# Feedback Control over Noisy Channels: Characterization of a General Equilibrium

Touraj Soleymani, John S. Baras, Sandra Hirche, and Karl H. Johansson

***Abstract*—In this article, we study an energy-regulation trade-off that delineates the fundamental performance bound of a feedback control system over a noisy channel in an unreliable communication regime. The channel and the process are modeled by an additive white Gaussian noise channel with fading and a partially observable Gauss-Markov process, respectively. Moreover, the feedback control loop is constructed by designing an encoder with a scheduler and a decoder with a controller. The scheduler and the controller are the decision makers deciding about the transmit power and the control input at each time, respectively. Associated with the energy-regulation trade-off, we characterize an equilibrium at which neither the scheduler nor the controller has a unilateral incentive to deviate from its policy. We argue that this equilibrium is a general one as it attains global optimality without any restrictions on the information structure or the policy structure, despite the presence of signaling and dual effects.**

***Index Terms*—communication channels, energy-regulation trade-off, feedback control, global optimality, packet loss, power adaptation, stochastic processes.**

## I. Introduction

WIRELESS COMMUNICATION can provide an effective solution for feedback control systems [1]. Exploiting the unique characteristics of wireless communication, one can realize unprecedented wireless control systems in which sensors are connected to actuators via wireless channels. Such control systems are envisioned to have abundant applications in automotive, automation, healthcare, and space exploration. Nevertheless, wireless channels, which are to close the feedback control loops in these systems, are highly subject to noise. A direct consequence of the channel noise in real-time tasks[1] is *packet loss*[2], which severely degrades the performance of the underlying control system or even yields instability. To decrease the packet error rate, for any fixed rate, bandwidth, and modulation, the transmit power needs to increase. This in turn raises the energy consumption of the transmitter, which is often afflicted with a constrained energy budget. Therefore, minimizing the cost of communication and minimizing the cost of control become conflicting objectives. Such a dilemma motivates us in the present article to study an *energy-regulation trade-off* that delineates the fundamental performance bound of a feedback control system over a noisy channel in an unreliable communication regime.

### A. Related Work

Previous research has already recognized the severe effects of packet loss on stability. Majority of works have considered independent and identically distributed (i.i.d.) erasure channels [2]–[7]. In a seminal work, Sinopoli *et al.* [2] studied mean-square stability of Kalman filtering over an i.i.d. erasure channel, and proved that there exists a critical point for the packet error rate above which the expected estimation error covariance is unbounded. Later, Schenato *et al.* [3] extended this work to optimal control, and showed that there exists a separation between estimation and control when packet acknowledgment is available. Moreover, several works have employed Gilbert-Elliott channels to capture the temporal correlation of wireless channels [8]–[11]. Notably, Wu *et al.* [8] addressed stability of Kalman filtering over a Gilbert-Elliott channel, and proved that there exists a critical region defined by the recovery and failure rates outside which the expected prediction error covariance is unbounded. The corresponding optimal control problem was addressed by Mo *et al.* [9] where they showed that the separation principle still holds when packet acknowledgment is available. Eventually, a number of works have employed fading channels in order to take into account the time variation of the strengths of wireless channels [12]–[14]. In particular, Quevedo *et al.* [12] investigated stability of Kalman filtering over a fading channel with correlated gains, and established a sufficient condition that ensures the exponential boundedness of the expected estimation error covariance. Besides, Elia [13] studied the stabilization problem in the robust mean-square stability sense over a fading channel by modeling the fading as stochastic model uncertainty, and designed a controller with the largest stability margin.

Power adaptation for energy efficient transmission of sensory information over noisy channels in estimation and control tasks has also been addressed in the literature, and various schedulers have been designed[3] [15]–[21]. In particu-

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[1]This implies that block codes or message retransmissions that cause delays more than a threshold are prohibited. Note that reliable communication in the capacity limit is attained when delay can be arbitrarily large.

[2]In the context of our study, a packet (or equivalently a message) is defined as a unit of bits corresponding to sensory information about the state of the process under control at each time. Moreover, packet loss refers to the phenomenon where one of these bits is detected erroneously.

[3]Throughout our study, schedulers and controllers refer to the entities that decide about transmit powers and control inputs, respectively. The former are also known as transmission power controllers in the literature.

lar, Leong *et al.* [15] studied the estimation of a Gauss-Markov process over a fading channel, and derived the optimal scheduling policy that minimizes the estimation outage probability subject to a constraint on the average total power. Quevedo *et al.* [16] investigated the estimation of a Gauss-Markov process over a fading channel, and derived the optimal scheduling policy that minimizes the average total power subject to a stability condition ensuring that the expected estimation error covariance is exponentially bounded. Later, Nourian *et al.* [17] and Li *et al.* [18] extended the above works, and obtained the optimal scheduling policy that minimizes the trace of the average expected estimation error covariance subject to an energy harvesting constraint. The fact is that the adopted scheduling policies in [15]–[18] depend on the estimation error covariances, and not on the outputs of the process. In contrast, scheduling policies that depend on the outputs of the process can obviously take advantage of all available sensory information. These policies, which are of interest to our study, have been considered in [19]–[21]. More specifically, Ren *et al.* [19] studied the estimation of a first-order Gauss-Markov process over a fading channel based on the common information approach, and proved that the optimal scheduling policy is deterministic symmetric and the optimal estimator is linear. Chakravorty and Mahajan [20] found a similar structural result for the estimation of a first-order autoregressive process with symmetric noise over a channel modeled by a finite-state Markov chain. In addition, Gatsis *et al.* [21] addressed the control of a first-order Gauss-Markov process over a fading channel by restricting the information structure such that a separation between estimation and control is achieved, and showed that the optimal scheduling policy is deterministic and the optimal control policy is certainty equivalent.

### B. Contributions and Outline

In this article, we study the energy-regulation trade-off without restricting the information structure or the policy structure. We model the channel and the process by an additive white Gaussian noise channel with fading and a partially observable Gauss-Markov process, respectively. The goal we seek in the energy-regulation trade-off, which is in general an intractable problem, is to find an optimal policy profile consisting of a scheduling policy and a control policy. Our study is different from that in [21] where the information structure is restricted, or from those in [15]–[18] where the policy structure is confined. It is also unlike the studies in [19], [20] where the results are restricted to first-order processes with no feedback control. In our study, the outputs of the process are subject to noise, and both the scheduler and the controller need to infer the state of the process. This model generalizes the model used in [19]–[21] where the scheduler observes the exact value of the state of the process. As a result, in contrast to the above studies, three types of estimation discrepancies can be considered here: the discrepancy between the state of the process and the state estimate at the scheduler, the discrepancy between the state of the process and the state estimate at the controller, and that between the state estimates at the scheduler and the controller.

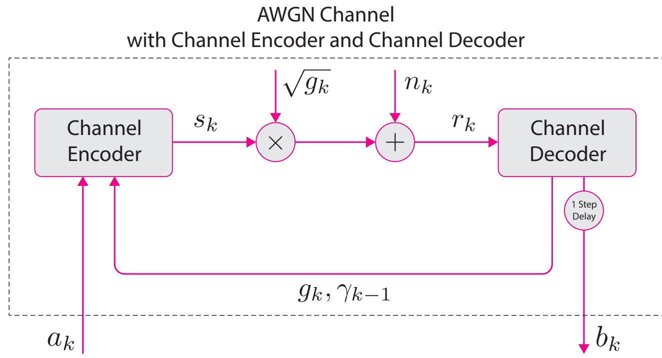


Fig. 1: Communication over an additive white Gaussian noise channel with fading. The input $a_k$ is transmitted over the channel, and the output $b_k$ is reconstructed.

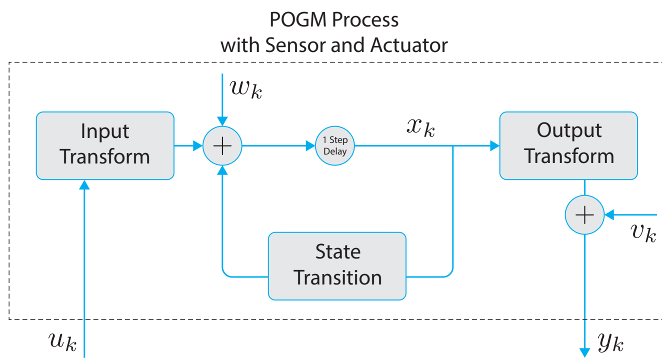


Fig. 2: Control of a partially observable Gauss-Markov process. The output $y_k$ is observed, and the input $u_k$ is applied to the process.

Our main contributions, in summary, are as follows. We characterize an equilibrium in the energy-regulation trade-off at which neither the scheduler nor the controller has a unilateral incentive to deviate from its policy. We argue that this equilibrium is a general one as it attains global optimality without any restrictions on the information structure or the policy structure, despite the presence of signaling[4] and dual effects. We show that at our equilibrium the scheduling policy is a deterministic symmetric policy and the control policy is a certainty-equivalent policy. As we will see, such structural attributes dramatically reduce the complexity of the design. Finally, we discuss the computational aspects of our equilibrium, and propose an approximation procedure for synthesizing a suboptimal scheduling policy with a probabilistic upper bound on its performance. Our analysis in this study is based on backward induction for dynamic games with asymmetric information (see e.g., [22]), and on the symmetric decreasing rearrangement of asymmetric measurable functions (see e.g., [23]).

The remainder of the article is organized in the following way. We introduce the models of the channel and the process, and formulate the energy-regulation trade-off in Section II. Then, we characterize an equilibrium in Section III, and

[4]Signaling here refers to the process of exchanging implicit information via actions.

prove its global optimality in Section IV. We discuss the computational aspects of the equilibrium and propose an approximation procedure in Section V, and provide a numerical example in Section VI. Finally, we make concluding remarks in Section VII.

### C. Preliminaries

In the sequel, the sets of real numbers and non-negative integers are denoted by $\mathbb{R}$ and $\mathbb{N}$, respectively. For $x, y \in \mathbb{N}$ and $x \leq y$, the set $\mathbb{N}_{[x,y]}$ denotes $\{z \in \mathbb{N} | x \leq z \leq y\}$. The sequence of vectors $x_0, \ldots, x_k$ is represented by $\mathbf{x}_k$. The symmetric decreasing rearrangement of a Borel measurable function $f(x)$ vanishing at infinity is represented by $f^*(x)$. The tail function of the standard Gaussian distribution is defined as $Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-y^2/2}\, dy$. The indicator function of a subset $\mathcal{A}$ of a set $\mathcal{X}$ is denoted by $\mathbb{1}_\mathcal{A} : \mathcal{X} \to \{0, 1\}$. The probability measure of a random variable $x$ is concisely represented by $\mathsf{P}(x)$, its probability density or probability mass function by $\mathsf{p}(x)$, and its expected value and covariance by $\mathsf{E}[x]$ and $\mathsf{cov}[x]$, respectively.

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space, and $x$ be an integrable random variable defined on this space. We will use conditional expectations of the form $\mathsf{E}[x|y, \gamma]$, where $y$ and $\gamma$ are random variables such that the latter takes on values in $\{0, 1\}$ and that $\sigma(y, \gamma) \subseteq \mathcal{F}$. By the Radon-Nikodym theorem and the Doob-Dynkin lemma, $z = \mathsf{E}[x|y, \gamma]$ satisfying $\mathsf{E}[(x - z)\mathbb{1}_\mathcal{G}] = 0$ for every $\mathcal{G} \in \sigma(y, \gamma)$ exists, and can be represented by a measurable function $\phi(y, \gamma)$. Accordingly, given a realization of $\gamma$, conditional expectations $\mathsf{E}[x|y, \gamma = 0]$ and $\mathsf{E}[x|y, \gamma = 1]$ also exist, and can be represented by $\phi(y, \gamma = 0)$ and $\phi(y, \gamma = 1)$, respectively.

We will adopt stochastic kernels to represent decision policies. Let $(\mathcal{X}, \mathcal{B}_\mathcal{X})$ and $(\mathcal{Y}, \mathcal{B}_\mathcal{Y})$ be two measurable spaces. A Borel measurable stochastic kernel $\mathsf{P} : \mathcal{B}_\mathcal{Y} \times \mathcal{X} \to [0, 1]$ is a mapping such that $\mathcal{A} \mapsto \mathsf{P}(\mathcal{A}|x)$ is a probability measure on $(\mathcal{Y}, \mathcal{B}_\mathcal{Y})$ for any $x \in \mathcal{X}$, and $x \mapsto \mathsf{P}(\mathcal{A}|x)$ is a Borel measurable function for any $\mathcal{A} \in \mathcal{B}_\mathcal{Y}$.

