Risk Budgeting under General Risk Measures

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Abstract

We provide an ample characterization for Risk Budgeting/Parity portfolios with general convex and homogeneous risk preferences for long-only portfolios, as well as for long-short portfolios. We propose a more general novel definition of Risk Budgeting/Parity portfolios that is less restrictive than the classical definition, and it guarantees their existence and uniqueness, at least for the long-only case. This case is shown to always be less risky than the Equal Weighted portfolio and a thorough mathematical characterization of Risk Budgeting/Parity portfolios is also provided. Equivalent properties are concluded for long-short risk budgeting portfolios under some additional conditions. We provide new insights about the Risk Budgeting/Parity portfolios, including that those portfolios are a rich subset of the newly coined set of Generalized Weighted Mean Constrained portfolios that, according to our knowledge, is defined for the first time in this paper. This new class of portfolios contains other portfolios with good performance, e.g., norm constrained and shortsale-constrained portfolios. Statistical inferences for Risk Budgeting portfolios are provided for volatility and Conditional-Value-at-Risk risk preferences, and a by-product of our work is

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the introduction of a novel Conditional-Value-at-Risk estimator. An extensive real data analysis shows that Risk Parity portfolios have an enhanced out-of-sample performance than its benchmark portfolios by reducing the risk, but also by better balancing the trade-off between risk and return that pays off during adverse and booming market conditions.
Keywords: Risk budgeting/parity portfolio, Portfolio theory, Risk measure.

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1. Introduction

One of the most important activities in financial markets is to construct an investment portfolio with good out-of-sample performance that is resistant to market downturns and recessionary periods. This paper focuses on explaining and characterizing the *Risk budgeting (RB)* and *Risk parity (RP)* portfolios that are recognised to perform well across various adverse market conditions (Cesarone and Colucci, 2018), though the theoretical and empirical evidence is still developing. The first RP fund recognized by practitioners is the Bridgewater's All Weather Fund in 1996 and there are reported examples of RP investments prior to the 2008 global financial crisis (Qian, 2005). However, in the aftermath of the global financial crisis RP portfolio construction was seen as a suitable solution to control risk, and the RP portfolio management sector reached an estimated USD 150-175 billion at year-end 2017[1], which motivated S&P Global to introduce the S&P Risk Parity Index Series. The RP portfolio is also known as *Equal Risk Contribution (ERC)* portfolio (Roncalli, 2013) and such portfolios achieve diversification through imposing equal individual risk contributions.

The first RP formulation can be traced back to Qian (2005). The initial RP/ERC implementation makes simplified assumptions from which the weights are inversely proportional to the asset-class risk position (known as IWP portfolios) when risk preferences are ordered by the *standard deviation (SD)*. That solution however only approximates RP portfolios, which was the practical way to perform RP-like evaluations before bespoke RP algorithms became available, and for this reason, the IWP and RP definitions are mistakenly assumed interchangeable. The first theoretical and practical contributions to understanding long-only RB/RP portfolios appeared after the 2008 financial crisis (Maillard et al., 2010; Roncalli, 2013; Spinu, 2013), but only for SD risk preferences; further extensions for other specific risk preferences are for *Conditional Value-at-Risk* (CVaR) (Mausser and Romanko, 2018) and expectiles (Bellini et al., 2021). Theoretical contributions to understanding the properties of long-only RB/RP portfo-

[1] See the 2020 S&P Global report from https://www.spglobal.com/spdji/en/documents/research/research-indexing-risk-parity-strategies.pdf

lios with general differentiable risk preferences have recently appeared in the mathematical finance and operations research literature (Cesarone et al., 2020; Cetingoz et al., 2024). The characterization of long-short RB portfolios is a challenging problem and very few papers have discussed this problem (Spinu, 2013; Bai et al., 2016), and these two references deal with RB portfolios only with SD risk preferences. Factor RP portfolios have been also considered in the literature (Roncalli and Weisang, 2016; Lassance et al., 2022; Cetingoz and Guéant, 2024).

The *Equal Weight (EW)* (also known as $1/N$) portfolio is a very good benchmark (DeMiguel et al., 2009b) due to its simplicity and lack of estimation error, and RP portfolios tend to outperform EW. The more recent literature has provided theoretical evidence in that respect for various settings (Roncalli, 2013; Bellini et al., 2021; Cetingoz et al., 2024) besides the empirical evidence that has been showcased in the grey literature. Note that independent of the RP literature, IWP portfolios with SD risk preferences are investigated under the name of *Volatility Timing portfolios* where they are shown to outperform EW (Kirby and Ostdiek, 2012). The norm constrained portfolios could outperform EW as well (Jagannathan and Ma, 2003; DeMiguel et al., 2009a,b), and our work explains this positive trait in a more general setting; that is, we show that RB/RP and the rich class of norm constrained portfolios are part of the same set of portfolios that we define in this paper and this large set of portfolios is shown to have this positive trait of being less risky than EW.

The *first* main contribution of this paper is to provide new insights about RB/RP, meaning that we provide a new mathematical RB/RP formulation that generalizes the classical definition for non-differentiable risk preferences, and CVaR is a well-known example. A new economic interpretation of RP for differentiable risk measures is given in terms of the portfolio risk position elasticities. We show how different the RP and IWP portfolios with SD risk preferences could be, which raises awareness that one should not rely on IWP formulations to simplify RP computations; this finding has practical significance as the IWP formulation is often misunderstood as the RP formulation. Finally, we discover new links between the class of RB/RP portfolios and an existing class of portfolios known as the norm constrained portfolios (DeMiguel et al., 2009a) which includes the shortsale-constrained portfolios (Jagannathan and Ma, 2003; DeMiguel et al., 2009b). We demonstrate that both RB/RP and norm constrained portfolios are rich sub-classes of the set of *Generalized Weighted Mean Constrained (GWMC)* portfolios that, according to our knowledge, is defined for the first time in this paper. We succinctly provide some properties of the GWMC class, one of which being that the long-only GWMC are always less risky than EW.

The *second* main contribution of this paper is to provide a mathematical characterization for RB/RP portfolios with general risk preferences when short selling is possible under the new and more general RB/RP definition that we coin in this paper. Our general theoretical results provide a comprehensive mathematical characterization of the existence and uniqueness of RB/RP portfolios when the long/short positions under general risk preferences. These findings confirm and generalize previous results that have been found in the literature for specific risk preferences; e.g., CVaR (Mausser and Romanko, 2018), expectiles (Bellini et al., 2021), SD (Maillard et al., 2010; Roncalli, 2013), which are mainly focused on long-only portfolios. In addition, our mathematical formulation generalizes two recent results (Cesarone et al., 2020; Cetingoz et al., 2024) that focus on the existence and uniqueness of long-only RB/RP portfolios under general risk preferences that are differentiable. Therefore, besides dropping the technical differentiability condition that could be problematic – e.g., see (Mausser and Romanko, 2018) for CVaR risk preferences – this paper is the first one to provide a mathematical characterization of long-short RB/RP portfolios with general risk preferences though some papers only discuss the long-short setting for a specific risk preference, namely, for SD (Spinu, 2013; Bai et al., 2016). Finally, we show that EW is riskier than any RB portfolio, which generalizes previous attempts in the literature showing that EW is riskier than the RP portfolio for specific risk preferences and general differentiable risk preferences. Such an important result is backed up by our ample data analyses.

The *third* main contribution of this paper is to provide statistical inferences and their asymptotic properties for RB/RP portfolios, which according to our knowledge is the first attempt in the literature. Our setting includes dependent data and we focus only on CVaR and SD preferences. A by-product of such mathematical statistics results is the introduction of a new CVaR estimator. These findings allow us to run an extensive data analysis for US financial time series data and to illustrate the resilience of RP investment portfolios as compared to standard benchmark portfolios during various adverse or favorable market conditions. We found that RP outperforms EW under favorable market conditions and it is no worse than EW otherwise; in addition, RP also reduces the portfolio losses under extremely unfavorable market conditions which implies that RP portfolios are not only risk conservative strategies, but also have a good trade-off between risk and return that pays off during adverse and booming market conditions.

The paper is organized as follows: Section 2 provides all definitions and notations and essential background, while new insights about RB/RP are provided in Section 3; Section 4 contains the main theoretical results of the RB/RP portfolios, while the statistical inferences theory is provided in Section 5; ample numerical evidence is provided in Section 6; the main conclusions and recommendations are provided in Section 7; some ancillary results are collected in Appendix A, while further empirical evidence is included in Appendix B.

2. Problem formulation

We first introduce some generic notations used throughout this paper. Note that all returns in this paper are in fact loss (minus returns) random variables.

Let Δ_d be the unit d-simplex $\Delta_d := \{\mathbf{x} \in \Re^d : \mathbf{1}^T\mathbf{x} = 1\}$ for any positive integer d. The non-negative d-simplex is denoted as $\Delta_d^+ := \{\mathbf{x} \in \Re_+^d : \mathbf{1}^T\mathbf{x} = 1\}$, where $\Re_+^d := \{\mathbf{x} \in \Re^d : \mathbf{x} \geq 0\}$ is the standard polyhedral cone of the non-negative quadrant of $\Re^d$. Similarly, the positive d-simplex is defined as $\Delta_d^{++} := \{\mathbf{x} \in \Re_{++}^d : \mathbf{1}^T\mathbf{x} = 1\}$, where $\Re_{++}^d := \{\mathbf{x} \in \Re^d : \mathbf{x} > 0\}$.

The financial field is represented by $(\Omega, \mathcal{F}, \mathbb{P})$, an atomless probability space, endowed with $L^0 := L^0(\Omega, \mathcal{F}, \mathbb{P})$, the set of all real-valued random variables on this probability space. Let L^q, $q \in [0, \infty)$, be the set of random variables with finite q^{th} moment, and L^∞ be the set of bounded random variables. A risk measure φ is a function that maps an element of L^0 to the real set, i.e. $\varphi : L^0 \to \Re \cup \{\pm\infty\}$; we then say that the investor's risk preferences are ordered by φ. We recall below some properties for a generic risk measure and generic random variable Y that represents the future loss of a financial asset. These properties are well-known in the literature (Föllmer and Schied, 2011), and are outlined below:

Convexity: $\varphi(aY_1 + (1-a)Y_2) \leq a\varphi(Y_1) + (1-a)\varphi(Y_2)$ for any $Y_1, Y_2 \in L^0$ and $a \in [0, 1]$;
Homogeneous of order $\tau > 0$: $\varphi(cY) = c^\tau \varphi(Y)$ for any $Y \in L^0$ and $c \geq 0$;
Shift invariance: $\varphi(Y + c) = \varphi(Y)$ for any $Y \in L^0$ and $c \in \Re$;
Translation invariance: $\varphi(Y + c) = \varphi(Y) + c$ for any $Y \in L^0$ and $c \in \Re$.

Four risk measures are often recalled in this paper: SD, var, *Value-at-Risk* (VaR) and CVaR. For any $p \in (0, 1)$, VaR at probability level p is $\mathrm{VaR}_p(Y) := \inf_x \{\mathbb{P}(Y \leq x) \geq p\}$, while CVaR at probability level p is $\mathrm{CVaR}_p(Y) := \min_\theta \left\{\theta + \frac{1}{1-p}\mathrm{E}(Y - \theta)_+\right\}$ with $(\cdot)_+ := \max(\cdot, 0)$ on $\Re$. There are other risk measures interrelated to those four choices; e.g., *Median Shortfall (MS)* (median of the tail distribution, i.e., a VaR-type risk measure).

The investor is assumed to invest in a given opportunity portfolio set containing $d > 1$ assets, and let $\mathbf{X} = (X_1, \ldots, X_d)$ be the vector of assets' losses. The investment strategy is uniquely determined by a vector of proportions $\boldsymbol{\alpha} \in \Delta_d$; that is, the portfolio loss/profit is $\boldsymbol{\alpha}^T\mathbf{X}$. Note that long-only portfolios are considered in Sections 4.1, which is a reasonable assumption; e.g., (Jagannathan and Ma, 2003; DeMiguel et al., 2009b) show that constraining the amount of short sales could improve the out-of-sample performance. Furthermore, we assume that the risk preferences of an investor are represented by the risk measure φ and therefore, the investor's perception of risk is given by $\mathcal{R}(\boldsymbol{\alpha}) := \varphi(\boldsymbol{\alpha}^T\mathbf{X})$. In our paper, we rely on the mathematical properties of various risk measures assumed to be homogeneous of order $\tau \in \mathbb{R}$. Hence, by Euler's

Homogeneous Function Theorem for differentiable $\mathcal{R}(\boldsymbol{\alpha})$, we have that

$$\mathcal{R}(\boldsymbol{\alpha}) = \frac{1}{\tau} \sum_{k=1}^{d} \alpha_k \frac{\partial \mathcal{R}(\boldsymbol{\alpha})}{\partial \alpha_k} = \sum_{k=1}^{d} \mathcal{RC}_k(\boldsymbol{\alpha}), \quad \text{where} \quad \mathcal{RC}_k\left(\boldsymbol{\alpha}\right) := \frac{\alpha_k}{\tau} \frac{\partial \varphi\left(\boldsymbol{\alpha}^T \mathbf{X}\right)}{\partial \alpha_k}. \tag{2.1}$$

By definition, $\mathcal{RC}_k(\boldsymbol{\alpha})$ is the *risk (or loss) contribution* of the k^{th} individual risk. Even though the risk contributions depend upon the risk measure choice, we do not add φ in $\mathcal{RC}_k(\boldsymbol{\alpha})$ so that the notations are kept as simple as possible. The classical RB definition says that an investing strategy $\boldsymbol{\alpha} \in \Delta_d$ is a RB portfolio with a budgeting vector $\mathbf{b} \in \Delta_d^{++}$ if

$$\mathcal{RC}_k\left(\boldsymbol{\alpha}\right) = b_k \varphi\left(\boldsymbol{\alpha}^T \mathbf{X}\right), \text{ for all } k \in \{1, 2, \ldots, d\}, \text{ where } \mathcal{RC}_k(\boldsymbol{\alpha}) \text{ is given in (2.1)}. \tag{2.2}$$

Note that this definition assumes $\mathcal{R}(\boldsymbol{\alpha})$ to have partial derivatives, and differentiability has been directly or indirectly assumed, which reduces the degree of generality; e.g., see Example 2.2. The homogeneity property of φ (and thus of $\mathcal{R}$) implies that $\mathcal{R}$ has partial derivatives almost everywhere, meaning that the set of points at which $\mathcal{R}$ does not admit partial derivatives is at most countable that could be finite or infinite. One could extend (2.2) by using the concept of *subdifferential* which is the set of all subgradients that generalizes the concept of gradient/differentiability. Any convex and proper function $\mathcal{R}$ on $\Re^d$ admits a non-null subgradient for any point in the relative interior of its domain, denoted as *relint*$(\mathcal{R})$; for details, see Rockafellar (1970). Theorem 2.2 in Hendrickson and Buehler (1971) extends the Euler's Homogeneous Function Theorem for non-differential functions and implies that for any $\boldsymbol{\alpha} \in$ *relint*$(\mathcal{R})$

$$\mathcal{R}(\boldsymbol{\alpha}) = \frac{1}{\tau} \boldsymbol{\alpha}^T \mathbf{a}, \quad \text{where } \mathbf{a} \in \partial \mathcal{R}(\boldsymbol{\alpha}), \tag{2.3}$$

provided that $\mathcal{R}$ is convex, proper and homogeneous of degree τ. We could now provide the generalized definition of RB/RP strategies, which is given in Definition 2.1. Note that we add φ in the notation of RB/RP strategies, so that one could distinguish RB/RP portfolios based on different risk preferences.

Definition 2.1. *Let* $\boldsymbol{b} := (b_1, \ldots, b_d)^T$ *be a given constant vector such that* $\boldsymbol{b} \in \Delta_d^{++}$*. An investing strategy* $\boldsymbol{\alpha} \in \Delta_d$ *is a solution to the RB problem based on the risk measure* φ *if there exists* $\boldsymbol{a} \in \partial \mathcal{R}(\boldsymbol{\alpha})$ *such that*

$$\frac{1}{\tau} \alpha_k a_k = b_k \varphi\left(\boldsymbol{\alpha}^T \boldsymbol{X}\right), \text{ for all } k \in \{1, 2, \ldots, d\}. \tag{2.4}$$

Let $\mathcal{RB}(\boldsymbol{b}, \varphi) := \{\boldsymbol{\alpha} \in \Delta_d : \boldsymbol{\alpha} \text{ satisfies (2.4) relative to } \varphi\}$ *be the set of RB portfolios for a given budgeting allocation vector* $\boldsymbol{b}$ *and a general risk measure* φ*. In particular,* $\mathcal{RB}(\frac{1}{d}\boldsymbol{1}, \varphi)$ *is the set of RP allocation strategies based on a general risk measure* φ*.*

Note that Definition 2.1 and (2.2) coincide if $\mathcal{R}$ is differentiable at any point in $\Re^d$. For a specific risk preference, e.g., $\varphi = SD$, any element of $\mathcal{RB}(\mathbf{b}, SD)$ (or $\mathcal{RB}(\frac{1}{d}\mathbf{1}, SD)$) is an RB (or RP) portfolio based on the SD risk measure, and we say that the portfolio is $RB - SD$ (or $RP - SD$). Table 1 summarizes the closed-form risk contributions for the four previously-mentioned risk measures and note that RB-SD and RB-var strategies are always equivalent. Further, we implicitly assume that the VaR risk allocations are well-defined, and a sufficient condition is for $\mathbf{X}$ to admit a joint probability density function; similarly, the simplified formulation of CVaR risk allocations in Table 1 is possible when $\boldsymbol{\alpha}^T\mathbf{X}$ has a continuous distribution. A more general discussion on CVaR risk allocations is presented in Hong and Liu (2009) for non-linear portfolios.

φ	$\mathcal{RC}_k(\boldsymbol{\alpha})$	τ
Standard deviation	$\frac{\mathrm{cov}\left(\alpha_k X_k, \boldsymbol{\alpha}^T\mathbf{X}\right)}{\sqrt{\mathrm{var}(\boldsymbol{\alpha}^T\mathbf{X})}}$	1
Variance	$\mathrm{cov}\left(\alpha_k X_k, \boldsymbol{\alpha}^T\mathbf{X}\right)$	2
Value-at-Risk at level $p \in (0,1)$	$\mathbb{E}\left[\alpha_k X_k \middle\vert \boldsymbol{\alpha}^T\mathbf{X} = \mathrm{VaR}_p\left(\boldsymbol{\alpha}^T\mathbf{X}\right)\right]$	1
Conditional Value-at-Risk at level $p \in (0,1)$	$\mathbb{E}\left[\alpha_k X_k \middle\vert \boldsymbol{\alpha}^T\mathbf{X} \geq \mathrm{VaR}_p\left(\boldsymbol{\alpha}^T\mathbf{X}\right)\right]$	1

Table 1: Individual risk contributions for some well-known risk measures.

Early versions of RB portfolios were reduced to approximations of RP-SD portfolios known in the literature as the inverse volatility weighted portfolio (Qian, 2005), which is a special case of Volatility Timing portfolios (Kirby and Ostdiek, 2012). Spinu (2013) showed that (2.2) could be written as an efficient convex optimization problem, which is a much simpler numerical problem than solving the system of non-linear equations in (2.2), whenever the aggregate risk position is measured by SD. Finding RP portfolios under CVaR risk preferences is discussed in Mausser and Romanko (2018), while Bellini et al. (2021) investigate the RP portfolios for expectiles. Both articles provide excellent computationally efficient algorithms that make their proposed investment strategies implementable even for a large number of assets.

