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# Life-cycle investment and housing decisions with longevity annuities and reverse mortgage

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## Abstract

We investigate optimal life-cycle consumption, investment, and housing decisions with longevity annuities and reverse mortgage. A risk-averse investor can hold financial assets, such as cash, bonds, and stocks, and can invest in housing, through renting and purchasing. Variable-rate mortgages are available. At retirement, the investor can release his housing equity through a reverse mortgage product. Longevity annuities are also available to support his income at advanced ages. We use multi-stage stochastic programming to solve the optimization problem numerically. Our numerical results show that longevity annuities can enhance non-housing consumption in retirement and the proceeds from reverse mortgage may raise not only housing and non-housing consumption in retirement but also non-housing consumption prior to retirement. Reverse mortgage also changes housing preference from renting to owning with a lower regular mortgage debt. Longevity annuities can help different risk-averse investors choose a suitable consumption stream in retirement. When social security cannot guarantee a high level of the replacement ratio, our model finds that longevity annuity and reverse mortgage can prevent housing and non-housing consumption drops in retirement. The housing ownership preference over rental however becomes more intense.

Keywords: Life-cycle investment, housing, longevity annuity, reverse mortgage, stochastic programming

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## I. Introduction

A certain level of consumption on housing is indispensable for a good quality of life, not just immediately but also for the future. An individual should carefully manage their housing asset in regard to lifetime consumption and retirement planning. Using a reverse mortgage product, the housing asset can generate retirement income. According to the OECD Affordable Housing Database, housing-related spending is about 22% of household expenditure in 2019, on average.[1] Historical house prices are highly uncertain. Over several years during the Covid-19 outbreak, for example, they have surged. Post-pandemic, house prices in many developed countries including South Korea are falling.[2] Despite the importance of the housing asset and reverse mortgages, there are not many academic studies which have investigated their use in the setting of a lifetime consumption and investment problem.

Optimal lifetime consumption and investment has been a decades-long research area in multiple academic disciplines, such as financial economics, operations research, and actuarial science. Using a stochastic optimal control approach, Merton (1969) and Samuelson (1969) derive an analytical solution for the optimal consumption and investment choices over lifetime in continuous and discrete time, respectively. They conclude that the optimal decision on risky asset weights over lifetime is “myopic”; it is constant and independent of wealth and time, subject to certain conditions. Thereafter, most academic studies in financial economics have focused on finding the conditions under which “non-myopic” investment choices hold (for example, Bodie et al., 1992; Campbell et al., 2003; Gomes et al., 2008; Kim and Omberg, 1996; Viceira, 2001). Recently, Kraft et al. (2018) introduce housing habits, unlike non-housing habit in Munk (2008). They deliver insights on understanding financial advisers’ conventional suggestions or empirically-observed investment decisions.

Most actuarial and insurance studies on lifetime investment and retirement examine optimal or practical annuitization and investment strategies for retirement plans with

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1 https://www.oecd.org/housing/data/affordable-housing-database/

2 A global house-price slump is coming. (n.d.). The Economist. Retrieved 16 November 2022, from https://www.economist.com/leaders/2022/10/20/a-global-house-price-slump-is-coming

different types of annuity. The annuities may vary by a waiting period (deferred or immediate), by payment level (variable or fixed), and by the existence of a guaranteed period (whole life or fixed-term). Representative studies are Blake et al. (2003), Cairns et al. (2006), Horneff et al. (2008, 2009), Koijen et al. (2011), Milevsky and Young (2007). Horneff et al. (2020) and Jang et al. (2022) show that longevity annuities can enhance retirement welfare. Their studies are related to the U.S. Treasury announcement in 2014, which allows 401(k) plans to include longevity annuities in default investments. Most of above-mentioned papers apply a dynamic programming method to search optimal solutions.

In the field of operations research, the practical features of lifetime investment problems have been studied. Stochastic programming can deliver implementable and immediately applicable solutions to financial and retirement planning problems, but require strong computational power. Therefore, many studies in this field adopt linear or quadratic objective functions (Berger and Mulvey, 1998; Consigli et al., 2012; Consiglio et al., 2004; Consiglio, Dempster and Medova, 2011). Also, the interpretability of results from a computational model depends on other statistical or numerical analyses. Using stochastic programming approaches enables us to examine optimal solutions with real-world features, such as income and capital gains taxes, transaction costs, and multiple goals and to suggest new product designs for retirement planning (Consigli et al., 2012; Dempster and Medova, 2011; Owadally et al., 2021a, b).

We develop a model to examine optimal lifetime consumption, investment, and housing decisions. Available assets include not only traditional financial products, such as bonds, stocks, and mortgages but also longevity annuities and reverse-mortgage products in retirement. Our model also incorporates practical features such as social security levels, transaction costs, and tax rules. Thus, the resulting strategies are immediately applicable and individually customizable to a real-world problem. Our results show that both non-housing and housing consumption may be higher in retirement when longevity annuities and reverse mortgages are available.

The paper is organized as follows. Section 2 specifies objectives and decisions of the life-cycle optimal consumption and investment problem. In Section 3, we define market models which govern price dynamics of financial assets, house, longevity annuities, and reverse mortgage. We formulate the multi-stage stochastic programming problem in Section 4. From numerical results using five different product mixes in Section 5, we investigate optimal decisions on housing and non-housing consumption, investments, and housing with longevity annuities and reverse mortgage. Section 6 concludes.

## II. Objectives and decisions

Consider an individual who plans a lifetime investment for a maximum actuarial lifespan $\tau$ and retirement at time $T < \tau$. The individual earns utility at time $t$ from non-housing consumption $C_t$ and housing consumption $H_t$, where housing consumption can be through either ownership or rental. We follow the approach of Kraft and Munk (2011), Kraft et al. (2018) and Jang et al. (2022). The housing market is divisible in a number of 'units'. A housing unit represents a quantum of housing in terms of size, quality and location.

Individuals can consume housing through a combination of buying and rental units, and $H_t$ is the number of such units consumed at time $t$. The individual can consume $H_t$ housing units directly by owning $X_{HA,t}$ units as an asset and also by renting $X_{HR,t}$ units from a landlord. If this represents a single house, then this is equivalent to shared ownership. Thus, $H_t = X_{HA,t} + X_{HR,t}$.

We assume a power utility function $u(x) = x^{1-\gamma}/(1-\gamma)$, with risk aversion coefficient $\gamma > 1$. The non-housing consumption is measured in monetary terms. The housing consumption is measurable in a standard unit of area such as a square metre. Let the minimum housing-consumption units be $\Lambda_{MinH}$, so $H_t \geq \Lambda_{MinH} > 0$.

We consider variable interest-only mortgage products with no penalty for early principal repayments. The individual can buy or sell housing units with or without a mortgage product. If the individual takes a new mortgage debt for $X_{HD,t}$ housing units, its interest payments should be made on the remaining mortgage balance. We assume that a flexible principal repayment of the mortgage $D_{P,t}$ is a separate decision that the investor can make dynamically.

The individual can invest in cash, bond and equity funds. Let the number of holding units in these funds be $X_{C,t}$, $X_{B,t}$ and $X_{E,t}$, respectively. At or after retirement in $t \in [T, \tau)$, he may purchase $X_{A(\psi),t}$ units of (deferred) longevity annuities, which pay \$1 p.a. from the pre-specified payment starting age $\psi$ till death. Another means of receiving retirement income is a reverse mortgage product. By entering this contract, the individual releases $X_{R,t}$ units of his housing equity in return for a lump sum payment.

If the individual dies between time $t$ and $t+1$, his wealth $W_{t+1}$ is bequeathed to his heirs. The wealth is the amount held in cash, bond and equity funds, as well as equity in housing, after taxes and fees.

Let $\mathcal{D}_t$ be the set of decision variables at time $t \in [0,\tau]$ over which expected utility is maximized. These decisions consist of non-housing consumption $C_t$, housing decisions collected in $\mathcal{H}_t$, decisions regarding the financial portfolio collected in $\mathcal{X}_t$, and decisions regarding the annuity portfolio collected in $\mathcal{A}_t$. Then, $\mathcal{D}_t = C_t \cup \mathcal{H}_t \cup \mathcal{X}_t \cup \mathcal{A}_t$, where $\mathcal{H}_t = \{X_{HA,t}, X_{HR,t}, X_{R,t}, X_{HD,t}, D_{P,t}\}$, $\mathcal{X}_t = \{X_{C,t}, X_{B,t}, X_{E,t}\}$ and $\mathcal{A}_t = \cup_{\psi} X_{A(\psi),t}$.

We use standard actuarial notation for survival and death probabilities: ${}_tp_\delta$ denotes the probability that a person aged $\delta$ years survives for $t$ years until age $\delta + t$, while $q_{\delta+t}$ denotes the probability that a $(\delta + t)$-year old person dies over the following year. These probabilities are evaluated in an actuarial life table.

The general objective function for the individual investor's problem can be stated as

$$\max_{\mathcal{D}_t, t\in[0,\tau]} \quad \mathbb{E}_0 \sum_{t\in[0,\tau)} \left( {}_tp_\delta e^{-\rho t} u\left(C_t^{1-\theta} H_t^{\theta}\right) + \ {}_tp_\delta q_{\delta+t} e^{-\rho(t+1)} \kappa^\gamma u(W_{t+1}) \right), \tag{1}$$

which comprises Cobb-Douglas utility over non-housing and housing consumption with $\theta \in (0,1)$ as well as bequest utility. A time preference coefficient $0 \le \rho \le 1$ represents the individual's preference for earlier consumption. A bequest preference coefficient $\kappa \ge 0$ captures the importance of bequest relative to housing and non-housing consumption.

