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SOLVENCY ANALYSIS OF DEFERRED ANNUITIES

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Abstract. While the solvency analysis of immediate life annuity portfolios has been extensively studied, the case of deferred annuities has received comparatively much less attention. We assess the importance and effect of stochastic mortality models and interest rates on the solvency analysis of a portfolio of deferred annuity contracts. Our analysis considers three steps: first, the benchmark case where mortality rates and interest rates are both deterministic; then, the case in which only mortality rates are stochastic is explored; finally, the full model where both mortality rates and interest rates are stochastic. The results demonstrate the model risk stemming from the uncertainty in the mortality models and its impact on the evaluation of solvency margins for life annuities. The role of the deferment period is thoroughly discussed and compared to the case of immediate annuities.

1. Introduction

Annuity products play a significant role in protecting policyholders from longevity risk, providing a minimum guaranteed income and reducing the risk of outliving their retirement assets. Regulators therefore have a strong

Key words and phrases. Deferred annuities, solvency, reserves, longevity risk, interest rate risk.

incentive to achieve a deep understanding of the solvability of annuity portfolios and the impact of systematic longevity risk. While immediate life annuities have been studied extensively, deferred annuities products play an equally important role as they are commonly sold, being relatively more affordable. The solvency of deferred annuities portfolios has not been considered thoroughly in the literature, and this paper aims at filling this gap.

Yaari (1965) demonstrates that a risk averse individual should convert all of his/her assets to an immediate annuity due to the fact that people who die earlier subsidise those who live longer. However, the evidence shows that retirees are reluctant to convert retirement savings into annuities voluntarily. Many theories have been brought forward to explain this so-called "annuity puzzle". As a result of the discounting effect and the possibility that annuitants may not survive the deferment period, a deferred annuity is much cheaper than an immediate annuity with identical benefits and therefore plays an important role in households' portfolios. Hu and Scott (2007) apply cumulative prospect theory to the analysis of annuities and demonstrate that it can explain the low demand for immediate annuities purchased at retirement and higher preference for deferred annuities. Chen et al. (2019) extend this analysis and show that immediate annuities are not attractive for retirees at all ages due to loss aversion and preferences for deferred annuities increase with the deferred period. Moreover, Chen et al. (2020) demonstrates that a hyperbolic discount model can explain the low demand of immediate annuities at retirement and greater attractiveness of long-term deferred annuities.

In this paper we analyse the solvency capital of deferred annuities portfolios using different assumptions and scenarios for longevity and investment risk. Hypotheses about underlying mortality and interest rate dynamics play an important role in the solvency assessment of life annuity contracts. By comparing the results for deterministic and stochastic mortality models we estimate the importance and effect of random changes in the mortality for solvency analysis. The systematic risk component is analysed in Coppola et al. (2000), Coppola et al. (2002), Olivieri (2001), and Pitacco (2004). Olivieri and Pitacco (2003) use a finite range of mortality scenarios and differentiate between pooling and non-pooling risks. In contrast, De Waegenaere et al. (2010) examine pension annuities in a generalized Lee-Carter model. The stochastic Gompertz-model is used in Christiansen and Helwich (2008) both for temporary life and pure endowment insurance. Bauer and Weber (2008) and Hari et al. (2008) calculate VaR and expected shortfall for immediate life annuities using different static hedging scenarios. Jarner and Møller (2015) consider a partial internal model for longevity risk, focusing in particular on the role of the unsystematic element related to the size of the portfolio. Gylys and Šiaulys (2019) compare, for a portfolio of term assurances, the run-off and 1-year Value-at-Risk under a Lee-Carter model to the results of the Solvency II standard formula

Several stochastic mortality models have been proposed in the literature, see Cairns et al. (2009) for a review of some of the most commonly used ones. The choice of a model requires significant judgement by the analyst and a change in the model can significantly impact solvency margins and the best estimate reserves (Richards and Currie, 2009). Therefore, model risk is particularly important, and its assessment requires the evaluation of many stochastic mortality framework.

We extend the solvency analysis approach of Olivieri and Pitacco (2003) to assess solvency margins for deferred annuities for a range of deferment periods. Deferred life annuities are policies providing life-long payments to the annuitants after a certain period of time, conditional on survival. Therefore, exposure to longevity risk and interest rate risk is an important factor due to the long term nature of the contract and the possibly long duration of contribution and benefit payments. Therefore, we aim at contributing to the existing literature on deferred annuities by estimating the combined effects of stochastic mortality and investment returns on the solvency of a portfolio of deferred annuities. The results are then compared to those obtained under a deterministic counterpart. Essentially, our analysis contains three risk scenarios: the benchmark case where mortality rates and interest rates are both deterministic; a second case where only mortality rates are stochastic; a third case where mortality rates and interest rates are both stochastic. To isolate the effect of these risks, we disregard expenses and inflation although these can be easily introduced in a simulation based approach.

We observe that the risk of random fluctuations in mortality decreases with increasing cohort size because fluctuations are smoothed in large homogeneous portfolios. Our results show the importance of systematic risk in comparison to the risk of random fluctuations. Moreover, the paper outlines the relevance of model risk and deferment period in the estimation of solvency margins.

The structure of this paper is as follows: Section 2 describes the general model for calculation of solvency margins. In Section 3 we review the approach of Olivieri and Pitacco (2003) and our adjustment to deferred annuities of the CMI reduction factor based approach, see (CMI, 2016). Section 4 discusses different stochastic mortality models and calculation of solvency margins for these models with deterministic interest rates, while Section 5 extends the previous analysis to the case of stochastic interest rates. Section 6 concludes the paper.

2. General model for solvency margins

We consider a portfolio of N_0 identical deferred life annuities-due with payments starting at $d \geq 0$ and continuing as long as the insured is alive. Without loss of generality, we can assume that payments will in any case stop after $n = \omega - x_0$ years, where x_0 is the common initial age of the insureds and ω denotes the biological age limit. A single premium P is paid

at contract inception, so that the insurer's premium income at time 0 is $N_0 P$. Insureds are assumed to be homogeneous, so their residual lifetimes can be taken to be identically distributed.

