



City Research Online

City St George's, University of London

Citation: Pantazopoulou, S., Tomas-Rodriguez, M., Kalogeropoulos, G. & Halikias, G. (2024). Pole Assignment for Symmetric Quadratic Dynamical Systems: An Algorithmic Method. WSEAS Transactions on Systems and Control, 19, pp. 227-233. doi: 10.37394/23203.2024.19.24 ISSN 1991-8763 doi: 10.37394/23203.2024.19.24

This is the accepted version of the paper.

This version of the publication may differ from the final published version. To cite this item please consult the publisher's version.

Permanent repository link: <https://openaccess.city.ac.uk/id/eprint/33202/>

Link to published version: <https://doi.org/10.37394/23203.2024.19.24>

Copyright and Reuse: Copyright and Moral Rights remain with the author(s) and/or copyright holders. Copies of full items can be used for personal research or study, educational, or not-for-profit purposes without prior permission or charge, unless otherwise indicated, provided that the authors, title and full bibliographic details are credited, a hyperlink and/or URL is given for the original metadata page and the content is not changed in any way. For full details of reuse please refer to [City Research Online policy](#).

Pole Assignment for Symmetric Quadratic Dynamical Systems: An Algorithmic Method

S. Pantazopoulou^a, M. Tomas-Rodriguez^a, G. Kalogeropoulos^b, G. Halikias^b

^a*School of Science and Technology, City University, London, Northampton Square, London, EC1V 0HB, UK*

^b*Department of Mathematics, National and Kapodistrian University of Athens, Panepistimiopolis, Athens, 15784, Greece*

Abstract

In this article an algorithmic method is proposed for the solution of the pole assignment problem which is associated with a symmetric quadratic dynamical system, in case it is completely controllable. The above problem is proved to be equivalent to two subproblems, one linear and the other multilinear. Solutions of the linear problem must be decomposable vectors, i.e. they must lie in an appropriate Grassmann variety. The proposed method computes a reduced set of quadratic Plucker relations with only three terms each, which describe completely the specific Grassmann variety. Using these relations one can solve the multilinear problem and consequently calculate the feedback matrices which give a solution to the pole assignment problem. Finally, an illustrative example of the proposed algorithmic procedure is given. The advantage of this approach, is that the complete set of feedback solutions is obtained, over which further optimisation can be carried out, if desired.

Keywords:

Control Theory, Pole assignment, Quadratic matrix pencils, Grassmann variety, Plucker relations, numerical algorithm

1. Problem definition and Methodology

The equation of motion for a matrix second-order system, e.g. a structural system with viscous damping and without externally applying forces is expressed as follows:

$$Mq''(t) + Dq'(t) + Kq(t) = 0 \quad (1)$$

Here, M , D , K are the $n \times n$ mass, damping and stiffness matrices respectively. Also $q(t)$ is the displacement vector, $q'(t)$ the velocity vector and $q''(t)$ the acceleration vector. In most applications M , D and K are symmetric matrices. Furthermore, M is positive definite and D , K are positive semi-definite or positive definite.

Separation of variables $q(t) = e^{\lambda t}c$ where c is a constant vector gives us the quadratic eigenvalue problem:

$$P(\lambda_j)c_j = 0, \quad j = 1, 2, \dots, 2n$$

in which $P(\lambda)$ denotes the quadratic matrix pencil: $P(\lambda) = \lambda^2 M + \lambda D + K$. The eigenvalues and eigenvectors of the pencil govern the free response of the system. The model can be used to identify poorly damped oscillations and resonance phenomena, which we may be able to avoid by the appropriate selection of the matrix parameters.

When a control force is applied the model described by the above equations becomes:

$$Mq''(t) + Dq'(t) + Kq(t) = F(t), \quad F(t) = Bu(t) \quad (2)$$

where $u(t)$ is the $m \times 1$ control input vector and B is an $n \times m$ input matrix.