We search the optimal decision set of the lifetime consumption investment problem using multi-stage stochastic programming (MSP). Decisions are made between time 0 and time $\mathcal{T}$, where $\mathcal{T}$ is a time point between the time of retirement $T$ and maximum lifespan $\tau$. Our planning horizon is therefore $[0, \mathcal{T}]$ with $T < \mathcal{T} < \tau$. Modelling statements of the optimization constraints are deferred to Section 4.

## III. Market models

In this section, we describe the assumptions made and models for financial, housing, labour, and retirement markets. In Section 4, we describe taxes and fees involved in each market because they are defined with variables used for MSP modelling. The last subsection defines outstanding balance rates of the reverse mortgage with fees. We follow the model notation and assumptions of Jang et al. (2022). It is noteworthy that Jang et al. (2022) investigate the optimal use of home reversion, but not of reverse mortgage.

## 1. Financial markets

An investor is allowed to invest in cash, bond funds and equity funds. We denote them by $C$, $B$ and $E$, respectively. Let $S_{i,t}$, $i \in \{C, B, E\}$ be the price of one unit at time $t$. In order to incorporate interest rate uncertainty into our financial and retirement market model, the Vasicek mean-reverting process is used to model the short rate $r_t$:

$$\mathrm{d}r_t = \nu(\tilde{\theta} - r_t)\mathrm{d}t + \sigma_0 \mathrm{d}W_{r,t}, \tag{2}$$

where $\nu > 0$ is the reversion speed to the mean level $\tilde{\theta}$, $\sigma_0$ is the volatility and $r_0 > 0$.

The stock price $S_{E,t}$ is given by

$$\mathrm{d}S_{E,t} = (r_t + \mu_E)S_{E,t}\mathrm{d}t + \sigma_E S_{E,t}\mathrm{d}W_{E,t}, \tag{3}$$

where $\mu_E$ is the risk premium, $\sigma_E > 0$ is the volatility. $W_{E,t}$ and $W_{r,t}$ are correlated Wiener processes, with correlation coefficient $-1 \leq \rho_{EB} \leq 1$:

$$\mathrm{d}W_{r,t} = \mathrm{d}\widetilde{W}_{1,t}, \quad \mathrm{d}W_{E,t} = \rho_{EB}\mathrm{d}\widetilde{W}_{1,t} + \sqrt{1-\rho_{EB}^2}\,\mathrm{d}\widetilde{W}_{2,t}, \tag{4}$$

where $\widetilde{W}_{1,t}$ and $\widetilde{W}_{2,t}$ are independent Wiener processes; we ignore any market regime switching that may affect optimal investment decisions (Park, 2023).

Assume that the individual can rebalance his portfolio at regular intervals of length $\Delta t$ years. There are $N \in \mathbb{N}$ such regular intervals in his planning horizon $[0, \mathcal{T}]$. Defining $R_{i,t}$ as the continuously compounded return of asset $i \in \{C, B, E\}$ from time $t - \Delta t$ to $t$, the price $S_{i,t}$ of asset $i$ evolves as follows:

$$S_{i,t} = S_{i,t-\Delta t} \cdot \exp(R_{i,t}), \tag{5}$$

where $S_{i,0} = 1$ without loss of generality.

Using the price of a zero-coupon bond in the Vasicek model, the continuously compounded return of the long-term bond fund with a maturity of $M$ years over a holding period of length $\Delta t$ from time $t - \Delta t$ to $t$ is approximated by

$$R_{B,t} = a(M - \Delta t) - a(M) - b(M - \Delta t)r_t + b(M)r_{t-\Delta t}, \tag{6}$$

where $b(M) = (1 - e^{-\nu M})/\nu$ and $a(M) = (\tilde{\theta} + \mu_0 - \sigma_0^2/2\nu^2)(b(M) - M) - \sigma_0^2\, b(M)^2/4\nu$, and $\mu_0$ relates to the market price of interest rate risk. The return of the cash fund over the time period $(t - \Delta t, t)$ is simply the accumulated spot rates defined by the Vasicek model, i.e. $R_{C,t} = \int_{t-\Delta t}^{t} r_s \mathrm{d}s$.

## 2. Housing market

In the lifetime consumption and investment literature, the geometric Brownian motion (GBM) is commonly used to model house prices (Kraft and Munk, 2011; Kraft et al., 2018). We ignore the short-run autocorrelation that is observed in the real estate markets (Case and Shiller, 1988; Gau, 1987). We assume that the market is efficient in the long run so that the GBM can be used for pricing a home equity release product (Szymanoski, 1994; Wang et al., 2008). The price $S_{H,t}$ of one unit of housing, say per square metre, is given by

$$\mathrm{d}S_{H,t} = (r_t + \mu_H)S_{H,t}\mathrm{d}t + \sigma_H S_{H,t}\mathrm{d}W_{H,t}, \tag{7}$$

where $\mu_H$ is the risk premium on the house price, $r_t$ is the short interest rate in Eq. (2), and $\sigma_H > 0$ is the constant price volatility. Let the correlation coefficients of housing returns with bond returns and equity returns be $-1 \leq \rho_{HB} \leq 1$ and $-1 \leq \rho_{HE} \leq 1$, respectively. The Wiener processes $W_{r,t}$, $W_{E,t}$ and $W_{H,t}$ in (2), (3) and (7) are correlated as follows:

$$\mathrm{d}W_{H,t} \;=\; \rho_{HB}\mathrm{d}\widetilde{W}_{1,t} \;+\; \hat{\rho}_{HE}\mathrm{d}\widetilde{W}_{2,t} \;+\; \sqrt{1-\rho_{HB}^2-\hat{\rho}_{HE}^2}\;\mathrm{d}\widetilde{W}_{3,t}, \tag{8}$$

where $\hat{\rho}_{HE} = (\rho_{HE} - \rho_{EB}\rho_{HB})/\sqrt{(1-\rho_{EB}^2)}$, and $\widetilde{W}_{1,t}$, $\widetilde{W}_{2,t}$ and $\widetilde{W}_{3,t}$ are independent Wiener processes.

## 3. Labour and retirement income markets

(1) Labour income

Wage $L_t$ is deterministic and positive in our model and, upon retirement at time $T$, it is replaced by a social security benefit of $L_T\Upsilon_S$, where $\Upsilon_S$ is the social security replacement ratio. The social security benefit is one of the retirement income sources that we consider in our model.

$$L_t > 0 \quad \text{for } \mathrm{t} \in [0, \mathrm{T}), \tag{9a}$$

$$L_t = L_T\Upsilon_S \quad \text{for } \mathrm{t} \in [\mathrm{T}, \tau]. \tag{9b}$$

(2) Annuity

Annuities and reverse mortgages constitute two other sources of retirement income. An annuity makes a regular stream of payments to the annuity-holder while he is alive. Longevity annuities comprise a deferment period between the annuity purchase date and its first payment date. If the deferment period is zero, the longevity annuity is regarded as an immediate annuity. Longevity annuities are sold by life insurers to help individuals plan their retirement. These annuities provide an income for life and mitigate retirees' longevity risk, i.e. the risk that they outlive their savings.

The longevity annuities are priced in terms of the term structure model described in Section 3.1. Suppose that the investor is aged $\delta$ at time $0$ and cannot live past the limiting age $\omega$, which is the maximum age in an actuarial life table. A longevity annuity paying an annual benefit of \$1 from age $\psi$ until death is denoted by $A(\psi)$. Let the price at time $t$ of such an annuity be $S_{A(\psi),t}$. For a policyholder aged $\delta + t$ at time $t$, the fair actuarial price of the longevity annuity contract is

$$S_{A(\psi),t} = \sum_{m=\psi-\delta-t}^{\tau-t-1} {}_{m}p_{\delta+t} \exp[a(m) - b(m)r_t], \tag{10}$$

where functions $a(\cdot)$ and $b(\cdot)$ are defined in Eq. (6) We assume static pricing mortality rates here and ignore any loading and other expenses.[3]

(3) Reverse mortgage

The final retirement income product that we consider is a reverse mortgage. Similar to regular mortgages, the reverse mortgage involves housing debt. Let $D_{R,t}$ be the outstanding balance of the reverse mortgage contracts for a single-house individual. Assume the individual decides to release his home equity only at retirement time $T$. With the debt principal $S_{H,T}X_{R,T}$, the outstanding balance at time $t$ will grow at a variable rate of $r_t + \alpha_R$, where $\alpha_R$ is a reverse mortgage rate premium.

The reverse mortgage provider offers a guarantee called the "no negative equity guarantee" (NNEG), also known as non-recourse provision. This protects the policyholder from having the loan balance exceeding the sale price of the house. Usually, the reverse mortgage provider charges the guarantee fee to the policyholder, and the fee amount is proportional to their housing value and the outstanding value of the reverse mortgage debt. We assume two different rates to account for the guarantee fee: an entering fee at rate $\varphi_R^u$ and a regular guarantee fee at rate $\varphi_R^m$ are added on top of the debt outstanding.