Besides, we will use two different notions of optimality. For a given team game with two decision makers, let $\gamma^1 \in \mathcal{G}^1$ and $\gamma^2 \in \mathcal{G}^2$ be the decision policies of the decision makers, where $\mathcal{G}^1$ and $\mathcal{G}^2$ are the sets of admissible policies, and $L(\gamma^1, \gamma^2)$ be the associated loss function. A policy profile $(\gamma^{1\star}, \gamma^{2\star})$ represents a Nash equilibrium if

$$L(\gamma^{1\star}, \gamma^{2\star}) \leq L(\gamma^1, \gamma^{2\star}), \text{ for all } \gamma^1 \in \mathcal{G}^1,$$

$$L(\gamma^{1\star}, \gamma^{2\star}) \leq L(\gamma^{1\star}, \gamma^2), \text{ for all } \gamma^2 \in \mathcal{G}^2.$$

However, a policy profile $(\gamma^{1\star}, \gamma^{2\star})$ is a globally optimal solution if

$$L(\gamma^{1\star}, \gamma^{2\star}) \leq L(\gamma^1, \gamma^2), \text{ for all } \gamma^1 \in \mathcal{G}^1, \gamma^2 \in \mathcal{G}^2.$$

Clearly, a globally optimal solution is necessarily a Nash equilibrium, but the converse need not hold.

## II. Energy-Regulation Trade-off

Consider an additive white Gaussian noise (AWGN) channel with fading with the discrete-time input-output relation

$$r_k = \sqrt{g_k} s_k + n_k, \tag{1}$$

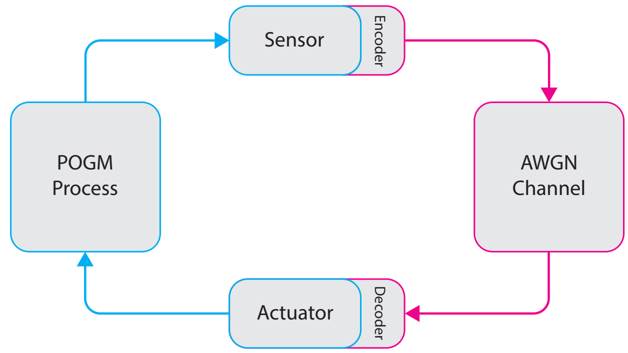


Fig. 3: Feedback control over a noisy channel. The channel is additive white Gaussian noise with fading, and the process is partially observable Gauss-Markov. The encoder consists of a filter, a scheduler, and a channel encoder. The decoder consists of a channel decoder, a filter, and a controller.

for $k \in \mathbb{N}_{[0,N]}$, where $r_k$ is the channel output, $g_k \geq 0$ is the channel gain, $s_k$ is the channel input, $n_k$ is a white Gaussian noise with zero mean and power spectral density $N_0$, and $N$ is a finite time horizon. The channel gain $g_k$ is a random variable representing the effects of path loss, shadowing, and multipath, which can change at each time with or without correlation over time according to any probability distribution satisfying the Markov property. The bit sequence corresponding to a message $a_k$ is modulated by the encoder into the carrier signal, and is transmitted over the channel. The signal is then detected by the decoder, and the message $b_k$ is reconstructed after one step delay (see Fig. 1). It is assumed that the channel is block fading, that the channel gain $g_k$ is known at both the decoder and the encoder before transmission at time $k$ given a feedback channel, and that the quantization error is negligible. For our purpose, we focus on uncoded square M-ary quadrature amplitude modulation (MQAM) signaling[5] with $M \in \{4, 16, 64, \ldots\}$ for which the packet error rate at time $k$ is determined exactly as

$$\mathsf{per}_k = 1 - \left(1 - c_0 Q\left(\sqrt{c_1 E_k / N_0}\right)\right)^{2L/b}, \tag{2}$$

with parameters $c_0 = 2(1 - 2^{-b/2})$, $c_1 = 3b/(2^b - 1)$, and $b = \log_2 M$, where $\mathsf{per}_k \in \mathcal{C} = [0, 1 - 2^{-L}]$ is the packet error rate, $E_k$ is the received average energy per bit, and $L$ is the packet length in bits. The MQAM signaling is desirable for its high spectral efficiency. However, given a mapping between the packet error rate and the received average energy per bit, any other signaling with or without coding can be adopted. Then, from (1) and (2), we can obtain the required transmit power at time $k$ for a given packet error rate as

$$p_k = \tfrac{N_0 R}{c_1 g_k} \left(Q^{-1}\left(\tfrac{1}{c_0} - \tfrac{1}{c_0}(1 - \mathsf{per}_k)^{b/2L}\right)\right)^2, \tag{3}$$

where $p_k$ is the transmit power, $R$ is the communication rate, and we used the fact that $E_k = g_k p_k / R$. Note that the function in (3) is decreasing in terms of $\mathsf{per}_k$, and that there exists a transmit power $p_k^r$ at each time $k$ for which $\mathsf{per}_k = \epsilon$, where

[5] Signaling here refers to the process of mapping digital sequences to signals.

$\epsilon$ is a negligible probability. In addition, from the definition of $\mathsf{per}_k$, we can model packet loss according to a random variable $\gamma_k$ such that $\gamma_k = 1$ if the message $a_k$ is successfully received after one time step and $\gamma_k = 0$ otherwise, and that the probability of $\gamma_k = 0$ is $\mathsf{per}_k$. Therefore, we have

$$b_{k+1} = \begin{cases} a_k, & \text{if } \gamma_k = 1, \\ \varnothing, & \text{otherwise}, \end{cases} \tag{4}$$

for $k \in \mathbb{N}_{[0,N]}$ with $b_0 = \varnothing$. Note that $\gamma_k$ for all $k \in \mathbb{N}_{[0,N]}$ are conditionally independent given all the previous and current channel gains and transmit powers. It is assumed that the acknowledgment of a message that is successfully received at time $k$ is available at the encoder at the same time via the feedback channel.

Now, consider a partially observable Gauss-Markov (POGM) process with the discrete-time state and output equations

$$x_{k+1} = A_k x_k + B_k u_k + w_k, \tag{5}$$

$$y_k = C_k x_k + v_k, \tag{6}$$

for $k \in \mathbb{N}_{[0,N]}$ with initial condition $x_0$, where $x_k \in \mathbb{R}^n$ is the state of the process, $A_k \in \mathbb{R}^{n\times n}$ is the state matrix, $B_k \in \mathbb{R}^{n\times m}$ is the input matrix, $u_k \in \mathbb{R}^m$ is the control input, $w_k \in \mathbb{R}^n$ is a Gaussian white noise with zero mean and covariance $W_k \succ 0$, $y_k \in \mathbb{R}^p$ is the output of the process, $C_k \in \mathbb{R}^{p\times n}$ is the output matrix, and $v_k \in \mathbb{R}^p$ is a Gaussian white noise with zero mean and covariance $V_k \succ 0$. The output $y_k$ is observed by a sensor, and the input $u_k$ is applied to the process by an actuator (see Fig. 2). It is assumed that $x_0$ is a Gaussian vector with mean $m_0$ and covariance $M_0$, and that $x_0$, $w_k$, and $v_k$ are mutually independent for all $k \in \mathbb{N}_{[0,N]}$.

The sensor is connected to the actuator via the channel. Fig. 3 illustrates a schematic view of the system of interest in which the encoder consists of a filter, a scheduler, and a channel encoder, and the decoder consists of a channel decoder, a filter, and a controller. In this system, the scheduler and the controller are the decision makers deciding about the transmit power and the control input at each time, respectively. The filters should be required since the process is partially observable. The message that is transmitted to the controller at time $k$, i.e., $a_k$, is the minimum mean-square-error (MMSE) state estimate at the scheduler at time $k$. This state estimate condenses all the previous and current outputs of the process into a single message. This implies that from the MMSE perspective the controller is able to develop a state estimate upon the receipt of a message that would be the same if it had all the previous outputs of the process, which is in fact the best possible case. Finally, the location of the controller in the system is nominal. The case in which the controller and the actuator are connected via another channel can essentially be converted to the case in which those are collocated [24]. The reason is that the information that would be transmitted from the controller to the actuator should be processed again at the actuator, and from the data-processing inequality (see e.g., [25]), it is always better to process the transmitted MMSE state estimate directly at the actuator. Hence, the two channels can in effect be modeled by a single channel.

The decision variables of the scheduler and the controller at time $k$ are $\mathsf{per}_k$[6] and $u_k$, respectively. These decisions are decided based on the causal information sets of the scheduler and the controller, which are expressed by

$$\mathcal{I}_k^s = \left\{ y_t, b_t, g_t, \mathsf{per}_{t'}, \gamma_{t'}, u_{t'} \;\middle|\; t \in \mathbb{N}_{[0,k]}, t' \in \mathbb{N}_{[0,k-1]} \right\},$$

$$\mathcal{I}_k^c = \left\{ b_t, g_t, \gamma_{t'}, u_{t'} \;\middle|\; t \in \mathbb{N}_{[0,k]}, t' \in \mathbb{N}_{[0,k-1]} \right\},$$

respectively. Clearly, $\mathcal{I}_k^c \subset \mathcal{I}_k^s$. We say that a policy profile $(\pi, \mu)$ consisting of a scheduling policy $\pi$ and a control policy $\mu$ is admissible if $\pi = \{\mathsf{P}(\gamma_k|\mathcal{I}_k^s)\}_{k=0}^N$ and $\mu = \{\mathsf{P}(u_k|\mathcal{I}_k^c)\}_{k=0}^N$, where $\mathsf{P}(\gamma_k|\mathcal{I}_k^s)$ and $\mathsf{P}(u_k|\mathcal{I}_k^c)$ are Borel measurable stochastic kernels. We represent the set of admissible policy profiles by $\mathcal{P} \times \mathcal{M}$, where $\mathcal{P}$ and $\mathcal{M}$ are the sets of admissible scheduling policies and admissible control policies, respectively. For the system described above, we are interested in an energy-regulation trade-off that is cast as an optimization problem with the loss function

$$\chi(\pi, \mu) := (1 - \lambda) E(\pi, \mu) + \lambda J(\pi, \mu), \tag{7}$$

over the space of admissible policy profiles $(\pi, \mu) \in \mathcal{P} \times \mathcal{M}$, given a trade-off multiplier $\lambda \in (0, 1)$, and for

$$E(\pi, \mu) := \tfrac{1}{N+1} \mathsf{E}\left[ \textstyle\sum_{k=0}^{N} \ell_k p_k \right], \tag{8}$$

$$J(\pi, \mu) := \tfrac{1}{N+1} \mathsf{E}\left[ \textstyle\sum_{k=0}^{N+1} x_k^T Q_k x_k + \sum_{k=0}^{N} u_k^T R_k u_k \right], \tag{9}$$

where $\ell_k \geq 0$ is a weighting coefficient, and $Q_k \succeq 0$ and $R_k \succ 0$ are weighting matrices.

*Remark 1:* The energy-regulation trade-off, which is formulated based on the weighted sum approach (see e.g., [26]), is a trade-off between two objective functions. The objective function in (8) penalizes the transmit power per packet, while the objective function in (9) penalizes the state deviation and the control effort. Note that the associated optimization problem is in general an intractable problem due to a non-classical information structure, a signaling effect, and a dual effect of the control. These issues prohibit the direct application of the traditional methods in stochastic optimal control. Despite these difficulties, in the subsequent sections, we develop a new method for the characterization of a solution $(\pi^\star, \mu^\star)$ to this problem. Although the problem we study is over a finite time horizon, the extension of our results to an infinite time horizon is straightforward provided the channel gain has a stationary distribution and the process is time-invariant, controllable, and observable.

## III. Existence of an Equilibrium

Certainly, the main technical obstacle to the characterization of any solution in the energy-regulation trade-off is that the design of the stochastic kernels $\mathsf{P}(\gamma_k|\mathcal{I}_k^s)$ and $\mathsf{P}(u_k|\mathcal{I}_k^c)$ is in general intertwined with the structures of the conditional distributions $\mathsf{P}(x_k|\mathcal{I}_k^s)$ and $\mathsf{P}(x_k|\mathcal{I}_k^c)$. Our goal in the following is to overcome this obstacle by investigating a separation in the design of these stochastic kernels. Let $\check{x}_k$ and $\hat{x}_k$, unless otherwise stated, denote the MMSE state estimates[7] at the

[6] Note that according to (3), given $g_k$ and $\mathsf{per}_k$, one can find $p_k$.

