}=\tilde{U}\equiv I_n \tag{8.4}$$

and

$$\Xi=\int_0^T e^{J\sigma}d\sigma=\mathrm{diag}\left\{\underbrace{T,...,T}_{\nu_1},\frac{\hat{\lambda}_{\nu_1+1}-1}{\lambda_{\nu_1+1}},...,\frac{\hat{\lambda}_n-1}{\lambda_n}\right\}=\mathrm{diag}\left\{\xi_i\right\} \tag{8.5}$$

where ν_1 is the number of eigenvalues of A equal to 0 ($\lambda_1 = \ldots = \lambda_{\nu_1} = 0$). Then as none of the eigenvalues is located on the imaginary axis it is $\hat{\lambda}_{\nu_1+1}, \hat{\lambda}_{\nu_1+2}, \ldots, \hat{\lambda}_n \neq 1$ and proposition is proved. □

Remark 43 *The parameters ξ_i in $\Xi = \operatorname{diag}\{\xi_i, i \in \mathbf{n}\}$ have the values $\xi_i = T$, if $\lambda_i = 0$ and $\xi_i = \frac{e^{\lambda_i T} - 1}{\lambda_i}$ if $\lambda_i \neq 0$.* □

8.2.1 Calculation of Coefficients of discretised zero polynomial

As for the case of continuous systems (relation 3.78), it can be proved that the zero polynomial of such a model is also of $n - m$ degree and it is of the form:

$$\hat{z}(s) = \hat{\alpha}_{n-m} z^{n-m} + \hat{a}_{n-m-1} z^{n-m-1} + \ldots + \hat{a}_1 z + \hat{a}_0 \tag{8.6}$$

where the coefficients of the discrete zero polynomial are also given, as in the continuous case (relations 3.81), by the relations,

$$\begin{aligned}
\hat{\alpha}_{n-m} &= \left\langle \hat{\boldsymbol{\gamma}}^{\mathsf{T}} \hat{\boldsymbol{\beta}} \right\rangle \\
\hat{\alpha}_{n-m-1} &= \left\{ \sum_{\omega_1} \hat{\lambda}_{\omega_1} \left\langle \hat{\boldsymbol{\gamma}}^{\mathsf{T}}_{\omega_1)} \hat{\boldsymbol{\beta}}^{\omega_1)} \right\rangle \right\} \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
\hat{\alpha}_{n-m-j} &= \left\{ \sum_{\omega_j} \hat{\lambda}_{\omega_j} \left\langle \hat{\boldsymbol{\gamma}}^{\mathsf{T}}_{\omega_j)} \hat{\boldsymbol{\beta}}^{\omega_j)} \right\rangle \right\} \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
\hat{\alpha}_0 &= \left\{ \sum_{\omega_{n-m}} \hat{\lambda}_{\omega_{n-m}} \left\langle \hat{\boldsymbol{\gamma}}^{\mathsf{T}}_{\omega_{n-m})} \hat{\boldsymbol{\beta}}^{\omega_{n-m})} \right\rangle \right\}
\end{aligned} \tag{8.7}$$

where as for the continuous system the bold letters $\hat{\boldsymbol{\gamma}}^{\mathsf{T}}, \hat{\boldsymbol{\beta}}$ denote the Grassman products of the rows of $\hat{\Gamma}$, columns of $\hat{\mathcal{B}}$ respectively.

Proposition 66 *Let the proper linear system $S(A, B, C)$ strict equivalent to the $S_J(J, \mathcal{B}, \Gamma)$, where J is in simple structure, if for the discretized model $S(\hat{A}, \hat{B}, \hat{C})$ it is $\hat{\lambda}_i \neq 1$, $i = \nu_1 + 1, \ldots, n$ then for every value of the sampling period T, the coefficients of the discretized zero polynomial* *are defined as functions of the corresponding continuous system parameters, that is :*

$$\begin{aligned}
\hat{\alpha}_{n-m} &= \left\langle \hat{\boldsymbol{\gamma}}^{\top} \hat{\boldsymbol{\beta}} \right\rangle = \langle \boldsymbol{\gamma}^{\top} \mathfrak{C}_{\mathbf{m}}(\Xi) \boldsymbol{\beta} \rangle \\
\hat{\alpha}_{n-m-1} &= \left\{ \sum_{\omega_1} \hat{\lambda}_{\omega_1} \left\langle \boldsymbol{\gamma}^{\top}_{\omega_1)} \mathfrak{C}_{\mathbf{m}}(\Xi_{\omega_1)}) \boldsymbol{\beta}^{\omega_1)} \right\rangle \right\} \\
&\ldots\ldots\ldots\ldots\ldots \\
\hat{\alpha}_{n-m-j} &= \sum_{\omega_j} \hat{\lambda}_{\omega_j} \left\langle \boldsymbol{\gamma}^{\top}_{\omega_j)} \mathfrak{C}_{\mathbf{m}}(\Xi_{\omega_j)}) \boldsymbol{\beta}^{\omega_j)} \right\rangle \\
&\ldots\ldots\ldots\ldots\ldots \\
\hat{\alpha}_0 &= \left\{ \sum_{\omega_{m-n}} \hat{\lambda}_{\omega_{m-n}} \left\langle \boldsymbol{\gamma}^{\top}_{\omega_{m-n})} \mathfrak{C}_{\mathbf{m}}(\Xi_{\omega_{m-n})}) \boldsymbol{\beta}^{\omega_{m-n})} \right\rangle \right\}
\end{aligned} \tag{8.8}$$

Proof. From the simple structure of $\hat{J}$ it is :

$$\begin{aligned}
\hat{\Gamma} &= \begin{bmatrix} \hat{\underline{\gamma}}_1^{\top} \\ \hat{\underline{\gamma}}_2^{\top} \\ \cdots \\ \hat{\underline{\gamma}}_m^{\top} \end{bmatrix} = \begin{bmatrix} \underline{\gamma}_1^{\top} \\ \underline{\gamma}_2^{\top} \\ \cdots \\ \underline{\gamma}_m^{\top} \end{bmatrix} = \Gamma \\
\hat{\mathcal{B}} &= \Xi \mathcal{B} = \left[\Xi \underline{\beta}_1, \Xi \underline{\beta}_2, \ldots, \Xi \underline{\beta}_m \right]
\end{aligned}$$

so,

$$\hat{\boldsymbol{\gamma}}^{\top} = \underline{\gamma}_1^{\top} \wedge \underline{\gamma}_2^{\top} \wedge \ldots \wedge \underline{\gamma}_m^{\top} = \boldsymbol{\gamma}^{\top}, \ \hat{\boldsymbol{\beta}} = \Xi \underline{\beta}_1 \wedge \Xi \underline{\beta}_2 \wedge \ldots \wedge \Xi \underline{\beta}_m$$