We end this section with an example that shows the benefits of using Definition 2.1 that generalizes (2.2), which is given as Example 2.2 and is inspired by Example 2 in Bellini et al. (2021). Our main results in Section 4 take advantage of this less restrictive definition of RB/RP portfolios that removes the differentiability condition which has been assumed in the existing literature. The impact of non-differentiability is discussed in Mausser and Romanko (2018) for RP-CVaR portfolios, and a solution is provided in that particular setting; this issue is resolved in Section 4 for generalized risk preferences and any RB (not only RP) portfolios.

Example 2.2. *Let* (X_1, X_2) *be two loss variables such that* $\Pr(X_1 = x_1, X_2 = x_2) = 1/6$ *for any* $(x_1, x_2) \in \{0, 1\}$ *and* $\Pr(X_1 = X_2 = 2) = 1/3$. *Thus,* $\mathrm{VaR}_{40\%}(\alpha_1 X_1 + \alpha_2 X_2) =$

$\max(\alpha_1,\alpha_2)$ *and* $\mathrm{CVaR}_{40\%}(\alpha_1 X_1+\alpha_2 X_2) = \frac{500}{36}(\alpha_1+\alpha_2)+\frac{1}{6}\max(\alpha_1,\alpha_2)$ *for any* $\alpha_1,\alpha_2>0$, *and the formulae in Table 1 for* $\varphi \in \{\mathrm{VaR}_{40\%}, \mathrm{CVaR}_{40\%}\}$ *do not apply.*

Assume $\varphi = \mathrm{VaR}_{40\%}$. *There exists exactly one RB/RP long-only portfolio for any* $b \in (0,1)$ *if using Definition 2.1, i.e.,* $\mathcal{RB}\big((b,1-b),\mathrm{VaR}_{40\%}\big) \cap \Delta_2^{++}$ *has one element for any* $b \in (0,1)$. *Further, the set of long-only portfolios in* Δ_2^{++} *satisfying (2.2) is empty for any* $b \in (0,1)$.

Assume $\varphi = \mathrm{CVaR}_{40\%}$. *There exists exactly one RB/RP long-only portfolio for any* $b \in (0,1)$ *if using Definition 2.1, i.e.,* $\mathcal{RB}\big((b,1-b),\mathrm{CVaR}_{40\%}\big) \cap \Delta_2^{++}$ *has one element for any* $b \in (0,1)$. *The set of long-only portfolios in* Δ_2^{++} *satisfying (2.2) is empty if* $\frac{250}{503} \le b \le \frac{253}{503}$, *and has exactly one element if* $b \in \big(0,\frac{250}{503}\big) \cup \big(\frac{253}{503},1\big)$.

In summary, Definition 2.1 always leads to a unique long-only RB strategy, while RB strategies based on (2.2) may not exist; e.g., there is no RP strategy if one relies on (2.2).

3. Some new insights about RB/RP

A series of new insights are summarized in this section and according to our knowledge, such insights have not been discussed in the literature. *First*, we provide in Section 3.1 a novel economic interpretation of RP portfolios. *Second*, the newly coined set of constrained portfolios is defined in Section 3.2, and we called this rich class as the GWMC set. *Third*, a succinct list of facts about the IWP set of portfolios is discussed in Section 3.3 given that IWP and RP are often interchangeable in the literature.

3.1. *Economic interpretation of RP*

There are various interpretations of RB/RP portfolios, and we would like to contribute with a new economic interpretation of RP portfolios for differentiable risk measures. That is, if φ is differentiable, then (2.2) implies that

$$\frac{\partial \mathcal{R}(\boldsymbol{\alpha})}{\partial \alpha_k}\left(\frac{\mathcal{R}(\boldsymbol{\alpha})}{\alpha_k}\right)^{-1} = \frac{\partial \mathcal{R}(\boldsymbol{\alpha})}{\partial \alpha_l}\left(\frac{\mathcal{R}(\boldsymbol{\alpha})}{\alpha_l}\right)^{-1} \quad \text{for all } 1 \le k < l \le d \tag{3.1}$$

if $\boldsymbol{\alpha} \in \mathcal{RB}(\frac{1}{d}\mathbf{1},\varphi)$. Thus, RP portfolios have equal elasticity of the portfolio risk position with respect to each asset's weight, which makes the aggregate risk position to be equally sensitive to each weight.

3.2. *GWMC portfolios*

We next demonstrate that RB/RP portfolios are elements of the set of GWMC portfolios that is further defined. The generalized weighted mean for a vector $\mathbf{x} \in \Re^d$ with a weighting vector $\mathbf{b}$ is denoted as

$$m_p(\mathbf{x};\mathbf{b}) := \left(\sum_{k=1}^{d} b_k |x_k|^p\right)^{1/p} \quad \text{for any } p \in \Re \cup \{\pm\infty\}.$$

It is well-known that

$$m_p(\mathbf{x};\mathbf{b}) \le m_q(\mathbf{x};\mathbf{b}) \quad \text{for all } -\infty \le p < q \le \infty \text{ and } \mathbf{b} \in \Delta_d^{++}, \tag{3.2}$$

where the limiting cases are for $p = 0$ (known as the weighted geometric mean)

$$m_0\left(\mathbf{x};\mathbf{b}\right) := \lim_{p\to 0} m_p\left(\mathbf{x};\mathbf{b}\right) = \prod_{k=1}^{d} |x_k|^{b_k} \quad \text{for any } \mathbf{b} \in \Delta_d^{++},$$

and $p = \pm\infty$

$$m_{-\infty}\left(\mathbf{x};\mathbf{b}\right) = \min_{1\le k\le d} |x_k| \quad \text{and} \quad m_{\infty}\left(\mathbf{x};\mathbf{b}\right) = \max_{1\le k\le d} |x_k| \quad \text{for any } \mathbf{b} \in \Delta_d^{++}. \tag{3.3}$$

The case $p = -1$ yields the weighted harmonic mean. Clearly, the equal weights case, $\mathbf{b} = \frac{1}{d}\mathbf{1}$, implies that $m_p\left(\mathbf{x};\frac{1}{d}\mathbf{1}\right) = d^{-1/p}||\mathbf{x}||_p$, where $||\cdot||_p$ is the usual p-norm; by convention, $||\mathbf{x}||_{-\infty} := \min_{1\le k\le d}|x_k|$ and $||\mathbf{x}||_{\infty} := \max_{1\le k\le d}|x_k|$ for any $\mathbf{x} \in \Re^d$.
Let $\Delta_d(\boldsymbol{\delta}) := \{\mathbf{x} \in \Delta_d : \boldsymbol{\delta} \circ \mathbf{x} \in \Re^d_{++}\}$ with $\circ$ being the usual Hadamard product, and $\boldsymbol{\delta} \in \Re^d \setminus \{-\mathbf{1}\}$ such that $\boldsymbol{\delta} \circ \boldsymbol{\delta} = \mathbf{1}$. Clearly, $\Delta_d(\mathbf{1}) = \Delta_d^{++}$ is the long-only case. GWMC portfolios are defined as follows:

$$\min_{\mathbf{x}\in\Delta_d(\boldsymbol{\delta})} \mathcal{R}(\mathbf{x}) \quad \text{such that } m_p(\mathbf{x};\mathbf{b}) \le \epsilon \quad \text{with} \quad \epsilon \in \Re_{++} \quad \text{if } p \ge 1 \tag{3.4}$$

and

$$\min_{\mathbf{x}\in\Delta_d(\boldsymbol{\delta})} \mathcal{R}(\mathbf{x}) \quad \text{such that } m_p(\mathbf{x};\mathbf{b}) \ge \epsilon \quad \text{with} \quad \epsilon \in \Re_{++} \quad \text{if } p < 1, \tag{3.5}$$

and

$$\min_{\mathbf{x}\in\Delta_d} \mathcal{R}(\mathbf{x}) \quad \text{such that } m_p(\mathbf{x};\mathbf{b}) \le \epsilon \quad \text{with} \quad \epsilon \in \Re_{++} \quad \text{if } p \in \mathbb{N}^*, \tag{3.6}$$

since $m_p(\mathbf{x};\mathbf{b})$ is convex in $\mathbf{x}$ on $\Delta_d(\boldsymbol{\delta})$ if $p \ge 1$ and concave if $p < 1$ for any $\boldsymbol{\delta} \in \Re^d\setminus\{-\mathbf{1}\}$, while $m_p(\mathbf{x};\mathbf{b})$ is convex in $\mathbf{x}$ on Δ_d if $p \in \mathbb{N}^*$.
The rich class of norm constrained portfolios (DeMiguel et al., 2009a) is a special case of (3.4) and (3.6) with $\varphi = SD$ and $\mathbf{b} = \frac{1}{d}\mathbf{1}$, where portfolios with $p \in \{1,2\}$ show good performance as compared to the EW benchmark. Note that $p = 1$ recovers some shortsale-constrained portfolios from (Jagannathan and Ma, 2003; DeMiguel et al., 2009b); for details, see (DeMiguel et al., 2009a). It is shown in (DeMiguel et al., 2009a) that EW is a special case of the norm constrained class of portfolios with $p = 2$, but one may show that EW is a special case of GWMC class from (3.4)– (3.6) with $\epsilon = 1/d$ and $\mathbf{b} = \frac{1}{d}\mathbf{1}$ for any $p \in \Re \cap \{\pm\infty\} \setminus \{1\}$ and any risk measure φ due to (3.11). Long-only RB

portfolios have been shown to be *logarithmic constrained* risk minimization problems, and we show that this is true for long-short RB portfolios as well; see Theorems 4.1 and 4.5. One may find RB portfolios through solving (3.5) with $p = 0$, which shows that RB/RP are GWMC portfolios. One difficulty with GWMC class is that ϵ is a tuning parameter, though cross-validation is one way to overcome this drawback, but EW and RP do not have this implementation issue.
It is clear that GWMC portfolios defined in (3.4)–(3.6) are convex instances if φ is a convex risk measure, and therefore, their optimal solutions always exist as long as ϵ is chosen such that the generalized weighted mean constraint is not infeasible in the corresponding instance from (3.4)–(3.6), which is assumed from now on. We first investigate when their optimal solutions are well-behaved by looking whether or not the GWMC portfolios are boundary solutions. These findings are summarized in Proposition 3.1.

Proposition 3.1. *Let* $\boldsymbol{b} \in \Delta_d^{++}$ *and* $\epsilon \in \Re_{++}$, *and assume that* φ *is a convex risk measure.*

a) Any optimal solution $\boldsymbol{x}^*$ *of* (3.4) *satisfies* $||\boldsymbol{x}^*||_\infty < \infty$ *for any* $\boldsymbol{\delta} \in \Re^d \setminus \{-\boldsymbol{1}\}$ *and* $p \geq 1$.

b) Any optimal solution $\boldsymbol{x}^*$ *of* (3.5) *satisfies* $||\boldsymbol{x}^*||_\infty < \infty$ *for any* $\boldsymbol{\delta} \in \Re^d \setminus \{-\boldsymbol{1}\}$ *and* $0 < p < 1$ *provided that* φ *is a homogeneous risk measure of order* $\tau \geq 1$ *and*

$$\min_{\substack{\boldsymbol{x} \in \Re^d \setminus \{\boldsymbol{0}\} \\ \boldsymbol{\delta} \circ \boldsymbol{x} \in \Re^d_+}} \mathcal{R}(\boldsymbol{x}) > 0. \tag{3.7}$$

c) Any optimal solution $\boldsymbol{x}^*$ *of* (3.5) *satisfies* $||\boldsymbol{x}^*||_\infty < \infty$ *and* $||\boldsymbol{x}^*||_{-\infty} > 0$ *for any* $\boldsymbol{\delta} \in \Re^d \setminus \{-\boldsymbol{1}\}$ *and* $p \leq 0$ *provided that* φ *is a homogeneous risk measure of order* $\tau \geq 1$ *and* (3.7) *holds.*

d) Any optimal solution $\boldsymbol{x}^*$ *of* (3.6) *satisfies* $||\boldsymbol{x}^*||_\infty < \infty$ *for any* $p \in \mathbb{N}^*$.

Proof. Parts a) and d) could be proved by noting that there exists $M_0 > 0$ sufficiently large such that $\mathbf{x} \in \Delta_d(\boldsymbol{\delta})$ is infeasible whenever $||\mathbf{x}||_\infty > M_0$.
Part b) is now proved. For any $\epsilon \in \Re_{++}$ and sufficiently large $M_0 > 0$, any $\mathbf{x} \in \Delta_d(\boldsymbol{\delta})$ such that $||\mathbf{x}||_\infty > M_0$ is feasible in (3.5). The homogeneity assumption implies that $\mathcal{R}(\mathbf{x}) = \mathcal{R}\left(\frac{\mathbf{x}}{||\mathbf{x}||_\infty}\right) ||\mathbf{x}||^\tau_\infty \to \infty$ whenever $||\mathbf{x}||_\infty \to \infty$ due to (3.7). This concludes part b).
Finally, we show part c). First, note that there exists $M_0 > 0$ sufficiently small such that $\mathbf{x} \in \Delta_d(\boldsymbol{\delta})$ is infeasible whenever $||\mathbf{x}||_{-\infty} < M_0$, since $p \leq 0$, and in turn, $||\mathbf{x}^*||_{-\infty} > 0$ holds. One may show that $||\mathbf{x}^*||_\infty < \infty$ by using the same arguments as in the proof of part b). The proof is now complete. ∎
Proposition 3.1 outlines the sufficient conditions under which the GWMC portfolio solutions are bounded away from neighborhoods of $\pm\infty$. Note that Proposition 3.1 c) with $p = 0$ confirms our results in Theorem 4.1 a) and Theorem 4.5 a).

The well-behaved optimal solutions guaranteed by Proposition 3.1 have some appealing properties, and they are summarized in Proposition 3.2. For a given $\boldsymbol{\delta}$, denote by d_+/d_- the number of positive/negative elements of $\boldsymbol{\delta}$.

Proposition 3.2. *Let $\boldsymbol{b} \in \Delta_d^{++}$ and $\epsilon \in \Re_{++}$, and assume that φ is a convex risk measure.*

a) Let $\boldsymbol{x}^*(\epsilon;p)$ and $\boldsymbol{x}^*(\epsilon;q)$ be an optimal solution of (3.4) with a generalized weighted mean constraint $m_p(\boldsymbol{x};\boldsymbol{b}) \le \epsilon$ and $m_q(\boldsymbol{x};\boldsymbol{b}) \le \epsilon$, respectively. Then,

$$\mathcal{R}\big(\boldsymbol{x}^*(\epsilon;p)\big) \le \mathcal{R}\big(\boldsymbol{x}^*(\epsilon;q)\big) \quad \text{for any } 1 \le p < q. \tag{3.8}$$

b) Assume that φ is a homogeneous risk measure of order $\tau \ge 1$ and (3.7) holds. Let $\boldsymbol{x}^*(\epsilon;p)$ and $\boldsymbol{x}^*(\epsilon;q)$ be an optimal solution of (3.5) with a generalized weighted mean constraint $m_p(\boldsymbol{x};\boldsymbol{b}) \ge \epsilon$ and $m_q(\boldsymbol{x};\boldsymbol{b}) \ge \epsilon$, respectively. Then,

$$\mathcal{R}\big(\boldsymbol{x}^*(\epsilon;q)\big) \le \mathcal{R}\big(\boldsymbol{x}^*(\epsilon;p)\big) \quad \text{for any } p < q < 1. \tag{3.9}$$

c) Let $\boldsymbol{x}^*(\epsilon;p)$ and $\boldsymbol{x}^*(\epsilon;q)$ be an optimal solution of (3.6) with a generalized weighted mean constraint $m_p(\boldsymbol{x};\boldsymbol{b}) \le \epsilon$ and $m_q(\boldsymbol{x};\boldsymbol{b}) \le \epsilon$, respectively. Then,

$$\mathcal{R}\big(\boldsymbol{x}^*(\epsilon;p)\big) \le \mathcal{R}\big(\boldsymbol{x}^*(\epsilon;q)\big) \quad \text{for any integers } 1 \le p < q. \tag{3.10}$$

d) Assume that $d_+ > d_-$ and let $\boldsymbol{x}^*(\epsilon;p)$ be an optimal solution of (3.4) with a generalized weighted mean constraint $m_p\big(\boldsymbol{x};\frac{1}{d}\boldsymbol{1}\big) \le \epsilon$. Then, $\mathcal{R}\big(\boldsymbol{x}^*(\epsilon;p)\big) \le \mathcal{R}\left(\frac{1}{2d_+-d}\boldsymbol{\delta}\circ\boldsymbol{1}\right)$.

e) Assume that φ is a homogeneous risk measure of order $\tau \ge 1$ and (3.7) holds. Further, $d_+ > d_-$ and let $\boldsymbol{x}^*(\epsilon;p)$ be an optimal solution of (3.5) with a generalized weighted mean constraint $m_p\big(\boldsymbol{x};\frac{1}{d}\boldsymbol{1}\big) \le \epsilon$. Then, $\mathcal{R}\big(\boldsymbol{x}^*(\epsilon;p)\big) \le \mathcal{R}\left(\frac{1}{2d_+-d}\boldsymbol{\delta}\circ\boldsymbol{1}\right)$.

f) Let $\boldsymbol{x}^*(\epsilon;p)$ be an optimal solution of (3.6) with a generalized weighted mean constraint $m_p\big(\boldsymbol{x};\frac{1}{d}\boldsymbol{1}\big) \le \epsilon$. Then, $\mathcal{R}\big(\boldsymbol{x}^*(\epsilon;p)\big) \le \min_{\{\boldsymbol{\delta}:d_+>d_-\}} \mathcal{R}\left(\frac{1}{2d_+-d}\boldsymbol{\delta}\circ\boldsymbol{1}\right)$.

Proof. One may show parts a)–c) by using (3.2) since the generalized weighted mean constraint reduces the feasibility set in parts a) and c) when q increases and it reduces the feasibility set in part b) when p decreases. Similar arguments could be used to demonstrate parts d)–f) by recalling that

$$\arg\min_{\mathbf{x}\in\Delta_d(\boldsymbol{\delta})} m_p\big(\mathbf{x};\frac{1}{d}\mathbf{1}\big) = \frac{1}{2d_+ - d}\boldsymbol{\delta}\circ\mathbf{1} = \arg\max_{\mathbf{x}\in\Delta_d(\boldsymbol{\delta})} m_q\big(\mathbf{x};\frac{1}{d}\mathbf{1}\big), \tag{3.11}$$

for all $-\infty \le q < 1 \le p \le \infty$. This completes the proof. ∎

One interesting conclusion of Proposition 3.2 is that GWMC long-only portfolios are less risky than EW, a very competitive benchmark portfolio. Further, Proposition 3.2 a)–c) tells us that the GWMC portfolios with $p = 1$ are very effective in

reducing the portfolio risk, which confirms the good performance of the shortsale-constrained portfolios from (Jagannathan and Ma, 2003; DeMiguel et al., 2009a,b). Furthermore, Proposition 3.2 e) with $p = 0$ confirms our results in Theorem 4.1 c) and Theorem 4.5 b). Finally, recall that $\frac{1}{2d_+-d}\boldsymbol{\delta} \circ \mathbf{1}$ is the *generalized equal weight (GEW)* portfolio that is an element of $\Delta_d(\boldsymbol{\delta})$, where EW is a particular case when $\boldsymbol{\delta} = \mathbf{1}$.
We conclude this section by inferring that the GWMC class of portfolios is very rich which could be linked to the class of RB portfolios that is the main aim of this paper.