Let N_t denote the random number of individuals alive at time t (that is at age $x_0 + t$), so the random number of deaths in the time interval $[t, t+1)$ is $D_t = N_t - N_{t+1}$. The portfolio reserve at time $t = 0, \ldots, n$ is denoted by $\mathcal{V}_t$ and given by

$$\mathcal{V}_t = N_t V_t,$$

where V_t is the time t individual reserve, calculated according to a given basis. For simplicity, in the following it will be assumed that the reserve and premium basis coincide, so that in particular $V_0 = P$.

The random portfolio fund at time $t = 0, \ldots, n$ is denoted by Z_t and given by the following relations:

$$Z_0 = N_0 P + C \tag{1}$$

$$Z_t = Z_{t-1}(1 + I_t) - \mathcal{R}_t, \quad t = 1, 2, \ldots, n, \tag{2}$$

where I_t is the (possibly random) rate of return earned by the fund over the period $[t-1, t)$, C is the initial capital and $\mathcal{R}_t$ is the portfolio's annuity outflow at time t, defined by

$$\mathcal{R}_t = \begin{cases} 0 & t < d \\ N_t R & t \geq d \end{cases},$$

where R is the annuity instalment.[1] In practice, the initial capital C can be partially financed by an additional premium loading. The inclusion of a safety loading in the premium calculation, combined with a best estimate basis for the reserves, would have the effect of shifting capital requirements for all the mortality models considered eventually in Sections 3 and 4.

The quantity $Z_t - \mathcal{V}_t$ is called *free portfolio fund* at time t. Inspired by the different possible solvency criteria set out in Olivieri and Pitacco (2003), we focus on the minimum amount of initial capital necessary for the free portfolio fund to be positive at a given horizon, with a given confidence level. More precisely, the *solvency margin* or *risk-based capital* RBC[2] is the minimum initial capital C, required at time 0 on top of the premium income, guaranteeing the condition

$$\Pr\{Z_T - \mathcal{V}_T \geq 0\} \geq 1 - \varepsilon, \tag{3}$$

[1] In practice, there will be variability of the annuity instalments across the portfolio. This heterogeneity will result in higher capital requirements to those that will be calculated in the following sections. In Clemente et al. (2022) a model to quantify demographic risk over an annual time horizon, allowing for the variability of the sum assured and consistent with Solvency II, is proposed.

[2] In the rest of the paper we use the term solvency margin not to be confused with the *solvency capital requirement* (SCR), which, in the context of Solvency II, refers to a 1 year horizon capital requirement.

where ε is the ruin probability. This correspond to a Value-at-risk calculation at level $1-\varepsilon$ for the free portfolio fund at the target horizon. Since the premium and reserve basis agree, the initial free fund and the solvency margin coincide:

$$Z_0 - \mathcal{V}_0 = Z_0 - N_0 P = C = \text{RBC}.$$

The *relative solvency margin* is $\text{RBC}_{\%} = \frac{\text{RBC}}{\mathcal{V}_0}$ gives the required solvency capital as a percentage of the initial reserve/premium income.

While the Value-at-Risk has been widely employed in the financial and insurance industry, its use has been frequently challenged and some alternative measures have been proposed, see (McNeil et al., 2015). In particular, the expected shortfall (ES) has found consistent use in the financial sector. The expected shortfall is a coherent risk measure, meaning that if several portfolios are combined, the total expected shortfall decreases, hence reflecting the benefit of diversification. Further, the ES is more conservative than the Value-at-Risk, when calculated with the same ruin probability. In the context of this paper, the expected shortfall is calculated, for a given ruin probability ε, as

$$\text{ES} = E[Z_T - \mathcal{V}_T | Z_T - \mathcal{V}_T \leq \text{RBC}].$$

Similarly, the expected shortfall can be expressed in percentage of the initial reserve/premium income as $\text{ES}_{\%} = \frac{ES}{\mathcal{V}_0}$.

In general, we assume that there are two sources of mortality uncertainty, one due to process risk, stemming from the individual times of death and eliminable through pooling, and another one due to the systematic risk linked to the uncertainty in future death rates. Therefore, we model the evolution of the number of survivors through

$$D_t | N_t, q_{x_0+t,t} \sim \text{Bin}(N_t, q_{x_0+t}),\ t = 0, \ldots, \omega - x_0,$$

where $q_{y,t}$ is the one-year death probability for an insured aged y prevailing at time t. These probabilities are drawn from a (possibly stochastic) best estimate mortality model used for the calculation of solvency margins. Implicit in the above assumption is that insured lifetimes are independent and identically distributed conditional on knowledge of the future death probabilities.

We analyse portfolios of identical immediate annuities, with annual instalment of $R = 100$, paid to policyholders with initial age $x_0 = 65$. Different deferment periods $d = 0$, 5, 10, 20, 30, the case $d = 0$ being that of immediate annuities as in Olivieri and Pitacco (2003), and of solvency horizons $T = 5$, 10, n, where $n = \omega - x_0$, are considered. When $T = n$ is used, a run-off perspective is adopted where solvency is analysed after the cohort of annuitants has become extinct and the free portfolio fund is given by the fund as the reserve is then 0. Unless otherwise specified, the probability of ruin is set at $\varepsilon = 0.025$ and the initial portfolio sizes $N_0 = 10^h$, $2 \leq h \leq 5$ are used. The constant annual interest rate used for the calculation of the

premium and the reserves is $i = 0.03$, while the corresponding life table, together with the best estimate mortality model for the solvency margin calculation, will be specified in the next Section.

We calculate absolute and relative solvency margins (the latter reported simply as 'solvency margin' in the y-axis label in Figures 1-5) for immediate and deferred annuities portfolios under different sets of assumptions. Section 3 considers the case of best estimate mortality rates drawn from a finite set of scenarios, as in Olivieri and Pitacco (2003). Section 4 extends the analysis to a range of full stochastic mortality models. In these Sections the return earned by the fund is deterministic and supposed to coincide with the technical rate used for premium and reserves, $I_t = 0.03$ for all t. This assumption is then dropped in Section 5.

