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On abelian subcategories of triangulated categories

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Abstract

The stable module category of a selfinjective algebra is triangulated, but need not have any nontrivial t -structures, and in particular, full abelian subcategories need not arise as hearts of a t -structure. The purpose of this paper is to investigate full abelian subcategories of triangulated categories whose exact structures are related, and more precisely, to explore relations between invariants of finite-dimensional selfinjective algebras and full abelian subcategories of their stable module categories.

Keywords Triangulated · Abelian · Stable module categories

Mathematics Subject Classification 18E30 · 18E10 · 16D90

1 Introduction

Definition 1.1 Let \mathcal{C} be a triangulated category with shift functor Σ . A *distinguished abelian subcategory* of \mathcal{C} is a full additive subcategory \mathcal{D} of \mathcal{C} which is abelian, such that for any short exact sequence

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

in \mathcal{D} there exists a morphism $h : Z \rightarrow \Sigma(X)$ in \mathcal{C} such that the triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma(X)$$

is exact in \mathcal{C} .

Proper abelian subcategories, introduced in [18, Def. 1.2], admissible abelian subcategories, from [5, Def. 1.2.5], and thus hearts of t -structures are distinguished abelian subcategories, but not all distinguished abelian subcategories are proper abelian subcategories. Full abelian subcategories of a triangulated category are not necessarily distinguished; see Example 9.5.

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The main motivation for considering this definition is the abundance of distinguished abelian categories in stable module categories of finite-dimensional selfinjective algebras, and the hope that these may therefore shed light on the invariants of selfinjective algebras in terms of their stable module categories. A conjecture of Auslander–Reiten predicts that for A a finite-dimensional algebra over a field, the stable category $\underline{\text{mod}}(A)$ of finitely generated left A -modules should determine the number of isomorphism classes of nonprojective simple A -modules. If this conjecture were true for blocks of finite group algebras, it would imply some cases of Alperin’s weight conjecture. By a result of Martínez–Villa [29], it would suffice to prove the Auslander–Reiten conjecture for selfinjective algebras. If A is selfinjective, then $\underline{\text{mod}}(A)$ is triangulated.

The following result recasts the Auslander–Reiten conjecture for selfinjective algebras in terms of maximal distinguished abelian subcategories of $\underline{\text{mod}}(A)$. For \mathcal{D} an abelian category, we denote by $\ell(\mathcal{D})$ the number of isomorphism classes of simple objects, with the convention $\ell(\mathcal{D}) = \infty$ if \mathcal{D} has infinitely many isomorphism classes of simple objects. For A a finite-dimensional algebra over a field, we write $\ell(A) = \ell(\text{mod}(A))$; that is, $\ell(A)$ is the number of isomorphism classes of simple A -modules.

Theorem 1.2 *Let A be a finite-dimensional selfinjective algebra over a field such that all simple A -modules are nonprojective. The following hold.*

- (i) *If \mathcal{D} is a distinguished abelian subcategory of $\underline{\text{mod}}(A)$ containing all simple A -modules, then the simple A -modules are exactly the simple objects in \mathcal{D} . In particular, in that case we have $\ell(A) = \ell(\mathcal{D})$.*
- (ii) *The stable module category $\underline{\text{mod}}(A)$ has a maximal distinguished abelian subcategory \mathcal{D} satisfying $\ell(\mathcal{D}) = \ell(A)$.*

Statement (i) of this theorem is Theorem 3.8, and statement (ii) will be proved in Sect. 6. The maximal distinguished abelian subcategories in (ii) are in general far from unique. Note that $\underline{\text{mod}}(A)$ may have distinguished abelian subcategories \mathcal{D} satisfying $\ell(\mathcal{D}) = \infty$; see Example 9.3. We describe a simple construction principle of distinguished abelian subcategories in stable module categories of selfinjective algebras.

Theorem 1.3 *Let A be a finite-dimensional selfinjective algebra over a field, and let I be a proper ideal in A . Denote by $r(I)$ the right annihilator of I in A . The canonical map $A \rightarrow A/I$ induces an embedding $\text{mod}(A/I) \rightarrow \underline{\text{mod}}(A)$ as a distinguished abelian subcategory in $\underline{\text{mod}}(A)$ if and only if $r(I) \subseteq I$.*

This will be proved in Sect. 3 as a consequence of Theorem 3.1, itself a consequence of the more general Theorem 2.1.

A subcategory \mathcal{D} of a triangulated category \mathcal{C} is called *extension closed* if for any exact triangle $U \rightarrow V \rightarrow W \rightarrow \Sigma(U)$ in \mathcal{C} with U, W belonging to \mathcal{D} , the object V is isomorphic to an object in \mathcal{D} . Hearts of t -structures on a triangulated category \mathcal{C} are extension closed distinguished abelian subcategories. In general, distinguished abelian subcategories need not be extension closed.

Theorem 1.4 *Let A be a finite-dimensional selfinjective algebra over a field, and let I be an ideal in A such that $r(I) \subseteq I \subseteq J(A)$. The following are equivalent.*

- (i) *The distinguished abelian subcategory $\text{mod}(A/I)$ of $\underline{\text{mod}}(A)$ is extension closed.*
- (ii) *The canonical functor $\text{mod}(A/I) \rightarrow \underline{\text{mod}}(A)$ is an equivalence of k -linear categories.*
- (iii) *The algebra A is a Nakayama algebra such that all projective indecomposable A -modules have composition length 2.*

(iv) We have $\text{soc}(A) = r(I) = I = J(A)$.

This will be proved in Sect. 3.

If a triangulated category \mathcal{C} carries a structure of a monoidal category and if \mathcal{D} is a distinguished abelian subcategory of \mathcal{C} which is also a monoidal subcategory of \mathcal{C} , we call \mathcal{D} a *monoidal distinguished abelian subcategory* of \mathcal{C} . Stable module categories of finite group algebras provide examples of monoidal distinguished abelian subcategories which do not arise as heart of a t -structure. For G a finite group and p a prime, we denote by $O^p(G)$ the smallest normal subgroup of G such that $G/O^p(G)$ is a p -group.

Theorem 1.5 *Let k be a field of prime characteristic p and G a finite group. Let N be a normal subgroup of G of order divisible by p .*

- (i) *Restriction along the canonical surjection $G \rightarrow G/N$ induces a full embedding of $\text{mod}(kG/N)$ as a symmetric monoidal distinguished abelian subcategory in $\underline{\text{mod}}(kG)$.*
- (ii) *The heart of any t -structure on $\underline{\text{mod}}(kG)$ is zero. In particular, $\text{mod}(kG/N)$ is not the heart of a t -structure on $\underline{\text{mod}}(kG)$.*
- (iii) *If $O^p(N) = N$, then the distinguished abelian subcategory $\text{mod}(kG/N)$ of $\underline{\text{mod}}(kG)$ is extension closed.*

This Theorem will be proved alongside more precise results: statement (i) will be proved in Theorem 4.2, statement (ii) follows from Corollaries 2.8, 2.9, and statement (iii) will be proved as part of Proposition 7.8. For a partial converse of statement (iii) see Proposition 7.9.

For k a field of prime characteristic p and P a finite p -group, the Auslander–Reiten conjecture is known to hold for kP (cf. [25, Theorem 3.4]). We use this to classify the distinguished abelian subcategories of $\underline{\text{mod}}(kP)$ which are equivalent to the module categories of split finite-dimensional algebras in Theorem 4.4.

Hearts of t -structures intersect trivially their shifts. This need not be the case for arbitrary distinguished abelian subcategories. This following result describes the intersection of a distinguished abelian subcategory and its shift in the ambient triangulated category.

Theorem 1.6 *Let \mathcal{C} be a triangulated category and let \mathcal{D} be a distinguished abelian subcategory of \mathcal{C} . Let W be an object in \mathcal{D} such that $\Sigma(W)$ is an object in \mathcal{D} . Then W is injective in \mathcal{D} and $\Sigma(W)$ is projective in \mathcal{D} .*

This is proved in Corollary 5.4. This result is used in Proposition 6.1 to identify precisely in what way a distinguished abelian subcategory need not be proper in the sense of [18, Def. 1.2].

Section 2 describes some construction principles of distinguished abelian subcategories of stable module categories. Section 3 describes distinguished abelian subcategories of $\underline{\text{mod}}(A)$ whose simple objects are the simple A -modules, where A is a finite-dimensional selfinjective algebra. Section 4 specialises previous results to distinguished abelian subcategories in finite group algebras over a field of prime characteristic p , and includes a proof of the first statement of Theorem 1.5. Section 5 contains some general facts on distinguished abelian subcategories. In particular, it is shown in Proposition 5.1 that the morphism h in Definition 1.1 is unique. Section 6 contains technicalities, needed for the proof of Theorem 1.2, on the interplay between short exact sequences in $\text{mod}(A)$ and short exact sequences in a distinguished abelian subcategory \mathcal{D} of $\underline{\text{mod}}(A)$. The main result of Sect. 7 is a criterion on extension closure of distinguished abelian subcategories, needed for the last part of Theorem 1.5. Section 8 relates embeddings of module categories of selfinjective algebras to a result of Cabanes. Section 9 contains examples and further remarks.

Remark 1.7 The present paper, investigating abelian subcategories of triangulated categories in situations where there are no t -structures with a nontrivial heart, started out as a speculation about a possible analogue of stability spaces (cf. [7]) for stable module categories of finite-dimensional selfinjective algebras. Another interesting angle to pursue would be connections with abelian *quotient* categories of triangulated categories, which appear in numerous sources, for instance, in [14, 17, 22], in the context of torsion and mutation pairs in triangulated categories. See also [12], which explores this topic with a particular emphasis on stable module categories of finite-dimensional selfinjective algebras.

Notation 1.8 Throughout this paper, k is a field. Modules are unital left modules, unless stated otherwise, and algebras are nonzero unital associative. Let A be a finite-dimensional k -algebra. We denote by $\text{mod}(A)$ the abelian category of finitely generated A -modules. We denote by $\underline{\text{mod}}(A)$ the stable module category of $\text{mod}(A)$. That is, the objects of $\underline{\text{mod}}(A)$ are the same as in $\text{mod}(A)$, and for any two finitely generated A -modules U, V , the morphism space in $\underline{\text{mod}}(A)$ from U to V is the k -space $\underline{\text{Hom}}_A(U, V) = \text{Hom}_A(U, V)/\text{Hom}_A^{\text{pr}}(U, V)$, where $\text{Hom}_A^{\text{pr}}(U, V)$ is the space of all A -homomorphisms from U to V which factor through a projective A -module. Composition in $\underline{\text{mod}}(A)$ is induced by that in $\text{mod}(A)$. We write $\underline{\text{End}}_A(U) = \underline{\text{Hom}}_A(U, U)$ and $\text{End}_A^{\text{pr}}(U) = \text{Hom}_A^{\text{pr}}(U, U)$. See [27, §2.13] for more details. The Nakayama functor of A is the functor $\nu = A^\vee \otimes_A -$ on $\text{mod}(A)$, where $A^\vee = \text{Hom}_k(A, k)$ is the k -dual of A regarded as an A - A -bimodule. We say that the algebra A is *split* if $\text{End}_A(S) \cong k$ for every simple A -module S , or equivalently, if $A/J(A)$ is isomorphic to a direct product of matrix algebras over k .

The algebra A is called *selfinjective* if A is injective as a left (or right) A -module. Equivalently, A is selfinjective if the classes of finitely generated projective and injective A -modules coincide. By results of Happel [15], if A is selfinjective, then $\underline{\text{mod}}(A)$ is a triangulated category, with shift functor Σ induced by the operator sending an A -module U to the cokernel of an injective envelope $U \rightarrow I_U$, and with exact triangles in $\underline{\text{mod}}(A)$ induced by short exact sequences in $\text{mod}(A)$. If A is selfinjective, then the Nakayama functor ν on $\text{mod}(A)$ is an equivalence and induces an equivalence on $\underline{\text{mod}}(A)$, and $\tau = \Sigma^{-2} \circ \nu$ is the Auslander–Reiten translate. The algebra A is a *Frobenius algebra* if A is isomorphic to its k -dual A^\vee as a left or right A -module. A Frobenius algebra is selfinjective. The algebra A is called *symmetric* if A is isomorphic to its k -dual A^\vee as an A - A -bimodule. The image s in A^\vee of 1_A under some bimodule isomorphism $A \cong A^\vee$ is called a *symmetrising form* of A . If A is symmetric, then A is a Frobenius algebra, hence selfinjective, and the Nakayama functor is isomorphic to the identity functor on $\text{mod}(A)$. Finite group algebras, their blocks, and Iwahori–Hecke algebras are symmetric. See for instance [38, Ch. III], [27, §§2.11, 2.14] and [28, Appendix A.3] for more background.

For I a left ideal in A , its right annihilator $r(I) = \{a \in A \mid Ia = 0\}$ is a right ideal, and for J a right ideal in A , its left annihilator $l(J) = \{a \in A \mid aJ = 0\}$ is a left ideal. If I is an ideal in A , then so are $r(I)$ and $l(I)$. By results of Nakayama in [30, 31], if A is selfinjective, then the correspondences $I \mapsto r(I)$ and $J \mapsto l(J)$ are inclusion reversing bijections between the sets of left and right ideals in A . These bijections are inverse to each other and restrict to bijections on the set of ideals in A . In particular, for any ideal I in A we have $r(l(I)) = I = l(r(I))$. Moreover, still for A selfinjective, the socle $\text{soc}(A)$ of A as a left A -module is equal to the socle of A as a right A -module, and we have $r(J(A)) = l(J(A)) = \text{soc}(A)$. If A is a Frobenius algebra, then $\dim_k(I) + \dim_k(r(I)) = \dim_k(A)$, and if A is symmetric, then $r(I) = l(I)$ for any ideal I in A . See [38, Chapter IV, Section 6] for details.

We will make use without further comment of the standard Tensor-Hom adjunction.

2 Distinguished abelian subcategories in stable module categories

The stable module category of a finite-dimensional non-semisimple selfinjective k -algebra A need not have any t -structures with a nontrivial heart (see Proposition 2.7 and Corollary 2.8), but it always has distinguished abelian subcategories, and these tend to come in varieties (see Proposition 2.6). The first result in this section describes those distinguished abelian subcategories of $\underline{\text{mod}}(A)$ which arise as image of a full exact embedding $\text{mod}(D) \rightarrow \text{mod}(A)$ for some other finite-dimensional k -algebra D . By the Eilenberg–Watts Theorem, any full exact embedding $\text{mod}(D) \rightarrow \text{mod}(A)$ is of the form $Y \otimes_D -$ for some A - D -bimodule which is finitely generated projective as a right D -module. Not any embedding of $\text{mod}(D)$ as a distinguished abelian category in $\underline{\text{mod}}(A)$ is, however, induced by a full exact embedding $\text{mod}(D) \rightarrow \text{mod}(A)$. In particular, we do not know whether in general an embedding $\text{mod}(D) \rightarrow \underline{\text{mod}}(A)$ as a distinguished abelian subcategory is induced by tensoring with a suitable A - D -bimodule (and we expect this not to be the case). See Remark 2.10.

Theorem 2.1 *Let A be a finite-dimensional selfinjective k -algebra, and let D be a finite-dimensional k -algebra. Let Y be a finitely generated A - D -bimodule. The functor $Y \otimes_D -$ is a full exact embedding of $\text{mod}(D)$ into $\text{mod}(A)$ and induces an embedding of $\text{mod}(D)$ as a distinguished abelian subcategory of $\underline{\text{mod}}(A)$ if and only if the following conditions hold.*

- (1) $\text{End}_A^{\text{pr}}(Y) = \{0\}$,
- (2) Y is projective as a right D -module, and
- (3) the Tensor-Hom adjunction unit maps $V \rightarrow \text{Hom}_A(Y, Y \otimes_D V)$, $v \mapsto (y \mapsto y \otimes v)$, are isomorphisms, for all finitely generated D -modules V .

We state some parts of the proof of Theorem 2.1 as separate lemmas in slightly greater generality.

Lemma 2.2 *Let A be a finite-dimensional selfinjective k -algebra. Let \mathcal{D} be a full abelian subcategory of $\text{mod}(A)$ such that $\text{Hom}_A^{\text{pr}}(U, V) = \{0\}$ for all A -modules U, V in \mathcal{D} . Then the image of \mathcal{D} in $\underline{\text{mod}}(A)$ is a distinguished abelian subcategory of $\underline{\text{mod}}(A)$, which as an abelian category, is equivalent to \mathcal{D} .*

Proof The fact that \mathcal{D} is a full subcategory of $\text{mod}(A)$, together with the hypothesis $\text{Hom}_A^{\text{pr}}(U, V) = \{0\}$ for all U, V in \mathcal{D} , implies that the image of \mathcal{D} in $\underline{\text{mod}}(A)$ is a full subcategory of $\underline{\text{mod}}(A)$ which is equivalent to \mathcal{D} . By the assumptions on \mathcal{D} , exact sequences in \mathcal{D} remain exact in $\text{mod}(A)$. Since distinguished triangles in $\underline{\text{mod}}(A)$ are induced by short exact sequences in $\text{mod}(A)$, it follows that the image of \mathcal{D} in $\underline{\text{mod}}(A)$ is a distinguished abelian subcategory. □

Lemma 2.3 *Let A be a finite-dimensional k -algebra and Y a finitely generated A -module such that $\text{End}_A^{\text{pr}}(Y) = \{0\}$. Set $D = \text{End}_A(Y)^{\text{op}}$. The following hold.*

- (i) *Let m, n be positive integers and let U, V be quotients of the A -modules Y^m, Y^n , respectively. Then $\text{Hom}_A^{\text{pr}}(U, V) = \{0\}$.*
- (ii) *For any two finitely generated D -modules M, N we have $\text{Hom}_A^{\text{pr}}(Y \otimes_D M, Y \otimes_D N) = \{0\}$.*

Proof With the assumptions in (i), there are surjective A -homomorphisms $\alpha : Y^m \rightarrow U$ and $\beta : Y^n \rightarrow V$. Let $\psi : U \rightarrow V$ be an A -homomorphism which factors through a projective A -module P . Let $\gamma : U \rightarrow P$ and $\delta : P \rightarrow V$ be A -homomorphisms such that $\psi = \delta \circ \gamma$.

Since P is projective and β is surjective, there is an A -homomorphism $\epsilon : P \rightarrow Y^n$ such that $\beta \circ \epsilon = \delta$. Note that the homomorphism $\epsilon \circ \gamma \circ \alpha : Y^m \rightarrow Y^n$ factors through P , hence is zero by the assumptions on Y . Thus $\psi \circ \alpha = \delta \circ \gamma \circ \alpha = \beta \circ \epsilon \circ \gamma \circ \alpha = 0$, and hence $\psi = 0$ as α is surjective. This shows that $\text{Hom}_A^{\text{pr}}(U, V) = \{0\}$ as stated. Let M, N be as in (ii). As a left A -module, we have $Y \otimes_k M \cong Y^m$, where $m = \dim_k(M)$, and we have a canonical surjection of A -modules $Y \otimes_k M \rightarrow Y \otimes_D M$; similarly for $Y \otimes_k N$. Thus (ii) is a special case of (i). \square

The following observation is well-known (see the papers [1] and [32] on static and adstatic modules). We include a short proof for convenience.

Lemma 2.4 *Let A, D be finite-dimensional k -algebras, and let Y be a finitely generated A - D -bimodule. The functor $Y \otimes_D - : \text{mod}(D) \rightarrow \text{mod}(A)$ is a full k -linear embedding if and only if the adjunction unit $V \rightarrow \text{Hom}_A(Y, Y \otimes_D V)$, $v \mapsto (y \mapsto y \otimes v)$ is an isomorphism, for every finitely generated D -module V .*

Proof The functor $Y \otimes_D -$ is a full embedding if and only if for any two finitely generated D -modules U, V , the map $\text{Hom}_D(U, V) \rightarrow \text{Hom}_A(Y \otimes_D U, Y \otimes_D V)$ induced by $Y \otimes_D -$ is an isomorphism, hence if and only if the canonical map $\text{Hom}_D(U, V) \rightarrow \text{Hom}_D(U, \text{Hom}_A(Y, Y \otimes_D V))$ is an isomorphism. By considering the case $U = D$, one sees that this is the case if and only if the adjunction map $V \rightarrow \text{Hom}_A(Y, Y \otimes_D V)$ itself is an isomorphism, whence the result. \square

Proof of Theorem 2.1 By Lemma 2.4, the functor $Y \otimes_D - : \text{mod}(D) \rightarrow \text{mod}(A)$ is a full embedding, if and only if the condition (3) holds. This embedding is exact if and only if Y is flat as a right D -module. Since Y is finitely generated, this is equivalent to requiring condition (2). It follows from the Lemmas 2.2 and 2.3 that the composition with the canonical functor $\text{mod}(A) \rightarrow \underline{\text{mod}}(A)$ yields an embedding of $\text{mod}(D)$ as a distinguished abelian subcategory in $\underline{\text{mod}}(A)$ if and only if (1) holds as well. This concludes the proof of Theorem 2.1. \square

If both A and D are selfinjective, then Theorem 2.1 yields the following result.