The reverse mortgage outstanding balance $D_{R,t}$ cannot exceed the housing value with

[3] The annuities are priced as in (10) with the S1PML mortality table based on 2000–2006 experience ($\omega = 120$): see [IFOA(2019)].

the NNEG option:[4]

$$D_{R,t} = X_{R,T} S_{H,T} R_t^{NNEG}, \quad t \geq T. \tag{11}$$

The balance at time $t$ accumulates from time $T$ at the rate of $R_t^{NNEG}$. It takes into account the NNEG option as follows:

$$R_t^{NNEG} = \min\left\{(1 + \varphi_R^u/\Lambda_{LTV(R)}) \prod_{s=T}^{t-\Delta t} (1 + \varphi_R^m)^{\Delta t} \exp[(r_s + \alpha_R)\Delta t], \frac{S_{H,t}}{S_{H,T}\Lambda_{LTV(R)}}\right\}, \tag{12}$$

where $t > T$ and $R_t^{NNEG} = \left(1 + \varphi_R^u/\Lambda_{LTV(R)}\right)$ if $t = T$. This implies that the value of an investor's house ownership at retirement time $T$ is equal to $S_{H,T} X_{R,T}/\Lambda_{LTV(R)}$ and that its balance $D_{R,t}$ with the entering fee $\varphi_R^u$ grows at the rate of $r_s + \alpha_R$ with the regular fee $\varphi_R^m$, but limited to the owned housing value $\frac{X_{R,T} S_{H,t}}{\Lambda_{LTV(R)}}$.

## IV. MSP model formulation

Using multi-stage stochastic programming (MSP), we formulate the lifetime investment problem defined with the objective function and constraints. The sparse discretization of the MSP formulation inevitably leads to some simplifications, as with all modeling exercises. We introduce other constraints to incorporate real-world features, such as transaction costs, management fees and taxes.

(1) Scenario tree

The MSP model is defined upon a non-recombining scenario tree (Birge and Louveaux, 2011). The number of nodes grows exponentially with the number of stages, which is an equivalent term to the number of time periods in our problem. To minimize intensive and prolonged computation, we discretize an investor's planning period $[0, \mathcal{T}]$ with $\Delta t$-long time periods, where $T < \mathcal{T} < \tau$. An $\delta$-year-old investor at $t = 0$, for example, is expected to retire at $t = T$. His planning period is over $t \in [0, \mathcal{T}]$, although he may survive for $\tau$ years to age $\delta + \tau$. If $\delta$, $\mathcal{T}$, and $\Delta t$ are 30, 80, and 10 respectively, the

[4] Putting it another way, there is a risk that there is a positive difference between the outstanding balance on the reverse mortgage and the housing value when the policyholder of the reverse mortgage contract dies. The reverse mortgage provider receives the NNEG premium(s) and can use this to transfer the risk to an insurer who pays out the difference if the risk event occurs.

model has five time intervals and six stages. Let the number of children nodes at each node be six, then the number of leaf nodes at the sixth stage is $6^5 = 7{,}776$.

The state space should also be discretized over the planning period within the scenario tree structure. Owadally et al. (2021a) and and Jang et al. (2022b) describe our scenario generation method and the characteristics of a generated scenario tree. It is worth emphasizing that the scenario tree is arbitrage-free; otherwise, the resulting solution will be biased or bounded to the limit.

Here, we define the notation used to model our problem on the scenario tree. The scenario tree starts from one root node $n_0$. The set of nodes in the tree at time $t$ is denoted by $\mathcal{N}_t$. Then, $\mathcal{N} = \bigcup_{t=0}^{\mathcal{T}} \mathcal{N}_t$ can be the set of all the nodes in the tree. The unconditional probability that a node $n$ occurs is $\mathbf{pr}_n$ and $\sum_{n\in\mathcal{N}_t} \mathbf{pr}_n = 1$. The parent node of a node $n$ is denoted by $n^-$. In the MSP model, all the variables previously indexed by time $t$ may now be indexed by node $n$.

(2) Objective function

The objective function in Eq. (1) is rewritten in a nodal form as follows:

$$\max_{\mathcal{D}_n, n\in\mathcal{N}} \left[ \sum_{t\in[0,\mathcal{T}]} \sum_{n\in\mathcal{N}_t} \left( \sum_{s=t}^{t+\Delta t-1} {}_{s}p_{\delta}\, e^{-\rho s}\, u\left(C_n^{1-\theta} H_n^{\theta}\right) \right) \mathbf{pr}_n \right.$$

$$+ \sum_{t\in[0,\mathcal{T})} \sum_{n\in\mathcal{N}_{t+\Delta t}} {}_{t}p_{\delta}\ {}_{\Delta t}q_{\delta+t}\ e^{-\rho(t+\Delta t)} \kappa^{\gamma} u(W_n) \mathbf{pr}_n$$

$$\left. + \sum_{s=\mathcal{T}}^{\tau-1} \sum_{n\in\mathcal{N}_{\mathcal{T}}} {}_{s}p_{\delta}\ {}_{\Delta s}q_{\delta+s}\ e^{-\rho(s+\Delta s)} \kappa^{\gamma} u\left(X_{W,n}\right) \mathbf{pr}_n \right], \qquad (13)$$

where $t$ occurs over the time stages during the planning horizon and $s$ denotes every year after the planning phase. The term ${}_{\Delta t}q_{\delta+t}$ denotes the probability that a $(\delta + t)$-year old person dies over the following $\Delta t$ years.

The first summation component in Eq. (13) shows that housing and non-housing consumption utilities for each $\Delta t$-long period are evaluated at every node $n \in \mathcal{N}$. The second component concerns bequest utility also during the planning phase. At the planning end $\mathcal{T}$, there must be a way to fulfill the individual's bequest demand. The final summation component in Eq. (13) concerns the bequest utility from a single-premium whole-life insurance policy which the individual can buy only at the planning end. This guarantees the monetary amount of $X_{W,\mathcal{T}}$ for $S_{W,\mathcal{T}}X_{W,\mathcal{T}}$, where $S_{W,\mathcal{T}}$ indicates the single-premium for a \$1 life benefit at the planning horizon end.

Similarly, the non-housing and housing consumption utilities from the planning end $\mathcal{T}$ to the life end $\tau$ are determined only by $C_{n_\mathcal{T}}$ and $H_{n_\mathcal{T}}$. The non-housing consumption amount comes only from annuities, which are directly and indirectly determined by the decision variables $\mathcal{D}_n = \mathcal{C}_n \cup \mathcal{H}_n \cup \mathcal{X}_n \cup \mathcal{A}_n,\ n \in \mathcal{N}$. As mentioned in Section 2, this MSP decision set is a shrunken form of the original one.

We separate buy and sell decisions. The trading decisions on cash and investments accounts are now $\mathcal{X}_n = \cup_{i=\{C,B,E\}} X_{i,n}^{buy}, X_{i,n}^{sell}$. Likewise, housing is the set of decisions with housing ownership, rental, reverse mortgage, and mortgages: $\mathcal{H}_n = \{X_{HA,n}^{buy}, X_{HA,n}^{sell}, X_{HR,n}, X_{R,n}^{buy}, X_{HD,n}, D_{P,n}\}$. Annuities can be bought, so $\mathcal{A}_n = \{X_{A(\psi),n}^{buy};\ \psi \in \{60,70,80\}\}$. Note that reverse mortgage and annuity purchases cannot be undone, so there are no $X_{R,n}^{sell}$ and $X_{A(\cdot),n}^{sell}$.

(3) Transaction costs and income taxes

Purchases, sales, and maintenance of any financial asset, housing unit, or deferred annuity are subject to the relevant upfront (buying), selling, and management fees. Various upfront, selling, and management fees were introduced in <Table 1>. We also consider taxes on labour income and annuity income.

The fees and taxes are controlled within cash balance and asset inventory constraints. The cash balance constraint controls cash inflows and outflows, as well as transaction costs. For $n \in \mathcal{N}$,

$$
\begin{aligned}
&\mathbb{1}_{\{n=n_0\}} w_0 + L_n \ddot{a}_n (1-\Upsilon_L) + I_{A,n} \ddot{a}_n (1-\Upsilon_A) + \mathbb{1}_{\{n \in \mathcal{N}_T\}} X_{R,n} S_{H,n} (1-\varphi_R^u) \\
&\quad + X_{HD,n} S_{H,n} (1-\varphi_D^u) + \sum_{i \in \{C,B,E,HA\}} X_{i,n}^{sell} S_{i,n} (1-\varphi_i^s) \\
&= C_n \ddot{a}_n + X_{HR,n} S_{H,n} \varphi_{HR}^u \ddot{a}_n + D_{P,n} + D_{I,n} \\
&\quad + \sum_{i \in \{C,B,E,HA,A(60),A(70),A(80)\}} X_{i,n}^{buy} S_{i,n} (1+\varphi_i^u) \\
&\quad + X_{HA,n} S_{H,n} \varphi_{HA}^m \ddot{a}_n + \mathbb{1}_{\{n \in \mathcal{N}_T\}} S_{W,n} X_{W,n} \qquad (14)
\end{aligned}
$$

Components on the l.h.s of Eq. (14) are related to cash inflows and ones on the r.h.s are to cash outflows. The investor has initial wealth $w_0 > 0$ at the root node $n_0$. He receives labour income subject to a constant tax rate $\Upsilon_L$, also applied to his social security benefits after retirement. Assume that income for the next $\Delta t$ years is paid in advance at node $n$, conditional on survival. We use the $\ddot{a}_n$ annuity factor to evaluate the labour income flow for a $\Delta t$-period as if it is a stock value. Using the standard actuarial symbols, the notations could be the term-life annuity $\ddot{a}_{x_n:\Delta t}$ at a node $n \in \mathcal{N} \backslash \mathcal{N}_{\mathcal{T}}$ and the whole-life annuity $\ddot{a}_{x_n}$ at a node $n \in \mathcal{N}_{\mathcal{T}}$, where $x_n$ is the corresponding age at that node $n$. This approach is applied to non-housing and housing consumption flows too, see $I_{A,n}$, $C_n$, $X_{HR,n}$, and $X_{HA,n}$ in Eq. (14).