and from the definition of the Grassman product it is:

$$\hat{\boldsymbol{\beta}} = \mathfrak{C}_{\mathbf{m}}(\Xi \mathcal{B}) = \mathfrak{C}_{\mathbf{m}}(\Xi) \mathfrak{C}_{\mathbf{m}}(\mathcal{B}) = \mathfrak{C}_{\mathbf{m}}(\Xi) \boldsymbol{\beta}$$

$$\hat{\boldsymbol{\beta}}^{\omega_j)} = \mathfrak{C}_{\mathbf{m}}(\Xi_{\omega_j)} \mathcal{B}^{\omega_j)}) = \mathfrak{C}_{\mathbf{m}}(\Xi_{\omega_j)}) \boldsymbol{\beta}^{\omega_j)}$$

and the proposition is proved. □

Remark 44 *The multiorthogonality property of the vectors $\hat{\boldsymbol{\gamma}}^{\top}, \hat{\boldsymbol{\beta}}$, introduced in Chapter 3, generally is the same as the corresponding notion for vectors of the continuous system $\boldsymbol{\gamma}^{\top}, \boldsymbol{\beta}$.* □

From the conditions 8.8 and Proposition 65 we conclude that Ξ is a diagonal and non singular matrix. Then $\mathfrak{C}_{\mathbf{m}}(\Xi)$ is also a $\binom{n}{m} \times \binom{n}{m}$ diagonal and non singular matrix and a such matrix generally does not affects the multiorthogonality property. The computation of the discretised zero polynomial is shown by the following example.

Example 12 *Consider the continuous system with simple structure given in Example 2 which is discretised with ZOH. Then, as matrix A has a simple structure Jordan form, we have*

$$\hat{J} = e^{JT} = \begin{bmatrix} e^{-6T} & 0 & 0 & 0 & 0 \\ 0 & e^{(-2.0-2.0i)T} & 0 & 0 & 0 \\ 0 & 0 & e^{(-2.0+2.0i)T} & 0 & 0 \\ 0 & 0 & 0 & e^{(-2.0+4.0i)T} & 0 \\ 0 & 0 & 0 & 0 & e^{(-2.0-4.0i)T} \end{bmatrix}$$

$$\Xi = \begin{bmatrix} \frac{e^{-6T}-1}{-6} & 0 & 0 & 0 & 0 \\ 0 & \frac{e^{(-2.0-2.0i)T}-1}{-2.0-2.0i} & 0 & 0 & 0 \\ 0 & 0 & \frac{e^{(-2.0+2.0i)T}-1}{-2.0+2.0i} & 0 & 0 \\ 0 & 0 & 0 & \frac{e^{(-2.0+4.0i)T}-1}{-2.0+4.0i} & 0 \\ 0 & 0 & 0 & 0 & \frac{e^{(-2.0-4.0i)T}-1}{-2.0-4.0i} \end{bmatrix}$$

or

$$\xi_1 = \frac{e^{-6T}-1}{-6}, \; \xi_2 = \frac{e^{(-2.0-2.0i)T}-1}{-2.0-2.0i}, \; \xi_3 = \frac{e^{(-2.0+2.0i)T}-1}{-2.0+2.0i},$$
$$\xi_4 = \frac{e^{(-2.0+4.0i)T}-1}{-2.0+4.0i}, \; \xi_5 = \frac{e^{(-2.0-4.0i)T}-1}{-2.0-4.0i}$$

and consequently the parameters $\widehat{\mathcal{B}}$ and Γ of the Jordan equivalent discretised system are,

$$\widehat{\mathcal{B}} = \Xi \mathcal{B} = \Xi \begin{bmatrix} 0 & 1 \\ 1-3i & -2-4i \\ 1+3i & -2+4i \\ 3-i & 2+4i \\ 3+i & 2-4i \end{bmatrix}$$

$$\hat{\Gamma} = \Gamma = \begin{bmatrix} -7 & \frac{1}{4}+\frac{1}{4}i & \frac{1}{4}-\frac{1}{4}i & -\frac{1}{4}-\frac{1}{4}i & -\frac{1}{4}+\frac{1}{4}i \\ 3 & -1-\frac{1}{4}i & -1+\frac{1}{4}i & -2-\frac{3}{4}i & -2+\frac{3}{4}i \end{bmatrix}$$

it is

$$\Xi = \operatorname{diag}\left\{ \xi_1 \quad \xi_2 \quad \xi_3 \quad \xi_4 \quad \xi_5 \right\} \Rightarrow$$

$$\mathfrak{C}_2(\Xi) = \operatorname{diag}\left\{ \xi_1\xi_2 \quad \xi_1\xi_3 \quad \xi_1\xi_4 \quad \xi_1\xi_5 \quad \xi_2\xi_3 \quad \xi_2\xi_4 \quad \xi_2\xi_5 \quad \xi_3\xi_4 \quad \xi_3\xi_5 \quad \xi_4\xi_5 \right\}$$

$$\begin{aligned}
\mathfrak{C}_2(\Xi^{1)}) &= \operatorname{diag}\left\{ \xi_2\xi_3 \quad \xi_2\xi_4 \quad \xi_2\xi_5 \quad \xi_3\xi_4 \quad \xi_3\xi_5 \quad \xi_4\xi_5 \right\} \\
\mathfrak{C}_2(\Xi^{2)}) &= \operatorname{diag}\left\{ \xi_1\xi_3 \quad \xi_1\xi_4 \quad \xi_1\xi_5 \quad \xi_3\xi_4 \quad \xi_3\xi_5 \quad \xi_4\xi_5 \right\} \\
\mathfrak{C}_2(\Xi^{3)}) &= \operatorname{diag}\left\{ \xi_1\xi_2 \quad \xi_1\xi_4 \quad \xi_1\xi_5 \quad \xi_2\xi_4 \quad \xi_2\xi_5 \quad \xi_4\xi_5 \right\} \\
\mathfrak{C}_2(\Xi^{4)}) &= \operatorname{diag}\left\{ \xi_1\xi_2 \quad \xi_1\xi_3 \quad \xi_1\xi_5 \quad \xi_2\xi_3 \quad \xi_2\xi_5 \quad \xi_3\xi_5 \right\} \\
\mathfrak{C}_2(\Xi^{5)}) &= \operatorname{diag}\left\{ \xi_1\xi_2 \quad \xi_1\xi_3 \quad \xi_1\xi_4 \quad \xi_2\xi_3 \quad \xi_2\xi_4 \quad \xi_3\xi_4 \right\}
\end{aligned}$$