Theorem 2.5 *Let A be a finite-dimensional selfinjective k -algebra. Let Y be a finitely generated A -module. Suppose that $\text{End}_A(Y)$ is selfinjective. Set $D = \text{End}_A(Y)^{\text{op}}$, and regard Y as an A - D -bimodule. The following are equivalent.*

- (i) *The functor $Y \otimes_D - : \text{mod}(D) \rightarrow \text{mod}(A)$ is a full exact embedding and induces an embedding of $\text{mod}(D)$ as a distinguished abelian subcategory in $\underline{\text{mod}}(A)$.*
- (ii) *We have $\text{End}_A^{\text{pr}}(Y) = \{0\}$, and Y is projective as an $\text{End}_A(Y)$ -module.*

Proof If (i) holds, then (ii) holds by Theorem 2.1. Suppose that (ii) holds. Note that the hypotheses imply that $D = \text{End}_A(Y)^{\text{op}}$ is selfinjective and that Y is projective as a right D -module. Thus the conditions (1) and (2) in Theorem 2.1 are satisfied. We need to show that condition (3) in that Theorem holds as well. That is, given a finitely generated D -module V , we need to show that the adjunction map $V \rightarrow \text{Hom}_A(Y, Y \otimes_D V)$ is an isomorphism. Note that this is clear if $V = D$ as a consequence of the assumption $D = \text{End}_A(Y)^{\text{op}}$. Thus this is the case for V any free D -module of finite rank. In general, since D is selfinjective, V is isomorphic to a submodule of a free D -module of finite rank. Thus there is an exact sequence of D -modules of the form

$$0 \longrightarrow V \longrightarrow D^n \longrightarrow D^m$$

for some positive integers n and m . By the hypotheses in (ii), the functor $Y \otimes_D -$ is exact, and hence we have an exact sequence of A -modules of the form

$$0 \longrightarrow Y \otimes_D V \longrightarrow Y \otimes_D D^n \longrightarrow Y \otimes_D D^m$$

Since the functor $\text{Hom}_A(Y, -)$ is left exact, this yields an exact sequence of D -modules of the form

$$0 \longrightarrow \text{Hom}_A(Y, Y \otimes_D V) \longrightarrow \text{Hom}_A(Y, Y \otimes_D D^n) \longrightarrow \text{Hom}_A(Y, Y \otimes_D D^m)$$

By naturality of the adjunction maps, we get a commutative diagram of D -modules with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & V & \longrightarrow & D^n & \longrightarrow & D^m \\ & & \alpha \downarrow & & \downarrow \beta & & \downarrow \gamma \\ 0 & \longrightarrow & \text{Hom}_A(Y, Y \otimes_D V) & \longrightarrow & \text{Hom}_A(Y, Y \otimes_D D^n) & \longrightarrow & \text{Hom}_A(Y, Y \otimes_D D^m) \end{array}$$

where α, β, γ are the adjunction maps. By the above remarks, β and γ are isomorphisms. The exactness of the rows implies that α is an isomorphism as well. This shows that condition (3) in Theorem 2.1 holds as well, and hence the result follows from Theorem 2.1. \square

Using a Theorem of Cabanes [9, Theorem 2] one can identify the image of the functor $Y \otimes_D -$ in Theorem 2.5 more precisely; see Sect. 8.

If A is a Frobenius algebra over an algebraically closed field, then the ideals I containing their right annihilators form subvarieties of certain Grassmannians.

Proposition 2.6 *Let A be a finite-dimensional Frobenius algebra over k . Suppose that k is algebraically closed. The set of proper ideals I in A satisfying $r(I) \subseteq I$ is a projective variety whose connected components are subvarieties of the Grassmannians $\text{Gr}(n, A)$, where $\frac{\dim_k(A)}{2} \leq n < \dim_k(A)$.*

Proof If A is a Frobenius algebra, then, as a consequence of [38, Lemma IV.3.6], we have $\dim_k(A) = \dim_k(I) + \dim_k(r(I))$. Since $r(I) \subseteq I$, it follows that $\dim_k(A) \leq 2 \dim_k(I)$. Thus the ideals satisfying $r(I) \subseteq I$ satisfy $\frac{\dim_k(A)}{2} \leq \dim_k(I)$. In each dimension, they form subvarieties of the Grassmannians, since being an ideal with an annihilator of a fixed dimension is obviously a polynomial condition (obtained by fixing a k -basis of A). \square

If A is symmetric, then A is selfinjective, but none of the distinguished abelian subcategories constructed above arises as the heart of a t -structure. More precisely, we have the following result.

Proposition 2.7 *Let A be a finite-dimensional selfinjective k -algebra. Denote by ν the Nakayama functor on $\text{mod}(A)$. Then the heart of any ν -stable t -structure on $\text{mod}(A)$ is zero.*

Proof Let $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ be a t -structure on $\mathcal{C} = \text{mod}(A)$. Suppose that this t -structure is preserved by the Nakayama functor ν . For any A -module U in $\mathcal{C}^{\leq 0}$ and any A -module V in $\mathcal{C}^{\geq 0}$ we have $\underline{\text{Hom}}_A(U, \Sigma^{-1}(V)) = \{0\}$. Auslander–Reiten duality for selfinjective algebras yields a duality between the space $\underline{\text{Hom}}(U, \Sigma^{-1}(V)) \cong \underline{\text{Hom}}_A(\nu(U), \nu(\Sigma^{-1}(V)))$ and $\underline{\text{Hom}}_A(V, \nu(U))$ (see e. g. [38, Ch. III, Theorem 6.3]). Thus $\underline{\text{Hom}}_A(V, \nu(U)) = \{0\}$. Since the heart $\mathcal{C}^{\leq 0} \cap \mathcal{C}^{\geq 0}$ of the t -structure is ν -stable, it follows that all morphisms in the heart of this t -structure are zero. \square

Corollary 2.8 *Let A be a finite-dimensional symmetric k -algebra. Then the heart of any t -structure on $\underline{\text{mod}}(A)$ is zero.*

Proof The Nakayama functor of a symmetric algebra is isomorphic to the identity functor, and hence the statement is a special case of Proposition 2.7. \square

Note that this Corollary follows also from more general results on negative Calabi–Yau triangulated categories in [16, §5.1], combined with Tate duality for symmetric algebras (Tate duality for symmetric algebras is the specialisation of the aforementioned Auslander–Reiten duality to the case where the Nakayama functor is isomorphic to the identity). Since finite group algebras are symmetric, this implies in particular the second statement of Theorem 1.5:

Corollary 2.9 *Let G be a finite group. Then the heart of any t -structure on $\underline{\text{mod}}(kG)$ is zero.*

Remark 2.10 With the notation of Theorem 2.1, not every embedding of $\text{mod}(D)$ as a distinguished abelian subcategory of $\underline{\text{mod}}(A)$ lifts in general to a full embedding $\text{mod}(D) \rightarrow \text{mod}(A)$. Suppose that $Y \otimes_D - : \text{mod}(D) \rightarrow \text{mod}(A)$ is a full exact embedding and induces an embedding $\text{mod}(D) \rightarrow \underline{\text{mod}}(A)$ as distinguished abelian subcategory. Let M be an A - A -bimodule inducing a stable equivalence of Morita type on A . Then the functor $M \otimes_A Y \otimes_D - : \text{mod}(D) \rightarrow \text{mod}(A)$ is exact but no longer necessarily full. It induces still an embedding of $\text{mod}(D)$ as a distinguished abelian subcategory, because the functor $M \otimes_A -$ induces a triangulated equivalence on $\underline{\text{mod}}(A)$, hence permutes distinguished abelian subcategories. It is not clear whether an embedding $\text{mod}(D) \rightarrow \underline{\text{mod}}(A)$ as a distinguished abelian subcategory is necessarily induced by tensoring with a suitable A - D -bimodule.

Remark 2.11 Let A be a finite-dimensional selfinjective k -algebra, D a finite-dimensional k -algebra, and Y an A - D -bimodule which is finitely generated projective as a right D -module. Then the functor $\text{mod}(D) \rightarrow \underline{\text{mod}}(A)$ induced by $Y \otimes_D -$ extends to a functor of triangulated categories $D^b(\text{mod}(D)) \rightarrow \underline{\text{mod}}(A)$. Indeed, since Y is finitely generated projective as a right D -module, it follows that $Y \otimes_D -$ induces a functor $D^b(\text{mod}(D)) \rightarrow D^b(\text{mod}(A))$. Composed with the canonical functor $D^b(\text{mod}(A)) \rightarrow \underline{\text{mod}}(A)$ from [36, Theorem 2.1] or [8, Theorem 4.4.1], this yields a functor $D^b(\text{mod}(D)) \rightarrow \underline{\text{mod}}(A)$.

3 Simple modules in distinguished abelian subcategories

We consider in this section distinguished abelian subcategories of $\underline{\text{mod}}(A)$ whose simple objects are simple A -modules, where A is a finite-dimensional selfinjective k -algebra.

Theorem 3.1 *Let A be a finite-dimensional selfinjective k -algebra and let I be a proper ideal in A . The following statements are equivalent.*

- (i) *The composition of canonical functors $\text{mod}(A/I) \rightarrow \text{mod}(A) \rightarrow \underline{\text{mod}}(A)$ is an embedding of $\text{mod}(A/I)$ as a distinguished abelian subcategory in $\underline{\text{mod}}(A)$.*
- (ii) *The ideal I contains its right annihilator $r(I)$.*
- (iii) *We have $\text{End}_A^{\text{pr}}(A/I) = \{0\}$.*
- (iv) *For any two finitely generated A/I -modules U, V , we have $\text{Hom}_A^{\text{pr}}(U, V) = \{0\}$.*
- (v) *We have $r(I)^2 = \{0\}$.*

Any full abelian subcategory of a distinguished abelian subcategory of a triangulated category is clearly again a distinguished abelian subcategory. In particular, if the canonical

functor $\text{mod}(A/I) \rightarrow \underline{\text{mod}}(A)$ is an embedding as a distinguished abelian subcategory, then so is the canonical functor $\text{mod}(A/J) \rightarrow \underline{\text{mod}}(A)$ for any ideal J which contains I , because this factors through the embedding $\text{mod}(A/J) \rightarrow \text{mod}(A/I)$ induced by the canonical surjection $A/I \rightarrow A/J$.

Every ideal which squares to zero gives rise to a distinguished abelian subcategory in the stable module category of a selfinjective algebra.

Corollary 3.2 *Let A be a finite-dimensional selfinjective k -algebra, and let J be an ideal in A such that $J^2 = \{0\}$. Set $I = l(J)$. Then the canonical surjection $A \rightarrow A/I$ induces an embedding $\text{mod}(A/I) \rightarrow \underline{\text{mod}}(A)$ of $\text{mod}(A/I)$ as a distinguished abelian subcategory in $\underline{\text{mod}}(A)$.*

Proof We have $r(I) = r(l(J)) = J$, hence $r(I)^2 = \{0\}$ by the assumptions. The result follows from the equivalence of the statements (i) and (v) in Theorem 3.1. □

If A is a finite-dimensional Hopf algebra, then $\text{mod}(A)$ is a monoidal abelian category, and A is selfinjective, by a result of Larson and Sweedler [24]. Given two A -modules U, V , if one of U, V is projective, then so is $U \otimes_k V$ (see e. g. [6, Proposition 3.1.5]). Thus $\underline{\text{mod}}(A)$ is a monoidal triangulated category. If I is a proper Hopf ideal in A , then A/I is a Hopf algebra, the canonical surjection $A \rightarrow A/I$ is a homomorphism of Hopf algebras, and hence induces a full embedding of monoidal categories $\text{mod}(A/I) \rightarrow \text{mod}(A)$. Thus Theorem 3.1 implies immediately the following observation.

Corollary 3.3 *Let A be a finite-dimensional Hopf algebra over k and let I be a proper Hopf ideal in A containing its right annihilator $r(I)$. Then the composition of canonical functors $\text{mod}(A/I) \rightarrow \text{mod}(A) \rightarrow \underline{\text{mod}}(A)$ is an embedding of $\text{mod}(A/I)$ as a monoidal distinguished abelian subcategory in the monoidal triangulated category $\underline{\text{mod}}(A)$.*

Some of the implications in Theorem 3.1 hold in slightly greater generality.

Lemma 3.4 *Let A be a finite-dimensional k -algebra, and let J be a proper left ideal in A . We have $\text{End}_A^{\text{pr}}(A/J) = \{0\}$ if and only if $r(J) \subseteq J$.*

Proof Note first that if $\beta : A/J \rightarrow A$ is an A -homomorphism, then $\beta \circ \pi$ is an A -endomorphism of A with kernel containing J , hence induced by right multiplication with an element $y \in r(J)$. Conversely, right multiplication with an element $y \in r(J)$ factors through π . Let $\alpha : A/J \rightarrow A/J$ be an endomorphism of A/J as a left A -module such that α factors through a projective A -module. Then α factors through the canonical surjection $\pi : A \rightarrow A/J$; that is, there is an A -homomorphism $\beta : A/J \rightarrow A$ such that $\alpha = \pi \circ \beta$. By the above, the endomorphism $\beta \circ \pi$ of A is induced by right multiplication with an element $y \in r(J)$. Since π is surjective, we have $\alpha = 0$ if and only if $\alpha \circ \pi = \pi \circ \beta \circ \pi = 0$, or equivalently, if and only if $\text{Im}(\beta \circ \pi) \subseteq \ker(\pi) = J$. Since the image of $\beta \circ \pi$ is Ay , it follows that $\alpha = 0$ if and only if $y \in J$. The result follows. □

Lemma 3.5 *Let A be a finite-dimensional k -algebra and let I be a proper ideal in A . Suppose that $r(I) \subseteq I$. Then for any two A/I -modules U, V we have $\text{Hom}_A^{\text{pr}}(U, V) = \{0\}$.*

Proof Set $Y = A/I$, regarded as an A - A/I -bimodule. Then $\text{End}_A^{\text{pr}}(Y) = \{0\}$ by Lemma 3.4, and we have $\text{End}_A(Y) \cong (A/I)^{\text{op}}$, hence $D = \text{End}_A(Y)^{\text{op}} \cong A/I$. Using this isomorphism, if U is an A/I -module, then $Y \otimes_{A/I} U = A/I \otimes_{A/I} U \cong U$, regarded as an A -module via the canonical surjection $A \rightarrow A/I$. The result follows from Lemma 2.3. □

Lemma 3.6 *Let A be a finite-dimensional selfinjective k -algebra, and let I be an ideal in A . The following are equivalent.*

- (i) *We have $r(I) \subseteq I$.*
- (ii) *We have $l(I) \subseteq I$.*
- (iii) *We have $r(I)^2 = 0$.*
- (iv) *We have $l(I)^2 = 0$.*

Proof If $r(I) \subseteq I$, then taking left annihilators yields $I = l(r(I)) \supseteq l(I)$, so (i) implies (ii). A similar argument shows that (ii) implies (i). Since $I \cdot r(I) = 0$, it follows that if $r(I) \subseteq I$, then $r(I)^2 = 0$. Thus (i) implies (iii). A similar argument shows that (ii) implies (iv). If $r(I)^2 = 0$, then $r(I) \subseteq l(r(I)) = I$, so (iii) implies (i), and a similar argument shows that (iv) implies (ii). □

Proof of Theorem 3.1 We are going to prove Theorem 3.1 as a special case of Theorem 2.1. Set $Y = A/I$, regarded as an A - A/I -bimodule. We have $\text{End}_A(Y) = \text{End}_{A/I}(Y) \cong (A/I)^{\text{op}}$. Clearly Y is projective as a right A/I -module. Given an A/I -module V , the adjunction unit $V \rightarrow \text{Hom}_A(A/I, A/I \otimes_{A/I} V)$ is trivially an isomorphism. Thus the A - A/I -bimodule satisfies the conditions (2) and (3) in Theorem 2.1. Therefore the composition of functors $\text{mod}(A/I) \rightarrow \text{mod}(A) \rightarrow \underline{\text{mod}}(A)$ is an embedding of $\text{mod}(A/I)$ as a distinguished abelian subcategory if and only if (1) holds, that is, if and only if $\text{End}_A^{\text{pr}}(A/I) = \{0\}$. This proves the equivalence of (i) and (iii). It follows from Lemma 3.4 that the statements (ii) and (iii) are equivalent. The implication (iii) \Rightarrow (iv) follows from Lemma 3.5. The implication (iv) \Rightarrow (i) follows from Lemma 2.2. The equivalence of (ii) and (v) holds by Lemma 3.6. □

Proof of Theorem 1.3 Theorem 1.3 follows from the equivalence of the statements (i) and (ii) in Theorem 3.1. □

Proof of Theorem 1.4 The hypothesis $r(I) \subseteq I$ and Theorem 1.3 imply that $\text{mod}(A/I)$ is indeed a distinguished abelian subcategory of $\underline{\text{mod}}(A)$. The inclusions $r(I) \subseteq I \subseteq J(A)$ imply that $\text{soc}(A) \subseteq r(I)$, and hence A has no simple projective modules.

Since $I \subseteq J(A)$, it follows that $\text{mod}(A/I)$ contains all simple A -modules. Thus the image of $\text{mod}(A/I)$ in $\underline{\text{mod}}(A)$ is extension closed if and only if $\text{mod}(A/I)$ contains all indecomposable non-projective A -modules, and hence if and only if the embedding $\text{mod}(A/I) \rightarrow \underline{\text{mod}}(A)$ is an equivalence. This shows that (i) and (ii) are equivalent. If (ii) holds, then $\underline{\text{mod}}(A)$ is an abelian category, hence semisimple. This forces that all indecomposable non-projective A -modules are simple, hence that all projective indecomposable A -modules have composition length 2 (here we use that A is selfinjective, so every projective indecomposable module has a simple top and socle). This implies that A is a Nakayama algebra all of whose projective indecomposable modules have composition length 2. Thus (ii) implies (iii). If (iii) holds, then every indecomposable non-projective A -module is simple, and hence (iii) implies (ii). Clearly A is a selfinjective Nakayama algebra with all projective indecomposable modules of composition length 2 if and only if $\text{soc}(A) = J(A)$, whence the equivalence between (iii) and (iv). □

Remark 3.7 Lemma 3.6 implies that working with left or right modules yields equivalent statements. To illustrate this point, by Theorem 3.1, we have $r(I) \subseteq I$ if and only if we have a full embedding $\text{mod}(A/I) \rightarrow \underline{\text{mod}}(A)$. There is an obvious right module analogue which states that $l(I) \subseteq I$ if and only if we have a full embedding $\text{mod}((A/I)^{\text{op}}) \rightarrow \underline{\text{mod}}(A^{\text{op}})$. Thus Lemma 3.6 implies that we have a full embedding $\text{mod}(A/I) \rightarrow \underline{\text{mod}}(A)$ if and only if we have a full embedding $\text{mod}((A/I)^{\text{op}}) \rightarrow \underline{\text{mod}}(A^{\text{op}})$. In other words, the full

distinguished abelian subcategories in $\underline{\text{mod}}(A)$ and $\underline{\text{mod}}(A^{\text{op}})$ constructed in Theorem 3.1 and its right module analogue correspond bijectively to each other.

The following result is Theorem 1.2 (i).

Theorem 3.8 *Let A be a finite-dimensional selfinjective k -algebra such that all simple A -modules are nonprojective. Let \mathcal{D} be a distinguished abelian subcategory of $\underline{\text{mod}}(A)$. Suppose that \mathcal{D} contains all simple A -modules. The simple A -modules are exactly all simple objects in \mathcal{D} . In particular, we have $\ell(\mathcal{D}) = \ell(A)$.*

Proof Let U be an indecomposable nonprojective A -module belonging to \mathcal{D} , and let S be a simple A -module. Let $\psi : U \rightarrow S$ be an A -homomorphism, and denote by $\underline{\psi}$ the image of ψ in $\underline{\text{Hom}}_A(U, S)$. Note that U, S are both nonzero objects of \mathcal{D} , by the assumptions.

We are going to show first that if ψ is not an isomorphism in $\text{mod}(A)$, then $\underline{\psi}$ is not a monomorphism in \mathcal{D} . If $\underline{\psi}$ is zero, there is nothing to show. If $\underline{\psi}$ is nonzero, then ψ is surjective because S is a simple A -module. Assume that ψ is not an isomorphism. Then $\ker(\psi)$ is nonzero. Let T be a simple A -submodule of $\ker(\psi)$. The inclusion $T \rightarrow U$ is an injective A -homomorphism, hence its image in $\underline{\text{mod}}(A)$ is a nonzero morphism in $\underline{\text{mod}}(A)$. By construction, the composition $T \rightarrow U \rightarrow S$ is zero in $\underline{\text{mod}}(A)$. Since also T belongs to \mathcal{D} , it follows that $\underline{\psi}$ is not a monomorphism in \mathcal{D} .

This argument shows that S is a simple object of \mathcal{D} . Indeed, if not, there would have to be a monomorphism $U \rightarrow S$ in \mathcal{D} which is not an isomorphism. But by the first paragraph, any such monomorphism is induced by an isomorphism in $\text{mod}(A)$, so is an isomorphism in \mathcal{D} as well. This argument also shows that \mathcal{D} contains no other simple objects. Indeed, let U be an indecomposable nonprojective A -module which is a simple object in \mathcal{D} . Consider a surjective A -homomorphism $\psi : U \rightarrow S$ onto some simple A -module S . Then S belongs to \mathcal{D} , and the image $\underline{\psi}$ is a monomorphism in \mathcal{D} because U is simple in \mathcal{D} . But then ψ is an isomorphism by the first argument. Thus the simple A -modules are exactly the simple objects in \mathcal{D} , whence the result. \square

Corollary 3.9 *Let A be a finite-dimensional selfinjective k -algebra such that all simple A -modules are nonprojective. Let I be an ideal such that $r(I) \subseteq I \subseteq J(A)$. Let \mathcal{D} be a distinguished abelian subcategory of $\underline{\text{mod}}(A)$ containing $\text{mod}(A/I)$. Then the simple A -modules are exactly the simple objects in \mathcal{D} .*

Proof The hypothesis $r(I) \subseteq I$ implies, by Theorem 3.1, that $\text{mod}(A/I)$ is a distinguished abelian subcategory of $\underline{\text{mod}}(A)$. The hypothesis $I \subseteq J(A)$ implies that $\text{mod}(A/I)$ contains all simple A -modules. The result follows from Theorem 3.8. \square

Removing the reference to simple A -modules yields the following statement.