Annuity income $I_{A,n}$ subject to a constant tax rate $\Upsilon_A$, which may be regarded as a retirement income tax. Since the first payment of longevity annuities depends on the time interval $\Delta t$, available annuities are with only three different deferment periods. Their first payment starts at $\psi = \{60,\ 70,\ 80\}$. Their prices are given by Eq. (10). The life-annuity income $I_{A,n}$ of one unit of the longevity annuity paying \$1 p.a. is:

$$I_{A,n} \quad = \quad \begin{cases} X_{A(60),n} \ , & \text{if } n \in \mathcal{N}_{30}, \\ X_{A(60),n} + X_{A(70),n} \ , & \text{if } n \in \mathcal{N}_{40}, \\ X_{A(60),n} + X_{A(70),n} + X_{A(80),n} \ , & \text{if } n \in \mathcal{N}_{50}, \\ 0 \ , & \text{otherwise.} \end{cases} \tag{15}$$

The investor also receives a lump sum of $X_{R,n}S_{H,n}$ if he purchases a reverse mortgage contract subject to a upfront fee at rate $\varphi_R^u$. The contract is allowed only at retirement. He earns a mortgage advance if he takes out mortgage debt subject also to a fee at rate $\varphi_D^u$. Finally, there is a cash income if any financial asset or housing units are sold, subject to the relevant selling cost $\varphi_i^s$.

Cash outgo on the r.h.s. of (14) consists of the following: non-housing consumption $C_n$, rent $X_{HR,n}$ at a fixed rate of $\varphi_{HR}^u$ per housing unit price $S_{H,n}$, mortgage debt principal repayment $D_{P,n}$, and mortgage interest payment $D_{I,n}$. Unit purchases of any financial asset, housing unit, or longevity and immediate annuities are subject to the relevant upfront (buying) fee $\varphi_i^u$. There is also a management cost for the upkeep of housing units in ownership at a rate of $\varphi_{HA}^m$; this includes housing under a mortgage charge and housing that has been released to reverse mortgage but is still under occupation. Finally, whole-life insurance may be purchased at the planning end $\mathcal{T}$ if bequest yields utility as in the last component of Eq. (13).

We assume a fixed percentage investment management fee $0 \leq \varphi_i^m \ll 1$ for the asset $i$. Asset inventory constraints track the number $X_{i,n}$ of units of financial asset $i \in \{C, B, E\}$ held at node $n \in \mathcal{N}$ after the management fee:

$$X_{i,n} = \mathbb{1}_{\{n \neq n_0\}} X_{i,n-}(1 - \varphi_i^m) + X_{i,n}^{buy} - X_{i,n}^{sell}. \tag{16}$$

Likewise, the inventory constraints for annuity can track the number $X_{i,n}$ of units of annuities $i \in \{A(60), A(70), A(80)\}$ held at node $n \in \mathcal{N}$:

$$X_{i,n} = \mathbb{1}_{\{n \neq n_0\}} X_{i,n-} + X_{i,n}^{buy}. \tag{17}$$

Note that annuities can be bought but not sold and no management fee charged to the individual investor to hold them.

The fees may differ by investment styles and strategies or by tax rules on a specific account. Active equity funds, for example, tend to charge more management and selling fees than passive bond funds (French, 2008). Selected investment funds in a qualified pension plan may charge lower upfront and selling fees. This is because investors are encouraged to hold and rebalance their invesetment portfolios within their pension account for a longer period until or even after their retirement. In order to investigate the effects of housing and annuity choices, we do not differentiate the fee structures by investment styles, strategies, or pension plans.

(4) Mortgage and reverse mortgage

We assume that the mortgage is an interest-only mortgage with flexible repayments of the principal. At a node $n \in \mathcal{N}_t$, mortgage interest $D_{I,n}$ must be paid at a variable rate $r_t + \alpha_M$, where $\alpha_M$ is a mortgage rate premium. The individual is free to choose the principal repayment $D_{P,n}$, but all mortgage principal must be repaid in full at the planning end $\mathcal{T}$. The maturity of a newly issued mortgage debt $X_{HD,n}$, $n \in \mathcal{N}_t$ cannot be longer than a time period of $\mathcal{T} - t$. The mortgage outstanding balance over the planning horizon then can be tractable by the equation below.

$$D_{H,n} = \begin{cases} X_{HD,n} \ S_{H,n} & \text{if } n \in n_0 \\ D_{H,n^-} + X_{HD,n} \ S_{H,n} - D_{P,n} & \text{if } n \in \mathcal{N}_t, 0 < t < \mathcal{T} \\ D_{H,n^-} - D_{P,n} & \text{if } n \in \mathcal{N}_{\mathcal{T}}, \end{cases} \tag{18}$$

where $D_{H,n} = 0$, if $n \in \mathcal{N}_{\mathcal{T}}$.

At retirement, the individual has an option to release his housing asset by contracting with a reverse mortgage provider. Replacing $t$ with $n$ in Eq. (11), the outstanding balance of a reverse mortgage debt $D_{R,n}$ accumulates at a rate of $R_n^{NNEG}$. In addition, to keep the initial loan-to-value (LTV) condition hold in retirement, increasing the minimum housing ownership is required:

$$X_{HA,n} \geq \frac{X_{R,n_T}}{\Lambda_{LTV(R)}} \quad \text{if } n \in \mathcal{N}_t, t \geq T, \tag{19}$$

where $n_T$ belongs to a set of $\mathcal{N}_T$.

A regular mortgage provider imposes a LTV ratio of $\Lambda_{LTV}$, so $D_{H,t} \leq X_{HA,t}\, S_{H,t}\, \Lambda_{LTV}$, in the absence of reverse mortgage before retirement. Since the reverse mortgage provider requires the minimum housing ownership as in Eq. (19), the regular mortgage outstanding balance after retirement should take the reverse mortgage contract into account:

$$D_{H,n} \leq \left(X_{HA,n} - \mathbb{1}_{\{t \geq T\}} X_{R,n_T}\right) S_{H,n} \Lambda_{LTV} n \in \mathcal{N}_t, 0 \leq t < \mathcal{T}. \tag{20}$$

It is noteworthy that the reverse mortgage LTV ratio is usually lower than the regular mortgage's ($\Lambda_{LTV(R)} < \Lambda_{LTV}$), since the outstanding balance tends to increase in the reverse mortgage, but decrease in the regular mortgage.

(5) Wealth for bequest

The decision set $\mathcal{D}_t$ affects the individual's wealth through his lifetime. The bequest wealth includes financial assets hold and net house ownership. Decision on a reverse mortgage contract also affects wealth bequeathed to heirs. With reverse mortgage, the owner retains the right to live in his home until he dies (or moves permanently out), at which point the reverse mortgage debt is repaid by selling the house. Annuities are irreversible. If the investor dies at some time between $t - \Delta t$ and $t$, correspondingly nodes $n^-$ and $n$, the bequeathed wealth $W_n$ to heirs follows

$$W_n = \sum_{i \in C,B,E} X_{i,n^-}\, S_{i,n}\, (1 - \varphi_i^m) + X_{HA,n^-}\, S_{H,n}\, (1 - \varphi_{HA}^s)$$
$$- \left(D_{H,n^-} + D_{I,n} + 1_{n \in N_t, t > T} D_{R,n}\right), \quad n \in \mathcal{N} \backslash n_0. \tag{21}$$

<Table 1> Default values of market model parameters

| *Interest rate and bond* (2), (6) | | | *Equity* (3) | | |
|---|---|---|---|---|---|
| $r_0$ | short rate at time 0 | 0.02 | $\mu_E$ | equity return expectation | 0.05 |
| $\tilde{\theta}$ | long-term mean level of short rate | 0.02 | $\sigma_E$ | equity return volatility | 0.2 |
| | | | *Housing* (7) | | |
| $\nu$ | mean reversion speed | 0.2 | $\mu_H$ | house return expectation | -0.011 |
| $\sigma_0$ | short rate volatility | 0.015 | $\sigma_H$ | house return volatility | 0.12 |
| $M$ | long-term bond maturity | 20 | *Correlations* (4), (8) | | |
| $\mu_0$ | market price of interest rate risk | 0.0075 | $\rho_{EB}$ | equity and bond correlation coefficient | 0 |
| $S_{H,0}$ | initial housing price per square metre | \$2,500 | $\rho_{HB}$ | house and bond correlation coefficient | 0.65 |
| | | | $\rho_{HE}$ | house and equity correlation coefficient | 0.5 |

