A STOCHASTIC APPROACH TO FIND OPTIMUM REINSURANCE ARRANGEMENTS FOR LIFE INSURANCE COMPANIES ON THE BASIS OF MAXIMIZING THE UTILITY OF RETURNS

by

Pedro José P. R. da S. Gonçalves

Submitted for the degree of

Doctor of Philosophy at City University

Department of Actuarial Science and Statistics,

November 2000

Table of Contents

List of Tables

List of Figures

Acknowledgements

I would like to acknowledge especially the support and guidance provided by my Supervisor Dr Robert Chadburn from the Department of Actuarial Science and Statistics of the City University. A happy ending would not have been possible without all the help, especially during the most difficult moments, he provided.

I would also like to acknowledge all the support and understanding I have received during the entire duration of this project from my wife, parents and sister.

Declaration

I hereby grant powers of discretion to the University Librarian to allow the thesis to be copied in whole or in part without further reference to the author. This permission covers only single copies made for study purposes, subject to normal conditions of acknowledgements.

Abstract

The choice of the retention level in life assurance has always been a polemic subject. In most cases, companies will choose the amount that has stand the test of time.

In this project we will determine the optimal retention level that maximizes the utility of returns for a life insurance company. The approach used is to asses the utility of the extra return obtained by the shareholders, by utilizing their capital in support of the life portfolio's risk as opposed to investing it in a risk free way. The required level of capital is calculated using a ruin probability approach. It is also linked with the exposure to risk, and therefore it allows for lower levels of capital when, through the reinsurance treaty, risk is passed to the reinsurer.

Two approaches were considered. The first, uses an analytical approach and looks at a one-year scenario. The second, looks at a multi-year scenario by using a stochastic approach. Both scenarios look at portfolio of n-year term assurance policies where all policyholders share the same characteristics: age, term of policy, distribution of sum assured, assumptions of the reinsurance treaty, etc.. Variations in the initial set of assumptions are considered and the effect on the optimal retention level analysed.

The results obtained have shown that utility theory can be a way in which capital and reinsurance combinations can be chosen such that shareholders interests are optimized, while ensuring policyholders interests continue to be met. When we were trying to assess the retention level that optimizes expected utility of profits, while still imposing a ruin constraint, we were balancing both policyholders and shareholder´s interests. Different optimal retentions and capital levels were obtained as a function of the portfolio characteristics.

Also, we showed possible ways in which parameter risk can be allowed for. We would also note the relative insignificance of process risk compared with parameter risk. The effect of introducing this additional degree of risk (model and/or parameter), on retention level was dramatic. In this, the increase in risk was so dramatic, that even if we were to take into consideration for the trade-off, that must have occured, both capital and reinsurance needed to be increased. Therefore parameter risk is a much larger component of total risk, having a more significant effect on retention levels.

1. Introduction

The most difficult part of formulating a reinsurance program, is the fixing of retention limits. A retention has been defined as (Carter (1979)):

> "The amount that the company can and wants to put at stake for its own account when underwriting a single or group of risks".

Setting the amount will depend on the company's objectives. A possible approach, is to make use of a utility function through the decision making process. This way, the insurer will choose the reinsurance program that maximizes the utility of returns. Another approach can be based on a balance between expected profit and safety. A possible measure of safety, is the probability of ruin. Nevertheless, different objectives will set different figures.

An insurer needs therefore to take into consideration many factors when setting a retention limit.

Despite the mathematical models available that could be used, retention limits still tend to be fixed with the values that market practice has shown to stand the test of time.

Carter (1979) states that there is certainly a relation between a company's size and its capability to absorb a one-year's death strain. The same author presents a relation found by Mercantile and General Reinsurance Company between a company's net premium income $\boldsymbol{X}$ and its overall retention $\boldsymbol{Y}$. The relation found was:

$$\log Y = 1.7034 + 0.449 \log X \tag{1.1}$$

It can be seen that the net premium income grows faster than the retention, which makes sense in the way that the company needs to become first financially stronger in order to be capable to absorb higher levels of aggregate losses.

Spedding (1989) addresses the problem of the retention level by focusing initially on the coefficient of variance for a group of N lives, all subject to an assumed mortality rate q and a sum at risk S. A simple analysis of the coefficient of variation suggests that higher retention levels should be chosen for older lives. Spedding (1989) also states that, before choosing the retention level, the insurer should estimate the total expected mortality costs at different retention levels. The retention level is then determined by taking into account the amount of free assets and by setting a low probability of ruin.

The Swiss Reinsurance Co. published the following rule, for maximum retention per loss:

(i)	As a percentage of capital and free reserves (retention per risk and per loss)	1-5%
(ii)	As a percentage of retained premium income for the class of business	1-10%
(iii)	As a percentage of the company's liquid assets (retention per risk and per loss)	500%

If we compare the retentions coming out from (1.1) and compare them with the ones from the rules above, for the same net premium income, it can be seen that many do not match. So certainly other factors are of influence and are taken into account when fixing retentions.

The need for reinsurance lies in the problem of the variation of the aggregate claims cost, when compared with the expected cost, due to random fluctuations.

Let us consider an insurer that writes 2,000 one-year-term assurance policies each with £100 sum assured and where all policyholders are aged 45 and with an expected mortality rate of 0.0040. If expenses and interest are ignored, the insurer will charge each policyholder a premium of 0.40. The total premium income of 800 will be enough to pay the 8 expected claims. If less than 8 die, the insurer will have a mortality profit, whereas if more than 8 die it will suffer a loss.

It can be said, that the ability of an insurer to withstand the variations of the mortality experience, is a function of its capital. Moreover, the size of the capital an insurer will need is related to the probability distribution of its expected claims which will depend on the nature and size of its portfolio of business. As the company increases the size of its business, provided its portfolio is composed of independent risk units, its claims results will tend to become more stable and its probability of ruin smaller for any given level of capital. Even if a portfolio is composed of heterogeneous exposure units, as the number of units is increased loss experience will tend to become more stable *provided* loss exposures are independent (Carter (1979)).

The capital can be provided either by the shareholders or by a premium loading. A premium loading, at first sight, may not be the best solution because it would increase premiums, which could mean in a competitive market to lose business to competitors. Still, the shareholders expect a rate of return on their investment, and so increasing capital requires also an increase in premium loadings in order to service capital.

Carter (1979) shows the relation between the level of the reserves and the probability of ruin assuming that the random variable representing the number of claims from a portfolio of x lives is a binomial with parameters x and expected mortality rate replaced with the one from the example above. A similar example is shown in Table 1.1.

Table 1.1: Relationship between the number of claims and Probability of Ruin

On a portfolio of 2.000 lives			On a portfolio of 4.000 lives		
Number of claims		Cumulative	Number of Claims		Cumulative
Number	As % of expected number	Probability	Number	As % of expected number	probability
8	100.0	0.59222	16	100.0	0.56596
9	112.5	0.71654	18	112.5	0.74268
10	125.0	0.81595	20	125.0	0.86862
11	137.5	0.88180	22	137.5	0.94213
12	150.0	0.93626	24	150.0	0.97791
13	162.5	0.96579	26	162.5	0.99265
14	175.0	0.98262	28	175.0	0.99786
15	187.5	0.99157	30	187.5	0.99945
16	200.0	0.99602	32	200.0	0.99987

If the insurer only charges its policyholders the risk premiums, then in both cases it would face a probability of ruin in excess of 40%. In order to reduce the probability of ruin to 0.4%, for a portfolio of 2,000 lives, it would need to double it reserves in order to cover twice the expected number of deaths. This would mean an additional amount of 800. However, for a portfolio of 4.000 lives it only needs to increase the reserves by 75% to satisfy the required ruin probability. If the reserves are borrowed at a rate of interest of 7.5%, the insurer would need to include a loading of 7.5% on a portfolio of 2,000 lives and of 5.63% on a portfolio of 4,000 lives, in order to service the loan.

Let's us now see, on the example above, how reinsurance can help when a 50% quota share reinsurance on original terms is considered. In return for accepting the liability, the reinsurer will receive 50% of the original premium and pay the same share of the claims. The insurer position, for a portfolio of 2.000 lives, will change as follows:

	100% retention	50% retention
Total resources		
Capital	800	800
+ retained premium income	860	430
	1,660	1,230
Expected claims cost	800	400
Balance	860	830

The insurer's total resources are now enough to cover 24 deaths, which means that the probability of ruin is now 0.033%, based on the distribution function. But since the reinsurer took half of the premium loading it has available only 30 when he needs 60. Usually the reinsurer is willing to accept a smaller loading. Let's us assume a reinsurance treaty with a loading of 3%. The insurer's retained premium income would now be 448, which after the claims cost would leave enough money to service a capital of 640. For this amount of capital the probability of ruin would be 0.036%, based on the distribution function.

Although the simplest form of reinsurance has been used above, the same basic factors apply to the determination of retentions for other types of reinsurance too. Again Carter (1979), identifies that retentions are a function of the following variables:

(i) Size of the portfolio (*N*);

(ii) The number *n* of exposure units of size *x* included in the portfolio;

(iii) Probability of an exposure unit of size *x* incurring a loss in time t;

(iv) The size of loss z if a loss occurs;

(v) Ratio of capital and reserves to *N* (*A*);

(vi) Rate of return payable on *A*;

(vii) Premium loading;

(viii) Selected probability of ruin;

(ix) Price payable for reinsurance;

(x) Type of reinsurance;

(xi) The company's investment policy.

Now that the variables have been identified, it is easy to understand that the choice of a retention is a difficult task for every insurer. To illustrate the decision process, we have used in the above example, an objective of a balance between expected profit and safety. Another possible approach, is to make use of a utility function through the decision making process. This way, the insurer will choose the reinsurance program (retention level) that maximizes the utility of returns.

The purpose of this thesis is to show how the optimal retention level can be chosen by maximizing the utility of returns for a life insurance company. The approach used is to assess the utility of the extra return obtained by the shareholders, considering hypothetical real life situations and by utilizing their capital in support of the life portfolio's risk as opposed to investing it in a risk free way.

The required level of capital is calculated using a ruin probability approach. It is also linked with the exposure to risk, and therefore it allows for lower levels of capital when, through the reinsurance treaty, risk is passed to the reinsurer.

Chapter 2 starts with a description of the main features of a life office, and in particular what makes a life assurance company different from any other business company. It is followed by a brief description of the most common life assurance products. Afterwards, it describes the concept of life office risk management and finally by introducing the concept of the Individual Risk Model, it looks at the distribution of the total or aggregate claims amounts for a life office. The latter will be needed in Chapter 4.

Chapter 3, covers Utility Theory, which is the basis for the assessment of the optimal retention level that maximizes the returns. It begins with an historical introduction starting with Aristotle and then describes how utility theory changed through time. It is followed by a description of the most common utility functions and also by a critique of utility theory. It finishes with an overview of the key theoretical results regarding the determination of optimal rules for exchange of risks and constructing reinsurance treaties.

Chapters 4 and 5, initially describe the approach used in the study of the retention level that maximises the expected utility of returns, being these defined as the extra return obtained by shareholders when investing in an insurance company instead of in a risk free way. Two approaches were considered. The first, covered by Chapter 4, uses an analytical approach and looks at a one-year scenario. The second, in Chapter 5, looks at a multi-year scenario by using a stochastic approach. Both approaches look at a portfolio of n-year term assurance policies where all policyholders share the same characteristics: age and policy term. The portfolio is sub-divided into cohorts of different sums assured, where a pre-defined number of policyholders is assumed. The assumptions of the reinsurance treaty, particularly the premium rate, are also introduced and equal for the whole portfolio. Variations in the initial set of assumptions are considered and analysed the effect on the optimal retention level.

Chapters 4 and 5, focus on process risk, a component of the life office's total risk. In Chapter 6, the analysis is expanded by considering parameter risk and its implications on the optimal retention level. The approach used is identical to what is described for Chapter 5.

Chapter 7 introduces the general discussion of the results obtained in the previous chapters. It highlights the main points and gives an overview of the conclusions obtained in this project.

Chapter 8, lists all the sources referred in the text. The ninth and final chapter, the bibliography, lists all the sources studied in relation to the preparation of this thesis.

2. Life Insurance and Reinsurance

2.1. Introduction

As it was mentioned in Chapter 1, the purpose of this thesis is to show how optimal retention levels can be chosen by maximizing the utility of returns for a life insurance company. It is therefore important to develop an understanding of the main aspects of a life office.

Initially, the main characteristics of a life office are discussed (Section 2.2). In particular, the need for capital, an important concept because it will be taken into consideration, later on in the thesis, when choosing optimal retention levels that maximize the expected utility of returns. The performance of a life office and how it can be measured are also discussed. These topics are important, since we are considering using a measure of performance (returns), when choosing optimal retention levels.

Section 2.3, gives a brief description of the most common life assurance products, for some of them will later on (Chapters 5, 6 and 7) be considered, when defining the main characteristics of a model life portfolio, in order to assess optimal retention levels. Also important, is a description of the main concepts and forms of life reinsurance. The latter is the scope of Section 2.4.

Section 2.5 discusses the main sources of risk a life office faces and how they can be managed. In particular, it looks at process risk and parameter risk, the scope of Chapters 4, 5 and 6, respectively.

Finally in Section 2.6, we introduce an expression by Panjer & Willmott (1992), for the distribution of the aggregate claims amount for life insurance based on the individual risk model concept. This expression will be used in Chapter 4, when maximizing expected utility of returns.

2.2. Aspects of a life office

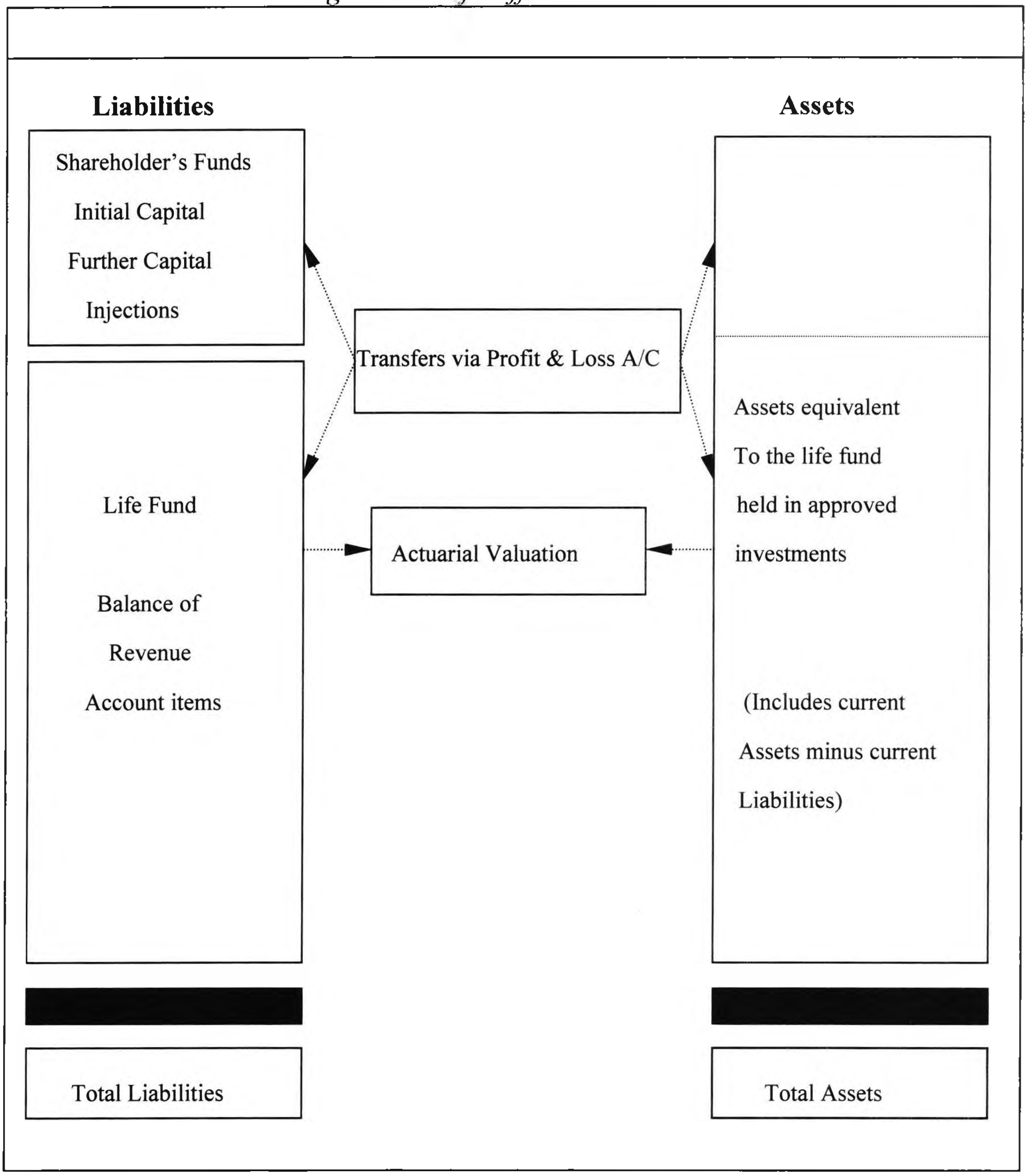
2.2.1. Characteristics of a life insurance company

A life office is a financial institution. The product it sells is primarily one which provides the client with a vehicle to ensure financial security over a period of time - which may be many years - both for the client and for his dependants.

At first sight, a life office will appear to operate very similarly to any other company selling a product to the public. It requires an administration system to deal with production and handling problems. The goods it produces will need to be marketed and its product range will have to fulfill the needs of the consumer. In order to operate, the life office will require working capital not only to cover the current production, but it will also need permanently invested capital.

Despite the similarities, a life office has some significant differences which make it uniquely different from a normal trading organization. A closer look at the balance sheet of a typical life assurance company will show that its financial structure differs in important ways from that of a normal trading company. Figure 2.1 shows a simple diagram of how a life office balance sheet is structured.

Figure 2.1: Life Office Balance Sheet



Looking at the liability side one can see that it is divided into two distinct parts: Shareholders' funds and Life fund (Policyholders' Fund).

In accounting terms, the policyholders' fund is nothing more than a bookkeeping entry and represents the accumulated balance of all previous revenue accounts. It is the up-to-date balance of all premiums minus all claims and expenses etc. paid to the life office together with the investment income of the life fund. The life fund is effectively held in trust for the benefit of the policyholders, and it exists to pay out the benefits when the policies eventually mature or become claims. No transfer can be made from the life fund unless an actuarial valuation has been made, which shows that assets are greater than liabilities. If for any reason the life fund assets are less than the life office's liabilities to the policyholders, the deficiency has to be made good from shareholder's funds. If there is an excess in the value of assets over the liabilities of the life fund, then an actuarial surplus arises which can be transferred as profits in whole or in part to shareholders.

The other important difference between a life office and a normal company lies in the way it operates and the effect it has on its cash flow.

Taking a normal trading company, when it sells a product it either receives the cash immediately or after a short delay. The payment it receives is used to cover its expenses and overheads, and the balance contributes towards profit.

The profit flow that a life office experiences after the sale of a policy is very different from what was described for a trading company. In this case, a negative profit flow will usually arise and instead of a positive contribution to profits, an immediate deficiency is created and shareholders' capital has to be transferred into the policyholders' fund. If the life office's policies are soundly constructed, then this capital may be returned after two or three years, but the shareholders are unlikely to earn a real profit on any one individual transaction for many years ahead. Such profits will be small at first and will emerge slowly over the future lifetime of the policy, which may be 10, 20 or even 50 years.

This feature of an immediate deficiency on the sale of a policy, slow recovery of capital and a long delay in profits is peculiar to a life office and is the second difference between a life office and almost any other type of company.

2.2.2. New business and need for capital

When a life office is first established, as well as when it sells new business, it needs initial capital to cover the costs of creating an administration system, establishing a sales force, recruiting management and training staff etc. During the first few years there will also be a period during which expenses will overrun the margins in the premiums until the volume of business written is sufficient to cover them. The demand on shareholders' funds will last longer than this initial period, on account of the financial strain writing new business puts on the life offices' capital requirements. A simple example of the revenue account for the first three years of a typical endowment plan is shown in Figure *2.2*.

Figure 2.2: Revenue Account for a Life Office

	Year 1		Year 2	Year3
	month1	In each month 2 - 12	In each month 13 – 24	In each month 25 - 36
Premium	25.00	25.00	25.00	25.00
Less				
Commission	12.50	12.50	0.62	0.62
Expected Expenses	180.00	1.75	1.75	1.75
Expected Mortality Cost	0.25	0.25	0.50	0.50
Add				
Expected Investment				
Earnings on Reserves	-	0.50	1.50	2.50
Cash Flow	-167.75	-11.00	23.63	24.63
Reserves Increase	11.00	11.00	11.00	22.00
Surplus	-178.75	-	12.63	2.63

A life office does not write one block of business in year 1 and let it run off, with the profit emerging over the years. It will continue to write new business in year 2, year 3 and so on. Each block of new business creates its own financial strain and the surplus from business written in the previous years may not be enough to cover the total strain from the new business written, while growth in new business occurs. This means that further transfers from the shareholders' fund are necessary in the first early years, in particular, for expanding portfolios. Writing new business, creates a strain that absorbs part or all of the surplus from previous years.

There will be a time when the surplus from previous years, will be enough to cover the strain from new business. At this time, the demand for shareholders' fund will stop, and they will start to recover slowly their capital back.

2.2.3. Measuring the performance of a life office

There are four main areas where the shareholders, regulators, policyholders, potential shareholders, potential policyholders require information on the progress of a developing life office:

(i) Is the life fund still sound and solvent despite changes that have happened in the recent past?

(ii) Is the life office still in a position to fulfill its long term objectives regarding capital requirements and profits?

(iii) Is the office performing well or badly compared to targets the directors and managers have set the office for the foreseeable future?

(iv) Is the day to day performance satisfactory and what is the trend of current performance?

All these four areas are interdependent, because an office can not meet its future objectives if its day to day performance is bad, as well if it is in financial difficulties.

The actuary's report is the most significant piece of information on the progress of the life office. The considerations that the actuary will include in this report are:

(i) The solvency of the fund at the valuation date;

(ii) The existing investments and matching requirements;

(iii) An outline of the factors affecting the fund - nature of contracts, guarantees, reinsurance arrangements, expense levels, mortality, marketing plans, etc..

This last analysis is of particular importance, as it is possible to evaluate from it, whether past performance is better or worse than previous forecasts.

Perhaps a logical way of viewing the long term future is to undertake a viability study:

(i) To estimate the capital requirements and the emergence of profits if the fund is closed to new business;

(ii) To assume that the fund is to be kept open for the foreseeable future (say five years or the length of the office's corporate plan), and then, closed to new business. The revised estimate of capital requirements and emergence of profits can be reassessed.

If the office's corporate plan is sufficient to improve the shareholders' profit expectations, then clearly the shareholders are best served by keeping the office open. If the position is not improved, then the corporate plan is inadequate and should be reconsidered and retested against various levels of expansion. In the unlikely event that no revised plan will improve expectations, serious consideration should be given to the alternative of closing the office to new business.

2.3. Life insurance products

2.3.1. Introduction

In terms of its age, the modern life insurance business is an infant when compared to many other industries. In terms of its size, however, the industry is among the world's largest. Life insurance products were not widely offered until the 1800s.

Insurance provides protection against some of the economic consequences of loss. Thus, insurance responds to the need of all persons for security. The insurance industry constantly designs, alters and updates its insurance policies to meet this need. However, despite these changes, the underlying purpose of these policies remains the same: providing economic protection against financial loss.

Life insurance provides a sum of money if the person who is insured dies while the policy is in force or the policyholder survives until maturity. This chapter develops some of the most common life assurance products. In what follows, a fixed interest rate is assumed.

2.3.2. Insurances payable at the moment of death

Let $\boldsymbol{T}$, represent the exact future lifetime of a person now aged x. This variable, due to its uncertainty, will be regarded as a random variable. The cumulative distribution function of $\boldsymbol{T}$ is given by:

$$F(t) = P[T \le t] = 1 -{}_t p_x \qquad (2.1)$$

for any t≥0 and where ${}_t p_x$ represents the probability of surviving until (x+t). Formulae for the expected value and variance of ***T***, see for example Bowers et al. (1986), are as follows:

$$E[T] = \int_0^\infty t.dF(t) = \int_0^\infty {}_t p_x dt \qquad (2.2)$$

$$V[T] = \int_0^\infty t^2.dF(t) - \left(E[T]\right)^2 = 2\int_0^\infty t.{}_t p_x.dt - \left(E[T]\right)^2 \qquad (2.3)$$

A **whole life insurance**, provides for a payment immediately after the death of the insured (x), at any time in the future. The present value of this benefit, is a random variable, ***g(T)***, that can be represented as:

$$g(T) = v^T, \quad v = \frac{1}{1+i} \qquad (2.4)$$

where ***i*** is the fixed interest rate. The expected value of g(T), can be derived by making use of the distribution of T ,being as follows:

$$E[g(T)] = E\left(v^T\right) = \int_0^\infty v^t.dF(t) = \bar{A}_x \qquad (2.5)$$

The variance is:

$$V[g(T)] = V\left[v^T\right] = \int_0^\infty v^{2t}.dF(t) - \left(\bar{A}_x\right)^2$$

$$= {}^2\bar{A}_x - \left(\bar{A}_x\right)^2 \qquad (2.6)$$

where ${}^2\bar{A}_x$ represents the second non-central moment of g(T). These results can be seen, for example, in Bowers et al (1986).

An **n-year term insurance**, provides a unit payment at the moment of death, only if the insured (x) dies during the n-year term of an insurance commencing at issue. The variable g(T), now becomes:

$$g(T)=\begin{cases} v^T & \text{if} \quad T \leq n \\ 0 & \text{if} \quad T > n \end{cases} \tag{2.7}$$

Expressions for the expected value and variance, can be seen in Bowers et al (1986), are:

$$E[g(T)] = \bar{A}^{\,1}_{x:\overline{n}|} \tag{2.8}$$

$$V[g(T)] = {}^2\bar{A}^{\,1}_{x:\overline{n}|} - \left(\bar{A}^{\,1}_{x:\overline{n}|}\right)^2 \tag{2.9}$$

Endowment policies, pay a benefit of 1 immediately on death if (x) dies within n years, or a payment of 1 at the end of n years if (x) survives for n years. The variable g(T), its expected value and variance, for an endowment, can be seen in Bowers et al (1986).

2.3.3. Insurances payable at the end of the year of death

In practice, the best information on the probability distribution of the random variable T is in the form of a discrete life table. This is the distribution of ***K***, the curtate future lifetime of an insured, when K is an integer. It can be said that:

$$K < T < K + 1 \tag{2.10}$$

With the aid of K, the models can be redefined to consider a payment at the end of the year of death. For example, consider a whole life policy for a person now aged x, where the death benefit is paid at the end of the year of death. The present value of this benefit is a function of K, that can be defined as:

$$g(K) = v^{K+1} \tag{2.11}$$

Hence:

$$E[g(K)] = E\left(v^{K+1}\right) = \sum_{k=0}^{\infty} v^{k+1} \cdot {}_{k|}q_x = A_x \tag{2.12}$$

$$V[g(K)] = V\left[v^{K+1}\right] = \sum_{k=0}^{\infty} v^{2(k+1)} \cdot {}_{k|}q_x - \left(A_x\right)^2$$

$$= {}^2A_x - \left(A_x\right)^2 \tag{2.13}$$

where ${}_{k|}q_x$ is the probability that (x) will survive for k years and will die in the following year. Expressions for the n-year term and endowment policies can be seen for example in Bowers et al (1986).

It is possible to establish a relation between insurances payable at the moment of death and at the end of the year of death. Bowers et al (1986) show for example the following relation:

$$\bar{A}_x = \frac{i}{\delta} A_x \tag{2.14}$$

where δ is the force of interest and it is assumed a uniform distribution of deaths over each year of age.

In what follows, we will focus on the discrete model. Expressions for the continuous model are easily obtained, or they can be seen in Bowers et al (1986).

2.3.4. Net Premiums

In the previous sections we discussed the actuarial present value of the payments of several forms of life insurance policies. In practice, life insurance is usually purchased in the form of a life annuity of ***gross premiums***. Gross premiums provide for the payment of the benefits and all the expenses related with the insurance. This section covers only net annual premiums that provide for the payment of benefits.

The present value of the loss to the insurer, represented by ***l(K,P)***, is a function of both ***K*** and ***P***, the net annual premium. This function for a whole life policy, issued to an insured (x), can be defined as:

$$l(K,P) = v^{K+1} - P.\ddot{a}_{\overline{K+1}|} \tag{2.15}$$

where $\ddot{a}_{\overline{K+1}|}$ is an annuity certain. For definition of an annuity certain, see for example Bowers et al. (1986). Its expected value and variance can be found in Bowers et al (1986), being equal to:

$$E[l(K,P)] = A_x - P.\ddot{a}_x \tag{2.16}$$

$$V[l(K,P)] = \left({}^2A_x - (A_x)^2\right).\left(1 + \frac{P}{d}\right)^2 \tag{2.17}$$

where d=1-v. There are several ways to determine P, one possible is that the expected profit to the insurance company should be zero. This gives:

$$P = P_x = \frac{A_x}{\ddot{a}_x} \tag{2.18}$$

Expressions for the net annual premium for the other models of life insurance, can be seen in Bowers et al (1986).

2.3.5. Reserves

In the previous section, we introduced the equivalence principle. Through it, the insured will pay a series of net premiums, equivalent at the time of issue of the policy, to the sum assured paid upon death.

After a period of time, this equivalence will disappear. The insured may still have or not to pay net premiums, but the insurer will always have to pay the sum assured upon death. A balancing item will be required, being a liability to the insurer and an asset to the insured. This balancing item is called the ***net premium reserve***.

Let us consider a whole life policy, issued to an insured (x), with net annual premium P_x. The reserve at the end of k years, will be represented by ${}_kV_x$. It is necessary to define a new random variable ***J***, as the curtate future lifetime of (x+k). The prospective loss, at the end of k years, ${}_k\boldsymbol{l(J)}$, is equal to:

$${}_kl(J) = v^{J+1} - P_x.\ddot{a}_{\overline{J+1}|} \tag{2.19}$$

The expected value (the reserve), in this case becomes:

$${}_kV_x = E\left[{}_kl(J)\right] = A_{x+k} - P_x.\ddot{a}_{x+k} \tag{2.20}$$

and

$$V\left[{}_kl(J)\right] = \left({}^2A_{x+k} - \left(A_{x+k}\right)^2\right)\left(1 + \frac{P_x}{d}\right)^2 \tag{2.21}$$

It can also be seen in Bowers et al (1986) a further development of this subject.

2.4. Life Reinsurance

2.4.1. Introduction

The proportion of total life assurance premiums which is reassured is small, when compared with most classes of non-life insurance. Life assurance has unique characteristics, that makes it completely different from non-life assurance. It differs in a great number of aspects and in particular:

(i) The long term character of most life policies;

(ii) The use of annual premiums;

(iii) The varying proportions of protection and savings elements in the different types of policies;

(iv) The pre-determined policy claim amount, as opposed to the uncertainty of non-life claims which are only determined by reference to incurred loss.

The primary demand for reinsurance is due to the mortality risk. Even though life insurers possess mortality tables that help to determine, with a certain degree of precision the number of expected deaths, this number is normally subject to random fluctuations. Still, this factor alone does not justify the need for reinsurance. The cost to a life office from a death is called the ***death strain***. The death strain, for any policy, is the difference between the sum assured plus any additional bonuses and the reserve for that policy. Therefore, the impact of mortality on a life office's results, depends not only on the number of deaths and its random fluctuations, but also on the incidence of claims with respect to the death strain at risk.

New life insurance companies tend to suffer from mortality experience due to the small dimension of its portfolio (high probability of ruin). This can also happen in large and established companies with classes with a small number of policyholders that suffer from the same problem. Still, the main problem for large companies lies with large policies that could alone reduce significantly the offices' results. Reinsurance can help, in different ways, to ease the impact of the mortality risk on the insurer's results.

Another reason for the demand of reinsurance is the new business strain. Since the costs involved in writing new business are so high, as it was mentioned above, some relief can be obtained by reassuring part of the new business.

In what follows, we will study the two main forms of reinsurance - ***proportional and non-proportional.***

2.4.2. Proportional reinsurance

This type of reinsurance can be arranged on two different terms. The first, usually called original terms, means that the reinsurer would be subject to the original terms of the policy, including surrender values, paid up values, bonus, etc.. The second would follow the premium scale that the reinsurer would apply to direct business. The latter means that the reinsurer's own terms, conditions and, when applicable bonuses apply to the reinsurance. Normally proportional reinsurance is arranged on original terms.

When the reinsurer accepts a liability, the insurer will pay a share of the premium. Usually a commission is allowed, being equal to the commission the ceding company pays its agent. If the ceding company does not use an agency force distribution, a commission can still be allowed to help the ceding office meet its expenses.

Through proportional reinsurance, the reinsurer accepts a fixed share of the liabilities assumed by the insurer. In the case of a quota share, a fixed proportion of all risks accepted by the insurer are ceded to the reinsurer. Under a surplus basis, only amounts accepted by the insurer above its own retention are ceded to the reinsurer.

Treaties on original terms are normally arranged on a surplus basis. Quota share are mostly used on group life, because this way the reinsurer will avoid antiselection. If the reinsurer were to use a surplus treaty, it would tend to get the older and less healthy elements of the population.

Another kind of proportional treaty is financed by risk premium method. It was designed to relieve the insurer from the impact of the death strain at risk. Each year the insurer will build up reserves for all policies in force. The basic idea is that part of the difference between the total sum total assured and the reserves be reassured. Since year after year the reserves will grow, the amount at risk will decrease and the same can happen to the reassured part. The part of the amount at risk that will be reassured, can be as Carter (1979) states:

(i) A percentage, which means that the reassured part will tend to decrease with the reserves increase;

(ii) A fixed amount of the amount at risk. In this way the reinsurance will only be necessary for a part of the term of the policy;

(iii) Decrease by a fixed and arbitrary amount over an agreed period of years.

The insurer will pay the reinsurer a premium equal to the expected reassured death strain plus any loading. The premium will be calculated every year to take into account the evolution of the reserves. It can be easily seen that the premiums are small when compared with the original premiums paid by the policyholder. This can be seen as an advantage. An obvious disadvantage is that by this method it is necessary to compute the premiums every year. Still, with the latest developments in computers this task is now quite easy to perform.

Risk premium reinsurance treaties provide for the payment of a profit commission. For the calculation of the profit commission, the insurer has to calculate the following account:

Income	**Outgo**
Reserve brought forward from previous year	Commissions
Premium income less retrocessions	Expenses
	Claims net of reinsurance recoveries
	Reserve at the end of year carried forward
(Loss carried forward)	Loss (if any) carried forward

The reserves for unexpired risk are taken as 50% of the premiums and expenses are a percentage of the premiums. If the balance is positive, the reinsurer will pay back a share of it. If a loss arises, it is carried forward until extinction.

2.4.3. Non proportional reinsurance

Under this class of reinsurance treaties, we find the ***excess of loss*** and the ***stop loss*** treaty.

Excess of loss treaties provide cover against the risk of a high number of deaths in one year. For example, if the company is highly exposed to the accumulation of deaths arising from air crashes, than there is a demand for an excess of loss cover. The insurer may have a typical proportional reinsurance treaty cover, but due to a sudden high number of deaths, he may find himself in a bad financial situation, even after receiving the reinsurer's share of the claims from the proportional treaty. This treaty is therefore designed to protect against accidents and natural catastrophes and it can be seen as a complement of the ones described in the previous section.

Under a stop loss treaty the reinsurer accepts liability for aggregate ultimate net losses in excess of a fixed amount, subject to an upper limit. This kind of treaty can also be seen as a complement of the proportional treaties, since they are also designed for natural disasters.

Although both types of treaties are attractive for a life office, they possess a big disadvantage. They are subject to annual reviews and even if they can be arranged to run for two years, they don't match the long term characteristic of life business. This is why this kind of treaty is not commonly used in life insurance in contrast with non-life insurance.

2.5. Life Office Risk Management

The nature of the business of a life insurer is to manage risk. There are three main parties that have an interest in the way the life office is managing risk, namely the current and future policyholders, the existing and potential future shareholders and the regulator, see Booth et al (1999).

From a policyholder's point of view the main risk he faces is that the life office will become insolvent and that it will not meet its obligations. Depending on the nature of the policy, if he holds a with-profits policy, he could also be concerned with the level of dividends the insurer is distributing to its policyholders.

The regulator's interest is to make sure that the life office meets its obligations. In this context, if the policyholders interest's are met, then the regulator's interest should also be met and vice versa.

A shareholder is mainly concerned with the level of dividends he is getting in relation to the amount of capital he has invested in the life office. He, therefore, shares similar concerns to a policyholder that has a with-profits policy. He is also concerned with the solvency of the life office as that will affect the level of dividends he will get.
The risk in a life office arises from three main sources: insurance risk, investment risk and business risk.