3. Solvency Margins under Scenario Based Mortality Projections

In this section we introduce a simplified scenario-based model used to project future mortality rates and the implications for solvency margins. As in Olivieri and Pitacco (2003), a finite set of scenarios - that is life tables - is considered, together with a subjective "degree of belief" assigned to each one of them. We choose three sets of projected mortality rates, denoted by $q^{[\text{low}]}$, $q^{[\text{avg}]}$ and $q^{[\text{high}]}$ corresponding to a small, medium and high mortality scenario, respectively. When a full degree of belief is attached to $q^{[\text{avg}]}$, the special case of a *deterministic* approach, where only process risk is considered, is retrieved. We choose $q^{[\text{low}]}$, $q^{[\text{avg}]}$ and $q^{[\text{high}]}$ using the mortality projection model developed by the IFoA's CMI (CMI, 2016), as follows. We consider England and Wales males data obtained from the Human Mortality Database (HMD, 2020), containing age-specific numbers of deaths and exposures for the period $1965 - 2011$ and ages $0 - 100$, calculate the corresponding death rates and use quadratic interpolation to close each life table and obtain death rates for the ages $101 - 120$ as well. We then calculate annual mortality improvement rates by single year of age[3] for the period 1965-2011 and observe that the maximum, average and minimum levels of such improvement rates, across all ages, have been 4.16%, 2.34% and 0.16% respectively.[4] Noting that current mortality trends (since 2011) imply that mortality improvement rates in the future are likely to be lower, see Murphy and Grundy (2022), we apply a 20% decrease to the maximum mortality improvement rate. This figure is subjective and acknowledges that, in this scenario, future improvement rates will be higher than experienced in the

[3]The improvement rate for a given age x and period $(t, t+1)$ is given by $\frac{q_{x,t} - q_{x,t+1}}{q_{x,t}}$, where $q_{x,t}$ is the one year death rate for age x prevailing at time t. See Richards et al. (2006), Baxter (2007), Haberman and Renshaw (2012), also for possible alternative definitions of the improvement rates.

[4]In this calculation, negative mortality improvement rates have been excluded.

last decade but still not as high as pre 2010. Finally, the maximum, average and minimum level of improvement rates are used as input for the CMI model to obtain the projected $q^{[\text{low}]}$, $q^{[\text{avg}]}$ and $q^{[\text{high}]}$, respectively. We assign weights $\rho^{[\text{low}]} = 0.2$, $\rho^{[\text{avg}]} = 0.6$ and $\rho^{[\text{high}]} = 0.2$ to the three scenarios. The single premium and the reserves are calculated according to the (deterministic) scenario $q^{[\text{avg}]}$. The choice of the weights is completely subjective and reflects the possible belief of an end user who attach most confidence to the average scenario, but feels that deviations from it resulting into higher or lower death trajectories are likely. Clearly, the numerical examples of this section are strongly dependent on the choice of the weight for each scenario.

The solvency margins using the deterministic mortality model are reported in Table 1-3 for different initial portfolio sizes, solvency horizons and ruin probabilities. The solvency capital obviously increases with the target solvency horizon, reflecting the higher uncertainty in fund results. As far as the size of the portfolio is concerned, Table 1 shows the extent of the pooling effect on the required margin. Recall that, using a deterministic mortality model, the insurance company is only exposed to diversifiable risk, which stems from an insufficiently large cohort. It can be observed that the relative solvency margins $\text{RBC}_{\%}$ are higher for the deferred annuity portfolios than the corresponding values for the immediate annuities, whereas the absolute solvency margins RBC are lower. Clearly, in any state of the world there are less benefit payments over the target period in the case of deferred annuities. Therefore, lower reserves are required to be held. However, the relative solvency margin $\text{RBC}_{\%}$ is higher due to the comparatively greater uncertainty associated with the postponed beginning of the benefit payments. Table 2 investigates the dependence of the solvency margin on the ruin probability, showing the extra capital necessary to satisfy a more stringent solvency criterion. While small portfolios are extremely penalized, the relative capital rapidly stabilize as the size of the portfolio grows to a reasonable size. Finally, Table 3 expands on Table 1 and demonstrates the dependence of the solvency margin on the deferment period. As hinted at before, a higher relative solvency margin is needed when the deferment period is extended. However, $\text{RBC}_{\%}$ is unchanged when the target solvency horizon precedes the annuity payout phase, that is $d \geq T$. This is intuitive, considering that there are no annuity cash flows before solvency is checked, therefore, at contract inception the company needs to hold the same relative solvency margin in order to be solvent.

By using a deterministic life table as a model for best estimate future mortality rates, the insurance company essentially assumes that demographic risk can be diversified away completely by pooling a large number of similar individuals. To include non-pooling risk affecting each individual's mortality in the same manner, a stochastic mortality model is needed.

Table 4 reports relative solvency margins for different combinations of portfolio sizes, solvency target horizons and deferment periods. The required capital now reflects both systematic (longevity) and diversifiable risk, and

TABLE 1. Absolute (in £1 000) and relative solvency margins for immediate ($d = 0$) and 5 years deferred ($d = 5$) annuities under the deterministic mortality model, for different solvency horizons and initial portfolio sizes.

	immediate annuities				deferred annuities			
	$T = n$		$T = 5$		$T = n$		$T = 5$	
N_0	RBC	$\text{RBC}_{\%}$	RBC	$\text{RBC}_{\%}$	RBC	$\text{RBC}_{\%}$	RBC	$\text{RBC}_{\%}$
10^2	10.3	7.47%	6.3	4.55%	9.6	10.21%	4.9	5.24%
10^3	33.0	2.39%	21.3	1.54%	30.7	3.26%	17.4	1.84%
10^4	106.5	0.77%	69.1	0.50%	98.9	1.05%	56.8	0.60%
10^5	334.1	0.24%	21.8	0.16%	31.0	0.33%	18.0	0.19%
10^6	1054.6	0.08%	69.8	0.05%	98.2	0.10%	57.8	0.06%

TABLE 2. Relative solvency margin ($\text{RBC}_{\%}$) for 5 years deferred annuities under the deterministic mortality model, for different solvency horizons, ruin probabilities and initial portfolio sizes.