One of the major concerns for the control engineer is to ensure stability of the control system. Thus, the behaviour of the system is usually modified by applying state feedback control to relocate the troublesome eigenvalues in the complex plane. If the system is (open-loop) unstable, we aim, as a minimum requirement, to stabilise it. If the system is stable, it may be desirable to maintain some degree of relative stability and robustness margins, i.e. ensure that stability is maintained under realistic uncertainty conditions. The notion of stabilization is connected with the problem of relocation of troublesome eigenvalues of the system in equation (2), otherwise called pole placement, and relocation can always be achieved if the above system is completely state controllable. In that case, for any choice of poles, suitable real $m \times n$ matrices F_1 and F_2 can be found such that under the state feedback:

$$u(t) = -F_1q(t) - F_2q'(t) \quad (3)$$

the corresponding closed-loop system described by the equation

$$Mq''(t) + (D + BF_2)q'(t) + (K + BF_1)q(t) = 0 \quad (4)$$

has the chosen set of poles. Equivalently, if

$$f(s) = a_0 + a_1 s + \dots + a_{2n-1} s^{2n-1} + s^{2n} = e_{2n}^t(s) \tilde{a} \quad (5)$$

is the required closed-loop polynomial, $e_{2n}^t(s) = [1 \ s \ s^2 \ \dots \ s^{2n}]$ and $\tilde{a} = [a_0 \ a_1 \ a_2 \ \dots \ a_{2n-1} \ 1]$, we want to compute the matrices F_1 and F_2 such that the characteristic polynomial

$$\tilde{\varphi}(s) = \det(s^2 M + s(D + B F_2) + (K + B F_1)) \quad (6)$$

of the quadratic pencil

$$P_c := s^2 M + s(D + B F_2) + (K + B F_1) \quad (7)$$

associated with the closed-loop system (4) is equal to:

$$\tilde{\varphi}(s) = (\det M) f(s) \quad (8)$$

Define:

$$Q(s) := [s^2 M + s D + K \mid s B \mid B] \quad (9)$$

and

$$V = \begin{bmatrix} I_n \\ \dots \\ F_2 \\ \dots \\ F_1 \end{bmatrix} \quad (10)$$

Then we can rewrite equation (7) as:

$$P_c(s) = Q(s)V \quad (11)$$

Using compound matrices and applying the Binet-Cauchy theorem we can write,

$$\tilde{\varphi}(s) = \det(P_c(s)) \equiv C_n(P_c(s)) = C_n(Q(s)V) = C_n(Q(s))C_n(V) \quad (12)$$

We now define:

$$P := P(M, D, K, B) \in \mathbb{R}^{(2n+1) \times \ell} \quad (13)$$

Since $C_n(Q(s)) := e_{2n}^t(s)P$, the columns of P are the coefficient vectors of the elements of $C_n(Q(s))$, which are polynomials of maximum degree $2n$ and

$$g := C_n(V) = C_n \left(\begin{bmatrix} I_n \\ \dots \\ F_2 \\ \dots \\ F_1 \end{bmatrix} \right) \in \mathbb{R}^\ell \quad (14)$$

where $\ell := \binom{n+2m}{n}$.

Combining equations (5), (8) and (12) we get

$$e_{2n}^t(s)Pg = e_{2n}^t(s)(\det M)\tilde{a} := e_{2n}^t(s)a$$

which in turn implies that:

$$Pg = a \quad (15)$$

2. Techniques from algebraic geometry and algorithmic solution

Before we state the theorems on which the construction of the algorithm is based, we need the following notation and definitions [21], [24], [25]. Let the binomial coefficient k be

$$k := \binom{n}{m}, n, m \in \mathbb{N}, n \geq m$$

Denote the set of the first n natural numbers as

$$T_n := \{\kappa \in \mathbb{N}^* : 1 \leq \kappa \leq n\}$$

with \mathbb{N}^* the set of naturals without zero.