The insurance risk is associated with the possibility that the actual claim costs deviates from the expected costs that were assumed on the premium basis. Insurance claims are normally due to unexpected events defined in the policy terms and conditions, such as death, survival and sickness, in the case of a life office. The expected cost is generally estimated using a statistical model. The actual claims cost, can deviate from the assumed due to, see Booth et al (1999):

a) **Process error**: random variation around the expected mortality cost;

b) **Parameter error**: for example actual claim´s cost is higher than assumed expected mortality cost;

c) **Specification or model error**: error in the choice of the model structure to estimate expected claim's cost.

In general, the claim amount under a life policy is generally fixed at the date the policy is issued. In the case of general insurance, the amount of a claim is very much dependent on the extent of the damage, which can not be assessed before the actual claim occurs, and also it is possible for a single policy to have more than one claim. The level of insurance risk for a life office is therefore reduced in comparison.

However, claim amounts still contribute to the insurance risk a life office is facing. This could be the case when there is a large concentration of risk in a small number of policyholders. The variance of the total expected claim amount is greater the more concentrated the risk is, even though the expected claim cost might be the same. Hence in these cases the higher is the probability of insolvency.

A life office can control the claims experience by performing underwriting of an applicant for a life insurance policy. This process allows the insurer to classifiy how "risky" the applicant might be to the office and to decide, whether or not, the applicant is insurable. If the applicant is insurable, there might be the need to attach special conditions to the policy, such as charging a higher premium or reducing the level of benefits provided. The number of policyholders that fall under the latter category should be small in a portfolio.

Reinsurance is an effective way of reducing the exposure of the office to the claims cost by sharing it with one or more life offices, (see also Chapter 1). It is also this method of controlling insurance risk which is the fundamental subject of this thesis. The investigation will focus on the control of both process and parameter elements of insurance risk by reinsurance.

Business risk, as described by Booth et al (1999), can be subdivided into expense risk, discontinuance risk, valuation strain risk and taxation. All with the exception of valuation strain risk, not generally controlled by reinsurance and will not be investigated in this thesis.

The valuation strain risk is associated with the need that the life office has to meet the solvency margin defined by the regulator. If the office is selling new business then it is to be expected that it will create some strain on the office´s free assets. The more new business is selling the more strain it will be imposed and the solvency could therefore be put at risk (see Section 2.2.2). This can be controlled by financial reinsurance, that is, arrangements which exchange future income for initial capital in order to relieve new business strain, and has little to do with the management of insurance risk. This, too, therefore falls outside the scope of this thesis.

The actuary has a key role in the risk management of a life office. In order to carry out this task, it needs to develop a model office that will allow him to build possible scenarios regarding the future development of the office. From these scenarios, the actuary can assess how key variables, such as solvency, level of profits, etc., will develop in the future. This model and its outcomes can generally form the basis for management decision making.

The build up of an office model starts with the profit test of a single block of business. Through this process, based on a cashflow approach, the actuary estimates the expected future level of profits. This is normally done before launching a new product, in order to assess its profitability. Looking in isolation into a single block business, does not give credit to the fact that different departments are shared by different blocks of business and therefore the expected level of expenses needs to be assessed from an office's point of view. The required level of capital is also measured for the whole office. These and other features justify the need for an office model, where the profit test with the necessary adjustments will be included.

The model office and its underlying assumptions, is updated and revised periodically to reflect the real experience of the life office. This process is identified as the control cycle, see Booth et al (1999). Through this process, the actuary, seeks to monitor its experience, to set premium rates and reserves and as a basis for other management decisions.

The model office approach will be adopted here in order to identify methods of establishing optimal reinsurance strategies. These models are developed in Chapters 4, 5 and 6. While this approach can help identify the impact of process and parameter risk, it is less effective at allowing for the effect of specification error, though this, too, will be explored to an extent in Chapter 6.

2.6. Individual Risk Model

2.6.1. Introduction

For an insurance company, the total or aggregate, amount of claims on the whole or for a part of its risks is a random variable. In this section we will be looking at the **Individual Risk Model,** which can be seen as a particular case of the Collective Risk Models (see Pentikainen et al (1989)). This model is generally accepted as appropriate for life insurance analysis.

2.6.2. Risk Model

Panjer & Willmot (1992) derived an expression for the distribution of the aggregate claims amount for life insurance, which is given bellow, based on the Individual Risk Model.

Let us consider a life portfolio split into classes as a function of the sum assured, and the probability that a claim should occur and represent by n_{ij} the number of policies with a sum assured i and a probability that a claim should occur q_j where i=1,...,r; j=1,...,m. The probability generating function of the total aggregate claim amount is defined by:

$$P_S(z)=\prod_{i=1}^{r}\prod_{j=1}^{m}\left(1-q_j+q_j\cdot z^i\right)^{n_{ij}} \tag{2.22}$$

the logarithm of (2.22) is equal to:

$$\log P_S(z)=\sum_{i=1}^{r}\sum_{j=1}^{m} n_{ij}\cdot\log\left(1-q_j+q_j\cdot z^i\right) \tag{2.23}$$

the first derivative in order of z of the previous formula is given by:

$$P_S'(z)=\left(\sum_{i=1}^{r}\sum_{j=1}^{m} i\cdot q_j\cdot n_{ij}\cdot z^{i-1}\cdot\left(1-q_j+q_j\cdot z^i\right)^{-1}\right)\cdot P_S(z) \tag{2.24}$$

which can be rewritten as

$$z\cdot P_S'(z)=\left(\sum_{i=1}^{r}\sum_{j=1}^{m} i\cdot n_{ij}\cdot\left(\frac{q_j}{1-q_j}\cdot z^i\right)\cdot\left(1+\frac{q_j}{1-q_j}\cdot z^i\right)^{-1}\right)\cdot P_S(z) \tag{2.25}$$

$$=\left(\sum_{i=1}^{r}\sum_{j=1}^{m} i\cdot n_{ij}\cdot\sum_{k=1}^{\infty}(-1)^{k-1}\cdot\left(\frac{q_j}{1-q_j}\right)^k\cdot z^{ik}\right)\cdot P_S(z) \tag{2.26}$$

for $|z|<\min_{i,j}\left(q^{-1}\cdot\left(1-q_j\right)\right)^{\frac{1}{i}}$ and where the term $\left(\frac{q_j}{1-q_j}\cdot z^i\right)\cdot\left(1+\frac{q_j}{1-q_j}\cdot z^i\right)^{-1}$ has been replaced by a binomial series. If we represent by:

$$h(i,k)=i\cdot(-1)^{k-1}\cdot\sum_{j=1}^{m} n_{ij}\cdot\left(\frac{q_j}{1-q_j}\right)^k \tag{2.27}$$

then (2.26) becomes

$$z \cdot P_S'(z) = \left(\sum_{i=1}^{r} \sum_{k=1}^{\infty} h(i,k) \cdot z^{ik} \right) \cdot P_S(z) \tag{2.28}$$

The coefficient of z^x on the left hand side of (2.28) is equal to $x.f_S(x)$, whilst in $P_S(z)$ the coefficient is $f_S(x)$, where $f_S(x)$is the density function of S. If we note that the right hand side of (2.28) is a convolution, then the coefficient of z^x is given by:

$$\sum_{i \cdot k \leq x} h(i,k) \cdot f_S(x - i \cdot k) \tag{2.29}$$

Finally and since h(i,k)=0 for i>k, due to its definition, if we make the coefficients of z^x equal on both sides of the equation we obtain:

$$f_S(x) = \frac{1}{x} \cdot \sum_{i=1}^{\min(x,r)} \sum_{k=1}^{\left[\frac{x}{i}\right]} h(i,k) \cdot f_S(x - i \cdot k); \quad n \geq 1 \tag{2.30}$$

where [x] is the biggest integer contained in x. Therefore:

$$f_S(0) = P_S(0) = \prod_{i=1}^{r} \prod_{j=1}^{m} (1 - q_j)^{n_{ij}} \tag{2.31}$$

and the probabilities $\{f_S(x); x = 1,2,\ldots\}$ can be calculated using formula (2.30).

Expression (2.29) is a weighted sum of the k-th power of $\left(\frac{q_j}{1-q_j} \right)$. When q_j is close to zero, $\left(\frac{q_j}{1-q_j} \right)^k$ is small and as a consequence the magnitude of h(i,k) decreases as the value of k increases. This fact suggests that we limit index k in formula (2.30) to a maximum **K** terms, or in other words:

$$f_S^{(K)}(x)=\frac{1}{x}\cdot\sum_{i=1}^{\min(x,r)}\sum_{k=1}^{\min\left(K,\left[\frac{x}{i}\right]\right)}h(i,k)\cdot f_S(x-i\cdot k);\quad n\geq 1 \tag{2.32}$$

De Pril (1988) shows that if $q_j<1/2$, j=1,...,m then:

$$\sum_{x=0}^{M}\left|f_S(x)-f_S^{(K)}(x)\right|<e^{\delta(K)}-1 \tag{2.33}$$

where

$$\delta(K)=\frac{1}{K+1}\cdot\sum_{i=1}^{r}\sum_{j=1}^{m}n_{ij}\cdot\frac{1-q_j}{1-2\cdot q_j}\cdot\left(\frac{q_j}{1-q_j}\right)^{K+1} \tag{2.34}$$

and $M=\sum_{i=1}^{r}\sum_{j=1}^{m}i\cdot n_{ij}$ is the maximum possible value for the aggregate claims amount.

Example 2.1.

Let us consider a group of 15 individuals with the following characteristics:

Age	$1000.q_x$	Sum Assured
68	42.183	1,000
22	1.480	2,000
65	32.545	3,000
20	1.351	4,000
25	1.602	5,000
32	2.108	6,000
30	1.826	10,000
42	4.589	11,000
65	32.545	12,000
45	5.874	13,000
40	3.903	14,000
50	8.943	15,000
52	10.603	20,000
60	21.110	21,000
55	13.712	22,000

In this example, n_{ij}, the number of individuals with a sum assured of i and probability that a claim should occur q_j, is always equal to 1.

If we make K=4 in (2.34), we obtain δ(K)=5.36E-08 which allows us to obtain an approximation with seven significant decimals. The values of h(i,k), for the group in question are shown in the next table:

h(i,k)

i	k=1	k=2	k=3	k=4
1,000	44.0407677	-1.9395893	0.085210	-0.0037620
2,000	2.9643872	-0.0043938	6.512456E-06	-9.652721E-09
3,000	100.9194107	-3.3949091	0.1142041	-0.0038418
4,000	5.4113102	-0.073206	9.903469E-06	-1.339769E-08
5,000	8.0228529	-0.0128732	2.065601E-05	-3.314402E-08
6,000	12.6747179	-0.0267747	5.65604E-05	-1.194812E-07
10,000	18.29334036	-0.0334649	6.121862E-05	-1.119897E-07
11,000	50.7117195	-0.2337890	0.0010778	-4.968843E-06
12,000	403.6776428	-13.5796366	0.4568163	-0.0153672
13,000	76.8132019	-0.4538668	0.0026818	-0.0153672
14,000	54.8561058	-0.2149423	0.0008422	-3.300014E-06
15,000	135.3554840	-1.2214071	0.0110216	-9.945569E-05
20,000	214.3325500	-2.2969222	0.0246153	-0.0002638
21,000	452.8700867	-9.7662435	0.2106116	-0.0045419
22,000	305.8579407	-4.2522306	0.0591172	-0.0008219

The density function of the aggregate claim amount is given by:

x	$f_S(x)$	$F_S(x)$	x	$f_S(x)$	$F_S(x)$
0	0.08295825	0.8295825	**26,000**	2.400524E-04	0.9963483
1,000	3.653545E-02	0.866118	**27,000**	3.392642E-04	0.9966875
2,000	1.229602E-03	0.8673475	**28,000**	9.161303E-05	0.9967791
3,000	2.796115E-02	0.8953087	**29,000**	4.079732E-05	0.9968199
4,000	2.351327E-03	0.89766	**30,000**	2.967499E-05	0.9968496
5,000	1.421886E-03	0.8990819	**31,000**	7.843477E-05	0.996928
6,000	1.814563E-03	0.9008964	**32,000**	4.079852E-04	0.0.997336
7,000	1.16979E-04	0.9010134	**33,000**	7.269813E-04	0.998063
8,000	4.912487E-05	0.9010625	**34,000**	5.630769E-04	0.9986261
9,000	6.289539E-05	0.9011254	**35,000**	2.570363E-04	0.9988831
10,000	1.522704E-03	0.9026482	**36,000**	2.432578E-04	0.9991264
11,000	3.89435E-03	0.9065425	**37,000**	1.352254E-04	0.9992616
12,000	2.877787E-02	0.9346203	**38,000**	1.724717E-05	0.9992788
13,000	6.118771E-03	0.9408081	**39,000**	1.125388E-05	0.9992901
14,000	3.641093E-03	0.9444491	**40,000**	7.455673E-06	0.9992976
15,000	8.590375E-03	0.9530395	**41,000**	1.941813E-04	0.9994918
16,000	5.886899E-04	0.9536282	**42,000**	1.337501E-04	0.9996255
17,000	1.909602E-04	0.9538192	**43,000**	2.575383E-04	0.9998831
18,000	3.315985E-04	0.9541508	**44,000**	2.179604E-05	0.9999049
19,000	4.178743E-05	0.9541926	**45,000**	9.923481E-06	0.9999148
20,000	8.912911E-03	0.9631055	**46,000**	1.549351E-06	0.9999303
21,000	1.830795E-02	0.9814134	**47,000**	1.04589E-05	0.9999407
22,000	1.238762E-02	0.9938011	**48,000**	9.610801E-06	0.9999503
23,000	9.748083E-04	0.9947759	**49,000**	6.222188E-06	0.9999565
24,000	6.808111E-04	0.9954567	**50,000**	1.872262E-06	0.9999583
25,000	6.515165E-04	0.9961082			

3. Utility Theory

3.1. Introduction

In this chapter, we describe the subject of utility theory. As it can be seen later in the thesis, we have used the maximization of expected utility of profits as a criterion for determinig optimal reinsurance strategies. But first of all, it it is important to establish the basis and justification for the use of this criterion.

3.2. Historical Introduction

An elaborate theory that provides insights into decision making in the face of uncertainty has been developed. This body of knowledge is known as **Utility Theory**.

Adam Smith, in his *Wealth of Nations* in 1776, believed that the hidden hand of Providence must be guiding economic action in order to ensure just prices. Fair prices are reached if the amount of labour in exchange of goods is the same. Like many other defenders of the labour theory, Adam Smith made use of the Aristotelean theory of fair price (see Kauder (1965)).

It is generally accepted that Aristotle was the first who created the concept of the value-in-use. Whether he had a far-reaching knowledge of this field is generally unknown (see Kauder (1965)). Economic goods, as Aristotle pointed out, derive their economic value from individual utility, scarcity and costs. If the amount of goods is increasing, the value decreases and can become even negative. Aristotle had, at least, some knowledge of the law of diminishing utility. Aristotle claimed in the *Topics* - a work not often read by economists- that the value of one good can be best judged if we remove or add it to a given group of commodities. The greater the loss which we suffer from a destruction of this good, the more "desirable" is this commodity. Also, the more we gain by the addition of a thing the higher is its value. The context makes it quite clear that Aristotle applies his argument to economic goods (see Kauder (1965)).

Aristotle had laid down the foundations for the later value discussion. His explanation was accepted by medieval scholars and by theologians. Buridanus and the Italian economists between the sixteenth and eighteenth centuries added new ideas to the Aristotelian heritage.

In three ways, Joannes Buridanus (about 1295-1366) improved the understanding of economic value. First, he explained the law of diminishing utility better than Aristotle. He wrote that the rich man attaches small value to goods with which he can even gratify his demand for luxury, while other people can only satisfy their most urgent desires. Second, value and price are not identical. Third, regarding the market, it is not the needs of each individual but the needs of those persons who can afford to trade that determine the price (see Kauder (1965)).

Gian Francesco Lottini (1548), was the earliest member of the Italian group of economists. Lottini's first point of departure is the traditionally Aristotelian dichotomy between the common good ("il bene publico") and the goods serving individual needs ("bene in particulare"). Common welfare and individual well being, are not identical but related. The common good is the foundation of the citizen's personal welfare. For instance, if the citizen loses his property, he can have it back with the help of the state. Private needs are satisfied with goods and these goods produce pleasure (see Kauder (1965)).

Lottini apparently was not aware of the fact that he was dealing with economic subjects. His younger comtemporary Davanzati, and his followers, formed a school of economists, because they had a common program: the application of utility value to other economic subjects.

Ferdinando Galliani (1728-1787) wrote his *Treatise on Money*, in which he surpassed the older Italian economists. He repeated the traditional formula that value is dependent on utility and scarcity, but he broadened the field of application for this formula. It is not labour which determines value, but value-in-use which causes the price of labour. Great Generals are so rare that they can receive high remunerations. Galiani almost discovered the principle of marginal utility and he almost visualized the law of the equalizing of utility (see Kauder (1965)).

Galiani's contemporaries appreciated the new vistas which he opened. The French economist and statesman Robert Turgot, developed a price theory along Galiani's lines. Turgot uses an example to explain the exchange mechanism. Two men are living on an isolated island. One *A*, has corn, the other, *B*, owns kindling wood. *A* freezes to death if he has no kindling wood, and *B* is starving if he has only wood but not food. *A* is willing to trade his corn for kindling wood and *B* wants to exchange his wood for corn. Both plan to keep the maximum of the other's good. A wants to give 3 measures of corn for 6 armfuls of wood and *B* wants to exchange 6 armfuls of wood for 9 measures of corn. Eventually a point agreement is reached, where the individual value of the offered good is still lower than the commodity received (see Kauder (1965)).

The Galiani's school never went beyond the very promising start indicated in Turgot's unfinished work. Adam Smith had an unfortunate influence on the further development of the value explanation. After reading the *Wealth of Nations*, many economists reached the conclusion that a further discussion of the value-in-use was meaningless, because they accepted his verdict.

In the following century there was a rapid development in economic theory. The three main centers of this development were Cambridge, Lausanne and Vienna. At the time that the theories developed at these centers, they were considered as different "schools". Today the difference seems less fundamental, and it is normal to refer to these schools as the "neo-classical" theory (see Kauder (1965)).

In Cambridge, Alfred Marshall came to an important conclusion with regard to the nature of decision makers. In his *Principles* (1890) he discusses insurance premiums as the price one has to pay to get rid of the "evils of uncertainty". Marshall wrote about the "evil of risk", and believed that people were willing to get rid of this evil. He noted that businessmen paid insurance premiums "which they know are calculated on a scale sufficiently above the true actuarial value of the risk to pay the companies' great expenses of advertising and working, and yet a surplus of net profits". This meant that Marshall believed that the important decision makers in the economy were "risk averse" (see Kauder (1965)).

The basic problem in the Austrian school in Vienna, was to assign utility to a collection of goods – which is informally referred to as "market baskets". If there was a choice, one would choose the basket with the highest utility. In a situation with uncertainty, the choice is not between different market baskets, but between probability distributions over sets of market baskets. This may sound fairly simple, but the amount of difficulty that the problem has caused in economic theory is really surprising, particularly when we realize that the solution was suggested by Daniel Bernoulli as early as 1738 (see Bernoulli (1954)).

Expected Utility Theory has been used for many years as a model of rational preferences in decision making under risk. In the beginning of the development of probability theory, the choice of a risky enterprise was based on the expected value of the outcomes. In the light of the two following games:

Game A	**Game B**
£2,500 with probability 0.33	£2,400 with probability 0.34
0 with probability 0.67	0 with probability 0.66

probability theory would recommend the choice of Game A, because its expected value £825 is higher than the expected value for Game B, £816.

The first challenge to the decision process based on the expected value of returns appears in 1738 by Daniel Bernoulli (see Bernoulli (1954)). Daniel Bernoulli stated that a person's subjective value v(w) of wealth w does not increase linearly with w but increases at a decreasing rate. This concept would later on be called in the economic literature by the principle of diminishing marginal utility of wealth. Bernoulli further stated that a risky prospect should be measured by its subjective value on levels of wealth w.

Bernoulli reached this conclusions from his work on the famous St. Petersburg game. In such a game, a fair coin is tossed until a head appears. If n tosses are required, then a sum of 2^n is won. The expected payoff to this game is:

$$\left(\frac{1}{2}\right)\cdot 2+\left(\frac{1}{4}\right)\cdot 2^2+\left(\frac{1}{8}\right)\cdot 2^3+\cdots=1+1+\cdots \qquad (3.1)$$

The expected value is infinite, though most people would offer this game at a finite price. Bernoulli and Cramer in 1728 explained the St. Petersburg paradox by proposing that it is the expected value of the subjective value of the outcomes that should be considered. Bernoulli suggested a logarithmic utility function, arguing that the rate of increase is inversely proportional to w, where as Cramer proposed a power utility function (see Fishburn (1988)).

It was almost two centuries later that a revival in the interest of utility reappeared. In the meantime there was a big debate about the measurability of utility. It was with the development of axiomatic mathematics that this was possible. This was based on the concept that people make absolute choices all the time. For example, we would prefer a £50 to £0, and therefore let us make $v(50)=v_1$ and $v(0)=v_0$ where $v_1>v_0$. If for example the point x where the preference of £50 over x equals the preference of x over £0 is equal to say £30, then $v(30)=(v_0+v_1)/2$. This process can be repeated to build the complete utility function. This idea of measurable utility can be made precise by a set of axioms of a binary relation of preference (see Fishburn (1988)).

In 1944 Von Neumann and Morgenstern introduced a new expected utility theory which presented a big change from Bernoulli' theory. Their theory begins with a binary preference relation defined on a convex state which is assumed to behave according to the order preserving and linearity property. Similarities can be found between the two theories, such as, the order of preference being preserved and in both v is unique up to a positive linear transformation. The difference lies in that in this case expected utility is derived from these axioms, rather than being merely stated as was the case with the Bernoulli theory. In the example above, for Von Neumann and Morgenstern x is the value at which we are indifferent between receiving x as a certain thing and playing out the lottery that pays either £50 or £0 with probability 1/2.

The following axioms differ slightly from the originals and can be seen in Fishburn (1988). It will be assumed that P is a non empty set of probability measures p, q, ... defined on a Boolean algebra $\boldsymbol{A}$ of subsets X. Given this structure for P, let > be a binary relation on P, interpreted as strict preference. The indifference relation ~ on P and the preference-or- indifference relation ≥ are defined by:

$$\begin{array}{llllll} p \sim q & \text{if} & \text{neither} & p > q & \text{nor} & q > p \\ p \geq q & \text{if} & \text{either} & p > q & \text{or} & p \sim q \end{array} \qquad (3.2)$$

The same author gives the three axioms in the following form:

A1. Order: > on p is a weak order

A2. Independence: $p > q \Rightarrow \lambda.p+(1-\lambda).r > \lambda.q+(1-\lambda).r$

A3. Continuity: $\{p > q, q > r\} \Rightarrow (\alpha.p+(1-\alpha).r > q$ and $q > \beta.p+(1-\beta).r$ for some α and β in $(0,1))$

A binary relation is said to be a weak order if it is asymmetric and negatively transitive. Further details can be seen in Fishburn (1988).

The first axiom A1 is known as the **ordering axiom**. Violations of A1 and particularly on the transitivity property, are seen as the author says " aberrations that any reasonable person would gladly "correct" if informed of his or her "error."". The second axiom known as the **linearity axiom** simply states that if $p > q$ then any convex combination of p and r is preferred to any similar combination of q and r. The last axiom, the **continuity or Archimedean axiom** prevents one measure from being infinitely preferred to another.

Theorem 3.1

Suppose P is a non empty convex set of probability measures defined on a Boolean algebra of subsets of X, and > is a binary relation on P. Then axioms A1, A2 and A3 hold if and only if there is a linear functional u on P such that, for all $p, q \in P$, $p > q \Leftrightarrow u(p) > u(q)$. Moreover, such a u is unique up to positive linear transformations, or:

$$u(\lambda \cdot p + (1-\lambda) \cdot q) = \lambda \cdot u(p) + (1-\lambda) \cdot u(q) \tag{3.3}$$

Proof of the above theorem can be seen in Fishburn (1988).

Special topics that have been developed in the context of Von Neumann and Morgenstern (1944) include the theory of risk attitudes, stochastic dominance and multiattribute utility theory.

The theory of risk attitudes developed by Pratt (1964) and Arrow (1974) is concerned with the curvature of u on X, when X is an interval of monetary amounts interpreted either as wealth levels or gains and losses around a given present wealth. It was designed to study economic behaviour in risky situations as a function of the curvature of u and other properties of u on X within the Von Neumann and Morgenstern framework of maximizing expected utility. It has been observed through the application of Pratt-Arrow's theory to changes in present wealth, that people tend to be risk averse in gains and risk seeking to losses.

Stochastic dominance is concerned with the curvature of u on X. It looks at comparative aspects of measures p and q and with classes of utility functions whose members have the same preference implication between p and q. An extensive bibliography on the subject can be seen in Bawa (1982).

Multiattribute theory, as the name suggests, studies decisions under risk which involve multiattribute outcomes of the form $x=(x_1, x_2, ..., x_n)$ with $X = X_1 \times X_2 \times \cdots \times X_n$. It has focused on special assumptions that simplify assessment by decomposing $u(x_1, x_2, ..., x_n)$ into algebraic combinations of functions of the individual variables and on interactive techniques that allow decision makers to maximize expected utility without having to assess all of *u*. A broad introduction is given by Fishburn (1988).

3.3. Utility Functions

Utility functions, despite the different functional forms they may take, share a few general properties. Firstly, the utility function should be monotonically increasing, ie the first derivative is always positive. This property means that individuals prefer more wealth to less. Secondly, they should be concave or the second derivative non positive, which corresponds to investors being risk averse. Also it is normal for them to be continuous functions of the wealth. Nevertheless, it is possible in some situations to include discontinuities.

The risk premium an individual requires before considering a given gamble reflects his risk aversion characteristics. Pratt (1964) showed that the risk premium is approximately $r(.) = -u''(.)/u'(.)$, where $r(.)$ is defined as the measure of risk aversion. The larger this value the more risk averse the investor is to risk.

The simplest utility function is such that $u''(.) = 0$ which gives:

$$u(x) = a + b \cdot x \quad b > 0 \tag{3.4}$$

An investor will try to maximize the expected value of the function u(x), where x represents the wealth after the outcomes of the investment. If we consider a starting value of wealth *w* and the random return of an investment *i*, since the investment is the only random component, the strategy for an investor in this case is the one that maximizes the expected investment return. This is because this utility function corresponds to an investor that is risk neutral.

Let us now consider the quadratic utility function:

$$u(x) = a \cdot x^2 + b \cdot x + c \quad c < 0, b > 0 \tag{3.5}$$

The first derivative and second derivative are $u'(x) = 2 \cdot a \cdot x + b$ and $u''(x) = 2 \cdot a$ respectively. In order for the second derivative to be negative, ie to show risk aversion it is necessary for a<0. This assumption implies that the function is only increasing when $x < -b/2 \cdot a$. This means that the quadratic utility function is limited to a range defined by the latter condition.

The risk aversion measure, for the quadratic utility function, is equal to $r(x) = -1/(x + b/2 \cdot a)$, which means that within the range of possible values of x, the level of risk aversion increases with x. The investor will therefore require higher risk premiums the highest its level of wealth.

From an analysis of the quadratic utility function, we can see that what we will maximize are the first and second central moments of *i*, the random return on the investment.

The logarithmic utility function, Bernoulli (1954) introduced, is given by:

$$u(x) = \ln(x) \quad x > 0 \tag{3.6}$$

Apart from the restriction on the value of x, the logarithmic function satisfies the first and second derivative conditions. The measure of risk aversion is equal to $r(x) = 1/x$. This means that the investor has a decreasing risk aversion and also a constant relative risk aversion $\rho(x) = x \cdot r(x)$, concept introduced by Pratt (1964), which means that a given investor will base its decisions on the proportion of wealth invested and not on the level of initial wealth.

The exponential function is expressed as:

$$u(x) = 1 - \exp(-r \cdot x) \quad r > 0 \tag{3.7}$$

and satisfies the first and second derivatives conditions for all values of x. The measure of risk aversion, in this case, is equal to r. This characteristic indicates constant absolute risk aversion. This property means that investors will take exactly the same decisions if they invest the same amount of money, regardless of the initial level of wealth.

The only other utility function that exhibits constant risk aversion is the power function:

$$u(x) = x^c \quad 0 < c < 1 \tag{3.8}$$

The measure of relative risk aversion is equal to 1-c and is therefore less than one. As a consequence this function is less risk averse than the logarithmic utility function. When c=1, this is a simple linear function and the relative measure of risk aversion is zero, or in other words it corresponds to an investor that is is risk neutrality.

3.4. Utility in a Multiperiod

Methods have been introduced to reflect utility time preferences in the economics literature. However, it is unclear how such time preferences can be observed and measured. In a simple one-year investment, the capital is provided at the start of the period and the surplus is assessed at the end of the period. This seems to suggest that a discounting factor is required between both points in time. Samuelson (1969) and Sherris et al (1992) suggested that the objective function to be maximized should be of the form:

$$u(-K)+(1-d)\cdot u(S) \tag{3.9}$$

where d is the rate of discount for the single period, K the amount of capital invested and S the surplus generated over the period. The same authors presented the generalization for the multiperiod. The objective function becomes:

$$U(-K)+\sum_{t=1}^{n} U(S_t)\cdot\prod_{j=1}^{t}(1-d_j) \tag{3.10}$$

where d_j is the rate of discount in year j. In order to simplify matters it is possible to consider a constant rate of discount.

3.5. Critique of Expected Utility

In order to present all that has been put against expected utility theory, it is necessary to introduce the concept of normative and descriptive status. Expected utility theory has been generally accepted as a normative model of rational choice and an applied descriptive model of economic behaviour (see for example Kahneman and Tversky (1979)).

A descriptive model would try to identify patterns in individual's preferences and develop models based on those patterns to predict future choices. A normative approach would be interested in the rationality of the preference patterns, which should be, as in expected utility theory, set forth as axioms.

It has been widely recognized that the way a question is posed to an individual can lead him to answer differently. This has been illustrated by Kahneman and Tversky (1979) where they refer to it as **Framing Effects**.

This phenomenon involves the asymmetry property from axiom A1, which states that if p > q then not (q > p). By posing the comparison between p and q in different frames it is possible to obtain a preference of p over q or vice versa. Well known examples which can be seen with the same authors, involve situations of life and death. They consider a situation paraphrased as follows:

- Six hundred people have contracted a potentially fatal disease. Two treatment programs are possible. If program 1 is adopted, 400 hundred people will die and 200 will live. If program 2 is adopted, either 600 will die, with probability 2/3, or all will live with probability 1/3. When the problem was posed to two different groups of respondents, one preferred program 1 over program 2 by a ratio of 2.6 to 1, when the two programs were stated in terms of lives saved: 200 saved versus 600 saved with probability 1/3 and nobody saved with probability 2/3. The other group preferred program 2 over program 1 by a ratio of 3.5 to 1 in the lives lost frame: 400 die versus nobody dies with probability 1/3 and 600 die with probability 2/3.

The ability to prefer p over q or q over p depending on the frame is referred to as a violation of invariance. According to the Invariance Principle, different representations of the same choice problem should yield the same preference. That is, the preference between options should be independent of their description (Kahneman and Tversky, 1986).

We will now consider the implications of the violation of the transitivity property. We will start by looking at nontransitive indifference. A definition of a nontransitive indifference relation can be seen in Fishburn (1988). This can be understood with an example suggested by Luce (1956). A person who likes sugarless coffee, will be indifferent between x and x+1 grains of sugar in his coffee, between x+1 and x+2 grains,..., but for each x there will come a smallest y, as a function of x, at which $x > y$ or x is preferred to y. The latter means that the indifference relation is not transitive.

Another form of breaking the transitivity relation is through preference cycles and money pumps. The next example, seen in Fishburn (1988) was due to May (1954). College students, a total of 62, were asked to make binary comparisons between hypothetical marriage partners x, y, z who were characterized by three attributes:

	Intelligence	Looks	Wealth
x:	Very Intelligent	Plain	Well off
y:	Intelligent	Very good looking	Poor
z:	Fairly intelligent	Good looking	Rich

Seventeen of the 62 students had cyclic choices which violated the transitivity property. Their pattern of preference was $x > y > z > x$, and the original author noted that "the intransitivity pattern is easily explained as the result of choosing the alternative that is superior in two out of three criteria".

The concept of money pumps show how irrational are cyclic preferences. Let us suppose that someone has the cycle $p > q > r > p$ and that presently holds p. Since r is preferred to p, it would make sense to pay something to change p for r. Now given that q is preferred to r, then logically it would make sense to pay something to change r for q. Finally it will also again make sense to pay something to change q for p. Thus the initial and final position are the same but are poorer in the process, hence a "money pump".

Another form of intransitivity is called the preference reversal phenomenon. To illustrate it consider the following example given in Fishburn (1988):

p: £30 with probability 0.9, nothing otherwise;

q: £100 with probability 0.3, nothing otherwise.

Let the amount c(p) be the minimum amount the individual would accept in exchange for title p and similarly for c(q). This example has been used with several groups and the majority shows preference of p over q, with c(p) about £27 and c(q) about £30. It reflects a predominant theme of experiments on preference reversals that use a "p-bet" with a high chance for modest winnings (p) and a "q-bet" with a lower chance for large winnings (q). When the lotteries are turned round and stated in terms of losses, then the preference goes the other way.

Of the axioms presented in the previous section, the one most often denied, relaxed or abandoned is A2, the independence axiom. An extensive list of examples of violations has been given by different authors, including Allais (1953, 1979a), Kahneman and Tversky (1979, 1981). We will show violation of the independence axiom through a phenomenon called the certainty effect, which states that people overweight outcomes which are considered certain, relative to outcomes which are merely probable. The best well known example of the certainty effect is attributed to Allais (1953). The choice problem is a variation of Allais' example and can be seen in Kahneman and Tversky (1979).

Problem 1: Choose between

A:	£2,500 with probability	0.33
	£2,400 with probability	0.66
	£0 with probability	0.01
B:	£2,400 with certainty	

From a group of 72 respondents, 82% choose option B.

Problem 2: Choose between

C:	£2,500 with probability	0.33
	£0	0.67
D:	£2,400 with probability	0.34
	£0 with probability	0.66

When facing this problem, 83% from the same group of respondents choose option C. The pattern of preferences seen in the two problems violates, in the same manner described originally by Allais (1979a), expected utility theory axiom A2. From the first problem, with u(0)=0, we have that:

$$u(2{,}400)>0.33.u(2{,}500)+0.66.u(2{,}400)$$

or

$$0.34.u(2{,}400)>0.33.u(2{,}500)$$

and from the second problem the same inequality is reversed.

The last axiom A3 says that if p is preferred over q and q to r, then some nontrivial convex combination of p and r is preferred over q and q to some nontrivial convex combination of p and r. There are some plausible examples in Chipman (1960) among others, but there is almost no practical evidence. Let us consider facing a choice between (A) receive £10,00,020 if the first head in a series of flips of a fair coin comes before the nth toss, £0 otherwise; (B) receive £10,000,000 with certainty. If (B) is preferred over (A) regardless of how big n is, then A3 is violated.

3.6. Elements of a Theory of Reinsurance

The risk situation of an insurance company can be defined by the following three elements:

(i) Its technical reserves P, where $P = \int_0^\infty x dF(x)$;

(ii) The underwriting responsibility, represented by F(x), that the amounts of claims paid under the contracts in the portfolio shall not exceed x, the total claim cost;

(iii) Its free reserves R;

(iv) The function u(x), an operator establishing ordering over the set of all possible probability functions.

The expected utility attached at its present situation is given by:

$$\int_0^\infty u(R + P - x) \cdot dF(x) \qquad (3.11)$$

If there exists a reinsurance market where the company can obtain any kind of coverage by paying a determined price, it will seek to maximize its expected utility. The simplest example, is the case of two companies who seek an agreement of exchange of risks that benefits both parties. In the "Theory of Games", this situation is defined as a "Two-person Co-operative Game". Let both companies, be identified by $F_1(x_1)$, R_1, P_1, $u_1(x)$ and $F_2(x_2)$, R_2, P_2, $u_2(x)$. The variables x_1 and x_2 represent the claims occurring in the two portfolios and are assumed to be stochastically independent.