$$\begin{aligned}
\mathfrak{C}_2(\Xi^{1,2)}) &= \operatorname{diag}\left\{ \xi_3\xi_4 \quad \xi_3\xi_5 \quad \xi_4\xi_5 \right\} & \mathfrak{C}_2(\Xi^{1,3)}) &= \operatorname{diag}\left\{ \xi_2\xi_4 \quad \xi_2\xi_5 \quad \xi_4\xi_5 \right\} \\
\mathfrak{C}_2(\Xi^{1,4)}) &= \operatorname{diag}\left\{ \xi_2\xi_3 \quad \xi_2\xi_5 \quad \xi_3\xi_5 \right\} & \mathfrak{C}_2(\Xi^{1,5)}) &= \operatorname{diag}\left\{ \xi_2\xi_3 \quad \xi_2\xi_4 \quad \xi_3\xi_4 \right\} \\
\mathfrak{C}_2(\Xi^{2,3)}) &= \operatorname{diag}\left\{ \xi_1\xi_4 \quad \xi_1\xi_5 \quad \xi_4\xi_5 \right\} & \mathfrak{C}_2(\Xi^{2,4)}) &= \operatorname{diag}\left\{ \xi_1\xi_3 \quad \xi_1\xi_5 \quad \xi_3\xi_5 \right\} \\
\mathfrak{C}_2(\Xi^{2,5)}) &= \operatorname{diag}\left\{ \xi_1\xi_3 \quad \xi_1\xi_4 \quad \xi_3\xi_4 \right\} & \mathfrak{C}_2(\Xi^{3,4)}) &= \operatorname{diag}\left\{ \xi_1\xi_2 \quad \xi_1\xi_5 \quad \xi_2\xi_5 \right\} \\
\mathfrak{C}_2(\Xi^{3,5)}) &= \operatorname{diag}\left\{ \xi_1\xi_2 \quad \xi_1\xi_4 \quad \xi_2\xi_4 \right\} & \mathfrak{C}_2(\Xi^{4,5)}) &= \operatorname{diag}\left\{ \xi_1\xi_2 \quad \xi_1\xi_3 \quad \xi_2\xi_3 \right\}
\end{aligned}$$

$$\begin{aligned}
\mathfrak{C}_2(\Xi^{1,2,3)}) &= \xi_4\xi_5, \mathfrak{C}_2(\Xi^{1,2,4)}) = \xi_3\xi_5, \mathfrak{C}_2(\Xi^{1,2,5)}) = \xi_3\xi_4, \mathfrak{C}_2(\Xi^{1,3,4)}) = \xi_2\xi_5, \mathfrak{C}_2(\Xi^{1,3,5)}) = \xi_2\xi_4 \\
\mathfrak{C}_2(\Xi^{1,4,5)}) &= \xi_2\xi_3, \mathfrak{C}_2(\Xi^{2,3,4)}) = \xi_1\xi_5, \mathfrak{C}_2(\Xi^{2,3,5)}) = \xi_1\xi_4, \mathfrak{C}_2(\Xi^{2,4,5)}) = \xi_1\xi_3, \mathfrak{C}_2(\Xi^{3,4,5)}) = \xi_1\xi_2
\end{aligned}$$

from the above and the corresponding calculations of the continuous time Example 2 we have the following expression for the coefficients of the zero polynomial :

$$\hat{a}_3 = \boldsymbol{\gamma}\mathfrak{C}_2(\Xi)\boldsymbol{\beta} \tag{8.9}$$

where

$$\beta = \begin{bmatrix} -1+3i & -1-3i & -3+i & -3-i & 20i & 24+8i & -8+4i & -8-4i & 24-8i & -28i \end{bmatrix}^T$$

$$\gamma = \begin{bmatrix} \frac{25}{4}+i & \frac{25}{4}-i & \frac{59}{4}+6i & \frac{59}{4}-6i & -\frac{3}{8}i & -\frac{1}{2}-i & -1-\frac{1}{8}i & -1+\frac{1}{8}i & -\frac{1}{2}+i & \frac{5}{8}i \end{bmatrix}$$

then

$$\begin{aligned} \hat{a}_3 \;=\; & -\frac{37}{4}\xi_1\xi_2 - \frac{37}{4}\xi_1\xi_3 - \frac{201}{4}\xi_1\xi_4 - \frac{201}{4}\xi_1\xi_5 + \frac{15}{2}\xi_2\xi_3 - 4\xi_2\xi_4 + \frac{17}{2}\xi_2\xi_5 + \frac{17}{2}\xi_3\xi_4 - 4\xi_3\xi_5 + \\ & + i\left(\frac{71}{4}\xi_1\xi_2 - \frac{71}{4}\xi_1\xi_3 - \frac{13}{4}\xi_1\xi_4 + \frac{13}{4}\xi_1\xi_5 - 28\xi_2\xi_4 - 3\xi_2\xi_5 + 3\xi_3\xi_4 + 28\xi_3\xi_5\right) \end{aligned} \tag{8.10}$$

Let a regular value of the sampling period be $T = \frac{\pi}{10}$. We may proceed to the numerical calculation of the coefficients for the above system. From the expression (8.10) we have :

$$\hat{a}_3 = -1.6498$$

For the calculation of $\hat{a}_2$ we have:

$$\left(e^{-6.0T}\right) \begin{bmatrix} -\frac{3}{8}i & -\frac{1}{2}-i & -1-\frac{1}{8}i & -1+\frac{1}{8}i & -\frac{1}{2}+i & \frac{5}{8}i \end{bmatrix} \mathfrak{C}_2(\Xi^1)) \begin{bmatrix} 20i \\ 24+8i \\ -8+4i \\ -8-4i \\ 24-8i \\ -28i \end{bmatrix} = 0.29819$$

$$\left(e^{(-2.0-2.0i)T}\right) \begin{bmatrix} \frac{25}{4}-i & \frac{59}{4}+6i & \frac{59}{4}-6i & -1+\frac{1}{8}i & -\frac{1}{2}+i & \frac{5}{8}i \end{bmatrix} \mathfrak{C}_2(\Xi^2)) \begin{bmatrix} -1-3i \\ -3+i \\ -3-i \\ -8-4i \\ 24-8i \\ -28i \end{bmatrix} =$$

$$= -0.25265 + 0.96608i$$

$$\left(e^{(-2.0+2.0i)T}\right)\begin{bmatrix} \frac{25}{4}+i & \frac{59}{4}+6i & \frac{59}{4}-6i & -\frac{1}{2}-i & -1-\frac{1}{8}i & \frac{5}{8}i \end{bmatrix}\mathfrak{C}_2(\Xi^{3)})\begin{bmatrix} -1+3i \\ -3+i \\ -3-i \\ 24+8i \\ -8+4i \\ -28i \end{bmatrix} =$$