	$T = n$			$T = 5$		
N_0	$\varepsilon = 0.01$	$\varepsilon = 0.025$	$\varepsilon = 0.05$	$\varepsilon = 0.01$	$\varepsilon = 0.025$	$\varepsilon = 0.05$
10^2	12.12%	10.21%	8.61%	6.34%	5.24%	5.11%
10^3	3.85%	3.26%	2.74%	2.17%	1.84%	1.62%
10^4	1.24%	1.05%	0.88%	0.71%	0.60%	0.51%
10^5	0.39%	0.33%	0.28%	0.23%	0.19%	0.16%
10^6	0.12%	0.10%	0.09%	0.07%	0.06%	0.05%

TABLE 3. Relative solvency margin ($\text{RBC}_{\%}$) for immediate and deferred annuities under the deterministic mortality model, for different solvency horizons and initial portfolio sizes.

	$N_0 = 100$			$N_0 = 10\,000$			$N_0 = 1\,000\,000$		
d	$T = n$	$T = 10$	$T = 5$	$T = n$	$T = 10$	$T = 5$	$T = n$	$T = 10$	$T = 5$
0	7.47%	6.02%	4.55%	0.77%	0.64%	0.50%	0.08%	0.06%	0.05%
5	10.21%	7.95%	5.24%	1.05%	0.83%	0.60%	0.10%	0.08%	0.06%
10	13.60%	8.88%	5.24%	1.39%	0.96%	0.60%	0.14%	0.10%	0.06%
20	18.22%	8.88%	5.24%	1.84%	0.96%	0.60%	0.18%	0.10%	0.06%
30	39.54%	8.88%	5.24%	3.81%	0.96%	0.60%	0.38%	0.10%	0.06%

a comparison with Table 3 provides an insight on the relevance of these components as a function of the deferment period. It can be seen how the contribution of non pooling risk exacerbates as the portfolio size increases, and this effect compounds with the length of the deferment period. In relative terms, the insurer is better off with a smaller portfolio size. Also, the effect of systematic risk is sensibly lower when a run-off perspective is adopted, as solvency is assessed when the cohort of insured is extinct.

4. Solvency margins under fully stochastic mortality models

The scenario based stochastic mortality model in Section 3 is "static" as, one of the scenarios is drawn, the corresponding mortality table is used for

TABLE 4. Relative solvency margins ($\text{RBC}_{\%}$) for immediate and deferred annuities under the scenario based stochastic mortality model, for different solvency horizons and initial portfolio sizes.

	$N_0 = 100$			$N_0 = 10\,000$			$N_0 = 1\,000\,000$		
d	$T = n$	$T = 10$	$T = 5$	$T = n$	$T = 10$	$T = 5$	$T = n$	$T = 10$	$T = 5$
0	8.56%	6.51%	4.75%	4.04%	2.69%	1.53%	3.65%	2.37%	1.29%
5	11.78%	8.58%	5.24%	5.56%	3.57%	1.86%	5.03%	3.16%	1.57%
10	15.72%	10.12%	5.24%	7.31%	4.13%	1.86%	6.61%	3.66%	1.57%
20	21.66%	10.12%	5.24%	8.99%	4.13%	1.86%	8.04%	3.66%	1.57%
30	41.70%	10.12%	5.24%	12.74%	4.13%	1.86%	10.61%	3.66%	1.57%

all future rates. Therefore, there is no allowance for a dynamic evolution of future mortality rates, in particular for changes in mortality trend over time. Also, results depend on the choice of scenario weights, which is inevitably subjective. This section considers a set of stochastic mortality projection models which are fitted on past data and used to generate future patterns of mortality rates. These models are essentially capable of generating infinitely many mortality scenarios. Further, the degree of subjectivity is limited to the model specification and its forecasting component. Nonetheless each model has its specific features and assumptions, and will generate different solvency capital requirements. The results in this section are also useful in order to understand the model risk implicit in such an exercise, therefore providing an invaluable yardstick for the range of solvency capital required to an insurance company.

Consistently with the choice made in Section 2, the number of deaths at age x and calendar year t in the general population is considered to follow a Binomial distribution:

$$D_{xt} \sim \text{Bin}(E^0_{xt}, q_{x,t})$$

where $q_{x,t}$ is the one year death rate for age x prevailing at time t and $E^0_{x,t}$ is the population matching initial exposed to risk. Following (Villegas et al., 2018), the death rates are modelled through

$$\log \frac{q_{x,t}}{1 - q_{x,t}} = \eta_{xt},$$

where η_{xt} is the predictor which is then characterized by a given combination of age and time specific terms and/or cohort terms. For the purpose of this paper we use six representative models having distinct features such as the presence of a cohort term, the number of time indices and age modulating terms and whether age is treated as a quantitative or categorical variable. The models and corresponding equations are listed below.

- Lee-Carter (LC) model, see Lee and Carter (1992):

$$\eta_{xt} = \alpha_x + \beta^{(1)}_x k^{(1)}_t.$$

- Renshaw-Haberman (RH) model, see Renshaw and Haberman (2006), Haberman and Renshaw (2011), which generalises the LC model by

incorporating a cohort effect:

$$\eta_{xt} = \alpha_x + \beta_x^{(1)} k_t^{(1)} + \gamma_{t-x}.$$

- Age-period-cohort (APC) model, which is a special case of the RH model when $\beta_x^{(1)} = 1$:

$$\eta_{xt} = \alpha_x + k_t^{(1)} + \gamma_{t-x}.$$

- CBD model, see Cairns et al. (2006), with two age-period terms and pre-specified age-modulating parameters:

$$\eta_{xt} = k_t^{(1)} + (x - \bar{x}) k_t^{(2)},$$

where $\bar{x}$ is the average age in the data set.

