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Application of the Fractal Brownian Motion to the Athens Stock Exchange

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Abstract: The Athens Stock Exchange (ASE) is a dynamic financial market with complex interactions and inherent volatility. Traditional models often fall short in capturing the intricate dependencies and long memory effects observed in real-world financial data. In this study, we explore the application of fractional Brownian motion (fBm) to model stock price dynamics within the ASE, specifically utilizing the Athens General Composite (ATG) index. The ATG is considered a key barometer of the overall health of the Greek stock market. Investors and analysts monitor the index to gauge investor sentiment, economic trends, and potential investment opportunities in Greek companies. We find that the Hurst exponent falls outside the range typically associated with fractal Brownian motion. This, combined with the established non-normality of increments, disfavors both geometric Brownian motion and fractal Brownian motion models for the ATG index.

Keywords: geometric Brownian motion; fractional Brownian motion; Athens Stock Exchange

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1. Introduction

Predicting future behavior in all aspects of social–financial behavior has long become a field of intense study [1,2]. The financial markets have long been a subject of fascination for researchers and practitioners alike. Understanding the underlying dynamics of stock prices and predicting their behavior is crucial for investors, traders, and policymakers. Traditional models, such as The Black–Scholes–Merton model [3–5] framework, assume that stock prices follow geometric Brownian motion, where the logarithm of the price exhibits independent increments. However, this assumption often falls short in capturing the intricate dependencies and long memory effects observed in real-world financial data.

The fractional Brownian motion (fBm) is a stochastic process that has gained prominence in recent years due to its ability to model correlated and persistent behavior. Unlike standard Brownian motion, fBm allows for dependent increments, making it a powerful tool for capturing long-range dependence and volatility clustering. One of the key parameters in fBm is the Hurst exponent H, which characterizes the degree of memory in the process. In this paper, we delve into the theory behind fractional Brownian motion, exploring its mathematical properties and applications. Specifically, we focus on its relevance to the Athens Stock Exchange (ASE). By incorporating fBm into the modeling of stock price dynamics, we aim to enhance our understanding of the ASE's behavior.

Our objectives include the following:

- Theoretical Foundations: We provide a rigorous treatment of fBm, emphasizing its Gaussian nature and the role of the Hurst exponent. We discuss how fBm differs from standard Brownian motion and its implications for financial modeling.

- ATG Index Data: We analyze historical data from the ATG index, which represents the performance of the top companies listed on the ASE. By applying fBm, we aim to uncover hidden patterns, long-range dependencies, and volatility clustering within the index.
- Modeling Approach: We construct a geometric fractional Brownian motion model tailored to the ATG index. Through parameter estimation, we simulate index behavior and compare it with traditional models.
- Empirical Results: Our analysis reveals that both geometric Brownian motion and fractal Brownian motion models are disfavored for the ATG index.

The challenge is to model efficiently the dynamics of the ATG index in the Athens. Stock Exchange. The main result of our paper is a negative one. Contrary to expectations, the ATG index of ASE is not modeled efficiently by fBm. We think that despite the fact that our result is negative, it does carry important information for readers. fBm has been used extensively in the literature to model similar dynamics. We mention in passing that the main result in Fama's seminal article "Efficient Capital Markets: A Review of Theory and Empirical Work" [6] is a negative one. Fama's work supports the idea that market prices already incorporate all available information, including past stock prices. Therefore, attempting to predict future prices solely based on historical data (such as technical analysis) is unlikely to yield consistent profits. Fama's efficient markets hypothesis (EMH) suggests that stock prices adjust rapidly to new information, making it difficult for investors to consistently outperform the market by analyzing historical data. The novelty of our work is that, as far as we know, the ATG index of the Athens Stock Exchange has not been modeled before by geometric or fBm. The advantage of our approach lies in the fact that we demonstrate that fBm, which has been used extensively to capture the dynamics of similar indices, is not sufficient to describe the dynamics of the ATG index in ASE.

This paper is organized as follows: In Section 2, we highlight the particulars of Black–Scholes–Merton model. We emphasize the importance of independent increments inherent in geometric Brownian motion assumed in the Black–Scholes–Merton model. Moreover, we give the essentials of fBm. In Section 3, we explain the importance of the ATG index on ASE, and we present historical data of the ATG index. In Section 4, we show that the dynamics of the ATG index cannot be modeled by geometric Brownian motion. In Section 5, we study the serial dependence of ATG returns. In Section 6, we show that the dynamics of the ATG index cannot be modeled by fBm either, and we outline avenues for further research.