$$= --0.25265 - 0.96608i$$

$$\left(e^{(-2.0+4.0i)T}\right)\begin{bmatrix} \frac{25}{4}+i & \frac{25}{4}-i & \frac{59}{4}-6i & -\frac{3}{8}i & -1-\frac{1}{8}i & -\frac{1}{2}+i \end{bmatrix}\mathfrak{C}_2(\Xi^{4)})\begin{bmatrix} -1+3i \\ -1-3i \\ -3-i \\ 20i \\ -8+4i \\ 24-8i \end{bmatrix} =$$

$$= -1.0875 - 7.0641 \times 10^{-2}i$$

$$\left(e^{(-2.0-4.0i)T}\right)\begin{bmatrix} \frac{25}{4}+i & \frac{25}{4}-i & \frac{59}{4}+6i & -\frac{3}{8}i & -\frac{1}{2}-i & -1+\frac{1}{8}i \end{bmatrix}\mathfrak{C}_2(\Xi^{5)})\begin{bmatrix} -1+3i \\ -1-3i \\ -3+i \\ 20i \\ 24+8i \\ -8-4i \end{bmatrix} =$$

$$= -1.0875 + 7.0641 \times 10^{-2}i$$

$$\Longrightarrow \hat{a}_2 = 2.3821$$

For the calculation of $\hat{a}_1$ we have:

$$\left(e^{-6.0T}\right)e^{(-2.0-2.0i)T}\begin{bmatrix} -1+\frac{1}{8}i & -\frac{1}{2}+i & \frac{5}{8}i \end{bmatrix}\mathfrak{C}_2(\Xi^{1,2)})\begin{bmatrix} -8-4i \\ 24-8i \\ -28i \end{bmatrix} = .16559+6.2242\times10^{-2}i$$

$$= .16559 + 6.2242 \times 10^{-2} i$$

$$\left(e^{-6.0T}\right)\left(e^{(-2.0+2.0i)T}\right)\begin{bmatrix} -\frac{1}{2}-i & -1-\frac{1}{8}i & \frac{5}{8}i \end{bmatrix} \mathfrak{C}_2(\Xi^{1,3})\begin{bmatrix} 24+8i \\ -8+4i \\ -28i \end{bmatrix} =$$

$$= .16559 - 6.2242 \times 10^{-2} i$$

$$\left(e^{-6.0T}\right) e^{(-2.0+4.0i)T}\begin{bmatrix} -\frac{3}{8}i & -1-\frac{1}{8}i & -\frac{1}{2}+i \end{bmatrix} \mathfrak{C}_2(\Xi^{1,4})\begin{bmatrix} 20i \\ -8+4i \\ 24-8i \end{bmatrix} =$$

$$= -5.6946 \times 10^{-2} + 8.3379 \times 10^{-2} i$$

$$\left(e^{-6.0T}\right) e^{(-2.0-4.0i)T}\begin{bmatrix} -\frac{3}{8}i & -\frac{1}{2}-i & -1+\frac{1}{8}i \end{bmatrix} \mathfrak{C}_2(\Xi^{1,5})\begin{bmatrix} 20i \\ 24+8i \\ -8-4i \end{bmatrix} =$$

$$= -5.6946 \times 10^{-2} - 8.3379 \times 10^{-2} i$$

$$e^{(-2.0-2.0i)T} e^{(-2.0+2.0i)T}\begin{bmatrix} \frac{59}{4}+6i & \frac{59}{4}-6i & \frac{5}{8}i \end{bmatrix} \mathfrak{C}_2(\Xi^{2,3})\begin{bmatrix} -3+i \\ -3-i \\ -28i \end{bmatrix} = -.47977$$

$$e^{(-2.0-2.0i)T} e^{(-2.0+4.0i)T}\begin{bmatrix} \frac{25}{4}-i & \frac{59}{4}-6i & -\frac{1}{2}+i \end{bmatrix} \mathfrak{C}_2(\Xi^{2,4})\begin{bmatrix} -1-3i \\ -3-i \\ 24-8i \end{bmatrix}$$

$$= -.55585 + .18697i$$

$$e^{(-2.0-2.0i)T} e^{(-2.0-4.0i)T}\begin{bmatrix} \frac{25}{4}-i & \frac{59}{4}+6i & -1+\frac{1}{8}i \end{bmatrix} \mathfrak{C}_2(\Xi^{2,5})\begin{bmatrix} -1-3i \\ -3+i \\ -8-4i \end{bmatrix} = -.1986 + .42891i$$

$$e^{(-2.0+2.0i)T} e^{(-2.0+4.0i)T}\begin{bmatrix} \frac{25}{4}+i & \frac{59}{4}-6i & -1-\frac{1}{8}i \end{bmatrix} \mathfrak{C}_2(\Xi^{3,4})\begin{bmatrix} -1+3i \\ -3-i \\ -8+4i \end{bmatrix} = -.1986 - .42891i$$

$$e^{(-2.0+2.0i)T}e^{(-2.0-4.0i)T}\begin{bmatrix}\frac{25}{4}+i & \frac{59}{4}+6i & -\frac{1}{2}-i\end{bmatrix}\mathfrak{C}_2(\Xi^{3,5})\begin{bmatrix}-1+3i\\ -3+i\\ 24+8i\end{bmatrix} = -.55585-.18697i$$

$$e^{(-2.0+4.0i)T}e^{(-2.0-4.0i)T}\begin{bmatrix}\frac{25}{4}+i & \frac{25}{4}-i & -\frac{3}{8}i\end{bmatrix}\mathfrak{C}_2(\Xi^{4,5})\begin{bmatrix}-1+3i\\ -1-3i\\ 20i\end{bmatrix} = 3.9314\times 10^{-2}$$

$$\Longrightarrow \hat{a}_1 = -1.7321$$

and for the calculation of $\hat{a}_0$ we have correspondingly :

$$e^{(-6.0)T}e^{(-2.0-2.0i)T}e^{(-2.0+2.0i)T}\,(-28i)\,\xi_4\xi_5\left(\frac{5}{8}i\right) = 3.6107\times 10^{-2}$$

$$e^{(-6.0)T}e^{(-2.0-2.0i)T}e^{(-2.0+4.0i)T}\,(24-8i)\,\xi_3\xi_5\left(-\frac{1}{2}+i\right) = -2.8862\times 10^{-2}+5.4076\times 10^{-2}i$$

$$e^{(-6.0)T}e^{(-2.0-2.0i)T}e^{(-2.0-4.0i)T}\left(-1+\frac{1}{8}i\right)\xi_3\xi_4\,(-8-4i) = 1.4901\times 10^{-2}-1.2632\times 10^{-2}i$$

$$e^{(-6.0)T}e^{(-2.0+2.0i)T}e^{(-2.0+4.0i)T}\left(-1-\frac{1}{8}i\right)\xi_2\xi_5\,(-8+4i) = 1.4901\times 10^{-2}+1.2632\times 10^{-2}i$$