- M7 model, see Cairns et al. (2009), which expands the CBD model by adding a quadratic term and a cohort effects:

$$\eta_{xt} = k_t^{(1)} + (x - \bar{x}) k_t^{(2)} + ((x - \bar{x})^2 - \hat{\sigma}_x^2) k_t^{(3)} + \gamma_{t-x},$$

where $\hat{\sigma}_x^2$ is the average value of $(x - \bar{x})^2$.

- Plat model, see Plat (2009), in which features of the LC and CBD models are combined:

$$\eta_{xt} = \alpha_x + k_t^{(1)} + (x - \bar{x}) k_t^{(2)} + (x - \bar{x})^+ k_t^{(3)} + \gamma_{t-x},$$

where $z^+ = \max\{0, z\}$.

Some of the models listed above have been presented under slightly different assumptions - for instance the number of deaths is frequently modelled using a Poisson distribution rather than Binomial. Also, several alternative models could be employed, but the structures considered here are a representative sample as they include some key features, see Table 5 for a summary and (Villegas et al., 2018) for a general approach to the construction, calibration and projection of stochastic mortality models.

TABLE 5. The stochastic mortality models considered in the paper classified according to the number of time factors, the presence of a cohort term, whether the age terms are categorical (c) or quantitative (q) and if they can be used to fit the entire age range.

model	time factors	cohort	age term	age range
LC	1	—	c	✓
RH	1	✓	c	✓
APC	1	✓	c	✓
CBD	2	—	q	—
M7	3	✓	q	—
Plat	3	✓	c/q	✓

Once models are estimated with maximum likelihood, the time indices and cohort term are fitted and projected into the future so that simulation of future mortality rates can be obtained. Consistently with the prevailing

literature, see (Villegas et al., 2018), the time indexes are projected using a multivariate random walk with drift while the cohort term, when present, is modelled as an ARIMA(1,1,0), independently of the time indexes.

Figure 1 shows relative solvency margins, with associated 95% confidence intervals, for the 6 stochastic mortality models and different portfolio sizes. It can be seen that solvency margins decrease with the portfolio size, converging to an asymptotic value reflecting the non-pooling effect of longevity risk. Table 6 presents the relative required margins for all the models, different portfolio sizes and deferment periods when $T = n$. For each combination of contract parameters, it can be seen how the required capital varies through the range of models, exemplifying the model risk. For a longer deferment period, the range of required capital remains wide regardless of the portfolio size. This is expanded in Figure 2, where relative solvency margins are plotted against the deferment period, showing how model risk widens with d. It can be seen that models M7 and CBD imply higher solvency margins, whereas the LC model has consistently the lowest values.

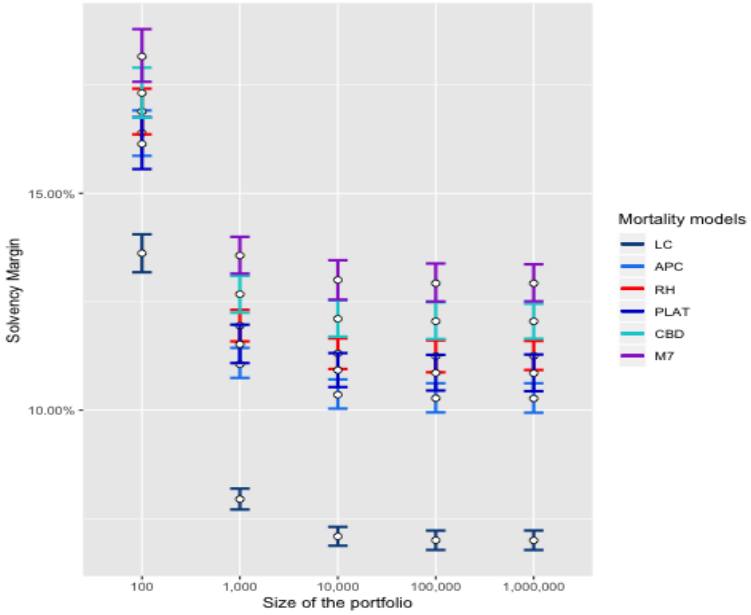

FIGURE 1. Relative solvency margins ($\text{RBC}_{\%}$) for different stochastic mortality models and associated 95% confidence intervals, 5 years deferment period and $T = n$.

Tables 7, 8, 9, 10, 11 and 12 provide relative solvency margins and expected shortfalls for different target solvency horizons and deferment periods for LC, RH, APC, CBD, M7 and Plat, respectively. For each model, the solvency margin increases with the target horizon due to the greater uncertainty. For a given choice of contract parameters, there is a remarkable spread between the results obtained in Section 3 with the deterministic and

	$d = 0$					
N_0	LC	APC	RH	PLAT	CBD	M7
10^2	9.79%	11.92%	12.45%	11.34%	12.25%	12.84%
10^3	5.59%	7.88%	8.64%	7.89%	8.77%	9.40%
10^4	4.93%	7.34%	8.13%	7.44%	8.34%	8.97%
10^5	4.86%	7.27%	8.08%	7.39%	8.29%	8.91%
10^6	4.86%	7.27%	8.08%	7.39%	8.29%	8.91%

	$d = 5$					
N_0	LC	APC	RH	PLAT	CBD	M7
10^2	9.79%	16.39%	12.45%	16.14%	17.31%	18.16%
10^3	7.95%	11.06%	11.95%	11.52%	12.67%	13.57%
10^4	7.09%	10.36%	11.31%	10.92%	12.11%	13.00%
10^5	7.00%	10.28%	11.24%	10.86%	12.05%	12.93%
10^6	7.00%	10.27%	11.25%	10.85%	12.05%	12.92%

	$d = 10$					
N_0	LC	APC	RH	PLAT	CBD	M7
10^2	18.84%	22.44%	22.77%	23.57%	25.06%	26.25%
10^3	11.42%	15.72%	16.72%	17.70%	19.19%	20.46%
10^4	10.35%	14.86%	15.97%	16.97%	18.50%	19.78%
10^5	10.24%	14.77%	15.90%	16.89%	18.43%	19.68%
10^6	10.24%	14.76%	15.91%	16.88%	18.43%	19.68%

TABLE 6. Relative solvency margins $\text{RBC}_{\%}$ for different mortality models and deferred period when $T = n$.

