Is the VIX just volatility? The devil is in the (de)tails

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The year 2023 was a year of anniversaries interestingly connected to each other: 50 years since the publication of the celebrated Black and Scholes (1973) option pricing formula, 30 years since the introduction of the VIX index, and 20 years since the release of its new design (see Cboe, 2023, for details).

The Black-Scholes formula postulates constant volatility; however, as this assumption is clearly not satisfied in real financial markets, traders delta hedging their option books still have to come to terms with the volatility risk that the hedge itself is exposed to. The initial version of the VIX index that the CBOE introduced in 1993 aimed at measuring an average Black-Scholes 30 days ahead implied volatility extracted from ATM options on the S&P100 index. In the attempt to transform the VIX into a potential instrument for trading and hedging volatility, in 2003 the focus has been shifted on OTM options on the S&P500. The claim in the Cboe (2023) White Paper is in fact to provide an 'estimate of the expected volatility' of a weighted portfolio of these OTM options. An interesting account of the changes and the motivations behind is offered by Carr and Wu (2006).

However, this redesign poses the question: is the VIX about volatility only? As the saying goes, the devil is in the details.

In its current definition, the VIX index is given by

$$\bar{V}(0, \Delta_T) = 100 \times \sqrt{\frac{2}{\Delta_T} e^{r\Delta_T} \sum_i \frac{\Delta K_i}{K_i^2} O(K_i) - \frac{1}{\Delta_T} \left(\frac{F_S(0, \Delta_T)}{K_0} - 1 \right)^2},$$

where r is the risk free interest rate to expiration, $O(K_i)$ is the mid price of OTM call and put options on the S&P500 with strike K_i and time to maturity Δ_T fixed at 30 days, $F_S(0, \Delta_T)$ is the forward index level derived from index option prices, K_0 is the largest available strike below or equal to the forward index level, and ΔK_i is the interval between strikes computed as $(K_{i+1} - K_{i-1})/2$. As the VIX by construction is based on market quotes of options, its calculation does not depend on any model.

In the last 20 years though the VIX market has developed significantly with the introduction of VIX futures and VIX options, which nowadays are sought after for hedging purposes, and are therefore very liquid. These developments meant amongst other things that the VIX cannot be model independent any longer: the need of a consistent model for the pricing of these futures

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and options implies that a suitable model should be deployed. This latter task becomes more challenging once we realise that the VIX is a derivative in its own right, as its value depends intrinsically on the S&P500 index.

Attention has been devoted over the last few years to the so-called joint calibration problem, which indicates precisely the problem of capturing with one model, and in a consistent manner the implied volatilities of options on the S&P500 index and the VIX index, and the price of VIX futures, see for example Gatheral et al. (2020), Guyon (2020), Guyon and Lekeufack (2023), Abi Jaber et al. (2023) and Ballotta et al. (2023) just to mention a few of the contributions to this topic.

The fact that the VIX is a derivative on the S&P500 index (rather than on the options on the S&P500 index) is shown clearly by means of the log-contract and an application of the static replication formula, which follows from the Breeden and Litzenberger (1978) framework (see also Carr and Madan, 2001, and references therein). In the case of a contingent claim with payoff function $g(S(T))$, this formula reads

$$\mathbb{E}\left(e^{-rT}g(S(T))\right) = g(F_S(0,T))e^{-rT} + \int_0^{F_S(0,T)} g''(K)P(K)dK + \int_{F_S(0,T)}^{\infty} g''(K)C(K)dK, \quad (1)$$

with $P(K)$ and $C(K)$ denoting the prices of put and call options respectively. An application of equation (1) to the log-contract with payoff at maturity T given by $g(S(T)) = \ln\left(S(T)/F_S(0,T)\right)$ returns the so-called log-strip formula

$$\mathbb{E}\left(e^{-rT}\ln\frac{S(T)}{F_S(0,T)}\right) = -\int_0^{F_S(0,T)} \frac{P(K)}{K^2}dK - \int_{F_S(0,T)}^{\infty} \frac{C(K)}{K^2}dK.$$

A few steps of maths (involving integration, the put-call parity and the Taylor expansion up to the second order of the log function, see for example Ballotta et al., 2023), and the recognition that a continuum of strikes is in reality not available in the market lead to expressing the price of the log-contract as

$$\mathbb{E}\left(e^{-rT}\ln\frac{S(T)}{F_S(0,T)}\right) \simeq -\sum_i \frac{O(K_i)}{K_i^2}\Delta K_i + \frac{e^{-rT}}{2}\left(\frac{F_S(0,T)}{K_0} - 1\right)^2. \quad (2)$$

By comparing equation (2) with the Cboe definition of the VIX index, we can deduce that

$$\bar{V}(0,\Delta_T) = 100 \times \sqrt{-\frac{2}{\Delta_T}\mathbb{E}\left(\ln\frac{S(\Delta_T)}{F_S(0,\Delta_T)}\right)}, \quad (3)$$

where it is understood that the equality holds up to the approximations highlighted above.