<Table 2> Default values of fees and market rules

| | | |
|---|---|---|
| *Fees for financial assets* | | |
| $\varphi_C^u, \varphi_B^u \varphi_E^u$ | upfront fees for cash, bond, equity funds | {0.0, 0.005,0.005} |
| $\varphi_C^s, \varphi_B^s \varphi_E^s$ | selling fees for cash, bond, equity funds | {0.0, 0.005,0.005} |
| $\varphi_C^m, \varphi_B^m \varphi_E^m$ | management fees for cash, bond, equity funds | {0.0, 0.0, 0.0} |
| *Fees and rules for housing* | | |
| $\varphi_{HA}^u, \varphi_{HA}^s, \varphi_{HA}^m$ | upfront, selling, and management fees for housing ownership | {0.1,0.1,0.01} |
| $\varphi_{HR}^u$ | upfront fee for housing rental | 0.06 |
| $\varphi_D^u$ | upfront fee for regular mortgage issuance | 0.0 |
| $\alpha_M$ | regular mortgage rate premium | 0.02 |
| $\Lambda_{LTV}$ | loan-to-value ratio for regular mortgage | 0.8 |
| $\Lambda_{MinH}$ | minimum housing units in square meters | 4.0 |
| *Fees and rules for retirement income* | | |
| $\alpha_R$ | reverse mortgage rate premium | 0.01 |
| $\varphi_R^u$ | upfront NNEG premium on withdrawals | 0.0075 |
| $\varphi_R^m$ | regular NNEG premium on the outstanding balance | 0.01 |
| $\Lambda_{LTV(R)}$ | loan-to-value ratio for reverse mortgage | 0.5 |
| $\varphi_{A(\psi)}^u$ | upfront fees for longevity annuities | 0.03 |
| *Social security and income taxes* | | |
| $\Upsilon_S$ | social security replacement ratio | 0.6 |
| $\Upsilon_L$ | labour income tax | 0.0 |
| $\Upsilon_A$ | annuity income tax | 0.0 |

<Table 3> Wealth, labour income and personal preferences

| Item | Description | Value |
|---|---|---|
| $w_0$ | initial wealth | \$40,000 |
| $l_0$ | annual wage until retirement | \$40,000 |
| $\gamma$ | risk aversion | 5.0 |
| $\rho$ | time preference | 0.03 |
| $\kappa$ | bequest preference | 4.0 |

## V. Numerical results

In order to investigate optimal decisions on investment, housing and annuitization, we solve the life-cycle investment and consumption problem for an individual who can invest in financial assets and buy as well rent a house. The MSP problem described earlier is to optimize the objective function in (13) subject to the constraints (14) to (21); all variables are non-negative.[5,6] He has access to a mortgage. In retirement, the individual can also buy immediate and longevity annuities to provide him with income in retirement. He can also release equity in housing using a reverse mortgage contract. We are particularly interested in the advantages that reverse mortgage and longevity annuities can confer in retirement, so we construct five separate cases with and without combinations of these products, see <Table 4>.

<Table 4> Availability of products

| | Immediate annuity at retirement | Longevity annuities | Reverse mortgage |
|---|---|---|---|
| Benchmark | No | No | No |
| Case A | Yes | No | No |
| Case B | Yes | Yes | No |
| Case C | Yes | No | Yes |
| Case D | Yes | Yes | Yes |

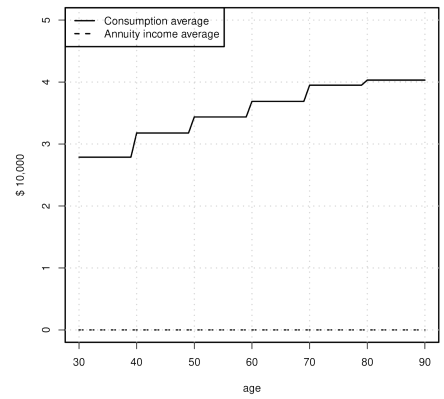
## 1. Benchmark case

We construct a benchmark case in which a 30-year-old individual expects to retire at age 60, i.e., $\delta = 30$, $T = 30$. His lifetime consumption plan is as described in Sections 2 and 4, with decisions taken every $\Delta t = 10$ years and a planning horizon of $\mathcal{T} = 50$ years.[7] In the benchmark case, the individual is male, with risk aversion coefficient $\gamma = 5$, time preference $\rho = 0.03$, and bequest parameter $\kappa = 4.0$, as in <Table 2>. His annual wage is fixed at $l_0 = \$40{,}000$ until retirement, whereupon he receives a social security benefit of $l_0 \cdot \Upsilon_S$ for lifetime, where $\Upsilon_S = 0.6$ is a replacement ratio. In the benchmark case, annuities and reverse mortgage are not available. Various upfront, selling and management fees are introduced in Section 4. <Table 2> summarises the relevant parameter values, as well as other parameter values related to housing, regular mortgage and reverse mortgage, in Eqs. (14)–(21). We adopt the parameter values of Kraft and Munk (2011) for the financial and housing markets: see <Table 1>.

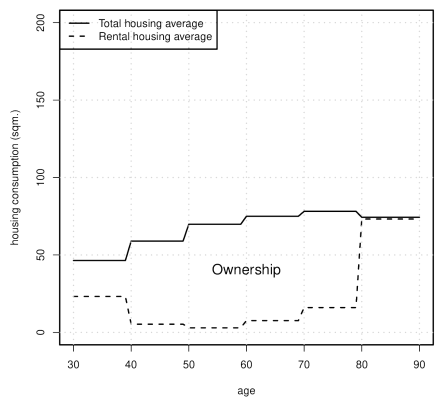
[5] To search the optimal decisions, we use a non-linear solver, MOSEK. It uses an interior point algorithm which is known as an efficient method to solve a large-scale optimization problem with a non-linear objective function with linear constraints (Nemirovsky and Todd, 2008).
[6] Given that our constructed scenario tree is arbitrage-free, the set of constraints guarantees optimal decisions admissible and tractable.
[7] The 10-year intervals are chosen to collect numerical results at a reasonable amount of time; each run takes about 30 seconds.

(1) Consumption and bequest

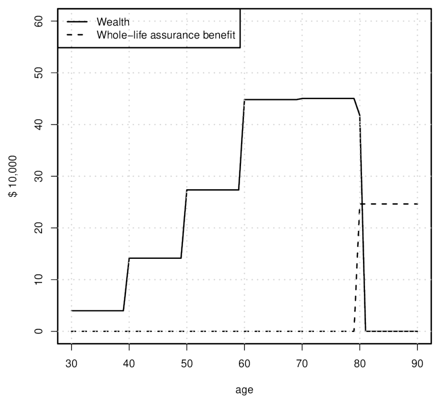
Maximizing the utilities of non-housing and housing consumption, as well as wealth for bequest are the individual's objectives of this life-cycle investment problem. Results from the benchmark case, as in <Figure 1>(a), show a clear upward pattern of average non-housing consumption over lifetime. Average housing consumption levels in <Figure 1>(b) show a similar pattern, but decreasing at the end of planning horizon. This may be because the bequest motive becomes stronger as his mortality is greater. At early and older ages, renting is preferred to owning. Wealth for bequest in <Figure 1>(c) is increasing with age, but relatively flat between ages 60 and 80. Our model assumes that the bequest demand after the planning horizon should be fulfilled by whole life assurance, which the investor can buy only at the planning end. It may be concerned that the financial assets and housing ownership are not considered as bequest wealth after the planning end.




(a) Non-housing consumption




(b) Housing consumption

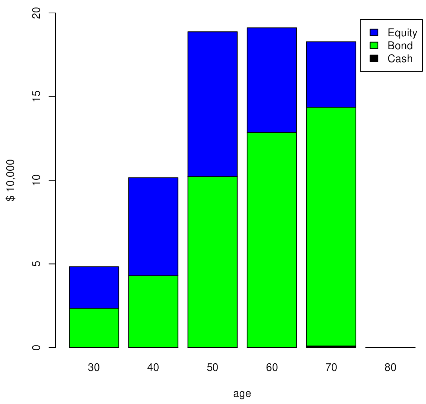


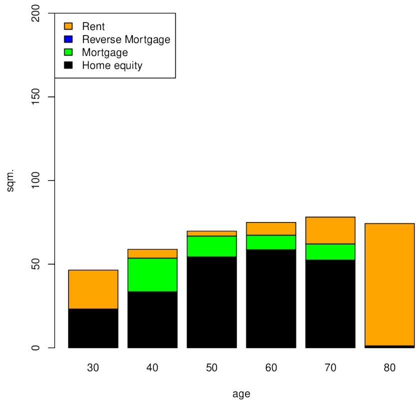

(c) Wealth for bequest

<Figure 1> Average number of housing units ($\times\, m^2$) over lifetime for Benchmark case.

(2) Financial asset allocation

<Figure 2>(a) shows after-rebalancing financial asset allocations among cash, bond and equity. Their levels differ from wealth for bequest in <Figure 1>(c), which is before-rebalancing wealth. The benchmark shows a hump shape of equity allocations over lifetime: 51.2% at age 30, 57.7%, 45.8%, 32.7%, and 21.4% at age 70. Overall, equity allocation decreases and bond allocation increases over lifetime. At age 70, the bond allocation is also affected by the bequest and consumption demands after age 80. They are fulfilled only by purchasing a whole life assurance $S_{W,n}X_{W,n}$ and by increasing the consumption factor $C_n\ddot{a}_n$ respectively, see Eq. (14). The wealth at age 80 is zero due to all spent on the annuitized non-housing and housing consumption, as well as on the whole life assurance.




(a) Financial asset allocation (b) Housing compositions

<Figure 2> Average number of housing units ($\times m^2$) over lifetime for Benchmark case.