Corollary 3.10 *Let A be a finite-dimensional selfinjective k -algebra such that all simple A -modules are nonprojective. Then $\underline{\text{mod}}(A)$ has a semisimple distinguished abelian subcategory \mathcal{D} such that $\ell(\mathcal{D})$ is finite and such that for any distinguished abelian subcategory \mathcal{D}' of $\underline{\text{mod}}(A)$ containing \mathcal{D} we have $\ell(\mathcal{D}') = \ell(\mathcal{D})$.*

Proof Let \mathcal{D} be the full subcategory of $\underline{\text{mod}}(A)$ consisting of all semisimple modules in $\text{mod}(A)$. The result follows from Theorem 3.8. \square

The distinguished abelian subcategories of $\underline{\text{mod}}(A)$ of the form $\text{mod}(A/I)$ in Theorem 3.1 have the property that the simple objects in $\text{mod}(A/I)$ remain simple in $\text{mod}(A)$. The next result explores the question under what circumstances a distinguished abelian subcategory of $\underline{\text{mod}}(A)$ whose simple objects correspond to simple A -modules is of the form $\text{mod}(A/I)$ for some ideal I in A .

Theorem 3.11 *Let A be a finite-dimensional selfinjective k -algebra, and let D be a finite-dimensional k -algebra. Let Y be a finitely generated A - D -bimodule such that Y is projective as a right D -module and such that Y has no nonzero projective direct summand as a left A -module. Suppose that for any simple D -module T the A -module $Y \otimes_D T$ is simple and that the functor $Y \otimes_D -$ induces a full embedding of $\text{mod}(D)$ as a distinguished abelian subcategory of $\text{mod}(A)$. Let I be the annihilator in A of Y as a left A -module. Then $r(I) \subseteq I$, and $Y \otimes_D -$ induces an equivalence $\text{mod}(D) \cong \text{mod}(A/I)$.*

For the proof, we will need the following elementary observation, which is a sufficient criterion for an epimorphism in the category of k -algebras to be an isomorphism.

Lemma 3.12 *Let D be a finite-dimensional k -algebra and A a unital subalgebra of D . Suppose that the restriction functor $\text{Res}_A^D : \text{mod}(D) \rightarrow \text{mod}(A)$ is a full embedding which sends every simple D -module to a simple A -module. Then $A = D$.*

Proof We first show that $J(A) = A \cap J(D)$. Since simple D -modules restrict to simple A -modules, it follows that $J(A)$ annihilates every simple D -module, and hence $J(A) \subseteq A \cap J(D)$. Since $A \cap J(D)$ is a nilpotent ideal in A , we have the other inclusion as well, whence the equality $J(A) = A \cap J(D)$. Thus the inclusion $A \subseteq D$ induces an injective algebra homomorphism $A/J(A) \rightarrow D/J(D)$. Since $A/J(A)$ is semisimple, every $A/J(A)$ -module is injective, and hence $A/J(A)$ is isomorphic to a direct summand of $D/J(A)$ as a right $A/J(A)$ -module. Thus, for any simple A -module S , the $D/J(D)$ -module $D/J(D) \otimes_{A/J(A)} S$ is nonzero. Regarded as a left D -module, this is a quotient of $D \otimes_A S$. In particular $D \otimes_A S$ is nonzero. Let T be a simple quotient of $D \otimes_A S$ and let $D \otimes_A S \rightarrow T$ be a nonzero D -homomorphism. The standard adjunction yields a nonzero A -module homomorphism $S \rightarrow \text{Res}_A^D(T)$. Since S and $\text{Res}_A^D(T)$ are both simple A -modules, it follows that $S \cong \text{Res}_A^D(T)$. This shows that Res_A^D induces a bijection between the isomorphism classes of simple D -modules and simple A -modules. Since Res_A^D is a full embedding, we also have $\text{End}_D(T) = \text{End}_A(T)$. Thus the simple modules for D and A which correspond to each other through the bijection induced by Res_A^D have the same dimensions and isomorphic endomorphism rings. The Artin–Wedderburn Theorem implies that $A/J(A) \cong D/J(D)$, and hence $D = A + J(D)$.

We show next that every maximal A -submodule of D is in fact a maximal D -submodule. Indeed, let M be a maximal A -submodule of D . Then $S = D/M$ is a simple A -module. By the previous argument, there is a simple D -module T and an A -module isomorphism $S \cong \text{Res}_A^D(T)$. The composition of A -homomorphisms $D \rightarrow D/M = S \cong \text{Res}_A^D(T)$ belongs to $\text{Hom}_A(D, T) = \text{Hom}_D(D, T)$, hence this composition is a D -homomorphism. The kernel of this D -homomorphism is M , and hence M is a maximal D -submodule of D . Conversely, any maximal D -submodule N of D is a maximal A -submodule since $\text{Res}_A^D(D/N)$ remains simple. Taking the intersection of all maximal submodules of D as an A -module yields $J(A)D = J(D)$.

It follows that $D = A + J(D) = A + J(A)D$. Nakayama’s Lemma, applied to the A -module D , implies that $A = D$. □

The converse of Lemma 3.12 holds trivially. One cannot drop in this Lemma the hypothesis that Res_A^D sends simple modules to simple modules. Consider the subalgebra A of upper triangular matrices in $D = M_2(k)$. The restriction from D to A of the unique (up to isomorphism) simple D -module is the unique (up to isomorphism) projective indecomposable A -module of dimension 2. One verifies easily that Res_A^D is a full embedding. (The inclusion $A \rightarrow D$ is thus an epimorphism in the category of rings; see Stenström [39, Chapter XI, Proposition 1.2].)

Lemma 3.13 *Let A be a finite-dimensional selfinjective algebra over a field k , let D be a finite-dimensional k -algebra, and let Y be a finitely generated A - D -bimodule. Suppose that the functor $Y \otimes_D -$ induces a full embedding $\text{mod}(D) \rightarrow \underline{\text{mod}}(A)$. Then, for any finitely generated D -module V , the map $V \rightarrow \underline{\text{Hom}}_A(Y, Y \otimes_D V)$ induced by the adjunction unit is an isomorphism of D -modules. In particular, the functor $\underline{\text{Hom}}_A(Y, -) : \underline{\text{mod}}(A) \rightarrow \text{mod}(D)$ is a left inverse of the embedding $\text{mod}(D) \rightarrow \underline{\text{mod}}(A)$ induced by $Y \otimes_D -$.*

Proof By the assumptions, for any two finitely generated D -modules U, V the map $\text{Hom}_D(U, V) \rightarrow \underline{\text{Hom}}_A(Y \otimes_D U, Y \otimes_D V)$ induced by the functor $Y \otimes_D -$ is an isomorphism. Specialising this isomorphism to $U = D$ and combining it with the canonical isomorphism $V \cong \text{Hom}_D(D, V)$ yields the result. \square

The converse in Lemma 3.13 need not hold; the issue is that the functor $\underline{\text{Hom}}_A(Y, -)$ need not be right adjoint to the functor induced by $Y \otimes_D -$. The Tensor-Hom adjunction induces a natural transformation between the induced bifunctors at the level of the stable category $\underline{\text{mod}}(A)$ (cf. Lemma 9.12 and Remark 9.13), but this need not be an isomorphism.

Lemma 3.14 *Let A, D be finite-dimensional k -algebras and let $\Phi : \text{mod}(D) \rightarrow \text{mod}(A)$ be a full exact embedding sending simple D -modules to simple A -modules. Then Φ is isomorphic to a functor of the form $Y \otimes_D - : \text{mod}(D) \rightarrow \text{mod}(A)$, where Y is an A - D -bimodule which is a progenerator as a right D -module. Moreover, if I is the annihilator in A of Y as a left A -module, then Φ factors through an equivalence $\Psi : \text{mod}(D) \cong \text{mod}(A/I)$ and the inclusion functor $\text{mod}(A/I) \rightarrow \text{mod}(A)$.*

Proof As mentioned at the beginning of Sect. 2, the first statement is a special case of the Eilenberg–Watts Theorem: since Φ is a full exact embedding, it is induced by tensoring over D with an A - D -bimodule Y which is flat as a right D -module. Since this is a functor between categories of finite-dimensional modules, preserving simple modules, it follows that Y is a progenerator as a right D -module. Thus D is Morita equivalent to $D' = \text{End}_{D^{\text{op}}}(Y)$, via the functor from $\text{mod}(D)$ to $\text{mod}(D')$ induced by $Y \otimes_D -$, with Y here regarded as a D' - D -bimodule. The action of A on Y induces an algebra homomorphism $A \rightarrow D'$. Let I be the annihilator of Y in A . Then the algebra homomorphism $A/I \rightarrow D'$ induced by the action of A on Y is injective. The functor Φ is the composition of the Morita equivalence $Y \otimes_D - : \text{mod}(D) \rightarrow \text{mod}(D')$ followed by the restriction functor along the injective algebra homomorphism $A/I \rightarrow D'$. By the assumptions, Φ preserves simple modules. Since $Y \otimes_D -$ induces an equivalence $\text{mod}(D) \cong \text{mod}(D')$ it induces in particular a bijection between isomorphism classes of simple D -modules and simple D' -modules. It follows that simple D' -modules restrict to simple A/I -modules. Lemma 3.12 implies that $A/I \cong D'$. Thus Φ factors through an equivalence $\text{mod}(D) \rightarrow \text{mod}(A/I)$ as stated. \square

Proof of Theorem 3.11 Since $Y \otimes_D -$ induces a full embedding $\text{mod}(D) \rightarrow \underline{\text{mod}}(A)$, it follows from Lemma 3.13 that for any finitely generated D -modules V , we have an isomorphism

$$V \cong \text{Hom}_D(D, V) \cong \underline{\text{Hom}}_A(Y, Y \otimes_D V)$$

sending $v \in V$ to the image of the map $y \mapsto (y \otimes v)$, where $y \in Y$. Arguing by induction over $\dim_k(V)$, we will show that $\text{Hom}_A^{\text{pr}}(Y, Y \otimes_D V) = \{0\}$. If V is simple, then by the assumptions, $Y \otimes_D V$ is simple. Since $Y \otimes_D V$ is a quotient of Y and since Y has no nonzero projective summand as an A -module, it follows that the simple A -module $Y \otimes_D V$ is nonprojective and hence that $\text{Hom}_A^{\text{pr}}(Y, Y \otimes_D V) = \{0\}$. Let

$$0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0$$

be a short exact sequence of nonzero D -modules. This sequence is isomorphic to the short exact sequence

$$0 \longrightarrow \text{Hom}_D(D, U) \longrightarrow \text{Hom}_D(D, V) \longrightarrow \text{Hom}_D(D, W) \longrightarrow 0$$

Since $Y \otimes_D -$ induces a full embedding $\text{mod}(D) \rightarrow \underline{\text{mod}}(A)$, this yields an exact sequence

$$0 \longrightarrow \underline{\text{Hom}}_A(Y, Y \otimes_D U) \longrightarrow \underline{\text{Hom}}_A(Y, Y \otimes_D V) \longrightarrow \underline{\text{Hom}}_A(Y, Y \otimes_D W) \longrightarrow 0$$

Since $Y \otimes_D -$ is exact, the first exact sequence yields an exact sequence

$$0 \longrightarrow Y \otimes_D U \longrightarrow Y \otimes_D V \longrightarrow Y \otimes_D W \longrightarrow 0$$

and applying the left exact functor $\text{Hom}_A(Y, -)$ yields an exact sequence

$$0 \longrightarrow \text{Hom}_A(Y, Y \otimes_D U) \longrightarrow \text{Hom}_A(Y, Y \otimes_D V) \longrightarrow \text{Hom}_A(Y, Y \otimes_D W)$$

Thus we have a commutative exact diagram of the form

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Hom}_A(Y, Y \otimes_D U) & \longrightarrow & \text{Hom}_A(Y, Y \otimes_D V) & \longrightarrow & \text{Hom}_A(Y, Y \otimes_D W) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \underline{\text{Hom}}_A(Y, Y \otimes_D U) & \longrightarrow & \underline{\text{Hom}}_A(Y, Y \otimes_D V) & \longrightarrow & \underline{\text{Hom}}_A(Y, Y \otimes_D W) \longrightarrow 0 \end{array}$$

where the vertical maps are the canonical surjections. Arguing by induction, the left and right vertical maps are isomorphisms. Thus the top right horizontal map is surjective, and comparing dimensions implies that the middle vertical map is an isomorphism as well. This shows that $\text{Hom}_A^{\text{pr}}(Y, Y \otimes_D V) = \{0\}$ for all finitely generated D -modules V . Applied to $V = D$ this implies that $\text{End}_A^{\text{pr}}(Y) = \{0\}$. By the first paragraph, this also implies that the canonical map $V \rightarrow \text{Hom}_A(Y, Y \otimes_D V)$ is an isomorphism for all V . By Theorem 2.1, the functor $Y \otimes_D -$ induces a full embedding $\text{mod}(D) \rightarrow \text{mod}(A)$, and by the assumptions, this embedding sends simple D -modules to simple A -modules. Since I is the annihilator in A of Y , it follows from Lemma 3.14, that the full embedding $Y \otimes_D - : \text{mod}(D) \rightarrow \text{mod}(A)$ factors through an equivalence $\text{mod}(D) \cong \text{mod}(A/I)$. By the assumptions, the functor $Y \otimes_D -$ induces a full embedding $\text{mod}(D) \rightarrow \underline{\text{mod}}(A)$. Thus the inclusion $\text{mod}(A/I) \rightarrow \text{mod}(A)$ induces a full embedding $\text{mod}(A/I) \rightarrow \underline{\text{mod}}(A)$ as distinguished abelian subcategory. The inclusion $r(I) \subseteq I$ follows from Theorem 3.1, whence the result. \square

Example 3.15 Let A be a finite-dimensional selfinjective k -algebra. Suppose that the simple A -modules are non-projective. Then $J(A)$ contains its right annihilator $\text{soc}(A)$ in A . Thus the subcategory of all semisimple A -modules, which is equivalent to $\text{mod}(A/J(A))$, is a distinguished abelian subcategory of $\underline{\text{mod}}(A)$. We have $\ell(A) = \ell(A/J(A))$, so for trivial reasons, $\underline{\text{mod}}(A)$ has distinguished abelian subcategories \mathcal{D} whose number of isomorphism classes $\ell(\mathcal{D})$ of simple objects in \mathcal{D} is equal to the number $\ell(A)$ of isomorphism classes of simple A -modules.

Example 3.16 Let A be a finite-dimensional selfinjective k -algebra. Suppose that $\text{soc}^2(A) \subseteq J(A)^2$. Since $\text{soc}^2(A)$ is the right annihilator of $J(A)^2$, it follows from Theorem 3.1 that the composition of canonical functors $\text{mod}(A/J(A)^2) \rightarrow \text{mod}(A) \rightarrow \underline{\text{mod}}(A)$ is an embedding of $\text{mod}(A/J(A)^2)$ as a distinguished abelian subcategory in $\underline{\text{mod}}(A)$. If A is indecomposable as an algebra, then so is $A/J(A)^2$, and both have the same quiver. Therefore, in this situation, $\mathcal{C} = \underline{\text{mod}}(A)$ has a *connected* distinguished abelian subcategory $\mathcal{D} = \text{mod}(A/J(A)^2)$ satisfying $\ell(\mathcal{D}) = \ell(A)$ and $\text{Ext}_{\mathcal{D}}^1(S, T) \cong \text{Ext}_{\mathcal{C}}^1(S, T)$, for any two simple objects S, T in \mathcal{D} .

Remark 3.17 The property $\text{soc}^2(A) \subseteq J(A)^2$ in the previous Example is not invariant under stable equivalences, in fact, not even under derived equivalences. For instance, the Brauer tree algebra of a star with four edges has this property, but the Brauer tree algebra of a line with four edges (and no exceptional vertex) does not.

4 Distinguished abelian subcategories for finite group algebras

We describe special cases of the situation arising in Theorem 3.1 involving finite group algebras. We use without further comments standard properties of finite p -group algebras in prime characteristic p ; see e.g. [27, Section 1.11].

Theorem 4.1 *Let k be a field of prime characteristic p and A a finite-dimensional selfinjective k -algebra. Suppose that $Z(A)^\times$ has a nontrivial finite p -subgroup Z such that A is projective as a kZ -module. Set $I = I(kZ) \cdot A$, where $I(kZ)$ is the augmentation ideal of kZ . Then I contains its right annihilator in A . In particular, restriction along the canonical surjection $A \rightarrow A/I$ induces a full embedding $\text{mod}(A/I) \rightarrow \underline{\text{mod}}(A)$ of $\text{mod}(A/I)$ as a distinguished abelian subcategory in $\underline{\text{mod}}(A)$.*

Proof The right annihilator of $I(kZ)$ in kZ is the 1-dimensional ideal $\text{soc}(kZ) = (\sum_{z \in Z} z) \cdot kZ$, and we have $\text{soc}(kZ) \subseteq I(kZ)$. Since A is a free left or right kZ -module, an easy argument shows that the right annihilator of $I(kZ) \cdot A = A \cdot I(kZ)$ is therefore $\text{soc}(kZ) \cdot A = A \cdot \text{soc}(kZ)$, which is contained in $I(kZ) \cdot A$. Thus the statement is the special case of Theorem 3.1 with $I = I(kZ) \cdot A$. □

Let G be a finite group. The module category $\text{mod}(kG)$ of the finite group algebra kG over a field k is a symmetric monoidal category with respect to the tensor product $- \otimes_k -$ of kG -modules over k . It is well-known that if U, V are finitely generated kG -modules with at least one of U, V projective, then $U \otimes_k V$ is projective as well. Therefore the tensor product over k induces a commutative monoidal structure on the triangulated category $\underline{\text{mod}}(kG)$. If N is a normal subgroup of G , then the canonical surjection $G \rightarrow G/N$ induces an embedding of symmetric monoidal categories $\text{mod}(kG/N) \rightarrow \text{mod}(kG)$. The following result implies the first statement in Theorem 1.5.

Theorem 4.2 *Let k be a field of prime characteristic p and G a finite group. Let N be a normal subgroup of G . Restriction along the canonical surjection $G \rightarrow G/N$ induces a full embedding $\text{mod}(kG/N) \rightarrow \underline{\text{mod}}(kG)$ of $\text{mod}(kG/N)$ as a symmetric monoidal distinguished abelian subcategory in $\underline{\text{mod}}(kG)$ if and only if p divides the order of N .*

Proof The fact that the functor $\text{mod}(kG/N) \rightarrow \underline{\text{mod}}(kG)$ is a functor of symmetric monoidal categories is obvious (see the remarks preceding the Theorem). We need to show that this induces an embedding as a distinguished abelian subcategory in $\underline{\text{mod}}(kG)$ if and only if $|N|$ is divisible by p . The kernel of the canonical algebra homomorphism $kG \rightarrow kG/N$ is equal to $I = kG \cdot I(kN)$, where $I(kN)$ is the augmentation ideal of kN . Arguing as in the previous proof, the right annihilator of $I(kN)$ in kN is the 1-dimensional ideal $(\sum_{y \in N} y)kN$. This is contained in $I(kN)$ if and only if p divides $|N|$. Indeed, if p divides $|N|$, then $\sum_{y \in N} y = \sum_{y \in N} (y - 1) \in I(kN)$. If p does not divide $|N|$, then $(\sum_{y \in N} y)kN$ is a complement of $I(kN)$ in kN . Since kG is free as a right kN -module of rank $|G : N|$, it follows that the right annihilator of I is equal to $kG \cdot (\sum_{y \in N} y)$. Therefore, if p divides $|N|$, then the right annihilator of I is contained in I by the previous argument. The result follows in

that case from Theorem 3.1. If $|N|$ is prime to p , then kG/N is a projective kG -module, so $\underline{\text{End}}_{kG}(kG/N)$ vanishes, and in particular, the canonical functor $\text{mod}(kG/N) \rightarrow \underline{\text{mod}}(kG)$ is not an embedding.

Alternatively, one can also show this using a special case of Higman’s criterion. Let U, V be kG/N -modules. When regarded as kG -modules, the elements of N act as identity on U, V . Thus any k -linear map $\tau : U \rightarrow V$ is a kN -homomorphism, and $\text{Tr}_1^G(\tau) = |N| \text{Tr}_N^G(\tau)$. If p divides $|N|$, then this expression is zero in k . It follows from the special case [27, Proposition 2.13.11] of Higman’s criterion that $\text{Hom}_{kG}^{\text{pr}}(U, V) = \{0\}$. Equivalently, we have $\underline{\text{Hom}}_{kG}(U, V) \cong \text{Hom}_{kG}(U, V) = \text{Hom}_{kG/N}(U, V)$. This shows that if p divides $|N|$, then $\text{mod}(kG/N)$ can indeed be identified canonically with a full subcategory of $\underline{\text{mod}}(kG)$. Note that kG/N is a projective kG/N -module. Thus every kG/N -endomorphism of kG/N is equal to $\text{Tr}_1^{G/N}(\sigma)$ for some linear endomorphism σ of kG/N . Equivalently, every kG -endomorphism of kG/N is of the form $\text{Tr}_N^G(\sigma)$ for some kN -endomorphism σ of kG/N . If $|N|$ is coprime to p , then $\sigma = \frac{1}{|N|} \text{Tr}_1^N(\sigma)$, hence $\tau = \text{Tr}_1^G(\frac{1}{|N|}\sigma)$, which shows that τ factors through a projective kG -module. Equivalently, the canonical map $\text{End}_{kG/N}(kG/N) \rightarrow \underline{\text{End}}_{kG}(kG/N)$ is zero. This shows that if $|N|$ is coprime to p , then the canonical functor $\text{mod}(kG/N) \rightarrow \underline{\text{mod}}(kG)$ is not an embedding. \square

Remark 4.3 Let k be a field of prime characteristic p .

- (1) Let G be a finite group having a nontrivial normal p -subgroup Q . It is well-known that the kernel I of the canonical algebra homomorphism $kG \rightarrow kG/Q$ is contained in the radical $J(kG)$ and hence that $\ell(kG) = \ell(kG/Q)$. Thus Theorem 4.2 illustrates Theorem 3.8, constructing explicitly the distinguished abelian subcategory $\text{mod}(kG/Q)$ of $\underline{\text{mod}}(kG)$ whose number of isomorphism classes of simple objects is equal to that of $\text{mod}(kG)$.
- (2) Theorem 4.2 implies that if P is a nontrivial finite p -group, then any cyclic subgroup of $Z(P)$ yields a distinguished abelian subcategory of $\underline{\text{mod}}(kP)$. But then so does any shifted cyclic subgroup of $Z(P)$, suggesting that distinguished abelian subcategories should form varieties which are related to cohomology support varieties.