3.3. Short discussion about IWP and RP

We aim to compare IWP and RP portfolios, and show that these portfolios are not quite the same and do not share the same properties. For a given φ, $RP-\varphi$ is as in Definition 2.1 with $\mathbf{b} = \frac{1}{d}\mathbf{1}$ and it could be computed via Theorem 4.1 a), while $IWP-\varphi$ has the following weights

$$\frac{1/\varphi(X_k)}{\sum_{k=1}^{d} 1/\varphi(X_k)} \quad \text{for all } k \in \{1, 2, \dots, d\}. \tag{3.12}$$

The latter is often used to approximate $RP-\varphi$ in empirical studies, and therefore, the two are often interchanged, meaning that RP portfolios are computed with the simplified formula in (3.12). Note that IWP-SD is the same as RP-SD when all pairwise returns are uncorrelated; it is also true that RP-SD and IWP-SD coincide if the asset correlations are equal (Roncalli, 2013), but equivalent results are unknown for other risk preferences. Independent of the RP literature, IWP portfolios with SD risk preferences are investigated under the name of Volatility Timing portfolios, where is shown to outperform EW (Kirby and Ostdiek, 2012). Our empirical analyses in Section 6.2 – see Tables 3 and 4 – show that IWP and RP could be quite different, especially in periods with very poor market performance (Periods 1 and 3) and very stable market conditions (Period 2) where the IWP performance is quite poor. The same is observed in the synthetic Example 3.3.

Example 3.3. *Let (X_1, X_2, X_3) be three loss variables such that $SD(X_1) = 1.2$, $SD(X_2) = 1.1$, $SD(X_3) = 1$, $corr(X_1, X_2) = corr(X_1, X_3) = -a$ and $corr(X_2, X_3) = a$. We assume that the risk preferences are ordered via SD, and thus, $\mathcal{R} = SD$.*
Five portfolios are considered, but we report only four of them in Table 2 as two portfolios are identical. That is, we compute the MinVar *portfolio, which is the unconstrained minimum SD portfolio as in (3.6) with $\epsilon = \infty$ and $p = 1$; we also computed the long-only minimum SD portfolio as (3.4) with $\epsilon = 1$ and $p = 1$, but this portfolio is identical to MinVar for all settings displayed in Table 2. We also compute the long-only RP-SD, which is computed via Theorem 4.1 a) and is an element of (3.5) with $p = 0$. EW is another GWMC as explained previously, while IWP-SD is usually not an element of GWMC except for some very specific settings (only $a = 0$ in this example). All results are summarized in Table 2.*

a	MinVar	RP-SD	EW	IWP-SD
0.5	0.4733	0.4748	0.4978	0.5157
0.25	0.5676	0.5683	0.5715	0.5765
0	0.6298	0.6316	0.6368	0.6316
-0.25	0.6351	0.6618	0.6960	0.6822

Table 2: Portfolio SD with $d = 3$ assets as set in Example 3.3 for various values of a.

Table 2 shows that IWP and RP may not share the same properties; RP is always less risky than EW – see Theorem 4.1 c) – but IWP may be riskier ($a \in \{0.25, 0.5\}$) or less risky ($a \in \{-0.25, 0\}$) than EW.

4. Main theoretical results

The main theoretical results are included in this section. Long-only and long-short portfolios are investigated in Appendix A when the loss returns are assumed to be elliptically distributed. This parametric assumption is removed in this section, where a mathematical characterization of the long-only RB portfolio solutions – given by Definition 2.1 – is provided in Sections 4.1; such a mathematical characterization is then extended for long-short portfolios in Section 4.2.

4.1. *Long-only RB for generally distributed risks*

We are now ready to provide two methods of finding and characterizing long-only RB portfolios for a large class of risk measures without making any assumption on the underlying asset returns distribution. Two methods are investigated, which are known in the literature (e.g., see Roncalli (2013) and Bellini et al. (2021)) as the *logarithmic barrier* formulation in (4.2) and *logarithmic constraint* RB formulation in (4.3).

Theorem 4.1. *Let $\boldsymbol{b} \in \Delta_d^{++}$, and φ be a convex, homogeneous risk measure of order $\tau \geq 1$. Further, assume that*

$$\min_{\boldsymbol{x} \in \Delta_d^{+}} \mathcal{R}(\boldsymbol{x}) > 0. \tag{4.1}$$

a) For any given $\lambda > 0$, the following instance

$$\min_{\boldsymbol{x} \in \Re_{++}^d} \frac{1}{\tau}\mathcal{R}(\boldsymbol{x}) - \lambda \sum_{k=1}^{d} b_k \log x_k \tag{4.2}$$

admits a unique solution, denoted as $\boldsymbol{x}^*(\lambda, \boldsymbol{b})$, that is an interior point of $\Re_{++}^d$. Then, $\boldsymbol{\alpha}^*(\boldsymbol{b}) \in \mathcal{RB}(\boldsymbol{b}, \varphi) \cap \Delta_d^{++}$, where $\boldsymbol{\alpha}^*(\boldsymbol{b}) = \boldsymbol{x}^*(\lambda, \boldsymbol{b})/\boldsymbol{1}^T\boldsymbol{x}^*(\lambda, \boldsymbol{b})$. Moreover,

$$\boldsymbol{\alpha}^*(\boldsymbol{b}) = \boldsymbol{x}^*(\lambda^*, \boldsymbol{b}) = (\lambda^*)^{1/\tau}\boldsymbol{x}^*(1, \boldsymbol{b}), \text{ where } \lambda^* = \left(\boldsymbol{1}^T\boldsymbol{x}^*(1, \boldsymbol{b})\right)^{-\tau}.$$

b) *For any given $c \in \Re$, the following instance*

$$\min_{\boldsymbol{x} \in \Re^d_{++}} \mathcal{R}(\boldsymbol{x}) \quad \text{such that} \quad \sum_{k=1}^{d} b_k \log x_k \geq c \quad \text{with} \quad c \in \Re \tag{4.3}$$

admits a unique solution, denoted as $\boldsymbol{x}^{**}(c, \boldsymbol{b})$, that is an interior point of the feasibility set. Then, $\boldsymbol{\alpha}^{**}(\boldsymbol{b}) \in \mathcal{RB}(\boldsymbol{b}, \varphi) \cap \Delta_d^{++}$, where $\boldsymbol{\alpha}^{**}(\boldsymbol{b}) = \boldsymbol{x}^{**}(c, \boldsymbol{b}) / \boldsymbol{1}^T \boldsymbol{x}^{**}(c, \boldsymbol{b})$. Moreover,

$$\boldsymbol{\alpha}^{**}(\boldsymbol{b}) = \boldsymbol{x}^{**}(c^*, \boldsymbol{b}) = e^{c^*-1} \boldsymbol{x}^{**}(1, \boldsymbol{b}), \text{ where } c^* = 1 - \log\left(\boldsymbol{1}^T \boldsymbol{x}^{**}(1, \boldsymbol{b})\right).$$

Furthermore, strong duality holds in (4.3).

c) *For any $\boldsymbol{b}$, we have that $\boldsymbol{\alpha}^*(\boldsymbol{b}) = \boldsymbol{\alpha}^{**}(\boldsymbol{b})$,*

$$\min_{\boldsymbol{x} \in \Delta_d^{++}} \mathcal{R}(\boldsymbol{x}) \leq \mathcal{R}\big(\boldsymbol{\alpha}^*(\boldsymbol{b})\big) \leq \mathcal{R}(\boldsymbol{b}) \quad \text{and} \quad \mathcal{R}\big(\boldsymbol{\alpha}^*(\boldsymbol{b})\big) \leq \mathcal{R}\left(\frac{1}{d}\boldsymbol{1}\right). \tag{4.4}$$

Proof. We first prove part a). Let $F(\mathbf{x}; \lambda)$ be the objective function in (4.2). The first step is to show that the optimal solution in (4.2) exists and is an interior point of the feasible set. Now, for any $\mathbf{x} \in \Re^d_{++}$

$$\begin{aligned} F(\mathbf{x}; \lambda) &= \frac{1}{\tau} \mathcal{R}\left(\frac{1}{d \max\limits_{1 \leq k \leq d} x_k} \mathbf{x}\right) d^\tau \Big(\max_{1 \leq k \leq d} x_k\Big)^\tau - \lambda \sum_{k=1}^{d} b_k \log x_k \\ &\geq \frac{\delta^* d^\tau}{\tau} \Big(\max_{1 \leq k \leq d} x_k\Big)^\tau - \lambda \log\Big(\max_{1 \leq k \leq d} x_k\Big), \end{aligned} \tag{4.5}$$

since φ is homogeneous of order τ, where $\delta^* > 0$ does not depend upon $\mathbf{x}$ and its existence is guaranteed by (4.1). Since $\lim_{t \to \infty} \delta t^\tau - \lambda \log t = \infty$ for any $\delta, \lambda, \tau > 0$, then (4.5) implies that

$$F(\mathbf{x}; \lambda) \to \infty, \text{ whenever } ||\mathbf{x}||_\infty \to \infty. \tag{4.6}$$

We now prove that (4.6) holds on the boundary (of the feasibility set) regions away from infinity. That is, let $M > \epsilon > 0$; note that for any $\mathbf{x} \in \Re^d_{++}$ such that $||\mathbf{x}||_{-\infty} \leq \epsilon$ and $||\mathbf{x}||_\infty \leq M$, there exists $0 < b^* \leq 1$ such that $F(\mathbf{x}; \lambda) \geq -\lambda b^* \log \epsilon$ by using similar arguments as in (4.5). Thus,

$$F(\mathbf{x}; \lambda) \to \infty, \text{ whenever } ||\mathbf{x}||_{-\infty} \downarrow 0 \text{ and } ||\mathbf{x}||_\infty \leq M \text{ for any finite } M > 0. \tag{4.7}$$

Equations (4.6) and (4.7) imply that there exist an $a > 0$ and an $\epsilon \in (0, a]$ such that

$$\inf_{\mathbf{x} \in \Re^d_{++}} F(\mathbf{x}; \lambda) = \inf_{\mathbf{x} \in B_{a,\epsilon}} F(\mathbf{x}; \lambda), \quad \text{where } B_{a,\epsilon} := \{\mathbf{x} \in B_a : ||\mathbf{x}||_{-\infty} \geq \epsilon\}$$

with $B_a := \{\mathbf{x} \in \Re^d_{++} : ||\mathbf{x}||_2 \leq a\}$. Since $B_{a,\epsilon}$ is a compact set, the global minimum of $F(\cdot;\lambda)$ on $\Re^d_{++}$ is an interior point of the feasibility set for any given $\lambda > 0$. Thus, (4.2) must have an optimal solution that is an interior point of the feasible set.
The objective function in (4.2) is strictly convex in $\mathbf{x}$ over the convex cone $\Re^d_{++}$ for any given $\lambda > 0$, since the logarithmic barrier term $(-\lambda \sum_{k=1}^d b_k \log x_k)$ is convex and the fact that φ is a convex risk measure. Thus, (4.2) admits a unique solution.
It only remains to prove for part a) the relationship between the unique solution in (4.2) for various penalty parameters λ. Note that

$$F(\mathbf{x};\lambda) = \lambda F\left(\lambda^{-1/\tau}\mathbf{x};1\right) - \frac{\lambda}{\tau}\log\lambda, \text{ for any } \mathbf{x} \in \Re^d_{++} \text{ and } \lambda > 0,$$

and any given $\mathbf{b} \in \Delta_d^{++}$, since φ is a homogeneous risk measure of order τ, and in turn, $\mathbf{x}^*(\lambda,\mathbf{b}) = \lambda^{1/\tau}\mathbf{x}^*(1,\mathbf{b})$ for any $\lambda > 0$ and $\mathbf{b} \in \Delta_d^{++}$. The first-order conditions in (4.2) imply that $\mathbf{0} \in \partial F(\mathbf{x}^*(\lambda,\mathbf{b});\lambda)$, and thus, $\lambda\big(b_1/x_1^*(\lambda,\mathbf{b}),\ldots,b_d/x_d^*(\lambda,\mathbf{b})\big)^T \in \partial\mathcal{R}(\mathbf{x}^*(\lambda,\mathbf{b}))$, which in turn gives that $\mathbf{x}^*(\lambda,\mathbf{b}) \in \mathcal{RB}(\mathbf{b},\varphi)$ due to (2.3). Now, the homogeneity of φ implies $t^\tau\mathbf{a} \in \partial\mathcal{R}(t\mathbf{x})$ for any $t > 0$ if $\mathbf{a} \in \partial\mathcal{R}(\mathbf{x})$, and in turn, $\boldsymbol{\alpha}^*(\mathbf{b}) = \mathbf{x}^*(\lambda,\mathbf{b})/\mathbf{1}^T\mathbf{x}^*(\lambda,\mathbf{b}) \in \mathcal{RB}(\mathbf{b},\varphi) \cap \Delta_d^{++}$ for any $\mathbf{b} \in \Delta_d^{++}$. We finish the proof of part a) by noting that

$$\boldsymbol{\alpha}^*(\mathbf{b}) = \frac{\mathbf{x}^*(\lambda,\mathbf{b})}{\mathbf{1}^T\mathbf{x}^*(\lambda,\mathbf{b})} = \frac{\mathbf{x}^*(1,\mathbf{b})}{\mathbf{1}^T\mathbf{x}^*(1,\mathbf{b})} \quad \text{and} \quad \mathbf{x}^*(\lambda^*,\mathbf{b}) = \left(\lambda^*\right)^{1/\tau}\mathbf{x}^*(1,\mathbf{b}) = \frac{\mathbf{x}^*(1,\mathbf{b})}{\mathbf{1}^T\mathbf{x}^*(1,\mathbf{b})}.$$

We now prove part b). As before, we initially show that (4.3) admits a unique solution that is an interior point. The interior point could be proved as in Proposition 3.1 c) and we only show the uniqueness property. The homogeneity property of the objective function in (4.3) implies that any optimal solution of (4.3) satisfies

$$\sum_{k=1}^d b_k \log x_k^{**}(c,\mathbf{b}) = c. \tag{4.8}$$

If (4.8) does not hold, $(1-\epsilon)\mathbf{x}^{**}(c,\mathbf{b})$ is feasible for any $\epsilon > 0$ sufficiently small; further,

$$\mathcal{R}\big((1-\epsilon)\mathbf{x}^{**}(c,\mathbf{b})\big) = (1-\epsilon)^\tau\mathcal{R}\big(\mathbf{x}^{**}(c,\mathbf{b})\big) < \mathcal{R}\big(\mathbf{x}^{**}(c,\mathbf{b})\big)$$

due to the homogeneity of φ and the fact $\mathcal{R}\big(x_k^{**}(c,\mathbf{b})\big) > 0$ (see (4.1)), which contradicts our assumption and concludes (4.8). The optimal solution in (4.3) is unique, since the inequality constraint in (4.3) is strictly concave due to (4.8). One could show that by assuming a case in which there are two optimal solutions, $\mathbf{x}^{**}(c,\mathbf{b})$ and $\mathbf{y}^{**}(c,\mathbf{b})$. The latter implies that

$$\mathbf{z}^{**}(c,\mathbf{b}) := \gamma\mathbf{x}^{**}(c,\mathbf{b}) + (1-\gamma)\mathbf{y}^{**}(c,\mathbf{b})$$

is another optimal solution of (4.3) for any $0 < \gamma < 1$, since φ is a convex risk measure. Moreover,

$$\sum_{k=1}^{d} b_k \log z_k^{**} > \gamma \sum_{k=1}^{d} b_k \log x_k^{**} + (1-\gamma) \sum_{k=1}^{d} b_k \log y_k^{**} = c,$$

since the $\log$ function is strictly concave, which in turn contradicts that $\mathbf{z}^{**}(c, \mathbf{b})$ must satisfy (4.8). Therefore, (4.3) admits a unique optimal solution.

It only remains to prove for part b) the relationship between the unique solution in (4.3) for various penalty parameters c. We first show that

$$\mathbf{x}^{**}(c, \mathbf{b}) = e^{c-1}\mathbf{x}^{**}(1, \mathbf{b}) \quad \text{for any given } \mathbf{b} \in \Delta_d^{++}. \tag{4.9}$$

Again, we show this claim by contradiction and assume that $\mathbf{x}^{**}(1, \mathbf{b})$ solves (4.3) when $c = 1$, but there exists $c_0 \neq 1$ such that $e^{c_0-1}\mathbf{x}^{**}(1, \mathbf{b})$ does not solve (4.3) whenever $c = c_0$. Therefore, there exists $\mathbf{y} \in \Re^d_{++}$ such that

$$\mathcal{R}(\mathbf{y}) < \mathcal{R}\big(e^{c_0-1}\mathbf{x}^{**}(1, \mathbf{b})\big) \quad \text{and} \quad \sum_{k=1}^{d} b_k \log y_k = c_0.$$

Clearly, the above imply that $e^{1-c_0}\mathbf{y}$ is feasible in (4.3) when $c = 1$, and

$$\mathcal{R}\big(e^{1-c_0}\mathbf{y}\big) = e^{(1-c_0)\tau}\mathcal{R}\big(\mathbf{y}\big) < e^{(1-c_0)\tau}\mathcal{R}\big(e^{c_0-1}\mathbf{x}^{**}(1, \mathbf{b})\big) = \mathcal{R}\big(\mathbf{x}^{**}(1, \mathbf{b})\big)$$

by keeping in mind that φ is a homogeneous risk measure of order τ, which in turn contradicts our assumption and concludes (4.9). The relationships among various optimal solutions stated in part b) could be easily shown as in part a). Finally, the Slater's condition is clearly satisfied in (4.3), and therefore, the strong duality holds in (4.3). The proof of part b) is fully argued.

We show the claims from part c). Note that $\boldsymbol{\alpha}^*(\mathbf{b}) = \boldsymbol{\alpha}^{**}(\mathbf{b})$, which is true since

$$\boldsymbol{\alpha}^*(\mathbf{b}) = \mathbf{x}^*(\lambda, \mathbf{b})/\mathbf{1}^T\mathbf{x}^*(\lambda, \mathbf{b}), \quad \boldsymbol{\alpha}^{**}(\mathbf{b}) = \mathbf{x}^{**}(c, \mathbf{b})/\mathbf{1}^T\mathbf{x}^{**}(c, \mathbf{b}),$$

and the fact that there exists $\gamma^* > 0$ such that $\mathbf{x}^{**}(c, \mathbf{b}) = \gamma^*\mathbf{x}^*(\lambda, \mathbf{b})$ for all $\lambda > 0$ and any $c \in \Re$. The latter is a direct consequence of the fact that solving the primal optimal in (4.3) is the same as solving (4.2) with $\lambda = \gamma^*/\tau$, where γ^* is the dual optimal in (4.3) corresponding to the logarithmic constraint $\sum_{k=1}^{d} b_k \log x_k \geq c$.

The left-hand side inequality in (4.4) is trivial, and thus, we show now the right-hand side inequality in (4.4). The proof of part a) allows us to say that $\boldsymbol{\alpha}^*(\mathbf{b})$ solves

$$\min_{\mathbf{x} \in \Delta_d^{++}} \frac{1}{\tau}\mathcal{R}(\mathbf{x}) - \lambda^* \sum_{k=1}^{d} b_k \log x_k, \tag{4.10}$$

which implies that

$$\frac{1}{\tau}\Big(\mathcal{R}\big(\boldsymbol{\alpha}^*(\mathbf{b})\big) - \mathcal{R}(\mathbf{b})\Big) \leq \lambda^* \sum_{k=1}^{d} b_k \log\left(\frac{\boldsymbol{\alpha}_k^*(\mathbf{b})}{b_k}\right) = -\lambda^* \times D_{KL}\big(\mathbf{b}||\boldsymbol{\alpha}^*(\mathbf{b})\big) \leq 0$$

where $D_{KL}\big(\mathbf{b}||\boldsymbol{\alpha}^*(\mathbf{b})\big)$ is the Kullback-Leibler divergence between the probability distributions induced by (the probability vectors) $\mathbf{b}$ and $\boldsymbol{\alpha}^*(\mathbf{b})$. Thus, $\mathcal{R}\big(\boldsymbol{\alpha}^*(\mathbf{b})\big) \leq \mathcal{R}(\mathbf{b})$ for any $\mathbf{b}$. The very last step is to show $\mathcal{R}\big(\boldsymbol{\alpha}^*(\mathbf{b})\big) \leq \mathcal{R}\left(\frac{1}{d}\mathbf{1}\right)$. From (4.10) we get that

$$\begin{aligned}
\frac{1}{\tau}\left(\mathcal{R}\big(\boldsymbol{\alpha}^*(\mathbf{b})\big) - \mathcal{R}\left(\frac{1}{d}\mathbf{1}\right)\right) &\leq \lambda^*\left(\sum_{k=1}^{d} b_k \log \boldsymbol{\alpha}_k^*(\mathbf{b}) - \sum_{k=1}^{d} b_k \log\left(\frac{1}{d}\right)\right) \\
&\leq \lambda^*\left(\max_{\mathbf{1}^T\mathbf{x}=1} \sum_{k=1}^{d} b_k \log x_k + \log d\right) \\
&= \lambda^*\left(\sum_{k=1}^{d} b_k \log b_k + \log d\right) \\
&\leq \lambda^*\left(\max_{\mathbf{1}^T\mathbf{x}=1} \sum_{k=1}^{d} x_k \log x_k + \log d\right) \\
&= \lambda^*\left(\sum_{k=1}^{d} \frac{1}{d}\log\left(\frac{1}{d}\right) + \log d\right) \\
&= 0,
\end{aligned}$$

which implies that $\mathcal{R}\big(\boldsymbol{\alpha}^*(\mathbf{b})\big) \leq \mathcal{R}\left(\frac{1}{d}\mathbf{1}\right)$, and in turn, part c) is concluded. This completes the proof.