(3) Housing and mortgage

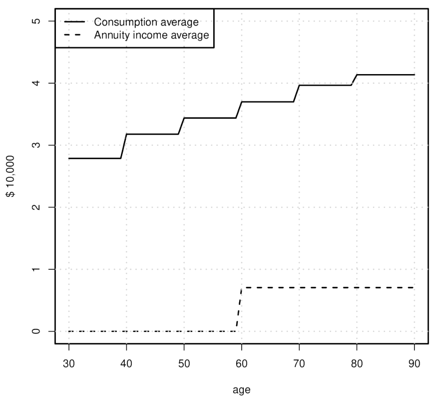
As mentioned above, <Figure 2>(b) confirms that the investor prefers rentals to ownership on housing at early and older ages. Loan-to-value ratios are significantly increasing from 0.0% to 37.6% at age 40, and decreasing from 18.6% at age 50 to 15.5% at age 70. The model enforces the individual to pay off the remaining mortgage balance by the end of the planning horizon.

## 2. Longevity annuities and reverse mortgage

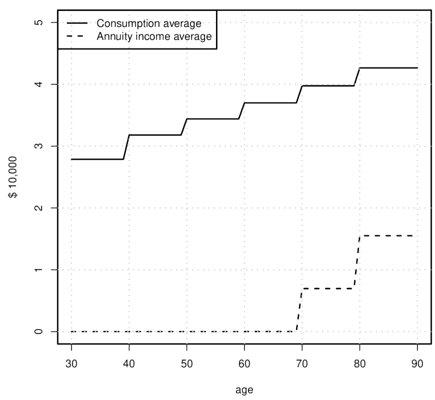
Availability of the longevity annuities and reverse mortgage may affect optimal investment and housing decisions to maximize the housing and non-housing consumption utility. Hereafter, we compare the five cases in <Table 4> to investigate changes in the levels and the decisions.

(1) Longevity annuities

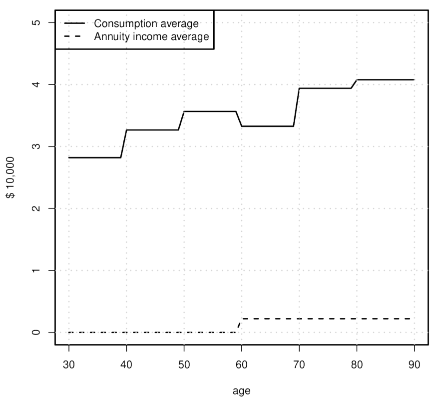
Comparing Case B with Case A or the benchmark, <Figure 3> shows higher average non-housing consumption levels at ages 70 and 80 with longevity annuities. Interestingly, the attractiveness of the longevity annuity $A(80)$ is stronger when a reverse mortgage contract is available.



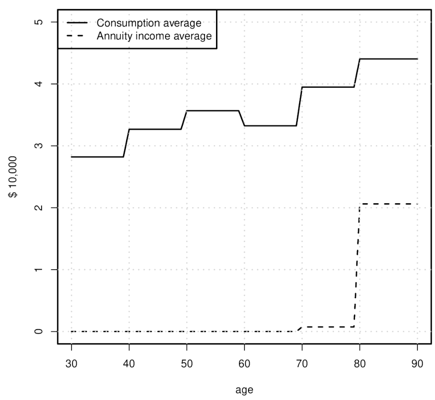

(a) Case A: LA(−), RM(−)




(b) Case B: LA(+), RM(−)

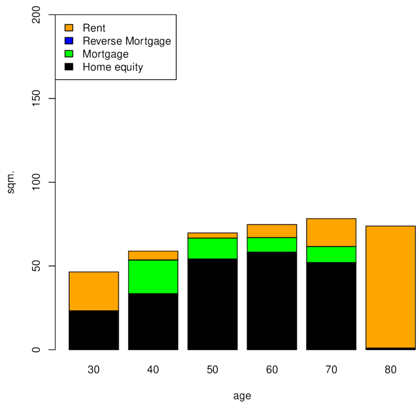



(c) Case C: LA(−), RM(+)

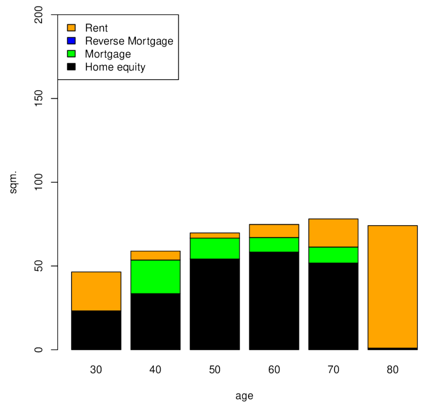



(d) Case D: LA(+), RM(+)

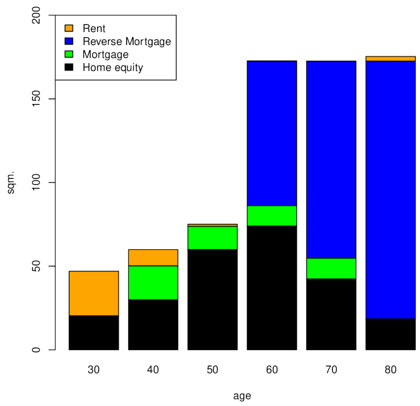
<Figure 3> Average non-housing consumption and annuity income ($\times$ \$ 10,000) over lifetime for cases A–D. Abbreviations: LA = Longevity annuities. RM = Reverse mortgage. $(+)$ = available. $(-)$ = not available.



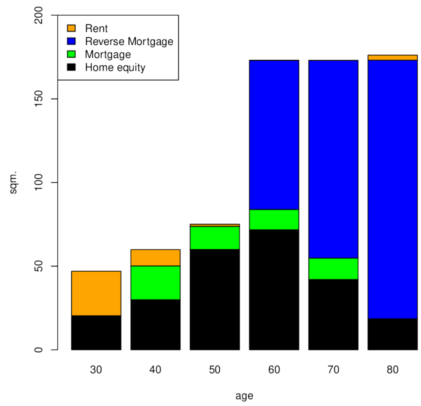

(a) Case A: LA(−), RM(−)




(b) Case B: LA(+), RM(−)




(c) Case C: LA(−), RM(+)




(d) Case D: LA(+), RM(+)

<Figure 4> Average number of housing units ($\times m^2$) over lifetime for cases A–D. Abbreviations: LA = Longevity annuities. RM = Reverse mortgage. (+) = available. (−) = not available.

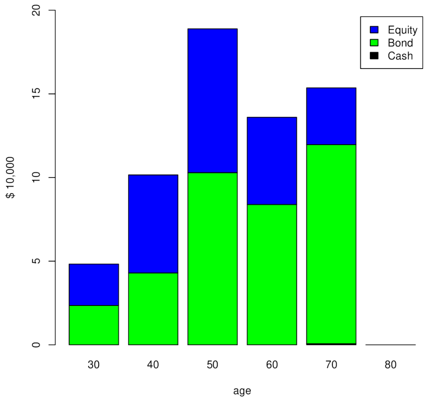
(2) Housing composition

Longevity annuities do not affect the housing decision between ownership and rent, on average. There are small increases in housing consumption from rent over lifetime on average. <Figures 4>(a) and (c) show that reverse mortgage significantly changes housing consumption levels and their compositions over lifetime. In early working ages in 30 and 40, renting is more preferred to ownership, but from age 50, the individual is pursuing housing ownership and keeps mortgage debt higher and longer in retirement. In case C, the housing-unit ratio of average reverse mortgage to housing asset is 50% and the ratio increases to 89% at age 80. Thanks to the NNEG option, it cannot be greater than 100%.

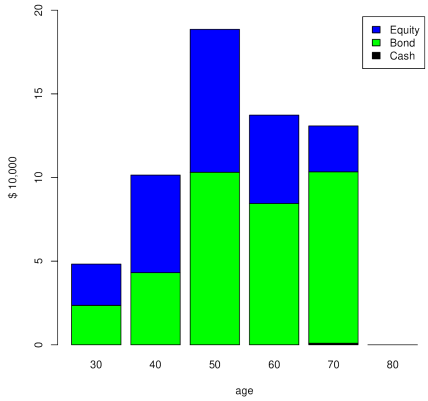
Housing units used to acquire the reverse mortgage agreement are varying, see the moderate risk averse in <Figure 7>(b). <Figure 7> shows the average outstanding balance of reverse mortgage over lifetime and its issuance distribution at retirement, both in housing units.