Combining Theorem 3.11, a result of J. F. Carlson [10, Theorem 1], and [25, Theorem 3.4] yields the following classification of those distinguished abelian subcategories of the stable module category of a finite p -group algebra in prime characteristic p which are equivalent to module categories of finite-dimensional split k -algebras (i.e. finite-dimensional k -algebras whose simple modules have 1-dimensional endomorphism algebras). If P is a finite p -group, then a finiteley generated kP -module V is called *endotrivial* if $V \otimes_k V^* \cong k \oplus U$ for some projective kP -module. If V is endotrivial, then $V \otimes_k -$ and $V^* \otimes_k -$ induce inverse equivalences on $\underline{\text{mod}}(kP)$. In particular, $V \otimes_k -$ sends in that case any distinguished abelian subcategory \mathcal{D} of $\underline{\text{mod}}(kP)$ to a distinguished abelian subcategory, denoted $V \otimes_k \mathcal{D}$, of $\underline{\text{mod}}(kP)$.

Theorem 4.4 Let p be a prime, P a nontrivial finite p -group and k a field of characteristic p . Let D be a finite-dimensional split basic k -algebra such that there is an embedding $\Phi : \text{mod}(D) \rightarrow \underline{\text{mod}}(kP)$ as distinguished abelian subcategory of $\underline{\text{mod}}(kP)$.

- (i) We have $\ell(D) = 1$; that is, D is split local.
- (ii) Let V is an indecomposable kP -module corresponding to a simple D -module under the functor Φ . Then V is an endotrivial kP -module.
- (iii) If Φ is induced by a functor $Y \otimes_D -$ for some finitely generated kP - D -bimodule Y which is projective as a right D -module, then there is an ideal I of kP containing its

right annihilator in kP such that $D \cong (kP)/I$ and such that Φ induces an equivalence between $\text{mod}(D)$ and the distinguished abelian subcategory $V \otimes_k \text{mod}((kP)/I)$.

The first statement of Theorem 4.4 holds slightly more generally, based on the following observation which is a consequence of the proof of [25, Theorem 3.4].

Lemma 4.5 *Let p be a prime, P a nontrivial finite p -group, and suppose that $\text{char}(k) = p$. Let \mathcal{D} be a distinguished abelian subcategory of $\text{mod}(kP)$ such that \mathcal{D} has a simple object. Suppose that for every simple object X in \mathcal{D} we have $\text{End}_{\mathcal{D}}(X) \cong k$. Then $\ell(\mathcal{D}) = 1$.*

Proof By the assumptions on \mathcal{D} we have $\ell(\mathcal{D}) \geq 1$ (we include here by convention the case where \mathcal{D} has infinitely many isomorphism classes of simple objects). Arguing by contradiction, suppose that $\ell(\mathcal{D}) \geq 2$. Thus \mathcal{D} has two nonisomorphic simple objects S, T . Since \mathcal{D} is a full subcategory of $\text{mod}(kP)$, the objects S, T remain nonisomorphic in $\text{mod}(kP)$. Again by the assumptions, we have $\text{End}_{kP}(S) \cong k \cong \text{End}_{kP}(T)$, and we have $\text{Hom}_{kP}(S, T) = \{0\} = \text{Hom}_{kP}(T, S)$. It is shown in the proof of [25, Theorem 3.4] that this is not possible. \square

Proof of Theorem 4.4 Denote by \mathcal{D} the distinguished abelian subcategory of $\text{mod}(kP)$ obtained from taking the closure under isomorphisms in $\text{mod}(kP)$ of the image of the embedding $\Phi : \text{mod}(D) \rightarrow \text{mod}(kP)$. By Lemma 4.5, the category \mathcal{D} has a unique isomorphism class of simple objects, whence (i). Let V be an indecomposable kP -module such that V is simple as an object in \mathcal{D} . Then in particular $\text{End}_{kP}(V) \cong k$. A result of J. F. Carlson [10, Theorem 1] implies that V is endotrivial, which shows (ii). The exact functor $V \otimes_k -$ on $\text{mod}(kP)$ induces an equivalence on $\text{mod}(kP)$, with inverse induced by the functor $V^* \otimes_k -$. Thus after replacing \mathcal{D} by the image of \mathcal{D} under the functor $V^* \otimes_k -$, we may (and do) assume that the trivial kP -module k belongs to \mathcal{D} , and is the—up to isomorphism unique—simple object of \mathcal{D} . In other words, with the notation and hypotheses of statement (iii), the kP - D -bimodule Y is projective as a right D -module and the functor $Y \otimes_D -$ sends a simple D -module to the trivial kP -module. Thus the hypotheses of Theorem 3.11 are satisfied, implying statement (iii). \square

Example 4.6 Let $p = 2$, let P be a finite 2-group of order at least 4, let Z be a subgroup of order 2 of $Z(P)$, and set $Q = P/Z$. By Theorem 4.2, the abelian category $\text{mod}(kQ)$ can be identified with a distinguished abelian subcategory of $\text{mod}(kP)$. This subcategory is not proper in the sense of [18, Def. 1.2]. Tensoring the obvious short exact sequence of kZ -modules

$$0 \longrightarrow k \longrightarrow kZ \longrightarrow k \longrightarrow 0$$

by $kP \otimes_{kZ} -$ yields a short exact sequence of kP -modules

$$0 \longrightarrow kQ \longrightarrow kP \longrightarrow kQ \longrightarrow 0 .$$

In other words, regarding kQ as a kP -module via the canonical surjection $P \rightarrow Q$ implies that $\Sigma(kQ) \cong kQ$. Thus we have a distinguished exact triangle in $\text{mod}(kP)$ of the form

$$kQ \longrightarrow 0 \longrightarrow kQ \longequal{\quad} kQ$$

The first three terms of this triangle belong to $\text{mod}(kQ)$ but do not form a short exact sequence in $\text{mod}(kQ)$.

5 Basic properties of distinguished abelian subcategories

We show that a short exact sequence in a distinguished abelian subcategory \mathcal{D} of a triangulated category \mathcal{C} determines a unique exact triangle in \mathcal{C} ; that is, we show that the morphism h in Definition 1.1 is unique. If not stated otherwise, the shift functor of a triangulated category is denoted by Σ .

Proposition 5.1 *Let \mathcal{C} be a triangulated category and let \mathcal{D} be a distinguished abelian subcategory of \mathcal{C} . Let*

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

be a short exact sequence in \mathcal{D} . There is a unique morphism $h : Z \rightarrow \Sigma(X)$ in \mathcal{C} such that

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma(X)$$

is an exact triangle in \mathcal{C} .

Proof The existence of h is clear by definition; we need to show the uniqueness. Let $h, h' : Z \rightarrow \Sigma(X)$ be morphisms in \mathcal{C} such that the triangles

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma(X)$$

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h'} \Sigma(X)$$

are exact. The pair of identity morphisms $(\text{Id}_X, \text{Id}_Y)$ can be completed to a morphism of triangles $(\text{Id}_X, \text{Id}_Y, a)$. That is, there is a morphism $a : Z \rightarrow Z$ satisfying $a \circ g = g$ and $h' \circ a = h$. Since Z belongs to \mathcal{D} and since \mathcal{D} is a full subcategory of \mathcal{C} , it follows that a is a morphism in the abelian category \mathcal{D} . Since g is an epimorphism in \mathcal{D} , this forces $a = \text{Id}_Z$, whence $h' = h$. \square

Remark 5.2 The definition of a distinguished abelian subcategory \mathcal{D} of a triangulated category \mathcal{C} does not require \mathcal{D} to be closed under isomorphisms in \mathcal{C} . One easily checks that the closure of \mathcal{D} under isomorphisms in \mathcal{C} is again a distinguished abelian subcategory of \mathcal{C} which is equivalent to \mathcal{D} as an abelian category.

Proposition 5.3 *Let \mathcal{C} be a triangulated category and let \mathcal{D} be a distinguished abelian subcategory of \mathcal{C} . Let*

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

be a short exact sequence in \mathcal{D} and let W be an object in \mathcal{C} . If W belongs to \mathcal{D} , then the maps

$$\text{Hom}_{\mathcal{C}}(\Sigma(W), Y) \longrightarrow \text{Hom}_{\mathcal{C}}(\Sigma(W), Z)$$

$$\text{Hom}_{\mathcal{C}}(\Sigma(Y), W) \longrightarrow \text{Hom}_{\mathcal{C}}(\Sigma(X), W)$$

induced by composition with g and precomposition with $\Sigma(f)$ are surjective.

Proof Since \mathcal{D} is a distinguished abelian subcategory in \mathcal{C} , there is a morphism $h : Z \rightarrow \Sigma(X)$ such that the triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma(X)$$

in \mathcal{C} is exact. Applying the functor $\text{Hom}_{\mathcal{C}}(W, -)$ yields a long exact sequence of the form

$$\cdots \longrightarrow \text{Hom}_{\mathcal{C}}(W, \Sigma^{-1}(Y)) \longrightarrow \text{Hom}_{\mathcal{C}}(W, \Sigma^{-1}(Z)) \longrightarrow \text{Hom}_{\mathcal{C}}(W, X) \longrightarrow \text{Hom}_{\mathcal{C}}(W, Y) \longrightarrow \cdots$$

The right map is induced by composing with the monomorphism f in \mathcal{D} . Thus if W belongs to \mathcal{D} , then the right map is injective. But then the map in the middle is zero, so the left map is surjective. Since Σ is an equivalence, it follows that the map $\text{Hom}_{\mathcal{C}}(\Sigma(W), Y) \longrightarrow \text{Hom}_{\mathcal{C}}(\Sigma(W), Z)$ is surjective. Similarly, we have a long exact sequence

$$\cdots \longrightarrow \text{Hom}_{\mathcal{C}}(\Sigma(Y), W) \longrightarrow \text{Hom}_{\mathcal{C}}(\Sigma(X), W) \longrightarrow \text{Hom}_{\mathcal{C}}(Z, W) \longrightarrow \text{Hom}_{\mathcal{C}}(Y, W) \longrightarrow \cdots$$

The right map is induced by precomposing with the epimorphism g in \mathcal{D} . Thus if W belongs to \mathcal{D} , then the right map is injective, hence the map in the middle is zero, and therefore the left map is surjective. This concludes the proof. \square

Unlike hearts of t -structures, distinguished abelian subcategories need not be disjoint from their shifts—they may contain periodic objects. The following consequence of Proposition 5.3—which is Theorem 1.6—shows that if \mathcal{D} is a distinguished abelian subcategory in a triangulated category \mathcal{C} , then $\mathcal{D} \cap \Sigma(\mathcal{D})$ is a subcategory of the additive category $\text{proj}(\mathcal{D})$ generated by the projective objects in \mathcal{D} .

Corollary 5.4 *Let \mathcal{C} be a triangulated category and let \mathcal{D} be a distinguished abelian subcategory of \mathcal{C} . Let W be an object in \mathcal{D} such that $\Sigma(W)$ is an object in \mathcal{D} . Then W is injective in \mathcal{D} and $\Sigma(W)$ is projective in \mathcal{D} .*

Proof By Proposition 5.3, if $g : Y \rightarrow Z$ is an epimorphism in \mathcal{D} , then every morphism $\Sigma(W) \rightarrow Z$ lifts through g . Since $\Sigma(W)$ belongs to \mathcal{D} , it follows that $\Sigma(W)$ is projective in \mathcal{D} . Similarly, by Proposition 5.3 (applied with $\Sigma(W)$ instead of W), if $f : X \rightarrow Y$ is a monomorphism in \mathcal{D} , then every morphism $X \rightarrow W$ factors through f , and hence W is injective in \mathcal{D} . \square

Remark 5.5 There is no converse to this Corollary 5.4: an injective object in a distinguished abelian subcategory \mathcal{D} of a triangulated category \mathcal{C} need not have the property that $\Sigma(W)$ belongs to \mathcal{D} . For instance, if k has prime characteristic p and P is a finite p -group of order at least 3, then $\mathcal{D} = \text{add}(k)$ is a semisimple distinguished abelian subcategory of $\mathcal{C} = \underline{\text{mod}}(kP)$, so the trivial kP -module k is injective in \mathcal{D} , but $\Sigma(k)$ is an indecomposable kP -module of dimension $|P| - 1$, so does not belong to \mathcal{D} . In the context of finite group algebras, the situation of Corollary 5.4 seems to be a characteristic 2 phenomenon—see Example 4.6.

Proposition 5.6 *Let \mathcal{C} be a triangulated category and let \mathcal{D} be a distinguished abelian subcategory of \mathcal{C} . Let*

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

be a short exact sequence in \mathcal{D} and let W be an object in \mathcal{C} .

(i) *If W is a projective object in \mathcal{D} , then the map*

$$\text{Hom}_{\mathcal{C}}(W, \Sigma(X)) \rightarrow \text{Hom}_{\mathcal{C}}(W, \Sigma(Y))$$

induced by composition with $\Sigma(f)$ is injective.

(ii) If W is an injective object in \mathcal{D} , then the map

$$\text{Hom}_{\mathcal{C}}(Z, \Sigma(W)) \rightarrow \text{Hom}_{\mathcal{C}}(Y, \Sigma(W))$$

induced by precomposition with g is injective.

Proof Since \mathcal{D} is a distinguished abelian subcategory in \mathcal{C} , there is a morphism $h : Z \rightarrow \Sigma(X)$ such that the triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma(X)$$

in \mathcal{C} is exact. Applying the functor $\text{Hom}_{\mathcal{C}}(W, -)$ yields a long exact sequence of the form

$$\dots \longrightarrow \text{Hom}_{\mathcal{C}}(W, Y) \xrightarrow{g_*} \text{Hom}_{\mathcal{C}}(W, Z) \xrightarrow{h_*} \text{Hom}_{\mathcal{C}}(W, \Sigma(X)) \xrightarrow{\Sigma(f)_*} \text{Hom}_{\mathcal{C}}(W, \Sigma(Y)) \longrightarrow \dots$$

If W is projective in \mathcal{D} , then g_* is surjective, hence h_* is zero. This implies that $\Sigma(f)_*$ is injective, proving (i). A dual argument, applying the functor $\text{Hom}_{\mathcal{C}}(-, W)$, and using the fact that Σ is an equivalence, shows (ii). \square

A morphism $f : X \rightarrow Y$ in a category \mathcal{D} is *split* if there exists a morphism $g : Y \rightarrow X$ in \mathcal{D} such that $f = f \circ g \circ f$. In that case, one can choose g such that $g = g \circ f \circ g$ (see e. g. [27, Proposition 1.12.21]). If \mathcal{D} is an abelian category, an easy verification shows that every morphism in \mathcal{D} is split if and only if every monomorphism (resp. every epimorphism) in \mathcal{D} is split. It is well-known that all epimorphisms and monomorphisms in a triangulated category are split (see e. g. [28, Proposition A.2.9]).

Proposition 5.7 *Let \mathcal{C} be a triangulated category and \mathcal{D} a distinguished abelian subcategory. If the inclusion functor $\mathcal{D} \subseteq \mathcal{C}$ has a left adjoint or a right adjoint as an additive functor, then every morphism in \mathcal{D} is split.*

Proof Suppose that the inclusion functor $\mathcal{D} \subseteq \mathcal{C}$ has a left adjoint Φ . That is, for any object U in \mathcal{D} and any object X in \mathcal{C} we have a natural isomorphism $\text{Hom}_{\mathcal{D}}(\Phi(X), U) \cong \text{Hom}_{\mathcal{C}}(X, U)$. Thus any monomorphism $U \rightarrow U'$ in \mathcal{D} induces an injective map $\text{Hom}_{\mathcal{C}}(X, U) \rightarrow \text{Hom}_{\mathcal{C}}(X, U')$. This shows that the morphism $U \rightarrow U'$ is a monomorphism in \mathcal{C} , hence split in \mathcal{C} . Since \mathcal{D} is a full subcategory of \mathcal{C} it follows that the monomorphism $U \rightarrow U'$ is split in \mathcal{D} , and hence every morphism in \mathcal{D} is split. A similar argument shows that if the inclusion functor $\mathcal{D} \subseteq \mathcal{C}$ has a right adjoint, then every epimorphism in \mathcal{D} is split, whence the result. \square

If D is a finite-dimensional k -algebra, then D is semisimple if and only if every morphism in $\text{mod}(D)$ is split. Thus Proposition 5.7 has the following immediate consequence.

Corollary 5.8 *Let A be a finite-dimensional selfinjective k -algebra, D a finite-dimensional k -algebra, and $\Phi : \text{mod}(D) \rightarrow \underline{\text{mod}}(A)$ a full embedding of $\text{mod}(D)$ as a distinguished abelian subcategory in $\underline{\text{mod}}(A)$. If Φ has a left adjoint or a right adjoint, then D is semisimple.*

This Corollary implies in particular that even if Φ is induced by tensoring with a suitable A - D -bimodule, the Tensor-Hom adjunction does not in general yield a right adjoint to Φ ; see Lemma 9.12 and Remark 9.13 for some more comments.

Remark 5.9 By a result of Balmer and Schlichting [4, Theorem 1.5], the idempotent completion $\hat{\mathcal{C}}$ of a triangulated category \mathcal{C} is triangulated in such a way that the canonical embedding $\mathcal{C} \rightarrow \hat{\mathcal{C}}$ is an exact functor. Since this embedding is full, it follows that a distinguished

abelian subcategory \mathcal{D} of \mathcal{C} remains a distinguished abelian subcategory of $\hat{\mathcal{C}}$. Since any abelian category is idempotent split, it follows that the indecomposable objects in \mathcal{D} remain indecomposable in $\hat{\mathcal{C}}$.

Example 5.10 Let k be a field of characteristic 2 and let P be a finite 2-group of order at least 4. Let Z be a central subgroup of order 2 of P . Set $Y = kP/Z$ as left kP -module. Then $\text{End}_{kP}^{\text{pr}}(Y) = \{0\}$ and $\text{End}_{kP}(Y) \cong (kP/Z)^{\text{op}}$. Clearly Y is a progenerator of the distinguished abelian subcategory $\text{mod}(kP/Z)$ of $\underline{\text{mod}}(kP)$, obtained from the restriction functor Φ given by the canonical surjection $kP \rightarrow kP/Z$. The functor Φ is trivially isomorphic to $Y \otimes_{kP/Z} -$. As mentioned in Example 4.6, we have $\Sigma(Y) \cong Y$, where Σ is the shift functor in $\underline{\text{mod}}(kP)$. In other words, as a kP -module, Y has period 1. Therefore, Y is also a progenerator of the distinguished abelian subcategory $\Sigma(\text{mod}(kP/Z))$. The subcategories $\text{mod}(kP/Z)$ and $\Sigma(\text{mod}(kP/Z))$ are different; in fact, their intersection is $\text{add}(Y)$ because of Proposition 5.4. In particular, the embedding $\Sigma \circ \Phi : \text{mod}(kP/Z) \rightarrow \underline{\text{mod}}(kP)$ is not induced by the functor $Y \otimes_{kP/Z} -$. It is, though, still induced by tensoring with a bimodule, namely the kP - kP/Z -bimodule $\Sigma_{P \times P/Z}(Y)$. This is because we have composed the embedding Φ with the self-equivalence Σ on $\underline{\text{mod}}(kP)$, which is a stable equivalence of Morita type, hence induced by tensoring with a suitable bimodule. One should expect that composing Φ with a stable equivalence on $\underline{\text{mod}}(kP)$ which is not of Morita type would yield embeddings $\text{mod}(kP/Z) \rightarrow \underline{\text{mod}}(kP)$ as distinguished abelian subcategories which are not induced by tensoring with any bimodule.

6 Exact sequences in distinguished abelian subcategories

Any exact triangle $X \rightarrow Y \rightarrow Z \rightarrow \Sigma(X)$ in a triangulated category \mathcal{C} such that X, Y, Z belong to the heart \mathcal{A} of a t -structure is in fact induced by a short exact sequence $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$ in \mathcal{A} . In an arbitrary distinguished abelian subcategory, this need not be the case. The case where a nonzero object W and its shift $\Sigma(W)$ belong to a distinguished abelian subcategory \mathcal{D} of \mathcal{C} yields an exact triangle

$$W \longrightarrow 0 \longrightarrow \Sigma(W) \longleftarrow \Sigma(W)$$

which is not induced by an exact sequence in \mathcal{D} . As noted in Corollary 5.4, in that situation W is injective in \mathcal{D} and $\Sigma(W)$ is projective in \mathcal{D} . This situation arises in Example 4.6.

The following result shows that these are essentially the only exact triangles with three terms in \mathcal{D} which can arise besides those induced by short exact sequences in \mathcal{D} . As before, we denote the shift functor in a triangulated category by Σ .

Proposition 6.1 *Let \mathcal{D} be a distinguished abelian subcategory in a triangulated category \mathcal{C} , and let*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma(X)$$

be an exact triangle in \mathcal{C} such that X, Y, Z belong to \mathcal{D} . Then this triangle is isomorphic to a direct sum of two exact triangles of the form

$$\begin{aligned} X' &\xrightarrow{f'} Y \xrightarrow{g'} Z' \xrightarrow{h'} \Sigma(X') \\ W &\longrightarrow 0 \longrightarrow \Sigma(W) \longleftarrow \Sigma(W) \end{aligned}$$

where $X', Z', W, \Sigma(W)$ are in \mathcal{D} , and where the sequence

$$0 \longrightarrow X' \xrightarrow{f'} Y \xrightarrow{g'} Z' \longrightarrow 0$$

is exact in \mathcal{D} . Moreover, W is injective in \mathcal{D} and $\Sigma(W)$ is projective in \mathcal{D} .