$$e^{(-6.0)T}e^{(-2.0+2.0i)T}e^{(-2.0-4.0i)T}\,(24+8i)\,\xi_2\xi_4\left(-\frac{1}{2}-i\right) = -2.8862\times 10^{-2}-5.4076\times 10^{-2}i$$

$$e^{(-6.0)T}e^{(-2.0+4.0i)T}e^{(-2.0-4.0i)T}\left(-\frac{3}{8}i\right)\xi_2\xi_3\,(20i) = 1.7073\times 10^{-2}$$

$$e^{(-2.0-2.0i)T}e^{(-2.0+2.0i)T}e^{(-2.0+4.0i)T}\,(-3-i)\,\xi_1\xi_5\left(\frac{59}{4}-6i\right) = -.19071-.13929i$$

$$e^{(-2.0-2.0i)T}e^{(-2.0+2.0i)T}e^{(-2.0-4.0i)T}\,(-3+i)\,\xi_1\xi_4\left(\frac{59}{4}+6i\right) = -.19071+.13929i$$

$$e^{(-2.0-2.0i)T}e^{(-2.0+4.0i)T}e^{(-2.0-4.0i)T}\,(-1-3i)\,\xi_1\xi_3\left(\frac{25}{4}-i\right) = -7.2591\times 10^{-2}-6.6727\times 10^{-2}i$$

$$e^{(-2.0+2.0i)T}e^{(-2.0+4.0i)T}e^{(-2.0-4.0i)T}\left(\frac{25}{4}+i\right)\xi_1\xi_2\,(-1+3i) = -7.2591\times 10^{-2}+6.6727\times 10^{-2}i$$

$$\Longrightarrow \hat{a}_0 = 0.50134$$

Thus the zero polynomial of the discretised system is :

$$\hat{z}(s) = -1.6498s^3 + 2.3821s^2 - 1.7321s + 0.50134$$

and the system invariant zeros are defined as,

$$\hat{z}_1 = 0.5414, \ \hat{z}_2 = 0.45124 + 0.59806i, \ \hat{z}_3 = 0.45124 - 0.59806i$$

$\square$

8.2.2 Migration of the Discretised Zeros to Infinity

If the continuous square system has a CB full rank, then according to Theorem 24 the system invariant zeros are $n-m$ in number. The corresponding discretised model has also $n-m$ invariant zeros if $\hat{C}\hat{B}$ is also full rank or equivalently if,

$$\hat{\alpha}_{n-m} = \left\langle \hat{\boldsymbol{\gamma}}^\top \hat{\boldsymbol{\beta}} \right\rangle = \langle \boldsymbol{\gamma}^\top \mathfrak{C}_m(\Xi)\boldsymbol{\beta} \rangle \neq 0$$

Example 13 *For the continuous time system of Example 2 we have that the first coefficient of the zero polynomial is $a_3 = \langle \boldsymbol{\gamma}^\top \boldsymbol{\beta} \rangle = -85$ and the first coefficient of the zero polynomial of the corresponding discretised system of Example 12 is given by Equation 8.10. The process of sampling may send a zero to infinity if $\hat{a}_3 = \boldsymbol{\gamma}\mathfrak{C}_2(\Xi)\boldsymbol{\beta} = 0$. This is equivalent to :*

$$-\tfrac{37}{4}\xi_1\xi_2 - \tfrac{37}{4}\xi_1\xi_3 - \tfrac{201}{4}\xi_1\xi_4 - \tfrac{201}{4}\xi_1\xi_5 + \tfrac{15}{2}\xi_2\xi_3 - 4\xi_2\xi_4 + \tfrac{17}{2}\xi_2\xi_5 + \tfrac{17}{2}\xi_3\xi_4 - 4\xi_3\xi_5$$
$$+i\left(\tfrac{71}{4}\xi_1\xi_2 - \tfrac{71}{4}\xi_1\xi_3 - \tfrac{13}{4}\xi_1\xi_4 + \tfrac{13}{4}\xi_1\xi_5 - 28\xi_2\xi_4 - 3\xi_2\xi_5 + 3\xi_3\xi_4 + 28\xi_3\xi_5\right) = 0$$

The above holds if and only if the following conditions hold true,

$$-37\xi_1\xi_2 - 37\xi_1\xi_3 - 201\xi_1\xi_4 - 201\xi_1\xi_5 + 30\xi_2\xi_3 - 16\xi_2\xi_4 + 34\xi_2\xi_5 + 34\xi_3\xi_4 - 16\xi_3\xi_5 = 0 \quad (8.11)$$

$$71\xi_1\xi_2 - 71\xi_1\xi_3 - 13\xi_1\xi_4 + 13\xi_1\xi_5 - 112\xi_2\xi_4 - 12\xi_2\xi_5 + 12\xi_3\xi_4 + 112\xi_3\xi_5 = 0 \quad (8.12)$$

The above conditions describe an algebraic variety (set of points defined as solutions of polynomial equations) with the further condition that the ξ_i must express their origin from the discretisation process that is: As it follows from (8.5) we have that $\xi_i = T$, if $\lambda_i = 0$, or

$\xi_i = \frac{e^{\lambda_i T}-1}{\lambda_i}$ if $\lambda_i \neq 0$. Investigating the existing of solutions of (8.9) is a difficult thing to test and thus we examine the special cases where collapsing occurs. Note that for $\xi_i = 0$, we must have $\lambda_i \neq 0$ and $e^{\lambda_i T} = 1$, which according to Proposition 29 and Definition 37 means that λ_i is located on the imaginary axis ($\lambda_i = 0 \pm j\omega$) and a real collapsing occurs for the sampling period T or $\lambda_i \in \Phi_0(A)$, where:

$$\Phi_0(A) = \{\forall \lambda_i \in \Phi(A), \mathrm{Re}(\lambda_i) = 0, \lambda_i \neq 0\} \tag{8.13}$$

For this specific example, since there are no eigenvalues on the imaginary axis for the continuous model, none of the ξ_i's is zero and thus conditions (8.11) and (8.12) are those needed to specify the values the values of T for which collapsing may occur. The analysis so far, as it is demonstrated by the Example 13 reveals the following properties regarding the migration of zeros at infinity as a function of sampling.