[7] We recall that given an information set $\mathcal{I}_k$ at time $k$, the MMSE state estimate at time $k$ is achieved by $\mathsf{E}[x_k|\mathcal{I}_k]$.

scheduler and the controller, respectively. Accordingly, we define

$$\check{e}_k := x_k - \check{x}_k, \tag{10}$$

$$\hat{e}_k := x_k - \hat{x}_k, \tag{11}$$

$$\tilde{e}_k := \check{x}_k - \hat{x}_k, \tag{12}$$

where $\check{e}_k$ is the estimation error from the perspective of the scheduler, $\hat{e}_k$ is the estimation error from the perspective of the controller, and $\tilde{e}_k$ is the estimation mismatch. The main result of this section is given by the next theorem, which characterizes a Nash equilibrium in the energy-regulation trade-off at which a separation in the design is guaranteed. The proof relies on backward induction for dynamic games with asymmetric information. For the statement of the theorem, we need the following lemma related to the dynamics of the conditional means and the conditional covariances, and the subsequent definition of two value functions with respect to the information sets.

*Lemma 1:* *The conditional mean $\check{x}_k = \mathsf{E}[x_k|\mathcal{I}_k^s]$ and the conditional covariance $Y_k = \mathsf{cov}[x_k|\mathcal{I}_k^s]$ satisfy*

$$\check{x}_{k+1} = m_{k+1} + K_{k+1}\big(y_{k+1} - C_{k+1}m_{k+1}\big), \tag{13}$$

$$m_{k+1} = A_k\check{x}_k + B_k u_k, \tag{14}$$

$$Y_{k+1} = \big(M_{k+1}^{-1} + C_{k+1}^T V_{k+1}^{-1} C_{k+1}\big)^{-1}, \tag{15}$$

$$M_{k+1} = A_k Y_k A_k^T + W_k, \tag{16}$$

*for $k \in \mathbb{N}_{[0,N]}$ with initial conditions $\check{x}_0 = m_0 + K_0(y_0 - C_0 m_0)$ and $Y_0 = (M_0^{-1} + C_0^T V_0^{-1} C_0)^{-1}$, where $K_k = Y_k C_k^T V_k^{-1}$, $m_k = \mathsf{E}[x_k|\mathcal{I}_{k-1}^s]$, and $M_k = \mathsf{cov}[x_k|\mathcal{I}_{k-1}^s]$. In addition, the conditional mean $\hat{x}_k = \mathsf{E}[x_k|\mathcal{I}_k^c]$ and the conditional covariance $P_k = \mathsf{cov}[x_k|\mathcal{I}_k^c]$ satisfy*

$$\hat{x}_{k+1} = A_k\hat{x}_k + B_k u_k + \gamma_k A_k \tilde{e}_k + (1-\gamma_k)\imath_k, \tag{17}$$

$$\begin{aligned} P_{k+1} = {} & A_k P_k A_k^T + W_k \\ & - \gamma_k A_k (P_k - Y_k) A_k^T - (1-\gamma_k)\Xi_k, \end{aligned} \tag{18}$$

*for $k \in \mathbb{N}_{[0,N]}$ with initial conditions $\hat{x}_0 = m_0$ and $P_0 = M_0$, where $\imath_k = A_k\,\mathsf{E}[\hat{e}_k|\mathcal{I}_k^c, \gamma_k = 0]$ and $\Xi_k = A_k(P_k - \mathsf{cov}[x_k|\mathcal{I}_k^c, \gamma_k = 0])A_k^T$.*

The proof of Lemma 1 is in Appendix A.

*Definition 1 (Value functions):* Let $S_k \succeq 0$ be a matrix satisfying the algebraic Riccati equation

$$\begin{aligned} S_k = {} & Q_k + A_k^T S_{k+1} A_k - A_k^T S_{k+1} B_k \\ & \times (B_k^T S_{k+1} B_k + R_k)^{-1} B_k^T S_{k+1} A_k, \end{aligned} \tag{19}$$

for $k \in \mathbb{N}_{[0,N]}$ with initial condition $S_{N+1} = Q_{N+1}$ and with the exception of $S_k = 0$ for $k \notin \mathbb{N}_{[0,N+1]}$. The value functions $V_k^s(\mathcal{I}_k^s)$ and $V_k^c(\mathcal{I}_k^c)$ are defined as

$$V_k^s(\mathcal{I}_k^s) := \min_{\pi\in\mathcal{P}:\mu=\mu^\star} \mathsf{E}\left[\textstyle\sum_{t=k}^N \theta_t p_t + \varsigma_{t+1}\middle|\mathcal{I}_k^s\right], \tag{20}$$

$$V_k^c(\mathcal{I}_k^c) := \min_{\mu\in\mathcal{M}:\pi=\pi^\star} \mathsf{E}\left[\textstyle\sum_{t=k}^N \theta_{t-1} p_{t-1} + \varsigma_t\middle|\mathcal{I}_k^c\right], \tag{21}$$

for $k \in \mathbb{N}_{[0,N]}$ given a policy profile $(\pi^\star, \mu^\star)$ where

$$\begin{aligned} \theta_k &:= \ell_k(1-\lambda)/\lambda, \\ \varsigma_k &:= \big(u_k + (B_k^T S_{k+1} B_k + R_k)^{-1} B_k^T S_{k+1} A_k x_k\big)^T \\ &\quad \times (B_k^T S_{k+1} B_k + R_k) \\ &\quad \times \big(u_k + (B_k^T S_{k+1} B_k + R_k)^{-1} B_k^T S_{k+1} A_k x_k\big), \end{aligned}$$

for $k \in \mathbb{N}_{[0,N]}$ with the exception of $\theta_k := 0$ and $\varsigma_k := 0$ for $k \notin \mathbb{N}_{[0,N]}$.

*Theorem 1:* *There exists at least one Nash equilibrium $(\pi^\star, \mu^\star)$ in the energy-regulation trade-off such that the scheduling policy $\pi^\star$ is a deterministic symmetric policy with respect to $\tilde{e}_k$ determined by*

$$\begin{aligned} \mathsf{per}_k^\star = \operatorname*{argmin}_{\mathsf{per}_k\in\mathcal{C}} \Big\{ & \mathsf{per}_k \left(\tilde{e}_k^T A_k^T \Gamma_{k+1} A_k \tilde{e}_k + \varrho_k\right) \\ & + \tfrac{\theta_k N_0 R}{c_1 g_k}\Big(Q^{-1}\Big(\tfrac{1}{c_0} - \tfrac{1}{c_0}(1-\mathsf{per}_k)^{b/2L}\Big)\Big)^2\Big\}, \end{aligned} \tag{22}$$

*where $\Gamma_k = A_k^T S_{k+1} B_k (B_k^T S_{k+1} B_k + R_k)^{-1} B_k^T S_{k+1} A_k$ and $\varrho_k = \mathsf{E}[V_{k+1}^s(\mathcal{I}_{k+1}^s)|\mathcal{I}_k^s, \gamma_k = 0] - \mathsf{E}[V_{k+1}^s(\mathcal{I}_{k+1}^s)|\mathcal{I}_k^s, \gamma_k = 1]$, and the control policy $\mu^\star$ is a certainty-equivalent policy determined by*

$$u_k^\star = -(B_k^T S_{k+1} B_k + R_k)^{-1} B_k^T S_{k+1} A_k \hat{x}_k, \tag{23}$$

*where $\hat{x}_k$ is the MMSE state estimate at the controller satisfying $\hat{x}_{k+1} = A_k\hat{x}_k + B_k u_k + \gamma_k A_k \tilde{e}_k$ for $k \in \mathbb{N}_{[0,N]}$ with initial condition $\hat{x}_0 = m_0$.*

The proof of Theorem 1 is in Appendix B.

*Remark 2:* Note that contrary to the conditional distribution $\mathsf{P}(x_k|\mathcal{I}_k^s)$, the conditional distribution $\mathsf{P}(x_k|\mathcal{I}_k^c)$ is non-Gaussian and is influenced by the signaling effect. According to Lemma 1, the existence of the signaling residuals $\imath_k$ and $\Xi_k$ in (17) and (18) implies that the controller might be able to decrease its uncertainty even when a packet loss occurs. However, the fact that at the equilibrium $(\pi^\star, \mu^\star)$ characterized in Theorem 1 the MMSE state estimate $\hat{x}_k$ satisfies (17) with $\imath_k = 0$ asserts that the controller's inference about the state of the process when a packet loss occurs has no contribution from the MMSE perspective. This is an important property as it consequently leads to a linear structure for the filter at the controller, to a separation in the design of the scheduler and the controller, and to the neutrality of the control (see e.g., [27]). It is also interesting to note that at the equilibrium $(\pi^\star, \mu^\star)$ the transmission of the MMSE state estimate $\check{x}_k$ becomes equivalent to the transmission of the estimation mismatch $\tilde{e}_k$ or the innovation $\nu_k := y_k - C_k\,\mathsf{E}[x_k|\mathcal{I}_{k-1}^s]$ because $\tilde{e}_k = \check{x}_k - \hat{x}_k = K_k \nu_k$ when $\gamma_{k-1} = 1$.

## IV. Global Optimality of the Equilibrium

Although Theorem 1 proves the existence of a Nash equilibrium, due to non-convexity, there might exist other Nash equilibria with better performance in the energy-regulation trade-off. Unfortunately, there is no direct way to the characterization of all these equilibria (if any). However, this is not required for our purpose if we could show that the equilibrium $(\pi^\star, \mu^\star)$ was globally optimal. The main result of this section is provided by the next theorem, which in fact proves that this equilibrium is dominant in the set of admissible policy profiles. The proof relies on the symmetric decreasing rearrangement of asymmetric measurable functions.

*Theorem 2: The Nash equilibrium $(\pi^\star, \mu^\star)$ characterized in Theorem 1 associated with the energy-regulation trade-off is globally optimal.*

The proof of Theorem 2 is in Appendix C.

*Remark 3:* The global optimality result in Theorem 2 is important as it guarantees that there exist no other equilibria in the energy-regulation trade-off that can outperform the equilibrium $(\pi^\star, \mu^\star)$ for any given $\lambda$. Note that the result does not rule out the possibility of existence of other equilibria with equal performance. However, even in that case, the equilibrium $(\pi^\star, \mu^\star)$ is preferable because as mentioned above it possesses unique structural attributes that dramatically reduce the complexity of the design. We should emphasize that the energy-regulation trade-off studied in this article can be reduced to a rate-regulation trade-off when $\mathsf{per}_k$ is restricted to take values only in $\{0,1\}$. In such a problem, which we have studied in [28], [29], instead of the energy the packet rate is penalized, and the scheduler's decision at each time is to transmit a message or not to transmit. Hence, our result here generalizes the result in [28], [29], where we found an optimal policy profile consisting of a symmetric threshold triggering policy and a certainty-equivalent control policy.

## V. Computation and Approximation

In this section, we look at the computational aspects of the equilibrium $(\pi^\star, \mu^\star)$. From Theorem 1, we see that there are some variables in the design of the optimal policies that can be computed offline, and some that must be computed online at the scheduler and/or the controller. In particular, the optimal control policy $\mu^\star$ can readily be computed based on the algebraic Riccati equation (19) and on the following linear recursive equation:

$$\hat{x}_{k+1} = A_k \hat{x}_k + B_k u_k + \gamma_k A_k \tilde{e}_k,$$

for $k \in \mathbb{N}_{[0,N]}$ with initial condition $\hat{x}_0 = m_0$. In addition, the optimal scheduling policy $\pi^\star$ can be computed with arbitrary accuracy by solving recursively and backward in time the following optimality equation:

$$\begin{aligned} V_k^s(\tilde{e}_k, g_k) = \min_{\mathsf{per}_k \in \mathcal{C}} \Big\{ & \theta_k p_k(\mathsf{per}_k, g_k) + \mathsf{per}_k\, \tilde{e}_k^T A_k^T \Gamma_{k+1} A_k \tilde{e}_k \\ & + \operatorname{tr}(A_k^T \Gamma_{k+1} A_k Y_k + \Gamma_{k+1} W_k) \\ & + \mathsf{per}_k \, \mathsf{E}\left[ V_{k+1}^s(\tilde{e}_{k+1}, g_{k+1}) \middle| \tilde{e}_k, g_k, \gamma_k = 0 \right] \\ & + (1-\mathsf{per}_k)\, \mathsf{E}\left[ V_{k+1}^s(\tilde{e}_{k+1}, g_{k+1}) \middle| \tilde{e}_k, g_k, \gamma_k = 1 \right] \Big\}, \end{aligned}$$

for $k \in \mathbb{N}_{[0,N]}$ with initial condition $V_{N+1}^s(\tilde{e}_{N+1}, g_{N+1}) = 0$ in conjunction with the probability distribution of the channel gain, and with the following linear recursive equation:

$$\tilde{e}_{k+1} = (1-\gamma_k) A_k \tilde{e}_k + K_{k+1} \nu_{k+1},$$

for $k \in \mathbb{N}_{[0,N]}$ with initial condition $\tilde{e}_0 = K_0 \nu_0$, where $\nu_k$ is a Gaussian white noise with zero mean and covariance $N_k = C_k M_k C_k^T + V_k$. Let $(\tilde{e}_k, g_k)$ and $\mathsf{per}_k$ be discretized in grids with $d_1^{n+1}$ and $d_2$ points, respectively, and the associated expected value be obtained based on a weighted sum of $d_3$ samples. The complexity of this computation is then $\mathcal{O}(N d_1^{n+1} d_2 d_3)$. Note that the associated computational requirements can be overwhelming especially when $n$ increases. In practice, one might be interested in a suboptimal scheduling policy with cheaper computation. The following proposition synthesizes such a policy with a probabilistic upper bound on its performance.