■

Theorem 4.1 suggests that $\mathcal{RB}(\mathbf{b},\varphi) \cap \Delta_d^{++}$ might have exactly one element under some conditions for a general risk measure choice. We clarify this point in Theorem 4.2.

Theorem 4.2. *Let $\boldsymbol{b} \in \Delta_d^{++}$, and φ be a convex, homogeneous risk measure of order $\tau \geq 1$. Also, (4.1) holds. Then, $\mathcal{RB}(\boldsymbol{b},\varphi) \cap \Delta_d^{++}$ and the set of parametric optimal solutions (in λ) of the surrogate problem (4.2) that also are in Δ_d^{++} coincide. Further, $\mathcal{RB}(\boldsymbol{b},\varphi) \cap \Delta_d^{++}$ has exactly one solution.*

Proof.

We first prove that $\mathcal{RB}(\mathbf{b},\varphi) \cap \Delta_d^{++}$ coincides with the set of parametric optimal solutions (in λ) of the surrogate convex problem (4.2) that are also in Δ_d^{++}. If $\mathbf{x}^* \in \mathcal{RB}(\mathbf{b},\varphi) \cap \Delta_d^{++}$, then $\mathbf{x}^*$ solves the surrogate problem (4.2) with $\lambda^* = \varphi\big(\mathbf{x}^{*T}\mathbf{X}\big)$ by applying the first-order Karush-Kuhn-Tucker conditions. The converse is also true for similar reasons.

We next show that $\mathcal{RB}(\mathbf{b},\varphi) \cap \Delta_d^{++}$ has exactly one solution. Since $\mathcal{RB}(\mathbf{b},\varphi) \cap \Delta_d^{++}$ coincides with the set of parametric optimal solutions in (4.2), then any element of

$\mathcal{RB}(\mathbf{b},\varphi) \cap \Delta_d^{++}$ solves (4.2) with $\lambda^* = \left(\mathbf{1}^T\mathbf{x}^*(1,\mathbf{b})\right)^{-\tau}$ due to Theorem 4.1 a). This yields the required results. The proof is now complete. ■

The positiveness property in (4.1) plays a crucial role, and we show that in Theorem 4.3 below. Specifically, if the positiveness property in (4.1) does not hold, then the objective function in (4.2) is unbounded from below, and thus, the *logarithmic barrier/constraint* RB formulations in (4.2)/(4.3) are not useful to find RB/RP portfolios. In addition, if all long-only portfolios have a non-negative risk position and some have a zero risk position, then portfolios with minimal risk are RB portfolios.

Theorem 4.3. *Let φ be a convex and homogeneous risk measure of order $\tau \geq 1$.*

a) Let $\boldsymbol{b} \in \Delta_d^{++}$. If there exists $\tilde{\boldsymbol{x}} \in \Delta_d^+$ such that $\mathcal{R}(\tilde{\boldsymbol{x}}) \leq 0$, then for any given $\lambda > 0$, the objective function in (4.2) satisfies

$$\inf_{\boldsymbol{x}\in\Re^d_{++}} F(\boldsymbol{x};\lambda) = -\infty. \tag{4.11}$$

b) Let $\boldsymbol{b} \in \Delta_d$. Assume that $\min_{\boldsymbol{x}\in\Delta_d^+}\mathcal{R}(\boldsymbol{x}) \geq 0$ and there exists $\tilde{\boldsymbol{x}} \in \Delta_d^+$ such that $\mathcal{R}(\tilde{\boldsymbol{x}}) = 0$. Then,

$$\boldsymbol{x}^* \in \arg\min_{\boldsymbol{x}\in\Delta_d^+}\mathcal{R}(\boldsymbol{x}) \Leftrightarrow \boldsymbol{x}^* \in \{\boldsymbol{x}\in\Delta_d^+ : \mathcal{R}(\boldsymbol{x}) = 0\} \Rightarrow \boldsymbol{x}^* \in \mathcal{RB}(\boldsymbol{b},\varphi). \tag{4.12}$$

Proof. We first prove part a). Without loss of generality $\mathcal{R}(\tilde{\mathbf{x}}) = 0$ is assumed, since the proof does not change if $\mathcal{R}(\tilde{\mathbf{x}}) < 0$. If $\tilde{\mathbf{x}} \in \Delta_d^{++}$, then

$$\begin{aligned} F(t\tilde{\mathbf{x}};\lambda) &= \frac{t^\tau}{\tau}\mathcal{R}(\tilde{\mathbf{x}}) - \lambda\sum_{k=1}^d b_k\log\tilde{x}_k - \lambda\log t \\ &= -\lambda\sum_{k=1}^d b_k\log\tilde{x}_k - \lambda\log t \to -\infty \quad \text{as } t\to\infty, \end{aligned}$$

since φ is homogeneous and the fact that $\mathbf{1}^T\mathbf{b} = 1$, which concludes (4.11). Assume now that $\tilde{\mathbf{x}} \in \Delta_d^+\setminus\Delta_d^{++}$; without loss of generality, assume that $\tilde{x}_k > 0$ for all $1 \leq k \leq d'$, and $\tilde{x}_k = 0$ for all $d'+1 \leq k \leq d$, where $1 \leq d' < d$. Let $\epsilon > 0$. Now,

$$\begin{aligned} &F\left(t^\epsilon\tilde{x}_1, t^\epsilon\tilde{x}_2,\ldots,t^\epsilon\tilde{x}_{d'},1/t,1/t,\ldots,1/t;\lambda\right) \\ &\quad= \frac{1}{\tau}\mathcal{R}\left(t^\epsilon\tilde{x}_1, t^\epsilon\tilde{x}_2,\ldots,t^\epsilon\tilde{x}_{d'},1/t,1/t,\ldots,1/t\right) - \lambda\sum_{k=1}^{d'} b_k\log\tilde{x}_k \\ &\qquad -\lambda\log t\left(\epsilon\sum_{k=1}^{d'} b_k - \sum_{k=d'+1}^{d} b_k\right) \\ &\quad\leq \frac{2^{\tau-1}}{\tau}\mathcal{R}\left(t^\epsilon\tilde{x}_1, t^\epsilon\tilde{x}_2,\ldots,t^\epsilon\tilde{x}_{d'},0,0,\ldots,0\right) + \frac{2^{\tau-1}}{\tau}\mathcal{R}\left(0,0,\ldots,0,1/t,1/t,\ldots,1/t\right) \end{aligned}$$

$$
\begin{aligned}
&-\lambda \log t\left(\epsilon \sum_{k=1}^{d'} b_k - \sum_{k=d'+1}^{d} b_k\right) - \lambda \sum_{k=1}^{d'} b_k \log \tilde{x}_k \\
&= \frac{2^{\tau-1}}{\tau t^{\tau}}\mathcal{R}\big(0,0,\dots,0,1,1,\dots,1\big) - \lambda \log t\left(\epsilon \sum_{k=1}^{d'} b_k - \sum_{k=d'+1}^{d} b_k\right) - \lambda \sum_{k=1}^{d'} b_k \log \tilde{x}_k \\
&\to -\infty \quad \text{as } t \to \infty \text{ for } \epsilon \text{ sufficiently large,}
\end{aligned}
$$

where the first inequality holds since φ is homogeneous and convex, while the latter identity is true due to the homogeneity property and the fact that $\mathcal{R}(\tilde{\mathbf{x}}) = 0$. Thus, (4.11) holds, and in turn, part a) is fully proved.

We now prove part b). If $\mathbf{x}^* \in \arg\min_{\mathbf{x}\in\Delta_d^+} \mathcal{R}(\mathbf{x})$, then $\mathbf{0} \in \partial\mathcal{R}(\mathbf{x}^*)$, and thus, $\mathcal{R}(\mathbf{x}^*) = 0$ due to (2.3). The reverse can be obtained in the same manner. Finally, $\mathbf{x}^* \in \arg\min_{\mathbf{x}\in\Delta_d^+} \mathcal{R}(\mathbf{x})$ implies that $\mathbf{x}^* \in \mathcal{RB}(\mathbf{b},\varphi)$, since $\mathbf{x}^* \in relint(\mathcal{R})$ and $\mathbf{0} \in \partial\mathcal{R}(\mathbf{x}^*)$. The proof is now complete. ■

Note 4.4. *Theorem 4.3 shows how important the positiveness assumption (4.1) is. Specifically, if the portfolio risk position may take non-positive values then Theorem 4.1 cannot be applied to find RB long-only portfolios, but this does not mean that long-only RB portfolios do not exist, or if exists, their number is finite. This is exemplified further for the RP-SD case.*

Assume that $\varphi = SD$ and the assets have equal variances. Let us consider the following three cases for which there exists $\tilde{\mathbf{x}} \in \Delta_d^+$ such that $\mathcal{R}(\tilde{\mathbf{x}}) = 0$:

i) Four assets ($d = 4$) with $corr(X_1, X_2) = 1$, $corr(X_3, X_4) = -1$ and all other pairs are uncorrelated; direct calculations show that

$$
\mathcal{RB}\left(\frac{1}{4}\mathbf{1}, SD\right) \cap \Delta_4^{++} = \varnothing \quad \textit{and} \quad \mathcal{RB}\left(\frac{1}{4}\mathbf{1}, SD\right) \cap \Delta_4^{+} = \big\{(0,0,1/2,1/2)\big\}.
$$

ii) Four assets ($d = 4$) with $corr(X_1, X_2) = -1$, $corr(X_3, X_4) = -1$ and all other pairs are uncorrelated; direct calculations show that

$$
\begin{aligned}
&\mathcal{RB}\left(\frac{1}{4}\mathbf{1}, SD\right) \cap \Delta_4^{++} = \left\{\left(t, t, \frac{1}{2}-t, \frac{1}{2}-t\right) : 0 < t < \frac{1}{2}\right\} \quad \textit{and} \\
&\mathcal{RB}\left(\frac{1}{4}\mathbf{1}, SD\right) \cap \Delta_4^{+} = \left\{\left(t, t, \frac{1}{2}-t, \frac{1}{2}-t\right) : 0 \le t \le \frac{1}{2}\right\}.
\end{aligned}
$$

iii) Three assets ($d = 3$) with $corr(X_1, X_2) = -0.5$, $corr(X_1, X_3) = -0.5$ and $corr(X_2, X_3) = -0.5$; direct calculations show that

$$
\mathcal{RB}\left(\frac{1}{3}\mathbf{1}, SD\right) \cap \Delta_3^{++} = \mathcal{RB}\left(\frac{1}{3}\mathbf{1}, SD\right) \cap \Delta_3^{+} = \big\{(1/3,1/3,1/3)\big\}.
$$

These confirm Theorem 4.3 b) and show that we may have zero, a unique or infinitely many

RB long-only portfolios if (4.1) *does not hold. This means that* (4.1) *is a sufficient (but not necessary) condition for having a unique long-only RB portfolio.*

Theorems 4.1 and 4.2 extend previous results for specific risk preferences such as $\varphi \in \{var, SD\}$ (Theorem 1.1 in Spinu (2013) and Lemma 2.2 in Bai et al. (2016)) and $\varphi = expectile$ (Theorem 4 in Bellini et al. (2021)), but also for general risk preferences for long-only RP (Theorem 2 in Cesarone et al. (2020)) and long-only RB (Theorem 1 in Cetingoz et al. (2024)) though our ample mathematical characterization helps in understanding the RB/RP strategies in more depth. Theorems 4.1 and 4.2 summarize a series of very interesting results that we next outline in a non-technical language.

First, Theorem 4.1 tells us through (4.2) and (4.3) that RB portfolios could be found under the positiveness condition (4.1) without requiring a differentiability condition of the portfolio risk position that is required in the current literature, even when the risk preferences are very general (Cesarone et al., 2020; Cetingoz et al., 2024). We also show the existence and uniqueness of RB/RP portfolios for any homogeneous risk preferences if (4.1) holds. The lack of differentiability for CVaR risk preferences is discussed in the literature (Mausser and Romanko, 2018) where it is pointed out that RP solutions based on the standard RB/RP definition in (2.2) may not exist even though (4.3) has a solution. This led us to redesign the RB/RP mathematical formulation as in Definition 2.1 that is also motivated by Example 2.2, and conclude the existence and uniqueness of RB/RP portfolios under very general assumptions.

Second, the technical condition in (4.1) is sufficient (but not necessary) to ensure that our RB portfolios are found without major computational issues, since (4.1) guarantees finite optimal solutions in (4.2) and (4.3). The uniqueness property in Proposition 4.2 is not guaranteed for long-only RB portfolios even if the portfolio risk position is always non-negative but not positive everywhere; if so, Note 4.4 shows that the RP/RB set may be a null set, consist of one element, or be an infinite set.

Third, the *logarithmic barrier* and *logarithmic constraint* RB formulations in (4.2) and (4.3), respectively, lead to the same RB portfolio that does not depend upon the normalizing parameters λ and c. This means that $\lambda = 1$ and $c = 0$ are recommended for numerical implementations, and thus, these parameters do not need any tuning, which is beneficial to establishing more powerful statistical inference results in Section 5.

Finally, we found that long-only RB/RP portfolios are always less risky than EW for general risk preferences. This confirms similar properties found in the literature for some particular risk preference choices (Roncalli, 2013; Bellini et al., 2021), which is confirmed by our numerical evidence in Section 6.2.

4.2. Long-short RB for generally distributed risks

Note that according to our knowledge, there is no attempt in the literature to characterize long-short RB/RB for general risk preferences. We now explain how the previous mathematical characterization could be extended to the case in which short-sales are permitted. That is, we look for RB strategies in $\mathcal{RB}(\mathbf{b},\varphi)\cap\Delta_d(\boldsymbol{\delta})$. The equivalent of Theorems 4.1 and 4.2 are given as Theorem 4.5 for which only the logarithmic barrier formulation in (4.2) is investigated, since the logarithmic constraint RB formulation in (4.3) could be dealt similarly.

Theorem 4.5. *Let $\boldsymbol{b}\in\Delta_d^{++}$, and φ be a convex, homogeneous risk measure of order $\tau\geq 1$. Further, let $\boldsymbol{\delta}\in\Re^d\setminus\{-\boldsymbol{1},\boldsymbol{1}\}$ such that $\boldsymbol{\delta}\circ\boldsymbol{\delta}=\boldsymbol{1}$. Furthermore, assume that*

$$\min_{\boldsymbol{x}\in\Delta_d^+}\mathcal{R}(\boldsymbol{\delta}\circ\boldsymbol{x})>0 \tag{4.13}$$

and let $\mathcal{K}(\boldsymbol{\delta}):=\{\boldsymbol{x}\in\Re^d:\boldsymbol{\delta}\circ\boldsymbol{x}\in\Re^d_{++}\}$ be the search cone.

a) For any given $\lambda>0$, the following instance

$$\min_{\boldsymbol{x}\in\mathcal{K}(\boldsymbol{\delta})}\frac{1}{\tau}\mathcal{R}(\boldsymbol{x})-\lambda\sum_{k=1}^{d}b_k\log\delta_k x_k \tag{4.14}$$

admits a unique solution, denoted as $\boldsymbol{x}^*(\lambda,\boldsymbol{b})$, that is an interior point of $\mathcal{K}(\boldsymbol{\delta})$. If $\boldsymbol{1}^T\boldsymbol{x}^*(1,\boldsymbol{b})>0$, then $\boldsymbol{\alpha}^*(\boldsymbol{b})\in\mathcal{RB}(\boldsymbol{b},\varphi)\cap\Delta_d(\boldsymbol{\delta})$, where $\boldsymbol{\alpha}^*(\boldsymbol{b})=\boldsymbol{x}^*(\lambda,\boldsymbol{b})/\boldsymbol{1}^T\boldsymbol{x}^*(\lambda,\boldsymbol{b})$. Moreover,

$$\boldsymbol{\alpha}^*(\boldsymbol{b})=\boldsymbol{x}^*(\lambda^*,\boldsymbol{b})=(\lambda^*)^{1/\tau}\boldsymbol{x}^*(1,\boldsymbol{b}),\ \textit{where}\ \lambda^*=\left(\boldsymbol{1}^T\boldsymbol{x}^*(1,\boldsymbol{b})\right)^{-\tau}.$$

b) Assume that $\boldsymbol{\alpha}^*(\boldsymbol{b})\in\mathcal{RB}(\boldsymbol{b},\varphi)\cap\Delta_d(\boldsymbol{\delta})$ from part a) exists. Then,

$$\min_{\boldsymbol{x}\in\Delta_d(\boldsymbol{\delta})}\mathcal{R}(\boldsymbol{x})\leq\mathcal{R}\big(\boldsymbol{\alpha}^*(\boldsymbol{b})\big)\leq\mathcal{R}\left(\frac{\boldsymbol{\delta}\circ\boldsymbol{b}}{\boldsymbol{1}^T\left(\boldsymbol{\delta}\circ\boldsymbol{b}\right)}\right)\ \textit{if}\ \boldsymbol{1}^T\left(\boldsymbol{\delta}\circ\boldsymbol{b}\right)>0. \tag{4.15}$$

Further, if $d_+>d/2$, where d_+ is the number of assets having a long position, then $\mathcal{R}\big(\boldsymbol{\alpha}^*(\boldsymbol{b})\big)\leq\mathcal{R}\left(\frac{1}{2d_+-d}\boldsymbol{\delta}\circ\boldsymbol{1}\right)$; the latter holds with strict inequality if $d_+<d$.

c) If (4.13) holds and $\boldsymbol{1}^T\boldsymbol{x}^*(1,\boldsymbol{b})>0$, then $\boldsymbol{\alpha}^*(\boldsymbol{b})=\boldsymbol{x}^*(\lambda,\boldsymbol{b})/\boldsymbol{1}^T\boldsymbol{x}^*(\lambda,\boldsymbol{b})$ is the only element of $\mathcal{RB}(\boldsymbol{b},\varphi)\cap\Delta_d(\boldsymbol{\delta})$. Define

$$\min_{\boldsymbol{x}\in\Delta_d^+}\mathcal{R}(\boldsymbol{\delta}\circ\boldsymbol{x})>0\quad\textit{and}\quad\mathcal{R}(-\boldsymbol{x})=\mathcal{R}(\boldsymbol{x})\ \textit{for all}\ \boldsymbol{x}\in\Re^d. \tag{4.16}$$