2. Black–Scholes–Merton Model

Developed by Fischer Black, Myron Scholes, and Robert Merton [3–5], the Black–Scholes–Merton model is a cornerstone of financial modeling. It provides a mathematical framework for pricing options, a type of derivative investment. Central to the model is the Black–Scholes equation, a differential equation of parabolic type that yields the Black–Scholes formula. This formula calculates the theoretical price of a European-style option based on underlying asset characteristics, including the volatility and expected return (represented by the risk-free interest rate). The model operates within a simplified market structure composed of at least one risky asset (typically a stock) and one risk-free asset (like cash or a bond). The assumptions which underpin these assets are as follows:

- Riskless rate: A risk-free investment exists, offering a constant rate of return known as the risk-free interest rate.
- Random walk: Stock prices follow a geometric Brownian motion, meaning they change randomly over time with a constant rate of growth (drift) and volatility. If these factors are not constant, adjustments to the model are necessary, provided volatility itself remains stable.
- Dividends: The stock being analyzed does not pay dividends.

- The Black–Scholes–Merton model operates under the following market conditions:
- No Arbitrage Opportunity: It is impossible to make risk-free profits.
- Perfect Market: Investors can borrow or lend any amount of money at the risk-free interest rate without restrictions.
- Frictionless Trading: There are no transaction costs or taxes on buying or selling stocks, including short selling.

Essentially, the model assumes a highly idealized market without imperfections.

The assumption which will be of main concern to us is the assumption about the random walk of the stock price. The stock price S, at time t, St, is assumed to follow geometric Brownian motion:

$$\zeta \frac{dS_t}{S_t} = \mu dt + \sigma dW_t \quad (1)$$

where Wt is a Wiener process or Brownian motion, and μ ('the percentage drift') and σ ('the percentage volatility') are constant.

2.1. The Assumption of Independent Increments

The Black–Scholes model assumes that stock price changes are independent, meaning past price movements do not influence future ones. However, this is unrealistic as market behavior is often influenced by historical trends. To address this limitation, researchers have explored models where price changes are dependent. One prominent example is the fractional Brownian motion, a statistical process that captures dependencies [7–11]. Given its Gaussian nature, it has been integrated into the Black–Scholes framework to create models that better reflect real-world market dynamics.

Thus, the stock price S, at time t, St, is now assumed to follow the stochastic differential equation:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_H(t) \quad (2)$$

where $B_H(t)$ is a generalization of Brownian motion called fractional (or fractal) Brownian motion. The fractional Brownian motion (fBm) $B_H(t)$ on $[0, T]$ is a continuous-time Gaussian process that starts at zero, has expectation zero for all t in $[0, T]$, and has the following covariance function:

$$E[B_H(t)\ B_H(s)] = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right) \quad (3)$$

where the Hurst exponent, denoted by H, is a crucial parameter in fractional Brownian motion (fBm). It ranges from 0 to 1 and determines the overall smoothness of the process. A higher H value indicates a smoother path, while a lower value results in a more jagged one. It was introduced by Mandelbrot and van Ness in [12].

The behavior of fBm is directly linked to the value of H:

- If $H = 1/2$ the process is equivalent to standard Brownian motion, where price changes are independent.
- If $H > 1/2$ the process exhibits persistence, meaning large price movements tend to be followed by similar large movements in the same direction.
- If $H < 1/2$ the process displays mean reversion, where large price changes are often followed by corrections in the opposite direction.

fBm exhibits a property called stationary increments. This means in particular that the differences $X(t) = B_H(s + t) - B_H(s)$ in the value of fBm at two points in time, regardless of the starting point, is always the same. This difference $X(t)$ is referred to as fractional Gaussian noise.