Proof Since \mathcal{D} is abelian, the morphism g has a kernel $f' : X' \rightarrow Y$ in \mathcal{D} . Then in particular $g \circ f' = 0$, and hence there is a morphism $v : X' \rightarrow X$ such that $f \circ v = f'$. Since f' is the kernel of g and since $g \circ f = 0$, there is a unique morphism $w : X \rightarrow X'$ such that $f = f' \circ w$. Thus

$$f' \circ w \circ v = f \circ v = f'$$

and since f' is a monomorphism, this forces $w \circ v = \text{Id}_{X'}$. Denote by $g' : Y \rightarrow Z'$ a cokernel of f' in \mathcal{D} , so that we get a short exact sequence

$$0 \longrightarrow X' \xrightarrow{f'} Y \xrightarrow{g'} Z' \longrightarrow 0$$

in \mathcal{D} . Since \mathcal{D} is distinguished, this can be completed to an exact triangle in \mathcal{C} with a morphism $h' : Z' \rightarrow \Sigma(X')$. The morphisms v and w yield morphisms of triangles

$$\begin{array}{ccccccc} X' & \xrightarrow{f'} & Y & \xrightarrow{g'} & Z' & \xrightarrow{h'} & \Sigma(X') \\ \downarrow v & & \parallel & & \downarrow a & & \downarrow \Sigma(v) \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma(X) \\ \downarrow w & & \parallel & & \downarrow b & & \downarrow \Sigma(w) \\ X' & \xrightarrow{f'} & Y & \xrightarrow{g'} & Z' & \xrightarrow{h'} & \Sigma(X') \end{array}$$

Since $w \circ v = \text{Id}_{X'}$, it follows that $b \circ a$ is an automorphism of Z' . Therefore $(\text{Id}_{X'}, \text{Id}_Y, b \circ a)$ is an automorphism of the third triangle, and hence so is its inverse. After replacing b by $(b \circ a)^{-1} \circ b$, we therefore may choose b in such a way that $b \circ a = \text{Id}_{Z'}$. It follows that the first triangle is a direct summand of the second, and that it has a complement isomorphic to

$$W \longrightarrow 0 \longrightarrow \Sigma(W) \xlongequal{\quad} \Sigma(W)$$

where W is the complement of X' in X determined by $\ker(w)$. The last statement on W (resp. $\Sigma(W)$) being injective (resp. projective) in \mathcal{D} follows from Corollary 5.4. \square

Corollary 6.2 *Let \mathcal{C} be a triangulated category, \mathcal{D} a distinguished abelian subcategory of \mathcal{C} , and \mathcal{T} a thick subcategory of \mathcal{C} . Suppose that $\mathcal{T} \subseteq \mathcal{D}$. Then all objects in \mathcal{T} are projective and injective in \mathcal{D} , and every morphism in \mathcal{T} is split.*

Proof Since \mathcal{T} is closed under powers of Σ , it follows from Corollary 5.4 that all objects in \mathcal{T} are projective and injective in \mathcal{D} . Let $f : X \rightarrow Y$ be a morphism in \mathcal{T} . Complete f to an exact triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma(X)$$

in \mathcal{T} (or equivalently, in \mathcal{C}). Since \mathcal{T} is contained in \mathcal{D} , it follows that (possibly after replacing Z by an isomorphic object) Z belongs to \mathcal{D} , and hence the morphism g belongs to \mathcal{D} . This

triangle is the direct sum of two triangles as in Proposition 6.1. All terms in these two triangles are in \mathcal{T} , hence projective and injective in \mathcal{D} . The first of the two triangles is induced by a short exact sequence in \mathcal{D} , and therefore split. The second of the two triangles is trivially split. The result follows. \square

Following Jørgensen [18, Definition 2.2], [19, Definition 0.1], a full additive subcategory \mathcal{D} of a triangulated category \mathcal{C} is called a *proper abelian subcategory* if it has the property that a triangle $X \rightarrow Y \rightarrow Z \rightarrow \Sigma(X)$ in \mathcal{C} with X, Y, Z in \mathcal{D} is exact in \mathcal{C} if and only if the sequence $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$ is exact in \mathcal{D} . We write $\mathcal{D} \cap \Sigma(\mathcal{D}) = 0$ if no nonzero object W in \mathcal{D} has the property that $\Sigma(W)$ is isomorphic to an object in \mathcal{D} .

Corollary 6.3 *Let \mathcal{C} be a triangulated category. A distinguished abelian subcategory \mathcal{D} of \mathcal{C} is proper if and only if $\mathcal{D} \cap \Sigma(\mathcal{D}) = 0$.*

Proof If $\mathcal{D} \cap \Sigma(\mathcal{D}) = 0$, then Proposition 6.1 implies that \mathcal{D} is proper. If $\mathcal{D} \cap \Sigma(\mathcal{D}) \neq 0$, then there is a nonzero object W in \mathcal{D} such that $\Sigma(W)$ belongs to \mathcal{D} . Thus all terms of the exact triangle $W \rightarrow 0 \rightarrow \Sigma(W) \rightarrow \Sigma(W)$ are in \mathcal{D} , but the sequence $0 \rightarrow W \rightarrow 0 \rightarrow \Sigma(W) \rightarrow 0$ is not exact in \mathcal{D} , and hence \mathcal{D} is not proper. \square

Remark 6.4 With the notation of Corollary 6.2, suppose that \mathcal{D} is equivalent to $\text{mod}(D)$ for some finite-dimensional symmetric k -algebra D and that \mathcal{T} is a thick subcategory of \mathcal{C} which is contained in \mathcal{D} . Then \mathcal{T} is generated, as an additive category, by indecomposable projective (or equivalently, injective) objects in \mathcal{D} . By the assumption on \mathcal{D} , each projective indecomposable object U in \mathcal{D} has an endomorphism with image the socle of U . This endomorphism is split, hence an isomorphism, and thus U is simple. Therefore, in this situation, \mathcal{T} consists of projective semisimple objects in \mathcal{D} which are permuted by Σ .

Proposition 6.5 *Let \mathcal{C} be a triangulated category and let $\mathcal{D}, \mathcal{D}'$ be distinguished abelian subcategories of \mathcal{C} such that $\mathcal{D} \subseteq \mathcal{D}'$. Then \mathcal{D} is an abelian subcategory of \mathcal{D}' ; that is, the inclusion functor $\mathcal{D} \subseteq \mathcal{D}'$ is exact.*

Proof Let

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

be a short exact sequence in \mathcal{D} . We need to show that this sequence remains exact in \mathcal{D}' . Since \mathcal{D} is a distinguished abelian subcategory of \mathcal{C} , it follows that there is a morphism $h : Z \rightarrow \Sigma(X)$ in \mathcal{D} such that the triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma(X)$$

in \mathcal{C} is exact. The objects X, Y, Z belong to \mathcal{D} , hence to \mathcal{D}' . By Proposition 6.1, this triangle is a direct sum of exact triangle of the form

$$\begin{aligned} X' &\xrightarrow{f'} Y \xrightarrow{g'} Z' \xrightarrow{h'} \Sigma(X') \\ W &\longrightarrow 0 \longrightarrow \Sigma(W) = \Sigma(W) \end{aligned}$$

where $X', Z', W, \Sigma(W)$ are in \mathcal{D}' , and where the sequence

$$0 \longrightarrow X' \xrightarrow{f'} Y \xrightarrow{g'} Z' \longrightarrow 0$$

is exact in \mathcal{D}' . Since \mathcal{D} is full in \mathcal{C} , hence in \mathcal{D}' and since any abelian category is idempotent complete, it follows that X', Z', W belong to \mathcal{D} (up to isomorphism). Since \mathcal{D} is a full subcategory of \mathcal{C} , hence of \mathcal{D}' , it follows that the morphisms f', g' belong to \mathcal{D} . Thus f is the direct sum in \mathcal{D} of f' and the zero morphism $W \rightarrow 0$. But f is also a monomorphism in \mathcal{D} , and hence $W = 0$. The result follows. \square

Proposition 6.6 *Let \mathcal{C} be an essentially small triangulated category. Every distinguished abelian subcategory of \mathcal{C} is contained in a maximal distinguished abelian subcategory of \mathcal{C} , with respect to the inclusion of subcategories.*

Proof We may assume that \mathcal{C} is small, so that the distinguished abelian subcategories form a set. Let \mathcal{T} be a totally ordered set of distinguished abelian subcategories of \mathcal{C} , where the order is by inclusion. In view of Zorn’s Lemma, we need to show that \mathcal{T} has an upper bound. We claim that $\mathcal{E} = \cup_{\mathcal{D} \in \mathcal{T}} \mathcal{D}$ is such an upper bound. We need to show that \mathcal{E} is a distinguished abelian subcategory. By construction, \mathcal{E} is a full subcategory of \mathcal{C} . We show next that \mathcal{E} is an abelian category. Let $f : X \rightarrow Y$ be a morphism in \mathcal{E} . Then there is $\mathcal{D} \in \mathcal{T}$ containing X, Y , and since \mathcal{D} is a full subcategory of \mathcal{C} , it follows that f is a morphism in \mathcal{D} . Thus f has a kernel $a : W \rightarrow X$ in \mathcal{D} . We are going to show that a is a kernel of f in \mathcal{E} . Let $g : Z \rightarrow X$ a morphism in \mathcal{E} such that $f \circ g = 0$. We need to show that g factors uniquely through a . Since \mathcal{T} is totally ordered, there is $\mathcal{D}' \in \mathcal{T}$ such that $\mathcal{D} \subseteq \mathcal{D}'$ and such that g is a morphism in \mathcal{D}' . By Proposition 6.5, the morphism a remains a kernel of f as a morphism in \mathcal{D}' . Thus there is a unique morphism $h : Z \rightarrow W$ in \mathcal{D}' such that $a \circ h = g$. We need to show that h is unique in \mathcal{E} with this property. Let $j : Z \rightarrow W$ be a morphism in \mathcal{E} such that $a \circ j = g$. Then j belongs to a category $\mathcal{D}'' \in \mathcal{T}$, which we may choose such that $\mathcal{D}' \subseteq \mathcal{D}''$. Again by Proposition 6.5, the morphism a remains a monomorphism in \mathcal{D}'' . Since $a \circ j = g = a \circ h$, it follows that $j = h$. This shows that the kernel of f in any subcategory $\mathcal{D} \in \mathcal{T}$ containing f is the kernel of f in \mathcal{E} . A similar argument shows that the cokernel of f in any subcategory $\mathcal{D} \in \mathcal{T}$ containing f is the cokernel of f in \mathcal{E} . This implies also that the canonical map $\text{coker}(\ker(f)) \rightarrow \ker(\text{coker}(f))$ in \mathcal{E} is an isomorphism, since it is an isomorphism in any subcategory $\mathcal{D} \in \mathcal{T}$ containing the morphism f . By the above arguments, any short exact sequence in \mathcal{E} is a short exact sequence in \mathcal{D} for some $\mathcal{D} \in \mathcal{T}$, hence can be completed to an exact triangle in \mathcal{C} . This shows that \mathcal{E} is a distinguished abelian subcategory in \mathcal{C} . Thus \mathcal{T} has an upper bound in the set of distinguished abelian subcategories of \mathcal{C} . Zorn’s Lemma implies the result. \square

Proof of Theorem 1.2 Statement (i) is Theorem 3.8. For statement (ii), let A be a finite-dimensional selfinjective algebra over a field k such that all simple A -modules are nonprojective. Then $J(A)$ contains its annihilator $\text{soc}(A)$. Thus $\text{mod}(A/J(A))$ is a distinguished abelian subcategory of $\underline{\text{mod}}(A)$ containing all simple A -modules such that $\ell(\text{mod}(A/J(A))) = \ell(A)$. By Proposition 6.6 there is a maximal distinguished abelian subcategory in \mathcal{C} which contains $\text{mod}(A/J(A))$. By Corollary 3.9 the simple A -modules are exactly the simple objects in \mathcal{D} . Thus $\ell(\mathcal{D}) = \ell(A)$, whence the result. \square

The next two Propositions are tools for passing between short exact sequences in $\text{mod}(A)$, for some finite-dimensional selfinjective algebra A , and short exact sequences in a distinguished abelian subcategory of the stable category $\underline{\text{mod}}(A)$.

Proposition 6.7 *Let A be a finite-dimensional selfinjective algebra over a field k , and let \mathcal{D} be a distinguished abelian subcategory of $\underline{\text{mod}}(A)$. Let*

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

be a short exact sequence in \mathcal{D} . Suppose that X, Y, Z have no nonzero projective direct summands as A -modules. The following hold.

- (i) There is a finitely generated projective A -module Q and a short exact sequence of A -modules

$$0 \longrightarrow X \xrightarrow{a} Y \oplus Q \xrightarrow{b} Z \longrightarrow 0$$

such that f and g are the images of a and b in $\text{mod}(A)$, respectively.

- (ii) In addition, if X or Z is simple as an A -module, then $Q = 0$ in the first statement.

Proof Since \mathcal{D} is a distinguished abelian subcategory of $\text{mod}(A)$, it follows that the given exact sequence in \mathcal{D} gives rise to an exact triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma(X)$$

in $\text{mod}(A)$, for some morphism h . By the construction of exact triangles in $\text{mod}(A)$, this exact triangle is induced by a short exact sequence of A -modules of the form

$$0 \longrightarrow X \oplus P \xrightarrow{a} Y \oplus Q \xrightarrow{b} Z \oplus R \longrightarrow 0$$

for some finitely generated projective A -modules P, Q, R , such that f and g are the images of a and b in $\text{mod}(A)$. Since a is injective, and since the A -module P is projective, hence also injective, it follows that $a(P) \cong P$ splits off the middle term $Y \oplus Q$. Since Y has no nonzero projective summand, it follows that we may assume $P = 0$. A similar argument shows that we may assume $R = 0$, whence the first statement.

For the second statement, assume first that Z is simple as an A -module. Write $b = (r, s)$, where $r = b|_Y : Y \rightarrow Z$ and $s = b|_Q : Q \rightarrow Z$. Since $g \neq 0$ in $\text{mod}(A)$, it follows that $r \neq 0$, hence r is surjective as Z is simple. Since Q is projective as an A -module, it follows that s factors through r . Write $s = r \circ t$ for some $t : Q \rightarrow Y$. Set $Q' = \left\{ \begin{pmatrix} -t(x) \\ x \end{pmatrix} \mid x \in Q \right\}$. This is a submodule of $Y \oplus Q$, isomorphic to Q , and contained in $\ker(b)$. Since Q' is also injective, it follows that Q' is isomorphic to a direct summand of X , hence $Q' = 0$ by the first statement, and so also $Q = 0$. Assume next that X is simple. Writing $a = \begin{pmatrix} u \\ v \end{pmatrix} : X \rightarrow Y \oplus Q$, we have that $u \neq 0$, so u is injective. Thus $a(X)$ is not contained in the summand Q , hence intersects this summand trivially since X is simple. Thus b sends Q to a submodule of Z isomorphic to Q . Since Q is also injective as an A -module, it follows that Q is isomorphic to a direct summand of Z , hence zero by the first statement. This proves the second statement. □

Proposition 6.8 *Let A be a finite-dimensional selfinjective algebra over a field k , and let \mathcal{D} be a distinguished abelian subcategory of $\text{mod}(A)$. Let*

$$0 \longrightarrow X \xrightarrow{a} Y \oplus Q \xrightarrow{b} Z \longrightarrow 0$$

be a short exact sequence of A -modules such that Q is a projective A -module, and such that the A -modules X, Y, Z belong to \mathcal{D} . Suppose that as an object in \mathcal{D} , X has no nonzero injective direct summand, or that as an object in \mathcal{D} , Z has no nonzero projective direct summand. Then the sequence

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

is exact in \mathcal{D} , where $f = \underline{a}$ and $g = \underline{b}$ are the images in $\underline{\text{mod}}(A)$ of a and b , respectively.

Proof By the construction of $\underline{\text{mod}}(A)$ as a triangulated category, the given short exact sequence of A -modules determines an exact triangle in $\underline{\text{mod}}(A)$ of the form

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma(X)$$

for some morphism h in $\underline{\text{mod}}(A)$. By Proposition 6.1, this triangle is isomorphic to a direct sum of two exact triangles of the form

$$\begin{array}{ccccccc} X' & \xrightarrow{f'} & Y & \xrightarrow{g'} & Z' & \xrightarrow{h'} & \Sigma(X') \\ W & \longrightarrow & 0 & \longrightarrow & \Sigma(W) & = & \Sigma(W) \end{array}$$

such that the sequence

$$0 \longrightarrow X' \xrightarrow{f'} Y \xrightarrow{g'} Z' \longrightarrow 0$$

is exact in \mathcal{D} , where W is an injective object in \mathcal{D} such that $\Sigma(W)$ is a projective object in \mathcal{D} . Thus f is a monomorphism in \mathcal{D} if and only if $W = 0$, which is equivalent to $\Sigma(W) = 0$, hence to g being an epimorphism in \mathcal{D} . The result follows. \square

Remark 6.9 Any commutative square in a triangulated category \mathcal{C}

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ a \downarrow & & \downarrow b \\ X' & \xrightarrow{f'} & Y' \end{array}$$

can be completed to a morphism of exact triangles

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma(X) \\ a \downarrow & & \downarrow b & & \downarrow c & & \downarrow \Sigma(a) \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & \Sigma(X') \end{array}$$

In general, c is not uniquely determined by (a, b) . If, however, the two exact triangles are determined by short exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \longrightarrow 0 \\ 0 & \longrightarrow & X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' \longrightarrow 0 \end{array}$$

in a distinguished abelian subcategory \mathcal{D} of \mathcal{C} , then a and c are both determined by b alone. Indeed, \mathcal{D} is a full subcategory of \mathcal{C} , so a, b, c all are morphisms in \mathcal{D} , and since f' is a monomorphism and g an epimorphism in \mathcal{D} , it follows that b determines both a and c . In particular, any endomorphism (a, b, c) of the exact triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma(Z)$$

is determined by the endomorphism b of Y , or equivalently, the algebra homomorphism from the endomorphism algebra of this triangle to $\text{End}_{\mathcal{C}}(Y)$ sending (a, b, c) to b is injective. This is a necessary criterion for an exact triangle to have the property that its components belong to a distinguished abelian subcategory of \mathcal{C} .

Remark 6.10 Remark 6.9 can be rephrased as stating that the inclusion functor of a distinguished abelian subcategory \mathcal{D} of a triangulated category \mathcal{C} sends morphisms of exact sequences in \mathcal{D} to morphisms of exact triangles in \mathcal{C} . Indeed, if

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & 0 \\
 & & \downarrow a & & \downarrow b & & \downarrow c & & \\
 0 & \longrightarrow & X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \longrightarrow & 0
 \end{array}$$

is a commutative exact diagram in \mathcal{D} , then c is uniquely determined by b , and hence the diagram

$$\begin{array}{ccccccc}
 X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma(X) \\
 \downarrow a & & \downarrow b & & \downarrow c & & \downarrow \Sigma(a) \\
 X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & \Sigma(X')
 \end{array}$$

in \mathcal{C} is commutative, where h, h' are the unique morphisms such that the rows are exact triangles.

In a similar vein, given two composable monomorphisms $X \rightarrow Y$ and $Y \rightarrow Z$ in \mathcal{D} , the obvious diagram in \mathcal{D}

$$\begin{array}{ccccc}
 X & \longrightarrow & Y & \longrightarrow & Y/X \\
 \parallel & & \downarrow & & \downarrow \\
 X & \longrightarrow & Z & \longrightarrow & Z/X \\
 & & \downarrow & & \downarrow \\
 & & Z/Y & \xlongequal{\quad} & Z/Y
 \end{array}$$

describing the third isomorphism theorem can be extended uniquely to an octahedral diagram in \mathcal{C} of the form

$$\begin{array}{ccccccc}
 X & \longrightarrow & Y & \longrightarrow & Y/X & \longrightarrow & \Sigma(X) \\
 \parallel & & \downarrow & & \downarrow & & \parallel \\
 X & \longrightarrow & Z & \longrightarrow & Z/X & \longrightarrow & \Sigma(X) \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & Z/Y & \xlongequal{\quad} & Z/Y & \longrightarrow & \Sigma(Y) \\
 & & \downarrow & & \downarrow & & \\
 & & \Sigma(Y) & \longrightarrow & \Sigma(Y/X) & &
 \end{array}$$

The kernel and cokernel of a morphism in a distinguished abelian subcategory are related with the third term of the exact triangle determined by that morphism, via the octahedron in \mathcal{C} associated with an epi-mono factorisation of the morphism.

Proposition 6.11 *Let \mathcal{D} be a distinguished abelian subcategory in a triangulated category \mathcal{C} . Every morphism $f : X \rightarrow Y$ in \mathcal{D} gives rise to an octahedron in \mathcal{C} of the form*

$$\begin{array}{ccccccc}
 X & \xrightarrow{u} & C & \longrightarrow & \Sigma(\ker(f)) & \longrightarrow & \Sigma(X) \\
 \parallel & & \downarrow v & & \downarrow & & \parallel \\
 X & \xrightarrow{f} & Y & \xrightarrow{g} & V & \xrightarrow{h} & \Sigma(X) \\
 & & \downarrow & & \downarrow & & \downarrow \Sigma(u) \\
 & & \text{coker}(f) & \xrightarrow{=} & \text{coker}(f) & \xrightarrow{w} & \Sigma(C) \\
 & & \downarrow w & & \downarrow & & \\
 & & \Sigma(C) & \longrightarrow & \Sigma^2(\ker(f)) & &
 \end{array}$$

where C is an object in \mathcal{D} , u an epimorphism in \mathcal{D} , v a monomorphism in \mathcal{D} , and where $\ker(f)$ and $\text{coker}(f)$ denote the kernel and cokernel of f in \mathcal{D} , respectively.