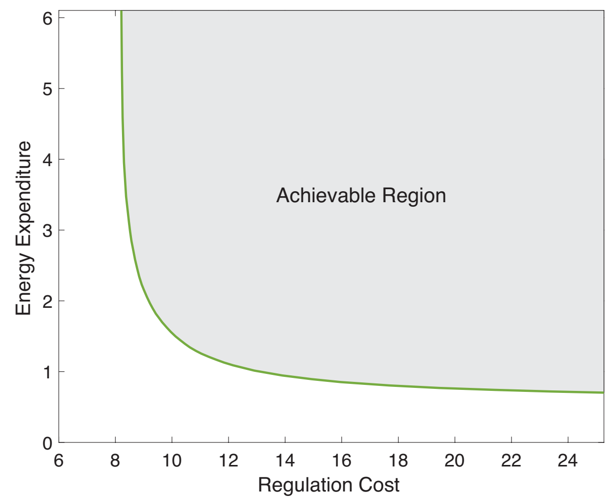


Fig. 4: The energy-regulation trade-off curve in feedback control over a noisy channel. The area above the trade-off curve represents the achievable region.

*Proposition 1: Let $\pi^+$ be a scheduling policy given by*

$$\begin{aligned} \mathsf{per}_k^+ = \operatorname*{argmin}_{\mathsf{per}_k \in \mathcal{C}} \Big\{ & \mathsf{per}_k\, \tilde{e}_k^T A_k^T \Gamma_{k+1} A_k \tilde{e}_k \\ & + \tfrac{\theta_k N_0 R}{c_1 g_k} \Big( Q^{-1}\Big( \tfrac{1}{c_0} - \tfrac{1}{c_0}(1-\mathsf{per}_k)^{b/2L} \Big) \Big)^2 \Big\}. \end{aligned} \tag{24}$$

*Then, the loss $\chi(\pi^+, \mu^\star)$ is upper bounded by*

$$\begin{aligned} \breve{\chi} := & \tfrac{1-\lambda}{N+1} \textstyle\sum_{k=0}^{N-1} \ell_k p_k^r + \tfrac{\lambda}{N+1} \Big\{ m_0^T S_0 m_0 \\ & + \operatorname{tr}(S_{N+1} M_{N+1}) + \textstyle\sum_{k=0}^{N} \operatorname{tr}\left(Q_k Y_k\right) \\ & + \textstyle\sum_{k=0}^{N} \operatorname{tr}\left(S_{k+1} K_k (C_k M_k C_k^T + V_k) K_k^T\right) \Big\}, \end{aligned} \tag{25}$$

with probability $(1-\epsilon)^N$.

The proof of Proposition 1 is in Appendix D.

## VI. Numerical Example

In this section, we provide a simple example to demonstrate the energy-regulation trade-off curve. In our example, we choose the parameters of the channel, the process, and the loss function as follows: the data rate $R = 4$ Kbps, noise power spectral density $N_0 = -120$ dB, modulation order $M = 16$, packet size $L = 128$ bits, state coefficient $A_k = 1.1$, input coefficient $B_k = 1$, output coefficient $C_k = 1$, process noise variance $W_k = 3$, output noise variance $V_k = 1$ for $k \in \mathbb{N}_{[0,N]}$, mean and variance of the initial condition $m_0 = 0$

and $M_0 = 1$, weighting coefficients $Q_{N+1} = 1$, $\ell_k = 1$, $Q_k = 1$, and $R_k = 0.1$ for $k \in \mathbb{N}_{[0,N]}$, and time horizon $N = 100$. In addition, we express the fading by the combined path loss and shadowing model

$$g_k = \left(\frac{4\pi f d_0}{c}\right)^{-2} \left(\frac{d}{d_0}\right)^{-\beta} 10^{\alpha_k/10},$$

for $k \in \mathbb{N}_{[0,N]}$, where $f = 2.4$ GHz is the carrier frequency, $d_0 = 1$ m is the reference distance, $c = 3 \times 10^5$ km/s is the speed of light, $d = 20$ m is the transmitter-receiver relative distance, $\beta = 3$ is the path loss exponent, and $\alpha_k$ is a Gaussian shadowing variable with zero mean and variance 5 dB. For this system, the energy-regulation trade-off curve was computed numerically using different values of the trade-off multiplier $\lambda \in (0, 1)$, and is depicted in Fig. 4. As specified, the area above the trade-off curve represents the achievable region. Note that the performance of any policy profile should be assessed with respect to the trade-off curve, and that there exists no policy profile with performance outside the achievable region.

## VII. Conclusion

In this article, we studied an energy-regulation trade-off that can express the fundamental performance bound of a feedback control system over a noisy channel in an unreliable communication regime. The central focus was on the characterization of an equilibrium at which the filter at the controller becomes linear, the design of the scheduler and the controller becomes separated, and the control becomes neutral. We proved that this equilibrium, which is composed of a deterministic symmetric scheduling policy and a certainty-equivalent control policy, cannot be outperformed by any other equilibria. This result can be interpreted as another manifestation of symmetry and certainty equivalence in the design of a class of stochastic systems with components that are widely used for modeling of physical phenomena in communication and control. We propose that future research should be undertaken on the extension of our study to wireless control systems with other models of the channel and the process. It would of course be interesting to see if any equilibria resemble to the one characterized here exist in other classes of systems.

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## Appendix A
## Proof of Lemma 1

*Proof:* For the first part of the claim, it is easy to verify that, given the information set of the scheduler $\mathcal{I}_k^s$, the conditional mean $\check{x}_k$ and the conditional covariance $Y_k$ satisfy the standard Kalman filter equations (see e.g., [30]).

Moreover, for the second part of the claim, given the information set of the controller $\mathcal{I}_k^c$ and from the state equation (5), we can obtain the propagation equations as

$$\hat{x}_{k+1} = A_k \, \mathsf{E}[x_k|\mathcal{I}_{k+1}^c] + B_k u_k, \tag{26}$$

$$P_{k+1} = A_k \, \mathsf{cov}[x_k|\mathcal{I}_{k+1}^c] A_k^T + W_k. \tag{27}$$

By definition, $\gamma_k$ at each time can be either one or zero. If $\gamma_k = 1$, the controller receives $\check{x}_k$ at time $k+1$. In this case, we have

$$\begin{aligned}\mathsf{p}(x_k|\mathcal{I}_{k+1}^c) &= \mathsf{p}(x_k|\mathcal{I}_k^c, b_{k+1} = \check{x}_k, g_{k+1}, \gamma_k = 1, u_k)\\ &= \mathsf{p}(x_k|\check{x}_k, Y_k)\\ &= \mathsf{p}(x_k|\mathcal{I}_k^s),\end{aligned}$$

where we used the fact that $\{\check{x}_k, Y_k\}$ is statistically equivalent to $\mathcal{I}_k^s$. Hence, we obtain $\mathsf{E}[x_k|\mathcal{I}_{k+1}^c] = \check{x}_k$ and $\mathsf{cov}[x_k|\mathcal{I}_{k+1}^c] = Y_k$. However, if $\gamma_k = 0$, the controller receives nothing at time $k+1$. In this case, we have

$$\begin{aligned}\mathsf{p}(x_k|\mathcal{I}_{k+1}^c) &= \mathsf{p}(x_k|\mathcal{I}_k^c, b_{k+1} = \varnothing, g_{k+1}, \gamma_k = 0, u_k)\\ &= \mathsf{p}(x_k|\mathcal{I}_k^c, \gamma_k = 0)\\ &= \frac{\mathsf{p}(\gamma_k = 0|\mathcal{I}_k^c, x_k)\, \mathsf{p}(x_k|\mathcal{I}_k^c)}{\mathsf{p}(\gamma_k = 0|\mathcal{I}_k^c)}.\end{aligned}$$

Note that for any admissible scheduling policy $\pi$, it is possible to calculate $\mathsf{p}(\gamma_k = 0|\mathcal{I}_k^c, x_k)$ and $\mathsf{p}(\gamma_k = 0|\mathcal{I}_k^c)$. Let us define $\hat{x}_k' := \mathsf{E}[x_k|\mathcal{I}_k^c, \gamma_k = 0] - \hat{x}_k$ and $P_k' := P_k - \mathsf{cov}[x_k|\mathcal{I}_k^c, \gamma_k = 0]$. As a result, for any value of $\gamma_k$, we can obtain the update equations as

$$\mathsf{E}[x_k|\mathcal{I}_{k+1}^c] = \hat{x}_k + \gamma_k(\check{x}_k - \hat{x}_k) + (1 - \gamma_k)\hat{x}_k', \tag{28}$$

$$\mathsf{cov}[x_k|\mathcal{I}_{k+1}^c] = P_k - \gamma_k(P_k - Y_k) - (1 - \gamma_k)P_k'. \tag{29}$$

Finally, we obtain the result by substituting (28) and (29) in (26) and (27), respectively, and by defining the signaling residuals $\imath_k := A_k\hat{x}_k'$ and $\Xi_k := A_k P_k' A_k^T$. ■

## Appendix B
## Proof of Theorem 1

*Proof:* Applying few operations on the state equation (5) and the algebraic Riccati equation (19), we see that

$$\begin{aligned}x_{k+1}^T S_{k+1} x_{k+1} = (A_k x_k + B_k u_k + w_k)^T\\ \times S_{k+1}(A_k x_k + B_k u_k + w_k),\end{aligned}$$

$$\begin{aligned}x_k^T S_k x_k = x_k^T \big(Q_k + A_k^T S_{k+1} A_k\\ - L_k^T (B_k^T S_{k+1} B_k + R_k) L_k\big) x_k,\end{aligned}$$

$$\begin{aligned}&x_{N+1}^T S_{N+1} x_{N+1} - x_0^T S_0 x_0\\ &\quad = \textstyle\sum_{k=0}^N x_{k+1}^T S_{k+1} x_{k+1} - \sum_{k=0}^N x_k^T S_k x_k.\end{aligned}$$

Let us now define the loss function $\chi'(\pi, \mu)$ as

$$\chi'(\pi,\mu) := \mathsf{E}\Big[\sum_{k=0}^{N}\Big\{\theta_k p_k(\mathsf{per}_k, g_k) + \big(u_k + (B_k^T S_{k+1} B_k + R_k)^{-1} B_k^T S_{k+1} A_k x_k\big)^T \times \big(B_k^T S_{k+1} B_k + R_k\big) \times \big(u_k + (B_k^T S_{k+1} B_k + R_k)^{-1} B_k^T S_{k+1} A_k x_k\big)\Big\}\Big].$$

Using the above identities, it is easy to see that $\chi'(\pi, \mu)$ is equivalent to $\chi(\pi, \mu)$ in the sense that it yields the same optimal policies. Hence, it suffices to show that the policy profile $(\pi^\star, \mu^\star)$ satisfies

$$\chi'(\pi^\star,\mu^\star) \leq \chi'(\pi,\mu^\star), \text{ for all } \pi \in \mathcal{P},$$

$$\chi'(\pi^\star,\mu^\star) \leq \chi'(\pi^\star,\mu), \text{ for all } \mu \in \mathcal{M}.$$