In the Black-Scholes model, the stock price follows a geometric Brownian motion, consequently the straightforward application of equation (3) returns

$$\bar{V}(0,\Delta_T) = 100 \times \sigma,$$

in other words, the VIX becomes the volatility of the log-returns σ, which could be extracted of course from option prices. However, as in the Black-Scholes model the implied volatility would be the same across all strikes, the information provided by OTM contracts would be lost.

And what would this information be? By their nature, option prices could be interpreted as the financial quantification of the probability mass in the tails of the distribution of the stock price, or equivalently of the log-returns. In other words, one might say that OTM calls and puts offer a view on the skewness and excess kurtosis of the underlying distribution; these are also the statistics which are reflected in the smile/smirk shape of the implied volatility, as illustrated for example by Ballotta (2023).

In a more general setting based on a Lévy process $L(t)$, the stock price under the risk neutral measure is

$$S(t) = S(0)\ exp\Big((r - \varphi(-\mathrm{i}))\, t + L(t) \Big)$$

with r representing the risk free rate of interest, and $\varphi(u)$ denoting the characteristic exponent of $L(t)$, i.e. $\mathbb{E}\left[exp\left(\mathrm{i}uL(t)\right)\right] = exp\left(\varphi(u)t\right)$. The resulting value of the VIX based on equation (3) is

$$\bar{V}(0, \Delta_T) = 100 \times \sqrt{2\left(\varphi(-\mathrm{i}) - \mathbb{E}\left(L(1)\right)\right)}. \tag{4}$$

(We note that the case of the formula under the Black-Scholes model can be recovered from equation (4) as a special case in which $L(t)$ is a standard Brownian motion rescaled by a volatility parameter σ, so that $\varphi(u) = -\sigma^2 u^2/2$.)

Under necessary integrability conditions which are satisfied by all Lévy processes commonly used in finance, the Lévy-Khintchine formula implies that

$$\varphi(u) = \mathrm{i}u\mathbb{E}\left(L(1)\right) - \frac{u^2}{2}\sigma^2 + \int_{\mathbb{R}} \left(e^{\mathrm{i}ux} - 1 - \mathrm{i}ux\right)\nu(dx), \tag{5}$$

in which σ is as usual the diffusion coefficient, i.e. the rescaling constant of the standard Brownian motion, and $\nu(\cdot)$ is the Lévy measure governing the discontinuous part of the process $L(t)$ (see Eberlein and Kallsen, 2019, Chap. 2, for full details). This component in particular provides the 'DNA' of the discontinuous part of $L(t)$, telling us whether this is Hyperbolic (Eberlein and Keller, 1995), Normal inverse Gaussian (Barndorff-Nielsen, 1997), Variance Gamma (Madan et al., 1998), or anything else. This component is also responsible for the generation of skewness and excess kurtosis in the distribution, as the Brownian motion has zero higher order cumulants $c_n(\cdot)$, $n > 2$. Indeed, repeated differentiation of the characteristic exponent shows that

$$\begin{aligned} \mathbb{V}ar(L(1)) &= \sigma^2 + \int_{\mathbb{R}} x^2 \nu(dx), \\ c_3(L(1)) &= \int_{\mathbb{R}} x^3 \nu(dx), \\ c_4(L(1)) &= \int_{\mathbb{R}} x^4 \nu(dx), \end{aligned}$$

from which it follows that the index of skewness is

$$sk(L(1)) = \frac{c_3(L(1))}{\mathbb{V}ar(L(1))^{3/2}},$$

and the index of excess kurtosis is

$$ek(L(1)) = \frac{c_4(L(1))}{\mathbb{V}ar(L(1))^2}.$$

An appropriate substitution of equation (5) into equation (4) returns

$$\bar{V}(0,\Delta_T) = 100 \times \sqrt{\sigma^2 + 2\int_{\mathbb{R}} (e^x - 1 - x)\,\nu(dx)}.$$

In virtue of the Taylor expansion of the exponential function, the above is equivalent to

$$\bar{V}(0,\Delta_T) = 100 \times \sqrt{\mathbb{V}ar(L(1)) + \frac{1}{3}c_3(L(1)) + \frac{1}{12}c_4(L(1)) + \cdots},$$

which shows that the VIX is much more than just (implied) volatility: it also contains higher order cumulants, and therefore (albeit indirectly) skewness and excess kurtosis.

In light of the above, the following considerations are in order. Firstly, the label 'fear index' attributed to the VIX stands to reason: fear - in the financial market - cannot be just volatility. Fear has also to do with high probabilities of significant market movements in the 'wrong' direction. But this is another way of speaking of tails of the distributions.