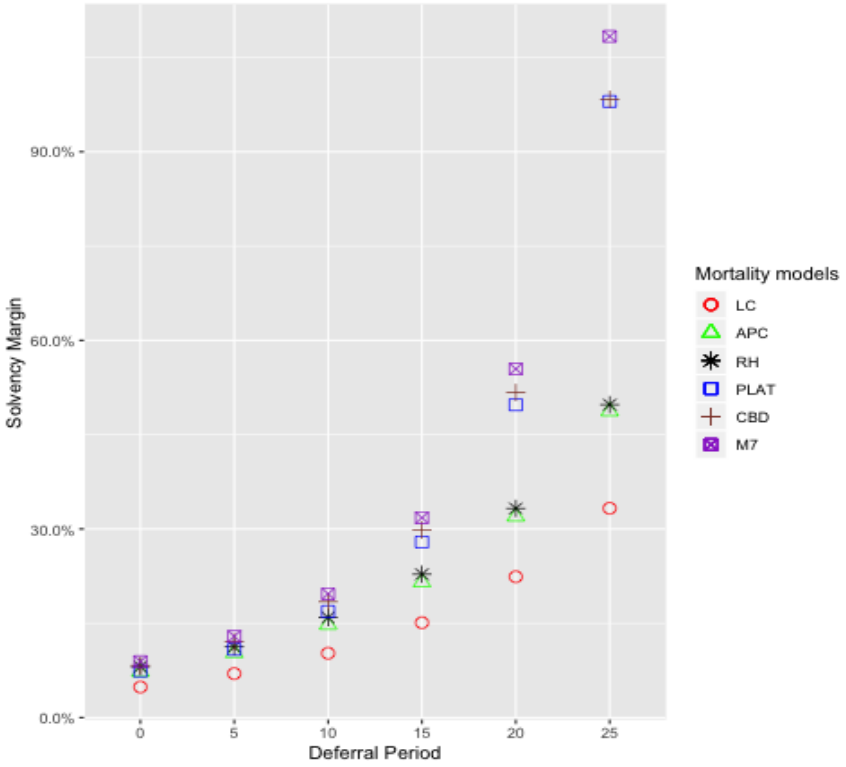

FIGURE 2. Relative solvency margins ($\text{RBC}_{\%}$) for different stochastic mortality models, deferment periods and $T = n$.

scenario mortality model and the results in the present section. These differences also vary depending on deferment period and target horizon. The tables display both the RBC and the expected shortfall ($\text{ES}_{\%}$), as the latter provides information about the extent of the loss when the free fund is negative at the target horizon. When the smallest target horizon is considered, i.e. $T = 5$, the results for the RH and APC models are quite similar and higher than those for the other models, with the CBD and PLAT models attaining the lowest values. Conversely, the PLAT model has one of the highest required capital when a run-off perspective is adopted, that is $T = n$. The LC model has the smallest gap between the $T = 5$ and $T = n$ cases. The choice of model often involves significant judgement by the analyst and, as we can see, a change in model can lead to material changes in solvency margins. The M7 model produces the largest solvency margins under both the VaR and ES. As such, it is the most conservative model and therefore it is of particular interest for regulators and supervisory authorities.

Summarizing the results, it seems that the most important model feature driving differences in solvency margins across the different dimensions is the model complexity as captured by the number of time factors with a

	$T=5$		$T=10$		$T=n$	
d	$\text{RBC}_{\%}$	$\text{ES}_{\%}$	$\text{RBC}_{\%}$	$\text{ES}_{\%}$	$\text{RBC}_{\%}$	$\text{ES}_{\%}$
0	0.82%	0.93%	1.85%	2.09%	4.86%	5.55%
5	1.04%	1.18%	2.56%	2.90%	7.00%	7.99%
10	1.04%	1.18%	3.08%	3.49%	10.24%	11.70%
20	1.04%	1.18%	3.08%	3.49%	22.41%	25.78%
30	1.04%	1.18%	3.08%	3.49%	49.04%	57.44%

TABLE 7. $\text{RBC}_{\%}$ and $\text{ES}_{\%}$ for different deferred periods and time horizons for LC model.

	$T=5$		$T=10$		$T=n$	
d	$\text{RBC}_{\%}$	$\text{ES}_{\%}$	$\text{RBC}_{\%}$	$\text{ES}_{\%}$	$\text{RBC}_{\%}$	$\text{ES}_{\%}$
0	1.02%	1.16%	2.12%	2.40%	8.08%	9.27%
5	1.24%	1.40%	2.80%	3.17%	11.25%	12.91%
10	1.24%	1.40%	3.24%	3.67%	15.91%	18.25%
20	1.24%	1.40%	3.24%	3.67%	33.26%	38.35%
30	1.24%	1.40%	3.24%	3.67%	77.18%	90.36%

TABLE 8. $\text{RBC}_{\%}$ and $\text{ES}_{\%}$ for different deferred periods and time horizons for RH model.

	$T=5$		$T=10$		$T=n$	
d	$\text{RBC}_{\%}$	$\text{ES}_{\%}$	$\text{RBC}_{\%}$	$\text{ES}_{\%}$	$\text{RBC}_{\%}$	$\text{ES}_{\%}$
0	1.01%	1.14%	2.14%	2.42%	7.27%	8.35%
5	1.24%	1.41%	2.88%	3.26%	10.27%	11.79%
10	1.24%	1.41%	3.38%	3.83%	14.76%	16.96%
20	1.24%	1.41%	3.38%	3.83%	31.99%	36.97%
30	1.24%	1.41%	3.38%	3.83%	76.62%	90.03%

TABLE 9. $\text{RBC}_{\%}$ and $\text{ES}_{\%}$ for different deferred periods and time horizons for APC model.