The process is defined by the following Weyl fractional integral:

$$B_H(t) = B_H(0) + \frac{1}{\Gamma\left(H+\frac{1}{2}\right)}\left\{\int_{-\infty}^{0}\left[(t-s)^{H-\frac{1}{2}} - (-s)^{H-\frac{1}{2}}\right]dB(s) + \int_{0}^{t}(t-s)^{H-\frac{1}{2}}dB(s)\right\}. \quad (4)$$

Zhang et al.'s work [13] provides a comprehensive overview of fractional Black–Scholes models, offering valuable insights into financial modeling and practical applications. Unlike standard Brownian motion, where price changes are independent, fractional Brownian motion allows for dependence between price movements. If H is greater than 0.5, prices tend to exhibit persistence, meaning past increases are likely to be followed by further increases. Conversely, if H is less than 0.5, prices tend to revert to the mean.

2.2. The Assumption of Constant Volatility

Financial models traditionally assumed constant volatility σ, implied in Equation (2). However, the discovery of the volatility smile challenged this assumption. This led to the development of numerous models that could capture the market's dynamic volatility and price derivatives more accurately.

One prominent approach involves stochastic volatility (SV) models. These models treat volatility as a random process, with the specific dynamics defining the model itself. The Heston, SABR, Hull–White, and Bergomi models are all examples of SV models, each with unique features suited for different pricing needs.

A significant limitation of current volatility models is their inability to simultaneously capture the true shape of the implied volatility surface and represent realistic volatility dynamics. The Rough Bergomi model, a leader in the field of rough volatility models, addresses this challenge. It achieves exceptional performance with only three parameters, outperforming conventional Brownian motion-based SV models. Rough volatility models, popularized in [14], replace Brownian motion with a more complex process called fractional Brownian motion (Equation (4)).

In this paper, we deviate from the assumption of fractional Brownian motion for volatility. Instead, we focus on the stock price itself, assuming it follows a fractional Brownian motion as described in Equation (2).

3. The ATG Index

To demonstrate the concepts and methods described in the present article we used financial data from the Athens General Composite (ATG).

The Athens General Composite (ATG) Index is the main stock market index for the Athens Stock Exchange (ASE) in Greece. The ATG is a capitalization-weighted index, meaning the companies with the largest market capitalization (stock price multiplied by number of shares outstanding) have a greater influence on the index's overall performance. The ATG has a base value of 100 set on December 31, 1980. This serves as a reference point to track the overall growth or decline of the Greek stock market over time. The ATG includes shares of companies listed on the ASE's "Big Cap" segment. These are typically large, well-established companies with significant market presence. The ATG is considered a key barometer of the overall health of the Greek stock market. Investors and analysts monitor the index to gauge investor sentiment, economic trends, and potential investment opportunities in Greek companies.

We used daily data of closing prices from 27 January 2020 to 22 March 2024, i.e., a total of 1037 observations. The evolution of the prices and their returns of ATG is depicted in Figures 1 and 2.