If (4.16) and $\boldsymbol{1}^T\boldsymbol{x}^*(1,\boldsymbol{b})>0$ hold, then $\boldsymbol{\alpha}^*(\boldsymbol{b})=\boldsymbol{x}^*(\lambda,\boldsymbol{b})/\boldsymbol{1}^T\boldsymbol{x}^*(\lambda,\boldsymbol{b})$ is the only element of $\mathcal{RB}(\boldsymbol{b},\varphi)\cap\Delta_d(\boldsymbol{\delta})$ and $\mathcal{RB}(\boldsymbol{b},\varphi)\cap\Delta_d(\boldsymbol{\delta}^C)=\varnothing$, where $\boldsymbol{\delta}^C=-\boldsymbol{\delta}$.
If (4.16) and $\boldsymbol{1}^T\boldsymbol{x}^*(1,\boldsymbol{b})<0$ hold, then $\boldsymbol{\alpha}^*(\boldsymbol{b})=\boldsymbol{x}^*(\lambda,\boldsymbol{b})/\boldsymbol{1}^T\boldsymbol{x}^*(\lambda,\boldsymbol{b})$ is the only element of $\mathcal{RB}(\boldsymbol{b},\varphi)\cap\Delta_d(\boldsymbol{\delta}^C)$ and $\mathcal{RB}(\boldsymbol{b},\varphi)\cap\Delta_d(\boldsymbol{\delta})=\varnothing$.

d) Let $\boldsymbol{b}\in\Delta_d$. Assume that $\min_{\boldsymbol{x}\in\Delta_d^+}\mathcal{R}(\boldsymbol{\delta}\circ\boldsymbol{x})\geq 0$ and there exists $\tilde{\boldsymbol{x}}\in\{\boldsymbol{\delta}\circ\boldsymbol{x}:\boldsymbol{x}\in\Delta_d^+\}$

such that $\mathcal{R}(\tilde{\boldsymbol{x}}) = 0$. Then,

$$\boldsymbol{x}^* \in \underset{\substack{\boldsymbol{x}\in\Delta_d \\ \boldsymbol{\delta}\circ\boldsymbol{x}\in\Re^d_+}}{\arg\min}\, \mathcal{R}(\boldsymbol{x}) \Leftrightarrow \boldsymbol{x}^* \in \{\boldsymbol{x} \in \Delta_d : \boldsymbol{\delta}\circ\boldsymbol{x} \in \Re^d_+, \mathcal{R}(\boldsymbol{x}) = 0\} \Rightarrow \boldsymbol{x}^* \in \mathcal{RB}(\boldsymbol{b}, \varphi). \quad (4.17)$$

Proof. Part a) could be proved in a similar way to the proof of Theorem 4.1 a), and we thus skip its proof. We now show part b), and as before, we only show the right-hand side inequality in (4.15). Equation (4.14) implies that $\boldsymbol{\alpha}^*(\mathbf{b})$ solves

$$\min_{\mathbf{x}\in\Delta_d(\boldsymbol{\delta})} \frac{1}{\tau}\mathcal{R}(\mathbf{x}) - \lambda^* \sum_{k=1}^{d} b_k \log \delta_k x_k, \quad (4.18)$$

which implies that

$$\begin{aligned}
\frac{1}{\tau}\left(\mathcal{R}\big(\boldsymbol{\alpha}^*(\mathbf{b})\big) - \mathcal{R}\left(\frac{\boldsymbol{\delta}\circ\mathbf{b}}{\mathbf{1}^T\big(\boldsymbol{\delta}\circ\mathbf{b}\big)}\right)\right) &\leq \lambda^*\left(\log\left(\mathbf{1}^T\big(\boldsymbol{\delta}\circ\mathbf{b}\big)\right) + \sum_{k=1}^{d} b_k \log\left(\frac{\delta_k \boldsymbol{\alpha}^*_k(\mathbf{b})}{b_k}\right)\right) \\
&= \lambda^*\left(\log\left(\mathbf{1}^T\big(\boldsymbol{\delta}\circ\mathbf{b}\big)\right) - D_{KL}\big(\mathbf{b}||\boldsymbol{\delta}\circ\boldsymbol{\alpha}^*(\mathbf{b})\big)\right) \\
&\leq 0,
\end{aligned}$$

since $\mathbf{1}^T\big(\boldsymbol{\delta}\circ\mathbf{b}\big) \leq \mathbf{1}^T\mathbf{b} \leq 1$, where $D_{KL}\big(\mathbf{b}||\boldsymbol{\delta}\circ\boldsymbol{\alpha}^*(\mathbf{b})\big)$ is the Kullback-Leibler divergence between the probability distributions induced by (the probability vectors) $\mathbf{b}$ and $\boldsymbol{\delta}\circ\boldsymbol{\alpha}^*(\mathbf{b})$; recall that $\boldsymbol{\alpha}^*(\mathbf{b}) \in \Delta_d(\boldsymbol{\delta})$, which in turn gives that $\boldsymbol{\delta}\circ\boldsymbol{\alpha}^*(\mathbf{b})$ is a proper probability vector.

The very last step is to show $\mathcal{R}\big(\boldsymbol{\alpha}^*(\mathbf{b})\big) \leq \mathcal{R}\left(\frac{1}{2d_+-d}\boldsymbol{\delta}\circ\mathbf{1}\right)$. Equation (4.18) implies that

$$\begin{aligned}
\frac{1}{\tau}\left(\mathcal{R}\big(\boldsymbol{\alpha}^*(\mathbf{b})\big) - \mathcal{R}\left(\frac{1}{2d_+-d}\boldsymbol{\delta}\circ\mathbf{1}\right)\right) &\leq \lambda^*\left(\sum_{k=1}^{d} b_k \log \boldsymbol{\alpha}^*_k(\mathbf{b}) - \sum_{k=1}^{d} b_k \log\left(\frac{1}{2d_+-d}\right)\right) \\
&\leq \lambda^*\left(\max_{\mathbf{1}^T\mathbf{x}=1} \sum_{k=1}^{d} b_k \log x_k + \log(2d_+-d)\right) \\
&= \lambda^*\left(\sum_{k=1}^{d} b_k \log b_k + \log(2d_+-d)\right) \\
&\leq \lambda^*\left(\max_{\mathbf{1}^T\mathbf{x}=1} \sum_{k=1}^{d} x_k \log x_k + \log(2d_+-d)\right) \\
&= \lambda^*\left(\log\frac{2d_+-d}{d}\right) \\
&\leq 0,
\end{aligned}$$

which becomes a strict inequality whenever $d_+ < d$. This concludes part b).

Part c) is clear by noting that $\mathbf{y} \in \mathcal{RB}(\mathbf{b}, \varphi)$ implies that $-\mathbf{y} \in \mathcal{RB}(\mathbf{b}, \varphi)$ as (4.16) holds and the fact that $\mathbf{a} \in \partial\mathcal{R}(\mathbf{y})$ if and only if $-\mathbf{a} \in \partial\mathcal{R}(-\mathbf{y})$ due to (4.16). Part d) could be proved as in Theorem 4.3 b), and its proof is thus omitted. The proof is now complete. ■

Before explaining the main results in Theorem 4.5, we provide Example 4.6 to shed some light over the importance of the positiveness condition (4.13) that is of crucial importance to understanding the properties of long-short RB/RP portfolios.

Example 4.6. *Let $\boldsymbol{X} = (X_1, X_2)$ be a bivariate normally distributed random vector (of negative returns) with correlation coefficient ρ, unit variances ($SD(X_1) = SD(X_2) = 1$), and mean losses of $\mathrm{E}(X_1) = \mu_1$ and $\mathrm{E}(X_2) = \mu_2$. The following three settings are considered:*

A) $\mu_1 = -1$, $\mu_2 = -1$ and $\rho = 0.5$;

B) $\mu_1 = -1$, $\mu_2 = -3$ and $\rho = 0.5$;

C) $\mu_1 = -1$, $\mu_2 = -3$ and $\rho = -0.9$.

Risk preferences are ordered via $\mathrm{CVaR}_{95\%}$ and due to (A.2), the portfolio's risk position is

$$\mathcal{R}(x_1, x_2) = \mu_1 x_1 + \mu_2 x_2 + \left(x_1^2 + x_2^2 + 2\rho x_1 x_2\right)^{1/2} \mathrm{CVaR}_{95\%}(Z_1), \tag{4.19}$$

where $\mathrm{CVaR}_{95\%}(Z_1) = 2.06271$ as Z_1 is a standard normal Gaussian random variable. Note that $\mathcal{R}(\cdot)$ is differentiable on $\Re^2$, and thus, we use (2.2) to find RP strategies. We first check whether (4.13) is true, and if so, we check if $\boldsymbol{1}^T\boldsymbol{x}^*(1, \boldsymbol{b}) > 0$ holds; recall that $\boldsymbol{x}^*(1, \boldsymbol{b})$ is the unique solution in (4.14) for a given cone $\mathcal{K}(\boldsymbol{\delta})$. Clearly, $\tau = 1$ and the risk contributions are

$$\mathcal{RC}_k(x_1, x_2) = \mu_k x_k + \frac{x_k^2 + \rho x_1 x_2}{\left(x_1^2 + x_2^2 + 2\rho x_1 x_2\right)^{1/2}} \mathrm{CVaR}_{95\%}(Z_1) \ \textit{for}\ k = 1, 2. \tag{4.20}$$

Assume setting A. *Direct computations for solving $\mathcal{RC}_1(x_1, x_2) = \mathcal{RC}_2(x_1, x_2)$ in $(x_1, x_2) \in \Delta_2$ show that there are exactly three RP strategies in $\mathcal{RB}\left(\frac{1}{2}\boldsymbol{1}, \mathrm{CVaR}_{95\%}\right) \cap \Delta_2$:*

$$\boldsymbol{x}^{*(A1)} = (0.5, 0.5),\ \boldsymbol{x}^{*(A2)} = (-1.3721, 2.3721) \ \textit{and}\ \boldsymbol{x}^{*(A3)} = (2.3721, -1.3721).$$

We recover these RP strategies by using Theorem 4.5; (4.13) holds for all four cones, and one may check that $\boldsymbol{1}^T\boldsymbol{x}^*(1, \boldsymbol{b}) > 0$ for all cones except the negative cone with $\boldsymbol{\delta} = (-1, -1)$. After standardizing the solutions in (4.14), one could find the three RP strategies, $\boldsymbol{x}^{*(A1)}$, $\boldsymbol{x}^{*(A2)}$ and $\boldsymbol{x}^{*(A3)}$, which are the unique RP strategies in the cone with $(1, 1)$, $(-1, 1)$ and $(1, -1)$, respectively. Even though (4.16) does not hold, standardizing the solutions in (4.14) when $\boldsymbol{\delta} = (-1, -1)$ leads to $\boldsymbol{x}^{*(A1)}$, which is just a coincidence triggered by the symmetry of $\mathcal{R}$, i.e., $\mathcal{R}(x_1, x_2) = \mathcal{R}(x_2, x_1)$ for $(x_1, x_2) \in \Re^2$.

Assume setting B. *As before, solving $\mathcal{RC}_1(x_1, x_2) = \mathcal{RC}_2(x_1, x_2)$ in $(x_1, x_2) \in \Delta_2$ shows that there is only one RP strategy in $\mathcal{RB}\left(\frac{1}{2}\boldsymbol{1}, \mathrm{CVaR}_{95\%}\right) \cap \Delta_2$, namely, $\boldsymbol{x}^{*(B)} = (1.5437, -0.5437)$. Now, we recover this RP strategy by using Theorem 4.5. Condition (4.13) holds only for the*

cones with $\boldsymbol{\delta} = (1, -1)$ *and* $\boldsymbol{\delta} = (-1, -1)$*, while* $\boldsymbol{1}^T \boldsymbol{x}^*(1, \boldsymbol{b}) > 0$ *is true only when* $\boldsymbol{\delta} = (1, -1)$*. After standardizing the solution in* (4.14) *for the cone with* $\boldsymbol{\delta} = (1, -1)$*, one could recover* $\boldsymbol{x}^{*(B)}$ *as the unique element of* $\mathcal{RB}\big(\frac{1}{2}\boldsymbol{1}, \mathrm{CVaR}_{95\%}\big) \cap \Delta_2(1, -1)$*. After standardizing the solution in* (4.14) *for the cone with* $\boldsymbol{\delta} = (-1, -1)$*, one gets* $\boldsymbol{x}^* = (0.6146, 0.3853)$*, which is not a RP strategy. This is not surprising, since* (4.16) *does not hold (*$\mathcal{R}$ *is not an even function).*
Assume setting C. *Once again, solving* $\mathcal{RC}_1(x_1, x_2) = \mathcal{RC}_2(x_1, x_2)$ *in* $(x_1, x_2) \in \Delta_2$ *shows that there are exactly three RP strategies in* $\mathcal{RB}\big(\frac{1}{2}\boldsymbol{1}, \mathrm{CVaR}_{95\%}\big) \cap \Delta_2$*:*

$$\boldsymbol{x}^{*(C1)} = (0.2816, 0.7184), \ \boldsymbol{x}^{*(C2)} = (0.3975, 0.6025) \text{ and } \boldsymbol{x}^{*(C3)} = (1.2733, -0.2733).$$

We can recover only the latter RP portfolio by the findings in Theorem 4.5*. Condition* (4.13) *holds only for the cones with* $\boldsymbol{\delta} = (1, -1)$ *and* $\boldsymbol{\delta} = (-1, -1)$*, while* $\boldsymbol{1}^T \boldsymbol{x}^*(1, \boldsymbol{b}) > 0$ *is true only for the cone with* $\boldsymbol{\delta} = (1, -1)$*. After standardizing the solution in* (4.14) *for the cone with* $\boldsymbol{\delta} = (1, -1)$*, one could recover* $\boldsymbol{x}^{*(C3)}$ *as the unique element of* $\mathcal{RB}\big(\frac{1}{2}\boldsymbol{1}, \mathrm{CVaR}_{95\%}\big) \cap \Delta_2(1, -1)$*. After standardizing the solution in* (4.14) *for the cone with* $\boldsymbol{\delta} = (-1, -1)$*, one gets* $\boldsymbol{x}^* = (0.5475, 0.4525)$*, which is not a RP strategy as* (4.16) *does not hold (*$\mathcal{R}$ *is not an even function). Unfortunately, the two RP portfolios in the positive cone (*$\boldsymbol{x}^{*(C1)}$ *and* $\boldsymbol{x}^{*(C2)}$*) could not be identified by the logarithmic barrier formulation as the main necessary condition in* (4.1) *(or its generalisation in* (4.13)*) is not satisfied. Further,*

$$\mathcal{R}\big(\boldsymbol{x}^{*(C2)}\big) < \mathcal{R}\Big(\frac{1}{2}\boldsymbol{1}\Big) < \mathcal{R}\big(\boldsymbol{x}^{*(C1)}\big),$$

which means that long-only RB portfolios may be riskier than the EW portfolio if (4.1) *does not hold; e.g.,* $\boldsymbol{x}^{*(C1)}$*. This is in contrast with our finding in Theorem* 4.1 *c), though the two do not contradict each other.*
In a nutshell, Example 4.6 *tells us that if* (4.13) *does not hold in a particular cone, then we may have no RB/RP portfolio (see setting B) or multiple RB/RP portfolios (see setting C) in that cone. Further, if* (4.13) *holds in a particular cone and* $\boldsymbol{1}^T \boldsymbol{x}^*(1, \boldsymbol{b}) < 0$*, then the standardised portfolio* $\boldsymbol{\alpha}^*(\boldsymbol{b}) = \boldsymbol{x}^*(\lambda, \boldsymbol{b}) / \boldsymbol{1}^T \boldsymbol{x}^*(\lambda, \boldsymbol{b})$ *may be (see setting A) or may not be (see settings B and C) a RB/RP portfolio in the complementary cone if* (4.16) *does not hold. Therefore, the standard logarithmic barrier formulation is helpful to identify RB/RP portfolios, but its use does not guarantee that all portfolios are found, since the logarithmic barrier procedure requires some regularity conditions.*

Theorem 4.5 is a replica of Theorems 4.1 and 4.2 for the long-short RB/RP case, and explains how to find RB/RP strategies in any cone, except the case in which there is no asset in a long position, which is an infeasible setting. A series of very interesting results are implied by Theorem 4.5, and we outline them in a non-technical language.
First, Theorem 4.5 provides the existence and uniqueness of long-short RB/RP portfolios within one given cone for general risk preferences, which is guaranteed if two

conditions are satisfied: i) the positiveness condition (4.13) and ii) $\mathbf{1}^T\mathbf{x}^*(1,\mathbf{b}) > 0$ without any differentiability condition. The positiveness condition is similar to our condition in Theorem 4.1, and ii) helps to preserve the cone after standardization (sum the weights to 1). If the second condition is not satisfied and $\mathbf{1}^T\mathbf{x}^*(1,\mathbf{b}) < 0$, then we require (4.16) (portfolio risk position is an even functional) to guarantee no RB/RP portfolio within the search cone and the existence and uniqueness of RB/RP portfolios within the complement cone. Finally, if $\mathbf{1}^T\mathbf{x}^*(1,\mathbf{b}) = 0$, the so-called "market-neutral" portfolios (as explained in Bai et al. (2016) when $\varphi \in \{SD, var\}$), is not tractable and no possible characterization is available.

Second, the technical positiveness condition i) (see (4.16)) is an essential condition to guarantee that RB/RP exists and is unique. Example 4.6 shows that without (4.16) RB/RP portfolios may not exist or multiple solutions are possible. If condition ii) does not hold, then finding RB/RP portfolios could be difficult. Note that (4.16) holds if $\varphi \in \{SD, var\}$ provided that the covariance matrix is positive definite, and thus, the RB/RP portfolio in the search cone ($\boldsymbol{\delta}$) and complementary cone ($\boldsymbol{\delta}^C$) exist in only one cone; in turn, we have at most 2^{d-1} RB/RP strategies in all feasible $2^d - 1$ search cones which recovers the discussion from Section 2.2 in Bai et al. (2016).

Third, finding RB/RP portfolios does not depend upon λ, and therefore, one can choose $\lambda = 1$ in the implementation phase as λ does not need any tuning.

Finally, we found that long-short RB/RP portfolios are always less risky than GEW for general risk preferences, where GEW is introduced in Section 3.2. Note that the GEW portfolio $\frac{1}{2d_+ - d}\boldsymbol{\delta} \circ \mathbf{1}$ is the equivalent of the EW portfolio in $\Delta_d(\boldsymbol{\delta})$ and $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)$ is an example for four assets with only the fourth one being in a short position.

5. Statistical inferences

The previous section explains how to find RB portfolios with the help of Theorems 4.1 and 4.5. We note that according to our knowledge, there are no statistical inferences for RB portfolios, which is the main aim of this section. Our statistical inferences are focused on two risk preferences, CVaR and SD, which are popular choices in practice. In this section, we observe $\left\{\boldsymbol{X}_t = (X_{t,1}, \ldots, X_{t,d})^T\right\}_{t=1}^n$ from the strictly stationary α-mixing sequence of $\left\{\boldsymbol{X}_t = (X_{t,1}, \ldots, X_{t,d})^T\right\}_{t=-\infty}^{\infty}$ satisfying

$$\alpha_{\boldsymbol{X}}(k) = \sup\left\{|\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)| : A \in \mathcal{F}_{-\infty}^i, B \in \mathcal{F}_{i+k}^{\infty}, -\infty < i < \infty\right\} \to 0$$

as $k \to \infty$, where $\mathcal{F}_a^b$ denotes the σ-field generated by $\{\boldsymbol{X}_t : a \leq t \leq b\}$. For statistical inferences, Theorem 4.1 suggests searching for a non-parametric estimator for $\mathcal{R}(\boldsymbol{x})$, which is convex and homogeneous.