Incorporating the control policy $\mu^\star$ in the loss function $\chi'(\pi, \mu)$ when $\hat{x}_k$ satisfies $\hat{x}_{k+1} = A_k\hat{x}_k + B_k u_k + \gamma_k A_k \tilde{e}_k$ for $k \in \mathbb{N}_{[0,N]}$ with initial condition $\hat{x}_0 = m_0$, we find

$$\chi'(\pi,\mu^\star) = \mathsf{E}\Big[\sum_{k=0}^{N}\Big\{\theta_k p_k(\mathsf{per}_k, g_k) + \hat{e}_k^T L_k^T (B_k^T S_{k+1} B_k + R_k) L_k \hat{e}_k\Big\}\Big],$$

where $L_k = (B_k^T S_{k+1} B_k + R_k)^{-1} B_k^T S_{k+1} A_k$. Pertaining to $\chi'(\pi, \mu^\star)$, we can write the value function $V_k^s(\mathcal{I}_k^s)$ as

$$V_k^s(\mathcal{I}_k^s) = \min_{\mathsf{P}(\gamma_k|\mathcal{I}_k^s)} \mathsf{E}\Big[\theta_k p_k(\mathsf{per}_k, g_k) + \hat{e}_{k+1}^T \Gamma_{k+1} \hat{e}_{k+1} + V_{k+1}^s(\mathcal{I}_{k+1}^s)\Big|\mathcal{I}_k^s\Big],$$

for $k \in \mathbb{N}_{[0,N]}$ with initial condition $V_{N+1}^s(\mathcal{I}_{N+1}^s) = 0$. We need to check that the solution of the above minimization is the scheduling policy $\pi^\star$. Moreover, incorporating the scheduling policy $\pi^\star$ in the loss function $\chi'(\pi, \mu)$ when $\hat{x}_k$ satisfies $\hat{x}_{k+1} = A_k\hat{x}_k + B_k u_k + \gamma_k A_k\tilde{e}_k + (1-\gamma_k)\imath_k$ for $k \in \mathbb{N}_{[0,N]}$ with initial condition $\hat{x}_0 = m_0$, we find

$$\chi'(\pi^\star,\mu) = \mathsf{E}\Big[\sum_{k=0}^{N}\Big\{\theta_k p_k(\tilde{e}_k, g_k) + (u_k + L_k x_k)^T \Lambda_k (u_k + L_k x_k)\Big\}\Big],$$

where $\Lambda_k = B_k^T S_{k+1} B_k + R_k$. Pertaining to $\chi'(\pi^\star, \mu)$, we can write the value function $V_k^c(\mathcal{I}_k^c)$ as

$$V_k^c(\mathcal{I}_k^c) = \min_{\mathsf{P}(u_k|\mathcal{I}_k^c)} \mathsf{E}\Big[\theta_{k-1} p_{k-1}(\tilde{e}_{k-1}, g_{k-1}) + (u_k + L_k x_k)^T \Lambda_k (u_k + L_k x_k) + V_{k+1}^c(\mathcal{I}_{k+1}^c)\Big|\mathcal{I}_k^c\Big],$$

for $k \in \mathbb{N}_{[0,N]}$ with initial condition $V_{N+1}^c(\mathcal{I}_{N+1}^c) = 0$. We need to check that the solution of the above minimization is the control policy $\mu^\star$.

First, we prove by induction that $V_k^s(\mathcal{I}_k^s)$ depends on $\tilde{e}_k$ and $g_k$, and is symmetric with respect to $\tilde{e}_k$. The claim is satisfied for time $N+1$. We assume that the claim holds at time $k+1$. Given the dynamics of $\hat{x}_k$ in this case, we observe that $\hat{e}_k$ and $\tilde{e}_k$ should satisfy

$$\hat{e}_{k+1} = A_k \hat{e}_k - \gamma_k A_k \tilde{e}_k + w_k, \tag{30}$$

$$\tilde{e}_{k+1} = (1-\gamma_k) A_k \tilde{e}_k + K_{k+1}\nu_{k+1}, \tag{31}$$

for $k \in \mathbb{N}_{[0,N]}$ with initial conditions $\hat{e}_0 = x_0 - m_0$ and $\tilde{e}_0 = K_0\nu_0$, where $\nu_k$ is a Gaussian white noise with zero mean and covariance $N_k = C_k M_k C_k^T + V_k$. It follows that

$$\mathsf{E}\Big[\hat{e}_{k+1}^T \Gamma_{k+1} \hat{e}_{k+1}\Big|\mathcal{I}_k^s\Big] = \mathop{\mathsf{E}}_{\mathsf{per}_k}\Big[\mathsf{per}_k\, \tilde{e}_k^T A_k^T \Gamma_{k+1} A_k \tilde{e}_k + \mathrm{tr}(A_k^T \Gamma_{k+1} A_k Y_k + \Gamma_{k+1} W_k)\Big],$$

where we used (30) and the facts that $\mathsf{E}[\hat{e}_k|\mathcal{I}_k^s] = \tilde{e}_k$, $\mathsf{cov}[\hat{e}_k|\mathcal{I}_k^s] = Y_k$, and $w_k$ is independent of $\hat{e}_k$. Moreover, applying the law of total expectation, we find

$$\mathsf{E}\Big[V_{k+1}^s(\mathcal{I}_{k+1}^s)\Big|\mathcal{I}_k^s\Big] = \mathop{\mathsf{E}}_{\mathsf{per}_k}\Big[\mathsf{per}_k\,\mathsf{E}[V_{k+1}^s(\mathcal{I}_{k+1}^s)|\mathcal{I}_k^s, \gamma_k = 0] + (1-\mathsf{per}_k)\,\mathsf{E}[V_{k+1}^s(\mathcal{I}_{k+1}^s)|\mathcal{I}_k^s, \gamma_k = 1]\Big].$$

Note that $\mathsf{E}[V_{k+1}^s|\mathcal{I}_k^s, \gamma_k = 0]$ and $\mathsf{E}[V_{k+1}^s|\mathcal{I}_k^s, \gamma_k = 1]$ are independent of $\mathsf{per}_k$. Accordingly, we deduce that

$$V_k^s(\mathcal{I}_k^s) = \min_{\mathsf{per}_k \in \mathcal{C}}\Big\{\theta_k p_k(\mathsf{per}_k, g_k) + \mathsf{per}_k\,\tilde{e}_k^T A_k^T \Gamma_{k+1} A_k \tilde{e}_k + \mathrm{tr}(A_k^T \Gamma_{k+1} A_k Y_k + \Gamma_{k+1} W_k) + \mathsf{per}_k\,\mathsf{E}[V_{k+1}^s(\mathcal{I}_{k+1}^s)|\mathcal{I}_k^s, \gamma_k = 0] + (1-\mathsf{per}_k)\,\mathsf{E}[V_{k+1}^s(\mathcal{I}_{k+1}^s)|\mathcal{I}_k^s, \gamma_k = 1]\Big\},$$

for $k \in \mathbb{N}_{[0,N]}$, where $Y_k$ and $W_k$ are independent of $\mathsf{per}_k$. Hence, the minimizer is obtained as

$$\mathsf{per}_k^\star = \mathop{\mathrm{argmin}}_{\mathsf{per}_k \in \mathcal{C}}\Big\{\theta_k p_k(\mathsf{per}_k, g_k) + \mathsf{per}_k\,\big(\tilde{e}_k^T A_k^T \Gamma_{k+1} A_k \tilde{e}_k + \varrho_k\big)\Big\},$$

where $\varrho_k = \mathsf{E}[V_{k+1}^s(\mathcal{I}_{k+1}^s)|\mathcal{I}_k^s, \gamma_k = 0] - \mathsf{E}[V_{k+1}^s(\mathcal{I}_{k+1}^s)|\mathcal{I}_k^s, \gamma_k = 1]$. In addition, we can write

$$\begin{aligned}
&\mathsf{E}\Big[V_{k+1}^s(\tilde{e}_{k+1}, g_{k+1})\Big|\mathcal{I}_k^s, \gamma_k\Big] \\
&= \mathsf{E}\Big[V_{k+1}^s\big((1-\gamma_k)A_k\tilde{e}_k + K_{k+1}\nu_{k+1}, g_{k+1}\big)\Big|\mathcal{I}_k^s, \gamma_k\Big] \\
&= \mathsf{E}\Big[V_{k+1}^s\big(-(1-\gamma_k)A_k\tilde{e}_k - K_{k+1}\nu_{k+1}, g_{k+1}\big)\Big|\mathcal{I}_k^s, \gamma_k\Big] \\
&= \mathsf{E}\Big[V_{k+1}^s\big(-(1-\gamma_k)A_k\tilde{e}_k + K_{k+1}\nu_{k+1}, g_{k+1}\big)\Big|\mathcal{I}_k^s, \gamma_k\Big],
\end{aligned}$$

where the first equality comes from (31), the second equality from the hypothesis assumption, and the last equality from the properties of $\nu_k$. Therefore, $\mathsf{E}[V_{k+1}^s(\mathcal{I}_{k+1}^s)|\mathcal{I}_k^s, \gamma_k]$ is symmetric with respect to $\tilde{e}_k$. This implies that $\mathsf{per}_k^\star$ is also symmetric with respect to $\tilde{e}_k$. In addition, note that $g_{k+1}$ depends only on $g_k$. Hence, we conclude that $V_k^s(\mathcal{I}_k^s)$ depends on $\tilde{e}_k$ and $g_k$, and is symmetric with respect to $\tilde{e}_k$. This completes the first part of the proof.

Now, we prove by induction that $V_k^c(\mathcal{I}_k^c)$ is independent of $\mathbf{u}_{k-1}$. The claim is satisfied for time $N+1$. We assume that the claim holds at time $k+1$. Given the dynamics of $\hat{x}_k$ in this case, we observe that $\hat{e}_k$ and $\tilde{e}_k$ should satisfy

$$\hat{e}_{k+1} = A_k\hat{e}_k - \gamma_k A_k\tilde{e}_k + w_k - (1-\gamma_k)\imath_k, \tag{32}$$

$$\tilde{e}_{k+1} = (1-\gamma_k)A_k\tilde{e}_k + K_{k+1}\nu_{k+1} - (1-\gamma_k)\imath_k, \tag{33}$$

for $k \in \mathbb{N}_{[0,N]}$ with initial conditions $\hat{e}_0 = x_0 - m_0$ and $\tilde{e}_0 = K_0\nu_0$, where $\imath_k = \mathsf{E}[\hat{e}_k|\mathcal{I}_k^c, \gamma_k = 0]$. Since $\gamma_k$ under $\pi^\star$ is a function of $\tilde{e}_k$, we recursively infer from (32) and (33) that $\hat{e}_k$ and $\tilde{e}_k$ are independent of the control inputs. Moreover, using the identity $x_k = \hat{x}_k + \hat{e}_k$, we find

$$\begin{aligned}&\mathsf{E}\left[(u_k + L_k x_k)^T\Lambda_k(u_k + L_k x_k)\middle|\mathcal{I}_k^c\right]\\ &\quad= \mathop{\mathsf{E}}_{u_k}\Big[\operatorname{tr}(\Gamma_k P_k) + (u_k + L_k\hat{x}_k)^T\Lambda_k(u_k + L_k\hat{x}_k)\Big],\end{aligned}$$

where we used the facts that $\mathsf{E}[\hat{x}_k|\mathcal{I}_k^c] = \hat{x}_k$ and $\mathsf{E}[\hat{e}_k|\mathcal{I}_k^c] = 0$. Accordingly, we deduce that

$$\begin{aligned}V_k^c(\mathcal{I}_k^c) = \min_{u_k\in\mathbb{R}^m}\Big\{&\theta_{k-1}\,\mathsf{E}[p_{k-1}(\tilde{e}_{k-1}, g_{k-1})|\mathcal{I}_k^c]\\ &+ \operatorname{tr}(\Gamma_k P_k) + (u_k + L_k\hat{x}_k)^T\Lambda_k\\ &\times(u_k + L_k\hat{x}_k) + \mathsf{E}[V_{k+1}^c(\mathcal{I}_{k+1}^c)|\mathcal{I}_k^c]\Big\},\end{aligned}$$

for $k \in \mathbb{N}_{[0,N]}$, where $p_{k-1}(\tilde{e}_{k-1}, g_{k-1})$ and $P_k = \mathsf{cov}[\hat{e}_k|\mathcal{I}_k^c]$ are independent of the control inputs because $\tilde{e}_{k-1}$ and $\hat{e}_k$ are independent of the control inputs, respectively. Hence, the minimizer is obtained as $u_k^\star = -L_k\hat{x}_k$, and we conclude that $V_k^c(\mathcal{I}_k^c)$ is independent of $\mathbf{u}_{k-1}$. We now proceed with the proof by showing that the signaling residual $\imath_k = 0$ for all $k \in \mathbb{N}_{[0,N]}$. Note that $\hat{e}_0$ and $\tilde{e}_0$ are Gaussian vectors with zero mean. We assume that $\imath_t = 0$ for all $t \in \mathbb{N}_{[0,k-1]}$. For any value of $\imath_k$, we have

$$\mathsf{p}(\tilde{e}_k|\mathcal{I}_k^c, \gamma_k = 0) \propto \mathsf{p}(\gamma_k = 0|\tilde{e}_k, \mathcal{I}_k^c)\,\mathsf{p}(\tilde{e}_k|\mathcal{I}_k^c). \tag{34}$$