Secondly, it is common in the literature to associate the square of the VIX with the variance swap rate up to an 'error term', or better a correction, for the jumps (see Carr and Wu, 2006 and Carr and Wu, 2009). However, according to the above derivation, this correction for the jumps is not an error term, and it is not about 'jumps' per se. It is really all about the distribution, tails of the distribution to be precise.

So, do these higher order cumulants matter? To illustrate the point, we consider the case of the CGMY model of Carr et al. (2002) calibrated to market quotes of options on the S&P500 with 30 days to maturity. The CGMY model has characteristic exponent

$$\varphi(u) = C\Gamma(-Y)((G + \mathrm{i}u)^Y - G^Y + (M - \mathrm{i}u)^Y - M^Y),$$

from which it follows that

$$\begin{aligned}\mathbb{V}ar(L(t)) &= C\Gamma(2-Y)(M^{Y-2} + G^{Y-2})t,\\ c_3(L(t)) &= C\Gamma(3-Y)(M^{Y-3} - G^{Y-3})t,\\ c_4(L(t)) &= C\Gamma(4-Y)(M^{Y-4} + G^{Y-4})t.\end{aligned}$$

For this example, we consider two observation dates reflecting two very different periods in the financial markets. The first date is March 18th 2020, when the VIX reached its historical high at 85.47 (and closed at 76.45) reflecting the significant uncertainty which followed the announcement on March 11th 2020 in which WHO declared the coronavirus (COVID-19) outbreak a global pandemic. The second date is June 21st 2023 and represents a period of relatively low values for the VIX: on this date in particular the VIX reached a high at 13.89 and closed at 13.20. The usual filters have been applied to the dataset, as well as SVI (see Gatheral and Jacquier, 2014, for full details).

We also calibrate the Black-Scholes model to the same dataset. The resulting implied volatilities are illustrated in Figure 1. Please note that the $y-$axis scale in the top two plots is very different. The calibrated parameters are reported in Table 1, together with the corresponding implied volatility root mean squared error (IVRMSE), average absolute error in prices (APE),

Figure 1: Implied volatilities from the CGMY model and the Black-Scholes model. Top panels: implied volatilities extracted from options with 30 days to maturity on the S&P500 observed on 18/03/2020 and 21/06/2023 (source: optionsdx). Bottom panels: CGMY model calibration errors.