First, we consider CVaR_p risk preferences with $0 < p < 1$, for which the portfolio risk position is measured as follows:

$$\inf_{\theta}\left\{\theta+\frac{1}{1-p}\mathbb{E}\big((\boldsymbol{x}^T\boldsymbol{X}_t-\theta)_+\big)\right\};$$

see Rockafellar and Uryasev (2002). Hence, the simple non-parametric estimator is

$$\widehat{\mathcal{R}}_{cvar}^{emp}(\boldsymbol{x}):=\inf_{\theta}\left\{\theta+\frac{1}{n(1-p)}\sum_{t=1}^{n}\left(\boldsymbol{x}^T\boldsymbol{X}_t-\theta\right)_+\right\},$$

which is convex, homogeneous, but not differentiable, though differentiable almost everywhere implied by the convexity. To derive the asymptotic properties of the RB estimator, one can use the smooth non-parametric estimation in Scaillet (2004) and Chen (2008), defined as

$$\widehat{\mathcal{R}}_{cvar}^{KD}(\boldsymbol{x}):=\frac{1}{n(1-p)}\sum_{t=1}^{n}\boldsymbol{x}^T\boldsymbol{X}_t\left\{1-K\left(\frac{\theta-\boldsymbol{x}^T\boldsymbol{X}_t}{h}\right)\right\},$$

where $\theta=\theta(\boldsymbol{x})$ solves

$$\frac{1}{n}\sum_{t=1}^{n}K\left(\frac{\theta-\boldsymbol{x}^T\boldsymbol{X}_t}{h}\right)=p,$$

$K(\cdot)$ is a smooth distribution function on $\Re$, and $h=h(n)>0$ is the kernel bandwidth. Unfortunately, we cannot ensure $\widehat{\mathcal{R}}_{cvar}^{KD}(\boldsymbol{x})$ to be convex and homogeneous. By writing that

$$\mathbb{E}\big((\boldsymbol{x}^T\boldsymbol{X}_t-\theta)_+\big)=\int(\boldsymbol{x}^T\boldsymbol{s}-\theta)_+f_{\boldsymbol{X}}(s_1,\cdots,s_d)\,d\mathbf{s},$$

where $\boldsymbol{s}=(s_1,\cdots,s_d)^T$ and $f_{\boldsymbol{X}}(\boldsymbol{s})$ is the density function of $\boldsymbol{X}_t$, we propose the following smooth non-parametric estimator

$$\widehat{\mathcal{R}}_{cvar}(\boldsymbol{x}):=\inf_{\theta}\left\{\theta+\frac{1}{n(1-p)}\sum_{t=1}^{n}\int(\boldsymbol{x}^T\boldsymbol{s}-\theta)_+\prod_{i=1}^{d}h_i^{-1}k\left(\frac{s_i-X_{t,i}}{h_i}\right)\,d\mathbf{s}\right\},$$

where $k(\cdot)=K'(\cdot)$ on $\Re$, and $h_i=h_i(n)>0$ is a bandwidth for all $i\in\{1,2,\ldots,d\}$. It is straightforward to verify that $\widehat{\mathcal{R}}_{cvar}(\boldsymbol{x})$ is convex, homogeneous with order one, and differentiable everywhere. Also,

$$\widehat{\mathcal{R}}_{cvar}(\boldsymbol{x})=\frac{1}{n(1-p)}\sum_{t=1}^{n}\int I(\boldsymbol{x}^T\boldsymbol{s}>\theta)\boldsymbol{x}^T\boldsymbol{s}\prod_{i=1}^{d}h_i^{-1}k\left(\frac{s_i-X_{t,i}}{h_i}\right)\,d\mathbf{s},\tag{5.1}$$

with $\theta = \theta(\boldsymbol{x})$ satisfying

$$1 - \frac{1}{n(1-p)} \sum_{t=1}^{n} \int I(\boldsymbol{x}^T \boldsymbol{s} > \theta) \prod_{i=1}^{d} h_i^{-1} k\left(\frac{s_i - X_{t,i}}{h_i}\right) d\mathbf{s} = 0 \tag{5.2}$$

and I denoting the indicator function with $I(A) = 1$ if A is true, and $I(A) = 0$ otherwise. Hence, using $\tau = 1$ for CVaR risk measure and taking $\lambda = 1$ in Theorem 4.1, we estimate $\boldsymbol{x}$ and $\boldsymbol{\alpha}$ by

$$\widehat{\boldsymbol{x}}_{cvar} = \arg\min_{\boldsymbol{x}\in\Re^d_{++}} \widehat{\mathcal{R}}_{cvar}(\boldsymbol{x}) - \sum_{i=1}^{d} b_i \log x_i \quad \text{and} \quad \widehat{\boldsymbol{\alpha}}_{cvar} = \widehat{\boldsymbol{x}}_{cvar}/\mathbf{1}^T\widehat{\boldsymbol{x}}_{cvar}.$$

That is, $\hat{\boldsymbol{x}}_{cvar}$ and $\hat{\theta}_{cvar} = \theta(\hat{\boldsymbol{x}}_{cvar})$ solve the system of equations for $\boldsymbol{x} > \mathbf{0}$:

$$\begin{cases} \dfrac{1}{n(1-p)} \displaystyle\sum_{t=1}^{n} \int I(\boldsymbol{x}^T \boldsymbol{s} > \theta(\boldsymbol{x})) s_i \prod_{j=1}^{d} h_j^{-1} k\left(\frac{s_j - X_{t,j}}{h_j}\right) d\mathbf{s} - \dfrac{b_i}{x_i} = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for } i \in \{1, 2, \dots, d\}, \\ 1 - \dfrac{1}{n(1-p)} \displaystyle\sum_{t=1}^{n} \int I(\boldsymbol{x}^T \boldsymbol{s} > \theta(\boldsymbol{x})) \prod_{j=1}^{d} h_j^{-1} k\left(\frac{s_j - X_{t,j}}{h_j}\right) d\mathbf{s} = 0. \end{cases} \tag{5.3}$$

On the other hand, the true values $\boldsymbol{x}_0$ and $\theta_0 = \theta(\boldsymbol{x}_0)$ solve

$$\mathbb{E}[\bar{\boldsymbol{Z}}_t(\boldsymbol{x}, \theta)] = \mathbf{0} \quad \text{for } \boldsymbol{x} > \mathbf{0}, \tag{5.4}$$

where $\bar{\boldsymbol{Z}}_t(\boldsymbol{x}, \theta) = \left(\bar{Z}_{t,1}(\boldsymbol{x}, \theta), \dots, \bar{Z}_{t,d+1}(\boldsymbol{x}, \theta)\right)^T$ is given by

$$\begin{cases} \bar{Z}_{t,i}(\boldsymbol{x}, \theta) = \frac{1}{1-p} X_{t,i} I\big(\boldsymbol{x}^T \boldsymbol{X}_t > \theta(\boldsymbol{x})\big) - \frac{b_i}{x_i} \text{ for all } i \in \{1, 2, \dots, d\}, \\ \bar{Z}_{t,d+1}(\boldsymbol{x}, \theta) = 1 - \frac{1}{1-p} I\big(\boldsymbol{x}^T \boldsymbol{X}_t > \theta(\boldsymbol{x})\big). \end{cases}$$

Define $\bar{\Gamma}(\boldsymbol{x}, \theta) = \mathbb{E}\bar{\boldsymbol{Z}}_1(\boldsymbol{x}, \theta)$ and denote the partial derivatives of $\bar{\Gamma}(\cdot, \cdot)$ by $\dot{\bar{\Gamma}}(\cdot, \cdot)$ on $\Re^d \times \Re$. We assume the following regularity conditions to derive the asymptotic limits of $\hat{\boldsymbol{x}}_{cvar}$, $\hat{\theta}_{cvar}$, and $\hat{\boldsymbol{\alpha}}_{cvar}$:

- C1) $\{\boldsymbol{X}_t\}_{t=-\infty}^{\infty}$ is a strictly stationary α-mixing sequence with $\alpha_{\boldsymbol{X}}(m) = O\big(a^m\big)$ for some $a \in (0, 1)$ as $m \to \infty$. Furthermore, assume $\mathbb{E}||\boldsymbol{X}_t||_2^{2+\delta} < \infty$ for some $\delta > 0$.
- C2) $(\boldsymbol{x}_0^T, \theta_0)^T$ is the unique solution to (5.4).
- C3) The probability density function of $\boldsymbol{X}_t$ has bounded second partial derivatives on the closure of $\Omega = \cup_{(\boldsymbol{x}^T,\theta)^T \in \Omega_0} \big\{\boldsymbol{s} \in \Re_r : \boldsymbol{x}^T \boldsymbol{s} \geq \theta\big\}$, where Ω_0 is an open set covering $(\boldsymbol{x}_0^T, \theta_0)^T$. For any $s \geq 1$, the joint density of $\boldsymbol{X}_t$ and $\boldsymbol{X}_{t+s}$ has bounded second partial derivatives on the closure of $\Omega \times \Omega$.
- C4) $k(\cdot)$ is a symmetric density function on $[-1, 1]$. For each $i \in \{1, 2, \dots, d\}$, $h_i =$

$c_i n^{-1/3}$ for some positive constant c_i.

The next theorem provides the result for deriving inference when risk preferences are ordered by the CVaR_p risk measure.

Theorem 5.1. *Assume conditions C1)–C4) hold and consider the case in which* $\varphi = \mathrm{CVaR}_p$ *with* $0 < p < 1$*. Then, there is a positive definite matrix* $\bar{\Sigma}$ *such that*

$$\mathbb{E}\left\{\bar{\boldsymbol{Z}}_1(\boldsymbol{x}_0, \theta_0)\bar{\boldsymbol{Z}}_1^T(\boldsymbol{x}_0, \theta_0)\right\} + 2\lim_{n\to\infty}\sum_{m=1}^{n-1}\mathbb{E}\left\{\bar{\boldsymbol{Z}}_1(\boldsymbol{x}_0, \theta_0)\bar{\boldsymbol{Z}}_{1+m}^T(\boldsymbol{x}_0, \theta_0)\right\} = \bar{\Sigma}. \tag{5.5}$$

Furthermore, as $n \to \infty$*,*

$$\sqrt{n}\big(\hat{\boldsymbol{x}}_{cvar}^T - \boldsymbol{x}_0^T, \hat{\theta}_{cvar} - \theta_0\big)^T \xrightarrow{w} N\Big(\boldsymbol{0}, \dot{\bar{\Gamma}}^{-1}(\boldsymbol{x}_0, \theta_0)\bar{\Sigma}\big(\dot{\bar{\Gamma}}^{-1}(\boldsymbol{x}_0, \theta_0)\big)^T\Big), \tag{5.6}$$

$$\sqrt{n}(\hat{\boldsymbol{\alpha}}_{cvar} - \boldsymbol{\alpha}_0) \xrightarrow{w} N\left(\boldsymbol{0}, \frac{\bar{\Sigma}_0}{(\boldsymbol{1}^T\boldsymbol{x}_0)^2} - \frac{2\boldsymbol{x}_0\boldsymbol{1}^T\bar{\Sigma}_0}{(\boldsymbol{1}^T\boldsymbol{x}_0)^3} + \frac{\boldsymbol{x}_0\boldsymbol{1}^T\bar{\Sigma}_0\boldsymbol{1}\boldsymbol{x}_0^T}{(\boldsymbol{1}^T\boldsymbol{x}_0)^4}\right), \tag{5.7}$$

where $\bar{\Sigma}_0$ *is the first* $d \times d$ *matrix of* $\dot{\bar{\Gamma}}^{-1}(\boldsymbol{x}_0, \theta_0)\bar{\Sigma}\big(\dot{\bar{\Gamma}}^{-1}(\boldsymbol{x}_0, \theta_0)\big)^T$.

Proof. For simplicity, assume $h_i = h$ for all $i \in \{1, \ldots, d\}$. Let $f_{1,1+r}(\boldsymbol{x}, \bar{\boldsymbol{x}})$ denote the joint density function of $\big(\boldsymbol{X}_t, \boldsymbol{X}_{t+r}\big)$. Put $\boldsymbol{s}, \bar{\boldsymbol{s}}, \boldsymbol{y}, \bar{\boldsymbol{y}}, \boldsymbol{x} \in \Re^d$, $\boldsymbol{Z}_t(\boldsymbol{x}, \theta) = \big(Z_{t,1}(\boldsymbol{x}, \theta), \ldots, Z_{t,d+1}(\boldsymbol{x}, \theta)\big)^T$,

$$\begin{cases} Z_{t,i}(\boldsymbol{x}, \theta) = \dfrac{1}{1-p}\displaystyle\int I\big(\boldsymbol{x}^T\boldsymbol{s} > \theta(\boldsymbol{x})\big)s_i\prod_{j=1}^d h_j^{-1}k\left(\frac{X_{t,j} - s_j}{h_j}\right)d\mathbf{s} - \dfrac{b_i}{x_i} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for all } i = 1, \ldots, d, \\ Z_{t,d+1}(\boldsymbol{x}, \theta) = 1 - \dfrac{1}{1-p}\displaystyle\int I(\boldsymbol{x}^T\boldsymbol{s} > \theta(\boldsymbol{x}))\prod_{j=1}^d h_j^{-1}k\left(\frac{X_{t,j} - s_j}{h_j}\right)d\mathbf{s}. \end{cases}$$

Then, (5.3) becomes

$$\frac{1}{n}\sum_{t=1}^n \boldsymbol{Z}_t(\hat{\boldsymbol{x}}_{cvar}, \hat{\theta}_{cvar}) = \boldsymbol{0}. \tag{5.8}$$

Define

$$\begin{aligned} \gamma_i(s; \boldsymbol{x}, \theta) \;\; = \;\; & E\left\{\big(Z_{t,i}(\boldsymbol{x}, \theta) - \bar{Z}_{t,i}(\boldsymbol{x}, \theta)\big)\big(Z_{t+s,i}(\boldsymbol{x}, \theta) - \bar{Z}_{t+s,i}(\boldsymbol{x}, \theta)\big)\right\} \\ & - \left\{E\big(Z_{t,i}(\boldsymbol{x}, \theta) - \bar{Z}_{t,i}(\boldsymbol{x}, \theta)\big)\right\}^2 \end{aligned}$$

for $i = 1, \cdots, d+1$ and nonnegative integer s. Write

$$\begin{aligned} & Z_{t,d+1}(\boldsymbol{x}, \theta) - \bar{Z}_{t,d+1}(\boldsymbol{x}, \theta) \\ & = \frac{1}{1-p}\int\left\{\prod_{j=1}^d k(s_j)\right\}\left\{I\big(\boldsymbol{x}^T\boldsymbol{X}_t + h\boldsymbol{x}^T\boldsymbol{s} > \theta(\boldsymbol{x})\big) - I\big(\boldsymbol{x}^T\boldsymbol{X}_t > \theta(\boldsymbol{x})\big)\right\}d\boldsymbol{s}, \end{aligned}$$

$$
\begin{aligned}
Z_{t,i}(\boldsymbol{x},\theta) - \bar{Z}_{t,i}(\boldsymbol{x},\theta) \;\; = \;\; & \frac{1}{1-p}\int \left\{\prod_{j=1}^{d} k(s_j)\right\}\left\{I\big(\boldsymbol{x}^T\boldsymbol{X}_t + h\boldsymbol{x}^T\boldsymbol{s} > \theta(\boldsymbol{x})\big)s_i h\right. \\
& \left. + X_{t,i}I\big(\boldsymbol{x}^T\boldsymbol{X}_t + h\boldsymbol{x}^T\boldsymbol{s} > \theta(\boldsymbol{x})\big) - X_{t,i}I\big(\boldsymbol{x}^T\boldsymbol{X}_t > \theta(\boldsymbol{x})\big)\right\}\, d\mathbf{s}
\end{aligned}
$$

for $i = 1, \cdots, d$. Then,

$$
E\{Z_{t,i}(\boldsymbol{x},\theta) - \bar{Z}_{t,i}(\boldsymbol{x},\theta)\} = \mathcal{O}(h^2),\; E\{Z_{t,i}(\boldsymbol{x},\theta) - \bar{Z}_{t,i}(\boldsymbol{x},\theta)\}^2 = \mathcal{O}(h^2) \tag{5.9}
$$

hold uniformly in $\{(\boldsymbol{x}^T,\theta)^T : ||\boldsymbol{x}-\boldsymbol{x}_0||_2 + |\theta-\theta_0| \le n^{-1/3}\}$ for all $i \in \{1,\ldots,d+1\}$, implying that

$$
|\gamma_i(0;\boldsymbol{x},\theta)| = \mathcal{O}(h^2) \tag{5.10}
$$

uniformly in $\{(\boldsymbol{x}^T,\theta)^T : ||\boldsymbol{x}-\boldsymbol{x}_0||_2 + |\theta-\theta_0| \le n^{-1/3}\}$ for all $i \in \{1,\ldots,d+1\}$. Here, $\mathcal{O}(h^2)$ means less than a constant times h^2. Using C2), we have that for any $r \ge 1$,

$$
\begin{aligned}
& E\left\{\int I\big(\boldsymbol{x}^T\boldsymbol{s} > \theta(\boldsymbol{x})\big)\prod_{j=1}^{d} h^{-1}k\left(\frac{X_{t,j}-s_j}{h}\right)d\mathbf{s}\right. \\
& \qquad \left.\times \int I\big(\boldsymbol{x}^T\boldsymbol{s} > \theta(\boldsymbol{x})\big)\prod_{j=1}^{d} h^{-1}k\left(\frac{X_{t+r,j}-s_j}{h}\right)d\mathbf{s}\right\} \\
& = \int I\big(\boldsymbol{x}^T\boldsymbol{s} > \theta(\boldsymbol{x})\big)I\big(\boldsymbol{x}^T\bar{\mathbf{s}} > \theta(\boldsymbol{x})\big)\left\{\prod_{j=1}^{d} h^{-1}k\left(\frac{y_j-s_j}{h}\right)\right\}\left\{\prod_{j=1}^{d} h^{-1}k(\frac{\bar{y}_j-\bar{s}_j}{h})\right\} \\
& \qquad \times f_{1,1+r}\big(\boldsymbol{y},\bar{\boldsymbol{y}}\big)\, d\mathbf{s}\, d\bar{\mathbf{s}}\, d\mathbf{y}\, d\bar{\mathbf{y}} \\
& = \int I\big(\boldsymbol{x}^T\boldsymbol{s} > \theta(\boldsymbol{x})\big)I\big(\boldsymbol{x}^T\bar{\mathbf{s}} > \theta(\boldsymbol{x})\big)\left\{\prod_{j=1}^{d} k(y_j)\right\} \\
& \qquad \times \left\{\prod_{j=1}^{d} k(\bar{y}_j)\right\} f_{1,1+r}(\boldsymbol{s}+h\boldsymbol{y}, \bar{\boldsymbol{s}}+h\bar{\boldsymbol{y}})\, d\mathbf{y}\, d\bar{\mathbf{y}}\, d\mathbf{s}\, d\bar{\mathbf{s}} \\
& = \int I\big(\boldsymbol{x}^T\boldsymbol{s} > \theta(\boldsymbol{x})\big)I\big(\boldsymbol{x}^T\bar{\mathbf{s}} > \theta(\boldsymbol{x})\big)\left\{\prod_{j=1}^{d} k(y_j)\right\}\left\{\prod_{j=1}^{d} k(\bar{y}_j)\right\}\left\{f_{1,1+r}(\boldsymbol{s},\bar{\boldsymbol{s}})\right. \\
& \qquad \left. + h\sum_{j=1}^{d}\frac{\partial}{\partial s_j}f_{1,1+r}(\boldsymbol{s},\bar{\boldsymbol{s}})y_j + h\sum_{j=1}^{d}\frac{\partial}{\partial \bar{s}_j}f_{1,1+r}(\boldsymbol{s},\bar{\boldsymbol{s}})\bar{y}_j + \mathcal{O}(h^2)\right\} d\mathbf{y}\, d\bar{\mathbf{y}}\, d\mathbf{s}\, d\bar{\mathbf{s}} \\
& = \int I\big(\boldsymbol{x}^T\boldsymbol{s} > \theta(\boldsymbol{x})\big)I\big(\boldsymbol{x}^T\bar{\mathbf{s}} > \theta(\boldsymbol{x})\big)\left\{\prod_{j=1}^{d} k(y_j)\right\}\left\{\prod_{j=1}^{d} k(\bar{y}_j)\right\} f_{1,1+r}(\boldsymbol{s},\bar{\boldsymbol{s}})\, d\mathbf{s}\, d\bar{\mathbf{s}} \\
& \qquad + \mathcal{O}(h^2)
\end{aligned}
$$