Theorem 54 *For a continuous square system that has $n-m$ finite invariant zeros the following properties hold true:*

(a) The discretised system has $k-1$ zeros migrating to infinity for some value of the sampling period T if and only if

$$\begin{gathered}
\left\langle \hat{\gamma}^{\mathsf{T}} \hat{\beta} \right\rangle = 0 \\
\left\{ \sum_{\omega_1} \hat{\lambda}_{\omega_1} \left\langle \hat{\gamma}^{\mathsf{T}}_{\omega_1)} \hat{\beta}^{\omega_1)} \right\rangle \right\} = 0 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\left\{ \sum_{\omega_j} \hat{\lambda}_{\omega_j} \left\langle \hat{\gamma}^{\mathsf{T}}_{\omega_j)} \hat{\beta}^{\omega_j)} \right\rangle \right\} = 0 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\left\{ \sum_{\omega_{n-m}} \hat{\lambda}_{\omega_{n-m}} \left\langle \hat{\gamma}^{\mathsf{T}}_{\omega_{n-m})} \hat{\beta}^{\omega_{n-m})} \right\rangle \right\} = 0
\end{gathered} \tag{8.14}$$

(b) For the generic value of the sampling period T the discretised model has also $n-m$ finite zeros. □

The result readily follows from the expression of the discretised zero polynomial. In fact, conditions (8.14) are equivalent to a set of homogeneous polynomial equations in ξ_i, which define a variety (proper variety of the corresponding projective space) and this leads to that for a generic T equation (8.14) cannot be satisfied.

For the case of continuous systems with eigenvalues on the imaginary axis, simple, or multiple collapsing of pure imaginary eigenvalues may occur to 1 (as this has been established in Chapter 3). This leads to that subset of the ξ_i's in the variety described by (8.14) becoming zero and as a result we may get migration of finite zeros to infinity, or even total system degeneracy. This is demonstrated by the following example.

Example 14 *Consider the continuous system $S(A, B, C)$:*

$$A = \begin{bmatrix} -6.0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3.0 & -3.0 & 1.0 \\ 0 & -1.5 & 0 & -1.0 & 1.5 \\ 0 & 1.5 & 1.0 & 0 & 1.5 \\ 0 & -1.0 & -3.0 & -3.0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 2 & 0 \\ 0 & -1 \\ 1 & 3 \\ 2 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} -7 & 0 & -2 & 0 & 0 \\ 3 & -6 & -3 & -1 & -2 \end{bmatrix}$$

with the following simple structure, Jordan form of A (all the four complex eigenvalues are located in the imaginary axis) :

$$J = \begin{bmatrix} -6 & 0 & 0 & 0 & 0 \\ 0 & -2i & 0 & 0 & 0 \\ 0 & 0 & 2i & 0 & 0 \\ 0 & 0 & 0 & 4i & 0 \\ 0 & 0 & 0 & 0 & -4i \end{bmatrix} = VAU$$

or $\Phi(A) = \{-6, -2i,\ 2i,\ 4i, -4i\}$, $\Phi_0(A) = \{-2i,\ 2i,\ 4i, -4i\}$ *and where,*

$$V = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1+i & -1-i & -i \\ 0 & 1 & -1-i & -1+i & i \\ 0 & 1 & 1-i & 1+i & -i \\ 0 & 1 & 1+i & 1-i & i \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & -\frac{1}{8}-\frac{1}{8}i & -\frac{1}{8}+\frac{1}{8}i & \frac{1}{8}+\frac{1}{8}i & \frac{1}{8}-\frac{1}{8}i \\ 0 & -\frac{1}{8}+\frac{1}{8}i & -\frac{1}{8}-\frac{1}{8}i & \frac{1}{8}-\frac{1}{8}i & \frac{1}{8}+\frac{1}{8}i \\ 0 & \frac{1}{4}i & -\frac{1}{4}i & \frac{1}{4}i & -\frac{1}{4}i \end{bmatrix}$$

Consequently the parameters $\mathcal{B}$ and Γ of the Jordan equivalent system are,

$$\mathcal{B} = VB = \begin{bmatrix} 0 & 1 \\ 1-3i & -2-4i \\ 1+3i & -2+4i \\ 3-i & 2+4i \\ 3+i & 2-4i \end{bmatrix}, \quad \Gamma = CU = \begin{bmatrix} -7 & \frac{1}{4}+\frac{1}{4}i & \frac{1}{4}-\frac{1}{4}i & -\frac{1}{4}-\frac{1}{4}i & -\frac{1}{4}+\frac{1}{4}i \\ 3 & -1-\frac{1}{4}i & -1+\frac{1}{4}i & -2-\frac{3}{4}i & -2+\frac{3}{4}i \end{bmatrix}$$

The zero polynomial of the above system, is of the form,

$$z(s) = a_3 z^3 + a_2 z^2 + a_1 z + a_0$$

As $\mathcal{B}$ and Γ are identical to the corresponding parameters $\mathcal{B}$ and Γ of the continuous time system of Example 2 it is also $a_3 = \langle \gamma^\top \beta \rangle = -85$.

From the calculation of the remaining coefficients, as in the case of Example 2 we have $a_2 = 377.0,\ a_1 = -252.0,\ a_0 - 1180.0$.

Let us now assume that the above continuous time system is discretised with ZOH. Then, the corresponding matrix A has a simple structure Jordan form, that is,

$$\Xi = \operatorname{diag}\left\{ \frac{e^{-6T}-1}{-6}, \frac{e^{(-2.0i)T}-1}{-2.0i}, \frac{e^{(2.0i)T}-1}{2.0i}, \frac{e^{(4.0i)T}-1}{4.0i}, \frac{e^{(-4.0i)T}-1}{-4.0i} \right\}$$

and

$$\xi_1 = \frac{e^{-6T}-1}{-6}, \xi_2 = \frac{e^{(-2.0i)T}-1}{-2.0i}, \xi_3 = \frac{e^{(2.0i)T}-1}{2.0i}, \xi_4 = \frac{e^{(4.0i)T}-1}{4.0i}, \xi_5 = \frac{e^{(-4.0i)T}-1}{-4.0i}$$

As in the previous Example 12 we have the following expressions for the coefficients of the zero polynomial :

$$\hat{a}_3 = \boldsymbol{\gamma}\mathfrak{C}_2(\Xi)\boldsymbol{\beta} \tag{8.15}$$

where

$$\boldsymbol{\beta} = \begin{bmatrix} -1+3i & -1-3i & -3+i & -3-i & 20i & 24+8i & -8+4i & -8-4i & 24-8i & -28i \end{bmatrix}^T$$

$$\boldsymbol{\gamma} = \begin{bmatrix} \frac{25}{4}+i & \frac{25}{4}-i & \frac{59}{4}+6i & \frac{59}{4}-6i & -\frac{3}{8}i & -\frac{1}{2}-i & -1-\frac{1}{8}i & -1+\frac{1}{8}i & -\frac{1}{2}+i & \frac{5}{8}i \end{bmatrix}$$

then

$$\begin{aligned} \hat{a}_3 \;=\; & -\frac{37}{4}\xi_1\xi_2 - \frac{37}{4}\xi_1\xi_3 - \frac{201}{4}\xi_1\xi_4 - \frac{201}{4}\xi_1\xi_5 + \frac{15}{2}\xi_2\xi_3 - 4\xi_2\xi_4 + \frac{17}{2}\xi_2\xi_5 + \frac{17}{2}\xi_3\xi_4 - 4\xi_3\xi_5 + \\ & +i\left(\frac{71}{4}\xi_1\xi_2 - \frac{71}{4}\xi_1\xi_3 - \frac{13}{4}\xi_1\xi_4 + \frac{13}{4}\xi_1\xi_5 - 28\xi_2\xi_4 - 3\xi_2\xi_5 + 3\xi_3\xi_4 + 28\xi_3\xi_5\right) \end{aligned}$$