Proof Since \mathcal{D} is abelian, we have a canonical isomorphism $C = \text{coker}(\ker(f)) \cong \ker(\text{coker}(f))$ in \mathcal{D} . Since \mathcal{D} is a distinguished abelian subcategory, the obvious short exact sequences

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \ker(f) & \longrightarrow & X & \xrightarrow{u} & C \longrightarrow 0 \\
 0 & \longrightarrow & C & \xrightarrow{v} & Y & \longrightarrow & \text{coker}(f) \longrightarrow 0
 \end{array}$$

can be completed to exact triangles

$$\begin{array}{ccccccc}
 \ker(f) & \longrightarrow & X & \xrightarrow{u} & C & \longrightarrow & \Sigma(\ker(f)) \\
 C & \xrightarrow{v} & Y & \longrightarrow & \text{coker}(f) & \xrightarrow{w} & \Sigma(C)
 \end{array}$$

Rotating the first of these two exact triangles yields an exact triangle

$$X \xrightarrow{u} C \longrightarrow \Sigma(\ker(f)) \longrightarrow \Sigma(X)$$

Thus an octahedron associated with the factorisation

$$\begin{array}{ccc}
 X & \xrightarrow{f} & Y \\
 & \searrow u & \nearrow v \\
 & C &
 \end{array}$$

has the form as stated. □

Remark 6.12 With the notation of Proposition 6.11, the kernel and cokernel of f and the factorisation of f via C are unique up to unique isomorphism. Once fixed, they determine the morphisms in the top horizontal and left vertical exact triangle uniquely, by Lemma 5.1.

7 Extension closed distinguished abelian subcategories

As mentioned in the introduction, unlike hearts of t -structures, distinguished abelian subcategories in a triangulated category need not be extension closed. We develop criteria for a distinguished abelian subcategory \mathcal{D} to be extension closed in a triangulated category \mathcal{C} in terms of Ext-bifunctors.

Let (\mathcal{C}, Σ) be a k -linear triangulated category and let \mathcal{D} be a distinguished abelian subcategory of \mathcal{C} such that $\mathcal{D} \cong \text{mod}(D)$ for some finite-dimensional k -algebra D . This hypothesis ensures that \mathcal{D} has enough injective objects. We are going to compare $\text{Ext}_{\mathcal{C}}^n(U, V) = \text{Hom}_{\mathcal{C}}(U, \Sigma^n(V))$ and $\text{Ext}_{\mathcal{D}}^n(U, V)$, where $n \geq 0$. For $n = 0$ these two spaces are equal since \mathcal{D} is full in \mathcal{C} . We investigate the case $n = 1$. In order to calculate $\text{Ext}_{\mathcal{D}}^1$, we will make use of the usual shift operator $\Sigma_{\mathcal{D}}$ on \mathcal{D} , defined as follows. For each object U in \mathcal{D} choose a (minimal) injective envelope $\iota_U : U \rightarrow I_U$ in \mathcal{D} , and set $\Sigma_{\mathcal{D}}(U) = \text{coker}(\iota_U)$. That is, we have a short exact sequence in \mathcal{D} of the form

$$0 \longrightarrow U \xrightarrow{\iota_U} I_U \longrightarrow \Sigma_{\mathcal{D}}(U) \longrightarrow 0$$

Definition 7.1 Let (\mathcal{C}, Σ) be a k -linear triangulated category and let \mathcal{D} be a distinguished abelian subcategory of \mathcal{C} such that $\mathcal{D} \cong \text{mod}(D)$ for some finite-dimensional k -algebra D . For each object U in \mathcal{D} and each nonnegative integer n , define a morphism

$$\sigma_{n,U} : \Sigma_{\mathcal{D}}^n(U) \rightarrow \Sigma^n(U)$$

in \mathcal{C} inductively as follows. We set $\sigma_{0,U} = \text{Id}_U$, assuming implicitly that $\Sigma_{\mathcal{D}}^0$ (resp. Σ^0) is the identity operator (resp. identity functor) on \mathcal{D} (resp. \mathcal{C}). We define

$$\sigma_{1,U} : \Sigma_{\mathcal{D}}(U) \rightarrow \Sigma(U)$$

as the unique morphism such that the triangle

$$U \xrightarrow{\iota_U} I_U \longrightarrow \Sigma_{\mathcal{D}}(U) \xrightarrow{\sigma_{1,U}} \Sigma(U)$$

in \mathcal{C} is exact. For $n \geq 2$, we define

$$\sigma_{n,U} = \Sigma(\sigma_{n-1,U}) \circ \sigma_{1,\Sigma_{\mathcal{D}}^{n-1}(U)}$$

where we identify $\Sigma_{\mathcal{D}} \circ \Sigma_{\mathcal{D}}^{n-1} = \Sigma_{\mathcal{D}}^n$ and $\Sigma \circ \Sigma^{n-1} = \Sigma^n$.

The uniqueness of the morphism $\sigma_{1,U}$, and hence of $\sigma_{n,U}$, in this Definition follows from Proposition 5.1. We are going to show that in the situation above, $\text{Ext}_{\mathcal{D}}^1(-, -)$ is a subbifunctor of $\text{Ext}_{\mathcal{C}}^1(-, -)$ restricted to \mathcal{D} .

Theorem 7.2 Let (\mathcal{C}, Σ) be a k -linear triangulated category and let \mathcal{D} be a distinguished abelian subcategory of \mathcal{C} such that $\mathcal{D} \cong \text{mod}(D)$ for some finite-dimensional k -algebra D . For any two objects U, V in \mathcal{D} , the morphism $\sigma_{1,V} : \Sigma_{\mathcal{D}}(V) \rightarrow \Sigma(V)$ induces an injective map $\text{Ext}_{\mathcal{D}}^1(U, V) \rightarrow \text{Ext}_{\mathcal{C}}^1(U, V)$ which is natural in U and V .

Proof We start with the standard description of calculating $\text{Ext}_{\mathcal{D}}^1(U, V)$ using an injective resolution of V

$$0 \longrightarrow V \xrightarrow{\iota_V} I^0 \xrightarrow{\delta^0} I^1 \xrightarrow{\delta^1} I^2 \longrightarrow \dots$$

with notation chosen such that $I^0 = I_V$, and $\text{Im}(\delta^0) = \ker(\delta^1) = \Sigma_{\mathcal{D}}(V)$. By definition, $\text{Ext}_{\mathcal{D}}^1(U, V)$ is the degree 1 cohomology of the cochain complex obtained from applying $\text{Hom}_{\mathcal{D}}(U, -)$ to the above injective resolution of V , of the form

$$\text{Hom}_{\mathcal{D}}(U, I^0) \longrightarrow \text{Hom}_{\mathcal{D}}(U, I^1) \longrightarrow \text{Hom}_{\mathcal{D}}(U, I^2) \longrightarrow \dots$$

where the first two maps are induced by composing with δ^0 and δ^1 , respectively. A morphism $\varphi : U \rightarrow I^1$ is in the kernel of the second map if and only if it factors through $\ker(\delta^1) = \text{Im}(\delta^0) = \Sigma_{\mathcal{D}}(V)$. Thus the kernel of the map

$$\text{Hom}_{\mathcal{D}}(U, I^1) \longrightarrow \text{Hom}_{\mathcal{D}}(U, I^2)$$

can be identified with $\text{Hom}_{\mathcal{D}}(U, \Sigma_{\mathcal{D}}(V))$, and hence $\text{Ext}_{\mathcal{D}}^1(U, V)$ is the cokernel of the map

$$\text{Hom}_{\mathcal{D}}(U, I^0) \longrightarrow \text{Hom}_{\mathcal{D}}(U, \Sigma_{\mathcal{D}}(V))$$

induced by composition with δ^0 .

Applying the functor $\text{Hom}_{\mathcal{C}}(U, -)$ to the exact triangle

$$V \xrightarrow{\iota_V} I_V \xrightarrow{\delta^0} \Sigma_{\mathcal{D}}(V) \xrightarrow{\sigma_{1,V}} \Sigma(V)$$

yields an exact sequence

$$\dots \longrightarrow \text{Hom}_{\mathcal{D}}(U, I_V) \longrightarrow \text{Hom}_{\mathcal{D}}(U, \Sigma_{\mathcal{D}}(V)) \longrightarrow \text{Hom}_{\mathcal{C}}(U, \Sigma(V)) \longrightarrow \dots$$

Taking the quotient of the middle term by the image of the left term yields a monomorphism $\text{Ext}_{\mathcal{D}}(U, V) \rightarrow \text{Hom}_{\mathcal{C}}(U, \Sigma(V)) = \text{Ext}_{\mathcal{C}}^1(U, V)$ as stated. We need to show the naturality. Since this map is defined by applying the functor $\text{Hom}_{\mathcal{C}}(U, -)$ to the above diagram, and since the Yoneda embedding $U \mapsto \text{Hom}_{\mathcal{C}}(U, -)$ is contravariantly functorial in U , it follows immediately that the map $\text{Ext}_{\mathcal{D}}(U, V) \rightarrow \text{Hom}_{\mathcal{C}}(U, \Sigma(V)) = \text{Ext}_{\mathcal{C}}^1(U, V)$ is contravariantly functorial in U .

In order to show functoriality in V , let $\psi : V \rightarrow W$ be a morphism in \mathcal{D} . Then ψ extends to a morphism $I_V \rightarrow I_W$, and hence there is a commutative diagram of exact triangles

$$\begin{array}{ccccccc} V & \longrightarrow & I_V & \longrightarrow & \Sigma_{\mathcal{D}}(V) & \xrightarrow{\sigma_{1,V}} & \Sigma(V) \\ \psi \downarrow & & \downarrow & & \downarrow \tau & & \downarrow \Sigma(\psi) \\ W & \longrightarrow & I_W & \longrightarrow & \Sigma_{\mathcal{D}}(W) & \xrightarrow{\sigma_{1,W}} & \Sigma(W) \end{array}$$

The morphism τ depends on the choice of an extension of ψ to $I_V \rightarrow I_W$. If τ, τ' are two morphisms making the above diagram commutative, then $\sigma_{1,W} \circ (\tau - \tau') = 0$, and hence $\tau - \tau'$ factors through the morphism $I_W \rightarrow \Sigma_{\mathcal{D}}(W)$ in the diagram. Thus applying $\text{Hom}_{\mathcal{D}}(U, -)$ to $\tau - \tau'$ induces the zero map $\text{Ext}_{\mathcal{D}}^1(U, V) \rightarrow \text{Ext}_{\mathcal{D}}^1(U, W)$, showing the functoriality in V . This proves the result. □

Corollary 7.3 *Let (\mathcal{C}, Σ) be a k -linear triangulated category and let \mathcal{D} be a distinguished abelian subcategory of \mathcal{C} such that $\mathcal{D} \cong \text{mod}(D)$ for some finite-dimensional k -algebra D . Suppose that $\text{Ext}_{\mathcal{C}}^1(X, Y) = 0$ for all X, Y in \mathcal{D} . Then the k -algebra D is semisimple.*

Proof The hypotheses and Theorem 7.2 imply that $\text{Ext}_{\mathcal{D}}^1(U, V) = 0$ for any two finitely generated D -modules U, V , whence the result. □

One may wonder whether the substantial literature on torsion pairs and mutation in [11, 12, 17, 18, 21], particularly in the context of simple-minded systems in stable categories of selfinjective algebras, allows for a broader theory including distinguished abelian subcategories.

The next result is a criterion when the canonical maps $\text{Ext}_{\mathcal{D}}^1(U, V) \rightarrow \text{Ext}_{\mathcal{C}}^1(U, V)$ in the previous Theorem yield an isomorphism of bifunctors on \mathcal{D} .

Theorem 7.4 *Let (\mathcal{C}, Σ) be a k -linear triangulated category and let \mathcal{D} be a distinguished abelian subcategory of \mathcal{C} such that $\mathcal{D} \cong \text{mod}(D)$ for some finite-dimensional k -algebra D . The following are equivalent.*

- (i) *The category \mathcal{D} is extension closed in \mathcal{C} and $\mathcal{D} \cap \Sigma(\mathcal{D}) = 0$.*
- (ii) *The morphisms $\sigma_{1,V}$ induce isomorphisms $\text{Ext}_{\mathcal{D}}^1(U, V) \cong \text{Ext}_{\mathcal{C}}^1(U, V)$ for all objects U, V in \mathcal{D} .*

Proof Suppose that (i) holds. Let U, V be objects in \mathcal{D} . By Theorem 7.2 we need to show that the map $\text{Ext}_{\mathcal{D}}^1(U, V) \rightarrow \text{Ext}_{\mathcal{C}}^1(U, V)$ induced by $\sigma_{1,V}$ is surjective. Let $\psi : V \rightarrow \Sigma(U)$ be a morphism in \mathcal{C} ; that is, $\psi \in \text{Ext}_{\mathcal{C}}^1(W, U)$. We need to show that there exists a morphism $\varphi : V \rightarrow \Sigma_{\mathcal{D}}(U)$ in \mathcal{C} (which is then automatically in \mathcal{D} as \mathcal{D} is full, thus representing an element in $\text{Ext}_{\mathcal{D}}^1(U, V)$) such that $\psi = \sigma_{1,V} \circ \varphi$. Complete ψ to an exact triangle in \mathcal{C} of the form

$$V \longrightarrow X \longrightarrow U \xrightarrow{\psi} \Sigma(V)$$

Since \mathcal{D} is extension closed, it follows that X can be chosen to belong to \mathcal{D} (possibly after replacing X by an isomorphic object). The morphism $V \rightarrow X$ belongs then to \mathcal{D} . Since $\mathcal{D} \cap \Sigma(\mathcal{D}) = 0$ it follows from Proposition 6.1 that this is a monomorphism in \mathcal{D} . Thus the morphism $\iota_V : V \rightarrow I_V$ extends to a morphism $X \rightarrow I_V$, and hence there exists a morphism of exact triangles

$$\begin{array}{ccccccc} V & \longrightarrow & X & \longrightarrow & U & \xrightarrow{\psi} & \Sigma(V) \\ \parallel & & \downarrow \tau & & \downarrow \varphi & & \parallel \\ V & \xrightarrow{\iota_V} & I_V & \longrightarrow & \Sigma_{\mathcal{D}}(V) & \xrightarrow{\sigma_{1,V}} & \Sigma(V) \end{array}$$

Thus $\psi = \sigma_{1,V} \circ \varphi$. This shows that (i) implies (ii).

Suppose conversely that (ii) holds. Then the map $\text{Ext}_{\mathcal{D}}^1(U, V) \rightarrow \text{Ext}_{\mathcal{C}}^1(U, V)$ is in particular surjective. Let $\psi \in \text{Ext}_{\mathcal{C}}^1(U, V) = \text{Hom}_{\mathcal{C}}(U, \Sigma(V))$. Then there is $\varphi \in \text{Hom}_{\mathcal{D}}(U, \Sigma_{\mathcal{D}}(V))$ such that the square

$$\begin{array}{ccc} U & \xrightarrow{\psi} & \Sigma(V) \\ \downarrow \varphi & & \parallel \\ \Sigma_{\mathcal{D}}(V) & \xrightarrow{\sigma_{1,V}} & \Sigma(V) \end{array}$$

is commutative in \mathcal{C} , hence can be completed to a morphism of triangles of the form

$$\begin{array}{ccccccc} V & \longrightarrow & X & \longrightarrow & U & \xrightarrow{\psi} & \Sigma(V) \\ \parallel & & \downarrow \tau & & \downarrow \varphi & & \parallel \\ V & \xrightarrow{\iota_V} & I_V & \longrightarrow & \Sigma_{\mathcal{D}}(V) & \xrightarrow{\sigma_{1,V}} & \Sigma(V) \end{array}$$

In order to show that \mathcal{D} is extension closed, we need to show that X is isomorphic to an object in \mathcal{D} . Consider a pullback diagram in \mathcal{D} of the form

$$\begin{array}{ccc} X' & \longrightarrow & U \\ \tau \downarrow & & \downarrow \varphi \\ I_V & \longrightarrow & \Sigma_{\mathcal{D}}(V) \end{array}$$

Since the morphism $I_V \rightarrow \Sigma_{\mathcal{D}}(V)$ is an epimorphism in \mathcal{D} , so is the morphism $X' \rightarrow U$ in the last square. Thus this square can be completed to an exact commutative diagram in \mathcal{D} of the form

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V & \longrightarrow & X' & \longrightarrow & U & \longrightarrow & 0 \\ & & \parallel & & \downarrow \tau & & \downarrow \varphi & & \\ 0 & \longrightarrow & V & \xrightarrow{\iota_V} & I_V & \longrightarrow & \Sigma_{\mathcal{D}}(V) & \longrightarrow & 0 \end{array}$$

This in turn can be completed to morphism of exact triangles

$$\begin{array}{ccccccc} V & \longrightarrow & X' & \longrightarrow & U & \xrightarrow{\psi'} & \Sigma(V) \\ \parallel & & \downarrow \tau' & & \downarrow \varphi & & \parallel \\ V & \xrightarrow{\iota_V} & I_V & \longrightarrow & \Sigma_{\mathcal{D}}(V) & \xrightarrow{\sigma_{1,V}} & \Sigma(V) \end{array}$$

But then $\psi' = \sigma_{1,V} \circ \varphi = \psi$, and this forces $X' \cong X$ in \mathcal{C} . Thus (ii) implies (i). This completes the proof. \square

Corollary 7.5 *Let (\mathcal{C}, Σ) be a k -linear triangulated category and let \mathcal{D} be a proper abelian subcategory of \mathcal{C} such that $\mathcal{D} \cong \text{mod}(D)$ for some finite-dimensional k -algebra D . Then \mathcal{D} is extension closed in \mathcal{C} if and only if the morphisms $\sigma_{1,V}$, with V running over the objects in \mathcal{D} , induce an isomorphism of bifunctors $\text{Ext}_{\mathcal{D}}^1(-, -) \cong \text{Ext}_{\mathcal{C}}^1(-, -)$.*

Proof This follows from Theorem 7.4 and Corollary 6.3. \square

If I is an injective module over a finite-dimensional k -algebra D , then $\text{Ext}_D^1(U, I) = \{0\}$ for any D -module U . Therefore, if a triangulated category \mathcal{C} has an extension closed distinguished abelian subcategory equivalent to $\text{mod}(D)$, then $\text{Ext}_{\mathcal{C}}^1(U, I)$ must also vanish thanks to the previous Theorem (where we identify U, I to their images in \mathcal{C}). This yields the following characterisation of extension closed distinguished abelian subcategories which are equivalent to $\text{mod}(D)$.

Theorem 7.6 *Let (\mathcal{C}, Σ) be a k -linear triangulated category and let \mathcal{D} be a distinguished abelian subcategory of \mathcal{C} such that $\mathcal{D} \cong \text{mod}(D)$ for some finite-dimensional k -algebra D . The following are equivalent.*

- (i) *The category \mathcal{D} is extension closed, and $\mathcal{D} \cap \Sigma(\mathcal{D}) = 0$.*
- (ii) *For any two objects U, Y in \mathcal{D} such that Y is injective in \mathcal{D} we have $\text{Ext}_{\mathcal{C}}^1(U, Y) = \{0\}$.*

Proof Suppose that \mathcal{D} is extension closed and satisfied $\mathcal{D} \cap \Sigma(\mathcal{D}) = 0$. Let U, Y be objects in \mathcal{D} such that Y is injective in \mathcal{D} . Then $\text{Ext}_D^1(U, Y) = \{0\}$. Theorem 7.4 implies $\text{Ext}_{\mathcal{C}}^1(U, Y) = \{0\}$.

Conversely, suppose that $\text{Ext}_{\mathcal{C}}^1(U, Y) = \{0\}$ for any two objects U, Y in \mathcal{D} such that Y is injective in \mathcal{D} . By Theorem 7.4, it suffices to show that there is an isomorphism $\text{Ext}_{\mathcal{D}}^1(U, V) \cong \text{Ext}_{\mathcal{C}}^1(U, V)$ induced by $\sigma_{1,V}$, for any two objects U, V in \mathcal{D} . Let U, V be objects in \mathcal{D} . Consider a short exact sequence in \mathcal{D} of the form

$$0 \longrightarrow V \xrightarrow{\iota_V} I_V \longrightarrow \Sigma_{\mathcal{D}}(V) \longrightarrow 0$$

for some injective object I_V in \mathcal{D} . Applying $\text{Hom}_{\mathcal{D}}(U, -)$ yields a long exact sequence of $\text{Ext}_{\mathcal{D}}$ -spaces. Since $\text{Ext}_{\mathcal{D}}^1(U, I_V) = \{0\}$, this long exact sequence yields in particular an exact 4-term sequence

$$0 \longrightarrow \text{Hom}_{\mathcal{D}}(U, V) \longrightarrow \text{Hom}_{\mathcal{D}}(U, I_V) \longrightarrow \text{Hom}_{\mathcal{D}}(U, \Sigma_{\mathcal{D}}(V)) \longrightarrow \text{Ext}_{\mathcal{D}}^1(U, V) \longrightarrow 0$$

Completing the previous short exact sequence to an exact triangle

$$V \xrightarrow{\iota_V} I_V \longrightarrow \Sigma_{\mathcal{D}}(V) \xrightarrow{\sigma_{1,V}} \Sigma(V)$$

and applying the functor $\text{Hom}_{\mathcal{C}}(U, -)$ yields a long exact sequence of $\text{Ext}_{\mathcal{C}}$ -spaces. Since $\text{Ext}_{\mathcal{C}}^1(U, I_V) = \{0\}$ by the hypotheses, this long exact sequence yields in particular an exact sequence

$$\text{Hom}_{\mathcal{C}}(U, V) \longrightarrow \text{Hom}_{\mathcal{C}}(U, I_V) \longrightarrow \text{Hom}_{\mathcal{C}}(U, \Sigma_{\mathcal{D}}(V)) \longrightarrow \text{Ext}_{\mathcal{C}}^1(U, V) \longrightarrow 0$$

The first three terms coincide with the first three nonzero terms in the previous 4-term exact sequence because \mathcal{D} is a full subcategory of \mathcal{C} , and hence the same is true for the fourth terms. By construction, the isomorphism $\text{Ext}_{\mathcal{D}}^1(U, V) \cong \text{Ext}_{\mathcal{C}}^1(U, V)$ arising in this way is induced by $\sigma_{1,V}$. The result follows from Theorem 7.4. □

We compare $\Sigma_{\mathcal{D}}, \Sigma_{\mathcal{E}}$ for distinguished abelian subcategories $\mathcal{D} \subseteq \mathcal{E}$ such that \mathcal{D}, \mathcal{E} are equivalent to module categories. As before, injective envelopes are understood to be minimal.