By the hypothesis assumption and using the scheduling policy $\pi^\star$, we see that $\mathsf{p}(\tilde{e}_k|\mathcal{I}_k^c)$ and $\mathsf{p}(\gamma_k = 0|\tilde{e}_k, \mathcal{I}_k^c)$ are symmetric with respect to $\tilde{e}_k$. Hence, $\mathsf{p}(\tilde{e}_k|\mathcal{I}_k^c, \gamma_k = 0)$ is also symmetric with respect to $\tilde{e}_k$. This implies that $\mathsf{E}[\tilde{e}_k|\mathcal{I}_k^c, \gamma_k = 0] = 0$. Note that we can write

$$\begin{aligned}\mathsf{E}\left[\hat{e}_k\middle|\mathcal{I}_k^c, \gamma_k\right] &= \mathsf{E}\left[\mathsf{E}[\hat{e}_k|\mathcal{I}_k^s, \gamma_k]\middle|\mathcal{I}_k^c, \gamma_k\right]\\ &= \mathsf{E}\left[\mathsf{E}[\hat{e}_k|\mathcal{I}_k^s]\middle|\mathcal{I}_k^c, \gamma_k\right]\\ &= \mathsf{E}\left[\tilde{e}_k\middle|\mathcal{I}_k^c, \gamma_k\right],\end{aligned}$$

where the first equality comes from the tower property of the conditional expectations and the second equality from the fact that $\gamma_k$ is a function of $\mathcal{I}_k^s$. Therefore,

$$\imath_k = A_k\,\mathsf{E}\left[\hat{e}_k\middle|\mathcal{I}_k^c, \gamma_k = 0\right] = 0.$$

This completes the second part of the proof, and establishes that $(\pi^\star, \mu^\star)$ is a Nash equilibrium. ■

## Appendix C
## Proof of Theorem 2

We shall need the following technical lemmas for the proof. For the proofs of these lemmas, see e.g., [31] and [32].

*Lemma 2 (Hardy-Littlewood inequality): Let $f$ and $g$ be non-negative functions defined on $\mathbb{R}^n$ that vanish at infinity. Then,*

$$\int_{\mathbb{R}^n} f(x)g(x)dx \le \int_{\mathbb{R}^n} f^*(x)g^*(x)dx. \tag{35}$$

*Lemma 3: Let $\mathcal{B}(r) \subseteq \mathbb{R}^n$ be a ball of radius $r$ centered at the origin, and $f$ and $g$ be non-negative functions defined on $\mathbb{R}^n$ that vanish at infinity and satisfy*

$$\int_{\mathcal{B}(r)} f^*(x)dx \le \int_{\mathcal{B}(r)} g^*(x)dx, \tag{36}$$

*for all $r \ge 0$. Then,*

$$\int_{\mathcal{B}(r)} h(x)f^*(x)dx \le \int_{\mathcal{B}(r)} h(x)g^*(x)dx, \tag{37}$$

*for all $r \ge 0$ and any symmetric non-increasing function $h$.*

We now present the proof of Theorem 2.

*Proof:* Without loss of generality, assume that $m_0 = 0$. For $m_0 \neq 0$, one can use a simple transformation, and find the same result. To prove global optimality of the equilibrium $(\pi^\star, \mu^\star)$, we need to show that

$$\chi(\pi^\star, \mu^\star) \le \chi(\pi, \mu) \text{ for all } \pi \in \mathcal{P}, \mu \in \mathcal{M}.$$

Let $(\pi^o, \mu^o)$ denote a globally optimal policy profile. In the light of Theorem 1, this policy profile indeed exists.

First, we will show that, given the control policy $\mu^o$, we can find an innovation-based scheduling policy $\sigma$ that is equivalent to $\pi^o$. From the definition of $\nu_k$, we have $\mathbf{y}_k = \boldsymbol{\nu}_k + E_k\check{\mathbf{x}}_{k-1} + F_k\mathbf{u}_{k-1}$, where $E_k$ and $F_k$ are matrices of proper dimensions. By Lemma 1, we have $\check{\mathbf{x}}_k = G_k\boldsymbol{\nu}_k + H_k\mathbf{u}_{k-1}$, where $G_k$ and $H_k$ are matrices of proper dimensions. Besides, from (4), we know that $\mathbf{b}_k$ depends on $\check{\mathbf{x}}_{k-1}$ and $\boldsymbol{\gamma}_{k-1}$. As a result, it is possible to write

$$\mathsf{p}_{\pi^o}(\gamma_k|\mathcal{I}_k^s) = \mathsf{p}_{\pi^o}(\gamma_k|\boldsymbol{\nu}_k, \boldsymbol{\gamma}_{k-1}, \mathbf{u}_{k-1}, \mathbf{g}_k),$$

$$\mathsf{p}_{\mu^o}(u_k|\mathcal{I}_k^c) = \mathsf{p}_{\mu^o}(u_k|\boldsymbol{\nu}_{k-1}, \boldsymbol{\gamma}_{k-1}, \mathbf{u}_{k-1}, \mathbf{g}_k).$$

Accordingly, any realizations of $\gamma_k$ and $u_k$ can be expressed as $\gamma_k = \gamma_k\big(\eta_k; \boldsymbol{\nu}_k, \boldsymbol{\gamma}_{k-1}, \mathbf{u}_{k-1}, \mathbf{g}_k\big)$ and $u_k = u_k\big(\zeta_k; \boldsymbol{\nu}_{k-1}, \boldsymbol{\gamma}_{k-1}, \mathbf{u}_{k-1}, \mathbf{g}_k\big)$, where $\eta_k$ and $\zeta_k$ represent random variables, independent of any other variables, that are used in the generation of the realizations of $\gamma_k$ and $u_k$, respectively. Therefore, it is possible to recursively construct $\mathsf{p}_\sigma(\gamma_k|\boldsymbol{\nu}_k, \boldsymbol{\gamma}_{k-1}, \boldsymbol{\zeta}_{k-1}, \mathbf{g}_k)$ such that it is equivalent to $\mathsf{p}_{\pi^o}(\gamma_k|\mathcal{I}_k^s)$. This establishes that $\chi(\sigma, \mu^o) = \chi(\pi^o, \mu^o)$. Note that although the scheduling policy $\sigma$ is constructed associated with the control policy $\mu^o$, it depends only on $\boldsymbol{\nu}_k$, $\boldsymbol{\gamma}_{k-1}$, $\boldsymbol{\zeta}_{k-1}$, and $\mathbf{g}_k$ at each time $k \in \mathbb{N}_{[0,N]}$.

Now, given the scheduling policy $\sigma$, we will find an optimal control policy $\xi$, and prove that $\xi$ is certainty equivalent. Recall that, by Lemma 1, $\hat{e}_k$ and $\tilde{e}_k$ in general satisfy

$$\hat{e}_{k+1} = A_k\hat{e}_k - \gamma_k A_k\tilde{e}_k + w_k - (1-\gamma_k)\imath_k,$$

$$\tilde{e}_{k+1} = (1-\gamma_k)A_k\tilde{e}_k + K_{k+1}\nu_{k+1} - (1-\gamma_k)\imath_k,$$

for $k \in \mathbb{N}_{[0,N]}$ with initial conditions $\hat{e}_0 = x_0$ and $\tilde{e}_0 = K_0\nu_0$, where $\imath_k = A_k \mathsf{E}[\hat{e}_k|\mathcal{I}_k^c, \gamma_k = 0]$. It is easy to see that $\hat{e}_k$ and $\tilde{e}_k$ are independent of the control inputs under $\sigma$. Then, by a similar argument used in the proof of Theorem 1, one can show that the value function $V_k^c(\mathcal{I}_k^c)$ under $\sigma$ should satisfy

$$V_k^c(\mathcal{I}_k^c) = \min_{u_k \in \mathbb{R}^m} \Big\{ \theta_{k-1} \mathsf{E}[p_{k-1}(\mathsf{per}_{k-1}, g_{k-1})|\mathcal{I}_k^c] + \mathrm{tr}(\Gamma_k P_k) + (u_k + L_k\hat{x}_k)^T \Lambda_k \times (u_k + L_k \hat{x}_k) + \mathsf{E}[V_{k+1}^c(\mathcal{I}_{k+1}^c)|\mathcal{I}_k^c] \Big\},$$

for $k \in \mathbb{N}_{[0,N]}$ with initial condition $V_{N+1}^c(\mathcal{I}_{N+1}^c) = 0$, where $p_{k-1}(\mathsf{per}_{k-1}, g_{k-1})$ and $P_k = \mathsf{cov}[\hat{e}_k|\mathcal{I}_k^c]$ are independent of the control inputs, and that the minimizer is obtained as $u_k^\star = -L_k\hat{x}_k$. This establishes that $\chi(\sigma, \xi) \leq \chi(\sigma, \mu^o)$.

Next, we will show that $\chi(\omega, \xi) \leq \chi(\sigma, \xi)$, where $\omega$ is a special type of $\sigma$ that is symmetric. Let $\mathcal{N}$ be the set on which $\nu_k$ is defined, $\mathcal{B}(r)$ be a ball of radius $r$ centered at the origin and of proper dimension, and $\bar{\nu}_k = T_k \boldsymbol{\nu}_k \in \mathcal{N}$ for a given $T_k$. For any fixed $\boldsymbol{\zeta}_{k-1}$ and $\mathbf{g}_k$[8], we construct $\omega$ with $\mathsf{p}_\omega(\bar{\nu}_k|\gamma_k = 0)$ as a radially symmetric function such that the following conditions are satisfied:

$$\int_{\mathcal{N}^k} \mathsf{p}_\omega(\gamma_k = 0|\boldsymbol{\nu}_k, \boldsymbol{\gamma}_{k-1} = 0)\, \mathsf{s}_k(\boldsymbol{\nu}_k) d\boldsymbol{\nu}_k = \int_{\mathcal{N}^k} \mathsf{p}_\sigma(\gamma_k = 0|\boldsymbol{\nu}_k, \boldsymbol{\gamma}_{k-1} = 0)\, \mathsf{q}_k(\boldsymbol{\nu}_k) d\boldsymbol{\nu}_k, \tag{38}$$

$$\int_{\mathcal{N}^k} p_k\big(\mathsf{p}_\omega(\gamma_k = 0|\boldsymbol{\nu}_k, \boldsymbol{\gamma}_{k-1} = 0)\big)\, \mathsf{s}_k(\boldsymbol{\nu}_k) d\boldsymbol{\nu}_k \leq \int_{\mathcal{N}^k} p_k\big(\mathsf{p}_\sigma(\gamma_k = 0|\boldsymbol{\nu}_k, \boldsymbol{\gamma}_{k-1} = 0)\big)\, \mathsf{q}_k(\boldsymbol{\nu}_k) d\boldsymbol{\nu}_k, \tag{39}$$

$$\int_{\mathcal{B}(r)} \big(\mathsf{p}_\omega(\gamma_k = 0|\bar{\nu}_k, \boldsymbol{\gamma}_{k-1} = 0)\, \mathsf{s}_k(\bar{\nu}_k)\big)^* d\bar{\nu}_k \geq \int_{\mathcal{B}(r)} \big(\mathsf{p}_\sigma(\gamma_k = 0|\bar{\nu}_k, \boldsymbol{\gamma}_{k-1} = 0)\, \mathsf{q}_k(\bar{\nu}_k)\big)^* d\bar{\nu}_k, \tag{40}$$

for $k \in \mathbb{N}_{[0,N]}$ and all $r \geq 0$, where $\mathsf{s}_k(\,.\,) := \mathsf{p}_\omega(\,.\,|\boldsymbol{\gamma}_{k-1} = 0)$ and $\mathsf{q}_k(\,.\,) := \mathsf{p}_\sigma(\,.\,|\boldsymbol{\gamma}_{k-1} = 0)$. Observe that

$$\mathsf{s}_{k+1}(\boldsymbol{\nu}_{k+1}) = \frac{1}{c_\omega} \mathsf{p}(\nu_{k+1}) \times \mathsf{p}_\omega(\gamma_k = 0|\boldsymbol{\nu}_k, \boldsymbol{\gamma}_{k-1} = 0)\, \mathsf{s}_k(\boldsymbol{\nu}_k),$$

$$\mathsf{q}_{k+1}(\boldsymbol{\nu}_{k+1}) = \frac{1}{c_\sigma} \mathsf{p}(\nu_{k+1}) \times \mathsf{p}_\sigma(\gamma_k = 0|\boldsymbol{\nu}_k, \boldsymbol{\gamma}_{k-1} = 0)\, \mathsf{q}_k(\boldsymbol{\nu}_k),$$