holds uniformly in $\{(\boldsymbol{x}^T,\theta)^T : ||\boldsymbol{x}-\boldsymbol{x}_0||_2 + |\theta-\theta_0| \le n^{-1/3}\}$. Similarly, we can show that

$$
|\gamma_i(s;\boldsymbol{x},\theta)| = \mathcal{O}(h^2) \tag{5.11}
$$

holds uniformly in positive integer s and $\{(\boldsymbol{x}^T,\theta)^T : ||\boldsymbol{x}-\boldsymbol{x}_0||_2 + |\theta-\theta_0| \le n^{-1/3}\}$ for all $i \in \{1, \cdots, d+1\}$. Using C1) and the Davydov inequality, we have

$$|\gamma_i(s; \boldsymbol{x}, \theta)| = \mathcal{O}\big(\{\alpha(s)\}^{1-2/(2+\delta)}\big) \tag{5.12}$$

uniformly in nonnegative integer s and $\big\{(\boldsymbol{x}^T, \theta)^T : ||\boldsymbol{x} - \boldsymbol{x}_0||_2 + |\theta - \theta_0| \le n^{-1/3}\big\}$ for all $i \in \{1, \cdots, d+1\}$. Hence, it follows from (5.10), (5.11), and (5.12) that for any given $\xi \in (1/2, 1)$,

$$|\gamma_i(s; \boldsymbol{x}, \theta)| = \mathcal{O}\big(h^{2\xi}\{\alpha(s)\}^{1-\xi-2(1-\xi)/(2+\delta)}\big) \tag{5.13}$$

uniformly in nonnegative integer s and $\big\{(\boldsymbol{x}^T, \theta)^T : ||\boldsymbol{x} - \boldsymbol{x}_0||_2 + |\theta - \theta_0| \le n^{-1/3}\big\}$ for all $i \in \{1, \cdots, d+1\}$. It follows from (5.9), (5.10), (5.13), and C1) that

$$\begin{aligned}
&E\left\{\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\big(Z_{t,i}(\boldsymbol{x}, \theta) - \bar{Z}_{t,i}(\boldsymbol{x}, \theta)\big)\right\}^2 \\
&\quad = \gamma_i(0; \boldsymbol{x}, \theta) + 2\sum_{m=1}^{n-1}(1 - m/n)\gamma_i(m; \boldsymbol{x}, \theta) + n\left\{E\big(Z_{1,i}(\boldsymbol{x}, \theta) - \bar{Z}_{1,i}(\boldsymbol{x}, \theta)\big)\right\}^2 \\
&\quad = \mathcal{O}(h^2) + h^{2\xi}\mathcal{O}\left(\sum_{m=1}^{n-1}\{\alpha(m)\}^{1-\xi-2(1-\xi)/(2+\delta)}\right) + \mathcal{O}(nh^4) \\
&\quad = o(1)
\end{aligned}$$

uniformly in $\big\{(\boldsymbol{x}^T, \theta)^T : ||\boldsymbol{x} - \boldsymbol{x}_0||_2 + |\theta - \theta_0| \le n^{-1/3}\big\}$ for all $i = 1, \cdots, d+1$, implying that

$$\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\big\{\boldsymbol{Z}_t(\boldsymbol{x}, \theta) - \bar{\boldsymbol{Z}}_t(\boldsymbol{x}, \theta)\big\} = o_p(1) \quad \text{as } n \to \infty \tag{5.14}$$

uniformly in $\big\{(\boldsymbol{x}^T, \theta)^T : ||\boldsymbol{x} - \boldsymbol{x}_0||_2 + |\theta - \theta_0| \le n^{-1/3}\big\}$.

For any constant $\boldsymbol{\lambda} \in \Re^{d+1} \backslash \{\mathbf{0}\}$, it follows from C1) that $\big\{\boldsymbol{\lambda}^T \bar{\boldsymbol{Z}}_t\big(\boldsymbol{x}_0, \theta_0\big)\big\}$ is a strictly stationary α-mixing sequence with $\alpha_{\boldsymbol{\lambda}^T\bar{\boldsymbol{Z}}}(m) = \mathcal{O}(a^m)$ as $m \to \infty$. Hence, using the Central Limit Theorem for α-mixing sequence (e.g., see Rosenblatt (1956)), (5.5) in Theorem 5.1 holds and

$$\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\boldsymbol{\lambda}^T\bar{\boldsymbol{Z}}_t(\boldsymbol{x}_0, \theta_0) \xrightarrow{w} N\big(0, \boldsymbol{\lambda}^T\bar{\Sigma}\boldsymbol{\lambda}\big) \quad \text{as } n \to \infty.$$

Using the Cramér-Wold device, we have that

$$\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\bar{\boldsymbol{Z}}_t(\boldsymbol{x}_0, \theta_0) \xrightarrow{w} N(\mathbf{0}, \bar{\Sigma}) \quad \text{as } n \to \infty. \tag{5.15}$$

Decomposing

$$\begin{aligned}
&\frac{1}{n}\sum_{t=1}^{n}\left\{\bar{\boldsymbol{Z}}_t(\boldsymbol{x},\theta)-\bar{\boldsymbol{Z}}_t(\boldsymbol{x}_0,\theta_0)-\Gamma(\boldsymbol{x},\theta)+\Gamma(\boldsymbol{x}_0,\theta_0)\right\}\\
&\quad=\frac{1}{n}\sum_{t=1}^{n}\left\{\bar{\boldsymbol{Z}}_t(\boldsymbol{x},\theta)-\bar{\boldsymbol{Z}}_t(\boldsymbol{x}_0,\theta)-\Gamma(\boldsymbol{x},\theta)+\Gamma(\boldsymbol{x}_0,\theta)\right\}\\
&\qquad\quad+\frac{1}{n}\sum_{t=1}^{n}\left\{\bar{\boldsymbol{Z}}_t(\boldsymbol{x}_0,\theta)-\bar{\boldsymbol{Z}}_t(\boldsymbol{x}_0,\theta_0)-\Gamma(\boldsymbol{x}_0,\theta)+\Gamma(\boldsymbol{x}_0,\theta_0)\right\}\\
&\quad:=I_1+I_2,
\end{aligned}$$

similar to the proofs of Lemmas 1 and 2 in Chen (2008), one can show that

$$I_1=o_p\big(||\boldsymbol{x}-\boldsymbol{x}_0||_2+|\theta-\theta_0|\big)\quad\text{and}\quad I_2=o_p\big(||\boldsymbol{x}-\boldsymbol{x}_0||_2+|\theta-\theta_0|\big)\quad\text{as } n\to\infty$$

uniformly in $\big\{(\boldsymbol{x}^T,\theta)^T:||\boldsymbol{x}-\boldsymbol{x}_0||_2+|\theta-\theta_0|\le n^{-1/3}\big\}$. That is,

$$\frac{1}{n}\sum_{t=1}^{n}\left\{\bar{\boldsymbol{Z}}_t(\boldsymbol{x},\theta)-\bar{\boldsymbol{Z}}_t(\boldsymbol{x}_0,\theta_0)-\Gamma(\boldsymbol{x},\theta)+\Gamma(\boldsymbol{x}_0,\theta_0)\right\}=o_p(||\boldsymbol{x}-\boldsymbol{x}_0||_2+|\theta-\theta_0|)\tag{5.16}$$

as $n\to\infty$ uniformly in $\big\{(\boldsymbol{x}^T,\theta)^T:||\boldsymbol{x}-\boldsymbol{x}_0||_2+|\theta-\theta_0|\le n^{-1/3}\big\}$. Therefore, it follows from (5.8)–(5.16) that

$$\begin{aligned}
\mathbf{0}&=\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\boldsymbol{Z}_t(\hat{\boldsymbol{x}}_{cvar},\hat{\theta}_{cvar})\\
&=\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\bar{\boldsymbol{Z}}_t(\hat{\boldsymbol{x}}_{cvar},\hat{\theta}_{cvar})+o_p(1)\\
&=\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\bar{\boldsymbol{Z}}_t(\boldsymbol{x}_0,\theta_0)+\sqrt{n}\frac{1}{n}\sum_{t=1}^{n}\left\{\bar{\boldsymbol{Z}}_t(\hat{\boldsymbol{x}}_{cvar},\hat{\theta}_{cvar})-\bar{\boldsymbol{Z}}_t(\boldsymbol{x}_0,\theta_0)\right\}+o_p(1)\\
&=\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\bar{\boldsymbol{Z}}_t(\boldsymbol{x}_0,\theta_0)+\sqrt{n}\left\{\bar{\Gamma}(\hat{\boldsymbol{x}}_{cvar},\hat{\theta}_{cvar})-\bar{\Gamma}(\boldsymbol{x}_0,\theta_0)\right\}+o_p(1)\\
&=\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\bar{\boldsymbol{Z}}_t(\boldsymbol{x}_0,\theta_0)+\sqrt{n}\dot{\bar{\Gamma}}(\boldsymbol{x}_0,\theta_0)\big(\hat{\boldsymbol{x}}_{cvar}^T-\boldsymbol{x}_0^T,\hat{\theta}_{cvar}-\theta_0\big)^T+o_p(1),
\end{aligned}$$

which implies (5.6). Equation (5.7) follows from (5.6) and the fact that

$$\sqrt{n}(\hat{\boldsymbol{\alpha}}_{cvar}-\boldsymbol{\alpha}_0)=\sqrt{n}\frac{\hat{\boldsymbol{x}}_{cvar}-\boldsymbol{x}_0}{\mathbf{1}^T\boldsymbol{x}_0}-\frac{\boldsymbol{x}_0}{(\mathbf{1}^T\boldsymbol{x}_0)^2}\mathbf{1}^T\sqrt{n}(\hat{\boldsymbol{x}}_{cvar}-\boldsymbol{x}_0)+o_p(1).$$

■

Next, we study SD risk preferences, which is equivalent to studying variance risk preferences, and thus, we assume $\varphi=var$ from now on. Clearly, the portfolio risk position is measured by the following non-parametric estimator:

$$\widehat{\mathcal{R}}_v(\boldsymbol{x}) = \frac{1}{n}\sum_{t=1}^{n} \boldsymbol{x}^T \boldsymbol{X}_t \boldsymbol{X}_t^T \boldsymbol{x} - \left(\frac{1}{n}\sum_{t=1}^{n} \boldsymbol{x}^T \boldsymbol{X}_t\right)^2,$$

which is convex, homogeneous, and differentiable. Using $\tau = 2$ for the variance risk measure and taking $\lambda = 1$ in Theorem 4.1, we estimate $\boldsymbol{x}$ and $\boldsymbol{\alpha}$ by

$$\widehat{\boldsymbol{x}}_v = \underset{\boldsymbol{x}\in\Re^d_{++}}{\arg\min}\ \frac{1}{2}\widehat{\mathcal{R}}_v(\boldsymbol{x}) - \sum_{i=1}^{d} b_i \log x_i \quad \text{and} \quad \widehat{\boldsymbol{\alpha}}_v = \widehat{\boldsymbol{x}}_v / \mathbf{1}^T \widehat{\boldsymbol{x}}_v.$$

That is, $\hat{\boldsymbol{x}}_v$ and $\hat{\theta}_v = \theta(\hat{\boldsymbol{x}}_v)$ solve the system of equations for $\boldsymbol{x} > \mathbf{0}$:

$$\begin{cases} \frac{1}{n}\sum_{t=1}^{n} X_{t,i}\{\boldsymbol{x}^T \boldsymbol{X}_t - \theta\} - \frac{b_i}{x_i} = 0 \text{ for } i \in \{1,2,\ldots,d\}, \\ \frac{1}{n}\sum_{t=1}^{n} \boldsymbol{x}^T \boldsymbol{X}_t = \theta. \end{cases} \tag{5.17}$$

On the other hand, the true values $\boldsymbol{x}_0$ and $\theta_0 = \theta(\boldsymbol{x}_0)$ solve

$$\mathbb{E}[\tilde{\boldsymbol{Z}}_t(\boldsymbol{x},\theta)] = \mathbf{0} \quad \text{for } \boldsymbol{x} > \mathbf{0}, \tag{5.18}$$

where $\tilde{\boldsymbol{Z}}_t(\boldsymbol{x},\theta) = \left(\tilde{Z}_{t,1}(\boldsymbol{x},\theta), \ldots, \tilde{Z}_{t,d+1}(\boldsymbol{x},\theta)\right)^T$ is given by

$$\begin{cases} \tilde{Z}_{t,i}(\boldsymbol{x},\theta) = X_{t,i}\{\boldsymbol{x}^T \boldsymbol{X}_t - \theta\} - \frac{b_i}{x_i} \text{ for all } i = k \in \{1,2,\ldots,d\}, \\ \tilde{Z}_{t,d+1}(\boldsymbol{x},\theta) = \boldsymbol{x}^T \boldsymbol{X}_t. \end{cases}$$

Define $\tilde{\Gamma}(\boldsymbol{x},\theta) = E\tilde{\boldsymbol{Z}}_1(\boldsymbol{x},\theta)$ and denote the partial derivatives of $\tilde{\Gamma}(\cdot,\cdot)$ by $\dot{\tilde{\Gamma}}(\cdot,\cdot)$ on $\Re^d \times \Re$.

The following regularity conditions are required for deriving the asymptotic behavior of our estimators, namely $\hat{\boldsymbol{x}}_v$, $\hat{\theta}_v$, and $\hat{\boldsymbol{\alpha}}_v$. These conditions are formalized below:

- C5) $\{\boldsymbol{X}_t\}_{t=-\infty}^{\infty}$ is a strictly stationary α-mixing sequence with $\alpha_{\boldsymbol{X}}(m) = O(a^m)$ for some $a \in (0,1)$ as $m \to \infty$. Furthermore, assume $E||\boldsymbol{X}_t||_2^{4+\delta} < \infty$ for some $\delta > 0$.
- C6) $(\boldsymbol{x}_0^T, \theta_0)^T$ is the unique solution to (5.18).

The next theorem provides the result for deriving inference when risk preferences are ordered by the SD/variance risk measure.

Theorem 5.2. *Assume conditions C5) and C6) hold and consider SD/variance risk preferences, i.e., $\varphi = var$. Then, there is a positive definite matrix $\tilde{\Sigma}$ such that*

$$\mathbb{E}\left\{\tilde{\boldsymbol{Z}}_1(\boldsymbol{x}_0,\theta_0)\tilde{\boldsymbol{Z}}_1^T(\boldsymbol{x}_0,\theta_0)\right\} + 2\lim_{n\to\infty}\sum_{m=1}^{n-1}\mathbb{E}\left\{\tilde{\boldsymbol{Z}}_1(\boldsymbol{x}_0,\theta_0)\tilde{\boldsymbol{Z}}_{1+m}^T(\boldsymbol{x}_0,\theta_0)\right\} = \tilde{\Sigma}. \tag{5.19}$$

Furthermore, as $n \to \infty$,

$$\sqrt{n}\big(\hat{\boldsymbol{x}}_v^T - \boldsymbol{x}_0^T, \hat{\theta}_v - \theta_0\big)^T \xrightarrow{w} N\Big(\mathbf{0}, \dot{\tilde{\Gamma}}^{-1}(\boldsymbol{x}_0, \theta_0)\tilde{\Sigma}\big(\dot{\tilde{\Gamma}}^{-1}(\boldsymbol{x}_0, \theta_0)\big)^T\Big), \tag{5.20}$$

and

$$\sqrt{n}(\hat{\boldsymbol{\alpha}}_v - \boldsymbol{\alpha}_0) \xrightarrow{w} N\left(\mathbf{0}, \frac{\tilde{\Sigma}_0}{(\mathbf{1}^T\boldsymbol{x}_0)^2} - \frac{2\boldsymbol{x}_0\mathbf{1}^T\tilde{\Sigma}_0}{(\mathbf{1}^T\boldsymbol{x}_0)^3} + \frac{\boldsymbol{x}_0\mathbf{1}^T\tilde{\Sigma}_0\mathbf{1}\boldsymbol{x}_0^T}{(\mathbf{1}^T\boldsymbol{x}_0)^4}\right), \tag{5.21}$$

where $\tilde{\Sigma}_0$ is the first $d \times d$ matrix of $\dot{\tilde{\Gamma}}^{-1}(\boldsymbol{x}_0, \theta_0)\tilde{\Sigma}\big(\dot{\tilde{\Gamma}}^{-1}(\boldsymbol{x}_0, \theta_0)\big)^T$.

Proof. It follows from the same arguments after (5.14) in the proof of Theorem 5.1, and thus, no specific derivations are further required. ■

6. Real data analyses

Our empirical analysis focuses on the US equity market from 01/01/2000 to 31/12/2023, and we identify ten structural break periods for the S&P500 index, which are explained in Appendix B. We also collect historical daily stock returns (with adjustments for dividends) for all S&P500 constituents between year 2000 and 2023, from the Wharton Research Data Services according to the unique PERMNO code in the CRSP dataset. A total of $1,070$ companies have been part of S&P500 during these 24 years and we identify the exact dates when these companies entered and/or exited S&P500. Our numerical section includes two data analyses: i) a fixed opportunity set of 441 companies selected from the $1,070$ S&P500 constituents (that continue to exist over the 24-year period) for which portfolios are rebalanced at the beginning of each structural break; this data analysis is recalled as *DA441* from now on; ii) a dynamic opportunity set of almost[2] 500 companies that consists of the full set of S&P500 constituents (that changes quarterly) for which portfolios are rebalanced at the beginning of each quarter; this data analysis is recalled as *DA500* from now on. We choose *DA500* to test the impact of survivorship bias in *DA441*, but the two data analyses show consistent results and both illustrate good performances of the RP-like portfolios. Section 6.1 provides a description of our long-only portfolios considered in *DA441* and *DA500*, while Section 6.2 compares the performance of these portfolios.

6.1. *Long-only Portfolios Description*

We investigate six long-only portfolios in our real-data analyses. The first one is EW that became a standard benchmark portfolio since the seminal paper of (DeMiguel et al., 2009b); since we focus on US stocks, the S&P500 benchmark index is included in our comparisons as well. The second and third portfolios are standard RP portfolios (namely, RP-SD and RP-$\mathrm{CVaR}_{95\%}$), while the fourth and fifth portfolios (namely,

[2]Rarely, there are fewer than 500 companies in S&P500 constituents at the beginning of a quarter.

IWP-SD and IWP-$\mathrm{CVaR}_{95\%}$) are discussed in Section 3.3. Further details about the six portfolios are given below:

1. *EW* portfolio (also known as $1/N$) with weights $\frac{1}{d}\mathbf{1}$;
2. *RP-SD*, an RP portfolio with $\varphi = SD$ by solving (4.2) with $\mathcal{R} = \widehat{\mathcal{R}}_v$;
3. RP-$\mathrm{CVaR}_{95\%}$, an RP portfolio with $\varphi = \mathrm{CVaR}_{95\%}$ by solving (4.3) with $\mathcal{R} = \widehat{\mathcal{R}}_{cvar}$;
4. IWP-SD, an *inverse weighted* portfolio as in (3.12) with $\varphi = SD$;
5. IWP-$\mathrm{CVaR}_{95\%}$, an *inverse weighted* portfolio as in (3.12) with $\varphi = \mathrm{CVaR}_{95\%}$, where the assets' $\mathrm{CVaR}_{95\%}$ are estimated via our $\widehat{\mathcal{R}}_{cvar}$ estimator;
6. *S&P500*.