Let the set of elements (sequences), where the elements $(a_j, 1 \leq j \leq m)$ are not necessarily different

$$T_m^n := \{(a_1, \dots, a_m) : a_j \in T_n, 1 \leq j \leq m\}$$

Also, let the set of elements (sequences), where the elements of every sequence are in strictly increasing order, be denoted as

$$\tilde{T}_m^n := \{(a_1, \dots, a_m) \in T_m^n : a_\kappa \neq a_\lambda, \text{ if } \kappa \neq \lambda\}$$

Denote by D_m^n all m -tuples $\langle a_1, a_2, \dots, a_m \rangle$ such that $(a_1, \dots, a_m) \in \tilde{T}_m^n$ and $a_\kappa < a_\lambda$ if $\kappa < \lambda$. We can order the elements of the set D_m^n as follows:

$$\langle a_1, a_2, \dots, a_m \rangle \prec \langle j_1, j_2, \dots, j_m \rangle$$

if and only if there exists $\kappa \in \mathcal{T}_n$ such that $a_\kappa < j_\kappa$ and $a_\lambda = j_\lambda$ for every $\lambda < \kappa$. The above relation is a total ordering called lexicographical ordering. Using the lexicographical ordering, we can now characterize the k elements of D_m^n by

$$\begin{aligned} a_0 &= \langle 1, 2, \dots, m-1, m \rangle \\ a_1 &= \langle 1, 2, \dots, m-1, m+1 \rangle \\ &\vdots \\ a_{k-1} &= \langle n-m+1, \dots, n-1, n \rangle \end{aligned}$$

Note that $a_0 \prec a_1 \prec \dots \prec a_{k-1}$. Next, we define the map

$$\delta : \tilde{T}_m^n \rightarrow N \times D_m^n$$

such that:

$$(a_1, a_2, \dots, a_m) \xrightarrow{\delta} (\lambda, \langle a'_1, a'_2, \dots, a'_m \rangle)$$

where λ is the number of permutations needed to order the elements of (a_1, a_2, \dots, a_m) in normal ordering $\langle a'_1, a'_2, \dots, a'_m \rangle$.

If $[x_0, x_1, \dots, x_{k-1}]^T \in \mathcal{P}^{k-1}(\mathbb{R})$, we define the following maps:

$$\phi : D_m^n \rightarrow [x_0, x_1, \dots, x_{k-1}]^T$$

by $a_\kappa \xrightarrow{\phi} x_\kappa$, $0 \leq \kappa \leq k-1$, where it is clear from the above definition of ϕ that $\phi^{-1}(x_\kappa) = a_\kappa$ and

$$\tilde{\phi} : N \times D_m^n \rightarrow \mathbb{R}$$

by

$$(\lambda, a_\kappa) \xrightarrow{\tilde{\phi}} (-1)^\lambda \phi(a_\kappa) = (-1)x_\kappa$$

Finally, if we denote $\beta := (a_1, a_2, \dots, a_m) \in \mathcal{T}_m^n$, we define the map

$$g : T_m^n \rightarrow \mathbb{R}$$

by

$$g : T_m^n \rightarrow \mathbb{R} : g(\beta) = \begin{cases} (\tilde{\phi} \circ \delta)(\beta), & \text{if } \beta \in \tilde{T}_m^n \\ 0, & \text{if } \beta \in T_m^n \setminus \tilde{T}_m^n \end{cases}$$

We can now construct the Plucker relations: For every group of $m-1$ indices

$$\langle t_1, t_2, \dots, t_{m-1} \rangle \in \tilde{T}_{m-1}^n$$

and from every group of $m+1$ indices

$$\langle p_1, p_2, \dots, p_{m+1} \rangle \in \tilde{T}_{m+1}^n$$

we define for $\kappa = 1, \dots, m+1$,

$$\beta_\kappa := (t_1, t_2, \dots, t_{m-1}, p_\kappa)$$

and

$$\gamma_\kappa := (p_1, p_2, \dots, p_{\kappa-1}, p_{\kappa+1}, \dots, p_{m+1}),$$

for $\kappa = 1, \dots, m+1$ and the corresponding Plucker relation is as follows

$$\sum_{\kappa=1}^{m+1} (-1)^\kappa g(\beta) g(\gamma_\kappa) = 0$$

Next we state the following theorems which extract from the whole set of quadratic Plucker relations a reduced set of relations which have simple form and describe completely the Grassmann variety of the corresponding projective space.