(3) Financial asset allocation

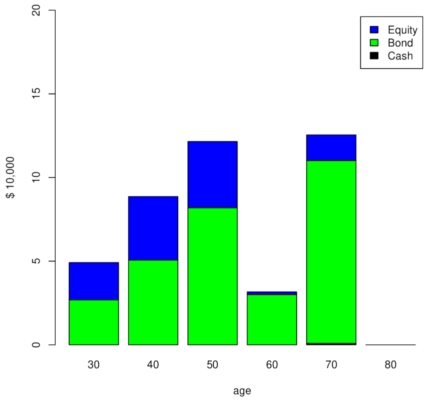
Availability of longevity annuities and reverse mortgage also affects the optimal financial asset allocation. When immediate annuity is available in Case A, the average allocation shows a marginally-lower equity weights in the working period and higher weights from retirement than the benchmark case. Case B with longevity annuities exhibits



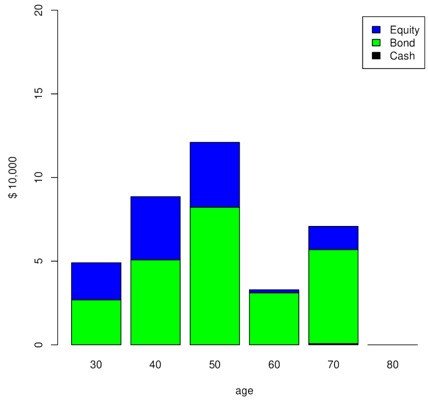

(a) Case A: LA(−), RM(−)




(b) Case B: LA(+), RM(−)




(c) Case C: LA(−), RM(+)



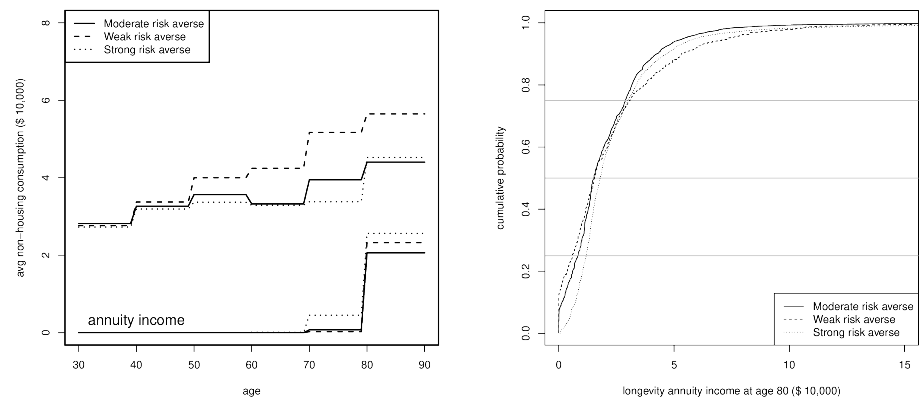

(d) Case D: LA(+), RM(+)

<Figure 5> Average financial asset allocation over lifetime for cases A–D. Abbreviations: LA = Longevity annuities. RM = Reverse mortgage. (+) = available. (−) = not available.

a clearer pattern like Case A, but the equity allocation at age 70 is lower than Case A as well as the benchmark. This can be explained by hedging behaviour on longevity-annuity prices. Cases C and D with reverse mortgage show much lower wealth levels from age 50 to 70. This is because reverse mortgage is higly expected and used to acquire a stable retirement income stream. Case C shows much lower equity allocations than Cases A and B. As reverse mortgage becomes a major source to buy the immediate annuity, the optimal investment strategy focuses on hedging the risk of long-term interest rates, which determine the annuity price. Case D shows even lower than Case C in the working period, but higher from retirement and still much lower than Case A and B.

## 3. Personal preference effects

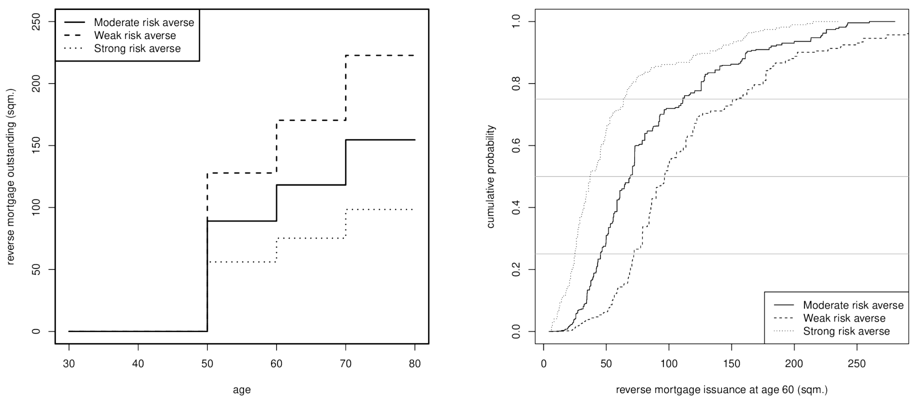
Personal preferences on risk aversion, time, bequest, and housing may change optimal choices on longevity annuities and reverse mortgage. As expected, <Figure 6>(a) shows that a relatively-strong risk-averse individual ($\gamma = 8$) to the moderate ($\gamma = 5$) chooses more stable and flat consumption levels over his lifetime. A weak risk-averse individual ($\gamma = 2$) chooses more volatile and fast growing consumption levels on average. We can verify the choice of income volatility by looking at the distribution of longevity annuity income at age 80, see <Figure 6>(b). Whereas the strong risk-averse's annuity income distribution is located on the right side of the moderate risk-averse's, the dotted-line for the weak risk-averse crosses the solid-line for the moderate.




(a) Average non-housing consumption (b) Longevity annuity income distribution

<Figure 6> Average non-housing consumption and annuity income over lifetime and longevity annuity income distribution at age 80.

When it comes to the optimal choices on reverse mortgage with different risk aversion coefficients, <Figure 7> shows a consistent pattern between the level and volatility. The weak risk-averse prefers a higher level and higher volatility of the reverse mortgage choice. On the other hand, the strong risk-averse selects the lower level and volatility.




(a) Average reverse motgage balance

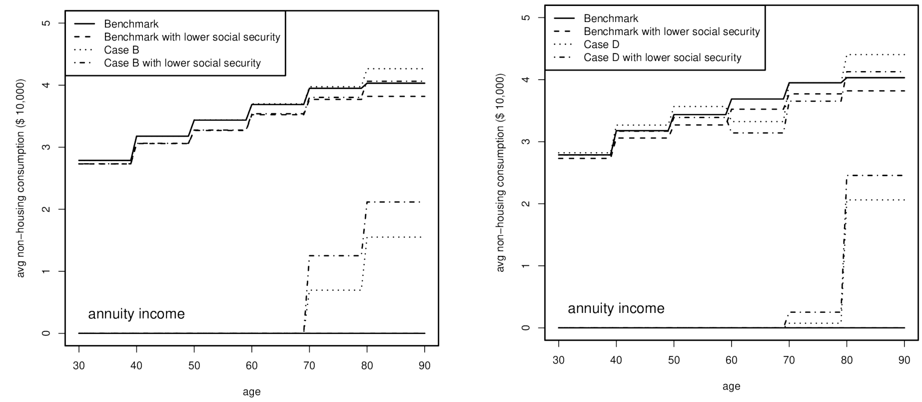
(b) Reverse mortgage issurance distribution

<Figure 7> Average reverse mortgage outstanding balance ($\times m^2$) over lifetime and reverse mortgage issuance distribution ($\times m^2$) at age 80 .

## 4. Social security effects

The replacement ratio is set to 60 % ($\Upsilon_S = 0.6$) in the previous cases. This high level security is raising a serious concern on its sustainability. Most developed countries are adjusting this number to some extent, for example by lowering the level itself or by deferring statutory retirement age, which results in a pension-liability decrease. We investigate the social security effects on the optimal longevity-annuity and reverse-mortgage decision, and their contributions to the optimized non-housing and housing consumption levels.

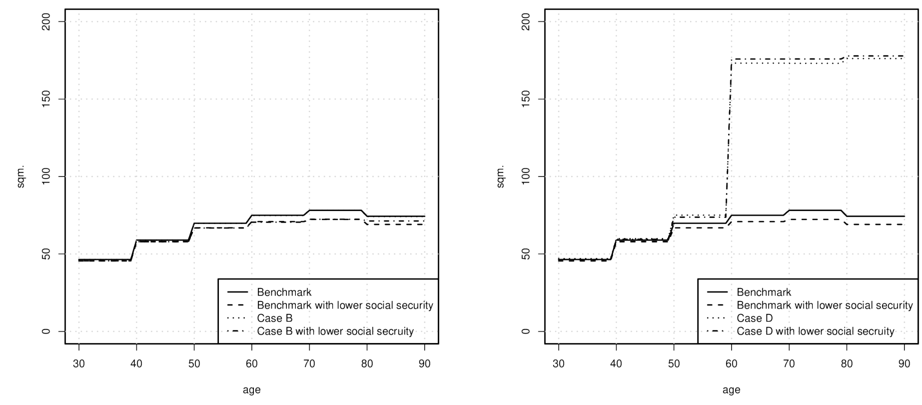
Longevity annuities in the lower social security could increase average consumption levels to ones with the higher social security in advanced ages. <Figure 8>(a) shows that before starting the income payments from the longevity annuities, the availability of them does not affect non-housing consumption levels on average, whether or not the social secruity benefit is lower ($\Upsilon_S = 0.4$). From age 70, however, the individual in “Case B with lower social seruity” receives the longevity annuity income. His average consumption levels are slightly higher than “Benchmark with lower social security” in age 70s and even than “Benchmark” in age 80s. With a combination of longevity annuities and reverse mortgage, the levels can be much higher before and after retirement, see <Figure 8>(b).




(a) Benchmark & Case B (b) Benchmark & Case D

<Figure 8> Average non-housing consumption and annuity income over lifetime and longevity annuity income distribution at age 80.

<Figure 9> shows average housing consumption units over lifetime when the social security level is lower. The individual in the benchmark with lower social security consumes housing units less than one in the original benchmark. When longevity annuity is available in <Figure 9>(a), there is a marginal increase in the housing consumption at age 80 and after. In contrast to the benchmark case, case D with reverse mortgage shows higher housing consumption units from retirement when the social security level is lower.




(a) Benchmark & Case B (b) Benchmark & Case D

<Figure 9> Average housing consumption units ($\times\ m^2$) over lifetime for benchmark, case B and case D with a lower social security level.