Proposition 7.7 *Let (\mathcal{C}, Σ) be a k -linear triangulated category, and let \mathcal{D}, \mathcal{E} be distinguished abelian subcategories of \mathcal{C} such that $\mathcal{D} \cong \text{mod}(D)$ and $\mathcal{E} \cong \text{mod}(E)$ for some finite-dimensional k -algebras D, E . Suppose that $\mathcal{D} \subseteq \mathcal{E}$ and that the simple objects of \mathcal{D} and \mathcal{E} coincide. We have a monomorphism $\Sigma_{\mathcal{D}}(X) \rightarrow \Sigma_{\mathcal{E}}(X)$ in \mathcal{E} , for any object X in \mathcal{D} .*

Proof Let $X \rightarrow I$ be an injective envelope of X in \mathcal{D} , and let $I \rightarrow J$ be an injective envelope of I in \mathcal{E} . By Proposition 6.5, \mathcal{D} is an abelian subcategory of \mathcal{E} , and hence $X \rightarrow I$ remains a monomorphism in \mathcal{E} . Thus the composition $X \rightarrow I \rightarrow J$ is a monomorphism in \mathcal{E} . We need to show that this is an injective envelope of X in \mathcal{E} . Since \mathcal{E} is a module category of a finite-dimensional algebra, we need to show that every simple subobject $S \rightarrow J$ of J in \mathcal{E} factors through the map $X \rightarrow J$. Note that $S \rightarrow J$ factors through $I \rightarrow J$ because J is an injective envelope of I in \mathcal{E} . Since S is also simple in \mathcal{D} and $X \rightarrow I$ an injective envelope of X in \mathcal{D} , it follows that this map factors indeed through $X \rightarrow J$. Thus the monomorphism $I \rightarrow J$ induces the required monomorphism via $\Sigma_{\mathcal{D}}(X) \cong I/X \rightarrow J/X \cong \Sigma_{\mathcal{E}}(X)$. □

The following result proves the last statement in Theorem 1.5.

Proposition 7.8 *Let k be a field of prime characteristic p , let G be a finite group, and let N be a normal subgroup of order divisible by p in G . If $O^p(N) = N$, then the canonical image of $\text{mod}(kG/N)$ in $\underline{\text{mod}}(kG)$ is a distinguished abelian subcategory which is extension closed.*

Proof The fact that $\text{mod}(kG/N)$ embeds as a distinguished abelian subcategory into $\underline{\text{mod}}(kG)$ follows from Theorem 4.2. Note that since k has characteristic p , it follows that $H^1(N; k) = \text{Hom}(N, k)$ is trivial if and only if $O^p(N) = N$. Suppose that $O^p(N) = N$. Then, as mentioned above, we have $H^1(N; k) = 0$. Let

$$U \longrightarrow V \longrightarrow W \longrightarrow \Sigma(U)$$

be an exact triangle in $\text{mod}(kG)$ such that U and W belong to $\text{mod}(kG/N)$, or equivalently, such that N acts trivially on U and W . By Proposition 6.1 there is a short exact sequence of kG/N -modules

$$0 \longrightarrow U' \longrightarrow V \longrightarrow W' \longrightarrow 0$$

for some direct summands U', W' of U, W , respectively. Since N acts trivially on U, W , hence on U', W' , and since $H^1(N, k) = 0$ it follows that the restriction to N of this sequence splits. But then N acts trivially on V , and hence V belongs to $\text{mod}(kG/N)$. Thus $\text{mod}(kG/N)$ is extension closed in $\underline{\text{mod}}(kG)$. \square

We have a partial converse of Proposition 7.8

Proposition 7.9 *Let k be a field of prime characteristic p , let G be a finite group, and let N be a normal subgroup of order divisible by p in G . Suppose that if $p = 2$, then the order of N is divisible by 4. If the canonical image of $\text{mod}(kG/N)$ is an extension closed distinguished abelian subcategory in $\underline{\text{mod}}(kG)$, then $O^p(N) = N$.*

Proof Assume that $\text{mod}(kG/N)$ is extension closed in $\underline{\text{mod}}(kG)$. Arguing by contradiction, suppose that $O^p(N)$ is a proper subgroup of N . Then $H^1(N; k) = \text{Ext}_{kN}^1(k, k)$ is nonzero. That is, we have a nonsplit exact sequence of kN -modules

$$0 \longrightarrow k \longrightarrow Z \longrightarrow k \longrightarrow 0$$

Applying the induction functor $kG \otimes_{kN} -$ yields a nonsplit exact sequence of kG -modules

$$0 \longrightarrow kG/N \longrightarrow X \longrightarrow kG/N \longrightarrow 0$$

such that the restriction to kN is nonsplit. Note that N does not act trivially on Z , hence also not on X . Since $\text{mod}(kG/N)$ is assumed to be extension closed, it follows that X is isomorphic, in $\underline{\text{mod}}(kG)$ to a kG/N -module. That is, $X = Y \oplus Y'$ with Y' a projective kG -module and Y a kG/N -module. Since N does not act trivially on X we have $Y' \neq 0$. But then $\text{Res}_N^G(Y')$ is a nonzero projective kN -module, and N acts trivially on $\text{Res}_N^G(Y)$. Since restriction to a Sylow p -subgroup P of N is injective on the cohomology of N with coefficients in k , it follows that the restriction to kP of the previous exact sequence is a nonsplit exact sequence of kP -modules

$$0 \longrightarrow U \longrightarrow V \oplus V' \longrightarrow W \longrightarrow 0$$

with U, V, W acted upon trivially by P and V' a nonzero projective kP -module. Since p is odd or 4 divides $|P|$ this is not possible. \square

Corollary 7.10 *Let k be a field of prime characteristic p , let G be a finite group, and let Q be a nontrivial normal p -subgroup of G of order at least 3. Then the canonical image of $\text{mod}(kG/Q)$ in $\underline{\text{mod}}(kG)$ is a distinguished abelian subcategory which is not extension closed.*

Proof Since $O^p(Q)$ is trivial but Q is not, this is a special case of Proposition 7.9. Alternatively, this follows also from Theorem 1.4: the kernel I of the canonical map $kG \rightarrow kG/Q$ is equal to $I = I(kQ)kG$, which is contained in $J(kG)$. Its annihilator in kG is the ideal $(\sum_{y \in Q} y) \cdot kG$. If Q has order at least 3, then this is strictly smaller than I , and hence Theorem 1.4 implies that $\text{mod}(kG/Q)$ is not extension closed in $\underline{\text{mod}}(kG)$. \square

The hypothesis $|Q| \geq 3$ is necessary: if $p = 2$ and $G = Q$ has order 2, then $\underline{\text{mod}}(kG)$ is semisimple, hence equivalent to the extension closed subcategory $\text{mod}(kG/Q) = \text{mod}(k)$.

Remark 7.11 A result in [13, §2.1, Theorem (b)] gives a sufficient criterion when an extension closed exact subcategory \mathcal{D} of a triangulated category \mathcal{C} has the property that $\text{Ext}_{\mathcal{D}}^1$ and $\text{Ext}_{\mathcal{C}}^1$ are isomorphic as bifunctors on \mathcal{D} .

8 Selfinjective distinguished abelian subcategories

Let A be a finite-dimensional k -algebra and Y a finitely generated A -module. We denote by $\text{mod}_Y(A)$ the full k -linear subcategory of $\text{mod}(A)$ of all A -modules which are isomorphic to $\text{Im}(\varphi)$ for some $\varphi \in \text{End}_A(Y^m)$ and some positive integer m . We denote by $\text{add}(Y)$ the full additive subcategory of $\text{mod}(A)$ of modules which are isomorphic to finite direct sums of direct summands of Y . Clearly $\text{mod}_Y(A)$ contains $\text{add}(Y)$. By a result of Cabanes [9, Theorem 2], if $E = \text{End}_A(Y)$ is selfinjective, then the canonical functor $\text{Hom}_A(Y, -) : \text{mod}(A) \rightarrow \text{mod}(E^{\text{op}})$ restricts to a k -linear equivalence $\text{mod}_Y(A) \cong \text{mod}(E^{\text{op}})$. We use the results and methods from Cabanes [9] to identify in a similar vein the distinguished abelian subcategories constructed earlier in Theorem 2.5. We denote by $\underline{\text{mod}}_Y(A)$ the image of $\text{mod}_Y(A)$ in $\underline{\text{mod}}(A)$. If Y has no nonzero projective direct summand, then no module in $\text{mod}_Y(A)$ has a nontrivial projective summand, and hence the canonical functor $\text{mod}_Y(A) \rightarrow \underline{\text{mod}}_Y(A)$ induces a bijection on isomorphism classes of objects.

Theorem 8.1 *Let A be a finite-dimensional selfinjective k -algebra. Let Y be a finitely generated A -module such that the algebra $E = \text{End}_A(Y)$ is selfinjective. Suppose that $\text{End}_A^{\text{pr}}(Y) = \{0\}$ and that Y is projective as an E -module. Set $D = E^{\text{op}}$.*

- (i) *The functor $Y \otimes_D - : \text{mod}(D) \rightarrow \text{mod}(A)$ induces a full embedding $\Phi_Y : \text{mod}(D) \rightarrow \underline{\text{mod}}(A)$ of $\text{mod}(D)$ as a distinguished abelian subcategory in $\underline{\text{mod}}(A)$, and moreover Φ_Y induces an equivalence of abelian categories $\text{mod}(D) \cong \underline{\text{mod}}_Y(A)$.*
- (ii) *The A -module Y , regarded as an object in the abelian category $\underline{\text{mod}}_Y(A)$, is a progenerator of $\underline{\text{mod}}_Y(A)$.*
- (iii) *The canonical functor $\text{mod}(A) \rightarrow \underline{\text{mod}}(A)$ induces an isomorphism of abelian categories $\text{mod}_Y(A) \cong \underline{\text{mod}}_Y(A)$.*

As mentioned earlier, not every distinguished abelian subcategory of $\underline{\text{mod}}(A)$ is of the form as described in Theorem 8.1, since any selfequivalence of $\underline{\text{mod}}(A)$ as a triangulated category induces a permutation on distinguished abelian subcategories which need not preserve the distinguished abelian subcategories of the form as described in Theorem 8.1. Note that the hypothesis $\text{End}_A^{\text{pr}}(Y) = \{0\}$ implies that Y has no nonzero projective direct summand. Therefore, the second statement of Theorem 8.1 implies that the isomorphism classes of indecomposable summands of the A -module Y are determined by $\underline{\text{mod}}_Y(A)$.

Corollary 8.2 *Let A be a finite-dimensional selfinjective k -algebra. Let Y, Y' be finitely generated A -modules which both satisfy the hypotheses on Y in Theorem 8.1. Then $\underline{\text{mod}}_Y(A) = \underline{\text{mod}}_{Y'}(A)$ if and only if $\text{add}(Y) = \text{add}(Y')$ in $\text{mod}(A)$.*

Remark 8.3 Corollary 8.2 does not imply that a distinguished abelian subcategory is necessarily determined by a progenerator—the same object in \mathcal{C} could be a progenerator of several distinguished abelian subcategories. This Corollary only asserts that the distinguished abelian subcategories obtained as in Theorem 8.1 are determined by their progenerators. See the Example 5.10.

We use the following notation from [9]. For any k -algebra A , any two A -modules Y, U , and any subset M of $\text{Hom}_A(Y, U)$, we set $M \cdot Y = \sum_{\mu \in M} \mu(Y)$; that is, $M \cdot Y$ is the A -submodule of U spanned by the sum of the images of the A -homomorphisms in M . Setting $E = \text{End}_A(Y)$, we consider Y as an A - E^{op} -bimodule in the obvious way. The following Proposition collects the technicalities for the proof of Theorem 8.1.

Proposition 8.4 *Let A be a finite-dimensional selfinjective k -algebra and Y a finitely generated A -module such that $E = \text{End}_A(Y)$ is selfinjective. Suppose that Y is projective as an E -module. Set $D = E^{\text{op}}$.*

(i) *Let n be a positive integer and let M be an E^{op} -submodule of $\text{Hom}_A(Y, Y^n)$. The canonical A -homomorphism*

$$\Psi : Y \otimes_D M \rightarrow M \cdot Y$$

sending $y \otimes \mu$ to $\mu(y)$, where $y \in Y$ and $\mu \in M$, is an isomorphism. In particular, $Y \otimes_D M$ belongs to $\text{mod}_Y(A)$.

(ii) *Let M be a finitely generated D -module. The canonical D -homomorphism*

$$M \rightarrow \text{Hom}_A(Y, Y \otimes_D M)$$

sending $m \in M$ to the map $y \mapsto y \otimes m$ for $y \in Y$ is an isomorphism.

(iii) *Let U be an A -module contained in $\text{mod}_Y(A)$. The canonical evaluation map*

$$\Phi : Y \otimes_D \text{Hom}_A(Y, U) \rightarrow U$$

sending $y \otimes \eta$ to $\eta(y)$, where $y \in Y$ and $\eta \in \text{Hom}_A(Y, U)$, is an isomorphism.

(iv) *The category $\text{mod}_Y(A)$ is an abelian subcategory of $\text{mod}(A)$, and the functor $\text{Hom}_A(Y, -) : \text{mod}(A) \rightarrow \text{mod}(D)$ restricts to an equivalence of abelian categories*

$$\text{mod}_Y(A) \cong \text{mod}(D)$$

with an inverse induced by the functor $Y \otimes_D - : \text{mod}(D) \rightarrow \text{mod}(A)$.

(v) *The equivalence $\text{mod}_Y(A) \cong \text{mod}(D)$ in (iv) sends Y to the regular D -module D . In particular, Y is a progenerator of the category $\text{mod}_Y(A)$.*

(vi) *If $\text{End}_A^{\text{pr}}(Y) = \{0\}$, then $\text{Hom}_A^{\text{pr}}(U, V) = \{0\}$ for any two A -modules U, V in $\text{mod}_Y(A)$.*

Proof There is clearly a well-defined A -homomorphism $\Psi : Y \otimes_D M \rightarrow M \cdot Y$ as described in (i), and this map is obviously surjective. To show that this map is also injective, consider the diagram

$$\begin{array}{ccc} Y \otimes_D M & \xrightarrow{\Psi} & M \cdot Y \\ \downarrow & & \downarrow \\ Y \otimes_D \text{Hom}_A(Y, Y^n) & \longrightarrow & Y^n \end{array}$$

where the vertical maps are induced by the inclusions $M \subseteq \text{Hom}_A(Y, Y^n)$ and $M \cdot Y \subseteq Y^n$, and where the bottom horizontal map is the obvious evaluation map. A trivial verification

shows that this diagram commutes. Since $\text{Hom}_A(Y, Y^n)$ is a free D -module of rank n , it follows that the bottom horizontal map is an isomorphism. Since Y is projective as a right D -module by the assumptions, it follows that the left vertical map is injective. This implies that Ψ is injective, whence (i). Since E , hence D , is selfinjective, we may assume that the D -module M in (ii) is a submodule of $\text{Hom}_A(Y, Y^n)$ for some positive integer n . Applying the functor $\text{Hom}_A(Y, -)$ to the isomorphism Ψ yields an isomorphism

$$\text{Hom}_A(Y, Y \otimes_D M) \cong \text{Hom}_A(Y, M \cdot Y)$$

By [9, Lemma 5], the right side in this isomorphism is equal to M (this is an equality of subsets of $\text{Hom}_A(Y, Y^n)$). The inverse of this isomorphism is the map described in (ii). For (iii), observe first that the map Φ is surjective, since U is in $\text{mod}_Y(A)$, hence a quotient of a finite direct sum of copies of Y . For the injectivity, again since U is in $\text{mod}_Y(A)$, hence isomorphic to a submodule of Y^n for some positive integer n , it follows that there is a commutative diagram of the form

$$\begin{array}{ccc} Y \otimes_D \text{Hom}_A(Y, U) & \xrightarrow{\Phi} & U \\ \downarrow & & \downarrow \\ Y \otimes_D \text{Hom}_A(Y, Y^n) & \longrightarrow & Y^n \end{array}$$

where the right vertical map is injective. The left vertical map is then injective, too, since Y is projective as an E^{op} -module, and the bottom horizontal map is an isomorphism. This shows that Φ is injective, whence (iii). One can prove (iii) also by applying (i) and (ii) with $U = M \cdot Y$. Statement (iv) follows from (ii) and (iii). Statement (v) is an immediate consequence of (iv). Statement (vi) follows from Lemma 3.5. \square

Proof of Theorem 8.1 In the situation of Theorem 8.1, Cabanes’ linear equivalence from [9, Theorem 2] is an equivalence of abelian categories, by Proposition 8.4 (iv). By construction, $\text{mod}_Y(A)$ is the image in $\text{mod}(A)$ of $\text{mod}_Y(A)$. Thus we need to show that the inclusion $\text{mod}_Y(A) \subseteq \text{mod}(A)$ composed with the canonical functor $\text{mod}(A) \rightarrow \underline{\text{mod}}(A)$ is still a full embedding. By Proposition 8.4 (vi) we have $\text{Hom}_A^{\text{pr}}(U, V) = \{0\}$ for any two A -modules U, V in $\text{mod}_Y(A)$. This implies that the canonical functor $\text{mod}_Y(A) \rightarrow \underline{\text{mod}}(A)$ is a full embedding. Since exact triangles in $\underline{\text{mod}}(A)$ are induced by short exact sequences in $\text{mod}(A)$, it follows that the image $\underline{\text{mod}}_Y(A)$ of $\text{mod}_Y(A)$ in $\underline{\text{mod}}(A)$ is a distinguished abelian subcategory in $\underline{\text{mod}}(A)$. Thus Theorem 8.1 follows from Proposition 8.4. \square

Proof of Corollary 8.2 Note that the hypothesis $\text{End}_A^{\text{pr}}(Y) = \{0\}$ implies that Y has no nonzero projective direct summand; similarly for Y' . Thus the linear subcategories $\text{add}(Y)$ and $\text{add}(Y')$ of $\text{mod}(A)$ are equal if and only if their images in $\underline{\text{mod}}(A)$ are equal. This equality is clearly equivalent to $\text{mod}_Y(A) = \text{mod}_{Y'}(A)$, whence the result. \square

9 Examples and further remarks

The following example illustrates that Theorems 2.5 and 8.1 cover some cases not covered by Theorem 3.1.

Example 9.1 Let A be a finite-dimensional selfinjective k -algebra. Let Y be a nonprojective uniserial A -module of length 2 with two non-isomorphic simple composition factors S and T . Then $\text{End}_A(Y) \cong \underline{\text{End}}_A(Y) \cong k$, and hence $\text{mod}_Y(A) = \text{add}(Y)$ is abelian semisimple,

but is not the category of a quotient of A . Indeed, such a quotient algebra would have to be semisimple, but $\text{mod}_Y(A)$ contains no simple A -module, because neither the simple quotient S of Y nor the simple submodule T of Y are contained in $\text{mod}_Y(A)$.

There are trivial examples of distinguished abelian subcategories beyond those constructed in Theorems 2.5 and 8.1.

Example 9.2 Let A be a split finite-dimensional selfinjective k -algebra. Let n be a positive integer and let $\{X_i \mid 1 \leq i \leq n\}$ be a set of A -modules which are pairwise orthogonal in $\text{mod}(A)$; that is, $\text{End}_A(X_i) \cong k$ and $\text{Hom}_A(X_i, X_j) = \{0\}$, where $1 \leq i, j \leq n, i \neq j$. Set $Y = \bigoplus_{i=1}^n X_i$. Then $\text{End}_A(Y)$ is a commutative split semisimple k -algebra, and the image of $\text{add}(Y)$ is a semisimple distinguished abelian subcategory of $\text{mod}(A)$, equivalent to $\text{mod}(\text{End}_A(Y))$. See for instance [12, 23, 34, 35, 37] for more details on orthogonal sets of modules in $\text{mod}(A)$.

There are examples of distinguished abelian subcategories of stable module categories that have infinitely many isomorphism classes of simple objects.

Example 9.3 Let A be a finite-dimensional selfinjective k -algebra. If A has two nonisomorphic simple modules S, T such that $\dim_k(\text{Ext}_A^1(S, T)) \geq 2$ and if k is infinite, then A has infinitely many pairwise non-isomorphic uniserial modules of length 2 with composition factors S and T , from top to bottom. The A -endomorphism algebra of any such module is 1-dimensional, and there is no nonzero A -homomorphism between any two non-isomorphic uniserial modules with these composition factors. Therefore the full additive subcategory of $\text{mod}(A)$ generated by these modules is a semisimple distinguished abelian subcategory with infinitely many isomorphism classes of simple objects.

The next example shows that the hypothesis on \mathcal{D} to contain all simple A -modules in the statement of Theorem 3.8 is necessary.

Example 9.4 Suppose that k is an algebraically closed field of characteristic 5. Consider the algebra $A = kD_{10} \cong k(C_5 \rtimes C_2)$. This is a symmetric Nakayama algebra with two nonisomorphic simple modules S, T and uniserial projective indecomposable modules of length 5. Let U be a uniserial module of length 2 with composition factors S and T . Then $\text{End}_A(U) = \text{End}_A(U) \cong k$. Thus the finite direct sums of modules isomorphic to U form a semisimple distinguished abelian subcategory \mathcal{D} of $\text{mod}(A)$ in which U is up to isomorphism the unique simple object. We have $\text{soc}^2(A) = r(J(A)^2) \subseteq J(A)^2$. Thus $\text{mod}(A/J(A)^2)$ is a distinguished abelian subcategory of $\text{mod}(A)$ containing the simple A -modules S and T and all uniserial modules of length 2. In particular, $\text{mod}(A/J(A)^2)$ contains the subcategory \mathcal{D} , but the simple object U in \mathcal{D} does not remain simple in $\text{mod}(A/J(A)^2)$.