If we represent by $y_1(x_1,x_2)$ the amount of claims paid by Company 1 as a function of x_1 and x_2, then it will try to maximize the following expression:

$$\int_0^\infty \int_0^\infty u_1\left(R_1 + P_1 - y_1(x_1, x_2)\right) \cdot dF_1(x_1) \cdot dF_2(x_2) = U_1(y) \tag{3.12}$$

Company 2 will also try to maximize:

$$\int_0^\infty \int_0^\infty u_1\left(R_2 + P_2 - x_1 - x_2 + y_1(x_1, x_2)\right) \cdot dF_1(x_1) \cdot dF_2(x_2) = U_2(y) \tag{3.13}$$

It is obvious that the objectives of the individual companies are opposed and they will have to negotiate a compromise. Borch (1960b) shows that a necessary and sufficient condition for $y_1(x_1,x_2)$ to be an efficient solution is that:

$$u_1'\left(R_1 + P_1 - y_1(x_1, x_2)\right) = k \cdot u_2'\left(R_2 + P_{21} - x_1 - x_2 + y_1(x_1, x_2)\right) \tag{3.14}$$

and gives its solution for the special case when the utility functions are of a quadratic form. The result gives a quota share treaty where Company 1 cedes to Company 2 a quota of 100h per cent of its net premium P_1. If claims amounting to x_1 occur in the portfolio of Company 1, a corresponding quota will be paid by Company 2. Company 1 itself will pay only the remainder $(1\text{-}h).x_1$. In the same way, Company 2 will cede a quota of 100.(1-h) per cent to Company 1. The parameter h is also defined in the same paper.

Benktander (1975) looks at the simplest possible market of one insurance company C and a reinsurer R and tries to optimize the situation of both C and R. The author uses the variance as a measure of exposure. The variance of the portfolio P written by C is represented by V. Company C tries to reduce the variance to a level V_c with a reinsurance treaty. The corresponding variance of R is V_R. Both the company and the reinsurer will look for solutions that will reduce substantially the variance. The result is expressed in the form of a loading addition to the pure premium risk that company C is prepared to pay to R and the minimum price acceptable to R.

The subject of optimal reinsurance has been dealt with before by various authors such as Borch (1960a), Kahn (1961) and Verbeek (1966). The first, tries to look at the problem of optimal reinsurance from the point of view of a company that acts as both an insurer and an reinsurer, where as the other two consider the point of view of the ceding company. In any case they reach the conclusion that a stop loss treaty is the optimal reinsurance arrangement from the ceding company point of view.

The determination of optimal rules for sharing risks and constructing reinsurance treaties has both practical and theoretical interest. Borch used the economic concept of utility to justify choosing pareto-optimal forms of risk exchange (see Borch 1990). In many cases, this leads to familiar linear quota-sharing of total pooled losses, or to stop-loss arrangements. However, this approach does not give a unique risk sharing agreement, and may lead to substantial fixed side payments. Buhlmann et al (1979), introduced the actuarial concept of long-run fairness to each participant in the risk exchange, in the sense that according to a commonly accepted premium principle, all participants agree that, over the long run, no company in the pool should profit at the expense of the others. The result is a unique a Pareto-optimal risk pool with "quota-sharing-by-layers" of the total losses.

The Pareto-optimality and the individual rationality conditions considered by Borch, do not allow for the possibility that a coalition of companies might be better off by seceding from the whole group. This problem is discussed in Lemaire and Baton (1981a) and subsequently generalized in Lemaire and Baton (1981b), where the negotiation process (bargaining process), is considered. Lemaire and Baton (1981b) also show that the theorem of Borch, characterizing Pareto-optimal treaties in a reinsurance market, is identical to the value of a cooperative non-transferable m person game.

The use of maximizing expected utility has also been used in other areas. Booth et al (1997) use stochastic simulation techniques to determine optimum strategic asset allocation decisions for life insurance companies, based on maximizing the expected utility of returns.

3.7. Summary

Utility theory, will be used later on the thesis, as a criterion for determining optimal retention levels. We have seen, in this Chapter, that violations of its axioms and underlying principles have been generated by certain experimental conditions and framing procedures. Virtually, any axiom or principle of choice can be violated by suitable framing in experiments on preference judgements and choice behaviour (Fishburn (1988)). Utility Theory, is no longer seen as an accurate descriptive theory for decision making in the face of uncertainty. Many generalizations of the Bernoullian and von Neumann-Morgenstern theories have been proposed to accomodate violations to those theories, see Fishburn (1988). Nevertheless, similarly to Booth et al (1997), we would argue that utility theory can be usefully employed to provide insights for decision making under uncertainty.

We have given various choices of utility functions (Section 4.2). Later on the thesis, we will be using the exponential utility function. Booth et al (1997) argue that the logarithmic utility function may have advantages over the exponential function, due to its property of constant relative aversion, provided that the output measures are positive real numbers. In the present case, the output measures can become negative which excludes the possibility of using the logarithmic function. The utility function used, does display a constant absolute risk aversion. The latter property, is in line with the expected behaviour of an invester of a life insurance company and it is also interesting from a theoretical point of view.

In Chapter 4, we will be looking at maximizing the expected utility of profits in a one-year scenario. This approach will be generalized in Chapters 5 and 6, when we consider maximizing the utility in a multi-period. In Section 3.4, we discussed the approach to be used in the multi-period scenario. The methodology described, assumes that each year's profits would be accumulated, at a given interest rate, until a fixed date and to take the utility of this accumulated amount. This approach does have the disadvantages of not allowing the shareholders to make any immediate use of the profits stream as it arises. It assumes that profits, as a whole, are re-invested in the financial market until a fixed date in the future.

In Section 3.6, we covered the topic of optimal rules for sharing risks and constructing reinsurance treaties. In particular, the economic concept of utility to justify choosing Pareto-optimal forms of risk exchange. We also drew a bridge between Pareto-optimal treaties in a reinsurance market and the value of a cooperative non-transferable m person game. The methodologies described identify optimal risk exchange strategies, considering at the same time the insurer and the reinsurer (in the most simple case). Later on the the thesis, we will be looking at optimal reinsurance strategies by looking at the insurer in isolation. This approach, therefore, assumes that reinsurance is available at the prices the insurer is looking for. However, we will be considering a wide range of prices in order to allow for the total possible range.

4. One-Year Scenario

4.1. Introduction

The total profit generated by an insurance company consists of a different number of components, namely: mortality profit, expense profit, investment profit and miscellaneous profit. We will be assessing the utility of the mortality profit in isolation from the total profit received by the shareholders.

Utility theory requires us to look at the utility of the whole profit. This is also the sole interest of the shareholders of the insurance company. However, it may be possible to assume, under certain conditions, that all other sources of profit are equal to zero or constant (eg, by assuming a constant rate of interest), apart from the mortality profit. This could be the case when the investment income component (for example in companies issuing pure protection contracts), is insignificant, or that investment is done at a group level and therefore outside the control of the insurance company. Also that all expense charges are equal to expense loadings in the office premium definition. The latter could happen, for a subsidiary of an insurance company selling business in a different country. It is becoming common, in these cases, to see the back-office of the subsidiary in the head office. The subsidiary is then charged a fixed amount every year for the cost of managing the portfolio at the head office. It is therefore easier in these cases, to include in the office premium structure, of the subsidiary, a loading approximately equal to the annual charge from the head office.

Investment profits tend to be used to compensate adverse results arising from the mortality experience in any year. They can have a large impact on the financial results, but if interest rate levels decrease, the relative importance of investment as opposed to mortality profit decreases.

Mortality profit is directly linked to the underwriting guidelines the company has put in place and to the pricing of its products. Controlling the mortality profit can be done through different ways and a possible one is with the help of reinsurance treaties. We have looked at the utility of using reinsurance arrangements in order to reduce the exposure to risk and volatility of mortality profits. It can be argued that, the sole purpose of insurance companies is to accept risk. Nevertheless, insurance companies have to make the most efficient use of the capital that is in support of the business they write. This twofold relation is the primary concern of insurance companies.

In addition we have also considered the level of capital that is in support of the mortality risk. The approach is to assess the utility of the extra return obtained by the shareholders, by utilizing their capital in support of the mortality risk, as opposed to investing in a risk-free way. The level of capital required is naturally linked with the exposure to risk. Reinsurance treaties reduce the exposure to risk and, as a natural consequence, the required level of capital that is in support of the mortality risk is reduced as a function of the amount of reinsurance bought.

Section 4.2 starts by introducing expressions for the calculation of the expected value and variance of the mortality profit, for a policyholder aged x. These results are then generalized to a more complex portfolio. The reinsurance treaty type to be used in the analysis is defined and the formulae are adapted to reflect the expected value and variance of the retained mortality profits. Finally for the chosen utility function, expressions for the expected value of the utility are derived.

In order to analyze the utility of the mortality profit, some hypothesis regarding the portfolio's characteristics have to be made. The distribution of sums assured in the portfolio, the number of policyholders per sum assured and the number of different ages are defined in Section 4.3. The initial level of capital is linked to the concept of ruin and the necessary formulae for its calculation are presented in this section. Finally, some comments regarding the utility function and identification of the basic parameters to be looked at in the process of maximization are made.

Section 4.4 presents the results obtained when an analytical approach and a one year time horizon is considered. It looks into the impact on the retention level when different sets of basic parameters are considered.

4.2. Theoretical Background

Consider a new policy, written by a policyholder aged **x**. The sum at risk during the first year, maybe represented as **S**. The first year's death strain (**DS**) is a random variable with probability density function:

$$P[DS = S] = q_x \quad P[DS = 0] = p_x \tag{4.1}$$

where $p_x=1-q_x$; and with expected value and variance:

$$E[DS] = P[DS = S] \cdot S + P[DS = 0] \cdot 0 = q_x \cdot S \tag{4.2}$$

$$Var[DS] = E[DS^2] - (E[DS])^2 = q_x \cdot S^2 - q_x^2 \cdot S^2 = q_x \cdot p_x \cdot S^2 \tag{4.3}$$

Insurance companies would normally charge policyholders an annual risk premium equal to the expected value of DS plus an additional safety loading, to protect against adverse mortality experience. This safety loading is normally expressed as a percentage of the expected value of DS and will be represented by α. If **ES** is equal to the annual risk premium charged to the policyholders and **MP** a random variable representing the first year's mortality profit, then the following relation holds:

$$MP = ES - DS \tag{4.4}$$

where the constant ES, can be set equal:

$$ES = (1+\alpha)\cdot E[DS] = (1+\alpha)\cdot q_x \cdot S \tag{4.5}$$

The expected value and variance of MP is:

$$E[MP] = ES - E[DS] = (1+\alpha)\cdot E[DS] - E[DS] = \alpha \cdot q_x \cdot S \tag{4. 6}$$

$$Var[MP] = Var[ES - DS] = Var[DS] = q_x \cdot p_x \cdot S^2 \tag{4.7}$$

Let us now consider a portfolio of **n** independent policies where all policyholders are aged **x** and $\mathbf{S_k}$ represents the sum at risk for the kth policy (k=1,...,n). If D_k is equal to the number of deaths (random) arising from the kth life, so that:

$$\begin{gathered} P[D_k = 1] = q_x \\ P[D_k = 0] = p_x \\ E[D_k] = q_x; \quad Var[D_k] = q_x \cdot p_x \end{gathered} \tag{4.8}$$

the mortality profit can be written as:

$$MP = \sum_{k=1}^{n} (1+\alpha)\cdot q_x \cdot S_k - \sum_{k=1}^{n} D_k \cdot S_k \tag{4.9}$$

where α is the safety loading introduced above and is the same for all policyholders. The expected value is now equal to:

$$E[MP] = (1+\alpha)\cdot q_x \cdot \sum_{k=1}^{n} S_k - q_x \cdot \sum_{k=1}^{n} S_k = \alpha \cdot q_x \cdot \sum_{k=1}^{n} S_k \tag{4.10}$$

and the variance:

$$\mathrm{Var}[MP] = \mathrm{Var}\left[\sum_{k=1}^{n} (-S_k) \cdot D_k\right] = \sum_{k=1}^{n} S_k^2 \cdot \mathrm{Var}[D_k] = q_x \cdot p_x \cdot \sum_{k=1}^{n} S_k^2 \qquad (4.11)$$

If we now are looking for the expected value of the utility of the mortality profit, we are trying to calculate the value of E[U(MP)], where U(.) represents the utility function. Let us represent by **S** the total aggregate claim amount for the portfolio and by $f_S(S)$ its density function. The expected value of the utility of the mortality profit for the portfolio in question can be calculated as follows:

$$E[U(MP)] = \sum_{y=0}^{M} f_S(y) \cdot U\left(\sum_{k=1}^{n} (1+\alpha) \cdot q_x \cdot S_k - y\right) \qquad (4.12)$$

where $M = \sum_{k=1}^{m} S_k$ is the maximum possible aggregate amount and $y = \sum_{k=1}^{n} D_k \cdot S_k$. The sum between brackets, excluding y, represents the total premium intake the insurer gets from the policyholders.

If for example the chosen utility function is:

$$U(MP) = 1 - \exp(-MP/r) \qquad (4.13)$$

where **r** is the risk aversion parameter, then the expected value of the utility of the mortality profit, becomes:

$$E[U(MP)] = \sum_{y=0}^{M} f_S(y) \cdot \left(1 - \exp\left(-\left(\sum_{k=1}^{n} (1+\alpha) \cdot q_x \cdot S_k - y\right) \Big/ r\right)\right) \qquad (4.14)$$

The density function of S is dependent on the different S_k existing in the portfolio. A recursive formula will be introduced later on when specific assumptions regarding the S_k have been made.

Let us introduce a reinsurance treaty by defining the premium rate from the reinsurer, which will be equal to $\mathbf{(1+\beta).q_x}$ and represent the fixed retention level by **L**. The margin β can be seen as the net effect of loadings less commissions. We will ignore for now other elements of a reinsurance treaty. From the **n** policies of the portfolio let **m** identify the number of policies out of the total n policies such that $S_i \le L$ (i=1,…,n). For the remaining **n-m** in the portfolio we have $S_i > L$. It will also be assumed in this split of the portfolio that all S_k (k=1,...,n) are known and that the first m indexes correspond to the m policies identified above. This means that the portfolio is ordered by increasing size of sum assured, and that each sum assured class includes just one policy.

We can rewrite MP as:

$$MP = (1+\alpha)\cdot q_x \cdot \sum_{k=1}^{n} S_k - (1+\beta)\cdot q_x \cdot \left(\sum_{k=m+1}^{n} S_k - (n-m)\cdot L \right) -$$

$$- \sum_{k=1}^{m} D_k \cdot S_k - L \cdot \sum_{k=m+1}^{n} D_k \tag{4.15}$$

The expected value of the retained MP now becomes:

$$E[MP] = (1+\alpha)\cdot q_x \cdot \sum_{k=1}^{n} S_k - (1+\beta)\cdot q_x \cdot \left(\sum_{k=m+1}^{n} S_k - (n-m)\cdot L \right) -$$

$$- q_x \cdot \sum_{k=1}^{m} S_k - L \cdot q_x \cdot (n-m) =$$

$$= q_x \cdot \left(\alpha \cdot \sum_{k=1}^{n} S_k - \beta \cdot \left(\sum_{k=m+1}^{n} S_k - (n-m)\cdot L \right) \right) \tag{4.16}$$

and the variance:

$$Var[MP] = Var\left[\sum_{k=1}^{m} (-S_k)\cdot D_k + \sum_{k=m+1}^{n} (-L)\cdot D_k \right] =$$

$$= \sum_{k=1}^{m} S_k^2 \cdot Var[D_k] + L^2 \cdot \sum_{k=m+1}^{n} Var[D_k] =$$

$$= q_x \cdot p_x \sum_{k=1}^{m} S_k^2 + q_x \cdot p_x \cdot (n - m) \cdot L^2 =$$

$$= q_x \cdot p_x \cdot \left(\sum_{k=1}^{m} S_k^2 + (n - m) \cdot L^2 \right) \tag{4.17}$$

If the retention level increases, so does the expected value of the retained mortality profit and its variance. The decision-maker has to find the balance between expected retained profit, which increases with L, and volatility of retained profits measured by the variance of profits, which also increases with L. The decision will depend on the level of risk aversion of the decision-maker.

The expected value of the utility of the retained mortality profit will obviously depend on the chosen retention level. In this case, S will represent the total aggregate retained amount, and $f_S(S)$ its corresponding density function. The resulting expression for the expected value of the utility of the mortality profit, for a given retention L, is equal to:

$$E[U(MP)] = \sum_{y=0}^{M} f_S(y) \cdot U\left((1+\alpha) \cdot q_x \cdot \sum_{k=1}^{n} S_k - (1+\beta) \cdot q_x \cdot \left(\sum_{k=m+1}^{n} S_k - (n-m) \cdot L \right) - y \right) \tag{4.18}$$

where $M = \sum_{k=1}^{n} S_k + (n-m) \cdot L$ is equal to the maximum possible retained aggregate amount for a given L. Also the expression within the outer-parentheses (excluding y), represent the total retained premium. If we consider again the exponential utility function, then:

$$E[U(MP)] = \sum_{y=0}^{M} f_S(y) \cdot \left(1 - \exp\left(-\left((1+\alpha) \cdot q_x \cdot \sum_{k=1}^{n} S_k - \right.\right.\right.$$

$$\left.\left.\left. - (1+\beta) \cdot \left(\sum_{k=m+1}^{n} S_k - (n-m) \cdot L\right) - y\right) \Big/ r\right)\right) \quad (4.19)$$

We will now consider the case where there is more then one policyholder per sum assured. If we represent by t_k the number of policyholders with sum assured S_k, then the expected value and variance of the retained mortality profit is now equal to:

$$E[MP] = q_x \cdot \left(\alpha \cdot \sum_{k=1}^{n} t_k \cdot S_k - \beta \cdot \left(\sum_{k=m+1}^{n} t_k \cdot (S_k - L)\right)\right) \quad (4.20)$$

$$Var[MP] = q_x \cdot p_x \cdot \left(\sum_{k=1}^{m} t_k \cdot S_k^2 + \sum_{k=m+1}^{n} t_k \cdot L^2\right) \quad (4.21)$$

In this case the expected value of the utility of the retained mortality profit, is given by:

$$E[U(MP)] = \sum_{y=0}^{M} f_S(y) \cdot \left(1 - \exp\left(-\left((1+\alpha) \cdot q_x \cdot \sum_{k=1}^{n} t_k \cdot S_k - \right.\right.\right.$$

$$\left.\left.\left. - (1+\beta) \cdot q_x \cdot \left(\sum t_k \cdot S_k - (n-m) \cdot L\right) - y\right) \Big/ r\right)\right) \quad (4.22)$$

where M and the density function of S should be recalculated to reflect the new composition of the portfolio.

Until now, we have not allowed for different ages in the portfolio. If this is the case, then the previous formulae only need a minor adjustment to reflect this new assumption. The adjustment in the expected value, for example, would consist in repeating formula (4.20) for each different age, and sum the results over all different ages. A similar approach could be used for the subsequent two formulae.

The retained premium calculation would have to consider all different ages and existing sums assureds in the portfolio. Also the density function of S would have to be recalculated. The resulting expression for the expected value of the utility of the mortality profit would be very similar.

When trying to identify from the formula above the retention level that maximizes the expected value of the utility of the retained mortality profit, we ought to consider in the model underlying formula (4.22), capital requirements and the question of when profits emerge during the year.

If we assume that profits emerge at the end of the year and that evaluation of the expected value of the utility of profits is done in that point of time, then formula (4.22) needs no further adjustment.

Let us introduce the total initial capital at moment zero, represented by C_0. We will first discuss how it should be taken into account, in the model discussed above, and then restate the formula for the expected utility of the retained mortality profit when capital requirements are included.

If a capital of 1 unit is invested in a risk-free asset, then the value of the interest to be paid at the end of the year is given by i, where i is the risk free rate of interest. For an invested amount equal to C_0, then the value of the interest to be paid at the end of the year, is equal to $i.C_0$.

The inclusion of capital gives an exact idea of the added value created by this investment to the shareholders of the company. Through this approach, we are considering that a specific amount of capital is allocated to support the mortality risk, which is essentially to protect the policyholders' interests, in other words, to ensure that an acceptably low probability of ruin is obtained. The shareholders of the company have the alternative possibility of investing this capital in a risk free way. In order to accept this sort of investment it has to generate a higher return, or in other words, an extra return. In the model above, we will therefore deduct from the value of the mortality profit the amount $i.C_0$. We do not use a risk discount rate in addition to i, because the stochastic risk is allowed for specifically by the variability of the mortality profit. In other words, it is the additional return from the mortality profit that corresponds to the additional risk rate of return. Through this approach, if the retention level goes up, so does C_0, in order that the overall mortality risk, in terms of the probability of ruin, is kept under control. Also the mortality profit will increase, but has at least to compensate, in utility terms, for the cost of the increase in $i.C_0$.

The shareholders will assess if the extra return is enough to compensate for the level of risk associated with it. The decision is mainly dependent on the level of risk aversion of the shareholders. Therefore for the same level of extra return the decision may vary from one group of shareholders to another, due to their different attitude towards risk.

Essentially we are assessing which strategy for controlling risk has the highest utility: use of capital or the use of reinsurance. The optimal compromise between the two is reached by choosing an appropriate level of retention.

If we go back to the calculation of the expected value of the utility of the retained mortality profit, formula (4.22) becomes:

$$E[U(MP)] = \sum_{y=0}^{M} f_S(y) \cdot \left(1 - \exp\left(-\left(\left((1+\alpha) \cdot q_x \cdot \sum_{k=1}^{n} t_k \cdot S_k - \right.\right.\right.\right.$$

$$\left.\left.\left.\left. - (1+\beta) \cdot q_x \cdot \left(\sum t_k \cdot S_k - (n-m) \cdot L\right) - y\right) - i \cdot C_0\right) \Big/ r\right)\right) \qquad (4.23)$$

In the formula above, we are calculating the difference between the retained mortality profit and the return the shareholders would get if they would invest their capital in a risk free way. The former is given by the difference between the sums deducted by the amount of total losses represented by y and the latter by $i.C_0$.

4.3. Assumptions

In order to analyze mortality profit, some hypothesis regarding the portfolio's characteristics have to be made. Let us assume that $S_k=k$ ($k=1,\ldots,n$), so that L becomes m. Also the number of policyholders per sum assured will be constant, equal to t_0 and all policyholders are aged x. In this case the expected value of the retained mortality profit is equal to:

$$E[MP] = q_x \cdot t_0 \cdot \left(\alpha \cdot \sum_{k=1}^{n} k - \beta \cdot \left(\sum_{k=m+1}^{n} k - (n-m) \cdot L\right)\right) =$$

$$= q_x \cdot t_0 \cdot \left(\alpha \cdot \sum_{k=1}^{n} k - \beta \cdot \sum_{k=1}^{n} k + \beta \cdot \sum_{k=1}^{m} k + \beta \cdot (n-m) \cdot m\right) =$$

$$= q_x \cdot t_0 \cdot \left(\alpha \cdot \frac{(n+1) \cdot n}{2} - \beta \cdot \frac{(n+1) \cdot n}{2} + \beta \cdot \frac{(m+1) \cdot m}{2} + \beta \cdot (n-m) \cdot m\right) =$$

$$= q_x \cdot t_0 \cdot \left((\alpha - \beta) \cdot \frac{(n+1) \cdot n}{2} + \beta \cdot \frac{(2 \cdot n - m + 1) \cdot m}{2} \right) \tag{4.24}$$

and the variance:

$$Var[MP] = q_x \cdot p_x \cdot \left(\sum_{k=1}^{m} k^2 + (n-m) \cdot m^2 \right) =$$

$$= q_x \cdot p_x \cdot t_0 \cdot \left(\frac{m \cdot (m+1) \cdot (2 \cdot m + 1)}{6} + (n-m) \cdot m^2 \right) =$$

$$= q_x \cdot p_x \cdot t_0 \cdot \left(\frac{-4 \cdot m^3 + 3 \cdot m^2 \cdot (1 + 2 \cdot n) + m}{6} \right) \tag{4.25}$$

The calculation of the utility of profits, see formula (4.23), needs the density function of S, the total aggregate claims amount. Panjer & Willmot (1992), derived a recursive formula for the density function of aggregate claims for a life portfolio from the probability generating function of S. This formula and its application is given in Section 2.6.2.

In what follows we will call the base scenario the one where all policyholders are aged 45, the number of policyholder per sum assured (t_0) is equal to 100, n=100, i=4%, α=5% and β=2%. With this assumption, highly risk averse investors could sense a guaranteed profit of $\alpha-\beta$ by 100%. But for others, there is still the cost of reinsurance relative to not reinsuring, so one would expect most utility maximizing decisions to have retentions higher than 0%. However, a whole range of β will be looked at, and the one chosen is simply one out of the total possible range.

The exponential utility function will be used to identify the retention level that maximizes expected utility of profits. The shape of the function is dependent on the value of r. A higher value of r identifies a less risk averse investor and vice versa.

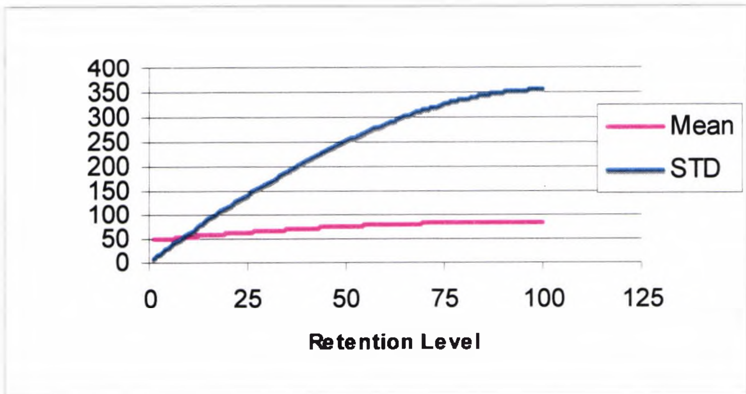
4.4. Results

Before presenting the results obtained using formula (4.23), that is, maximizing the expected value of utility, we will first develop an understanding for the basic scenario and its output variables, namely expected value, standard deviation, etc..

In many real life situations an investor has a fixed amount of capital to invest in the business. It may not be possible to invest more or less capital depending on the level of risk retained. Therefore, we will initially consider a fixed level of initial capital, equal to 325 units. This level of capital is approximately equal to what is required for a retention level equal to 50 and a probability of ruin of 5%.

The expected value and standard deviation of the net mortality profit (net of the cost of capital), for the basic scenario with a fixed level of capital of 325 is given by *Figure 4.1.*

Figure 4.1: Mean-Standard Deviation (STD) of profits as a function of the retention level



The expected value of profits net of the cost of capital increases with the retention level. We assumed a fixed level of initial capital. The return the shareholders would get by investing the capital in a risk free way is always the same, independently of the chosen retention level. As the retention level increases, so does the amount of retained risk and the expected value of the retained mortality profit. Therefore, the expected value of the net profit should also increase. The standard deviation is only dependent on the level of risk retained. It should increase as more business is retained.

From a mean-standard deviation pure analysis there is no optimal point, ie a retention level for which the expected value is maximized and the standard deviation is minimized. The more business is retained, the higher the expected value but also the higher the standard deviation. Also it is not possible to find retention levels that are preferred to others. This would be the case, if, for example, for two retention levels with the same expected value one had a lower standard deviation than the other, and for the same standard deviation, one had a higher expected value than the other.

Figure 4.2 :Probability of ruin as a function of the retention level

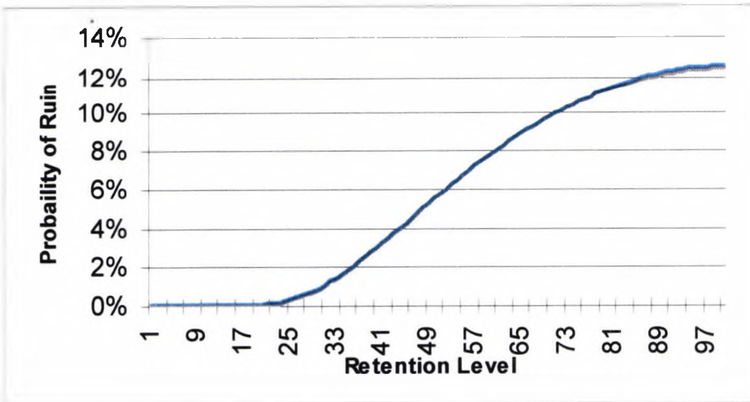


Figure 4.2 illustrates the probability of ruin as a function of the retention level. We would note that the insurance company has a fixed level of capital in support of the business. Since the volatility of profits increases the more business is retained, then it should be expected that the probability of ruin should increase the higher the retention level.

We next considered different levels of capital and its impact on the expected value of profits, net of the cost of capital. The results are presented in *Figure 4.3*.

Figure 4.3: Mean of profits as a function of the initial fixed amount of capital

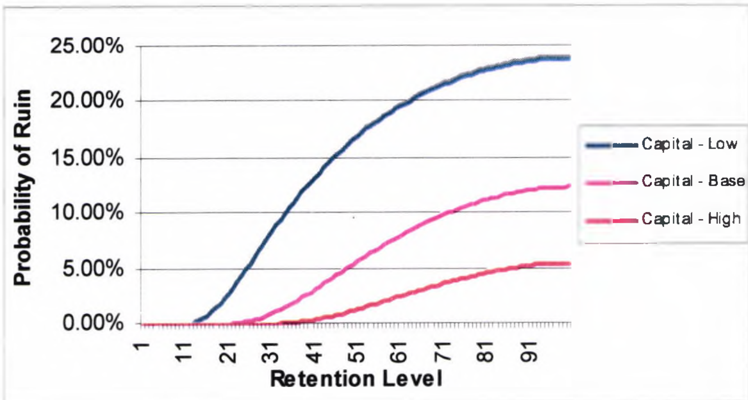
In Figure 4.3, “Capital – Base” identifies the level of capital introduced before, ie equal to 325. “Capital – High” and “Capital – Low” identify levels of capital 50% higher and lower than the base, respectively.

Everything else being equal, a higher level of capital means the shareholders would get a higher total return, if it had been invested in a risk free way. The opposite occurs if a lower level of capital is considered. As a consequence, the expected value of the profit net of the cost of capital should decrease with a higher capital and increase with a lower capital.

When considering a different level of capital we are not changing in any way the amount of business retained and/or ceded. Therefore the volatility of profits does not change with different levels of capital.

As far as the shape of the curves for mean and standard deviation is concerned, they are exactly the same. They are only moved upwards or downwards. From a pure mean and standard deviation analysis, we are still in the presence of a situation that, as the expected value increases, so does the volatility of profits.

Figure 4.4: Probability of ruin as a function of the initial level of fixed capital



The effect on the probability of ruin of considering different levels of capital is shown in Figure 4.4. If the insurer holds a higher capital it is in a less dangerous position than another insurer with less capital. It is clear that the probability of ruin is very sensitive to the level of capital, in considerable contrast to the relative insensitivity of the expected return. It is also possible to see that the insurer has available two ways to control the solvency risk. It can, on the one hand, hold more capital which, as a result, will reduce the probability of ruin, and, on the other hand, reduce the retention level which will also reduce the risk of insolvency.

We will now find the retention level that maximizes the expected value of the utility of profits, but still considering a fixed level of capital and equal to 325 units. We will assume a value of r equal to 2,000.

Figure 4.5: Expected utility of profits as a function of the retention level

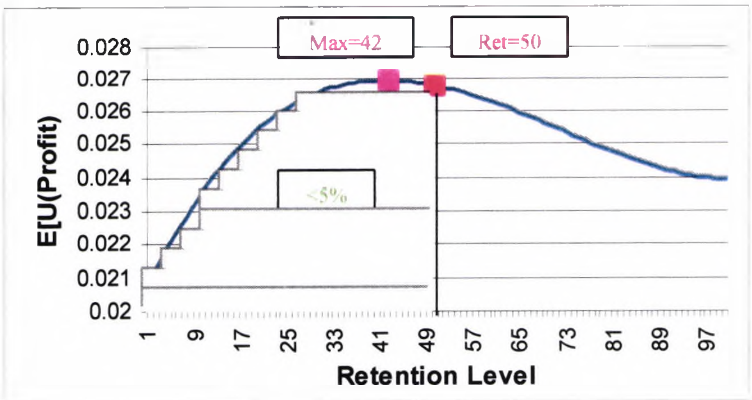


Figure 4.5 above displays the expected value of the utility of profits as a function of the retention level. The shaded area identifies retention levels for which the probability of ruin is less than or equal to 5%. The optimal retention level found, for a value of r equal to 2,000, was 42. The optimal retention level is not subject to any constraint. If we were looking for a retention level with an associated probability of ruin of 5%, given the existing capital, then the chosen retention level would be 50. Looking at the results, the retention level obtained from the maximization of expected utility of profits does satisfy the constraint. This example shows that the application of utility theory has produced a decision which produces an expected value of profit somewhat lower than the maximum possible value subject to a 5% ruin constraint.

Figure 4.6: Expected utility of profits as a function of r

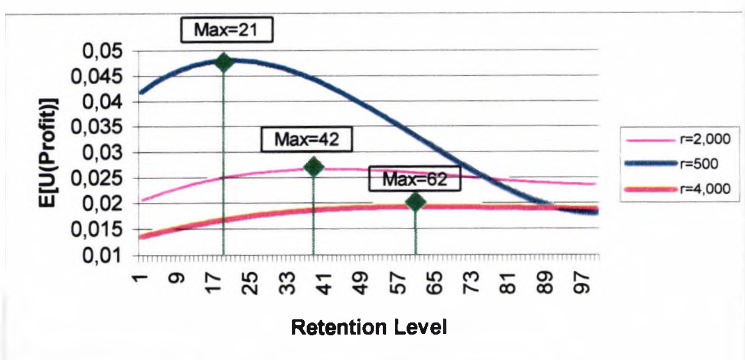


Figure 4.6 shows the impact on the expected value of the utility of profits by considering a different value of r, whilst still keeping the capital fixed and equal to 325. A lower value of r identifies a more risk averse investor and vice-versa.

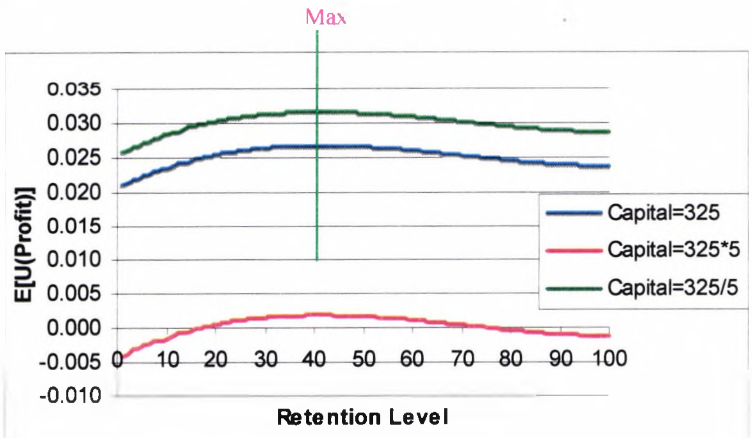
For a lower value of r, or a more risk averse investor, we obtain a lower optimal retention value. In the case of r=500 the optimal retention level is 21. The choice of a retention level of 21 would also meet the probability of ruin constraint of 5%, for the given amount of capital. Hence for this (risk averse) investor, the optimal utility decision also produces a satisfactory probability of ruin.

A higher value of r, equal to 4,000, produces an optimal retention level of 62 which is clearly in line with a less risk averse investor. However, he will be subject to a ruin probability of approximately 8.32%.

The choice of a retention level that does not meet the ruin constraint, does not cause an immediate problem for the shareholders if this retention level is the result of a maximization of the utility of profits. Clearly, risk tolerant investors can tolerate higher ruin frequencies. However, this situation represents a problem for policyholders interests as they face a riskier situation. Hence shareholders interests may have to be sacrificed in order to meet policyholders requirements for solvency.

We will now look at the impact on the expected utility maximization process when different fixed levels of capital are considered. We will assume a value of r equal to 2,000.

Figure 4.7: Expected utility of profits as a function of the initial capital for r=2,000



Looking at Figure 4.7, the first result to note is that the retention level that maximizes expected utility is not affected by the level of capital considered. In absolute terms the expected utility curves move up or down, depending on whether we are considering a lower or higher capital level, respectively. The latter is a straightforward consequence of the fact that the expected level of profits net of the cost of capital, increases (decreases) if we consider a lower (higher) capital and as a consequence the absolute value of the utility of profits should also go up (go down).

From an investor's point of view, the volatility of profits seems to be the key driving force for determining the retention level, regardless of the amount he has invested in the business. If this is the case, then when choosing the optimal retention level for the shareholders, the company does not take into account the level of capital invested simply because it bears no connection with the amount of risk it is taking, in other words, it does not reduce or increase the volatility of profits. The decision is made independently of the level of its initial wealth. Nevertheless, it is quite clear that, overall, expected utility increases the less capital is used, as would be expected.