<Table 5> compares average housing composition ratios over lifetime when the social security level is high (left columns) and low (right columns). As the replacement ratio of the social security reduces by 20 %p from $\Upsilon_S = 60\%$ to $40\%$, the individual prefers holding a housing asset to rent, on average, in any cases. The rent composition ratios are about 15%p lower at age 30 in cases A and B with the lower social security level. Also, he takes less mortgage debts in his working ages from 30's to 50's on average, see the third row of "(B)/(A+B+C)". This behaviour becomes clearer when a reverse mortgage product is available. In cases C and D, the lower social security level do not make a significant change in the reverse mortgage decision, in terms of the composition ratios, see the fifth row of "(C)/(A+B+C)".

## VI. Conclusion

We solve a life-cycle optimal investment and housing decision problem for a risk-averse individual to maximize his utility of non-housing and housing consumption. Our research focus is on the effects of longevity annuities and reverse mortgage on the individual's optimized housing and non-housing consumption, as well as optimal decisions on investment and housing composition. Jang et al. (2022) incorporate only home reversion into their model; home reversion is a type of home equity release products.

According to our numerical results with different combinations of product availabilities, longevity annuities help the individual achieve higher retirement income and non-housing consumption in advanced ages, on average. A reverse mortgage raises non-housing consumption before retirement and helps support income after retirement. The home-equity release product significantly affects the housing decision, whereas the longevity annuity does not. Expecting the reverse mortgage in use at retirement, the investor prefers housing ownership to renting. The decision on reverse mortgage is almost bounded to its loan-to-value (LTV) limit, which is 50% in the numerical cases that we consider. This is because all costs and fees accumulate on top of the outstanding balance on the reverse mortgage with the no negative equity guarantee (NNEG) option.

The availability of longevity annuities and reverse mortgage affects life-cycle optimal financial asset allocations. With longevity annuities, it is optimal to weight long-term bonds more heavily than without longevity annuities, because these bonds are a hedging instrument to price change in the longevity annuities. With reverse mortgage, financial wealth levels over lifetime are much lower than cases without reverse mortgage, as wealth accumulation is concentrated on housing ownership; allocation to long-term bonds also increases.

Longevity annuities can help an individual investor smooth their consumption over their lifetime. A less risk-averse individual, for example, purchases more longevity annuity paying out from age 80 than one paying out from age 70. On the other hand, a more risk-averse individual purchases more of the age-70 longevity annuity than the less risk-averse individual. Risk preferences also govern the amount that is borrowed in a reverse mortgage.

When the replacement ratio of social security to pre-retirement income drops from 60% to 40%, average non-housing and housing consumption levels are also reduced. Our model finds that longevity annuities and reverse mortgage can substitute partially for social security and support consumption, especially at advanced ages. Individuals prefer owning a house to renting and pay down their mortgage debt whilst they are still working. Our numerical results show the importance of alternative retirement-income instruments in the event that the social security net fails.

Although our numerical results demonstrate the advantages of using longevity annuities and reverse mortgage, there are some limitations to our model and to our analyses. Reverse mortgage products are available after retirement, but also before. In this case, house price autocorrelation in the short term may affect housing decisions and reverse mortgage choice, and this is a feature that we did not consider. We also neglect mortgage default risk. Labour income could be stochastic and correlated with other assets, such as property prices (Kraft and Munk, 2011). The timing of retirement and the supply of labour (flexibility over working hours) are also decision variables (Gomes et al., 2008), whereas we fixed them in our model. Life-cycle investment choices could be affected by fiscal policies on tax-exempt or tax-advantaged accounts (Gomes et al., 2009). Lastly, we disregard any external and unexpected adjustments to the annuity benefits level (Park, 2021). We will explore these limitations and issues in future work.

<Table 5> Average houing composition ratios (%) over lifetime.

| | Case A: LA(−), RM(−) & $\Upsilon_S$ = 60% | | | | | | Case A: LA(−), RM(−) & $\Upsilon_S$ = 40% | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Age | 30 | 40 | 50 | 60 | 70 | 80 | 30 | 40 | 50 | 60 | 70 | 80 |
| Home Equity (A) | 50.05 | 56.78 | 77.72 | 77.96 | 66.50 | 1.39 | 66.67 | 62.82 | 81.60 | 81.63 | 70.07 | 0.66 |
| Mortgage (B) | 0.00 | 34.17 | 17.83 | 11.68 | 12.20 | 0.00 | 0.00 | 34.63 | 17.24 | 11.99 | 13.08 | 0.00 |
| (B) / (A+B+C) | 0.00 | 0.38 | 0.19 | 0.13 | 0.16 | 0.00 | 0.00 | 0.36 | 0.17 | 0.13 | 0.16 | 0.00 |
| Reverse Mortgage (C) | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| (C) / (A+B+C) | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| Rent | 49.95 | 9.05 | 4.45 | 10.36 | 21.30 | 98.61 | 33.33 | 2.56 | 1.17 | 6.38 | 16.85 | 99.34 |
| | Case B: LA(+), RM(−) & $\Upsilon_S$ = 60% | | | | | | Case B: LA(+), RM(−) & $\Upsilon_S$ = 40% | | | | | |
| Age | 30 | 40 | 50 | 60 | 70 | 80 | 30 | 40 | 50 | 60 | 70 | 80 |
| Home Equity (A) | 50.03 | 56.77 | 77.67 | 77.95 | 66.27 | 1.37 | 66.43 | 62.75 | 81.56 | 81.63 | 70.18 | 0.57 |
| Mortgage (B) | 0.00 | 34.17 | 17.83 | 11.67 | 12.24 | 0.00 | 0.00 | 34.62 | 17.24 | 11.98 | 13.12 | 0.00 |
| (B) / (A + B + C) | 0.00 | 0.38 | 0.19 | 0.13 | 0.16 | 0.00 | 0.00 | 0.36 | 0.17 | 0.13 | 0.16 | 0.00 |
| Reverse Mortgage (C) | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| (C) / (A + B + C) | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| Rent | 49.97 | 9.07 | 4.51 | 10.38 | 21.49 | 98.63 | 33.57 | 2.62 | 1.20 | 6.39 | 16.70 | 99.43 |
| | Case C: LA(−), RM(+) & $\Upsilon_S$ = 60% | | | | | | Case C: LA(−), RM(+) & $\Upsilon_S$ = 40% | | | | | |
| Age | 30 | 40 | 50 | 60 | 70 | 80 | 30 | 40 | 50 | 60 | 70 | 80 |
| Home Equity (A) | 43.38 | 50.00 | 79.90 | 42.85 | 24.57 | 10.60 | 51.53 | 60.38 | 81.82 | 41.54 | 24.80 | 10.94 |
| Mortgage (B) | 0.00 | 33.81 | 18.44 | 7.08 | 7.16 | 0.00 | 0.00 | 34.39 | 18.18 | 6.96 | 7.12 | 0.00 |
| (B) / (A + B + C) | 0.00 | 0.40 | 0.19 | 0.07 | 0.07 | 0.00 | 0.00 | 0.36 | 0.18 | 0.07 | 0.07 | 0.00 |
| Reverse Mortgage (C) | 0.00 | 0.00 | 0.00 | 49.92 | 68.23 | 87.81 | 0.00 | 0.00 | 0.00 | 51.50 | 68.05 | 88.11 |
| (C) / (A + B + C) | 0.00 | 0.00 | 0.00 | 0.50 | 0.68 | 0.89 | 0.00 | 0.00 | 0.00 | 0.51 | 0.68 | 0.89 |
| Rent | 56.62 | 16.19 | 1.65 | 0.16 | 0.05 | 1.60 | 48.47 | 5.24 | 0.00 | 0.00 | 0.03 | 0.95 |
| | Case D: LA(+), RM(+) & $\Upsilon_S$ = 60% | | | | | | Case D: LA(+), RM(+) & $\Upsilon_S$ = 40% | | | | | |
| Age | 30 | 40 | 50 | 60 | 70 | 80 | 30 | 40 | 50 | 60 | 70 | 80 |
| Home Equity (A) | 43.36 | 49.91 | 79.86 | 41.43 | 24.29 | 10.53 | 51.32 | 60.16 | 81.81 | 41.66 | 24.55 | 10.85 |
| Mortgage (B) | 0.00 | 33.81 | 18.44 | 7.01 | 7.38 | 0.00 | 0.00 | 34.38 | 18.18 | 6.84 | 7.32 | 0.00 |
| (B) / (A + B + C) | 0.00 | 0.40 | 0.19 | 0.07 | 0.07 | 0.00 | 0.00 | 0.36 | 0.18 | 0.07 | 0.07 | 0.00 |
| Reverse Mortgage (C) | 0.00 | 0.00 | 0.00 | 51.43 | 68.30 | 87.76 | 0.00 | 0.00 | 0.00 | 51.50 | 68.12 | 88.11 |
| (C) / (A + B + C) | 0.00 | 0.00 | 0.00 | 0.51 | 0.68 | 0.89 | 0.00 | 0.00 | 0.00 | 0.51 | 0.68 | 0.89 |
| Rent | 56.64 | 16.27 | 1.70 | 0.12 | 0.03 | 1.71 | 48.68 | 5.46 | 0.00 | 0.00 | 0.01 | 1.04 |

Abbreviation: Abbreviations: LA = Longevity annuities. RM = Reverse mortgage. (+) = available. (−) = not available.

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