The following example shows the existence of full abelian subcategories in a triangulated category which are not distinguished.

Example 9.5 The idea to construct a full abelian subcategory which is not distinguished in a triangulated category is simply this: start with a non-exact full embedding of an abelian category \mathcal{D} into another abelian category \mathcal{E} , and then embed \mathcal{E} as a distinguished abelian subcategory into a triangulated category \mathcal{T} . Then \mathcal{D} is a full abelian subcategory of \mathcal{T} , but not distinguished. If \mathcal{E} is the module category of a ring, one could take for \mathcal{T} the derived category of that ring and embed \mathcal{E} as the heart of the canonical t -structure on \mathcal{T} . Here is a

concrete example in the context of stable module categories of finite-dimensional selfinjective algebras.

Suppose that $\text{char}(k) = 7$, and set $A = k(C_7 \rtimes C_2)$. This is a symmetric Nakayama algebra with two isomorphism classes of simple modules S and T such that the projective covers of S and T are uniserial with composition series of length 7. Let Y be the three-dimensional quotient of a projective cover P_S of S . The unique composition series of Y has composition factors isomorphic to S, T, S , from top to bottom. Thus $\text{End}_A(Y) \cong k[x]/(x^2)$. The full additive subcategory $\mathcal{D} = \text{add}\{Y, S\}$ of $\text{mod}(A)$ is equivalent to the module category of the 2-dimensional local algebra $k[x]/(x^2)$, with S corresponding to the trivial $k[x]/(x^2)$ -module and Y corresponding to the regular $k[x]/(x^2)$ -module. The inclusion functor $\mathcal{D} \subset \text{mod}(A)$ is full but not exact: the module Y has composition series S, T, S , and hence we have a sequence

$$0 \longrightarrow S \longrightarrow Y \longrightarrow S \longrightarrow 0$$

with nonzero maps to and from Y . This sequence is exact in \mathcal{D} , but not in $\text{mod}(A)$. Since the objects in \mathcal{D} are annihilated by $J(A)^3$, we can regard \mathcal{D} as a full subcategory of $\text{mod}(A/J(A)^3)$, and then by the above, the embedding $\mathcal{D} \rightarrow \text{mod}(A/J(A)^3)$ is not exact. Since $J(A)^3$ contains its annihilator $J(A)^4$ in A , it follows from Theorem 3.1 that the canonical surjection $A \rightarrow A/J(A)^3$ induces a full embedding of $\text{mod}(A/J(A)^3)$ as a distinguished abelian subcategory of $\underline{\text{mod}}(A)$. The image of \mathcal{D} in $\underline{\text{mod}}(A)$ is a full abelian subcategory which is not distinguished, since the above sequence which is exact in \mathcal{D} but not in $\text{mod}(A/J(A)^3)$ cannot be completed to an exact triangle in $\underline{\text{mod}}(A)$.

Let A be a symmetric k -algebra and I a proper ideal in A . By a result of Nakayama [30, Theorem 13], the quotient algebra A/I is symmetric if and only if $I = \text{ann}(z)$ for some $z \in Z(A)$. In that case, if s is a symmetrising form on A , then $z \cdot s$ has kernel I and induces a symmetrising form on A/I . The following Proposition shows that the elements $z \in Z(A)$ satisfying $z^2 = 0$ parametrise the symmetric quotients A/I of A satisfying $\text{End}_A^{\text{pr}}(A/I) = \{0\}$ through the correspondence $z \mapsto \text{ann}(z)$.

Proposition 9.6 *Let A be a symmetric k -algebra and I a proper ideal in A such that $I = \text{ann}(z)$ for some $z \in Z(A)$. We have $\text{End}_A^{\text{pr}}(A/I) = \{0\}$ if and only if $z^2 = 0$.*

Proof By the assumptions on I and z , multiplication by z induces an A - A -bimodule isomorphism $A/I \cong Az$ mapping $a + I$ to az , where $a \in A$. An endomorphism of Az as a left A -module factors through a projective module if and only if it factors through the map $A \rightarrow Az$ given by multiplication with z . Any A -homomorphism $Az \rightarrow A$ extends to an endomorphism of A (because A is symmetric) hence is induced by right multiplication with an element c . Composing the two maps $Az \rightarrow A \rightarrow Az$ given by right multiplication with c and z , respectively, yields the endomorphism of Az given by right multiplication with cz , and the image of this endomorphism is Az^2c . It follows that this endomorphism is zero for all $c \in A$ if and only if $z^2 = 0$. The result follows. \square

Different elements in $Z(A)$ may have the same annihilators. If $z, z' \in Z(A)$ such that $I = \text{ann}(z) = \text{ann}(z')$, and if s is a symmetrising form for A , then both $z \cdot s$ and $z' \cdot s$ induce symmetrising forms on A/I . Thus there exists an element $y \in A$ such that $y + I \in Z(A/I)^\times$ and such that $z' = yz$. Specialising Theorem 3.1 to symmetric quotients of symmetric algebras yields the following result.

Proposition 9.7 *Let A be an indecomposable nonsimple symmetric k -algebra. Let $z \in Z(A)$ such that $z^2 = 0$ and such that $\text{soc}^2(A) \subseteq Az$. Set $I = \text{ann}(z)$. Then $I \subseteq J(A)^2$, the algebras A and A/I have the same quiver, A/I is symmetric, and the canonical functor $\text{mod}(A/I) \rightarrow \text{mod}(A)$ induces an embedding $\text{mod}(A/I) \rightarrow \underline{\text{mod}}(A)$ as distinguished abelian subcategory. In particular, $\underline{\text{mod}}(A)$ has a connected distinguished abelian subcategory \mathcal{D} satisfying $\ell(\mathcal{D}) = \ell(A)$.*

Proof By [30, Theorem 13] the algebra A/I is symmetric. By the assumptions, we have $\text{soc}^2(A) \subseteq Az$. Taking annihilators yields $I \subseteq J(A)^2$, and hence A and A/I have the same quiver. In particular, $\ell(A) = \ell(A/I)$. Since $z^2 = 0$, it follows from Proposition 9.6 that $\text{End}_A^{\text{pr}}(A/I) = \{0\}$. Theorem 3.1 implies the result. \square

Example 9.8 Let p be a prime number such that $p \geq 7$, and let k be a field of characteristic p . Let

$$A = k\langle x, y \mid x^p = y^p = 0, xy = -yx \rangle$$

Then A is a split local symmetric algebra of dimension p^2 with basis $\{x^i y^j \mid 0 \leq i, j \leq p-1\}$. The linear map sending $x^{p-1} y^{p-1}$ to 1 and all other monomials in this basis to 0 is a symmetrising form for A . The monomials $x^i y^j$ with either both i, j even or one of them equal to $p-1$ form a basis of $Z(A)$. The monomial $x^{p-1} y^{p-1}$ is a basis of $\text{soc}(A)$, and the set $\{x^{p-2} y^{p-1}, x^{p-1} y^{p-2}, x^{p-1} y^{p-1}\}$ is a basis of $\text{soc}^2(A)$. The element $z = x^{p-3} y^{p-3}$ belongs to $Z(A)$, satisfies $z^2 = 0$ (this is where we use $p \geq 7$), and we have $\text{soc}^2(A) \subseteq Az$. Thus A and z satisfy the assumptions in Proposition 9.7. Therefore, setting $I = \text{ann}(z)$, we have a full embedding $\text{mod}(A/I) \rightarrow \underline{\text{mod}}(A)$ as distinguished abelian subcategory, and the algebras A and A/I are both symmetric and have the same quiver. More precisely, we have

$$A/I \cong k\langle x, y \mid x^3 = y^3 = 0, xy = -yx \rangle$$

which is a 9-dimensional quantum complete intersection, with basis the image of the set of monomials $\{1, x, y, x^2, xy, y^2, x^2y, xy^2, x^2y^2\}$. Indeed, z is annihilated, in A , by a monomial $x^i y^j$ if and only if at least one of i, j is greater or equal to 3.

Remark 9.9 The center of an essentially small k -linear category \mathcal{C} is the k -algebra $Z(\mathcal{C})$ of k -linear natural transformations on the identity functor $\text{Id}_{\mathcal{C}}$ on \mathcal{C} . Let \mathcal{C} be an essentially small k -linear triangulated category, and let \mathcal{D} be a k -linear distinguished abelian subcategory of \mathcal{C} . Restriction to objects in \mathcal{D} induces a k -algebra homomorphism $Z(\mathcal{C}) \rightarrow Z(\mathcal{D})$. If $\mathcal{D} \cong \text{mod}(D)$ for some k -algebra D , then this induces a k -algebra homomorphism $Z(\mathcal{C}) \rightarrow Z(D)$. If in addition D a finite-dimensional k -algebra, then this yields finite-dimensional k -algebra quotients of $Z(\mathcal{C})$. Finally, if $\mathcal{C} = \underline{\text{mod}}(A)$ for some finite-dimensional selfinjective k -algebra A , then the canonical isomorphism $Z(A) \cong Z(\text{mod}(A))$ induces an algebra homomorphism $\underline{Z}(A) \rightarrow Z(\underline{\text{mod}}(A))$, where $\underline{Z}(A)$ is the stable center of A . Thus restriction to a distinguished abelian subcategory of $\underline{\text{mod}}(A)$ which is equivalent to $\text{mod}(D)$ for some finite-dimensional k -algebra D yields a k -algebra homomorphism $\underline{Z}(A) \rightarrow Z(D)$. Such a homomorphism is in general neither injective nor surjective.

Remark 9.10 Let \mathcal{C} be an essentially small triangulated category, and let \mathcal{D} be a distinguished abelian subcategory of \mathcal{C} . The Grothendieck group $K(\mathcal{D})$ of \mathcal{D} is the abelian group generated by the isomorphism classes $[X]$ of objects X in \mathcal{D} subject to the relations $[X] - [Y] + [Z]$ for any short exact sequence $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$ in \mathcal{D} . The Grothendieck group $K(\mathcal{C})$ of \mathcal{C} is the abelian group generated by the isomorphism classes $[X]$ of objects X in \mathcal{C} subject to the relations $[X] - [Y] + [Z]$ for any exact triangle $X \rightarrow Y \rightarrow Z \rightarrow \Sigma(X)$

in \mathcal{C} . Since short exact sequences in \mathcal{D} can be completed to exact triangles in \mathcal{D} , it follows that the inclusion $\mathcal{D} \rightarrow \mathcal{C}$ induces a canonical group homomorphism $K(\mathcal{D}) \rightarrow K(\mathcal{C})$. In general, this group homomorphism need not be injective or surjective. If \mathcal{C} is monoidal and \mathcal{D} a monoidal distinguished abelian subcategory such that tensor products with objects in \mathcal{C} and \mathcal{D} preserve exact triangles in \mathcal{C} and short exact sequences in \mathcal{D} , respectively, then the canonical map $K(\mathcal{D}) \rightarrow K(\mathcal{C})$ is a ring homomorphism. If $\mathcal{C} = \underline{\text{mod}}(A)$ for some finite-dimensional selfinjective k -algebra A , then $K(\mathcal{C})$ is the quotient of $K(\text{mod}(A))$ by the subgroup generated by the images in $K(\text{mod}(A))$ of finitely generated projective A -modules (see e.g. [40, Proposition 1]), and hence $K(\mathcal{C})$ is finite precisely if the Cartan matrix of A is nonsingular. If \mathcal{D} is a distinguished abelian subcategory of \mathcal{C} which contains all simple A -modules, then the canonical group homomorphism $K(\mathcal{D}) \rightarrow K(\mathcal{C})$ is surjective.

Remark 9.11 Let A be a finite-dimensional selfinjective algebra over a field k and let I be an ideal in A which contains its right annihilator $r(I)$. Then the distinguished abelian subcategory $\text{mod}(A/I)$ is functorially finite in $\underline{\text{mod}}(A)$ (cf. [2, §3], [3]). Indeed, let U be an A -module. Then U/IU and the annihilator U_I of I in U are in $\text{mod}(A/I)$. The canonical map $U \rightarrow U/IU$, regarded as a morphism in $\underline{\text{mod}}(A)$, is a left approximation of U , and the inclusion $U_I \rightarrow U$, again regarded as a morphism in $\underline{\text{mod}}(A)$, is a right approximation of U . (Of course, $\text{mod}(A/I)$ is also functorially finite in $\text{mod}(A)$, by the same argument.)

The Tensor-Hom adjunction induces a natural transformation of bifunctors at the level of stable categories, but this need not be an isomorphism (cf. Proposition 5.7).

Lemma 9.12 *Let A be a finite-dimensional selfinjective k -algebra, and let D be a finite-dimensional k -algebra. Let Y be a finitely generated A - D -bimodule, U a finitely generated A -module and V a finitely generated D -module. The Tensor-Hom adjunction*

$$\Psi : \text{Hom}_A(Y \otimes_D V, U) \cong \text{Hom}_D(V, \text{Hom}_A(Y, U))$$

sends $\text{Hom}_A^{\text{pr}}(Y \otimes_D V, U)$ to $\text{Hom}_D(V, \text{Hom}_A^{\text{pr}}(Y, U))$ and induces a natural map

$$\underline{\Psi} : \underline{\text{Hom}}_A(Y \otimes_D V, U) \rightarrow \text{Hom}_D(V, \underline{\text{Hom}}_A(Y, U)) .$$

Proof We need to show that Ψ sends $\text{Hom}_A^{\text{pr}}(Y \otimes_D V, U)$ to $\text{Hom}_D(V, \text{Hom}_A^{\text{pr}}(Y, U))$. Let $\pi : P \rightarrow U$ be a projective cover of U . Then any A -homomorphism ending at U which factors through a projective A -module factors through π . Let $\varphi : Y \otimes_D V \rightarrow U$ be an A -homomorphism which factors through π . That is, there is an A -homomorphism $\alpha : Y \otimes_D V \rightarrow P$ such that $\varphi = \pi \circ \alpha$. For $v \in V$, denote by $\alpha_v : Y \rightarrow P$ the A -homomorphism defined by $\alpha_v(y) = \alpha(y \otimes v)$, for all $y \in Y$. Through the adjunction Ψ , the homomorphism φ corresponds to the map $v \mapsto (y \mapsto \varphi(y \otimes v))$ in $\text{Hom}_D(V, \text{Hom}_A(Y, U))$. Now $\varphi(y \otimes v) = \pi(\alpha(y \otimes v)) = \pi(\alpha_v(y))$, hence the map $y \mapsto (\varphi(y \otimes v))$ is equal to $\pi \circ \alpha_v$, hence belongs to $\text{Hom}_A^{\text{pr}}(Y, U)$. This shows that Ψ induces a map $\underline{\Psi}$ as stated \square

Remark 9.13 The map $\underline{\Psi}$ in Lemma 9.12 need not be an isomorphism. Consider for instance the case $D = A$ and $Y = A$ regarded as an A - A -bimodule. Then Φ is the canonical functor $\text{mod}(A) \rightarrow \underline{\text{mod}}(A)$. The map $\underline{\Psi}$ is zero for all U, V (because Y is projective as a left A -module), but if $U = V$ is nonprojective, then the left side in the map $\underline{\Psi}$ is $\underline{\text{End}}_A(U)$, hence nonzero.

Remark 9.14 Let \mathcal{D} be a distinguished abelian subcategory in a triangulated category \mathcal{C} , and let $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$ be a short exact sequence in \mathcal{D} . If the associated exact triangle $X \rightarrow Y \rightarrow Z \rightarrow \Sigma(X)$ in \mathcal{C} is an Auslander–Reiten triangle, then the short exact sequence above clearly is an Auslander–Reiten sequence in \mathcal{D} . The converse need not hold.

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References

1. Alperin, J.L.: Static modules and non-normal Clifford theory. *J. Aust. Math. Soc. (Ser. A)* **49**, 347–353 (1990)
2. Auslander, M., Smalø, S.O.: Preprojective modules over Artin algebras. *J. Algebra* **66**, 61–122 (1980)
3. Auslander, M., Smalø, S.O.: Almost split sequences in subcategories. *J. Algebra* **69**, 426–454 (1981)
4. Balmer, P., Schlichting, M.: Idempotent completion of triangulated categories. *J. Algebra* **236**, 819–834 (2001)
5. Beilinson, A.A., Bernstein, J., Deligne, P.: Faisceaux pervers. *Astérisque* **100** (1982) (Proceedings of the Conference 'Analysis and Topology of Singular Spaces', Vol. 1, Luminy, 1981)
6. Benson, D.J.: Representations and Cohomology, Vol. I: Cohomology of Groups and Modules. Cambridge Studies in Advanced Mathematics 30. Cambridge University Press, Cambridge (1991)
7. Bridgeland, T.: Stability conditions on triangulated categories. *Ann. Math.* **166**, 317–345 (2007)
8. Buchweitz, R.-O.: Maximal Cohen–Macaulay modules and Tate-cohomology. In: With appendices and an introduction by Avramov, L.L., Briggs, B., Iyengar, S.B., Letz, J.C. (eds.) *Mathematical Surveys and Monographs*, vol. 262. American Mathematical Society, Providence, RI (2021)
9. Cabanes, M.: A criterion for complete reducibility and some applications. *Astérisque* **181–182**, 93–112 (1990)
10. Carlson, J.F.: A characterization of endotrivial modules over p -groups. *Manuscr. Math.* **97**, 303–307 (1998)
11. Coelho Simões, R., Pauksztello, D.: Simple-minded systems and reduction for negative Calabi–Yau triangulated categories. *Trans. Am. Math. Soc.* **373**, 2463–2498 (2020)
12. Dugas, A.: Torsion pairs and simple-minded systems in triangulated categories. *Appl. Categ. Struct.* **23**, 507–526 (2015)
13. Dyer, M.J.: Exact subcategories of triangulated categories. <https://www3.nd.edu/~dyer/papers/extri.pdf> (2005)
14. Grimeland, B., Jacobsen, K.M.: Abelian quotients of triangulated categories. *J. Algebra* **439**, 110–133 (2015)
15. Happel, D.: *Triangulated Categories in the Representation Theory of Finite-Dimensional Algebras*. London Mathematical Society. Lecture Notes Series 119, Cambridge University Press, Cambridge (1988)
16. Holm, T., Jørgensen, P., Yang, D.: Sparseness of t -structures and negative Calabi–Yau dimension in triangulated categories generated by a spherical object. *Bull. Lond. Math. Soc.* **45**, 120–130 (2013)
17. Iyama, O., Yoshino, Y.: Mutation in triangulated categories and rigid Cohen–Macaulay modules. *Invent. Math.* **172**, 117–168 (2008)
18. Jørgensen, P.: Abelian subcategories of triangulated categories induced by simple minded systems. *Math. Z.* **301**, 565–592 (2022)
19. Jørgensen, P.: Proper abelian subcategories of triangulated categories and their tilting theory. Preprint [arXiv:2109.01395](https://arxiv.org/abs/2109.01395) (2021)
20. Keller, B., Neeman, A.: The connection between May's axioms for a triangulated tensor product and Happel's description of the derived category of the quiver $D4$. *Doc. Math.* **7**, 535–560 (2002)
21. König, S., Liu, Y.: Simple-minded systems in stable module categories. *Quart. J. Math.* **63**, 653–674 (2012)
22. König, S., Zhou, B.: From triangulated categories to abelian categories: cluster tilting in a general framework. *Math. Z.* **258**, 143–160 (2008)
23. König, S., Liu, Y.: Simple-minded systems in stable module categories. *Quart. J. Math.* **63**, 653–674 (2012)

24. Larson, R.G., Sweedler, M.: An associative orthogonal bilinear form for Hopf algebras. *Am. J. Math.* **91**, 75–94 (1969)
25. Linckelmann, M.: Stable equivalences of Morita type for self-injective algebras and p -groups. *Math. Z.* **223**, 87–100 (1996)
26. Linckelmann, M.: Tate duality and transfer in Hochschild cohomology. *J. Pure Appl. Algebra* **217**, 2387–2399 (2013)
27. Linckelmann, M.: *The Block Theory of Finite Group Algebras I*. London Mathematical Society Student Texts 91, Cambridge University Press, Cambridge (2018)
28. Linckelmann, M.: *The Block Theory of Finite Group Algebras II*. London Mathematical Society Student Texts 92, Cambridge University Press, Cambridge (2018)
29. Martínez-Villa, R.: Properties that are left invariant under stable equivalence. *Commun. Algebra* **18**, 4141–4169 (1991)
30. Nakayama, T.: On Frobeniusean algebras I. *Ann. Math.* **40**, 611–633 (1939)
31. Nakayama, T.: On Frobeniusean algebras II. *Ann. Math.* **42**, 1–21 (1941)
32. Nauman, S.K.: Static modules and stable Clifford theory. *J. Algebra* **128**, 497–509 (1990)
33. Neeman, A.: *Triangulated Categories*, Annals of Mathematics Studies 148. Princeton University Press, Princeton (2001)
34. Pogorzały, Z.: On the stable Grothendieck groups. *Can. Math. Soc. Conf. Proc.* **14**, 393–406 (1993)
35. Pogorzały, Z.: Algebras stably equivalent to self-injective special biserial algebras. *Commun. Algebra* **22**(4), 1127–1160 (1994)
36. Rickard, J.: Derived categories and stable equivalence. *J. Pure Appl. Algebra* **61**, 303–317 (1989)
37. Rickard, J., Rouquier, R.: Stable categories and reconstruction. *J. Algebra* **475**, 287–307 (2017)
38. Skowróński, A., Yamagata, K.: *Frobenius Algebras I*. Textbooks in Mathematics. European Mathematical Society, Zürich (2012)
39. Stenström, B.: *Rings of quotients*. Die Grundlehren der Mathematischen Wissenschaften, Band 217. An Introduction to Methods of Ring Theory. Springer, New York (1975)
40. Tachikawa, H., Wakamatsu, T.: Cartan matrices and Grothendieck groups of stable categories. *J. Algebra* **144**, 390–398 (1991)

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