Figure 4.8: Expected utility of profits for optimal retentions and its corresponding probability of ruin, as a function of the initial capital for r=2,000

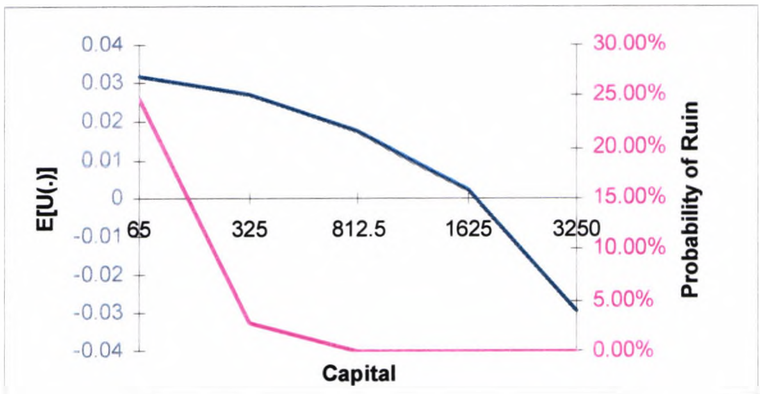


Figure 4.8 gives the expected utility values for optimal retention levels, when we consider different levels of initial capital, and the corresponding probability of ruin.

Even though we have obtained the same optimal retention level when considering different levels of initial capital, for the same value of r, the absolute value of the expected utility of profits does decrease as the level of capital increases. The probability of ruin decreases as the capital goes up simply because the insurer is in a better position to cope with the volatility of profits. From a policyholder's point of view the higher the capital the stronger the insurer is, and the less likely he is expected to face an insolvency situation. However, there is the danger that in order to have a low probability of ruin there is capital being inefficiently used.

Figure 4.8, gives an interesting guide for decision-making. For example, one would choose capital of 325 not 65 because the fall in probability of ruin has more weight than the fall in expected utility. However, to increase capital to 812.5 does not seem worth the fall in expected utility, as it only reduces probability of ruin by a small amount and which is already at an acceptable level. Essentially, the procedure is to maximize expected utility, whilst meeting the ruin constraint. Therefore prefer capital equal to 325.

Figure 4.9: Expected utility of profits for retention levels with a probability of ruin of 5%, as a function of the initial capital for r=2,000

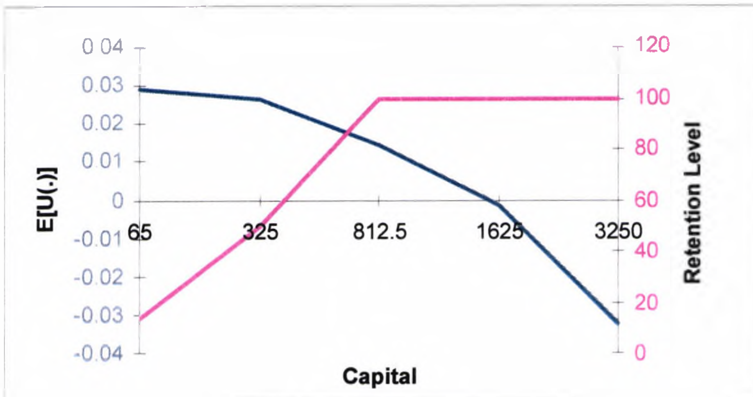


Figure 4.9 above gives the constraint results (probability of ruin less than 5%), as a function of the initial capital and where we have selected r=2,000. The lower the level of capital the insurer has, the less business he can retain in order to meet the ruin constraint. Because all we are doing is increasing the capital, as it increases the retention level also goes up. For the scenario in question, if the insurer retains all the business he can do so provided he puts up an initial capital of 812.5 units. However, in utility terms, it is better for the shareholders to consider low retention levels rather to put in more capital, because it results in a higher expected utility.

A choice of the retention level that maximizes the expected utility of profits does not always meet the ruin probability constraint. This is dependent on the existing level of capital in support of the business. It may not be possible for the investor to raise additional capital to meet the safety condition. The investor may not even be interested, because in essence he is maximizing the utility of profits, which might just be his main concern. Also his risk aversion nature may tolerate higher levels of ruin probability than the one considered. However, he must be interested because a high insolvency risk will reduce the policyholder's security and therefore policyholders will not buy insurance from a risky insurer.

Also, the regulator who looks out for the policyholder's interests may impose security or solvency constraints. This type of situation may force the investor to lock in the business a higher amount of capital than he would normally wish to do. The capital invested in an insurance company is normally determined by the regulator of the legal environment where an insurance company operates. In order to protect the policyholders interests, the regulator would normally require higher levels of capital than would otherwise be done by a normal investor to protect the volatility of the business where he has invested its money.

Figure 4.10: Expected utility of profits as a function of the initial capital and for different values of r (risk aversion coefficient)

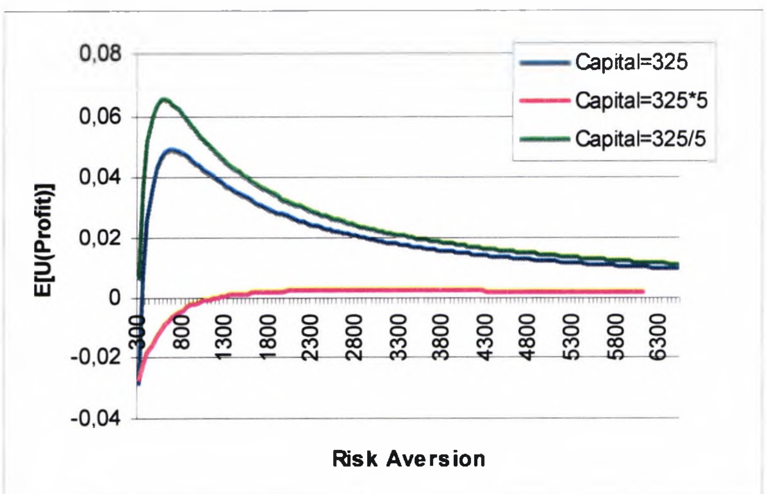


Figure *4.10* gives the expected utility of profits as a function of the initial capital for different values of the risk aversion parameter r. For a given amount of capital it is possible to identify the value of r that maximizes the expected utility of profits, in other words to define the type of investor that would maximize the utility of profits. If we calculate the retention level, for that particular value of r, that maximizes the expected utility of profits, we have identified both the retention level and investor that best suit us. However, it is possible that for the given level of capital, this optimal choice (retention and investor) does not meet the solvency constraint. Some adjustments might have to be considered in order to meet the solvency constraint. This could be done by either reviewing the level of reinsurance or the type of investor.

Figure 4.11: Expected utility of profits for an optimal retenion level and an optimal risk aversion parameter as a function of the initial capital

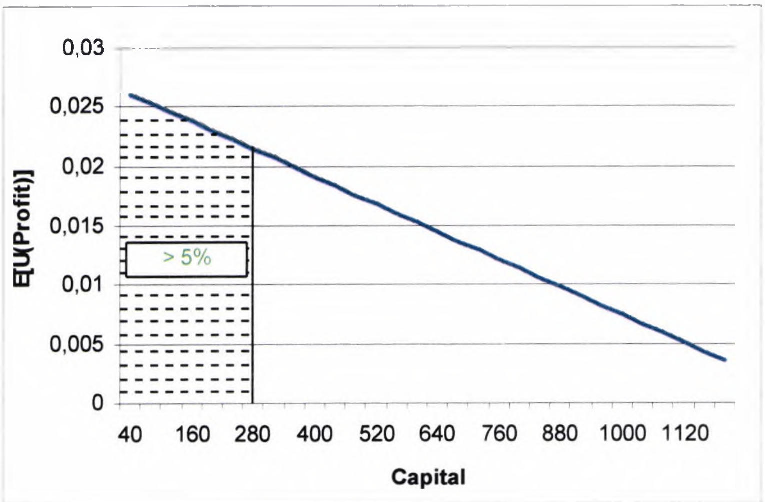


Figure 4.11 shows how the expected utility of profits changes as a function of the initial level of capital and where the shaded area identifies a probability of ruin higher than 5%. Having identified the optimal r and retention level, we are left with the problem of the right choice of capital. This choice should meet the solvency constraint, in ruin probability terms, and also maximize the utility of profits so that

capital is not being used inefficiently. In the example above, one would choose a capital of 280. This approach could be followed for every set of optimal r and retention level. This is the type of optimum which is chosen in Table 4.1 and beyond.

So far we have assumed that the investor has limited resources of capital. For a given level of capital, we have looked at the volatility and expected value of profits, and its associated probability of ruin. We have also seen what would be the retention level for different investors when we were trying to maximize the expected utility of profits and whether their choice satisfied the ruin constraint. Additionally, for a given investor we considered different levels of capital to study its impact on the optimal retention level.

The approach used focused on the set of options available to an investor in the short term. In the long term the investor would expect to invest an amount of capital that is related to the level of risk he is assuming, or in other words the more business he retains the more capital is required and vice-versa. The regulator when assessing the capital requirements, always takes into consideration the amount of business retained by the insurance company. This is generally one of the reasons why insurance companies take reinsurance coverage, to reduce capital requirements. Still the regulator when allowing for lower levels of capital will always bear in mind the necessary solvency requirements, which can be imposed by the ruin probability constraint and will not allow for reinsurance to be taken into account at a 100% level.

In the next set of results, we have allowed for the capital to vary freely as a function of the amount of business retained, along with the retention level, subject to a ruin constraint of 5%. We will find the retention level and the level of capital which together maximize the expected value of profits for different values of r, which identifies different types of investors. Ruin will occur if $MP < -C_0 \cdot (1+i)$ and the associated probability can be represented by:

$$P[\text{Ruin}] = P[MP < -C_0 \cdot (1+i)] = P\left[Z < \frac{-C_0 \cdot (1+i) - E[MP]}{\text{Var}[MP]^{1/2}}\right] \quad (4.26)$$

where E[MP] and Var[MP] are equal to the expected value and variance of the retained mortality profit. The variable Z follows a standard normal distribution. If we use 5% as an acceptable level of probability of ruin, then C_0 is such that:

$$-C_0 \cdot (1+i) = E[MP] - 1.644 \times \text{Var}[MP]^{1/2} \Leftrightarrow C_0 = \frac{-E[MP] + 1.644 \times \text{Var}[MP]^{1/2}}{(1+i)} \quad (4.27)$$

Using the values of the expected value and variance of the retained mortality profit, we are making C_0 dependent on the retention level. It is natural to assume that if more risks are retained, then the volatility of profits increases (increase in the variance) and therefore higher levels of capital should be considered. The latter follows from the way C_0 was defined above.

We will now present the results obtained when using formula (4.23). This formula was used to calculate the retention level and the amount of capital that maximizes the expected value of the utility of retained profits net of the cost of capital, subject to the ruin constraint, when the exponential function is chosen.

In our assumptions we have defined a distribution of sum assured, or in other words the different values of sum assured considered in the portfolio. The retention level to be considered in the portfolio will be equal to one of the possible sums assured.

Formula (4.23) was calculated a number of times equal to the total number of different sums assured in the portfolio. In each time the value of L, would be set equal to a sum assured value and so on until all the possibilities for L were exhausted. By comparison of the results obtained the retention level and the level of capital that maximizes the expected value of the retained mortality profit would be identified.

Table 4.1 shows the results from the base scenario for different values of r.

Table 4.1: Retention level and associated capital expressed as a % of premiums for different vaues of r

r=1,000	r=1,400	r=1,800	r=2,200	r=2,600	r=3,000	r=3,400	r=3,800	r=4,200	r=4,600
Retention Level									
10	14	17	21	24	28	31	34	37	39
Capital as a % of Premiums									
7.5%	11.6%	13.7%	15.7%	16.8%	18.0%	18.7%	19.3%	19.9%	20.2%

The optimal retention level is a function of the parameter r. A higher r identifies a more risk tolerant investor and vice-versa. It is therefore natural to expect that the retention level that maximizes the expected value of the utility of retained profits should increase with r.

From the results obtained before, the optimal retention level was 42 for a capital level equal to 325 and where r=2,000. In Table 4.1, the optimal retention level for a similar value of r would be approximately half of the previous optimal retention level of 42. Clearly, what is now changing the result is the fact that the investor is assumed to hold less capital. Hence a company holding more capital than the optimal amount should find other profitable uses for that excess capital rather than holding it towards this business.

There is an obvious trade-off between capital and reinsurance which is related to how risk tolerant the investor in question is. The higher the value of r or the more risk tolerant the investor is, more capital is preferred as opposed to reinsurance.

Figure 4.12: Mean-Standard Deviation of the return as a function of the level of capital (expressed as a % of premium) for optimal retention levels

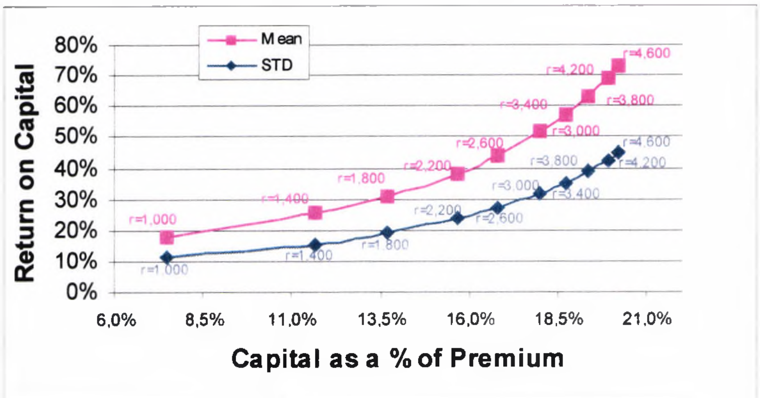


Figure 4.12 shows together the relationship between the mean and standard deviation of the return as a function of the initial capital, capital being expressed as a percentage of premium. In this context and throughout this chapter, return is equal to the ratio between expected mortality profits and initial capital. All the points represented are optimal (utility maximizing) decision points, for a given capital level. It shows that higher risk tolerance leads to decisions which produce higher expected return on capital as well as variance of the return. The return on capital may seem at a first glance high, but they are a consequence of the way capital was calculated (capital was defined as a function of the mean and standard deviation in order to meet the solvency constraint and therefore can be seen as a number of times the mean of total profit).

The required capital to backup the assumed mortality risk increases as the insurer retains more. When the insurer chooses to retain more risk, it faces a situation where retained profits are more volatile and to compensate for this adversity it needs to hold more capital. It looks as if this decision, with the immediate consequence of having to hold more capital, is not balanced in equal terms by an increase on the expected rate of return. On the contrary, if the insurer decides to retain more, it should expect a decrease on the expected level of the rate of return.

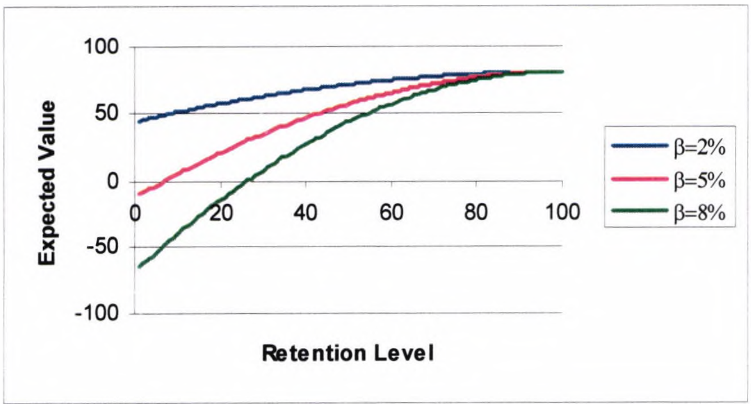
Also it can be argued, that a more risk tolerant investor would have greater tendency to prefer capital as opposed to buying reinsurance in order to meet the solvency constraint, due to the fact that a higher volatility of profits from the retained insurance portfolio has less disutility than before. Sometimes, the insurer might be faced with not enough levels of capital in which case it might be forced to choose lower levels of retention anyway.

4.4.1. Reinsurance Premium Rates

Let us consider once again the basic scenario with a fixed initial capital equal to 325 units, the same amount considered in the previous section.

The impact of different reinsurance premium rates on the expected value of the retained profit net of the cost of capital, is shown in Figure 4.13.

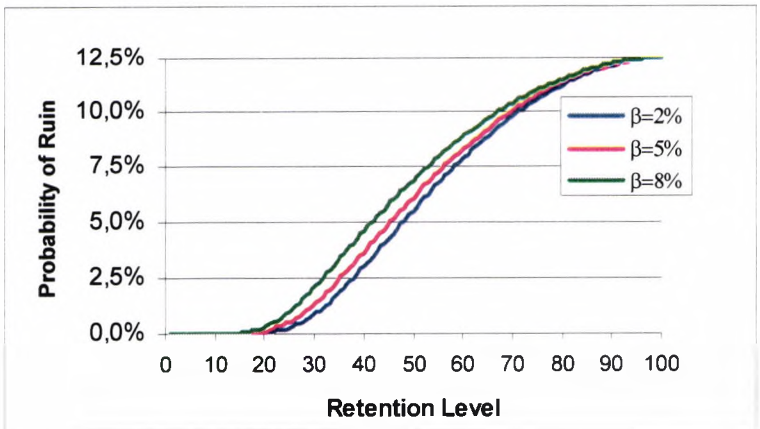
Figure 4.13: Expected value of profits net of the cost of capital as a function of the reinsurance premium rate



The expected value is obviously influenced by the terms of the reinsurance treaty which is in force when business is ceded away. The higher the reinsurance premium rate the lower is the expected profit on ceded business. This is why the curves start to assume lower values, for the same retention level, as the cost of reinsurance increases. Also the difference between different curves decreases for higher retention levels. This is a consequence that, as we move to higher retention levels, less business is reinsured and therefore the loss on ceded business decreases.

The standard deviation is independent of the reinsurance premium rate. If we consider any given level, all values of β lead to the same standard deviation of returns. It is clear, and indeed obvious, that the insurer would prefer minimum cost of reinsurance.

Figure 4.14: Probability of ruin as a function of the reinsurance premium rate for a fixed capital



The effect on the probability of ruin of considering different reinsurance premium rates is shown in Figure 4.14. For a given retention level, a higher β decreases the expected value of retained profits, but has no effect on the level of volatility of profits. Therefore, if the insurer is considering buying more expensive reinsurance at the same level of protection, it should expect to see a slight increase on its insolvency risk, measured in this case by the probability of ruin.

Figure 4.15: Expected utility of profits as a function of the reinsurance premium rate for a fixed capital and r=2,000

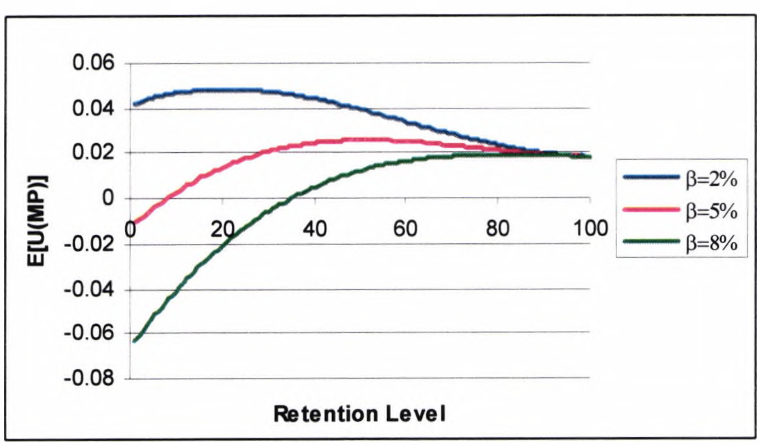


Figure 4.15 above gives the expected value of the utility of profits as a function of the reinsurance premium rate for a fixed level of capital and where r was set equal to 2,000. As we increase the reinsurance premium rate, the retention level that maximizes the expected value of the utility of profits also increases, which is intuitive. For the chosen utility function, the optimal retention level, is the point associated with an absolute amount of expected profits. As the reinsurance premium rate increases, the expected value of retained profits decreases for the same retention level. It is therefore to be expected that the absolute amount of expected profits should be reached at a higher retention level. This result is also due to the fact that the volatility of profits is not affected by different reinsurance premium rates.

It is interesting to see that an optimal retention less than 100 is still obtained even where there is a considerable loss on reinsured business. The value of α used, is equal to 5% and therefore a value of β higher than 5% means that the net profit margin the insurer retains from the annual risk premium charged to policyholders is negative, or in other words the insurer is actually in a net loss position on ceded business. In term assurances, it may well happen that insurers need to sell policies at a very low profit margin. When the reinsurance market is soft the insurer can get better deals and will therefore look for positive net profit margins. If the market is hard then the insurer, still in need of protection, will have to buy reinsurance at a net loss on ceded business.

The retention level increases with β, which is an intuitive consequence of the net profit margin being reduced. This means that even though the same level of protection may still be needed, the financial terms of the agreement do not justify buying the same level of protection as before. Therefore, for the same attitude towards risk the trade off between risk and return is a key decision aspect in the decision buying process of reinsurance, which is quantified by the use of the maximizing utility approach.

We have looked at the impact of considering higher or lower levels of fixed capital for a given reinsurance premium rate. The results showed that the optimal retention levels were not affected with a different capital level. We will again optimize for both the capital and retention level, such that the probability of ruin is not greater than 5%. *Table 4.2* shows the optimal results obtained when β, the reinsurance premium takes different values.

Table 4.2: Retention level and associated capital as a function of the reinsurance premium rate and Capital expressed as a % of Premiums

	r=1,000	r=1,400	r=1,800	r=2,200	r=2,600	r=3,000	r=3,400	r=3,800	r=4,200	r=4,600
					Retention Level					
β=2%	10	14	17	21	24	28	31	34	37	39
β=3%	20	28	35	42	49	55	61	66	71	76
β=4%	30	42	52	62	71	80	87	94	99	99
β=5%	40	55	69	81	92	99	99	99	99	99
β=6%	50	68	85	99	99	99	99	99	99	99
β=7%	60	81	99	99	99	99	99	99	99	99
β=8%	69	93	99	99	99	99	99	99	99	99
					Capital as a % of Premiums					
β=2%	7.5%	11.6%	13.7%	15.7%	16.8%	18.0%	18.7%	19.3%	19.9%	20.2%
β=3%	17.0%	19.1%	20.3%	21.2%	22.0%	22.5%	23.0%	23.4%	23.7%	24.0%
β=4%	20.6%	21.8%	22.6%	23.3%	23.8%	24.3%	24.6%	24.8%	24.9%	24.9%
β=5%	22.3%	23.1%	23.8%	24.4%	24.7%	24.9%	24.9%	24.9%	24.9%	24.9%
β=6%	23.2%	23.9%	24.6%	24.9%	24.9%	24.9%	24.9%	24.9%	24.9%	24.9%
β=7%	23.8%	24.5%	24.9%	24.9%	24.9%	24.9%	24.9%	24.9%	24.9%	24.9%
β=8%	24.2%	24.8%	24.9%	24.9%	24.9%	24.9%	24.9%	24.9%	24.9%	24.9%

There is again a relationship between the retention level, or the amount of reinsurance bought by the insurer and the level of capital. Given a group of investors, the more risk tolerant will choose to invest higher amounts of capital as opposed to reinsurance buying. However, an individual investor would also rather invest more capital than to buy reinsurance under unfavorable terms and conditions. In market terms, this could mean that when the reinsurance market is hard the shareholders are much better off putting more of their capital into the company, if they can afford to do that. When the market is soft it should happen exactly the opposite, i.e. to buy additional reinsurance protection.

Figure 4.16: Optimal retention levels as a function of the reinsurance premium rate and the value of r

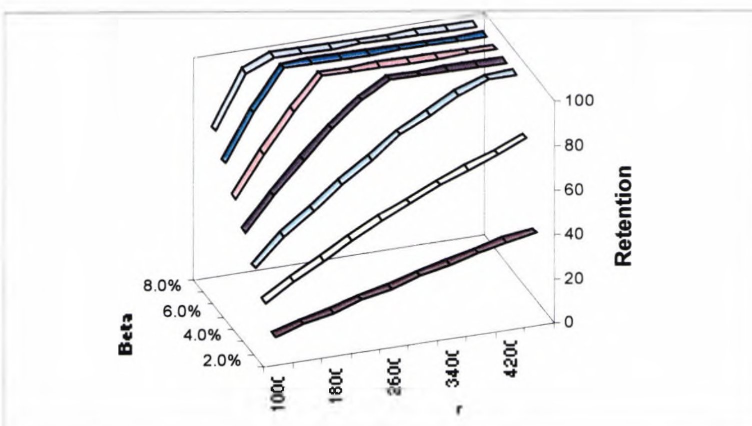


Figure 4.16 above gives the relationship between the retention level that maximizes expected utility of profits, the parameter r, which measures the investor's attitude towards risk and also the reinsurance cost. To a higher value of r corresponds a less risk averse investor and therefore it is natural to expect that it should correspond to higher retention levels. The effect of having a higher reinsurance cost also affects the

retention level. Also the less risk averse investor moves faster to higher retention levels than the opposite type of investor.

Figure 4.17: Mean of profits net and gross of capital cost as a function of the reinsurance premium rate for optimal results at r=1,800

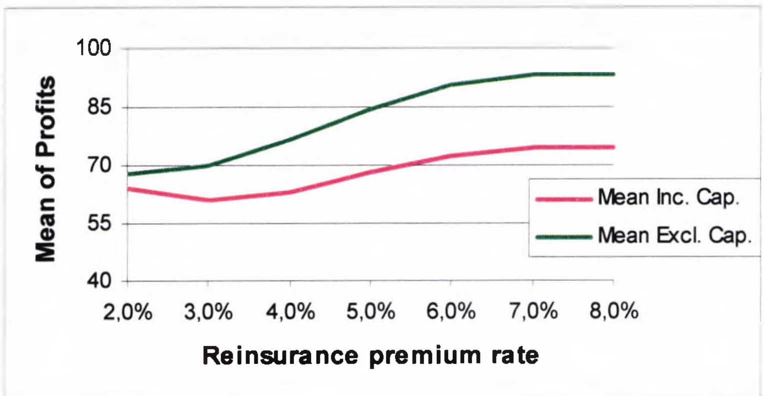


Figure 4.17 above illustrates the relationship between the expected value of profits net and gross of the cost of capital, for optimal retention levels, when the value of β changes and r is equal to 1,800. The cost of reinsurance increases sharply the variability of profits by the insurer preferring to increase the retention level. Also from a reinsurance premium cost of 2% to 3% there is a decrease in the mean value of retained profits net of the cost of capital. An increase in the retention implies incurring a higher cost of capital, because the insurer is required to hold more capital to balance a higher volatility, which is clearly causing a reduction in the expected value of profits.

If we fix the value of r, at say 1,800, we could be looking at the results from the point of view of one insurance company or a decision-maker. The increase in the mean only reflects that the company is prepared to cede less and less as the net loss position increases. As a consequence, the volatility of profits increases, which is reflected by an increase in the standard deviation, see *Figure 4.18*.

Figure 4.18: Mean-Standard Deviation of the return as a function of the reinsurance premium rate and the initial capital (expressed as a % of premium) for r=1,800

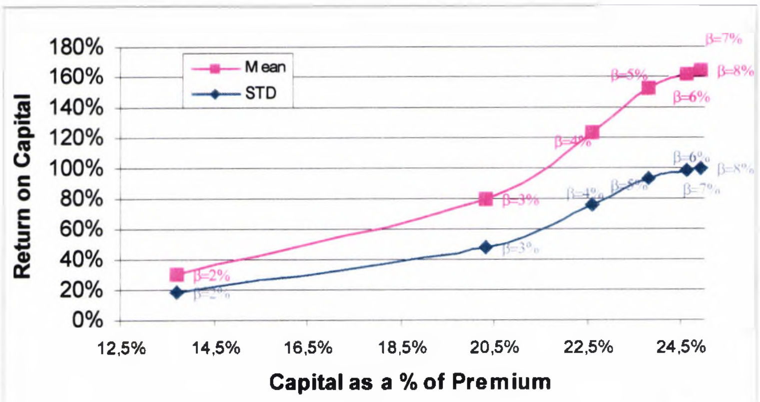
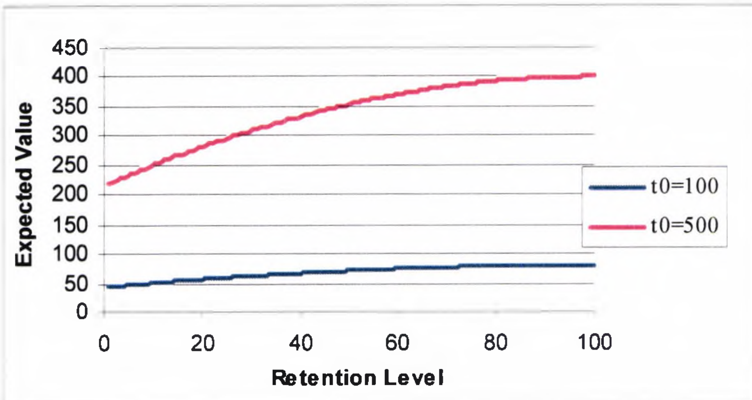


Figure 4.18 shows how the capital requirements, expressed as a % of premium, are related with the change in the mean and standard deviation of the rate of return. Since the insurance company will consider higher retentions as β increases, for the same value of r, it will also need to hold higher levels of capital to compensate for the increase in volatility. Therefore, it is to be expected to see an increase in both the mean and standard deviation of the rate of return.

4.4.2. Portfolio Size

We will initially study the impact on the mean and standard deviation of the profit net of the cost of capital when we change the number of policyholders per sum assured. Also we will consider the initial level of capital to be fixed and equal to 325 units. However, when the number of policyholders per sum assured changes, the fixed amount of capital is recalculated such that the ratio between premium and capital remains the same in all cases.

Figure 4.19: Expected value of profits net of the cost of capital as a function of the policyholders per sum assured



The simple increase in the number of policyholders per sum assured has the expected effect of increasing the expected value of profits gross of the cost of capital, simply because for all possible sums assured we have 500 instead of 100 policyholders. However, when we look at the value of profits net of the cost of capital, the increase is not proportional to the increase in the number of policyholders, because of the effect of the cost of capital that has a greater impact when the size of the portfolio is low.

The cost of the capital borrowed by management of an insurance company from the shareholders to run the company, needs to be paid back at the end of the year. It is therefore, in the best interest of the management to write as much business as it can because it can more easily service the capital. This attitude can put the company in a risk of insolvency, which is normally avoided through the ruin constraints imposed by the regulator. Therefore, even though the management might be compelled to write more and more business, there is a limit to what it can and should accept. The reinsurance treaty it has in force, can also be used as a way to increase the amount of business accepted because, part of it, is transferred to the reinsurer, and also because it reduces the capital requirements.

Figure 4.20: Coefficient of Variation as a function of the retention level and number of policyholders per sum assured

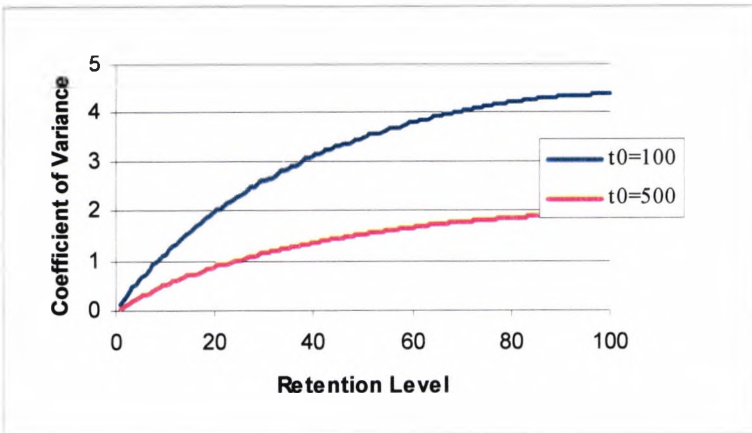
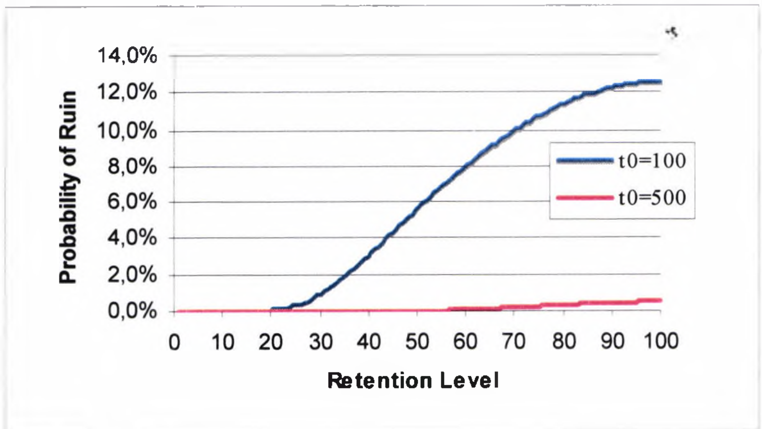


Figure 4.20 shows the coefficient of variance, the ratio between the standard deviation and mean of profits, as a function of the number of policyholders and the retention level. The standard deviation of the retained profits naturally increases with the size of the portfolio. When measured as a percentage of the expected value, the standard deviation decreases as the portfolio increases. This means that, as the size of the portfolio increases, the insurer gets more stability in retained profits.

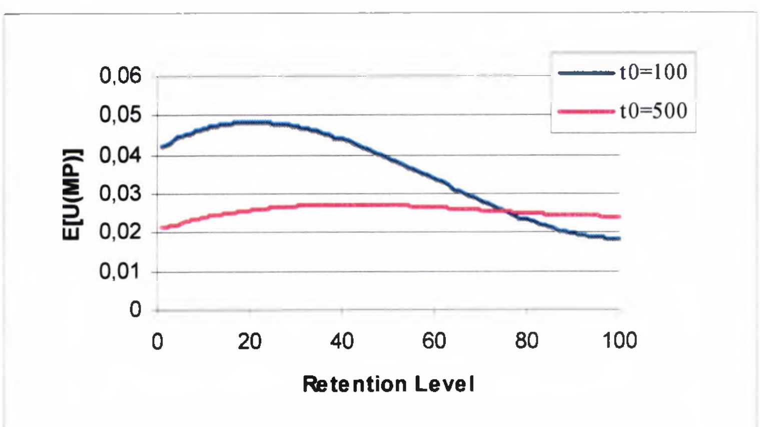
The management has therefore two advantages in writing more business. On the one hand, it is easier to service the capital with a bigger portfolio and, on the other hand, the stability of profits is much better. As mentioned before, the only negative aspect is that the capital resources limit the amount of business the company is allowed to write.

Figure 4.21: Probability of ruin as a function of the number of policyholders per sum assured



The probability of ruin as a function of the size of the portfolio is displayed in Figure 4.21. We have seen that an increase in the size of the portfolio leads to a decrease in the coefficient of variation which measures the level of volatility in the retained portfolio. Since we are keeping the same level of relative capital in both cases, then the probability of ruin should decrease as the number of policyholders per sum assured increases.

Figure 4.22: Expected utility of profits as a function of the number of policyholders per sum assured for fixed capital and r=1,000



The expected utility of profits, as a function of the number of policyholders per sum assured, is shown in the graph above for r equal to 1,000. If we increase the number of policyholders per sum assured, the stability of profits increases, then the insurer can choose a higher retention level in return for the reduction in the volatility of profits.

In the long term, the investor will relate the amount of capital invested with the amount of business retained. This is achieved by letting the capital vary as a function of the chosen retention level.

The next table looks at the effect on the optimal retention level, of increasing the number of policyholders per sum assured, whilst keeping everything else to be as in the basic scenario and subject to the constraint that the probability of ruin is ≤5%.

Table 4.3: Retention level as a function of the number of policyholders per sum assured and Capital expressed as a % of Premiums

	r=1,400	r=1,800	r=2,200	r=2,600	r=3,000	r=3,400	r=3,800	r=4,200	r=4,600	r=5,000
					Retention Level					
t_0=100	14	17	21	24	28	31	34	37	39	42
t_0=200	18	23	28	33	37	42	46	50	54	58
t_0=300	20	26	32	37	42	47	52	56	61	65
t_0=400	22	28	33	39	45	50	55	60	65	70
t_0=500	23	29	35	41	47	52	58	63	68	73
t_0=600	23	30	36	42	48	54	60	65	70	75
t_0=700	24	30	37	43	49	55	61	67	72	77
					Capital as a % of Premium					
t_0=100	11.6%	13.7%	15.7%	16.8%	18.0%	18.7%	19.3%	19.9%	20.2%	20.7%
t_0=200	7.3%	9.2%	10.5%	11.5%	12.2%	12.9%	13.3%	13.7%	14.1%	14.4%
t_0=300	5.0%	6.8%	7.8%	8.8%	9.4%	9.9%	10.3%	10.6%	11.0%	11.2%
t_0=400	3.8%	5.3%	6.2%	7.0%	7.6%	8.1%	8.4%	8.7%	9.0%	9.3%
t_0=500	2.8%	4.1%	5.1%	5.8%	6.4%	6.8%	7.2%	7.4%	7.7%	7.9%
t_0=600	1.8%	3.3%	4.2%	4.9%	5.4%	5.8%	6.2%	6.4%	6.7%	6.8%
t_0=700	1.3%	2.5%	3.5%	4.2%	4.6%	5.1%	5.4%	5.7%	5.9%	6.0%

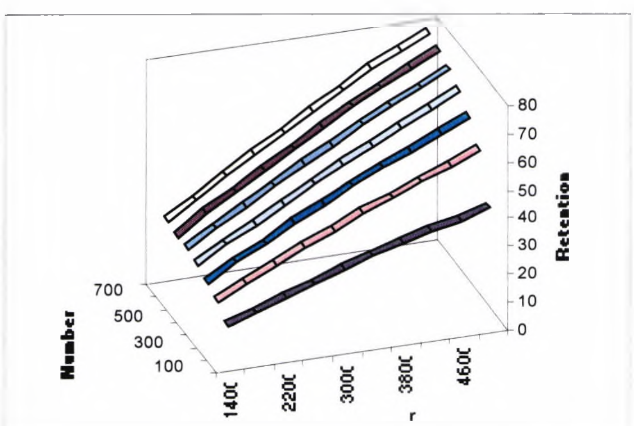
As the size of the portfolio gets bigger, the volatility of profits is reduced when measured through the coefficient of variation. The results obtained indicate that it is possible to increase retention as t_0 increases. The increase in the retention is very gradual.