Theorem 1 [25]: Assume that $x = [x_0, x_1, \dots, x_{k-1}]^t \in \Omega(m, n)$ and $\phi^{-1}(x_\kappa) = \langle a_1, a_2, \dots, a_m \rangle \in D_m^n$ with $a_2 < n - m + 1$. Then, there exists a three-term Plucker relation $\sigma(x_\kappa, x_{\kappa_1}, x_{\kappa_2}, x_{\kappa_3}, x_{\kappa_4}, x_{\kappa_5}) = 0$ where $\kappa < \kappa_i$, $i = 1, 2, \dots, 5$.

Theorem 2 [25]: Assume that $x = [x_0, x_1, \dots, x_{k-1}]^t \in \Omega(m, n)$. The full number of the coordinates x_κ having the property $\phi^{-1}(x_\kappa) = \langle a_1, a_2, \dots, a_m \rangle$ with $a_2 < n - m + 1$ is

$$r := \binom{n}{m} - (n - m)m - 1$$

Corollary 1 [25]: For every $x = [x_0, x_1, \dots, x_{k-1}]^t \in \Omega(m, n)$, there exists a set S of three-term quadratic Plucker relations, of the form

$$\sigma_i(x_{\kappa_i}, x_{\lambda_i}, x_{\mu_i}, x_{\nu_i}, x_{\xi_i}, x_{\rho_i}) = 0$$

where $1 \leq i \leq r$ with

$$\kappa_i \leq \min(\lambda_i, \mu_i, \nu_i, \xi_i, \rho_i), \text{ for } 1 \leq i \leq r,$$

and

$$\kappa_j < \kappa_{j+1}, \quad 1 \leq j \leq r-1.$$

Theorem 3 [25]: The three-term quadratic Plucker relations given by the set S of Corollary 1 describe completely the Grassmann variety $\Omega(m, n)$ of the projective space $P^{k-1}(\mathbb{R})$.

Next, we propose the following algorithm which computes the Reduced Set of Quadratic Plucker Relations (RSQPR).

Algorithm RSQPR

- Step 1: Read the dimensions n, m .
- Step 2: Compute $k = \binom{n}{m}$.
- Step 3: Repeat for $k = 0, 1, \dots, k-1$.
 - a. Find the κ -th order multi-index $\langle a_1, a_2, \dots, a_m \rangle$
 - b. If $\lambda_2 < n - m + 1$ then
 - b1. Find indices $j_1, j_2 \in T_n - \{a_1, a_2, \dots, a_m\}$ such that $j_1 > a_2$ and $j_2 > a_2$.
 - b2. Define $a_{m+1} := j_2$, $\beta_\rho := (a_3, \dots, a_m, j_1, a_\rho)$ and $\gamma_\rho := (a_1, \dots, a_{\rho-1}, a_{\rho+1}, \dots, a_{m+1})$.
 - b3. Type Plucker relation

$$\sum (-1)^\rho \cdot g(\beta_\rho) \cdot g(\gamma_\rho) = 0$$

- Step 4: End.

Finally, suppose we have computed the coordinates of a decomposable vector as it is seemed below

$$x = [x_0, x_1, \dots, x_{k-1}]^t \in \Omega(m, n)$$

using the quadratic Plucker relations given by the above algorithm. Then we would like to reconstruct $H \in \mathbb{R}^{n \times m}$ which has the property

$$C_m(H) = x \quad (16)$$

The following Proposition is helpful.

Proposition 1 [25]: Let $x = [x_0, x_1, \dots, x_{k-1}]^t \in \mathbb{R}^k$ be a decomposable vector and $x_p \neq 0$ for an index p , $1 \leq p \leq k-1$, such that

$$\phi^{-1}(x_p) = \langle a_1, a_2, \dots, a_m \rangle$$

Then, the elements h_{ij} of matrix $H \in \mathbb{R}^{n \times m}$ which satisfies equation (16) are given by

$$h_{ij} = g((a_1, a_2, \dots, a_{j-1}, i, a_{j+1}, \dots, a_m)) \quad (17)$$

for $1 \leq i \leq n$ and $1 \leq j \leq m$.