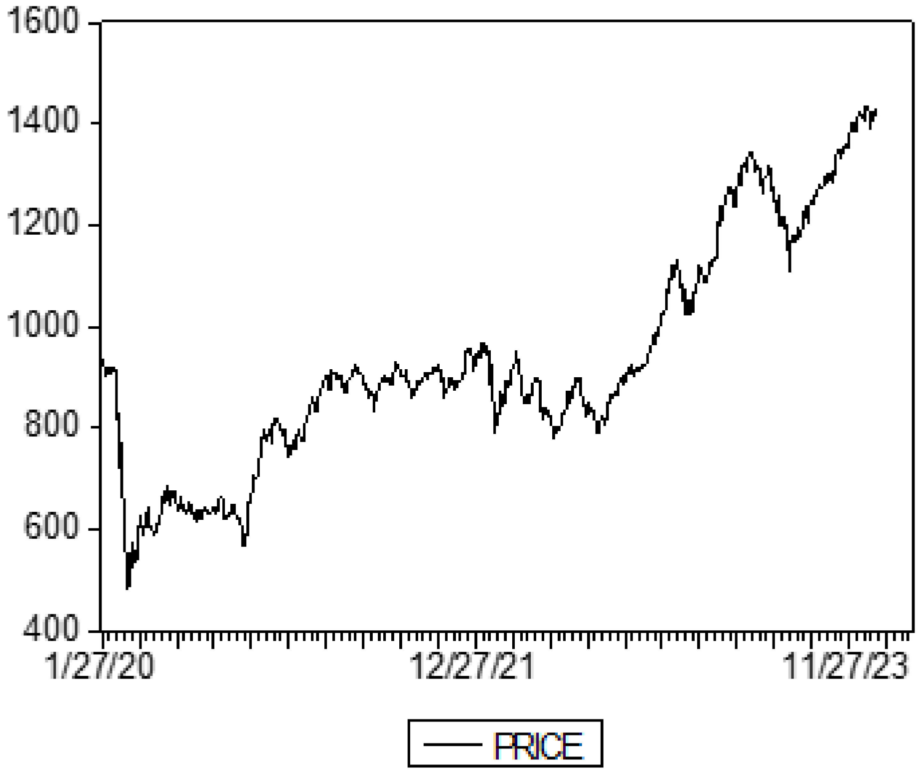

Figure 1. Athens composite general index of the athens stock exchange—price.

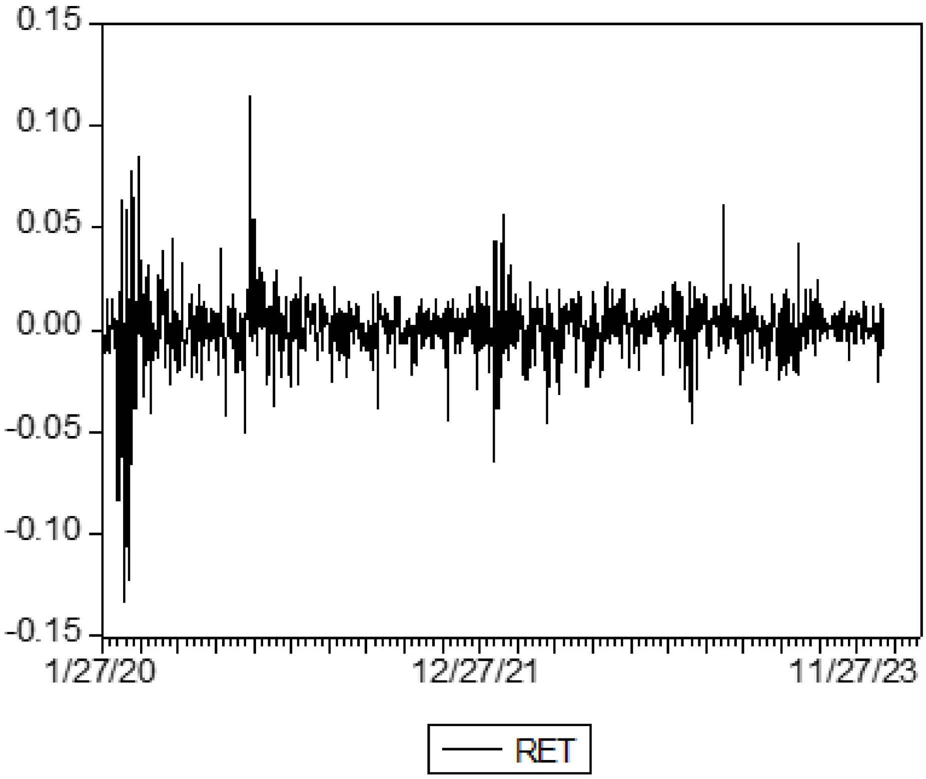

Figure 2. Athens composite general index of the athens stock exchange—returns.

4. The ATG Index and Geometric Brownian Motion

To test whether the value *Vt* of the index ATG follows a geometric Brownian motion:

$$dV_t = \mu V_t dt + \sigma V_t dW_t \tag{5}$$

we analyze the distribution of returns $\frac{dV_t}{V_t}$ of ATG in which case they have to follow a normal distribution, whose mean and variance can be used to estimate the parameters μ, σ of the above Geometric Brownian model. The distribution of returns of ATG is given in Figure 3.