Both capital and retention seem to behave in a similar way. But as r increases, the optimal retention rises more quickly than optimal capital (latter seems to level off as r rises). This reflects the reduced disutility from the volatility of returns the higher the value of r, so making reinsurance less popular.

When imposing a probability of ruin constraint, it is possible to arrive at meeting that constraint by any of a whole spectrum of capital/ reinsurance combinations. We have chosen one according to the expected utility of the profits to the shareholders. Hence, a good demonstration of the use of utility theory in decision making.

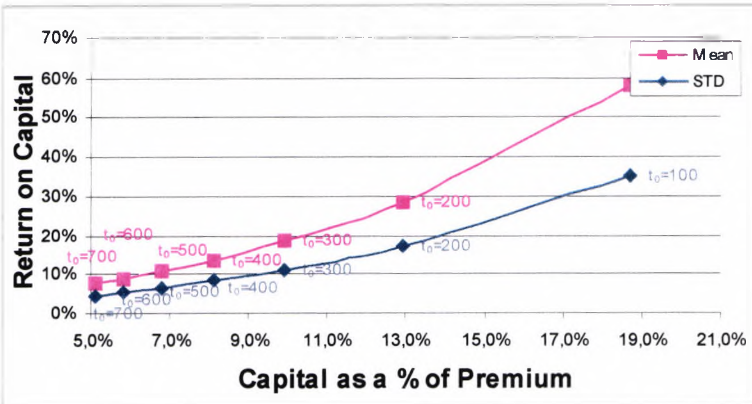
Table 4.3 is next displayed graphically:

Figure 4.23: Optimal retention levels as a function of the number of policyholders per sum assured and the value of r



The rate of increase of the retention level as a function of both the size of the portfolio and the parameter r does not change significantly. There is only a slight increase when we are moving towards higher values of t_0.

Figure 4.24: Mean-Standard Deviation of the return as a function of the initial capital (expressed as a % of premium) and the number of policyholders per sum assured for r=3,400



From Figure 4.24, we can see that as the size of the portfolio increases, there is a reduction in the required level of capital in relative terms. This confirms the idea already put forward, that with a larger portfolio we have a lower profit variation. Also by expecting less variation in profits the insurer can also expect to achieve a lower rate of return.

4.4.3. Policyholder Age

We will initially study the effect on the mean and standard deviation of considering a different age from the one that was assumed in the basic scenario. Also, as it was done in the previous sections, we will initially assume that the initial capital is fixed and equal to 325. However, when the policyholders' age changes, the fixed amount of capital is recalculated such that the ratio between premium and capital remains the same in all cases.

Figure 4.25: Expected value of profits as a function of the policyholder's age

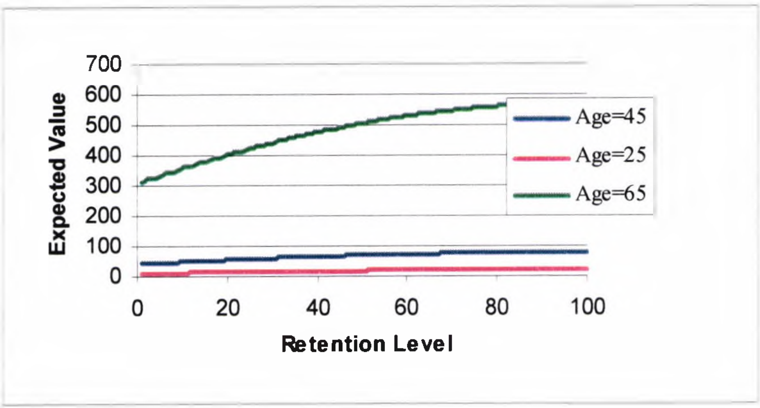
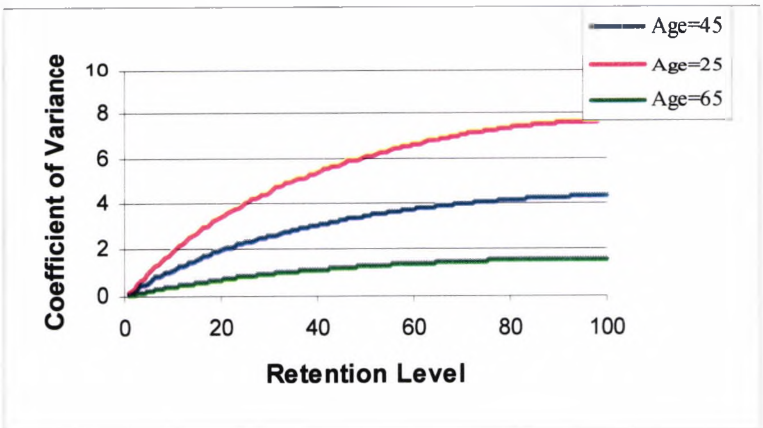


Figure 4.25 shows the expected value of profits net of the cost of capital. The expected value of profits increases with age and vice-versa. This is a consequence of the change in the underlying q_x for different ages, whilst keeping everything else constant. It is also for the same reason that the standard deviation of profits is affected by the fact of considering different ages. In fact, the standard deviation will increase with the policyholders age. Nevertheless, if we look at the coefficient of variance (ratio between the standard deviation to the mean of profits), the result is quite the opposite. This can be seen in Figure 4.26.

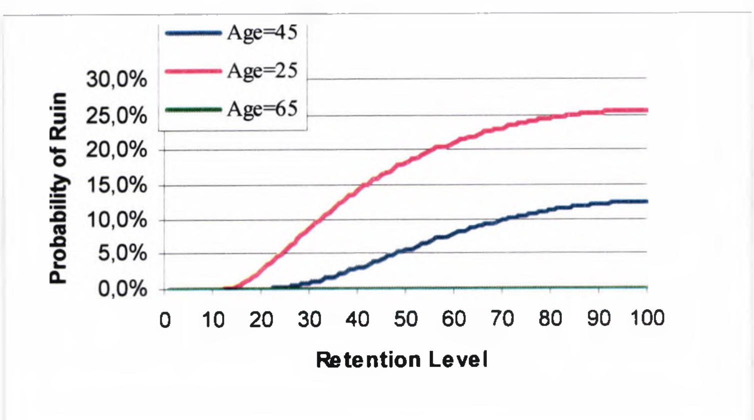
Figure 4.26: Coefficient of Variation as a function of the policyholder's age



If we are facing an increase in the value of q_x we are in fact increasing the premium volume intake for the insurer. As a side effect, the volatility of profits has decreased when measured against the premium volume. This means that a claim of any given size will affect the insurer's results differently, depending on the policyholders age. The older the policyholders are in the portfolio, the lesser the impact will be, in relative terms.

The probability of ruin when different ages are considered is shown in Figure 4.27.

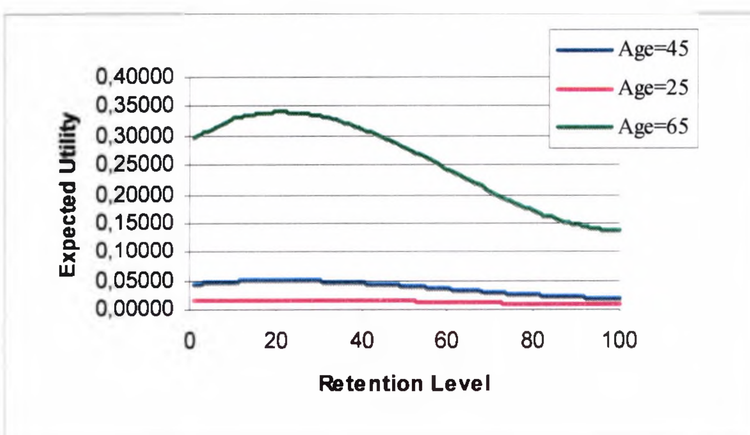
Figure 4.27: Probability of ruin as a function of the policyholder's age with a fixed level of initial capital



The probability of ruin decreases as a higher policyholder's age is considered. As it was seen before, both the expected value and the standard deviation of profits are very sensitive to the underlying age assumption. Since we are assuming a level of capital that reflects the different levels of volatility (different age assumptions and ratio between premium and capital constant), the probability of ruin should increase when the volatility of profits increases and vice-versa. In the case where we have an older portfolio with a lower coefficient of variance (lower volatility), than the risk of ruin should also be the lowest.

The insurer is facing an increasing probability of ruin as the policyholder's age decreases. If the insurer is facing a higher probability of ruin and has limited capital resources then it has to lower its retention to bring the ruin constraint to more acceptable levels.

Figure 4.28: Expected utility as a function of the policyholder's age with a fixed level of initial capital and r=2,000



The difference between optimal retentions, as a function of the policyholders age, displayed by Figure 4.28, is marginal. This seems to suggest that the use of relative capital levels, in expected utility terms, has the consequence of diluting the effect of the cost of capital in the maximization of expected utility of profits.

The results of the optimization process when different ages are considered and where capital varies as a function of retained business, can be seen in Table 4.4:

Table 4.4: Optimal retention and capital levels as a function of the policyholder's age and Capital expressed as a % of Premium, subject to Pr(ruin)≤5%

	r=2,500	r=2,900	r=3,300	r=3,700	r=4,100	r=4,500	r=4,900	r=5,300	r=5,700	r=6,100
					Retention Level					
x=25	4	5	5	6	6	7	7	8	8	8
x=30	5	6	7	7	8	8	9	10	10	11
x=35	8	10	11	12	13	14	15	15	16	17
x=40	16	18	20	22	24	26	28	29	31	32
x=45	23	27	30	33	36	39	41	44	46	49
x=50	30	34	38	42	46	49	53	56	59	62
x=55	35	40	45	49	54	58	62	66	70	74
					Capital as a % of Premiums					
x=25	4.3%	9.7%	9.7%	13.8%	13.8%	17.0%	17.0%	19.7%	19.7%	19.7%
x=30	8.9%	13.0%	16.2%	16.2%	18.8%	18.8%	21.0%	22.8%	22.8%	24.4%
x=35	16.1%	20.0%	21.5%	22.8%	24.0%	25.0%	25.9%	25.9%	26.7%	27.5%
x=40	19.9%	21.2%	22.3%	23.2%	24.0%	24.7%	25.3%	25.6%	26.1%	26.4%
x=45	16.4%	17.7%	18.5%	19.1%	19.7%	20.2%	20.5%	21.0%	21.2%	21.6%
x=50	12.8%	13.5%	14.2%	14.7%	15.2%	15.5%	15.9%	16.2%	16.4%	16.7%
x=55	9.3%	10.0%	10.6%	11.0%	11.4%	11.7%	11.9%	12.2%	12.4%	12.6%

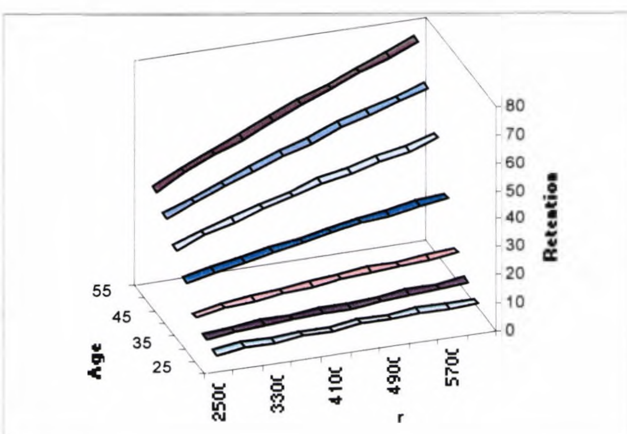
From the table above, we can see that the optimal retention level increases with age. This is explained by the fact that each individual death, for a given sum assured, in the case of a younger portfolio has a much bigger relative effect on profits than in the case of an older portfolio, because its cost is that much greater in relation to the size of the premium.

If we look at the results obtained for the level of capital expressed as a percentage of premiums for optimal retention levels, then it is possible to identify two factors influencing the pattern shown. To an older portfolio corresponds a higher retention level and as a consequence the volatility of retained profits increases. The insurer needs therefore to hold higher levels of capital in relative terms. However, there is a point where the premium increase is enough to justify, that in relative terms, the insurer starts to lower the capital level. This is a similar situation to what happen when we were looking at the effect of increasing the number of policyholders.

Another interesting aspect to look at, would be what would have happened if premium income had been kept constant and simply changed the sum assured distribution with age. In this case, we would be considering the same level of expected profits independently of the policyholders age. However, to an older portfolio would correspond a lower total amount at risk for the same level of premium income, in other words a lower degree of volatility of profits. It would seem intuitive to see, in this case, also for older ages higher optimal retention levels.

Table 4.4 is next displayed graphically.

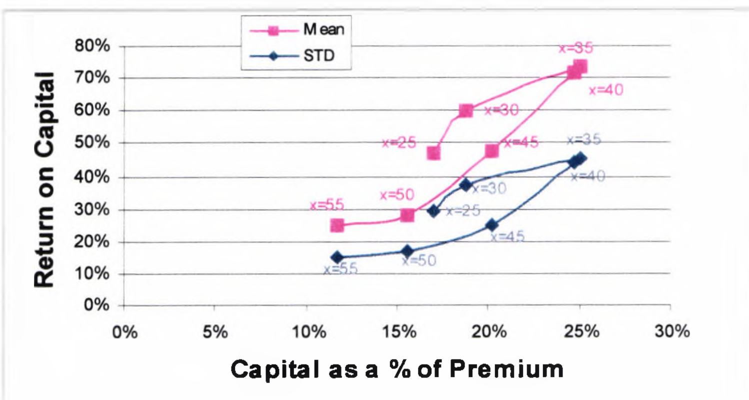
Figure 4.29: Optimal retention levels as a function of the policyholder's age and the value of r



The rate of increase of the retention level is highly influenced by the policyholders age. The older the age the steeper is the rate of increase and vice-versa. This shows that the investor is more cautious when dealing with younger portfolios than with older ones.

In this case we can see that the effect of assuming a different age has clearly changed the way a risk averse investor chooses the optimal retention level. As we consider older ages the investor moves quicker to higher retention levels. In an older portfolio we have an increase in premium volume and a decrease in the coefficient of variance. Figure 4.30 displays the relationship between the mean-standard deviation and the capital.

Figure 4.30: Mean-Standard Deviation of the return as a function of the initial capital (expressed as a % of premium) and policyholder's age for r=4,500



Both the expected value and standard deviation of the return the insurer gets, as a function of the policyholders age, increases until age 35 and then decreases until age 55. This is a consequence of the fact that for optimal retention levels, capital as a percentage of premiums also displays the same pattern, as it can be seen in Table 4.4.

Therefore for higher capital levels and less retained business, the insurer should expect a higher rate of return.

4.5. Summary

In this chapter we looked at optimal retention levels obtained when an analytical approach and a one year time horizon is considered. In particular, we looked at the impact on optimal retention levels when different sets of assumptions of basic parameters are considered.

For a given (fixed) level of capital the optimal retention may be less than that which secures the 5% ruin constraint (Figure 4.5). However, for the same retention level, the expected utility can be increased by holding less capital (assuming that capital can be controlled=variable) (Figure 4.7). Note that the optimal retention level does not change when capital is varied, only the probability of ruin does. Capital is therefore being used inefficiently and the company should reduce capital at least to the point where the probability of ruin becomes equal to the constraint of 5%.

Alternatively, for a given level of capital the optimal retention level may be higher than that which secures the 5% ruin constraint (Figure 4.6). This is done by considering different values of r, the risk aversion parameter. Also for the same hypothesis, if we vary the value of r, we identify the type of investor (value of r) that maximizes the utility of profits (Figure 4.10). Having identified an optimal r and retention level, we were left with the choice of the "right" amount of capital, in such a way that it meets the solvency constraint and is being used efficiently. The choice is shown in Figure 4.11.

The same approach was used when changes in the distribution of the basic portfolio were introduced, namely: reinsurance premium rate, number of policyholders per sum assured and policyholders'age.

From the results obtained, it was shown that the insurer has two ways to control solvency. It can hold more capital or buy more reinsurance protection, because both will reduce the probability of ruin.

Further, when assessing the effect on the optimal retention level of considering higher/lower capital levels, the results demonstrated that in some cases capital will start to be used inefficiently and this, is something to keep in mind when choosing the capital level. Finally, different optimal retentions were obtained when the portfolio characteristics change.

However, the approach used is restricted to a one year case and therefore does not allow for more realistic long term policies to be considered. In order to do that, a stochastic model of a life insurance company needs to be considered, so that, it is possible to study what would be the optimal retention level that optimizes expected utility of profits for policies with longer terms. The latter is the purpose and scope of Chapters 5 and 6.

5. Multi-Year Scenario

5.1. Introduction

A stochastic model of a life insurance company has been developed which simulates the evolution of a life portfolio for a given time horizon. It assumes a standard portfolio of temporary assurance policies without profit sharing and a typical surplus risk premium reinsurance treaty. Within the model, all payments are at the beginning of the year, except for claim payments and profit commission from the reinsurance treaty, which are assumed to happen at the end of the year.

Section 5.2 starts by introducing the model on a gross basis, when the reinsurance treaty is excluded. This is done by a series of expressions which are used to derive the mortality profit in any point in time. Expressions for the different sources of mortality profit are also given. Finally, the reinsurance treaty assumptions and definitions are introduced and the previous expressions are generalized.

Section 5.3 introduces the calculation of the stochastic claim experience. The number of policyholders who die in any year follows a binomial distribution. The model simulates independently the number of deaths using a binomial distribution.

The expected utility maximization procedure is covered by Section 5.4. In any year the model calculates the mortality profits. They are accumulated at a given interest rate to the valuation date. This value is deducted of the interest generated by the initial capital if it was invested in a risk free way throughout the investigation period. The utility of this "net" value can then be calculated. This process is repeated a chosen number of times for a given retention level, and the average of all the utility values is calculated. By repeating it for all the possible retention levels the best retention level can be identified.

The basic assumptions of the model are described in Section 5.5 These are the basic parameters, for each class, in the initial distribution.

Section 5.6 concentrates on testing the stochastic model. This is done by comparing the results obtained in the analytical approach with the ones coming out of the stochastic model. Also, a test is done to assess the number of times we need to repeat the evolution of the portfolio until the best retention can be identified.

Section 5.7 starts by presenting the results for a closed portfolio. Different sets of assumptions are considered in order to test their effect on the retention level. The results for an open portfolio are shown in Section 5.8 in the same lines as for the closed portfolio.

5.2. The Stochastic Model

The stochastic model projects the evolution of a portfolio of life assurance policies. It assumes that all payments are at the beginning of the year, except for claim payments and profit commission from the reinsurance treaty. We will start by introducing some notation.

Notation

n	Number of different sums assured in the portfolio;
a	Number of different ages in the portfolio;
j	Valuation date, i.e. contracts anniversary no. j or the j'th year;
i_j	Valuation rate of interest in year j;
S_k	kth sum assured;
$t_{s,j,k}$	Number of policyholders aged s with a sum assured S_k at the beginning of year j, (s=1,..., a; k=1,..., n);
$V_{s,j}$	Pure net premium reserve at the end of year j for a policyholder aged s for a sum assured of one (s=1,..., a) (see Section 2.3.5);
MP_j	Mortality profit in year j for all policyholders in the portfolio;
$E_{s,j,k}$	Administration expenses in year j for policyholders aged s with a sum assured S_k;
$C_{s,j,k}$	Claims paid during year j for policyholders aged s with a sum assured S_k;
$q_{s,j}$	Premium basis rate of mortality in year j for a policyholder aged s;
$\ddot{q}_{s,j}$	Actual rate of mortality in year j for a policyholder aged s;
L	Retention level;
$Pr_{s,j}$	Reinsurance premium rate in year j for policyholders aged s;
$Cr_{s,j,k}$	Reinsurer's share of paid claims during year j for policyholders aged s with a sum assured S_k;
$Er_{s,j,k}$	Reinsurer's share of expenses (commissions) in year j for policyholders aged s with a sum assured S_k;
Pre_j	Reinsurance expenses in the profit commission account calculation in year j;
LF	Maximum number of years a loss is allowed to be carried forward in the profit commission account;
LCF_j	Loss to be carried forward in year j in the profit commission account calculation;
$PC\%$	Profit commission percentage;

PC_j ; Profit commission in year j

g_j Profit commission calculation excluding loss carried forward;

AMP Accumulated value of future mortality profit;

AMPc Accumulated value of future mortality profit including capital effect;

PMP Present value of future mortality profit;

PMPc Present value of future mortality profit including capital effect;

C_0 Initial capital;

U(.) Utility function;

r Risk aversion parameter;

α,β Mortality loadings in office and reinsurance premiums.

In the notation introduced above, we made no reference to the term of the policy and to the number of years the evolution of the portfolio is to be projected. For presentation purposes, we will, for now, assume that all policies in the portfolio have the same term, equal to **p**, are all newly issued, and also that we will simulate the portfolio's evolution for p years, or in other words until all risks expire. Variations of this assumption will be dealt with later.

Let a class in the portfolio be defined as a group of policyholders all with the same age. If we consider the class of the portfolio where all policyholders are aged i, then the total mortality profit in year j can be defined as follows:

$$MP_j = \sum_{s=1}^{a}\sum_{k=1}^{n} t_{s,j,k} \cdot q_{s,j} \cdot [S_k - V_{s,j}] - \sum_{s=1}^{a}\sum_{k=1}^{n} t_{s,j,k} \cdot q'_{s,j} \cdot [S_k - V_{i,j}] \qquad (5.1)$$

The reinsurance policy to be used, will be a typical surplus risk premium treaty where the reinsurer pays the insurer the amount above the retention level L, on every claim. The annual reinsurance risk premium is based on the net amount at risk for each policy.

We will start by deriving an expression for the profit commission account. The calculation can be done at a class level, but generally profit commission accounts are done for the whole portfolio (portfolio in this context meaning a group of policies ceded through the same treaty). The calculation of the profit commission is subject to numerous variations and in this context is assumed to comprise the following elements:

Income	*Outgo*
Premiums	Claims
Reserves at the beginning of year	Commissions
Interest on reserve deposit	Reserves at year-end
Loss to be carried forward, if any	Loss brought forward from previous years, if any
	Reinsurer's expenses
	Profit, if any

Let the profit commission account in year j, be represented by the function g_j. The function g_j can be defined analytically in the following way:

$$g_j = \sum_{s=1}^{a} \sum_{k=1}^{n} \Big\{ t_{s,j,k} \cdot \max\{0, S_k - V_{s,j} - L\} \cdot Pr_{s,j} \times$$

$$\times \left[1 - Pr\,e_j\right] - Cr_{s,j,k} - Er_{s,j,k} \Big\} \qquad (5.2)$$

where $Cr_{s,j,k} = t_{s,j,k} \cdot q'_{s,j} \cdot \max\{0, S_k - V_{s,j} - L\}$.

From the expression above, the unearned premium reserves have been excluded because since it was assumed that premiums are paid at the beginning of the year, by the year end all risks in force have expired and therefore no unearned premium reserves are needed. This also causes that no interest on reserve deposit is generated on this type of reserve. In this type of treaty no other type of reserves are normally considered apart from the unearned premium reserves.

In the formula above, two types of expenses have been considered, namely: commissions and the reinsurer's expenses. Both are in effect expenses incurred by the reinsurer, but they should be considered separately because of their different nature. The commissions paid back by the reinsurer to the insurer are a contribution to reduce the insurer's financial strain incurred when writing business. The other source of expenses are the actual expenses incurred by the reinsurer through managing the treaty, but only reflected in the calculation of the profit commission calculation. Their level is normally low at approximately 1% of ceded premiums.

The balance to be carried forward in the profit commission account, can be defined in the following way:

$$LCF_j = -\min\left\{0, \sum_{t=j-LF}^{j-1} g_t, \sum_{t=j-LF+1}^{j-1} g_t, \ldots, g_{t-1}\right\} \tag{5.3}$$

and the profit commission in year j, is equal to:

$$PC_j = \max\{0, PC\% \cdot [g_j - LCF_j]\} \tag{5.4}$$

The total mortality profit, at the end of year j when the reinsurance treaty is considered is given by:

$$\sum_{s=1}^{a}\sum_{k=1}^{n} t_{s,j,k} \cdot q_{s,j} \cdot [S_k - V_{s,j}] - \sum_{s=1}^{a}\sum_{k=1}^{n} t_{s,j,k} \cdot q'_{s,j} \cdot [S_k - V_{s,j}] -$$

$$-\sum_{i=1}^{a}\sum_{k=1}^{n} t_{s,j,k} \cdot \max\{0, S_k - V_{s,j} - L\} \cdot Pr_{s,j} \cdot [1 + i_j] +$$

$$+\sum_{i=1}^{a}\sum_{k=1}^{n} Cr_{s,j,k} + PC_j \qquad (5.5)$$

The mortality profit can further be split into:

Mortality profit from retained risk

$$\sum_{s=1}^{a}\sum_{k=1}^{n} t_{s,j,k} \cdot q_{s,j} \cdot \min\{S_k - V_{s,j}, L\} -$$

$$-\sum_{s=1}^{a}\sum_{k=1}^{n} t_{s,j,k} \cdot q'_{s,j} \cdot \min\{S_k - V_{s,j}, L \qquad (5.6)$$

Mortality profit from ceded risk

$$\sum_{i=1}^{a}\sum_{k=1}^{n} t_{s,j,k} \cdot q_{s,j} \cdot \max\{0, S_k - V_{s,j} - L\} -$$

$$-\sum_{s=1}^{a}\sum_{k=1}^{n} t_{s,j,k} \cdot \max\{0, S_k - V_{s,j} - L\} \cdot Pr_{s,j} \cdot [1 + i_j] + PC_j \qquad (5.7)$$

The model can assume either an open or a closed portfolio, depending on the set of assumptions defined at the start. If the portfolio is closed, therefore not allowing for new policyholders, then $t_{s,j,k}$ represents the number of policyholders left aged i with a sum assured S_k at the beginning of year j.

If the portfolio allows for new policyholders or is open, then $t_{i,j,k}$ will be equal to the number of policyholders left which are aged i with a sum assured S_k, at the end of year j-1, plus the new policyholders joining this class at the beginning of year j.

It is also possible to consider a start up operation or an on going operation. The difference will be reflected in the distribution of $t_{i,0,k}$ (s=1,...,a; k=1,...,n) and the way we project the portfolio's evolution. The different scenarios and assumptions will be dealt with in more detail in the results section where each situation will be analyzed separately.

At the beginning of each projection, the number of years to project the evolution of the portfolio is defined. It excludes from the portfolio at the start of year j all policies with a term less than j and this will be immediately reflected in the distribution of $t_{s,j,k}$.

In all different cases, the model will project the portfolio's evolution until all the risks have expired, in other words until the portfolio is extinguished. In an open portfolio situation it is possible to introduce the assumption that the portfolio becomes closed after a chosen number of years. This approach deals with the problem of having different terms in the portfolio and also sets the number of years to project the portfolio's evolution to be equal to the maximum term in the portfolio. Through this approach, it is possible to calculate the utility of accumulated profits plus the present value of future profits in any point in time. The valuation can be done at a chosen point in time which might be different from the value of the maximum term in the portfolio.

As mentioned before, the model will only focus on mortality profit. Therefore, the assumed hypothesis regarding $q_{i,j}$, $q_{i,j}$ and the reinsurance treaty terms and its conditions will play a major role in the results obtained from the model and as a consequence on the implied retention level that maximizes expected utility. As it was done in the Section 4.4, different sets of assumptions will be chosen to determine its influence in the mortality profit.

5.3. Stochastic claim experience

The policyholders in the portfolio were split into different **a.n** classes, according to the notation introduced above. The model simulates independently the number of deaths for each class in the portfolio.

Let us now consider the class of policyholders aged i with a sum assured S_k. From the previous notation, at the beginning of year j, the existing number of policyholders in this class is equal to $t_{s,j,k}$. The number of policyholders, from this class, who die during year j, is a random variable with a binomial distribution with parameters $t_{s,j,k}$ and $q'_{s,j}$. The distribution function F(x) for this random variable is equal to:

$$F_{s,k}(x) = \sum_{y=0}^{x} \binom{t_{s,j,k}}{y} \cdot [q'_{s,j}]^{y} \cdot [1 - q'_{s,j}]^{t_{s,j,k}-y} \tag{5.8}$$

with the expected value and variance $t_{s,j,k}.q'_{s,j,k}$ and $t_{s,j,k}.q'_{s,j,k}.[1- q'_{s,j,k}]$, respectively.

The model starts by computing the cumulative distribution $F_{s,k}(x)$ for the class in question, using the formula above. It will then sample from a uniform random variable distributed in the interval (0,1) to finally determine the exact number of deaths in the class in question. If the realized value of the uniform random variable is equal to **b**, then there exists x such that:

$$F(x') \leq b < F(x'+1) \tag{5.9}$$

The number of deaths in year j, for the class in question, is then set equal to x'. The same process is repeated separately for every class of the portfolio (k=1,...,n; s=1,...,a).

In order to simulate a uniformly distributed number over the interval (0,1) a linear congruential generator was used. All modern computers have random generators based on this method. These generators operate with integers and produce random observations on the unit interval by division. A third-order linear congruential generator was used. The following equation represents the process:

$$W_n = k_1 \cdot W_{n-1} + k_2 \cdot W_{n-2} + k_3 \cdot W_{n-3} \pmod{p} \tag{5.10}$$

where W_n represents the random integer at time n. The random integer W_n is divided by p to produce a random observation on the unit interval. The full length of the cycle is p^{s-1}. The values of p and k_1, k_2 and k_3 have to be chosen carefully. The chosen values were p=997,783, k_1=360,137, k_2=519,815 and k_3=616,087, see for example Hossack et al (1983).

5.4. Expected Utility Maximization

The total mortality profit in year j was defined by formula (5.5) where the retention level was set to L.

In the stochastic model we have also considered initial capital requirements. The methodology and rationale is identical to that of Chapter 4.

Let us consider that the maximum term for all policies in the portfolio is equal to p. The accumulated future expected mortality profits at the end of p years, represented by AMP, is equal to:

$$AMP = \sum_{j=1}^{p} MP(j) \cdot \prod_{k=j}^{p} [1 + i_k] \tag{5.11}$$

The total accumulated value of the mortality profit, net of the cost of capital is given by:

$$AMPc = AMP - C_0 \cdot \left[\prod_{j=1}^{p} [1 + i_j] - 1 \right] \tag{5.12}$$

where the sum on the right hand side represents the return that shareholders would get by investing the capital in a risk free way. In the formula above we have allowed for the risk free rate of interest to vary by year.

The utility of the expression above is equal to:

$$U(AMPc) = 1 - \exp\left(- \frac{AMPc}{r} \right) \tag{5.13}$$

where the exponential utility function has been used and **r** is the risk aversion parameter.

As mentioned before, the model projects the portfolio's evolution until the last policy expires. It accumulates, at a given interest rate, the mortality profit generated in any year until the end of the projection. At this point, it then calculates the utility of the accumulated mortality profit, net of capital cost, for the projection.

The model projects stochastically the portfolio's evolution several times. Each time the model calculates U(AMPc) and its value is stored in a database. At the end, the simple average of all projections is calculated for the chosen retention level L and the result is stored as the expected utility of the present value of mortality profit, for this given retention level.

The process is repeated for different possible retention levels and the value that maximizes the value of the expected utility can then be identified. The possible retentions used are all the existing S_k (k=1,...,n) in the portfolio.

5.5. Assumptions of the Model

The model projects a portfolio of life assurance policies. Initially, it looks at a portfolio of p-year term assurance (see Section 2.3) policies without profit sharing, and where all policyholders are aged x at the start of the projection. It assumes a predetermined number of new policyholders at moment zero, and then projects its evolution until maturity.

The distribution of the new policyholders is set at the start. The basic parameters, for each class, in the initial distribution are as follows:

- Age;
- Term of the policy;
- Sum assured;

- Number of policyholders;
- Expected value of the actual mortality rate.

In the previous developments, it was assumed that the expected value of the actual mortality rate would be constant within each class and between different classes, where a class was understood to be a group of policyholders sharing the same basic characteristics.

The way the initial distribution is defined allows for a further generalization. It is possible to link expected value of the actual mortality rate with sum assured or in other words to consider that the higher the sum assured the most likely a claim will occur. Other scenarios can also be looked at, such as underwriting cycles where the expected value of the actual mortality rate changes with time. Also, it is possible to consider that the expected value of the actual mortality rate is a random variable, etc..

The model uses the Swiss mortality table GKM-80. This mortality table is used for premium, reserves and all other actuarial factors needed during the projection. Premium and reserves are calculated using a 4% interest rate. A pure net premium reserve basis is also assumed (see Section 2.3).

The model considers a mortality profit loading, in line with the approach used in Chapter 4. The value of the expected value of the actual mortality rate will be standard and so we will be looking at the effects of considering different values of α.

As mentioned before, the initial reinsurance policy used is a typical surplus risk premium treaty where the reinsurer pays the insurer the amount above the chosen retention level. The reinsurance premium is based on the net amount at risk (sum assured minus reserves and retention level), for each policyholder and follows the annual risk premium method. The reinsurer will pay back a commission to the insurer, which will be expressed as a percentage of ceded premiums.

It will also be assumed that the reinsurance premium rate is based on the expected value of the actual rate of mortality with an additional security loading β. Again, here the same methodology used in Chapter 4 was followed.

For the calculation of the initial capital we need the formula of the expected value and variance of the present value of the mortality profit. Let us consider class s (s=1,...a), with all possible S_k (k=1,...,n) of the portfolio and assume that the number of policyholders per sum assured is equal to one, for illustration purposes. If out of the n policies, there are **m** such that $S_i \leq L$, for a given retention level L, the expected value of the present value of the mortality profit is equal to:

$$E[PMP] = A^{1}_{x:\overline{p}|} \cdot \left(\alpha \cdot \sum_{k=1}^{n} S_k - \beta \cdot \left(\sum_{k=m+1}^{n} S_k - (n-m) \cdot L \right) \right) \tag{5.14}$$

and the variance:

$$Var[PMP] = \left[{}^{2}A^{1}_{x:\overline{p}|} - \left(A^{1}_{x:\overline{p}|} \right)^2 \right] \cdot \left(\sum_{i=1}^{m} S_k^2 + (n-m) \cdot L^2 \right) \tag{5.15}$$

where

$$A^{1}_{x:\overline{p|}} = \sum_{k=0}^{p-1} v^{k+1} \cdot {}_k p_x \cdot q_{x+k} \tag{5.16}$$

and

$${}^{2}A^{1}_{x:\overline{p|}} = \sum_{k=0}^{p-1} v^{2\cdot(k+1)} \cdot {}_k p_x \cdot q_{x+k} \tag{5.17}$$

The last two formalae were first introduced in Sections 2.3.2 and 2.3.3. The generalization of the above formulae to include all classes of the portfolio, and where the number of policyholders per sum assured is different from one, is straight forward.

We have so far described the basic assumptions that underlie the stochastic model. When the results are presented, the assumptions underlying each scenario will be presented in further detail.

It was stated that the model uses a standard insurance policy and reinsurance type of treaty. Nevertheless, it allows for changes to the policy and treaty type, which will be done to test their effect on the retention level.

We will, once again, identify a base scenario, for future reference:

- All policyholders are aged 45 (a=1);
- The number of policyholders per sum assured is equal to 100 and n=100;
- A constant interest rate during the simulation equal to 4% p.a.;
- α=5%;

- $\beta=2\%$;

- Policy term is equal to 10, p=10;

- No profit commission is included.

5.6. Model Testing

The purpose of this section, is to compare the results coming out of the stochastic model against the ones obtained from the analytical approach presented in the previous Chapter. This comparison will allow a check of the performance of the stochastic model.

In order to be able to compare the results, some simplifications in the stochastic model have to be introduced, so that the set of assumptions in both cases are exactly the same. From what was described above, the major difference is in the treatment of how the sum at risk varies with time. In the analytical approach it was assumed that the sum at risk throughout the term of the policy was equal to the sum assured of the policy. This meant that reserves were not taken into consideration. However, in the stochastic model reserves are calculated every year and the sum at risk is therefore a function of the sum assured and reserves. Therefore, reserves will have to be set equal to zero in the latter model.