	$T=5$		$T=10$		$T=n$	
d	$\text{RBC}_{\%}$	$\text{ES}_{\%}$	$\text{RBC}_{\%}$	$\text{ES}_{\%}$	Var	$\text{ES}_{\%}$
0	0.64%	0.73%	1.60%	1.81%	8.29%	9.63%
5	0.81%	0.93%	2.23%	2.53%	12.05%	14.00%
10	0.81%	0.93%	2.73%	3.09%	18.43%	21.43%
20	0.81%	0.93%	2.73%	3.09%	51.71%	60.67%
30	0.81%	0.93%	2.73%	3.09%	215.80%	264.13%

TABLE 10. $\text{RBC}_{\%}$ and $\text{ES}_{\%}$ for different deferred periods and time horizons for CBD model.

second order effect due to the presence of the cohort term. The LC model, the simplest of all models considered, seems to consistently produce overly optimistic capital requirements for any portfolio size or deferment period. Next in rank are the RH and APC models. The former extends the LC with the inclusion of the cohort term, while the latter simplifies the RH model by constraining the age modulating parameter to a constant. The solvency

	$T=5$		$T=10$		$T=n$	
d	$\text{RBC}_{\%}$	$\text{ES}_{\%}$	$\text{RBC}_{\%}$	$\text{ES}_{\%}$	$\text{RBC}_{\%}$	$\text{ES}_{\%}$
0	0.85%	0.96%	1.88%	2.13%	8.91%	10.4%
5	1.06%	1.20%	2.59%	2.93%	12.92%	15.03%
10	1.06%	1.20%	3.12%	3.52%	19.68%	22.92%
20	1.06%	1.20%	3.12%	3.52%	55.45%	65.22%
30	1.06%	1.20%	3.12%	3.52%	255.67%	314.91%

TABLE 11. $\text{RBC}_{\%}$ and $\text{ES}_{\%}$ for different deferred periods and time horizons for the M7 model.

	$T=5$		$T=10$		$T=n$	
d	$\text{RBC}_{\%}$	$\text{ES}_{\%}$	$\text{RBC}_{\%}$	$\text{ES}_{\%}$	$\text{RBC}_{\%}$	$\text{ES}_{\%}$
0	0.61%	0.73%	1.50%	1.76%	7.39%	8.74%
5	0.78%	0.93%	2.12%	2.48%	10.85%	12.84%
10	0.78%	0.93%	2.61%	3.05%	16.88%	19.96%
20	0.78%	0.93%	2.61%	3.05%	49.76%	59.24%
30	0.78%	0.93%	2.61%	3.05%	225.18%	281.33%

TABLE 12. $\text{RBC}_{\%}$ and $\text{ES}_{\%}$ for different deferred periods and time horizons for the Plat model.

margins are higher than those implied by the LC with narrow differences between the two models. It should be noted here that the cohort term need not be simulated as, for the case considered of 65 years old pensioners in 2011, it is known. Finally, models with more than one time factor produce conservative capital requirements across all portfolio sizes and deferment period. Clearly, the driving force is the greater uncertainty stemming from the presence of two stochastic ingredients, further increasing with the simulation horizon. The hybrid Plat model seems to be a partial exception, as results are comparable to those of the – apparently simpler – CBD model.

5. Stochastic interest rates

This section compounds on the previous ones and adds stochastic returns to the fund dynamics, investigating the effects on capital requirements of this additional uncertainty layer. We use the Vasicek model, see Vasicek (1977), to generate simulations of the short interest rate. It is assumed that the short rate r_t, the unique source of interest rate risk in this model, follows the dynamics

$$dr_t = a(\gamma - r_t)d_t + \sigma dW_t,$$

where W_t is a standard Wiener process and a, γ and σ are positive constants representing the speed of mean reversion, the long term mean and the volatility of the short rate, respectively. The random return of the portfolio fund, see (2), is given by $I_t = e^{r_{t-1}} - 1$, the latter expression implicitly assuming that the short rate remains constant over $[t-1, t)$. The parameters for the Vasicek model are given in Table 13. These parameters were chosen so that they reflect adequate stochastic fluctuations from the rate of

return used in the previous sections and match the overall performance of assets, resulting from both market behaviour and the investment strategy of an insurance company. We also consider alternative values for the volatility σ. As for the mortality model, we only consider the M7 and Plat model as these were generating the largest solvency requirements in the previous section.

parameter	value
r_0	0.03
a	0.3
γ	0.03
σ	0.05, 0.1, 0.2

TABLE 13. Parameters for simulations of interest rates for Vasicek model.

Figures 3, 4, 5 plot the relative solvency requirements with associated 95% confidence intervals, with or without stochastic interest rates for the three volatility scenarios, respectively. The greater severity of solvency margins is due to the fact that assets, cumulating at a random rate, are compared with the reserve, calculated according to a deterministic financial hypotheses which could be significantly different from the actual investment performance. The solvency margins in M7 model combined with Vasicek model are higher and the corresponding confidence intervals are wider than in Plat model.