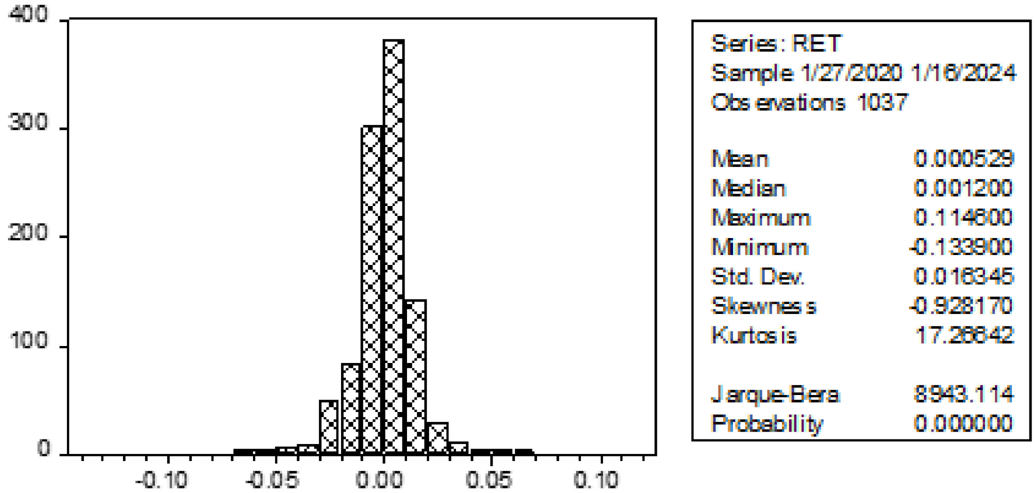

Figure 3. Distribution of returns of ATG.

Although the parameters of the Brownian model can be seen to be estimated as $\mu = 0$ and $\sigma = 1.63\%$, the distribution is far from being normal. This can be seen from the fact that the distribution of returns is negatively skewed (the normal distribution has skewness 0) and also its kurtosis is much larger than 3 (which is the kurtosis of the normal distribution). This is also manifested by the large value of the Jarque–Berra statistic (p-value = 0) and can be further demonstrated by the following $Q - Q$ plot of the distribution of returns against the normal distribution (Figure 4).

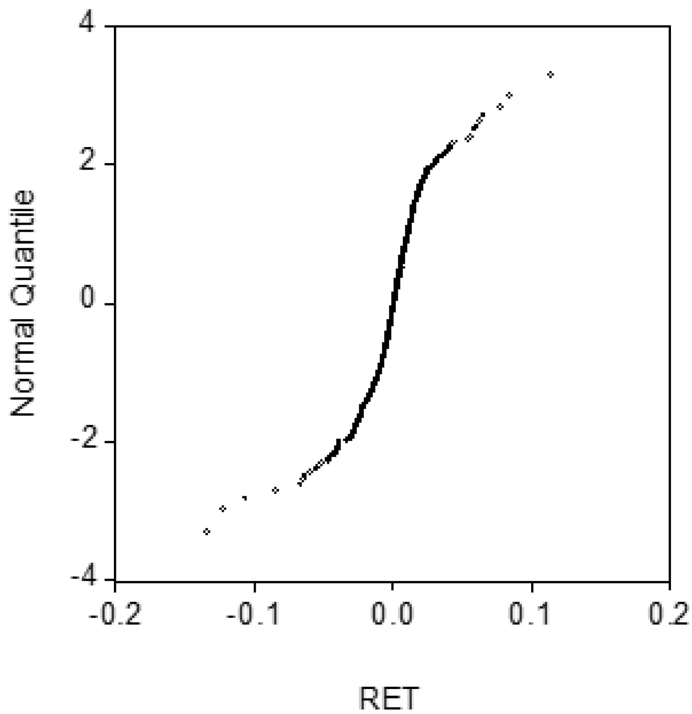

Figure 4. Distribution of returns against the normal distribution.

5. Serial Dependence of ATG Returns

Before diving into complex models like fractal Brownian motion, which relies on correlations between Brownian motion increments, let us analyze the serial dependence of ATG returns. We can calculate the auto-correlogram of the returns $\frac{dV_t}{V_t}$ to assess these dependencies. This visual tool will help us identify potential trends, seasonality, or volatility patterns by looking for spikes or dips at specific lags. Table 1 depicts the auto-correlogram.

Table 1. Autocorrelation (AC), Partial Autocorrelation (PAC) and Q-Statistics (Q-Stat) of ATG returns.