3. Numerical Example

The benefits of the proposed algorithmic procedure are clarified by the following example. We consider the system of the form (2), with

$$M = 10I_3, \quad K = \begin{bmatrix} 40 & -40 & 0 \\ -40 & 80 & -40 \\ 0 & -40 & 80 \end{bmatrix}, \quad D = 0_{3 \times 3}, \quad B = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}$$

The open-loop e-values are $\pm 3.6039i, \pm 2.4940i, \pm 0.8901i$. Suppose we want to shift these to $-1, -2, -3, -4, -5, -6$, using the state feedback law (3) where $F_1, F_2 \in \mathbb{R}^{1 \times 3}$. In this case the problem consists of finding matrices F_1, F_2 such that the closed-loop system (4) has a characteristic polynomial

$$\tilde{\phi}(s) = (\det M) f(s)$$

where $f(s) = \prod_{i=1}^6 (s - \lambda_i) = \prod_{i=1}^6 (s + i)$.

After some computations, we arrive at a linear system of the form (15) which is $Pg = a$, the solution of which is given by:

$$g_0 = 1, \quad g_1 = 2932.5, \quad g_2 = -4318.571, \quad g_3 = 3345$$

$$g_4 = -5737.143, g_6 = 1447.5, g_7 = -2705.714$$

Variables g_5, g_8, g_9 can be chosen arbitrarily since they do not appear in equations of the linear system. Now we must solve a multilinear subproblem, which consists of the following three equations, the so-called reduced set of quadratic Plucker relations:

$$g_4g_6 - g_3g_7 + g_0g_9 = 0$$

$$-g_5g_6 + g_3g_8 - g_1g_9 = 0$$

$$g_5g_7 - g_4g_8 + g_2g_9 = 0$$

These equations are generated by the proposed algorithm. Using a symbolic package i.e. Mathematica and substituting the solutions of the linear system to the above set of quadratic relations the following solutions are obtained:

$$g_5 = -2378500, g_8 = -1683375, g_9 = -746100$$

Now, from equation (15) in combination with Proposition 1, we obtain the feedback matrices

$$F_1 = [g_7, -g_4, g_2] = [-2705.714, 5737.143, -4318.571]$$

and

$$F_2 = [g_6, -g_3, g_1] = [1447.5, -3345, 2932.5]$$

In order to avoid cumbersome calculations, a numerical software package can be used to verify that the closed-loop system (4) has the following poles:

$$\tilde{\lambda}_1 = -1.000000000047105$$

$$\tilde{\lambda}_2 = -1.999999999594935$$

$$\tilde{\lambda}_3 = -3.000000000111104$$

$$\tilde{\lambda}_4 = -3.99999999929748$$

$$\tilde{\lambda}_5 = -5.000000000132230$$

$$\tilde{\lambda}_6 = -6.000000000120886$$

Note that whatever the starting guess for the approximation of the eigenvalues, the error $|\lambda_i - \tilde{\lambda}_i|$, for $i = 1, \dots, 6$, is always smaller than 10^{-9} .

4. Conclusion

An efficient algorithmic framework has been proposed for the solution of the pole-assignment problem of symmetric quadratic systems. This involves the computation of a reduced set of quadratic Plucker relations describing completely the Grassmann variety of the corresponding projective space. The extraction of this reduced set has been achieved by the use of a simple criterion based on the correspondence between the coordinates of a decomposable vector and lexicographical orderings. The minimum number of the linear independent quadratic Plucker relations which describe completely the Grassmann variety is:

$$V_{eq} = \binom{n}{m} - m(n - m) - 1$$

Each equation of this reduced set is homogeneous and contains only three terms. This fact has beneficial influence on the overall complexity of the problem which is mainly due to its non-linearity. The method can be extended to the solution of other problems of control theory having a similar multi-linear nature [1], [2], [3], [14].