for $k \in \mathbb{N}_{[0,N]}$, where $c_\omega = \mathsf{p}_\omega(\gamma_k = 0|\boldsymbol{\gamma}_{k-1} = 0)$ and $c_\sigma = \mathsf{p}_\sigma(\gamma_k = 0|\boldsymbol{\gamma}_{k-1} = 0)$ with initial conditions $\mathsf{s}_0(\nu_0) = \mathsf{q}_0(\nu_0) = \mathsf{p}(\nu_0)$. We can write

$$\begin{aligned} &\mathsf{p}_\sigma(\gamma_k = 0|\boldsymbol{\gamma}_{k-1} = 0) \\ &= \int_{\mathcal{N}^k} \mathsf{p}_\sigma(\gamma_k = 0|\boldsymbol{\nu}_k, \boldsymbol{\gamma}_{k-1} = 0)\, \mathsf{p}_\sigma(\boldsymbol{\nu}_k|\boldsymbol{\gamma}_{k-1} = 0) d\boldsymbol{\nu}_k \\ &= \int_{\mathcal{N}^k} \mathsf{p}_\omega(\gamma_k = 0|\boldsymbol{\nu}_k, \boldsymbol{\gamma}_{k-1} = 0)\, \mathsf{p}_\omega(\boldsymbol{\nu}_k|\boldsymbol{\gamma}_{k-1} = 0) d\boldsymbol{\nu}_k \\ &= \mathsf{p}_\omega(\gamma_k = 0|\boldsymbol{\gamma}_{k-1} = 0), \end{aligned}$$

where the second equality is by (38). Hence, $c_\sigma = c_\omega$. In addition, note that $\mathsf{s}_{k+1}(\bar{\nu}_{k+1})$ and $\mathsf{p}_\sigma(\gamma_k = 0|\bar{\nu}_k, \boldsymbol{\gamma}_{k-1} = 0)\, \mathsf{q}_k(\bar{\nu}_k)$ can be obtained based on $\mathsf{s}_{k+1}(\boldsymbol{\nu}_{k+1})$ and $\mathsf{q}_{k+1}(\boldsymbol{\nu}_{k+1})$, respectively.

[8]For brevity, hereafter we omit the dependency on $\boldsymbol{\zeta}_{k-1}$ and $\mathbf{g}_k$.

To make use of the above construction, we shall introduce an equivalent loss function. It is possible to write

$$\begin{aligned} \chi'(\sigma, \xi) &= \textstyle\sum_{k=0}^N \mathsf{E}\Big[\theta_k p_k(\mathsf{per}_k) + \hat{e}_k^T \Gamma_k \hat{e}_k\Big] \\ &= \textstyle\sum_{k=0}^N \mathsf{E}\Big[\theta_k p_k(\mathsf{per}_k) + \mathsf{E}[\hat{e}_k^T \Gamma_k \hat{e}_k|\mathcal{I}_k^s]\Big] \\ &= \textstyle\sum_{k=0}^N \mathsf{E}\Big[\theta_k p_k(\mathsf{per}_k) + \tilde{e}_k^T \Gamma_k \tilde{e}_k + \mathrm{tr}(\Gamma_k Y_k)\Big], \end{aligned}$$

where in the second equality we used the tower property of conditional expectations. As stated in the proof of Theorem 1, $\chi'(\sigma, \xi)$ is equivalent to $\chi(\sigma, \xi)$. Let us define the loss function $\Upsilon_\sigma^M(\tilde{e}_0)$ as

$$\Upsilon_\sigma^M(\tilde{e}_0) := \textstyle\sum_{k=0}^M \mathsf{E}_\sigma\Big[\theta_k p_k(\mathsf{per}_k) + \tilde{e}_k^T \Gamma_k \tilde{e}_k\Big],$$

for $M \in \mathbb{N}_{[0,N]}$. Since $Y_k$ is independent of $\sigma$, it is enough to prove that $\Upsilon_\omega^M(\tilde{e}_0) \leq \Upsilon_\sigma^M(\tilde{e}_0)$ for any $M \in \{0, \ldots, N\}$ and for any Gaussian vector $\tilde{e}_0$.

Note that $\tilde{e}_0 = K_0\nu_0$ under both $\sigma$ and $\omega$. Moreover, we have

$$\begin{aligned} \mathsf{E}_\sigma\Big[p_0(\mathsf{per}_0)\Big] &= \int_{\mathcal{N}} p_0\big(\mathsf{p}_\sigma(\gamma_0 = 0|\nu_0)\big)\, \mathsf{p}(\nu_0) d\nu_0 \\ &\geq \int_{\mathcal{N}} p_0\big(\mathsf{p}_\omega(\gamma_0 = 0|\nu_0)\big)\, \mathsf{p}(\nu_0) d\nu_0 \\ &= \mathsf{E}_\omega\Big[p_0(\mathsf{per}_0)\Big], \end{aligned}$$

where the inequality is by (39). Hence, the claim holds for the time horizon 0. We assume that it also holds for all the time horizons from 1 to $M-1$. Applying the law of total probability, we see that

$$\mathsf{p}_\sigma(\gamma_0 = 1) + \mathsf{p}_\sigma(\boldsymbol{\gamma}_t = 0) + \textstyle\sum_{k=1}^t \mathsf{p}_\sigma(\boldsymbol{\gamma}_{k-1} = 0, \gamma_k = 1) = 1,$$

for any $t \in \mathbb{N}_{[0,N]}$. Using the above identities, we can obtain

$$\begin{aligned} \Upsilon_\sigma^M(\tilde{e}_0) = \textstyle\sum_{k=0}^M \Big\{ &\theta_k\, \mathsf{p}_\sigma(\boldsymbol{\gamma}_{k-1} = 0)\, \mathsf{E}_\sigma[p_k(\mathsf{per}_k)|\boldsymbol{\gamma}_{k-1} = 0] \\ &+ \mathsf{p}_\sigma(\boldsymbol{\gamma}_{k-1} = 0)\, \mathsf{E}_\sigma[\tilde{e}_k^T \Gamma_k \tilde{e}_k|\boldsymbol{\gamma}_{k-1} = 0] \\ &+ \mathsf{p}_\sigma(\boldsymbol{\gamma}_{k-1} = 0, \gamma_k = 1) \\ &\times \mathsf{E}_\sigma[\Upsilon_\sigma^{k+1,M}(\tilde{e}_{k+1})|\boldsymbol{\gamma}_{k-1} = 0, \gamma_k = 1] \Big\}, \end{aligned}$$

where the cost-to-go is given by

$$\Upsilon_\sigma^{k,M}(\tilde{e}_k) = \textstyle\sum_{t=k}^M \mathsf{E}_\sigma\Big[\theta_t p_t(\mathsf{per}_t) + \tilde{e}_t^T \Gamma_t \tilde{e}_t\Big],$$

for $M \in \mathbb{N}_{[0,N]}$. In the following, we will compare the probability coefficients, the transmit power terms, the estimation mismatch terms, and the cost-to-go terms in the above loss function, which are under $\sigma$, with those under $\omega$.

Since $c_\sigma = c_\omega$, we have $\mathsf{p}_\sigma(\boldsymbol{\gamma}_{k-1} = 0) = \mathsf{p}_\omega(\boldsymbol{\gamma}_{k-1} = 0)$ and $\mathsf{p}_\sigma(\boldsymbol{\gamma}_{k-1} = 0, \gamma_k = 1) = \mathsf{p}_\omega(\boldsymbol{\gamma}_{k-1} = 0, \gamma_k = 1)$.

Hence, all the probability coefficients remain the same under $\omega$. Moreover, for the transmit power terms, we get

$$\begin{aligned}
&\mathsf{E}_\sigma\left[p_k(\mathsf{per}_k)\Big|\boldsymbol{\gamma}_{k-1}=0\right]\\
&\quad=\int_{\mathcal{N}^k} p_k\big(\,\mathsf{p}_\sigma(\gamma_k=0|\boldsymbol{\nu}_k,\boldsymbol{\gamma}_{k-1}=0)\big)\,\mathsf{q}_k(\boldsymbol{\nu}_k)d\boldsymbol{\nu}_k\\
&\quad\geq\int_{\mathcal{N}^k} p_k\big(\,\mathsf{p}_\omega(\gamma_k=0|\boldsymbol{\nu}_k,\boldsymbol{\gamma}_{k-1}=0)\big)\,\mathsf{s}_k(\boldsymbol{\nu}_k)d\boldsymbol{\nu}_k\\
&\quad=\mathsf{E}_\omega\left[p_k(\mathsf{per}_k)\Big|\boldsymbol{\gamma}_{k-1}=0\right],
\end{aligned}$$

where the inequality is by (39). We proceed with the proof for the estimation mismatch terms by first showing that the signaling residual $\imath_k=0$ for all $k\in\mathbb{N}_{[0,N]}$ under $\omega$. We assume that $\imath_t=0$ for all $t\in\mathbb{N}_{[0,k-1]}$. Let $\tau_k$ denote the time elapsed since the last successful delivery when we are at time $k$. By Lemma 1, we can express $\imath_k$ as

$$\begin{aligned}
\imath_k&=A_k\,\mathsf{E}_\omega\left[\textstyle\sum_{t=0}^{\tau_k}D_{k-t}\nu_{k-t}\Big|\gamma_{k-\tau_k}=0,\ldots,\gamma_k=0\right]\\
&=A_k\textstyle\sum_{t=0}^{\tau_k}D_{k-t}\,\mathsf{E}_\omega\left[\nu_{k-t}\Big|\gamma_{k-\tau_k}=0,\ldots,\gamma_k=0\right],
\end{aligned}$$

where $D_{k-t}$ is a matrix depending on $A_{t'}$ for $t'\in\mathbb{N}_{[k-t,k-1]}$ and $K_{k-t}$. As $\mathsf{p}_\omega(\boldsymbol{\nu}_k|\boldsymbol{\gamma}_k=0)$ has zero mean, we deduce that $\mathsf{p}_\omega(\nu_{k-\tau_k},\ldots,\nu_k|\gamma_{k-\tau_k}=0,\ldots,\gamma_k=0)$ has also zero mean. This implies that $\imath_k=0$ for all $k\in\mathbb{N}_{[0,N]}$ under $\omega$. Hence, given $\boldsymbol{\gamma}_{k-1}=0$, we find that $\tilde{e}_k=Z_k\boldsymbol{\nu}_{k-1}+K_k\nu_k+c_k$ under $\sigma$, and that $\tilde{e}_k=Z_k\boldsymbol{\nu}_{k-1}+K_k\nu_k$ under $\omega$ for a suitable matrix $Z_k$ and a suitable vector $c_k$ both independent of $\boldsymbol{\nu}_k$. Let us now use the decomposition $\Gamma_k=L_k^TU_kU_k^TL_k$, choose $T_{k-1}=U_k^TL_kZ_k$, and define functions $f_\sigma(\bar{\nu}_{k-1},\nu_k):=(\bar{\nu}_{k-1}+U_k^TL_kc_k)^T(\bar{\nu}_{k-1}+U_k^TL_kc_k)+\nu_k^TK_k^T\Gamma_kK_k\nu_k$, $f_\omega(\bar{\nu}_{k-1},\nu_k):=\bar{\nu}_{k-1}^T\bar{\nu}_{k-1}+\nu_k^TK_k^T\Gamma_kK_k\nu_k$, $g_\sigma(.):=z-\min_z\{z,f_\sigma(.)\}$, and $g_\omega(.):=z-\min_z\{z,f_\omega(.)\}$. Clearly, for any fixed $z$, $g_\sigma(\bar{\nu}_{k-1},\nu_k)$ and $g_\omega(\bar{\nu}_{k-1},\nu_k)$ vanish at infinity. It follows that

$$\begin{aligned}
\mathsf{E}_\sigma\left[\tilde{e}_k^T\Gamma_k\tilde{e}_k\Big|\boldsymbol{\gamma}_{k-1}=0\right]&=\int_{\mathcal{N}^2}f_\sigma(\bar{\nu}_{k-1},\nu_k)\\
&\qquad\times\mathsf{p}_\sigma(\bar{\nu}_{k-1}|\boldsymbol{\gamma}_{k-1}=0)\,\mathsf{p}(\nu_k)d\bar{\nu}_{k-1}d\nu_k.
\end{aligned}$$