Date: 03/26/24 Time: 15:57
Sample: 1/27/2020 3/22/2024
Included observations: 1037

Autocorrelation	Partial Correlation		AC	P AC	Q-Stat	Prob
. | |	. | |	1	−0.037	−0.037	1.3898	0.238
. | |	. | |	2	0.061	0.060	5.2929	0.071
. |* |	. |* |	3	0.082	0.087	12.327	0.006
. | |	. | |	4	−0.005	−0.003	12.355	0.015
. |* |	. |* |	5	0.148	0.139	35.207	0.000
* | |	* | |	6	−0.093	−0.092	44.277	0.000
. | |	. | |	7	0.058	0.037	47.749	0.000
. | |	. | |	8	0.040	0.030	49.391	0.000
. | |	. | |	9	−0.029	−0.018	50.265	0.000
. | |	. | |	10	0.065	0.035	54.745	0.000
. | |	. |* |	11	0.060	0.088	58.502	0.000
. | |	. | |	12	0.059	0.042	62.153	0.000
. | |	. | |	13	−0.004	−0.017	62.167	0.000
* | |	* | |	14	−0.061	−0.071	66.053	0.000
. | |	. | |	15	−0.002	−0.036	66.056	0.000
. | |	. | |	16	−0.015	−0.019	66.293	0.000
* | |	* | |	17	−0.096	−0.092	75.973	0.000
. | |	. | |	18	−0.025	−0.026	76.658	0.000
. | |	. |* |	19	0.062	0.088	80.722	0.000
* | |	. | |	20	−0.061	−0.050	84.725	0.000
. | |	. | |	21	−0.036	−0.044	86.129	0.000
. | |	. | |	22	0.009	0.020	86.213	0.000
. | |	. | |	23	−0.004	−0.010	86.230	0.000
. | |	. | |	24	0.019	0.012	86.622	0.000
* | |	. | |	25	−0.087	−0.043	94.727	0.000
. | |	. | |	26	0.029	0.029	95.603	0.000
. | |	. | |	27	0.024	0.037	96.216	0.000
* | |	. | |	28	−0.079	−0.048	102.95	0.000
. | |	. | |	29	0.023	0.009	103.50	0.000
* | |	. | |	30	−0.059	−0.045	107.23	0.000
. | |	. | |	31	0.041	0.018	109.05	0.000
. | |	. | |	32	−0.020	−0.010	109.47	0.000
. | |	. | |	33	−0.040	−0.006	111.22	0.000
. | |	. | |	34	−0.005	−0.052	111.25	0.000
. | |	. | |	35	0.017	0.043	111.56	0.000
. | |	. | |	36	0.020	0.033	111.99	0.000

Our auto-correlogram analysis reveals no significant autocorrelation at any lag, indicating an absence of simple dependencies within a reasonable timeframe for ATG returns.

6. The ATG Index and Fractal Brownian Motion

Let us now explore a crucial parameter for fractal Brownian motion, the Hurst exponent H. By calculating H for ATG returns, we can gain valuable insights into the stochastic process governing the stock market index. To calculate the Hurst exponent H, we plot the R/S value (across subseries of a specific length) on the y-axis (log scale) against the subseries length on the x-axis (log scale); R and S are defined in [12]. Then, we perform a linear regression on the log–log plot. The slope of the fitted line is an estimate of the Hurst exponent H. Hence, in our case we calculated the RS Statistic for subseries of the ATG returns, of length 10, 20, 30, . . ., 1000, 1020, 1030 with the resulting log–log plot depicted in Figure 5.