We have tested the stochastic model for the one-year scenario. In the analytical approach we have defined the same base scenario as above.

We simulated stochastically the one-year scenario using different values of β and where the number of simulations done for each scenario was 10,000. For comparison reasons, we only considered r equal to 1,000. The results and its comparison with the analytical results are shown in Table 5.1:

Table 5.1: Comparison of Optimal Results from the Analytical Approach versus the Stochastic Model

	Analytical Approach			Stochastic Approach		
	Retention Level	Expected Value	Standard Deviation	Retention Level	Expected Value	Standard Deviation
β=2%	10	62.34	64.39	9	59.57	63.59
β=3%	20	53.25	119.44	20	50.78	114.90
β=4%	30	49.39	167.97	32	47.11	163.94
β=5%	40	49.64	212.68	40	47.34	207.64
β=6%	50	52.85	252.21	49	51.46	240.13
β=7%	60	57.92	286.14	60	55.29	274.69
β=8%	69	62.96	311.46	71	61.84	301.30

The table above, gives the retention levels obtained from the stochastic and analytical approaches using the same set of assumptions. The results above have confirmed the performance of the stochastic model against the analytical approach. Overall, the stochastic model gives a good approximation of the optimal retention level, even though the values obtained for the expected value and standard deviation are generally slightly lower than the theoretical values.

We then tested for the base scenario on a 10 years horizon, what would be the minimum number of simulations that had to be run in order to achieve the optimal retention value with sufficient precision. This would also give an idea of how fast the stochastic model converges to the optimal retention value. Table 5.2 gives a picture of the rate of convergence of the stochastic model, for a value of r equal to 1,000 and for different values of β.

Table 5.2: Convergence of the Stochastic Model to the Optimum Retention Level for different values of β and r=1,000

Simulations	β=2%	β=4%	β=6%	β=8%
100	12	19	48	70
250	11	19	48	70
500	10	21	50	71
750	9	20	49	71
1,000	9	20	49	71
2,500	9	20	49	71
5,000	9	20	49	71
10,000	9	20	49	71

The results clearly indicate that from a number of simulations equal to 1,000 we have already achieved the optimal retention level. The performance for different values of r is similar. In what follows we will always run 1,000 simulations for each scenario.

5.7. Closed Portfolio Simulation

Having tested the performance of the stochastic model, we will in this section present and discuss the results for a closed portfolio. The latter means that the insurer is closed to new policyholders throughout the projection period.

Different scenarios will be looked at in order to identify the effect, on the optimum retention level, of using different values for the initial parameters defined for the basic scenario.

In what follows, it will be assumed that if, for a given retention level the capital, derived from the formula introduced in Chapter 4 is negative, then it will be set to zero.

In Section 5.7.1., we will look at the effect of considering higher policy terms and compare the results obtained with those presented in Chapter 4. We will initially look in isolation at the impact of a higher term and, then, of introducing the same changes in the basic scenario, namely: different ages, number of policyholders per sum assured and reinsurance premium rates. In the following sections we will study the impact of introducing additional changes in the basic scenario, whilst keeping a high policy term.

5.7.1. One Year Term versus Multi Year Term

We will start by presenting the results obtained for the basic scenario for different values of r. They are given in Table 5.3:

Table 5.3: Retention Levels for the Basic Scenario and corresponding capital as a % of Premiums

r=1,000	r=2,500	r=5,000	r=7,500	r=15,000	r=50,000
Retention Levels					
27	37	37	37	37	46
Capital as a % of Premium					
0.00%	0.05%	0.05%	0.05%	0.05%	0.86%

The optimal retention level more than doubled just by considering both a time horizon/term of all policies equal to 10. In Table 4.1, for a value of r equal to 1,000, the results indicated an optimal retention of 10 where now it is 27. Another interesting result is the effect on the level of capital, expressed as a percentage of premiums, when considering a ten year horizon. Year on year, the insurer now has a higher annual premium than for the one year case, because it´s a level premium to meet a rising risk. In this situation, it needs to borrow a much lower amount of capital from the shareholders, in relative terms.

Figure 5.1: Variability of profits for the basic scenario

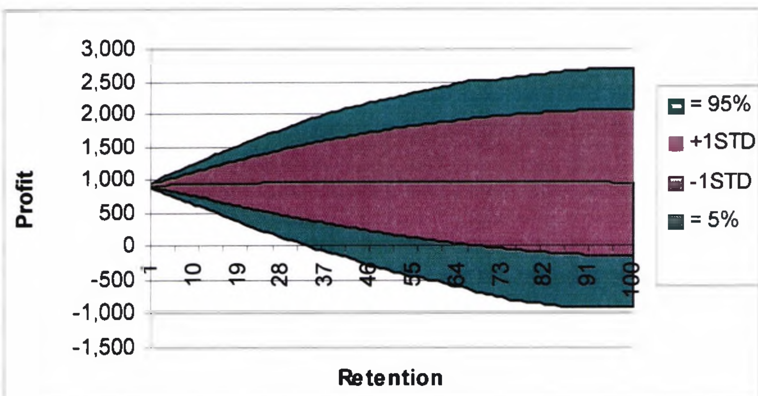


Figure *5.1* illustrates the variability of retained mortality profits as a function of the retention level. The profit shown is the extra return obtained by the shareholders, by utilizing capital in support of mortality risk.

The variability of profits clearly decreases as the retention level also decreases. This is a natural consequence of reducing the amount of retained risk. Also for retention levels lower than 37, the variability of profits is so low, that it is possible to say that the insurer will never be in a ruin position for a 5% level of confidence. Given our definition of capital, this is clearly the case and, therefore, there is no need to hold capital in support of the mortality risk for the chosen level of confidence. The insurance company will obviously need to hold some capital in practice, but it could be used to support the variability of other existing sources of profit. This explains the discontinuity in the progression of optimal retention levels with increasing r at around 37, over a wide range of r.

Figure 5.2: Expected Profit as a function of the Retention Level for the Basic Scenario

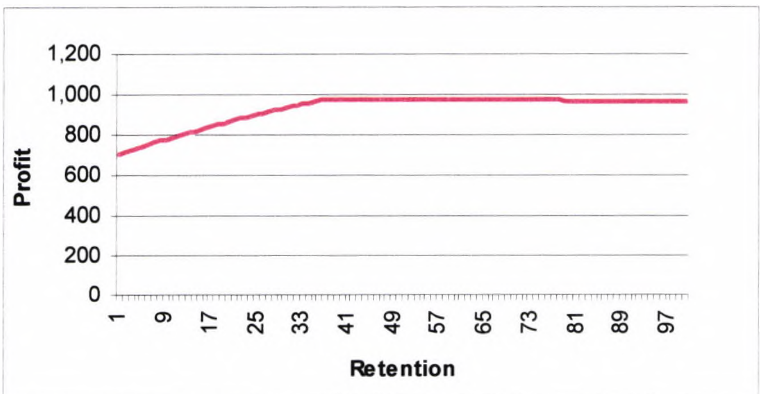


Figure 5.2, illustrates the profit as a function of the retention level. In this scenario, capital is only required for retention levels higher than 37. Also, retained profits display an increasing pattern until a retention level equal to 37. For retention levels higher than 37, the addtional cost of borrowing capital is bigger than the increase in retained profits. It is not surprising that maximum retentions are approximately 37. Increasing retention does not necessarily increase the expected profit.

In real life the insurer would never be allowed by the regulator to be in a position where he does not need to hold any capital. One option to deal with this situation, could be to force a need for capital by changing α/β. However, this may not be the case. By holding some capital, despite the "little" risk retained, the actual optimal retention will not change, but it will reduce the level of profitability and hence its utility.

The impact of using different reinsurance premium rates, or in other words when β takes different values, whilst keeping a 10 year horizon is shown by Table 5.4:

Table 5.4: Retention Levels as a function of the Reinsurance Premium Rate

	r=100	r=250	r=500	r=1,000	r=2,500	r=50,000
			Retention Level			
β=2%	3	6	13	25	36	46
β=3%	4	10	19	30	43	94
β=4%	5	13	20	36	72	100
β=5%	5	14	25	46	92	100
β=6%	6	16	30	58	99	100
β=7%	9	22	44	75	100	100
β=8%	10	25	50	86	100	100
			Capital as a % of Premiums			
β=2%	0.00%	0.00%	0.00%	0.00%	0.00%	0.05%
β=3%	0.00%	0.00%	0.00%	0.03%	1.08%	2.50%
β=4%	0.00%	0.00%	0.06%	1.27%	2.26%	2.52%
β=5%	1.60%	1.70%	1.81%	2.06%	2.50%	2.52%
β=6%	6.56%	3.32%	2.48%	2.40%	2.52%	2.52%
β=7%	9.57%	4.11%	2.76%	2.50%	2.52%	2.52%
β=8%	10.69%	4.39%	2.82%	2.50%	2.52%	2.52%

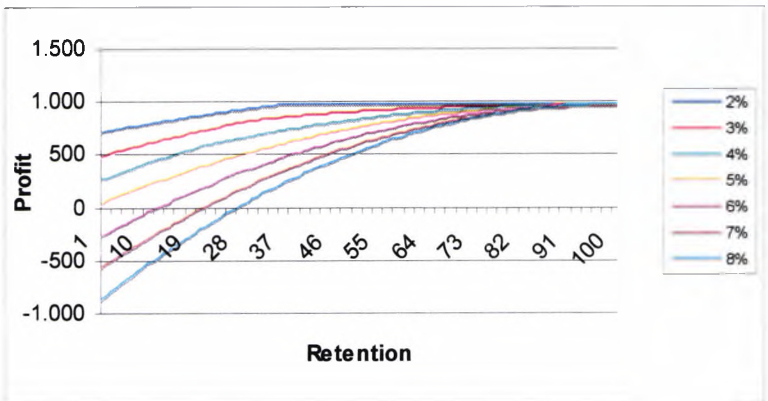
The optimal retention levels obtained for r equal to 1,000 and shown in Table 5.4, are higher than in the case when we were looking at a one-year scenario. For example, for a reinsurance premium equal to 2%, the optimal retention level is 25 against 10 in the one year case. Again the insurer in all cases chooses to increase the retention level when he is looking at a higher time horizon.

The behaviour of optimal retention and capital combinations is generally intuitive: there being an increased preference for capital at the expense of reinsurance as β increases, and hence as the cost of reinsurance increases. At very low values of r (r=100, 250), investors' preference for low variability still leads to very high levels of reinsurance, even though the resulting net losses will require significant capital input to cover the probability of ruin. Such risk averse investors prefer to transact net business at almost certain losses in preference to the risk of making even worse losses which would apply at higher retentions. Note that as shown in Figure 5.3, expected profits are negative for all optimal retentions for r=100 when β>5%.

The effect of considering a 10 year rather than a one year term risk has resulted in increased optimal retentions levels for a given level of risk (compare Tables 4.2 and 5.4). The riskiness of the business has been reduced by the increased duration of the policy, at least while no parameter risk is being considered.

An increase or any change in the value of β does not change the variance of retained mortality profits, because it is independent of β. The only change is in the expected value of retained profits, which is shown in the next graph:

Figure 5.3: Profit as a function of the Reinsurance Premium Rate



The results seen in Figure 5.3 are similar to what was seen in the one-year time horizon. In some scenarios, including the basic, the expected value of profits levels off at some retention. This should be the point at which all investors would "naturally" choose the maximum.

We will now look at the effect of increasing the size of the portfolio, by increasing the number of policyholders per sum assured, whilst keeping a time horizon of 10. We have used the basic scenario assumptions, except for the reinsurance premium rate β which was set equal to 3%, in order to allow for a wider range of retention results.

Table 5.5: Retention Levels as a function of the Number of Policyholders per Sum Assured and Capital Expressed as a % of Premiums

	r=500	r=750	r=1,000	r=1,500	r=2,500	r=50,000
			Retention Level			
t(i,0,k)=75	18	23	23	25	37	87
t(i,0,k)=100	18	26	30	30	43	94
t(i,0,k)=300	18	27	37	56	94	100
t(I,0,k)=500	19	30	40	58	95	100
t(i,0,k)=1,000	26	37	45	69	100	100
			Capital as a % of Premium			
t(i,0,k)=75	0.00%	0.01%	0.01%	0.36%	1.70%	3.55%
t(i,0,k)=100	0.00%	0.00%	0.03%	0.03%	1.08%	2.50%
t(i,0,k)=300	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
t(I,0,k)=500	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
t(i,0,k)=1,000	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%

The results presented in Table 4.3, for the one-year scenario, show that optimal retention level increases with the size of the portfolio. The same pattern is seen in Table 5.5 even though the optimal retention levels are higher. In this case, we have two factors that are influencing the change in the retention level.

The first, is the change in the portfolio size. As the size of the portfolio increases, the insurer will, as a consequence, choose higher retention levels. This is a natural consequence of having a more stable portfolio the bigger it is, which follows from the law of large numbers. The second change is the increase in the time horizon, which means that the insurer will expect losses to be spread over time. The two together in the eyes of the insurer reduce the existing level of risk and therefore the optimal retention level can be raised.

Another consequence is that the level of capital required to meet the solvency constraint is much lower in the ten year scenario than for the one year scenario. In most cases it is actually equal to zero. The tradeoff between capital and reinsurance seems to increase, not just with the size of portfolio, as it was seen in the one year scenario, but also with the time horizon.

Figure 5.4: Variability of Profits for Number of Policyholders per Sum Assured equal to 100

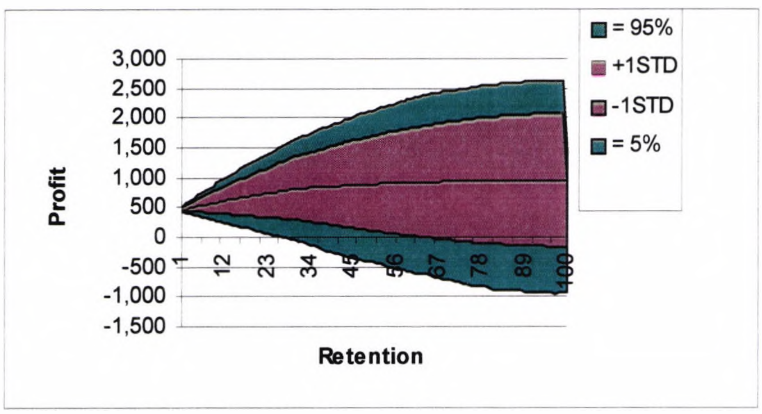


Figure 5.4 displays the volatility of profits, as a function of the retention level, when the number of policyholders per sum assured is equal to 100. The volatility of profits increases with the retention level. Also the bottom green area that represents the 5% confidence level stays above the x-axis up until a retention level of approximately 23, which, as a consequence, leads to the conclusion that in the case of retention levels lower than 23 there is no need to hold any capital to support the volatility of profits.

In all the results presented in this section, we assumed that in the portfolio all policyholders are aged 45. We, next looked, at the effect of considering different ages in the portfolio for a time horizon of 10. From the basic assumptions considered in the basic scenario, only the reinsurance premium rate was changed to 4%, to give a broader range of results.

Table 5.6: Retention Levels as a function of the Policyholder Age

	r=100	r=250	r=500	r=1,000	r=2,500	r=50,000
			Retention Level			
Age=25	4	6	6	6	11	27
Age=35	4	9	12	21	40	81
Age=45	5	13	20	36	72	99
Age=55	5	13	23	46	100	100
Age=65	6	14	28	53	100	100
			Capital as a % of Premiums			
Age=25	0.00%	0.41%	0.041%	0.41%	1.23%	1.90%
Age=35	0.00%	0.45%	0.96%	1.67%	2.21%	2.69%
Age=45	0.00%	0.00%	0.06%	1.27%	2.26%	2.52
Age=55	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
Age=65	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%

These results confirm the conclusions in Section 4.4.3, regarding the effect of age in the retention level and on the level of capital. However, the rate of increase, in the retention level as the age increases, seems to be much higher in this case. The fact that at the same time the insurer is faced with an increase in age and term/time horizon, gives an extra degree of comfort to the insurer which leads to choosing higher retention levels. It would seem natural to expect that a higher retention would mean higher capital, but this is not the case. The insurer is also making savings in holding less capital than before in relative terms.

Figure 5.5: Variability of Profits for Age=25

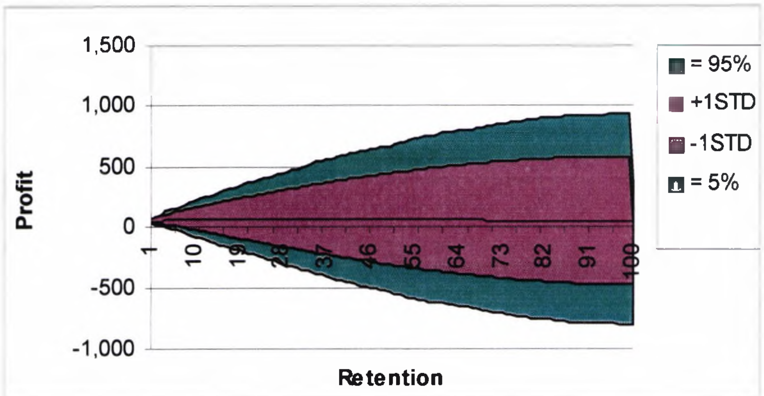


Figure 5.5 shows the variability of profits as a function of the retention level when all policyholders are aged 25. As it was mentioned before, the premium volume intake in this case is considerably lower when compared with older ages, everything kept constant. It can be seen that the line that represents the median of profits is close to the x-axis and that for almost every retention the 5% confidence level is on the negative side of the y-axis. This means that in this case it is to be expected that the insurer will always need to hold some capital to face the expected variability of profits.

In summary, as consequence of considering a higher term optimal retention levels are increased. The riskiness of the business decreased as a consequence of the increased duration of the policy.

5.7.2. Profit Commission

The previous investigations did not include profit commission arising from the reinsurance treaty, when calculating the year profit. We will now consider the profit commission and test the impact of different assumptions on the optimum retention level. We used the normal assumptions from the basic scenario, apart from the reinsurance premium rate β which was set equal to 4%. Also the level of capital as a function of the retention level does not take into account the different profit commission arrangements.

We have considered four different cases. The first three define PC% equal to 5%, 10% and 50% respectively, of PC(.) in any year. The last one defines PC% as a function of the ratio between the result of the profit commission account PC(.) to the ceded premiums in any year. If we define RATIO to be equal to the latter ratio, then PC% is equal to:

- 40% if RATIO<10%;
- 55% if RATIO <36%;
- 70% if RATIO<43%;
- 85% otherwise.

The last definition of the profit commission is a real case from a life insurance company in Portugal. All profit commission scenarios do not include any conditions to carry forward losses through the calculation.

The results are presented by the next table, where the different profit commission structures are identified by B, C, D and E respectively by the order they were described above. Also the basic structure (ie no profit commision), in this case is identified by A.

Table 5.7: Retention Levels as a function of the Profit Commission

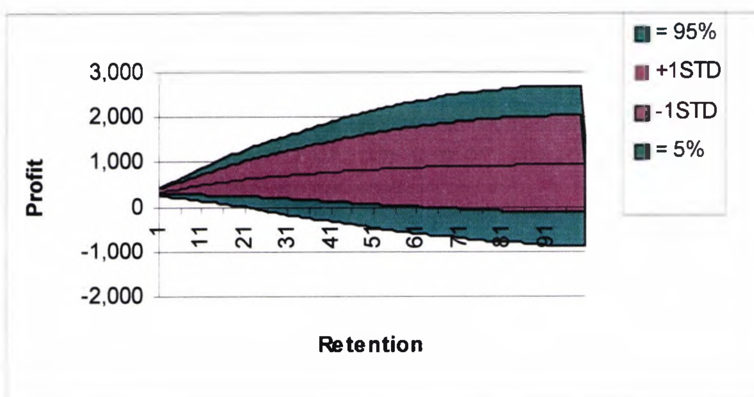
	r=1,500	r=2,500	r=7,500	r=10,000	r=25,000	r=50,000
A	51	72	98	99	100	100
B	44	66	91	93	96	96
C	37	57	82	85	89	90
D	3	10	19	19	19	19
E	2	10	19	19	19	19

The results obtained all show that, as a general rule, if the profit commission is included then the insurer should always consider choosing lower retention levels. The retention should decrease as the level of profit participation increases. Also from the results obtained, it looks as if the more sophisticated structure E can be thought as equivalent to D for retention level purposes. Structure E can be looked at as a combination of simple profit commission schemmes, which gives rise to the same expected utility as D. However, we would note, how a different combination of mean and standard deviation has been achieved in the two cases, even if they are equivalent in expected utility terms (Figure 5.8).

The insurer by having a profit commission scheme in place is sharing the reinsurer's results in any one year. If the results are bad in a given year, then the insurer benefits from the fact of having protection, but is not entitled to any profit commission. If the year is good the insurer makes a profit as well as the reinsurer. In this case, the profit commission amount increases as the amount of ceded business also increases. The inusurer may in this case decide to play a more conservative position, as he can afford to buy extra protection at no extra cost, or in other words by using the profit commission he is entitled to. The difference in net retained premium from A to any other option should be approximately similar to the loss in profit commission, used in buying extra protection.

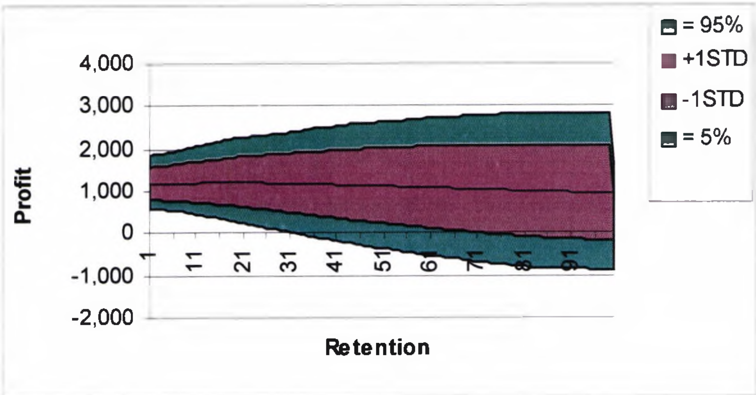
Also it is worth mentioning that the profit commission brings out the fact that the reinsurer is making profits/losses and may therefore also be playing an important role on the amount of business accepted. So far, we have only looked at the results from the point of view of the insurer. In practice the resulting retention level is also a function of the level of profits/losses the reinsurer is willing to share/accept with/from the insurer.

Figure 5.6: Variability of Profits for Structure B



The effect of having a low profit commission does not change materially the variability of profits when we compare it with the basic scenario (compare Figure 5.6 with Figure 6.1). Even though we have added an extra degree of variability to the profits, namely the profit commission, its size is still not big enough to influence it.

Figure 5.7: Variability of Profits for Structure D



Having a profit commission of 50% changed the shape of the graph significantly (Figure 5.7). The variability of profits for low up to mid retention levels and the mean level (the mean is highest at approximately a retention of 19), increased substantially. This effect is mainly due to the fact that the insurer's profits are now subject to an additional sensitivity, that the lower the retention level the higher its impact on its profits. Therefore more profit variability is to be expected.

Figure 5.8: Mean-Standard Deviation of profits for the optimal Retention Levels when r=1,500

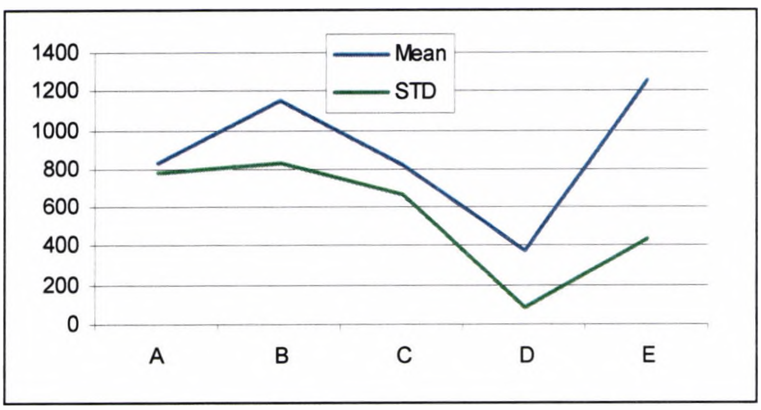


Figure 5.8, illustrates the mean and variance behaviour for the optimum retention levels when the value of r is equal to 1,500. The initial increase in the mean of profits

is due to the fact that the profit commission is for the first time included in the calculation of profits. Even though the optimal retention level does not change significantly when we move from A to B, still there is an increase in the mean of profits and standard deviation. As the optimal retention level decreases so does the mean of profits and associated standard deviation. The last increase, when we move from D to E, reflects the change from a 50% profit commission level to a more sophisticated profit commission scheme. In both cases the retention level is very similar, but the fact that a scaled profit commission was introduced, allows the insurer to achieve higher levels of profits commission in good years. This also results in a higher degree of volatility, because the level of profit commission is a function of the claims experience and hence itself volatile.

Structure D is also very interesting, because the optimal decision is one which has a distinctly lower standard deviation while the mean has been reduced by not so much. Also if we compare structures D and E, we have obtained a very similar retention but the commissions are far from equivalent. It was the change in mean that has reduced the retention. The change in variability, which would, otherwise, serve to increase the retention, has not been influential here. This is because the increase in mean is large enough to over-ride the increased variability. This example shows the "power" of utility theory in choosing between mean and variance.

All points in Figure 5.8 are optimal retention levels for the given level of r. However, from a pure mean-variance analysis, structure E is preferred to any of A, B or C, because it has the lowest standard deviation and the highest mean.

The previous results did not allow in the profit commission calculation for losses to be carried forward. It is very common for reinsurance treaties to include a clause that allows the insurer to carry forward losses, under certain terms and conditions. Generally, the insurer is allowed to carry forward losses for a fixed number of years. We will now consider and test the impact of allowing for losses to be carried forward a fixed number of years, on the optimum retention level.

We will assume that PC% is equal to 50% and consider two scenarios for the number of years that losses can be carried forward, namely three and five years. We have maintained the remaining assumptions from the basic scenario, apart from the reinsurance premium rate, which was set equal to 4% to give a broader range of results.

Table 5.8: Retention Levels as a Function of the Profit Commission with and without carry forward of losses condition

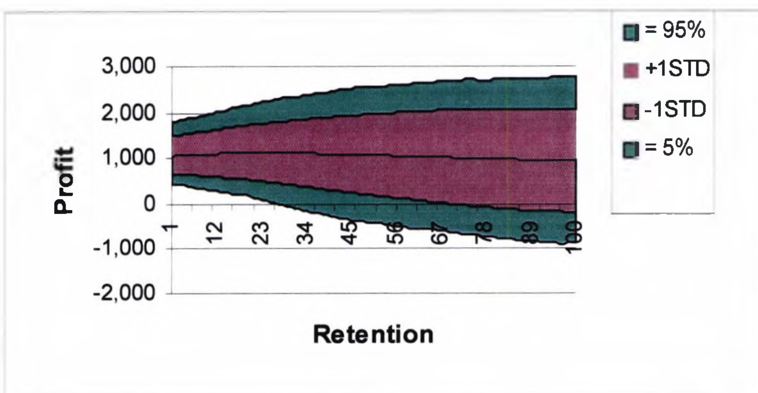
	r=1,500	r=2,500	r=7,500	r=10,000	r=25,000	r=50,000
A	51	72	98	99	100	100
D	3	10	19	19	19	19
F	7	16	19	20	20	20
G	12	19	20	20	20	20

In Table 5.8, we have identified the normal basic scenario without and with profit commission at a 50% level with no allowance to carry forward losses, by A and D to be consistent with Table 5.7. Letters F and G identify the two new scenarios, namely three and five years respectively.

If in the profit commission calculation is included a condition that allows the reinsurer to carry forward losses for a certain number of years, the reinsurer is deferring in time the sharing arrangement. If we compare structure D with F and G, the insurer, in the second case, is choosing less reinsurance. All that is happening is that the sharing of losses, in the second case, reduces the value of reinsurance to the insurer and hence optimal retentions are higher. In effect, in D the insurer was getting a benefit (commission) on a yearly basis, whilst now, in structures F and G, it takes longer before he receives the "same" benefit from the reinsurer.

There is little difference in the variability of profits with the introduction of a three year carry forward condition in the profit commission calculation. The results can be seen in Figure 5.9, which shows the variability of profits for structure F (a similar result would be obtained for structure G).

Figure 5.9: Variability of Profits for Scenario F



The introduction of the profit commission in all cases has had a considerable impact on the variability of profits in the low end of retentions. For retention levels lower than 20, the range of values that can be obtained is considerably bigger than in the basic scenario.

Figure 5.10: Mean-Standard Deviation of Profits for the Retention Levels when r=1,500

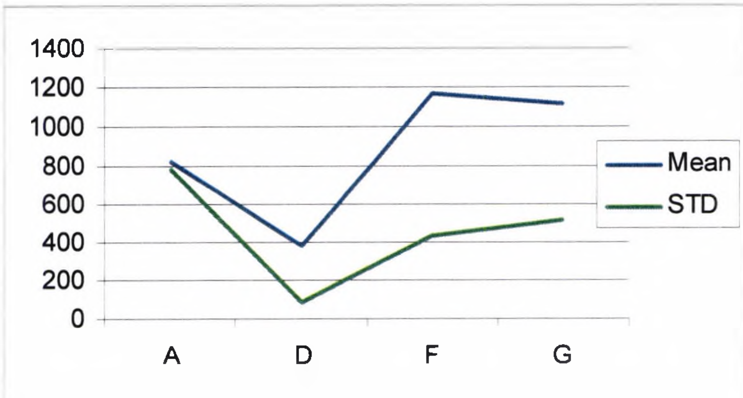


Figure 5.10, above illustrates the mean and variance behaviour for the optimum retention levels when the value of r is equal to 1,500, for scenario D, F and G. It should be read from left to right going through the line, the points corresponding to the scenario presented above.

Again, the main point is that the loss sharing (F and G compared with D) reduces the ability of the reinsurance to reduce losses. Hence retentions which lead to higher volatility are now preferred because the ability to reduce volatility has less utility than it had, compared with the ability to increase the mean return, since losses are now carried forward.

5.7.3. Endowment Policy

In all the previous results it was assumed that the portfolio only had term assurance policies. In this case, we have assumed that all policyholders in the portfolio have an endowment policy. If all other hypothesis from the basic assumptions are kept the same, in particular the reinsurance policy, this assumption only changes the way reserves at the year end are calculated. In the case of an endowment, stronger reserves are build up during the lifetime of the policy and therefore there is a reducing sum assured at risk. They have an impact on the amount of business ceded for a certain level of retention and may also influence the results.

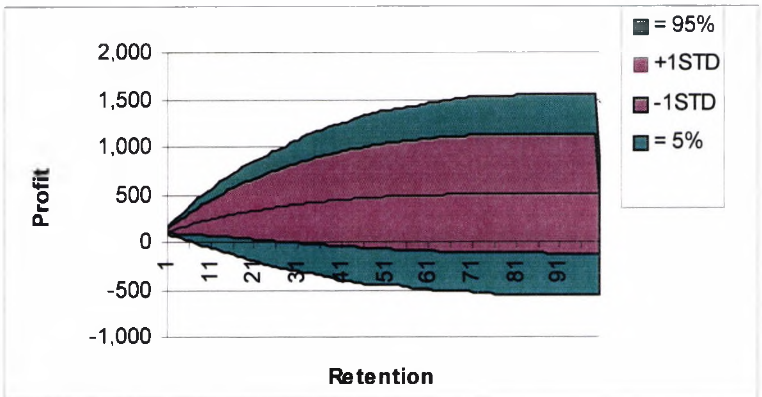
The next table presents the results obtained:

Table 5.9: Retention Levels Results for an Endowment Policy and Capital as a % of Premiums

	r=1,500	r=2,500	r=7,500	r=10,000	r=25,000	r=50,000
			Retention Level			
A	66	92	99	100	100	100
B	51	72	98	99	100	100
			Capital as a % of Premiums			
A	0.16%	0.16%	0.16%	0.16%	0.16%	0.16%
B	1.80%	2.26%	2.52%	2.52%	2.52%	2.52%

In the table above, scenario A illustrates the results for an endowment policy and B the results for the base, ie Term Assurance case, but where the reinsurance premium rate is 4%. The change of policy type seems to have an impact on the retention level. In the case of an endowment, the insurer faces a less risky situation due to the fact that the sum at risk reduces with policy duration, and the insurer has to pay the sum assured in any case, the only question being the moment of when it is going to happen. Also the fact that stronger reserves are built throughout the policy term, the amount at risk is gradually decreasing, leaving the insurer in a better off position. As it has been seen before, smaller variability implies higher retentions. Also by building up stronger reserves, the insurer becomes less and less exposed as the policy term reaches its end. The fact that the risk the insurer is facing is reduced, in the case of an endowment policy, is reflected also in the amount of capital the insurer needs to hold.

Figure 5.11: Variability of Profits for an Endowment Policy



The shape of the variability of profits is, as expected, lower than what was obtained for the basic scenario (Figure 5.1 shows the variability of profits for the base scenario, since the reinsurance premium rate does not influence the variability of profits, and so it can be used for comparison purposes).

5.7.4. Reinsurance Structure

The reinsurance treaty considered so far assumes that the insurer keeps on every risk a fixed amount, year on year. In some cases, where the insurer builds up increasing reserves, the ceded amount after a certain number of years becomes zero. This is the case in endowment policies, for example, where due to its survival guarantee the insurer has to build up reserves throughout the term of the policy. Depending on the chosen retention level, there might be situations where the ceded amount is actually equal to zero and, therefore, nothing is ceded to the reinsurer.

Another possible approach in a risk premium reinsurance treaty, is to assume that the ceded amount in any year is a fixed percentage of the amount at risk. The latter is normally defined to be equal to the policy sum assured minus held reserves.

In this section, we have tested the impact on the retention level of using this new reinsurance structure in a term assurance policy. Care must be used when comparing the results from both types of treaty structures due to the different way business is ceded.

Table 5.10, presents the results obtained, where the base assumptions were maintained apart from the reinsurance premium rate which was set equal to 4%.

Table 5.10: Retention Levels when a different Reinsurance Structure is tested and Capital expressed as a % of Premiums

	r=1,500	r=2,500	R=7,500	r=10,000	r=25,000	r=50,000
			Retention Level			
A	73%	100%	100%	100%	100%	100%
B	51	72	98	99	100	100
			Capital as a % of Premiums			
A	2.16%	2.52%	2.52%	2.52%	2.52%	2.52%
B	1.80%	2.26%	2.52%	2.52%	2.52%	2.52%

In the table above, scenario B again illustrates the results for the base. The results for the new treaty structure are shown as percentages. They determine the percentage amount used on every policy's sum at risk to calculate the retained amount.

In almost every cases there is an indication to retain 100% of the sum at risk in every policy. This is not the case in scenario B where, apart from the last two values of r the retention level obtained leads the insurer to cede business to the reinsurer. If the insurer chooses this new type of treaty structure, he is more likely to choose higher levels of retention.

The required level of capital in the new reinsurance structure is higher than for the basic scenario, which may be a consequence of the fact that the insurer always has some amount at risk, regardless of the amount of reserves.

If the reinsurance treaty in place defines retained amounts as a percentage of sums at risk, then the insurer for low ceding percentages is necessarily retaining less then for low retention levels in the basic scenario, where the retention level was defined to be a fixed amount. Even though the ceded amounts are now calculated in different ways, it is possible that the insurer now achieves the same level of variability in profits at "higher" retention levels. The insurer will therefore consider ceding less business away to the reinsurer.

Figure 5.12: Variability of Profits when a different Reinsurance Structure is tested

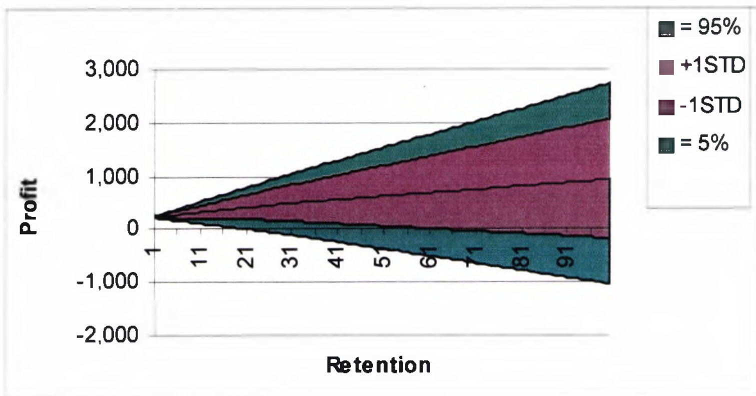


Figure 5.12, displays the variability of profits for the new reinsurance treaty structure. The shape is clearly different when compared with the results obtained so far. In this case, the insurer is subject to a lesser degree of variability of profits. This is confirmed by the fact that the cone is much thinner. However, because the insurer is subject to variability for a longer period of time the insurer is required to hold higher levels of capital.