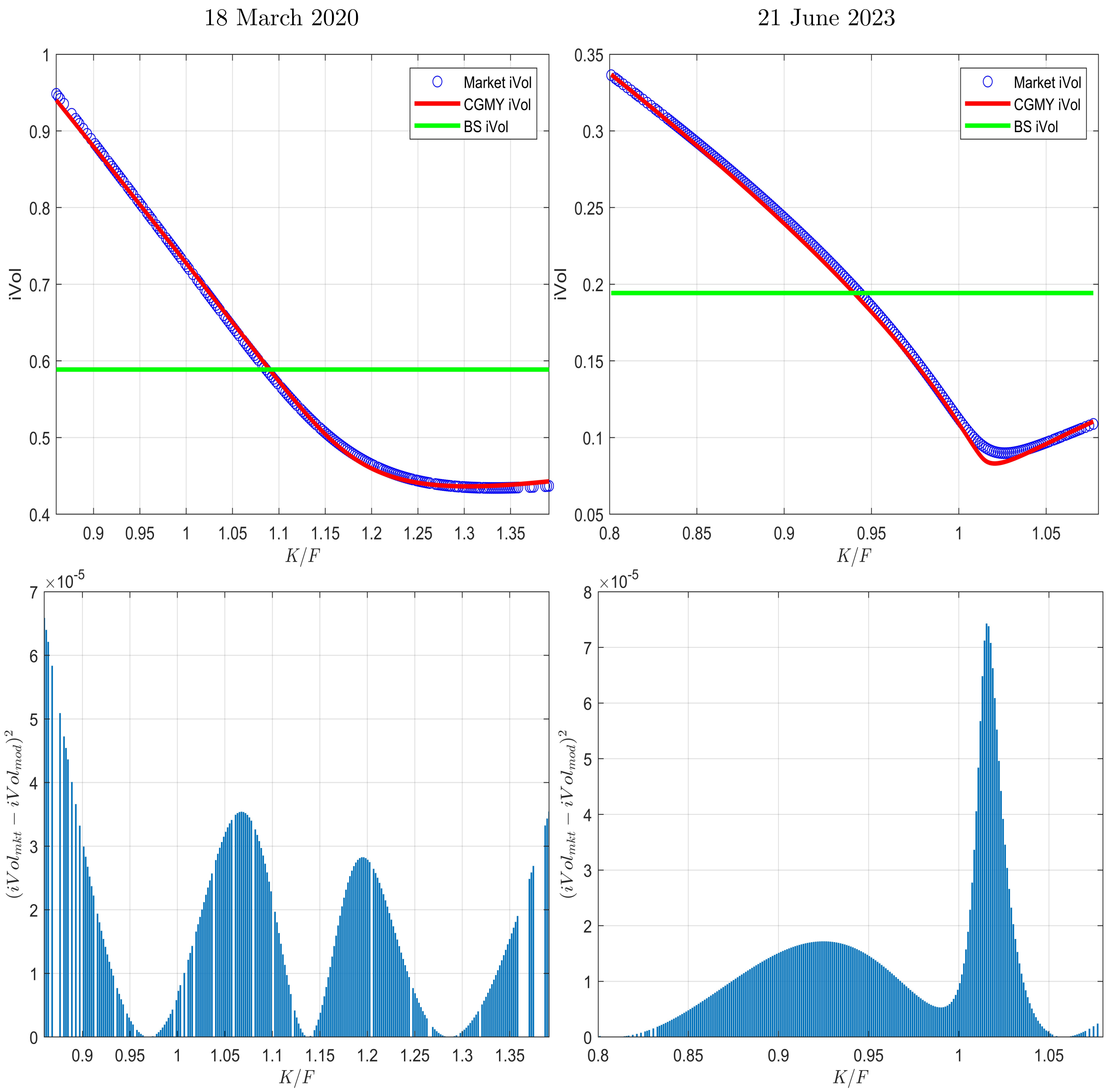

Table 1: Calibration of the CGMY model and the Black-Scholes model to options with 30 days to maturity on the S&P500 observed on 18/03/2020 and 21/06/2023 (source: optionsdx). Calibrated model parameters, statistics of the calibrated distributions and value of the VIX from the log-contract (equation (4)).

18/03/2020 – VIX close price: 76.45; high price: 85.47										
	Calibrated parameters				IV RMSE	APE Price	$std.dev.(L(1))$	$sk(L(1))$	$ek(L(1))$	VIX formula
(C, G, M, Y)	4.7303	2.3336	19.0884	0.1987	0.0040	0.0236	0.9897	-0.7605	0.9253	88.8666
σ	0.5887				0.1590	1.7254	0.5887	0.0000	0.0000	58.8739
21/03/2023 – VIX close price: 13.20; high price: 13.89										
	Calibrated parameters				IV RMSE	APE Price	$std.dev.(L(1))$	$sk(L(1))$	$ek(L(1))$	VIX formula
(C, G, M, Y)	0.9056	9.4954	41.5846	0.3669	0.0035	0.0627	0.1498	-1.0322	1.9600	14.6139
σ	0.1943				0.0797	2.9699	0.1943	0.0000	0.0000	19.4276

standard deviation, skewness and excess kurtosis of the calibrated (unit-time) distributions. Finally, we also report the values of the VIX returned by formula (4).

The value of the VIX generated by equation (4) for the CGMY model is not the same as the market quote, which is also a reflection of the different methodology with which the VIX is computed in reality (see the Cboe, 2023, White Paper for full details); nevertheless it is considerably closer than the estimate that the Black-Scholes model can generate. By looking at the calibrated value of the standard deviation, we can appreciate the contribution of the higher order cumulants in 'correcting' the square root of the log-contract (rescaled) price towards the quoted value of the VIX index. Thus, higher order cumulants do matter as they capture the smirk of the market implied volatility, especially the ATM skew.

The above example highlights a few points for consideration. The first one concerns the approximation in equation (3). As previously pointed out, the practical use of this result is to connect the VIX directly to the S&P500 index as a way of providing a grounded platform for the pricing of VIX derivatives. The validity of equation (3) depends on the calibrated model for the underlying, as well as the data used for the purpose. As the previous example shows, an excellent calibration performance might not lead to an exact match to the actual market value of the index. It is worth noticing that in this experiment we have calibrated to the same dataset also other models, such as the Variance Gamma, the Normal inverse Gaussian, the Hyperbolic and the Heston (1993) model, with very similar results. As mentioned above the discrepancy in values is due to the different methodology, and of course the different dataset. In addition, we have to bear in mind that equation (3) holds up to a few approximations.

Finally, admittedly the CGMY and the Black-Scholes models are relatively simple, especially the Black-Scholes one, as they do not allow for a dynamic of the VIX index which is sufficiently sophisticated for the pricing of VIX derivatives. This of course can be amended by moving to more realistic processes which allow for the S&P500/VIX joint calibration, such as for example time changed Lévy processes investigated in Ballotta et al. (2023).

What we learn from the analysis in this note is that the conditional dynamic of these processes needs to contain a non-Gaussian component capable of capturing skewness and excess kurtosis. The 'jump-induced error term' (Carr and Wu, 2006) is not an error term and is not about jumps: it speaks of distributions and their tails. Indeed, it is what lies in these tails that the market has to fear.

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