In addition, we can write

$$\begin{aligned}
&\int_{\mathcal{N}}g_\sigma(\bar{\nu}_{k-1},\nu_k)\\
&\quad\times\mathsf{p}_\sigma(\gamma_{k-1}=0|\bar{\nu}_{k-1},\boldsymbol{\gamma}_{k-2}=0)\,\mathsf{q}_{k-1}(\bar{\nu}_{k-1})d\bar{\nu}_{k-1}\\
&\leq\int_{\mathcal{N}}g_\sigma^*(\bar{\nu}_{k-1},\nu_k)\\
&\quad\times\big(\,\mathsf{p}_\sigma(\gamma_{k-1}=0|\bar{\nu}_{k-1},\boldsymbol{\gamma}_{k-2}=0)\,\mathsf{q}_{k-1}(\bar{\nu}_{k-1})\big)^*d\bar{\nu}_{k-1}\\
&=\int_{\mathcal{N}}g_\omega(\bar{\nu}_{k-1},\nu_k)\\
&\quad\times\big(\,\mathsf{p}_\sigma(\gamma_{k-1}=0|\bar{\nu}_{k-1},\boldsymbol{\gamma}_{k-2}=0)\,\mathsf{q}_{k-1}(\bar{\nu}_{k-1})\big)^*d\bar{\nu}_{k-1}\\
&\leq\int_{\mathcal{N}}g_\omega(\bar{\nu}_{k-1},\nu_k)\\
&\quad\times\mathsf{p}_\omega(\gamma_{k-1}=0|\bar{\nu}_{k-1},\boldsymbol{\gamma}_{k-2}=0)\,\mathsf{s}_{k-1}(\bar{\nu}_{k-1})d\bar{\nu}_{k-1},
\end{aligned}$$

where in the first inequality we used the Hardy-Littlewood inequality with respect to $\bar{\nu}_{k-1}$, in the equality the fact that $g_\sigma^*(\bar{\nu}_{k-1},\nu_k)=g_\omega(\bar{\nu}_{k-1},\nu_k)$, and in the second inequality Lemma 3 and (40). This implies that

$$\begin{aligned}
&\int_{\mathcal{N}}\min\nolimits_z\{z,f_\sigma(\bar{\nu}_{k-1},\nu_k)\}\,\mathsf{p}_\sigma(\bar{\nu}_{k-1}|\boldsymbol{\gamma}_{k-1}=0)d\bar{\nu}_{k-1}\\
&\geq\int_{\mathcal{N}}\min\nolimits_z\{z,f_\omega(\bar{\nu}_{k-1},\nu_k)\}\,\mathsf{p}_\omega(\bar{\nu}_{k-1}|\boldsymbol{\gamma}_{k-1}=0)d\bar{\nu}_{k-1}.
\end{aligned}$$

Taking $z$ to infinity, we conclude that

$$\begin{aligned}
&\int_{\mathcal{N}}f_\sigma(\bar{\nu}_{k-1},\nu_k)\,\mathsf{p}_\sigma(\bar{\nu}_{k-1}|\boldsymbol{\gamma}_{k-1}=0)d\bar{\nu}_{k-1}\\
&\geq\int_{\mathcal{N}}f_\omega(\bar{\nu}_{k-1},\nu_k)\,\mathsf{p}_\omega(\bar{\nu}_{k-1}|\boldsymbol{\gamma}_{k-1}=0)d\bar{\nu}_{k-1}.
\end{aligned}$$

Furthermore, for the cost-to-go terms, we find

$$\begin{aligned}
&\mathsf{E}_\sigma\left[\Upsilon_\sigma^{k+1,M}(\tilde{e}_{k+1})\Big|\boldsymbol{\gamma}_{k-1}=0,\gamma_k=1\right]\\
&\quad=\int_{\mathcal{N}^{k+1}}\Upsilon_\sigma^{k+1,M}(\tilde{e}_{k+1})\\
&\qquad\qquad\times\mathsf{p}_\sigma(\boldsymbol{\nu}_{k+1}|\boldsymbol{\gamma}_{k-1}=0,\gamma_k=1)d\boldsymbol{\nu}_{k+1}.
\end{aligned}$$

Note that $\tilde{e}_{k+1}=K_{k+1}\nu_{k+1}$ under both $\sigma$ and $\omega$ when $\gamma_k=1$. Let $\bar{\Upsilon}_\sigma^M(\tilde{e}_0)$ denote a loss function that is structurally similar to $\Upsilon_\sigma^M(\tilde{e}_0)$ but with different parameter values. Clearly, if $\Upsilon_\sigma^M(\tilde{e}_0)\geq\Upsilon_\omega^M(\tilde{e}_0)$, then $\bar{\Upsilon}_\sigma^M(\tilde{e}_0)\geq\bar{\Upsilon}_\omega^M(\tilde{e}_0)$. We can write

$$\begin{aligned}
&\int_{\mathcal{N}^{k+1}}\Upsilon_\sigma^{k+1,M}(K_{k+1}\nu_{k+1})\\
&\qquad\qquad\times\mathsf{p}_\sigma(\boldsymbol{\nu}_{k+1}|\boldsymbol{\gamma}_{k-1}=0,\gamma_k=1)d\boldsymbol{\nu}_{k+1}\\
&=\int_{\mathcal{N}}\bar{\Upsilon}_\sigma^{M-k-1}(K_{k+1}\nu_{k+1})\,\mathsf{p}(\nu_{k+1})d\nu_{k+1}\\
&\geq\int_{\mathcal{N}}\bar{\Upsilon}_\omega^{M-k-1}(K_{k+1}\nu_{k+1})\,\mathsf{p}(\nu_{k+1})d\nu_{k+1}\\
&=\int_{\mathcal{N}^{k+1}}\Upsilon_\omega^{k+1,M}(K_{k+1}\nu_{k+1})\\
&\qquad\qquad\times\mathsf{p}_\omega(\boldsymbol{\nu}_{k+1}|\boldsymbol{\gamma}_{k-1}=0,\gamma_k=1)d\boldsymbol{\nu}_{k+1},
\end{aligned}$$

where in the equalities we used the facts that $\bar{\Upsilon}_\sigma^{M-k-1}(\tilde{e})$ can be defined such that it is equal to $\Upsilon_\sigma^{k+1,M}(\tilde{e})$ for any Gaussian vector $\tilde{e}$, and that $\nu_{k+1}$ is independent of $\boldsymbol{\gamma}_k$, and the Fubini's theorem; and in the inequality we used the hypothesis $\Upsilon_\sigma^{M-k-1}(\tilde{e})\geq\Upsilon_\omega^{M-k-1}(\tilde{e})$ for any Gaussian vector $\tilde{e}$. This establishes that $\Upsilon_\omega^M(\tilde{e}_0)\leq\Upsilon_\sigma^M(\tilde{e}_0)$ and $\chi(\omega,\xi)\leq\chi(\sigma,\xi)$.

Finally, we will conclude that the equilibrium $\chi(\pi^\star,\mu^\star)$ is globally optimal. Note that by a similar argument used in the proof of Theorem 1, one can show that the value function $V_k^s(\mathcal{I}_k^s)$ under $\xi$ in conjunction with $\imath_k=0$ for $k\in\mathbb{N}_{[0,N]}$ should satisfy

$$\begin{aligned}
V_k^s(\mathcal{I}_k^s)=\min_{\mathsf{per}_k\in\mathcal{C}}\Big\{&\theta_kp_k(\mathsf{per}_k,g_k)+\mathsf{per}_k\,\tilde{e}_k^TA_k^T\Gamma_{k+1}A_k\tilde{e}_k\\
&+\mathrm{tr}(A_k^T\Gamma_{k+1}A_kY_k+\Gamma_{k+1}W_k)\\
&+\mathsf{per}_k\,\mathsf{E}[V_{k+1}^s(\mathcal{I}_{k+1}^s)|\mathcal{I}_k^s,\gamma_k=0]\\
&+(1-\mathsf{per}_k)\,\mathsf{E}[V_{k+1}^s(\mathcal{I}_{k+1}^s)|\mathcal{I}_k^s,\gamma_k=1]\Big\},
\end{aligned}$$

for $k\in\mathbb{N}_{[0,N]}$ with initial condition $V_{N+1}^s(\mathcal{I}_{N+1}^s)=0$, and that the minimizer is obtained as $\mathsf{per}_k^\star=\mathrm{argmin}_{\mathsf{per}_k\in\mathcal{C}}\{\theta_kp_k(\mathsf{per}_k,g_k)+\mathsf{per}_k(\tilde{e}_k^TA_k^T\Gamma_{k+1}A_k\tilde{e}_k+\varrho_k)\}$. This establishes that $\chi(\pi^\star,\mu^\star)\leq\chi(\omega,\xi)$, and hence completes the proof. ■

## Appendix D
## Proof of Proposition 1

*Proof:* Let $\breve{\pi}$ be a scheduling policy with $p_k = p_k^r$ for $k \in \mathbb{N}_{[0,N-1]}$, for which $\mathsf{per}_k = \epsilon$, and with $p_N = 0$. In addition, let $\pi^+$ be a scheduling policy that is obtained according to (22) in Theorem 1 except that $\varrho_k$ is now substituted with a new function based on $\breve{\pi}$, i.e., $\breve{\varrho}_k = \mathsf{E}[V_{k+1}^{\breve{\pi}}(\mathcal{I}_{k+1}^s)|\mathcal{I}_k^s, \gamma_k = 0] - \mathsf{E}[V_{k+1}^{\breve{\pi}}(\mathcal{I}_{k+1}^s)|\mathcal{I}_k^s, \gamma_k = 1]$, where $V_k^{\breve{\pi}}(\mathcal{I}_k^s)$ is the cost-to-go associated with $\chi(\breve{\pi}, \mu^\star)$. We shall prove that

$$\chi(\pi^+, \mu^\star) \leq \chi(\breve{\pi}, \mu^\star).$$

To do so, it suffices to show $V_k^{\pi^+}(\mathcal{I}_k^s) \leq V_k^{\breve{\pi}}(\mathcal{I}_k^s)$, where $V_k^{\pi^+}(\mathcal{I}_k^s)$ is the cost-to-go associated with $\chi(\pi^+, \mu^\star)$. Note that $V_{N+1}^{\pi^+}(\mathcal{I}_{N+1}^s) = V_{N+1}^{\breve{\pi}}(\mathcal{I}_{N+1}^s) = 0$. We assume that the claim holds for $k+1$. We can write

$$\begin{aligned}
&\mathsf{E}\Big[\theta_k p_k\big(\mathsf{p}_{\pi^+}(\gamma_k = 0|\mathcal{I}_k^s), g_k\big) \\
&\qquad + \hat{e}_{k+1}^T \Gamma_{k+1} \hat{e}_{k+1} + V_{k+1}^{\pi^+}(\mathcal{I}_{k+1}^s)\Big|\mathcal{I}_k^s\Big] \\
&\leq \mathsf{E}\Big[\theta_k p_k\big(\mathsf{p}_{\pi^+}(\gamma_k = 0|\mathcal{I}_k^s), g_k\big) \\
&\qquad + \hat{e}_{k+1}^T \Gamma_{k+1} \hat{e}_{k+1} + V_{k+1}^{\breve{\pi}}(\mathcal{I}_{k+1}^s)\Big|\mathcal{I}_k^s\Big] \\
&\leq \mathsf{E}\Big[\theta_k p_k\big(\mathsf{p}_{\breve{\pi}}(\gamma_k = 0|\mathcal{I}_k^s), g_k\big) \\
&\qquad + \hat{e}_{k+1}^T \Gamma_{k+1} \hat{e}_{k+1} + V_{k+1}^{\breve{\pi}}(\mathcal{I}_{k+1}^s)\Big|\mathcal{I}_k^s\Big],
\end{aligned}$$

where the first inequality comes from the induction hypothesis and the second inequality from the definition of the suboptimal policy $\pi^+$. This implies that $V_k^{\pi^+}(\mathcal{I}_k^s) \leq V_k^{\breve{\pi}}(\mathcal{I}_k^s)$.

Note that, under $\breve{\pi}$, $\gamma_k = 1$ for all $k \in \mathbb{N}_{[0,N-1]}$ with probability $(1-\epsilon)^N$. In that condition, it is easy to verify that $\chi(\breve{\pi}, \mu^\star) = \breve{\chi}$ (see e.g., [33]), and that $\hat{e}_t$ satisfies

$$\hat{e}_{t+1} = A_t \breve{e}_t + w_t,$$

for $t \in \mathbb{N}_{[k+1,N-1]}$. The latter implies that $\hat{e}_t$ for all $t \in \mathbb{N}_{[k+2,N]}$ are independent of $\gamma_k$. Hence, we get $\breve{\varrho}_k = 0$, and this completes the proof. ∎

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