The results also show that this new method of reinsurance is more efficient, in which case the insurer will probably need to buy much less reinsurance protection to achieve the described reduction in variability. It is also possible to optimize reinsurance strategy based on a expected utility approach. Given r=1,500 and an optimal retention level of 73% for A and 51 for B, a simple analysis reveals that to A corresponds the highest expected utility (A=0,896428; B=0,8913297). This way, we have determined an optimal reinsurance strategy, or at least a better one. Conventional wisdom, would suggest B to be the best (would imply less reinsurance for a given retention), but results have shown that a better strategy is to have higher retention, ie lower reinsurance, but to keep it going for longer.

5.8. Open Portfolio Simulation

In this section we will present and discuss the results for the open portfolio. This means that the insurer's portfolio is open to new policyholders throughout the projection period. Different scenarios were looked at in order to identify the effect of the open portfolio on the optimum retention level.

5.8.1. Growth Rates

An open portfolio has normally associated a growth rate. The growth rate of a portfolio is one of the key assumptions used by insurers in the business plan. Expense and investment levels are usually dependent on the size of the portfolio and also the way it is expanding.

The growth rate from year x to year x+1 is defined to be equal to the ratio between the average number of policies in year x+1 and the average number of policies in year x.

We have tested the impact on the retention level of assuming different growth rates and have used the normal assumptions from the basic scenario, apart from the term of all policies and the reinsurance premium rate that were set equal to 5 and 4%, respectively. Again, this change in the basic assumptions was done in order to obtain a wider range of results.

The structure identified in Table 5.11 by "Base" is where the portfolio is 5 year term but not open. In all other structures, we have assumed that the portfolio is open for 5 years and then becomes closed until the end of the remaining 5 years needed for the extinction of the whole portfolio. Table 5.11 presents the optimal retention levels for the "Base" and as a function of the growth rate.

Table 5.11: Retention Levels as a Function of the Growth Rate and Capital expressed as a % of Premiums

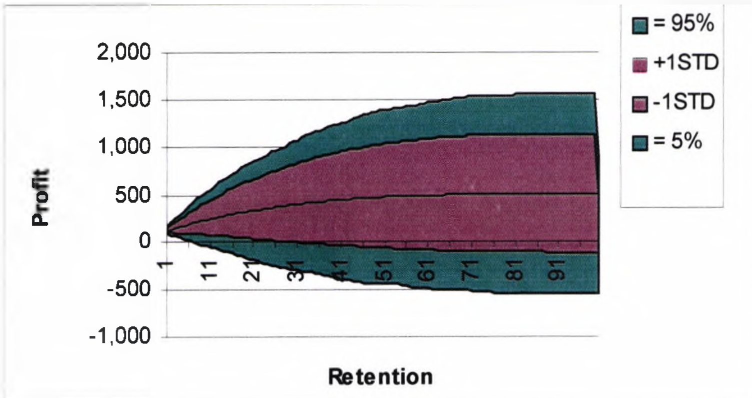
Growth	r=1,500	r=2,500	R=7,500	r=10,000	r=25,000	r=50,000
			Retention Level			
Base	38	54	74	80	95	99
0%	6	6	8	8	10	12
10%	8	8	10	11	13	15
30%	15	21	37	40	48	56
50%	31	44	73	78	87	97
			Capital as a % of Premiums			
Base	1.15%	1.60%	1.94%	2.02%	2.13%	2.14%
0%	1.32%	1.89%	2.31%	2.42%	2.82%	3.01%
10%	2.32%	2.32%	2.70%	2.85%	3.07%	3.25%
30%	4.22%	4.83%	6.43%	6.61%	7.04%	7.38%
50%	6.90%	7.89%	9.17%	9.32%	9.53%	9.66%

The retention level is influenced by the growth rate assumption. The higher the growth rate the higher the retention level the insurer should choose. A higher growth rate means that the portfolio is expanding faster and therefore the insurer is writing more risks. This conflicts against conventional wisdom, where the higher the growth, the more financial reinsurance is needed. In other words, reinsurance for the two purposes (financing as opposed to reducing mortality risk), are operating in contradictory directions.

Overall, the retention levels obtained for an open portfolio are lower than for the Base or, in other words, for a closed portfolio. This might suggest that the financial strain associated with a portfolio that is expanding in size is not balanced by the increase in premium volume, which otherwise might lead to higher retention levels. This balance will be a function of the way the portfolio is expanding. The higher the growth, the faster the insurer will probably be making profits and also the higher are expected profits and therefore the insurer will be able to choose higher retention levels.

The problem of expanding too fast is that it implies a higher investment from the shareholders, in other words, the amount of capital that needs to be invested to be backing up the volatility of the business, will increase with the rate of growth of the portfolio. This fact, is confirmed by the results obtained for the amount of capital expressed as a percentage of premiums, where the faster the portfolio is expanding the bigger this ratio is. Even if we consider the same retention level, for example 50, we obtain 1.51%, 5.48%, 7.13% and 8.23%, respectively for the base and as a function of the growth rate (0%, 10% and 30% respectively). There is, therefore, a higher demand for capital the faster the portfolio is expanding. Since the amount of capital an insurer has to invest is normally limited, therefore it will have to take this into consideration when expanding its portfolio. In order to compensate, the insurer uses reinsurance as a way to transfer to the reinsurer the higher need of capital which he may not be able to afford.

Figure 5.13: Variability of Profits for a Growth Rate of 10%



The shape of the cone in this first scenario, a 10% growth rate, did not change when compared with results obtained in the closed portfolio sections (Figure 5.13). The variability of profits is substantially higher which is a consequence of the insurer being subject to a growing portfolio. The suggested capital requirements are expected to be higher. The amount of capital required in the case of the closed portfolio was calculated as a function of that portfolio. In an open portfolio, the insurer gets a new closed portfolio every year.

5.8.2. Number of Open Years

In the previous results we have assumed that the portfolio remains open for five years and then becomes closed for an extra five years until the extinction of all policies in the portfolio. We have now tested the effect of letting the portfolio remain open for three years instead of five. After that, the portfolio becomes closed again for another five years until the extinction of the last policy in the portfolio.

We have used the normal assumptions from the basic scenario, apart from the term of all policies and the reinsurance premium rate were set equal to 5 and 4%, respectively. Also we have assumed a growth rate equal to 10%.

The results are presented in Table 5.12.

Table 5.12: Retention Levels for an Open Portfolio for three years

Growth	r=1,500	r=2,500	R=7,500	r=10,000	r=25,000	r=50,000
10%	14	20	35	39	47	50

The number of years the portfolio is open clearly influences the retention level. The lower it is open, the higher is the resulting retention level. If the portfolio is left open a smaller number of years then the risk the insurer is subject to, by writing new business, is reduced. In this case, it is therefore possible to raise the retention level.

However, when we increase the rate of growth for new business (so the risk is increased) the optimal retention level rises. This could be derived from the fact that for the same growth rate, the lower the risk (smaller number of open years), the higher the retention level. For different growth rates (same number of open years), the higher the rate the quicker volatility stabilizes and therefore the higher the retention level. Also if capital was not variable then lower retentions would, of course, be necessary.

5.9. Summary

This chapter was a generalization of Chapter 4, in the sense that it expanded the approach, methodology and results discussed in the latter chapter. The purpose was to study optimal retention levels that maximize the expected utility of profits over a time horizon bigger than one year.

For that purpose, a stochastic model of a life insurance company was developed, which simulates the evolution of a life portfolio for a given time horizon. The same approach as regards to capital needs was followed. The optimal retention levels were, then, assessed as a function of the portfolio's characteristics.

Initially we looked at the impact of considering a 10 year term as opposed to one year (Chapter 4). The effect of considering a higher term resulted in higher retention levels for a given level of r, or a given investor. The riskiness of the business seemed to be reduced by the increased duration of the policy. The behaviour of optimal retentions and capital combinations was generally intuitive.

The next set of results, looked at the effect of including the profit commission in the reinsurance treaty arrangement. Different structures were considered, in order to assess whether or not optimal retentions were influenced in any way by the profit commssion, and also if the influence changed whenever the profit commission structure also changed. The results obtained showed that if the profit commission was included, then the insurer should consider lower retention levels. Also, different optimal retention levels were obtained for different profit commission structures. This provided a particularly interesting illustration of the way utility theory can help distinguish between the merits of different strategies which produced materially different profit distributions.

In all previous results, it was assumed that the portfolio´s consisted of term assurance policies. The following results, considered endowment policies instead of term assurance. The change of policy type had a clear impact on optimal retention levels. In the case of an endowment, the insurer should choose higher retentions and also lower capital levels. This is a consequence, that under an endowment sums at risk reduce with policy duration, and the insurer has to pay the sum assured in any case. Therefore, in a less risky situation it is more likely that the insurer should choose higher retentions.

A different reinsurance structure was also studied and compared with the one underlying all previous results. This showed that the new method of reinsurance was more efficient due to an optimization exercise of reinsurance strategy based on an utility approach.

The results for an open portfolio were shown next. Different scenarios were considered in order to identify the effect of the open portfolio hypothesis on optimal retention levels. Overall the retention levels obtained for the open porfolio were lower than for a closed porfolio. When a porfolio is expanding, it requires a higher investment from shareholders in terms of more capital to backup the increased volatility of the business written. Since the amount of capital is normally limited, the insurer can use reinsurance as a way to transfer the higher need of capital.

Throughout this chapter we have seen that on a multi-year scenario, the insurer can control solvency, either through buying more reinsurance or capital. The same result was seen in Chapter 4. The focus of Chapters 4 and 5 was **Process Risk** and we have seen that it influences the level of reinsurance protection and capital. In other words, Process Risk is an important component of total risk for the life office and should always be considered to meet solvency targets. Another important component of total risk for the life office is parameter risk. This will be the purpose and scope of Chapter 6.

6. Parameter Risk

6.1. Introduction

When looking at the effect of the mortality rate in the retention, we have so far only considered one possible scenario, namely:

- In any given year, expected mortality rate is equal for all policyholders. The number of policyholders that die is a random variable with a binomial distribution with expected value equal to the implied mortality rate in the mortality table used for the common age of all policyholders.

It is simple and interesting to expand this assumption regarding the expected mortality and to study the effect it has on the optimal retention level.

A possible approach could be to consider that the expected mortality rate varies with the sum assured. The number of policyholders, for a given sum assured, that die would be a random variable with a binomial distribution with expected value equal to the implied mortality rate in the mortality table used for the age of all policyholders corrected with a factor to reflect the sum assured. This assumption means that the underlying expected mortality rate for a policyholder increases as he chooses a higher sum assured. A policyholder will choose a level of protection depending on its known medical conditions. If he is aware of any health problems he will try to get, from the insurer, the maximum level of protection. A "healthy" policyholder will more likely choose the level of protection depending on the price. This relates to the risk of adverse selection which financial underwriting attempts, in some way, to address.

It is possible to expand the first approach further, by considering that the expected value of the binomial distribution is itself a random variable. This possibility adds an extra dimension in the variability of the mortality profit. In all the previous results, we were charging a premium to the policyholders that would on average be correct, because we did not allow for any significant adverse experience in mortality. In real life policyholders are charged premiums based on estimates of expected mortality, which may or may not deviate materially from real experience. It is this phenomenon that we will try to introduce.

When considering the mortality risk as an additional degree of risk in the mortality profit we will compare the results with a base scenario with the following characteristics:

- All policyholders are aged 45 (a=1);
- The number of policyholders per sum assured is equal to 100 and n=100;
- A constant interest rate during the simulation equal to 4% p.a.;
- α=5%;
- β=4%;
- Term of the policy equal to 10 years;
- Closed Fund;

• No profit commission.

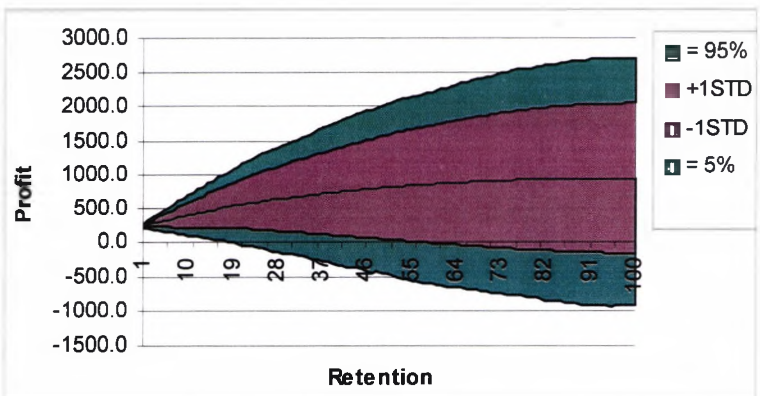
The only difference between this base scenario and the one considered in Chapter 5, is the reinsurance premium rate which is now equal to 4% to provide a wider range of results. The optimal retention levels and capital expressed as a percentage of premiums, are shown in Table 6.1 for reference purposes.

Table 6.1: Optimal Retention Levels and Capital expressed as a % of Premiums for the Base Scenario

r=1,500	r=2,500	r=7,500	r=10,000	r=25,000	r=50,000
Retention Level					
51	72	98	99	100	100
Capital as a % of Premiums					
1.80%	2.26%	2.52%	2.52%	2.52%	2.52%

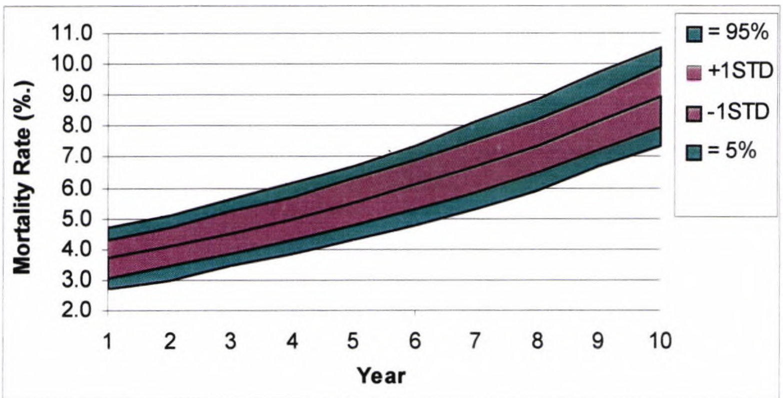
The variability of profits is shown next.

Figure 6.1: Variability of Mortality Profits for the Base Scenario



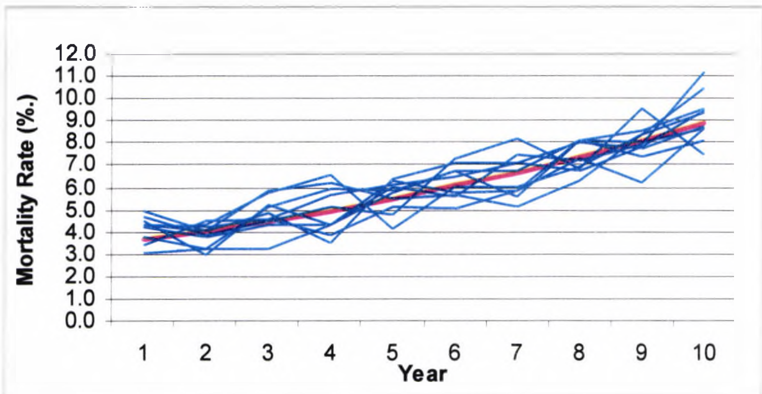
Because it will be important to analyse the effects on the variability of the mortality rate of new sets of assumptions, Figure 6.2 and Figure 6.3 give the variability for the basic set of assumptions and sample random paths, respectively for the mortality rate.

Figure 6.2: Variability of Mortality Rate for the Base Scenario



The variability increases as we move in time as a consequence of the portfolio becoming older in terms of average policyholder age.

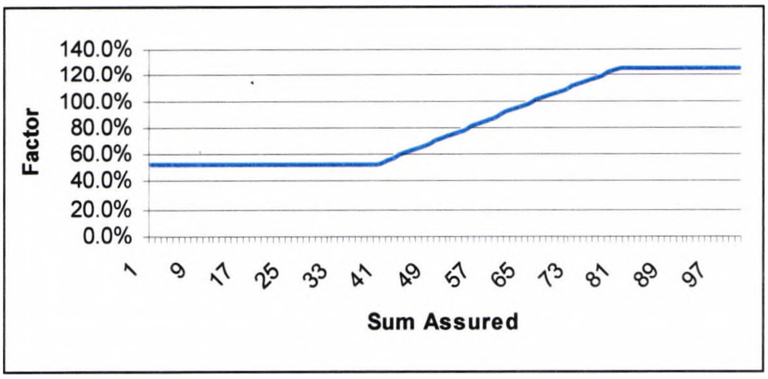
Figure 6.3: Mortality Rate Sample Paths for the Base Scenario



6.2. Expected Mortality Rate as a Function of the Sum Assured

We will now look into the case where expected mortality rate varies as a function of the sum assured. Figure 6.4 illustrates the factors that will be used for each sum assured, to reflect the way mortality varies with the sum assured:

Figure 6.4: Relationship between Actual Mortality Rate and Sum Assured



If we represent by q_x the mortality rate for a policyholder aged x, then if its sum assured is, for example, 33 and the correction factor 60% for this sum assured, then this policyholder's expected mortality rate is $0.6.q_x$.

The factors used were derived in such a way that the weighted average of the factors by sum assured were equal to 100%.

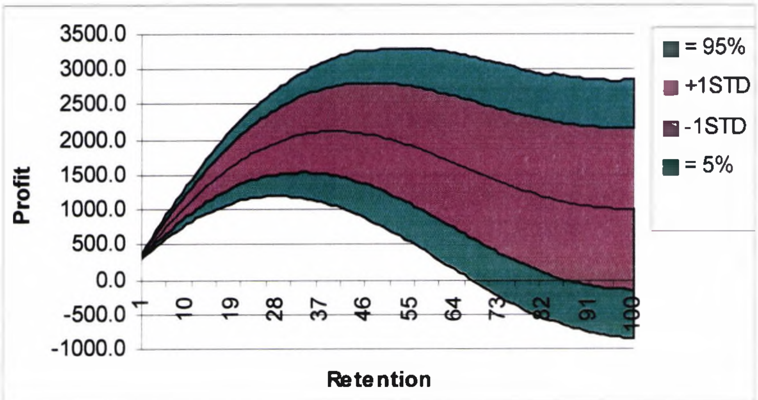
Table 6.2 presents the results obtained when the mortality rate varies as a function of the sum assured.

Table 6.2: Retention Levels Results when Actual Mortality Rate varies by Sum Assured

	r=1,500	r=2,500	r=7,500	r=10,000	r=25,000	r=50,000
A	50	71	98	99	100	100
B	37	38	39	39	40	40

In Table 6.2, scenario A illustrates the results for the base scenario. The introduction of a relationship between expected rate of mortality and sum assured has decreased the retention level. Also, a retention of 40 is a mean-variance efficient result. At higher retentions (than 40), one would get the same mean as at lower retentions, but for a much higher variance. However, it is reasonable, for more risk averse investors, to choose lower retentions (as they do), as these are mean-variance efficient decisions.

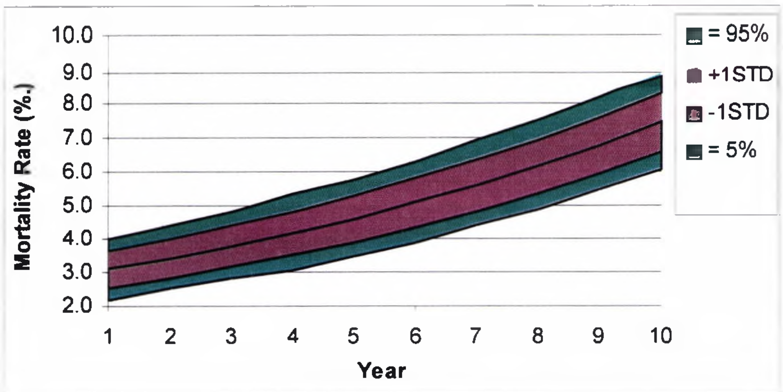
Figure 6.5: Variability of Profits when the Actual Mortality varies with the Sum Assured



If we compare Figure 6.1 with Figure 6.5, the downside variability is considerably worse in Figure 6.1 at low-medium retentions. At medium retentions there is a much lighter mortality, to which corresponds much higher profits with lower variability (the big "peak" in Figure 6.5). The latter makes lower retentions much more desirable, because it excludes all the higher risks.

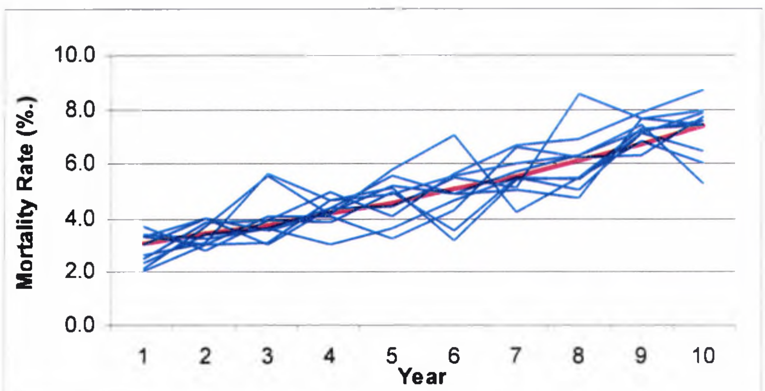
From Table 6.2, it is possible to conclude that a retention of 40 is optimal ever for very risk tolerant investors. Also, for that same retention level, appears to coincide almost exactly with the maximum expected profit. Clearly, no investor is going to prefer higher retentions than this, because for higher retentions, expected profits are lower and variance is higher.

Figure 6.6: Variability of Mortality Rate



Even though the factors used were derived in such a way that the weighted average of the factors by sum assured to be equal to 100%, the resulting expected mortality rate is lower by approximately 15%. Looking at the expected value line, it is possible to see a slight decrease when comparing with the basic scenario.

Figure 6.7: Mortality Rate Sample Paths



The sample paths also show a higher degree of variability and a small reduction in expected value.

In order to compensate for the change in the expected value, the same scenario was tested but where policyholders were charged 15% less than before, in order to keep the overall expected mortality at the same level. The new optimal retention levels obtained were approximately the same, which suggests that what is driving the choice of retention levels is the weight given to any sum assured and not the change in premium volume.

6.3. Expected Mortality Rate as a Random Variable with a Lognormal Distribution

We will now consider that the expected value of the binomial distribution is itself a random variable. Because we are looking on a ten year horizon, the random effect on the expected value of the binomial distribution that generates the number of policyholders that die in a given year, can be looked at from two different perspectives:

- The random effect will be the same for all the different years, ie the expected mortality in all years will either increase or decrease by the same amount;
- The random effect will vary by year, ie the expected mortality in all years will vary independently.

We will identify these two scenarios by A and B, respectively. We have used a lognormal random variable, to generate randomly the increase or decrease on the expected value. Let INC be random variable with a lognormal distribution with expected value μ and σ. The new expected mortality rate $\overline{q'}_x$, for a policyholder aged x, is therefore given by:

$$\overline{q'}_x = \overline{q}_x * \text{INC} \qquad (6.18)$$

where $\bar{q}_x$ is the mortality rate implied by the mortality table for a policyholder aged x.

In scenario A, the same value of INC is used for all projection years in a single sample path; in B the value of INC is independently calculated for each projection year of each sample path.

The results obtained for scenario A for different assumptions of both μ and σ are given in Table 6.3.

Table 6.3: Retention Levels Results when Expected Mortality Rate is a Random Variable and Capital expressed as a % of Premiums for Scenario A

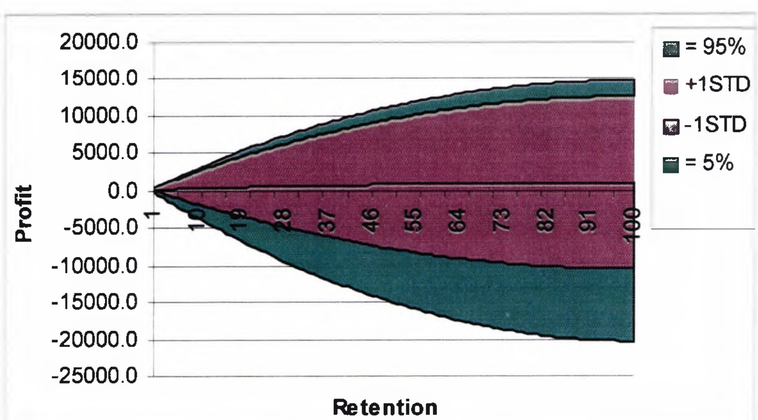
	r=1,500	r=2,500	r=7,500	r=10,000	r=25,000	r=50,000
			Retention Level			
Base	51	72	98	99	100	100
μ=1 & σ=0.5	1	1	3	4	10	19
μ=1 & σ=0.25	2	3	11	14	29	63
μ=1 & σ=0.05	32	43	61	85	88	92
			Capital as a % of Premiums			
Base	1.80%	2.26%	2.52%	2.52%	2.52%	2.52%
μ=1 & σ=0.5	24.60%	24.60%	58.23%	63.70%	74.22%	78.09%
μ=1 & σ=0.25	12.95%	20.66%	33.46%	34.64%	37.74%	39.53%
μ=1 & σ=0.05	3.23%	4.45%	15.01%	18.61%	19.79%	20.56%

There was a massive change in the optimal retention levels. This means that the variability of profits has now increased as a consequence of the effect of introducing mortality variability. Compared with the results obtained when the mortality rate varied as a function of the sum assured, the retention levels are much lower. By facing a higher degree of variability, it is natural to expect that the insurer should decrease the retention level. When the standard deviation of the lognormal decreases, then there is less variability and therefore higher retention levels are expected.

We would also note that capital requirement has also increased at the same time as retention decreasing. This is contrary to the usual situation (see Chapters 4 and 5), where a higher capital was associated with higher retention, ie we were trading-off one with the other. Here the increased risk is so "dramatic" , that even allowing for the trade-off which will have occurred, both reinsurance and capital need to be increased dramatically. Hence, we get the reverse patterns of association between capital and reinsurance. In this case it is optimal to increase both in order to cope with greatly increased risk.

The levels of parameter variability introduced above, were chosen in such a way that a wide range of results could be obtained. Another possible approach would be to look at historical mortality trends in order to derive the value of the parameter σ. The approach chosen, does have the advantage of showing the effects of different parameter variability and its direct effect on optimal retention levels.

Figure 6.8: Variability of Profits for μ=1 & σ=0.5



The variability of profits has increased massively (by an order of magnitude) as a result of the introduction of the assumption that the expected value of the mortality rate is a lognormal random variable. We would note the very asymmetrical distribution of profits, with a much downside risk, which reflects the skewed shape of the lognormal distribution (Figure 6.8).

In real life, there is a high probability that actual mortality rate will deviate from the theoretical mortality rate implied by the mortality table. Probably a bad year is no longer followed by a continuous series of bad years, because most insurance companies now have better control over mortality experience. This has been achieved with the development of more precise underwriting techniques, that allow the insurer to better assess the nature of the risk. However, we would note that in the case we are looking at, ie a closed portfolio, once the policies are issued, the insurer can do nothing about mortality experience, unless it can review its premium mid-term.

Nevertheless, the results seem to suggest that the insurer should be conservative and cede a great amount of written business. Even for less risk averse investors we are still obtaining low retention levels when compared with the basic scenario.

Figure 6.9: Variability of Mortality Rate for μ=1 & σ=0.5

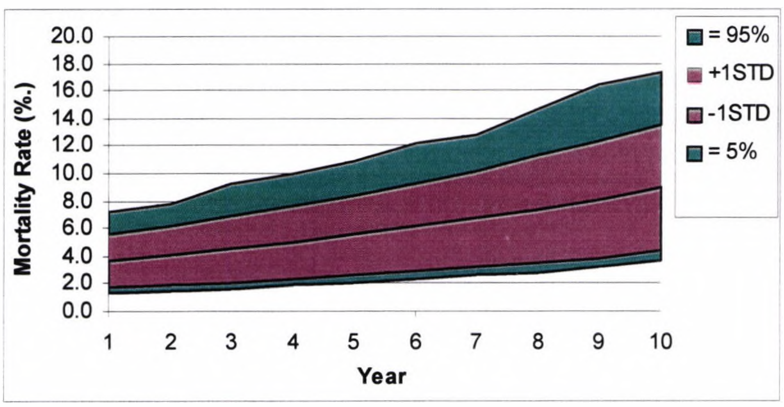
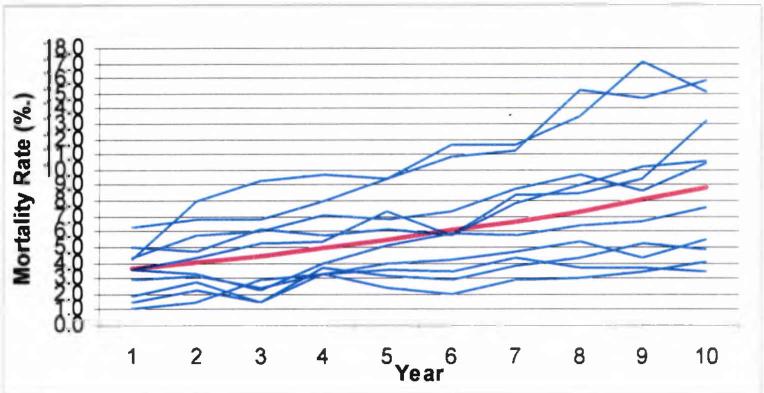


Figure 6.9 gives the variability of the mortality profit when the standard deviation of the lognormal distribution function is equal to 0.5. The variability is much higher and highly skewed which has resulted in a reduction on the optimal retention levels.

Figure 6.10: Mortality Rate Sample Paths for $\mu=1$ & $\sigma=0.5$



The mortality rate sample paths shown in Figure 6.10, when the standard deviation of the lognormal distribution is equal to 0.5, clearly show the extra variability.

We will now present the results obtained for scenario B, when again different assumptions of both μ and σ have been considered. These are shown in Table 6.4.

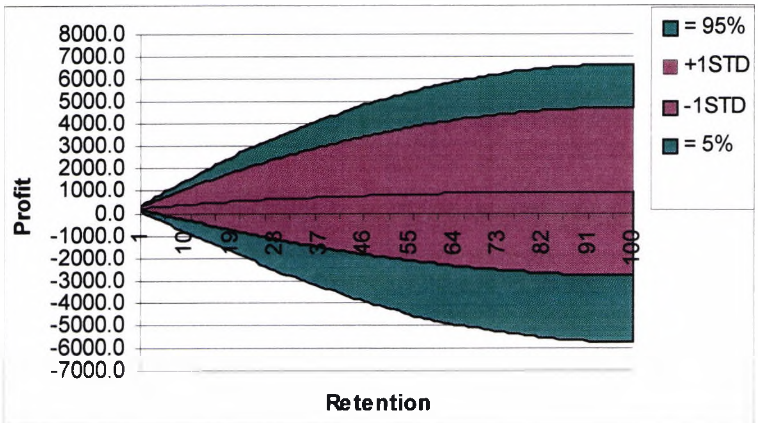
Table 6.4: Retention Levels Results when Expected Mortality Rate is a Random Variable and Capital expressed as a % of Premium

	r=1,500	r=2,500	r=7,500	r=10,000	r=25,000	r=50,000
			Retention Level			
Base	51	72	98	99	100	100
μ=1 & σ=0.5	5	8	21	28	76	98
μ=1 & σ=0.25	17	22	68	82	98	99
μ=1 & σ=0.05	45	54	72	95	99	99
			Capital as a % of Premium			
Base	1.80	2.26	2.52	2.52	2.52	2.52
μ=1 & σ=0.5	12.70	16.28	20.29	21.13	22.63	22.76
μ=1 & σ=0.25	7.32	7.99	10.15	10.40	10.43	10.44
μ=1 & σ=0.05	2.01%	3.22%	5.45%	6.75%	8.54%	8.88%

The retention and capital levels are again significantly influenced by the introduction of this extra dimension of variability. Considering a single impact (Scenario A), produces a greater change than independent impacts by year (Scenario B), even though B is still a dramatic effect. A single impact will affect the expected mortality rate in all years the same way, it can therefore be considered as a worse scenario. If one year is bad, then all years are bad. If we consider different impacts by year, then there is equal chance that a bad year may be followed by a good year as by another bad year. The probability that the results are better in this case is higher and therefore the insurer can consider higher retention levels. The variability introduced is in both cases big enough to lead the insurer to reduce the retention level compared with the base case. In practice it is likely to be somewhere between the two extremes.

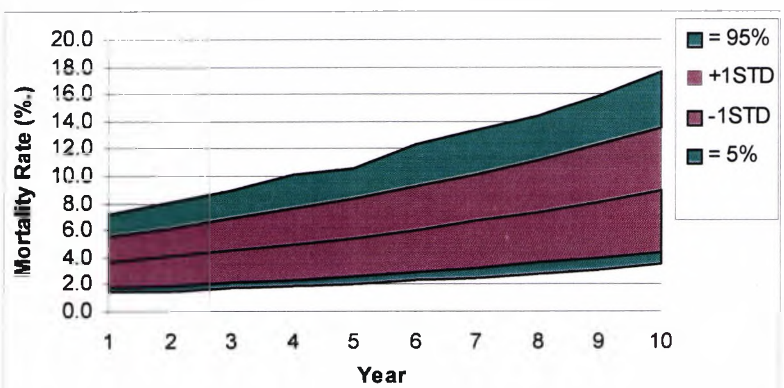
The situation where a bad year might be compensated with a good one is probably closer to real life situations if we exclude natural catastrophes (not excluded from this approach). A moderate risk averse investor will bear in mind the extra or normal variability that is likely to be expected in mortality and consider lower retention levels than we obtained for the basic scenario.

Figure 6.11: Variability of Profits for μ=1 & σ=0.5



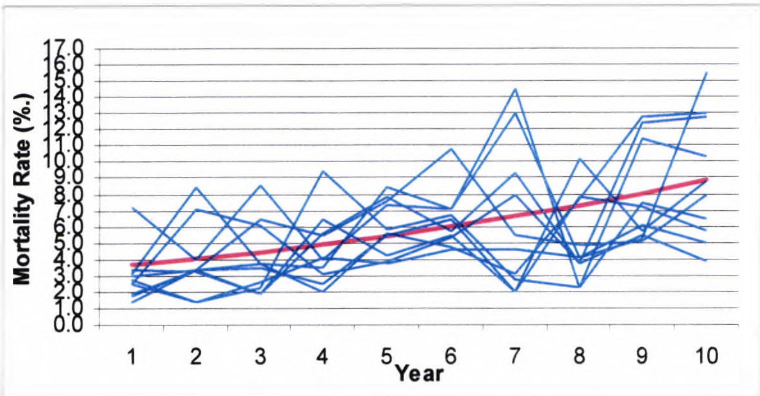
Even though the variability has decreased when we compare it with Figure 6.8 it is still a lot higher than in the basic scenario. As a consequence, the optimal retention levels are higher than for scenario B and lower for the basic scenario. A similar result is obtained when the standard deviation is 0.25.

Figure 6.12: Variability of Mortality Rate for μ=1 & σ=0.5



If we compare the variability of the mortality rate in Figure 6.12 with Figure 6.9 we can see that they are approximately the same, as it would be expected. We would note that in scenario A, there is a huge serial correlation between projection years, while in scenario B there is no serial correlation at all. Hence the variability of total profits, a function of all projection years combined, will be much higher for A than for B. This explains why Figure 6.8 is different from Figure 6.11 and Figure 6.12 is equal to Figure 6.9.

Figure 6.13: Mortality Rate Sample Paths for μ=1 & σ=0.5



By comparing Figure 6.13 with Figure 6.10 we can clearly see the difference between the scenario. In the former case, a good year in mortality rate can be followed by a good or bad year, whilst in the latter case a bad year is almost always followed by another bad one.

6.4. Expected Mortality Rate as a Random Variable with an Autoregressive Distribution

We will again consider that the expected value of the binomial distribution is itself a random variable. The expected value in any year, for a policyholder aged x, will be a function of the implied mortality rate in the mortality table for a policyholder aged x, and of last year's mortality experience. The new expected mortality rate $\overline{q'}_x$, for a policyholder aged x, is therefore given by:

$$\overline{q'}_x = \overline{q}_x * \left(\frac{\overline{q^o}_{x-1}}{\overline{q}_{x-1}} \right)^{\alpha}, \quad -1 < \alpha < 1 \tag{6.19}$$

where $\overline{q}_x$ is the mortality rate implied by the mortality table and $\overline{q^o}_x$ is the observed mortality rate, for a policyholder aged x. Based on the formula above, the expected value of the mortality rate in any year is dependent on how much in the previous year actual mortality rate deviated from expected. Expected in this context means as in the mortality table.

The results obtained for different values of α are given in Table 6.5.