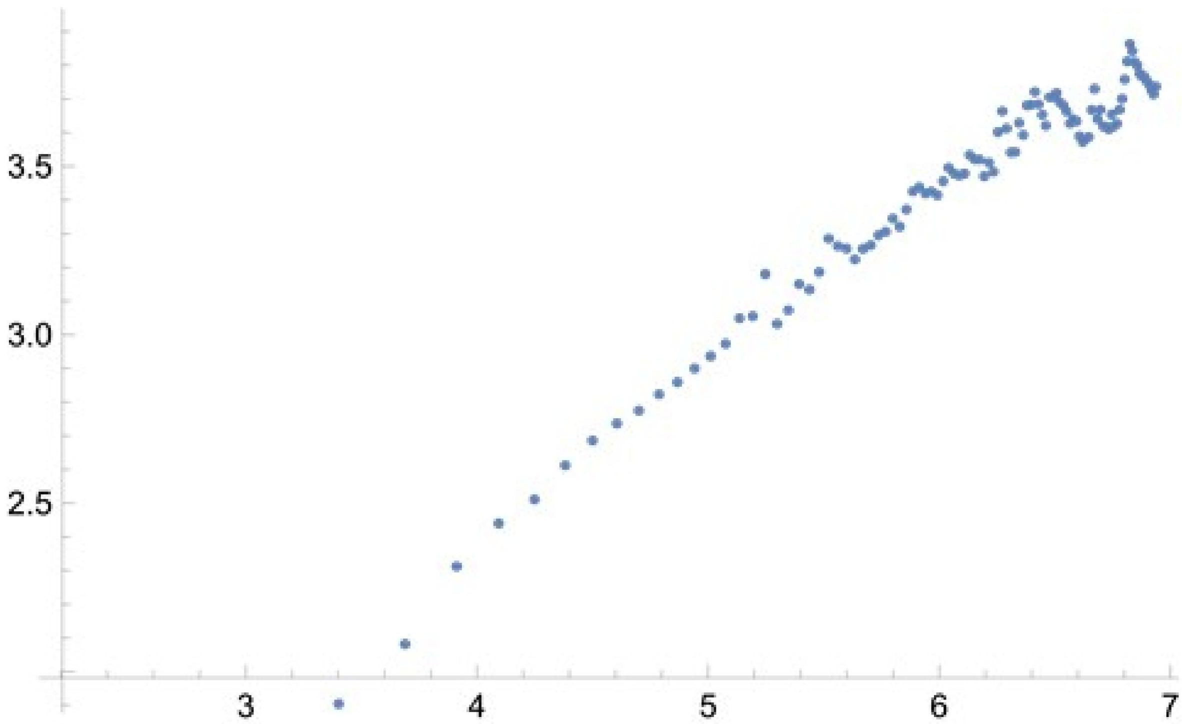

Figure 5. RS Statistic for subseries of the ATG returns.

This analysis reveals an almost linear pattern in the log–log plot of the R/S statistic for the ATG returns, suggesting the presence of a Hurst exponent. However, the estimated Hurst exponent ($H = 0.499$) falls outside the range typically associated with fractal Brownian motion ($H = 1/2$). This, combined with the previously established non-normality of increments, disfavors both geometric Brownian motion and fractal Brownian motion models for the ATG index. Given these limitations, we recommend exploring alternative stochastic processes to capture the behavior of the ATG index. Here are some potential avenues for further research:

- Long Memory Processes: Investigate models like Fractional ARIMA (Autoregressive Integrated Moving Average) or other long memory models that can account for persistent dependence structures observed in the data.
- Jump Diffusion Processes: Explore models that incorporate jumps or sudden price changes, potentially reflecting market events or news announcements that might not be captured by Brownian motion.
- Machine Learning Techniques: Consider employing machine learning algorithms like recurrent neural networks (RNNs) or long short-term memory (LSTM) networks to learn complex non-linear relationships within the data.
- Rough Volatility Models: Consider employing rough volatility in order to capture the true shape of the implied volatility surface and deliver realistic dynamics for the volatility surface.

Many financial instruments, such as currency exchange rates and stock prices, exhibit a pattern that differs from a random walk. This means their past values significantly influence future values, a characteristic known as long memory. Fractional Brownian motion (fBm) is a mathematical model that captures this long memory behavior. It uses a parameter called the Hurst exponent (H) to quantify the strength of these dependencies.

However, while fBm effectively models linear relationships [12], more complex patterns in financial data may require additional factors beyond the Hurst exponent to fully describe their behavior [15]. This justifies even further the need for future research. The avenues that are suggested are possible avenues whose real relevance and suitability is only going to be established after pursuing them.

7. Conclusions

The fBm is a continuous-time Gaussian process that extends the standard Brownian motion by introducing dependence between non-overlapping increments. Unlike classical Brownian motion, fBm increments need not be independent. It is characterized by a parameter called the Hurst exponent (H), which determines the degree of correlation between increments. We found that neither geometric Brownian motion nor fractal Brownian motion model efficiently the ATG index in the Athens Stock Exchange.

8. Limitations and Future Research

By construction, the current study seeks to model the dynamics of the ATG index in the Athens Stock Exchange by assuming geometric Brownian motion and fractal Brownian motion. It may be the case that this dynamic is captured more efficiently by rough volatility models where it is assumed that the volatility itself follows fBm. This and other avenues for future research which are outlined in Section 6 are left for future research.

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