References

- [1] B.N. Datta, Numerical Linear Algebra and Applications, Brooks/Call Publishing Company, 1963
- [2] B.N. Datta, Numerical methods for Linear Control Systems Design and Analysis, Elsevier Academic Press, 2004
- [3] B.N. Datta and F. Rinconi, Feedback stabilization of a second-order system: a nonmodal approach, *Lin. Alg. Appl.*, 188/189, (1993), 135-161
- [4] B.N. Datta, E. Elhay and Y. Ram, Orthogonality and Partial multi-input pole assignment of a second-order control system, *Proc. IEEE Conf. Dec. Contr.* 1996, 2025-2029
- [5] I. Gohberg, P. Lancaster, L. Rodman, Matrix Polynomials, Academic Press, New York, 1982

- [6] T. Williams and A. Laub, Orthogonal Canonical forms for Second-order Systems, *IEEE Trans. Aut. Control*, 37, (1992), 1050-1052
- [7] G. Kalogeropoulos, Matrix Pencils and Linear Systems Theory, Ph.D Thesis, City University, London, U.K., 1985
- [8] T. Kailath, *Linear Systems*, Prentice Hall, New Jersey, 1980
- [9] R.E. Skelton, *Dynamic Systems Control, Linear Systems Analysis and Synthesis*, John Wiley and Sons, Inc. N.Y., 1985
- [10] Ch. Giannakopoulos, Frequency Assignment Problems of Linear Multi-variable Problems: An Exterior Algebra and Algebraic Geometry Based Approach, Ph.D. Thesis, City University, London, U.K., 1984
- [11] F.R. Gantmacher, *The Theory of Matrices*, Vol. I, II, Chelsea, N.Y., 1959
- [12] N. Karcanias, Ch. Giannakopoulos, Grassmann invariants, almost zeros and the determinantal zero, pole assignment problems of linear systems, *Int. J. Control*, Vol. 40, 1984, 673-689
- [13] N. Karcanias, Ch. Giannakopoulos, Grassmann matrices, decomposability of multivectors and the determinantal assignment problem, in *Linear circuits, Systems and Signal Processing: Theory and Applications*, C. I. Byrnes et al.(eds.), North Holland, 1988, 307-312
- [14] N. Karcanias, Ch. Giannakopoulos, Necessary and sufficient conditions for zero assignment by constant squaring down, *Linear Algebra Apl.*, Special Issue on Control Theory, Vols. 122/123/124, 1989, 415-446
- [15] Kimura H., Pole assignment by gain output feedback, *IEEE Trans.*, *Autom. Control*, Vol AC-20, 1975, 509-516
- [16] Marcus M., *Finite Dimensional Multilinear Algebra* , Marcel Dekker, New York, 1973
- [17] Greub W. H., *Multilinear Algebra*, Springer Verlag, New York, 1967
- [18] Sain M. K., Seshadri V., Pole assignment and a theorem from Exterior Algebra, *Proc. 20th IEEE CDC*, San Diego, CA, 1981

- [19] Martin C. F., Hermann R., Applications of Algebraic Geometry to systems theory: The McMillan degree and Kronecker indices, SIAM J. Control Optim., Vol. 16, 1978
- [20] Hermann R., Martin C. F., Applications of Algebraic Geometry to system theory-Part I, IEEE. Trans. Autom. Control, Vol. AC-22, 1977, 19-25
- [21] Hodge W.V.D., Pedoe D., Methods of Algebraic Geometry, Cambridge University Press, Boston, 1952
- [22] Dai L., Lecture Notes in Control and Information Sciences, No 118, Springer-Verlang, 1989
- [23] C. Giannakopoulos, N. Karcanias, G. Kalogeropoulos, The theory of polynomial combinants in the context of linear systems, Multivariable Control New concept and tools, S.G. Tzafestas, D. Reidel Co., 1984, 27-41
- [24] C. Giannakopoulos, G. Kalogeropoulos, N. Karcanias, The Grassmann variety of non dynamic compensators, the determinantal assignment problem of Linear Systems, Bull of the Greek Mathematical Society, Vol. 24, 1983, 33-57
- [25] G. Kalogeropoulos, D. Kytasias, K. G. Arvanitis, On the Computation of a reduced set of quadratic Plucker Relations and their use in the solutions of the determinantal assignment problem, Systems Science, Vol. 26, 2000
- [26] D. Kytasias, Algorithm of computation of the plucker relations and its application in feedback problems of regular and singular systems, PhD Thesis, University of Athens, 1993
- [27] J. Leventides, Algebrogeometric and topological methods in control theory, PhD Thesis, City Univeristy, 1993