Table 6.5: Retention Levels Results when Expected Mortality Rate is a Random Variable

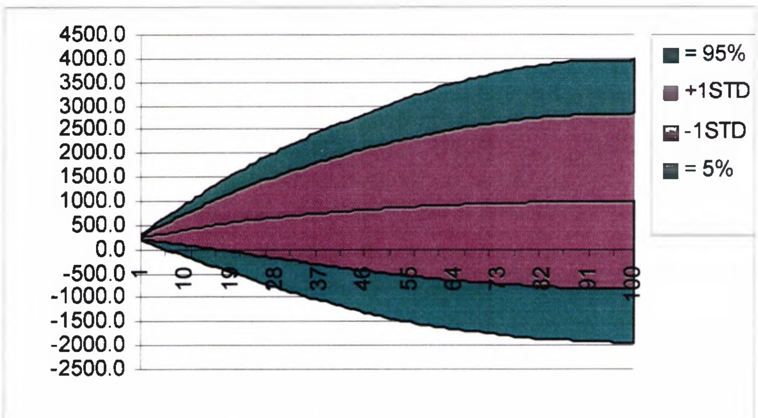
	r=1,500	r=2,500	r=7,500	r=10,000	r=25,000	r=50,000
			Retention Level			
α=80%	20	33	88	96	99	99
α=40%	37	57	98	99	99	99
α=0% (Base)	51	72	98	99	100	100
α=-40%	64	80	97	98	99	99
α=-80%	61	72	90	93	96	97
			Capital as a % of Premium			
α=80%	5.33%	6.62%	7.06%	7.11%	7.13%	7.13%
α=40%	4.33%	4.55%	4.94%	4.95%	4.95%	4.95%
α=0% (Base)	1.80%	2.26%	2.52%	2.52%	2.52%	2.52%
α=-40%	1.92%	2.92%	3.22%	3.22%	3.22%	3.22%
α=-80%	0.69%	1.00%	1.11%	1.11%	1.11%	1.11%

A higher positive value of α means that one year is more dependent on the previous one. As it approaches zero all years are independent and we are back to the base scenario, where, because of the independence a bad year is equally likely to be followed by a bad or by a good year.

When α increases from zero (ie α >0), retention falls and capital increases. This is a typical pattern seen in Scenario A/B examples from the previous section, implying overall increased risk. Bearing in mind the similarity between α >0 and Scenario A in Section 6.3, this is consistent.

When α decreases from zero to moderate levels (ie α <0 and equal to -40%), retention rises but so does capital. Therefore we are getting the more "usual" trade-off between reinsurance and capital for a situation with relatively low risk, and that is why reinsurance must be relatively less efficient than capital in controlling the risk at around α =-40%, than at α =0, hence the swoop. However, at very high negative α (-80%) retention falls again, ie reinsurance is more useful and preferred to using capital. This must mean that with such a high α, overall variability must be so increased as to require more protection.

Figure 6.14: Variability of Profits for α=80%



The variability of profits is considerably higher in Figure 6.14 then in Figure 6.1 and also the range of values is bigger in the former case. This is a consequence of having the experience in a year dependent on the previous one. This effect leads to achieving both higher positive and negative values. As a result the insurer will choose lower retention and higher capital values.

Figure 6.15: Variability of Profits for α=-40%

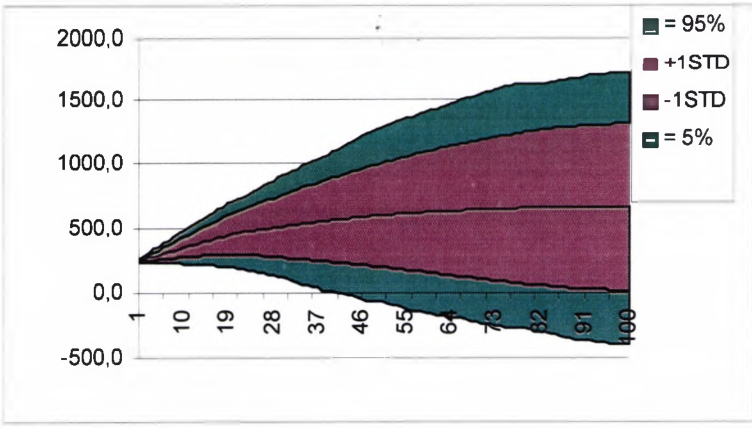
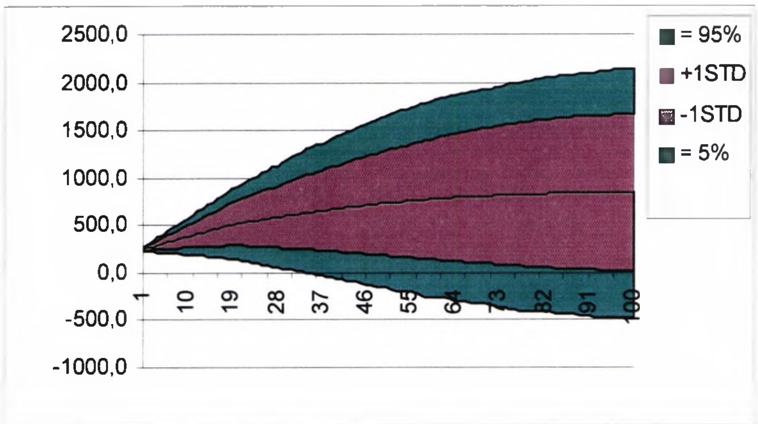


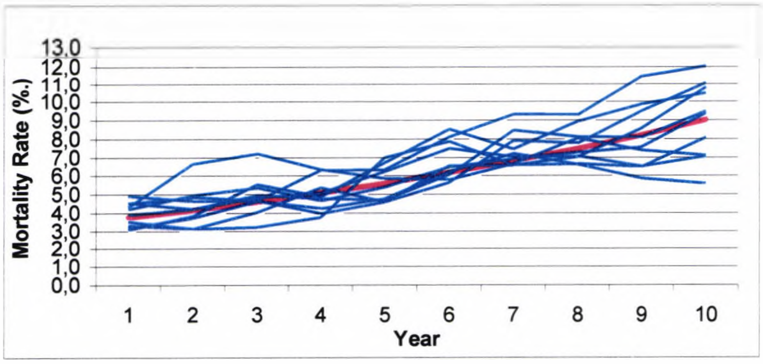
Figure 6.15 shows the variability of profits for α=-40%. It is evident from it, that we are now in a situation of low risk which justifies the increase in retention level seen in Table 6.5, as a consequence of reinsurance now being less efficient in controlling risk as opposed to using capital.

Figure 6.16: Variability of Profits for α=-80%



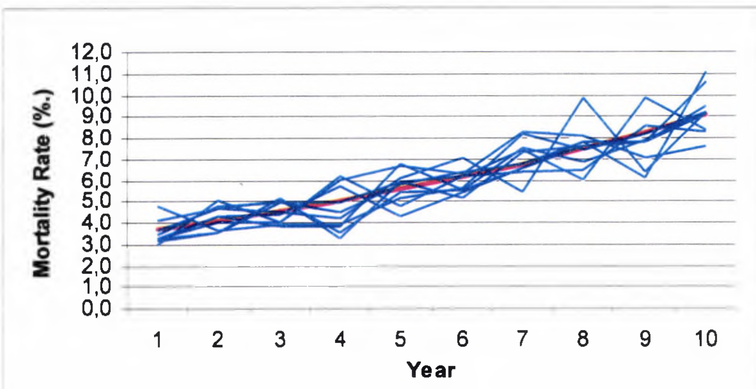
In Figure 6.16, we see an increase in the overall variability of profits as a consequence of considering a very negative α. This is in line of our discussion of the results from Table 6.5.

Figure 6.17: Mortality Rate Sample Paths for α=80%



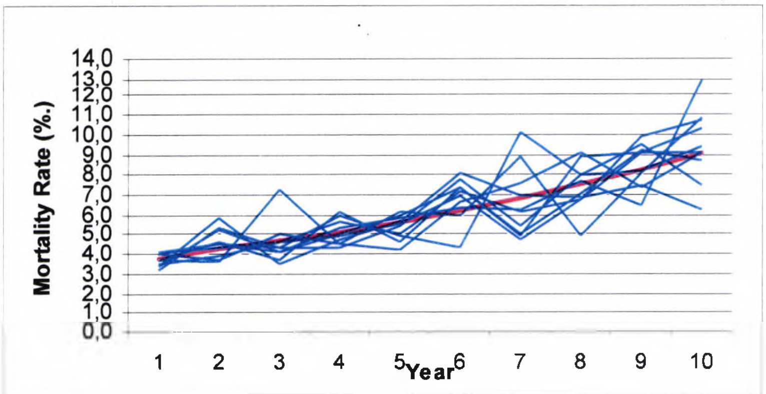
In Figure 6.17 we can see how the mortality experience in one year influences the following year. A bad year, is generally followed by another bad one and vice-versa.

Figure 6.18: Mortality Rate Sample Paths for α=-40%



The mortality experience for a negative value of α is seen in Figure 6.18. In this case a bad year is generally followed by a better year and vice-versa.

Figure 6.19: Mortality Rate Sample Paths for α=-80%



The mortality experience for a very high value of α is seen in Figure 6.19. It can be seen an increase in variability of mortality rates experience.

6.5. Summary

This chapter looked at parameter risk, a component of life office's total risk and its implications on retention levels. The purpose was to expand the scope of Chapter 5, which only looked at process risk and to assess how differently these two sources of risk influence optimal retention levels..

The methodology used, considered expanding the assumption regarding expected mortality, which had so far been assumed to be equal and fixed for all policyholders. A first approach assumed that the expected mortality rate varied as a function of the sum assured. In the second approach, the expected mortality rate was itself a random variable. Two models were considered for the latter case.

When we were looking in isolation at process risk we clearly identified a trade-off between capital and reinsurance. The insurer could hold more capital or buy more reinsurance protection, because both would help him meet solvency constraints. In the case of paramenter risk, with one exception, we got the reverse patterns. In this case, capital requirement increased at the same time retention was decreasing. The increase in risk was so dramatic that, even if we were to take into consideration for the trade-off, that must have occurred, both capital and reisurance needed to be increased. The optimal situation for the insurer was to increase both. In the exception mentioned above (see Section 7.4 for α=-40%), the increase in risk was not big enough to counter the usual trade-off between capital and reinsurance.

Hence, we would conclude that parameter risk is a much larger component of total risk than process risk, having therefore a more significant effect on retention levels. The nature of the distribution or variability of the parameter is important in setting retention and capital levels.

7. Discussion

The purpose of this project was to see if utility theory could provide a framework by which decisions involving reinsurance strategies can be assumed and compared. The methodology used was to asses the utility of the extra return obtained by the shareholders, by utilizing their capital in support of the life portfolio's risk as opposed to investing it in a risk free way.

Two approaches were considered: An analytical approach focusing on a one-year scenario (Chapter 4) and a multi-year scenario making use of a stochastic approach (Chapter 5 and 6). Both looked at portfolio of term assurance policies where all policyholders share the same characteristics: age, term of policy and reinsurance premium rate. Variations in the initial set of assumptions were considered and the effect on the optimal retention and capital level analysed.

When considering the profit generated by an insurance company, we have only considered the utility of the mortality profit in isolation from the total profit received by the shareholders. This is certainly one of the sources of risk which can be controlled by reinsurance (the other being the capital strain for new business). Hence it appears logical to focus on this source of risk when attempting to determine appropriate reinsurance strategies.

In addition we have also considered the level of capital that is in support of the mortality risk. The approach is to assess the utility of the extra return obtained by the shareholders, by utilising their capital in support of the mortality risk, as opposed to investing in a risk-free way. The level of capital required is naturally linked with the exposure to risk. Reinsurance treaties reduce the exposure to risk and as a natural consequence, the required level of capital that is in support of the mortality risk is reduced as a function of the amount of reinsurance bought.

The results obtained have shown that utility theory can be a way in which capital and reinsurance combinations can be chosen such that shareholders interests are optimized, while ensuring policyholders interests continue to be met. When we were trying to assess the retention level that optimizes expected utility of profits, while still imposing a ruin constraint, we were balancing both policyholder and shareholder´s interests. Different optimal retentions and capital levels were obtained as a function of the portfolio characteristics. Also, we showed possible ways in which parameter risk can be allowed for.

The choice of the retention level is very much dependent on the nature of the investor, the portfolio´s characteristics, the assumed risk model (ie the nature of q_x) and the type of business the insurance company is writing.

A very risk averse investor is more likely to choose lower retention levels then a more risk seeking one. One key element in the choice of an acceptable retention level for an investor is necessarily the amount of capital it has available to invest in the insurance company. An investor who is not so risk averse, but that has limited capital resources will be limited in the amount of business it will be able to retain. An investor with unlimited capital resources and willing to venture in risky situations, will not necessarily invest all its resources in an insurance company. It is necessary to service the capital and to pay shareholders its cost. Investing more than the volatility of the business requires means the investor is not fully optimizing the capital usage. The amount of capital invested in an insurance company should be just enough and not more than that necessary to cope with the business volatility.

The results presented could have been further expanded by also optimizing policyholder′s interests. The way policyholder′s interests were allowed for was by imposing a ruin constraint of 5%, when assessing the level of capital needed to backup the volatility of retained business. However, if this percentage was in itself the result of an utility optimizing exercise, then the resulting capital and reinsurance strategy would both optimize shareholders and policyholders interests.

The results obtained when the portfolio size is increased (retentions rise while required capital falls), the cost of reinsurance is increased (retentions rise and required capital rises) and the average age of the portfolio changes (older portfolios require less reinsurance, and beyond about age 40 less capital per £ of premium), were generally intuitive. The same can be said when we considered a higher policy term (retention levels increased as opposed to capital), an open instead of a closed portfolio (retentions decrease and required capital rises), and finally when considering an endowment instead of a term assurance policy (retentions rise and capital decreases).

We would also note the relative insignificance of process risk compared with model and parameter risk. The effect of introducing this additional degree of risk (model and/or parameter), on retention level was dramatic. In this, the increase in risk was so dramatic, that even if we were to take into consideration for the trade-off, that must have occured, both capital and reinsurance needed to be increased. Therefore paramenter risk is a much larger component of total risk, having a more significant effect on retention levels.

One key aspect that comes out from the results obtained is that the portfolio´s characteristics is fundamental for the assessment of the "best" combination of capital and reinsurance. This being fundamental, means that insurers should at all times devote time and effort to study their portfolio. However, this is not enough, because if model or parameter risk is present then this is what the insurer should be concentrating on. It is therefore, more important to investigate and estimate properly these sources of risk. This is an area where theorists can play a major role, because of their possibility of doing research at a market level. Looking at a macro level is better in order to estimate these sources of risk. We would note that the actual nature of the parameter variability (ie such as the assumed model for q_x), is clearly important in reaching important decisions.

The results presented above were based on a very simple form of reinsurance, which is the most common form of reinsurance treaty used in life reinsurance. This form of reinsurance can be formalized in two distinct ways. We compared the two and assessed which one was more efficient based on an utility approach. This result suggested, that the methodolgy followed could also be applied by the insurer when deciding between different reinsurance arrangements.

As stated in Chapter 1, according to Carter (1979), the choice of the retention level is a function of the following variables:

(i) Size of the portfolio (N);

(ii) The number n of exposure units of size x included in the portfolio;

(iii) Probability of an exposure unit of size x incurring a loss in time t;

(iv) The size of loss z if a loss occurs;

(v) Ratio of capital and reserves to N (A);

(vi) Rate of return payable on A;

(vii) Premium loading;

(viii) Selected probability of ruin;

(ix) Price payable for reinsurance;

(x) Type of reinsurance;

(xi) The company's investment policy.

Our analysis covered most of the variables defined by Carter (1979). We left out the company's investment policy and rate of return, because we were only focusing on the mortality profit. Also we did not cover directly variable (v), even though our analysis covered capital. Carter (1979), covered all sources of mortality risk in point (iii). However, our understanding is that he did not cover a precise specification of the nature of the source of mortality risk, ie in particular the nature of parameter risk, which was brought out in this thesis.

The retention levels were derived as a function of the characteristics of the investor, based on the exponential utility function. This function characterizes investors that take the same decisions if they invest the same amount of money, regardless of the initial level of wealth. It seems reasonable to assume that an investor when deciding whether or not to invest in an insurance company, it will look at this possibility in isolation from all its other investments, or initial wealth. This is particularly true when we try to consider the utility of the profit to all shareholders taken collectively, when the total amount of capital they invest is assumed to be variable. However, we did not fully test its appropriateness.

If we assume that the axioms of Von Neumann and Morgenstern hold, then in order to determine an investor's utility function, it is necessary to take him through a series of questions (refer to Chapter 3). It could be possible, from a theoretical point of view, to ask as many questions needed in order to obtain an investor's utility curve, or some points that would allows us to fit a curve. However, there is a limit to how much information we could get from an investor, and some allowance for error would have to be made. There is also the problem of whether a collection of investors can be adequately modelled by assuming they are a single investor, as done here.

Even though it is possible to argue the adequacy of the chosen utility function, it would be of interest to expand the results obtained, by looking at other utility functions and also to explore ways of estimating with more precison, possibly with questions, what would be an appropriate utility function for an investor of a life office. Also, in Section 3.5, we have drawn the attention to some of the weak points of utility theory. In particular, the violations of its axioms and underlying principles which were generated by certain conditions and framing procedures. Even though, utility theory is no longer seen as an accurate descriptive theory, it can still be used to provide useful insights for decision making under uncertainty.

The analysis presented can be further expanded in such a way that when choosing the retention level maximizes the expected utility of profits, profits now being equal to the total life office's profit. It is normal practice to look at capital, or solvency, for the insurance company. It would be interesting to see whether the trade-off between capital and reinsurance would still be the same.

Another area for improvement would be to include in the optimization process the reinsurer's "supply curve". The approach used considers that reinsurance is available at any given price. In real life this might not be the case, except when the market is soft and it is possible, in theory, to find reinsurance cover at almost any price. Reaching an agreement in the reinsurance price, is what Lemaire (1981a) calls the bargaining process.

Whilst we have looked at the insurer in isolation, Lemaire and many other authors (see Section 3.6 for further details), have looked at optimal rules for constructing reinsurance treaties, by considering at the same time the insurer and the reinsurer (in the most simple case). In particular, by using the economic concept of utility theory, they have managed to choose Pareto-optimal forms of risk exchange. It is therefore difficult to compare both results because the underlying hypothesis are different. Still, it was also possible to see that utility theory can be used as a tool to find optimal rules for exchange of risks. In other words, utility theory was a useful tool for decision making under uncertainty, a similar result to what we have seen in this thesis.

It would be of interest to study the impact on the optimal solutions and in the trade-off between reinsurance and capital, of including the reinsurer in the optimization process. Even though noone is ever alone in the market, the approach used in this thesis, does bring what what sould be the main concerns of the insurer, possibly before he goes into the market.

Another area of interest would also be in general insurance. In most classes of general insurance, retention levels have also tend to be fixed with the values that market practice has shown to stand the test of time. Since the characteristics are quite different it would be an area of interest the assessment of the factors that determine retention levels in general insurance. It is likely that many of the results obtained here (particularly the one year case), will be parallelled by similar results in general insurance.

8. References

Allais, M. (1953). Le comportement de l'homme rationnel devant le risque: Critique des postulats et axiomes de l'école américaine. *Econometrica* **21**, 503-46.

Allais, M. (1979a). The foundations of a positive theory of choice involving risk and a criticism of postulates and axioms of the American school. *Expected Utility Hypotheses and the Allais Paradox* (ed. M. Allais and O. Hagen), pp. 27-145. Dordrecht, Holland: Reidel. Translation of "Fondements d'une theorie positive des choix comportant un risque et critique des postulats et axiomes de l'école américaine", *Colloques Internationaux du Centre National de la Recherche Scientifique. XL. Econométrie*, Paris, 1953, pp. 257-332.

Arrow, K.J. (1974). Essays in the Theory of Risk Bearing. Amsterdam: North-Holland.

Bawa, V.S. (1982). Stochastic dominance: a research bibliography. *Management Science* **28**, 698-712.

Benktander, G. (1975). A note on optimal reinsurance. *ASTIN Bulletin* **8**, 174-163.

Bernoulli, D. (1954). Exposition of a new theory on true mesurement of risk. *Econometrica* **22**, 22-36. Translated by L. Sommer from "Specimen theorie novae de mensura sortis," Commentari Academiae Scientiarium Imperialis Petropolitanae **5** (1738), 175-92.

Booth, P., Chadburn, R., Cooper, D., Haberman, S. and James, D. (1999). Modern Actuarial Theory and Practice. Chapman & Hall/CRC.

Borch, K. H. (1960). Reciprocal reinsurance treaties seen as a two-person cooperative game. *Skandinavisk Aktuarietidskrif* **43**, 29-58.

Borch, K. H. (1975). Optimal insurance arrangements. *ASTIN Bulletin* **8**, 284-290.

Borch, K. H. (1990). Economics of Insurance. Elsevier Science Publishers B. V.: North- Holland.

Bowers, N. L., Gerber, H. U., Hickman, J. C., Jones, D. A. and Nesbitt, C. J. (1986) *Actuarial Mathematics*. Society of Actuaries, Chicago.

Buhlmann, H. and Jewell, W. (1979). Optimal Risk exchanges. *ASTIN Bulletin* **10**, 243-262.

Carter, R. L. (1979). Reinsurance. Kluwer Publishing Limited: UK.

Chipman, J. S. (1960). The foundations of utility. *Econometrica* **28**, 193-224.

Coutts, S. M. and Thomas, T. R. H. (1997). Modelling the impact of reinsurance on financial strength. *Journal of Institute of Actuaries* **3**, 583-653.

Fishburn, P. C. (1988). Nonlinear Preference and Utility Theory. Wheatsheaf Books.

Hossack, I.B., Pollard, J.H. and Zehnwirth, B. (1983). Introductory statistics with applications in general insurance. Cambridge University Press

Kahneman, D. and Tversky, A. (1979). Prospect Theory: an analysis of decision under risk. *Econometrica* **47**, 263-291.

Kahneman, D. and Tversky, A. (1981). The framing of decisions and psychology of choice. *Management Science* **211**, 453-58.

Kahneman, D. and Tversky, A. (1986). Rational choice and the framing of decisions. *Rational Choice* (ed. R.M. Hogarth and M.W. Reder), pp. 67-94. Chicago: University of Chicago Press.

Kauder, E. (1965). A History of Marginal Utility Theory. Princeton: Princeton University Press.

Khan, P.M. (1961). Some remarks on a recent paper by Borch. *ASTIN Bulletin* **1**, 265-272.

Lemaire, J. and Baton, B. (1981a). The core of a reinsurance market. *ASTIN Bulletin* **12**, 57- 71.

Lemaire, J. and Baton, B. (1981b). The bargaining set of a reinsurance market. *ASTIN Bulletin* **12**, 101-114.

Luce, R.D. (1956). Semiorders and a theory of utility discrimination. *Econometrica* **24**, 178-191.

May, K.O. (1954). Intransitivity, utility and aggregation of preference patterns. *Econometrica* **22**, 1-13.

Panjer, H. H. and Willmott, G. E. (1992). Insurance Risk Models. Society of Actuaries, Chicago.

Pentikainen, T., Rantala, J., Pesonen, M., Bonsdorff, H. and Ruohonen, M. (1989). Solvency and Financial Strength. Finnish Insurance Training and Publishing Company Ltd, Helsinki.

Pratt, J.W. (1964). Risk aversion in the small and in the large. *Econometrica* **32**, 122-136.

Samuelson, P. (1969). Lifetime Portfolio Selection by Dynamic Programming. *The Review of Economics and Statistcis* **51**, 239-246.

Sherris, M. (1992). Portfolio Selection and Matching: a synthesis. *Journal of the Institute of Actuaries* **119**, 87-105.

Spedding, A. (1989). Life Reinsurance. Life Assurance Monograph. Institute of Actuaries.

Verbeek, H.G. (1966). On optimal reinsurance. *ASTIN Bulletin* **4**, 29-38.

Von Neumann, J. and Morgenstern, O. (1944). Theory of Games and Economic Behaviour. Princeton, NJ: Princeton University Press; 3rd ed. 1953.

9. Bibliography

Agmon, T. and Kahane, Y. (1978). Balancing international insurance portfolios and exchange risks. 14th ASTIN Colloquium, Taormina, October 1978.

Allais, M. (1979b). The so-called Allais paradox and rational decisions under uncertainty. *Expected Utility Hypotheses and the Allais Paradox* (ed. M. Allais and O. Hagen), pp. 437-681. Dordrecht, Holland: Reidel.

Allais, M. (1986). The General Theory of Random Choices in Relation to the Invariant Cardinal Utility Function and the Specific Probability Function. Dordrecht, Holland: Reidel.

Andreadakis, M. and Waters, H.R. (1980). The effect of reinsurance on the degree of risk associated with an insurer's portfolio. *ASTIN Bulletin* **11**, 119-135.

Apostol, T. A. (1985). Cálculo: vol. I e II. Editorial Reverté.

Asmussen, S. (1984). Approximations for the probability of ruin within finite time. *Scandinavian Actuarial Journal* 31-57.

Babier, J. and Chan, B. (1992) Approximation of ruin probability by di-atomic or di-exponential claims. *ASTIN Bulletin* **22**, 235-246.

Bahr, B. (1975). Asymptotic ruin probabilities when exponential moments do not exist. *Scandinavian Actuarial Journal* 6-10.

Bahr, B. (1974). Ruin probabilities expressed in terms of ladder height distributions. *Scandinavian Actuarial Journal* 190-204.

Beard, R. E., Pentikainen, T. and Pesonen, E. (1984). Risk theory, the stochastic basis of insurance. 3rd edition. Chapman and Hall, London and New York.

Beekman, J. A. (1969). A ruin function approximation. *Transactions of Society of Actuaries, Volume XXI.* Meetings n. 59A and 59B.

Beekman, J. A. and Fuelling, C. P. (1980). Simulation of a Multirisk Collective Model. Department of Mathematical Sciences, Ball State University, Muncie, Indiana.

Beekman, J. A (1974). Two stochastic processes. Almqvist & Wiksell International, Stockholm, Sweden.

Beekman, J. A., and Clinton, P. F. (1981). Risk convolution calculations. *Scandinavian Actuarial Journal* 151-164.

Bjork, T. and Grandel, J. (1985): An insensivity property of the ruin probability. *Scandinavian Actuarial Journal* 148-156.

Booth, P. M., Chadburn, R. G. and Ong, A. S. (1997). Utility-maximisation and the control of solvency for life insurance funds. *Actuarial Research Paper* **94**. Department of Actuarial Science & Statistics, City University, London.

Booth, P. M. (1997). The analysis of actuarial investment risk. *Actuarial Research Paper* **93**. Department of Actuarial Science & Statistics, City University, London.

Borch, K. H. (1978). Problems in the economic theory of insurance. *ASTIN Bulletin* **10**, 1-11.

Buhlmann, H. and Gerber, H. (1978). Risk bearing and the insurance market. *ASTIN Bulletin* **10**, 12-24.

Chadburn, R. G. (1996). Use of a parametric risk measure in assessing risk based capital and insolvency constraints for with profits life insurance. *Actuarial Research Paper* **83**. Department of Actuarial Science & Statistics, City University, London.

Cox, D. R. and Miller, H. D. (1965). The Theory of Stochastic Processes. Chapman and Hall.

Davis, K. R. and Leverett, E. J. (1975). Evaluating the financial requirements of a life Insurance Company via Simulation. *Scandinavian Actuarial Journal* 54-58.

De Vylder, F. and Goovaerts, M. J. (1988). Recursive calculation of finite- time ruin probabilities. *Insurance: Mathematics and Economics* **7**, 1-8.

De Vylder, F. and Goovaerts, M. J. (1993). The solution of pratical ruin problems. Conference: Risk Theory, Oberwalfach.

De Pril, N. (1988). Improved approximations for the aggregate claims distribution of Life Insurance Portfolio. *Scandinavian Actuarial Journal* 61-68.

Dickson, D. C. M. and Gray, J. R. (1984a). Exact solutions for ruin probability in the presence of an absorbing upper barrier. *Scandinavian Actuarial Journal* 174-186.

Dickson, D. C. M. and Gray, J. R. (1984b). Approximation to ruin pobability in the presence of an upper absorbing barrier. *Scandinavian Actuarial Journal* 105-115.

Dickson, D. C. M. and Waters, H. R. (1991). Recursive calculation of survival probabilities. *ASTIN Bulletin* **21**, 199-221.

Dufesne, F. and Gerber, H. U. (1987). Three methods of calculating the probability of ruin. University of Lausanne, Switzerland.

Dufesne, F. (1988). Distribuitions stationnaires d'un système bonus-malus et probabilité de ruine. *ASTIN Bulletin* **18**, 31-46.

Feller, W. (1957). An intoduction to probability theory and its applications. 2nd edition. New York: Wiley.

Fishburn, P. C. (1965). Independence in utility theory with whole product sets. *Operations Research* **13**, 28-45.

Fishburn, P. C. (1967). Bounded expected utility. *Annals of Mathematical Statistics* **38**, 1054-60.

Fishburn, P. C. (1969). Weak qualitative probability on finite sets. *Annals of Mathematical Statistics* **40**, 2118-26.

Fishburn, P. C. (1970a). Intrasitive indifference in preference theory: a survey. *Operations Research* **18**, 207-28.

Fishburn, P. C. (1970b). Utility Theory for Decison Making. New York: Wiley.

Fishburn, P. C. (1971a). A study of lexigrographic expected utility. *Management Science* **17**, 672-78.

Fishburn, P. C. (1971b). One-way expected utility with finite consequence spaces. *Annals of Mathematical Statistics* **42,** 572-77.

Fishburn, P. C. (1972). Alternative axiomatizations of one-way expected utility. *Annals of Mathematical Statistics* **43**, 1648-51.

Fishburn, P. C. (1973a). A mixture-set axiomatization of conditional subjective expected utility. *Econometrica* **41**, 1-25.

Fishburn, P. C. (1973b). The Theory of Social Choice. Princeton: Princeton University Press.

Fishburn, P. C. (1974). Lexicographic orders, utilities, and decison rules: a survey. *Management Science* **20**, 1442-1471.

Fishburn, P. C. (1975a). Unbounded expected utility. *Annals of Statistics* **3**, 884-96.

Fishburn, P. C. (1975b). Weak comparative probability on infinite sets. *Annals of Probability* **3**, 889-93.

Fishburn, P. C. (1976a). Cardinal utility: an interpretive essay. *International Review of Economics and Business* **23**, 1102-14.

Fishburn, P. C. (1976b). Continua of stochastic dominance relations for bounded probability distributions. *Journal of Mathematical Economics* **3**, 295-311.

Fishburn, P. C. (1977a). Multiattribute utilities in expected utility theory. *Conflicting Objectives in Decisions* (ed. D. E. Bell, R. L. Keeney and H. Raiffa), p. 172-94. New York: Wiley.

Fishburn, P. C. (1977b). Condorcet social choice functions. *SIAM Journal on Applied Mathematics* **33**, 469-89.

Fishburn, P. C. (1978a). A survey of multiattribute/multicriterion evaluation theories. *Multiple Criteria Problem Solving* (ed. S. Zionts), pp. 181-224. Berlin: Springer-Verlag.

Fishburn, P. C. (1978b). On Handa's "new theory of cardinal utility" and the maximization of expected return. *Journal of Political Economy* **86**, 321-24.

Fishburn, P. C. (1979). On the nature of expected utility. *Expected Utility Hypotheses and the Allais Paradox* (ed. M. Allais and O. Hagen), pp. 243-57. Dordrecht, Holland: Reidel.

Fishburn, P. C. (1982). The Foundations of Expected Utility. Dordrecht, Holland: Reidel.

Gerber, H. U. (1995). Life Insurance Mathematics. 2nd ed. Springer-Verlag: Berlin.

Janssen, J. (1985). The equivalence of the infinite time ruin problems for positive and negative risk models. *Scandinavian Actuarial Journal* 86-93.

Karlin, S. and Taylor, H. M. (1975). A first course in stochastic processes. 2nd edition. Academic Press.

Kastelijn, W. M. and Remmerswaal, J. C. M. (1986). Solvency. *Surveys of Actuarial Studies* **1**. Nationale-Nederlanden N. V., Rotterdam.

Kremer, E. (1984). An asymptotic formula for the net premium of some reinsurance treaties. *Scandinavian Actuarial Journal* 11-22.

Kroll, Y., Levy, H. and Markowitz, H. M. (1984). Mean-variance versus direct utility maximization. *The Journal of Finance* **Vol. XXXIX**, 47-61.

Lemaire, J., and Quairíere, J. P. (1986). Chains of reinsurance revisited. *ASTIN Bulletin* **16,** 77-88.

Lemaire, J. (1977). Exchange de risques entre assureurs et theorie des jeux. *ASTIN Bulletin* **9**, 155-180.

Lemaire, J. (1979). A non symmetrical value for games without transferable utilities; application to reinsurance. *ASTIN Bulletin* **10**, 195-214.

Lemaire, J. and Lorea, M. (1980). Equilibrege d'un marche de reassurance. *ASTIN Bulletin* **11**, 107-118.

Levy, H. and Markowitz, H. M. (1979). Approximating expected utility by a function of mean and variance. *Management Science* **69**, 309-317.

Loubergé, H. (1991). Risk, Information and Insurance. Kluwer Academic Publishers: The Netherlands.

Mood, A. M., Grybill, F. A. and Boes, D. C. (1985). Introduction to the Theory of Statistics. International Student Edition. McGraw-Hill.

Panjer, H. H. and Willmott, G. E. (1982a). Recursions for compound distributions. *Working Paper Series in Actuarial Science*. Department of Statistics. University of Waterloo.

Panjer, H. H. and Willmott, G. E. (1982b). Random sum reinsurance models. *Working Paper Series in Actuarial Science*. Department of Statistics. University of Waterloo.

Panjer, H. H. and Willmott, G. E. (1982c). Pratical aspects of stop-loss calculations.*Working Paper Series in Actuarial Science*. Department of Statistics. University of Waterloo

Panjer, H. H. and Willmott, G. E. (1982d). Compound poisson models in actuarial risk theory.*Working Paper Series in Actuarial Science*. Department of Statistics. University of Waterloo.

Panjer, H. H (1986). Direct caculation of ruin probabilities. *The Journal of Risk and Insurance* **53**, 3, 521-529.

Panjer, H. H. and Willmott, G. E. (1993). On the stability of recursive formulas. *Working Paper Series in Actuarial Science*. Department of Statistics. University of Waterloo

Pechlivanides, P. M. (1978). Optimal reinsurance and dividend payment strategies. *ASTIN Bulletin* **14**, 34-36.

Pentikainen, T. (1975). A model of stochastic-dynamic prognosis. *Scandinavian Actuarial Journal* 29-53.

Pentikainen, T. and Rantala, J. (1982). Solvency of insurers and equalization reserves, *Vol. I and II*. Insurance Publishing Company Ltd., Helsinki.

Prelec, D. and Loewenstein, G. (1991). Decision making overtime and under uncertainty: a common approach. *Management Science* **37**, 770-786.

Pulley, L. B. (1983). Mean-variance approximations to expected logaritmic utility. *Operations Research* **31**, 685-696.

Pyle, D. H. and Turnovsky, J. (1970). Safety first and expected utility maximization in mean standard deviation portfolio analysis. *The review of economics and statistics* **52**, 75-81.

Ramsay, C. M. (1986). Ruin Probabilities and the Compound Poisson-Markov Chain. *Scandinavian Actuarial Journal* 3-12.

Reid, D. W. and Tew, B. V. (1986). Mean-variance versus direct utility maximization: a comment. *The Journal of Finance* **41**, 1177-1779.

Ross, S. M. (1989). Introduction to Probability Models. 4th edition. Academic Press.

Samuelson, P. A. (1977). St. Petersburgh paradoxes: defanged, dissected, and historically described. *Journal of Economic Literature* **15**, 24-55.

Seal, H. L. (1974). The numerical calculation of U(w,y), the probability of non-ruin in an interval (0,t). *Scandinavian Actuarial Journal* 121-139.

Seal, H. L. (1978). Survival Probabilities, the Goal of Risk Theory. John Wiley & Sons, Chichester, new York, Brisbane and Toronto.

Stigler, G. J. (1950). The development of Utility Theory: I, II. *Journal of Political Economy* **58**, 307-27, 373-96.

Straub, E. (1984). Actuarial remarks on planning and controlling in insurance. *ASTIN Bulletin* **14**, 183-191.

Stundt, B. and Jewell, W. S. (1981). Further Results on Recursive Evaluation of a Family of Compound Distributions. *ASTIN Bulletin* **12**, 27-39.

Torin, O. (1982). Probabilities of Ruin. *Scandinavian Actuarial Journal* 65-102.

Waters, H. R. (1983). Probability of Ruin for a Risk Process with Claims Cost Inflation. *Scandinavian Actuarial Journal* 148-164.

Xekalai, E. and Panaretos, J. (1983). Identifiability of Compound Poisson Distributions. *Scandinavian Actuarial Journal* 39-45.