The conditions for migration of a zero to infinity is $\hat{a}_3 = 0$ or equivalently,

$$\begin{aligned} -37\xi_1\xi_2 - 37\xi_1\xi_3 - 201\xi_1\xi_4 - 201\xi_1\xi_5 + 30\xi_2\xi_3 - 16\xi_2\xi_4 + 34\xi_2\xi_5 + 34\xi_3\xi_4 - 16\xi_3\xi_5 &= 0 \\ 71\xi_1\xi_2 - 71\xi_1\xi_3 - 13\xi_1\xi_4 + 13\xi_1\xi_5 - 112\xi_2\xi_4 - 12\xi_2\xi_5 + 12\xi_3\xi_4 + 112\xi_3\xi_5 &= 0 \end{aligned}$$

We note the following:

1. If collapsing of $\hat{\lambda}_2, \hat{\lambda}_3$ occurs, then $\xi_2 = 0, \xi_3 = 0$ and the above conditions become

   $$-201\xi_1\xi_4 - 210\xi_1\xi_5 = 0 \Rightarrow \xi_1(\xi_4 - \xi_5) = 0, \;\; -13\xi_1\xi_4 + 13\xi_1\xi_5 = 0 \Rightarrow \xi_1(\xi_4 - \xi_5) = 0$$

   Given that $\xi_1 \neq 0$, it follows that $\xi_4 = \xi_5$ which means that $\hat{\lambda}_4, \hat{\lambda}_5$ must collapse to a real value (since they are complex conjugate) and this implies that $\xi_4 = \xi_5 = 0$. With these values we can compute the zero polynomial of the discretised system. In fact let the irregular values of the sampling period $T = \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \pi$. Then the corresponding values of the coefficient and the zeros are summarized by the following Table:

T	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π
$\hat{\lambda}_1$	8.9833×10^{-3}	1.8674×10^{-3}	8.07×10^{-5}	6.5124×10^{-9}
$\hat{\lambda}_2$	$-1.0i$	$-.5 - .86603i$	-1.0	1.0
$\hat{\lambda}_3$	$+1.0i$	$-.5 + .86603i$	-1.0	1.0
$\hat{\lambda}_4$	-1.0	$-.5 - .86603i$	1.0	1.0
$\hat{\lambda}_5$	-1.0	$-.5 + .86603i$	1.0	1.0
ξ_1	$.16517$	$.16636$	$.16665$	$.16667$
ξ_2	$.5 - .5i$	$.43301 - .75i$	$-1.0i$	0
ξ_3	$.5 + .5i$	$.43301 + .75i$	$1.0i$	0
ξ_4	$.5i$	$-.21651 + .375i$	0	0
ξ_5	$-.5i$	$-.21651 - .375i$	0	0
$\hat{a}_3$	6.316	26.339	13.416	0
$\hat{a}_2$	2.6751	6.1647	-20.917	0
$\hat{a}_1$	$.27984$	-7.4792	1.5851	0
$\hat{a}_0$	5.0918	2.5797	5.9155	0
$\hat{z}_1$	$-.72848$	$-.76896$	1.0046	$-$
$\hat{z}_2$	$.28226 - .59053i$	$.26746 - .2363i$	$-.44092$	$-$
$\hat{z}_3$	$.28226 + .59053i$	$.26746 + .2363i$	$.99547$	$-$

In the above table of arithmetic results, it must be noted that :

(a) For the irregular values of $T = \frac{\pi}{4}$, and $\frac{\pi}{3}$ for which, no real collapsing occurs, all the coefficients of the zero polynomial are non zero and there exist no one migration of zeros to infinity.

(b) For the irregular value of $T = \frac{\pi}{2}$, we have a real collapsing between the eigenvalues $\hat{\lambda}_2$ and $\hat{\lambda}_3$ to -1 and a real collapsing between the eigenvalues $\hat{\lambda}_4$ and $\hat{\lambda}_5$ to 1. We have $\xi_4 = \xi_5 = 0$ but as ξ_1, ξ_2 and ξ_3 are different from zero, the matrix $\mathfrak{C}_2(\Xi)$ remains different from zero and so is $\hat{a}_3$ and so there is no one migration of zeros to infinity.

(c) For the irregular value of $T = \pi$, we have a real collapsing between the eigenvalues $\hat{\lambda}_2$, $\hat{\lambda}_3$, $\hat{\lambda}_4$ and $\hat{\lambda}_5$ to 1. As ξ_2, ξ_3, ξ_4, ξ_5 becomes zero, the matrix $\mathfrak{C}_2(\Xi)$ becomes also zero. If the matrix $\mathfrak{C}_2(\Xi)$ becomes zero, then all the submatrices of $\mathfrak{C}_2(\Xi)$ becomes also zero and consequently all the coefficients of the zero polynomial becomes also zero and this leads to complete degeneration of the zero polynomial, i.e. it becomes identically zero.

2. Consider the following regular values of the sampling period : $T = \frac{\pi}{9}, \frac{\pi}{7}, \frac{\pi}{5}$. We proceed to the numerical calculation of the coefficients for the above system and the results are summarized below:

T	$\frac{\pi}{9}$	$\frac{\pi}{7}$	$\frac{\pi}{5}$
$\hat{\lambda}_1$	$.12314$	6.7692×10^{-2}	2.3054×10^{-2}
$\hat{\lambda}_2$	$.76604 - .64279i$	$.62349 - .78183i$	$.30902 - .95106i$
$\hat{\lambda}_3$	$.76604 + .64279i$	$.62349 + .78183i$	$.30902 + .95106i$
$\hat{\lambda}_4$	$.17365 + .98481i$	$-.22252 + .97493i$	$-.80902 + .58779i$
$\hat{\lambda}_5$	$.17365 - .98481i$	$-.22252 - .97493i$	$-.80902 - .58779i$
ξ_1	$.14614$	$.15538$	$.16282$
ξ_2	$.32139 - .11698i$	$.39092 - .18826i$	$.47553 - .34549i$
ξ_3	$.32139 + .11698i$	$.39092 + .18826i$	$.47553 + .34549i$
ξ_4	$.2462 + .20659i$	$.24373 + .30563i$	$.14695 + .45225i$
ξ_5	$.2462 - .20659i$	$.24373 - .30563i$	$.14695 - .45225i$
$\hat{a}_3$	$.6455$	3.0518	9.515
$\hat{a}_2$	4.3788	2.8526	$.84229$
$\hat{a}_1$	-8.0737	-6.8808	-1.7602
$\hat{a}_0$	5.1331	6.2513	6.4125
$\hat{z}_1$	-8.3878	-2.3013	$-.97962$
$\hat{z}_2$	$.8021 - .55199i$	$.68327 - .65058i$	$.44555 - .6996i$
$\hat{z}_3$	$.8021 + .55199i$	$.68327 + .65058i$	$.44555 + .6996i$