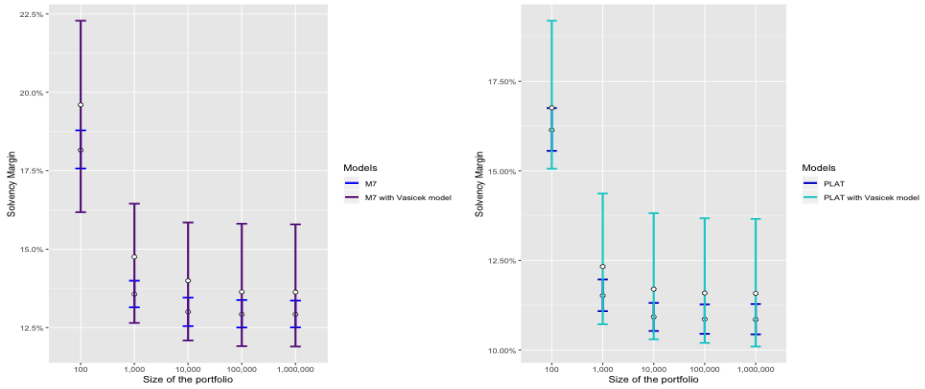

FIGURE 3. Relative solvency margins $\text{RBC}_{\%}$ and corresponding 95% confidence intervals for the M7 and Plat models, Vasicek interest rate model with $\sigma = 0.05$, $d = 5$ and $T = n$.

Further, the confidence intervals for both M7 and Plat models in case of stochastic interest rates are much wider than those when interest rates are deterministic. While confidence intervals narrow down with the portfolio size, the asymptotic width reflect the overall systematic interest rate and

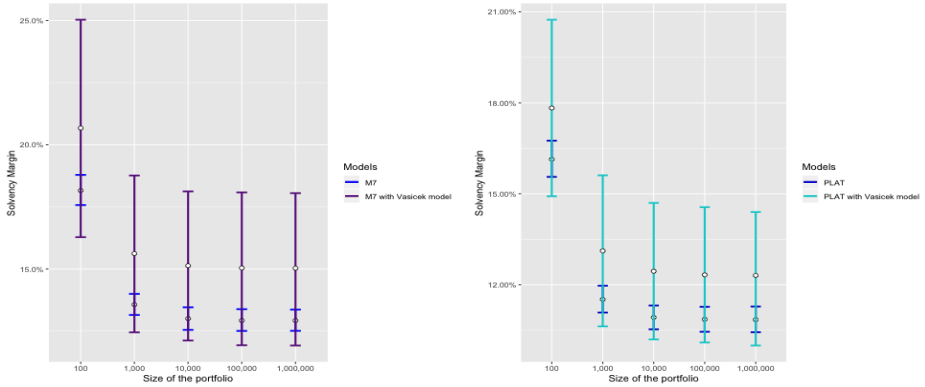

FIGURE 4. Relative solvency margins $\mathrm{RBC}_{\%}$ and corresponding 95% confidence intervals for the M7 and Plat models, Vasicek interest rate model with $\sigma = 0.1$, $d = 5$ and $T = n$.

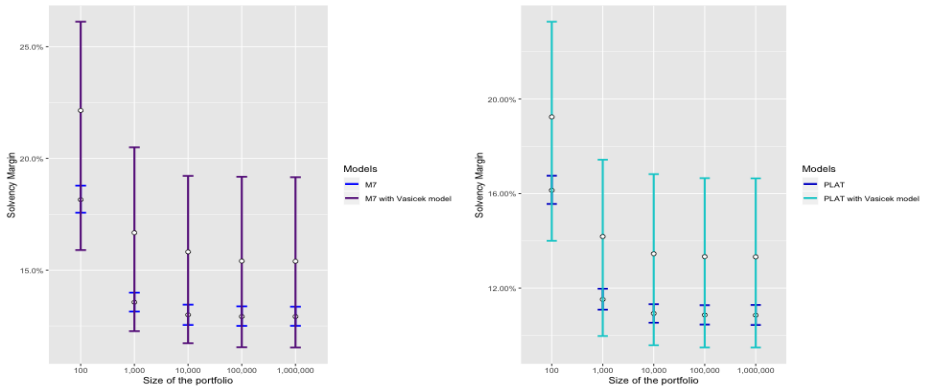

FIGURE 5. Relative solvency margins $\mathrm{RBC}_{\%}$ and corresponding 95% confidence intervals for the M7 and Plat models, Vasicek interest rate model with $\sigma = 0.2$, $d = 5$ and $T = n$.

mortality risk. The effect of stochastic interest rates on the overall width of confidence intervals in comparison with those in Section 4 is significant due to the non-pooling nature of interest rate risk. Moreover, it can be remarked that with greater interest rate volatility the solvency margins and their confidence intervals also increase reflecting higher levels of uncertainty.

6. Conclusions

Solvency requirements for deferred annuities have been analysed and compared to those of immediate annuities. Particular focus has been placed on the longevity risk. This has been incorporated to the analysis using different deterministic and stochastic models, particularly the family of generalized age-period-cohort stochastic mortality models. Several numerical examples illustrate solvency requirements produced by these different approaches to

longevity risk. The results demonstrate the dramatic importance of the mortality models on evaluation of solvency margins for life annuities and the fact that model risk is a prevalent issue when forecasting solvency margins. Another focus of this paper is the *deferment period* and its effect on solvency requirements. The margins increase with longer deferment periods which is caused by the extra longevity risk and the greater uncertainty about numbers of future survivors. Section 5 explores the impact of stochastic interest rates on solvency margins and corresponding confidence intervals.

The analysis could be extended, by considering some realistic features such as level premiums, return of premium guarantees in case of premature death and guaranteed annuity during the payout phase. Each one, or combinations, of such provisions will affect the solvency margin in a way that may not be easily anticipated.

While the paper has focused most of its attention on model risk, with all models calibrated on the same fixed sample, another dimension worth exploring would involve the dependence of results on the fitting period. It is well known, see for instance Cairns et al. (2011), Basellini et al. (2023), that mortality forecasts can be highly sensitive to the length of historical data employed in the estimation process. In turn, simulations and solvency margins may also be affected. Two routes seems viable in that respect. One may engage in a careful examination of available models, seeking for the most appropriate compromise between forecasting accuracy and robustness with respect to the fitting period. Alternatively, a fully fledged model averaging approach where a range of mortality models, trained over different periods, are combined through a weighing process to produce robust results, see for instance Benchimol et al. (2016), Shang (2012).

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