$\square$

The above example demonstrates that there exist cases of collapsing which lead not only to zeros migrating to infinity but even making the resulting discretised system degenerate. This is summarized as follows:

Corollary 9 *For continuous time systems with eigenvalues on the imaginary axis, there exist values of sampling that lead to discretised models for which zeros may migrate to infinity, or becoming degenerate.* □

The emergence of such phenomena is due to the event that a number of ξ_i becoming zero; in this case migration of zeros to infinity, or degeneracy is independent from the numerical values of the B, C matrices and they depend on the original eigenvalue pattern. We shall refer to such zero transformations as *structural transformations due to collapsing.*

8.3 Conclusions

The zero polynomial of the discretised square system has been defined; this expression allows the further study of discretised zeros under sampling and leads to conditions characterizing the migration of zeros at infinity as a function of the sampling. For the case of irregular sampling, it has been shown through examples that drastic changes to the overall system may occur for certain types of systems, which may even lead to total system degeneracy. The study here is of preliminary nature and the direction of more explicit results has to use the explicit structure of the Segré characteristic of the open loop system.

Chapter 9

CONCLUSIONS, FURTHER WORK

The problem of investigating the effect of sampling rate in the discretisation of continuous time linear systems under Zero Order Hold devices has been the main subject of this thesis. The main objective was to initiate research in the area of Model Based Theory of Sampling, which may act as a complement to the classical Signal based theory of Shannon [Sha., 1] and thus it has a significant role to play in the development of modern Computer Control methodologies. As such, the work here belongs, to the general area of "*Implementation of Digital Schemes*".

The motivation for the study undertaken here has been the original work by Kalman [Kal. Ho & Nar.] on the loss of controllability under certain values of sampling. This initial observation has been fully developed here and has led to the classification of sampling rates to regular and irregular. The main part of the work here has been the study of the effects of irregular sampling on a number of structural properties, such as Segrê characteristics, controllability, observability, Zero polynomial. As such , the results in this thesis form part of the study of the mapping of model based properties from the continuous time (A, B, C) model to the discrete time $(\hat{A}, \hat{B}, \hat{C})$ model as function of the sampling rate. More specifically, the following type of results have been derived..

The basis of the approach established here has been the classification of the set of distinct eigenvalues into groups having the same real part. For such sets it is shown that collapsing of eigenvalues under sampling may occur, if certain relationships hold between the imaginary parts of the eigenvalues and the sampling rate is appropriately chosen. Collapsing characterizes the case, where two distinct eigenvalues of the continuous model become equal for the discretised model; the special value of such sampling has been defined as irregular, whereas all other values are called regular. Under irregular types of sampling, phenomena such us merging of Segré Characteristics, generalized eigenspaces and corresponding Jordan forms merge and a detailed study of such phenomena has been given. The results in this area completely characterize the mapping of the structural properties from A to $\ddot{A}$.

The study of structural properties of the mapping from (A, B) to $(\hat{A}, \tilde{B})$ and (A, C) to $(\ddot{A}, \hat{C})$ has been considered next using the fundamental properties of the irregular sampling. It has been shown that regular sampling preserves the controllability and observability, but this is not necessarily the case for irregular sampling. The results in this area also indicate that the classical duality between controllability and observability do not completely carry over, as duality between properties of the discretised model.

The effect of collapsing under irregular sampling on controllability, has been examined in detail. This has led to the emergence of two distinct forms of loss of controllability; the first is of structural nature and depends on the merging of Segré characteristics, whereas the second depends on the numerical parameters of the corresponding model. Similar results are also established for observability, but their derivation is of simpler nature.

The effects of irregular sampling on controllability, observability of the discretised model has been further expanded by developing additional results for the problems of determining the dimension of controllable subspace (unobservable subspace), as well as determining the degrees of the newly formed input (output) decoupling zeros under irregular sampling. The first problem is based on the use of the cyclic invariant subspaces of $\ddot{A}$ and leads to tests defining the dimension of the corresponding spaces. Determining the degrees of the newly formed o.d.z., i.d.z. of the discretised model under irregular sampling is based on some new characterization of such zeros for linear systems. The spectral characterization of controllability (observability) together with results for the determination of degrees of divisors using properties of Piecewise Arithmetic Progression Sequences lead to a simple new test for determining the Segré characteristic of the newly formed decoupling zeros. These two types of results complement each other and complete the study of loss of controllability, observability under irregular sampling.

The last area examined in the thesis is that related to zeros of the discretised model. The derivation of an explicit form of the zero polynomial in terms of the state space parameters allows us to relate the effects of structural transformation of $\hat{A}, \hat{B}$ on the zero polynomial and thus provides a useful framework for studying the mapping of zeros problem from the continuous to the discrete domain. The effect of irregular sampling on the migration of finite zeros to infinity has been examined and it was demonstrated that for certain types of systems and sampling rate, finite zeros of the continuous model move to infinity. In certain cases, irregular sampling may even lead to total system degeneracy. A variety that characterizes the loss of zeros to infinity has been defined. The results here are of preliminary nature and by no means complete the study of the zero mapping problem, which from many aspects is still open.

The model based theory of sampling is an open area and there is a number of open issues which are subjects for future research. Amongst the topics of interest are:

(a) Investigate the effects of irregular sampling on the values of dynamic indices, such as controllability, observability, output nulling indices.

(b) Provide a detailed investigation of the zero mapping problem using the already derived expression for the zero polynomial.

(c) Examine the effects of irregular sampling on transfer function invariants such us Plucker matrices.

The above family of problems, as well as those considered in this thesis, deal with the mapping of invariants and the associated properties. Another family of problems is linked to the transformation of design indicators, that is:

(d) Study the effect of regular and irregular sampling on property, design indicators such as Nyquist, Bode diagrams, singular values, condition numbers etc.

Although the values of irregular sampling is a set with specific values and such nongeneric cases can be avoided, what happens to the system properties when the sampling is regular, but is value is "*close*" in some sense to irregular sampling values is an important issue. This leads to the following interesting family of problems:

(e) Investigate the system property indicators, such as degree of controllability, observability, Sensitivity properties of the discretised model etc., when the sampling rate is regular, but its value is close to some irregular sampling value.

The above are some of the topics for further research which form a natural extension of the structural methodology and approach developed here.

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