Note that only Portfolios 2 and 3 require bespoke algorithms, which are briefly explained. Specific numerical methods are required for finding the RP-SD and RP-$\mathrm{CVaR}_{95\%}$ portfolios. Recall that Theorem 4.1 provides two general methods – *logarithmic barrier* and *logarithmic constraint* – that could be applied to finding RB/RP portfolios. The logarithmic barrier formulation is used for SD risk preferences; Spinu (2013) is the first reference to show that RB-SD portfolios could be found via an efficient convex algorithm and it is implemented in the *riskParityPortfolio* R package that we rely on in our implementations. The logarithmic constraint formulation is useful to efficiently find RP-CVaR portfolios, while RB-CVaR (that are not RP-CVaR, i.e., $\mathbf{b} \neq \frac{1}{d}\mathbf{1}$) could be found only via general convex programming algorithms. RP-CVaR involves a hyperbolic constraint such as $-\frac{1}{d}\sum_{k=1}^{d}\log(\alpha_k) \leq 0$, which is second-order cone representable; thus, we compute the RP-$\mathrm{CVaR}_{95\%}$ via (4.3) with $c = 0$ and $\mathcal{R} = \widehat{\mathcal{R}}_{cvar}$ through the efficient SOCP implementation described in Mausser and Romanko (2018). Note that the multiple integrals in the $\widehat{\mathcal{R}}_{cvar}$ estimator are approximated via the Monte-Carlo method; further, the Epanechnikov kernel function and a bandwidth choice of $h_k = 0.2n^{-1/3}$ for all $k \in \{1, 2, \ldots, d\}$ is used for $\widehat{\mathcal{R}}_{cvar}$ estimations.

We now describe the rebalancing details for our portfolios except for S&P500 that is quarterly rebalanced by construction. DA441 has the same opportunity set over all ten periods and the five portfolios (EW, RP-SD, RP-$\mathrm{CVaR}_{95\%}$, IWP-SD and IWP-$\mathrm{CVaR}_{95\%}$) are rebalanced at the beginning of each structural break period. The portfolios' initial weights for each period are calculated by considering the historical stock returns from 01/01/2000 up to the day just before the first day of the considered period.

DA500 mimics the dynamic process of S&P500 and the three portfolios (EW, RP-SD and IWP-SD) are reset at the beginning of each quarter from 2002 to 2023 (88 quarters). The three portfolios (in a given quarter) consist of the S&P500 constituents from the first day of that quarter. The portfolios' initial weights are usually calculated by considering the historical stock returns from 01/01/2000 up to the day just before the first day of the considered quarter. Note that some S&P500 constituents (newly entries on NYSE or NASDAQ stock exchanges) have shorter periods of historical data than most of the S&P500 constituents (well-established firms), and these unequal sample sizes may lead to an ill conditioned sample covariance matrix estimate that may not even be positive definite if one uses the full information to estimate the pairwise covariance estimates based on the overlapping available data. We have not had such an issue in any of the 88 quarters, otherwise we would have used the covariance matrix shrinkage estimator (Ledoit and Wolf, 2004). RP-SD and IWP-SD have been implemented this way, since IWP-SD requires only the information from the main diagonal of the sample covariance matrix estimate. The main difference between DA500 and DA441 is that companies may be delisted in between two rebalancing points since S&P500 constituents could be delisted from the stock exchange during a quarter[3] and thus, we adjust the three portfolios accordingly. Specifically, we hold the investment part corresponding to a delisted company as cash (without earning any interest) from the trading day the company is delisted until the end of the quarter, i.e., the daily returns are zero. Thus, EW, RP-SD and IWP-SD account for delisted firms in this fashion. We could not compute RP-CVaR since such calculations require complete data and the only option would have been to consider the overlapping observation period for all 500 firms to compute the RP-CVaR initial weights. However, that would lead to results with low power as some new entries to S&P500 may be new entries on NYSE or NASDAQ stock exchanges with an observation period as short as one month. Therefore, RP-$\mathrm{CVaR}_{95\%}$ and IWP-$\mathrm{CVaR}_{95\%}$ portfolios are not included in DA500.

6.2. Portfolio performance comparisons

This section provides the DA441 and DA500 portfolio performances. Multiple performance measures are reported in Tables 3 and 4 for *DA441* and *DA500*, respectively; we compute the mean and SD return, *Sharpe ratio (SR)*, *skew-Adjusted Sharpe ratio (skew-Adj SR)* and *Calmar ratio* performance measures. Note that skew-Adj SR incorporates a penalty factor for negative skewness, while the Calmar ratio is defined as the ratio of annualized return over the absolute value of the maximum drawdown of an investment computed over each structural break period. A *diversification index (DI)* when the risk preferences are ordered by SD, $SD(\boldsymbol{\alpha}^T\mathbf{X})/\sum_{k=1}^{d}\alpha_k SD(X_k)$, is computed in Table 3. DI-SD is not tabulated in Table 4 since its average (over all quarters in a given period) would not be informative measure of performance; note that the lower (but positive) the DI-SD value is, the more diversified portfolio is.

The S&P500 daily return dynamic displayed in Figure B.2 and the summary S&P500 statistics (see Tables 3 or 4) show that the ten periods could be grouped as follows: i) stable market conditions (Periods 2, 6 and 7), ii) moderately volatile market condi-

[3]On average, less than three (eleven) firms are delisted per quarter (year) during the period 2002–2023; the fourth quarter of 2007 is an outlier with 10 firms being delisted at that time, and year 2007 is an outlier as well with 27 firms being delisted in that year.

tions (Periods 5, 8 and 10) and iii) turbulent market conditions (Periods 1, 3, 4 and 9). Our main findings are summarized in Tables 3 and 4, and their interpretations are discussed across the three groups of market conditions. Figure 1 is a granular like-for-like comparison of DA441 and DA500 in terms of SRs that helps to conclude that the overall trends on the two data analyses are very similar, and thus, our conclusions in DA441 are not affected by a possible survivorship bias that one may infer. To test the statistical significance of whether two portfolios are different (in terms of SR), we compute two-sided p-values with the circular-bootstrapping methods (Ledoit and Wolf, 2008); these computations rely on the R package *PeerPerformance*, i.e., function *sharpeTesting* with $nBoot = 500$ bootstrap resamples and a block length in the circular bootstrap of $bBoot = 5$.

Under turbulent market conditions – Periods 1, 3, 4 and 9 – Tables 3 and 4 show that EW has a slightly better performance in crises periods (Periods 4 and 9) in terms of expected returns than the other portfolios, which is not the case for all other performance measures; RP portfolios have a slightly better performance than EW in periods with very poor market performance (Periods 1 and 3) and RP strategies are shown to be very effective to reduce the overall loss or to even make a marginal profit while all other portfolios are loss-making. Further, the SR tests could not differentiate between EW and RP portfolios (by means of large p-values) in any of these four periods, which is also confirmed by Figure 1.

The mirror extreme case is under stable market conditions, which is seen in Periods 2, 6 and 7. We should note that Period 7 includes the US market dive in 2018; further, Period 7 was affected by the slowdown in global economic growth that was also signified by the historical low crude oil prices, which makes this period atypical and different than the other two. We note that RP-SD and RP-$\mathrm{CVaR}_{95\%}$ are well diversified and show a slightly better performance than their IWP equivalent, but outperform EW and S&P500 in Periods 2 and 6, while EW and RP-SD are statistically indistinguishable in the atypical Period 7 which is confirmed by Figure 1. Further, the SR tests comparing EW and RP-SD show small p-values for DA500 ($p = 0.024$ and $p = 0.01$ for Periods 2 and 6) and not as conclusive for DA441 ($p = 0.09$ for the first part of Period 2 – from 22/04/2003 until 11/01/2005 – and $p = 0.04$ for Period 6).

Under moderately volatile market conditions seen in Periods 5, 8 and 10, the conclusions are in line with the previous trends. Period 10 is affected by high inflation that triggers a stock market decline, and while EW shows a slightly better performance than the other portfolios, the SR tests conclude no statistical evidence to differentiate any of the six portfolios; this matches our findings for periods under turbulent market conditions, which is sensible given the traits of Period 10. The global economic factors were more favorable in Period 5 than Period 8, which explains why RP and IWP outperform EW in these two periods, though EW and RP-SD are statistically distin-

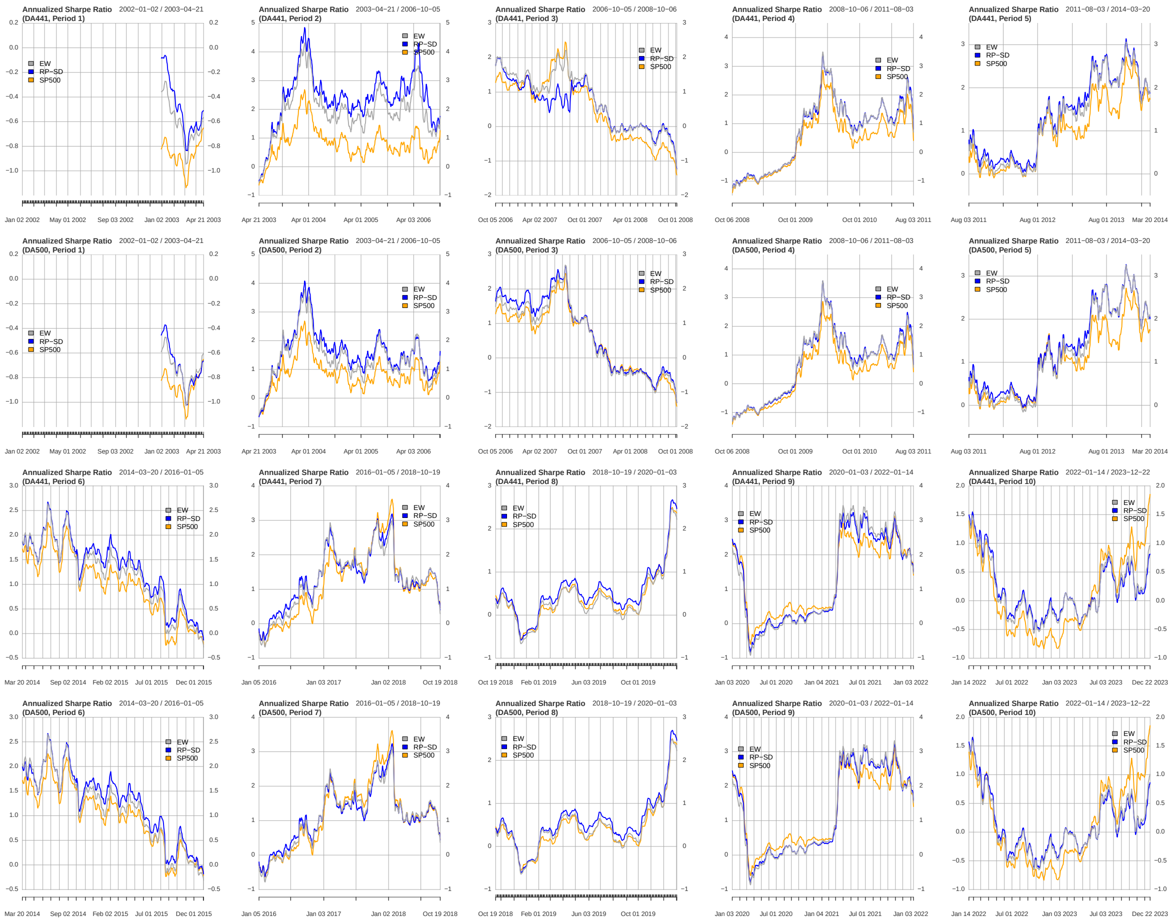

Figure 1: Smoothed annualized SRs (weekly averages) for DA441(first and third rows) and DA500 (second and fourth rows) are illustrated for EW, RP-SD and S&P500 over the ten structural break periods.

guishable only in DA500 for Period 5 ($p = 0.05$).

We conclude this section by recalling Theorem 4.1 c) where it is shown that RP-SD has a lower volatility than EW. This property is guaranteed to hold only for in-sample comparisons, but DA441 and DA500 show that this property holds in all settings for any out-of-sample comparisons except for DA441 for Period 2 though RP-SD has a higher SR than EW in that particular instance.

7. Conclusions

This paper discusses many aspects of RB/RP portfolios. We start by redefining them and argue why the new definition has more practical value than the classical definition. Based on that novel definition, we show the existence and uniqueness of long-only RB/RP portfolios under the least restrictive conditions possible that exist in the literature and replicate the same results for long-short RB/RP portfolios, which have not been previously attempted. We found that RB/RP are always less risky than the equivalent GEW (that is the same as EW for long-only portfolio setting). We coined a new very large class of portfolios and we named it as GWMC, and it is shown that

Table 3: Various performance measures in *DA441*

Structural Break	Portfolio	Mean	SD	SD-SR	skew-Adj SR	Calmar Ratio	DI-SD
Period 1	EW	-3.87%	0.2190	-0.1769	-0.2713	-0.2127	0.5068
01/01/2002	RP-SD	**0.51%**	0.1773	**0.0286**	**-0.0591**	**-0.0448**	0.4630
-	RP-CVaR$_{95\%}$	0.45%	**0.1746**	0.0255	-0.0609	-0.0448	**0.4551**
21/04/2003	IWP-SD	-1.80%	0.1976	-0.0908	-0.1833	-0.1445	0.5194
	IWP-CVaR$_{95\%}$	-1.85%	0.1982	-0.0935	-0.1861	-0.1474	0.5190
	S&P500	-16.24%	0.2538	-0.6398	-0.6520	-0.5236	NA
Period 2	EW	27.90%	0.1436	1.9433	2.0291	2.2894	0.5466
22/04/2003	RP-SD	**31.36%**	0.1544	2.0312	0.5957	2.2050	0.6223
-	RP-CVaR$_{95\%}$	30.14%	0.1478	**2.0387**	1.2813	2.3697	0.5938
05/10/2006	IWP-SD	25.85%	0.1303	1.9836	**2.0498**	**2.4075**	0.5445
	IWP-CVaR$_{95\%}$	25.73%	0.1304	1.9730	2.0426	2.3945	**0.5432**
	S&P500	12.68%	**0.1139**	1.1132	1.1165	1.5657	NA
Period 3	EW	-4.69%	0.2096	-0.2238	-0.3251	-0.2338	0.5803
06/10/2006	RP-SD	-3.46%	**0.1962**	-0.1766	-0.2732	-0.2065	**0.5711**
-	RP-CVaR$_{95\%}$	**-3.41%**	0.1993	**-0.1710**	**-0.2690**	**-0.2047**	0.5730
06/10/2008	IWP-SD	-4.39%	0.2056	-0.2136	-0.3124	-0.2368	0.5876
	IWP-CVaR$_{95\%}$	-4.38%	0.2062	-0.2124	-0.3116	-0.2363	0.5882
	S&P500	-10.26%	0.2052	-0.5002	-0.5628	-0.3587	NA
Period 4	EW	**19.46%**	0.3187	0.6105	0.4579	0.4044	0.6658
07/10/2008	RP-SD	17.84%	0.2938	0.6071	0.4617	0.4023	**0.6590**
-	RP-CVaR$_{95\%}$	18.12%	0.2946	**0.6152**	**0.4697**	**0.4092**	0.6612
03/08/2011	IWP-SD	17.79%	0.3077	0.5783	0.4302	0.3644	0.6689
	IWP-CVaR$_{95\%}$	17.83%	0.3075	0.5799	0.4318	0.3664	0.6696
	S&P500	10.27%	**0.2846**	0.3609	0.2237	0.1786	NA
Period 5	EW	**21.94%**	0.1912	1.1476	**0.6989**	1.4359	0.6726
04/08/2011	RP-SD	21.11%	0.1734	1.2175	0.6743	1.7233	0.6610
-	RP-CVaR$_{95\%}$	21.23%	0.1735	**1.2238**	0.6768	**1.7414**	**0.6603**
20/03/2014	IWP-SD	21.40%	0.1787	1.1976	0.6820	1.6301	0.6762
	IWP-CVaR$_{95\%}$	21.43%	0.1789	1.1984	0.6835	1.6310	0.6757
	S&P500	16.54%	**0.1692**	0.9776	0.6907	1.2757	NA
Period 6	EW	5.99%	0.1347	0.4448	0.3752	0.4560	0.5588
21/03/2014	RP-SD	7.60%	**0.1279**	0.5942	0.5188	0.6827	**0.5575**
-	RP-CVaR$_{95\%}$	**7.71%**	0.1281	**0.6019**	**0.5262**	**0.6951**	0.5586
05/01/2016	IWP-SD	7.15%	0.1296	0.5522	0.4798	0.6188	0.5691
	IWP-CVaR$_{95\%}$	7.12%	0.1297	0.5487	0.4765	0.6153	0.5688
	S&P500	5.10%	0.1375	0.3707	0.3020	0.3431	NA
Period 7	EW	**14.96%**	0.1273	1.1755	**0.8052**	**1.4977**	0.4994
06/01/2016	RP-SD	14.14%	**0.1145**	1.2343	0.7968	1.4652	**0.4747**
-	RP-CVaR$_{95\%}$	14.29%	0.1149	**1.2433**	0.7966	1.4827	0.4757
19/10/2018	IWP-SD	14.15%	0.1176	1.2025	0.7857	1.4457	0.4911
	IWP-CVaR$_{95\%}$	14.15%	0.1178	1.2011	0.7858	1.4451	0.4909
	S&P500	12.03%	0.1174	1.0243	0.6628	1.1811	NA
Period 8	EW	14.64%	0.1464	1.0004	0.8278	0.8801	0.5150
20/10/2018	RP-SD	14.87%	**0.1331**	1.1178	0.8781	0.9835	**0.4902**
-	RP-CVaR$_{95\%}$	15.06%	0.1336	**1.1274**	**0.8830**	**0.9966**	0.4918
02/01/2020	IWP-SD	15.05%	0.1363	1.1037	0.8716	0.9820	0.5080
	IWP-CVaR$_{95\%}$	**15.06%**	0.1365	1.1031	0.8719	0.9820	0.5078
	S&P500	14.80%	0.1517	0.9754	0.7938	0.8892	NA
Period 9	EW	**22.53%**	0.2851	0.7901	**0.4854**	0.5042	0.6323
03/01/2020	RP-SD	20.45%	0.2646	0.7730	0.4761	0.4880	0.6247
-	RP-CVaR$_{95\%}$	20.56%	0.2635	0.7803	0.4777	0.4954	**0.6236**
14/01/2022	IWP-SD	20.76%	0.2729	0.7606	0.4749	0.4771	0.6385
	IWP-CVaR$_{95\%}$	20.84%	0.2733	0.7628	0.4748	0.4787	0.6383
	S&P500	20.96%	**0.2588**	**0.8097**	0.4469	**0.5665**	NA
Period 10	EW	**4.01%**	0.1894	**0.2116**	**0.1182**	**0.1130**	0.5636
15/01/2022	RP-SD	3.30%	**0.1731**	0.1909	0.1052	0.0979	0.5471
-	RP-CVaR$_{95\%}$	3.31%	0.1734	0.1912	0.1054	0.0978	**0.5467**
31/12/2023	IWP-SD	3.53%	0.1772	0.1992	0.1117	0.1070	0.5612
	IWP-CVaR$_{95\%}$	3.56%	0.1776	0.2006	0.1129	0.1081	0.5609
	S&P500	3.08%	0.1958	0.1572	0.0597	0.0503	NA

Portfolio performance is illustrated for *DA441* (within each period and performance criterion), where the "**best**" portfolio is in bold and underlined. (Column 1) includes the ten periods identified by the Bai-Perron test. (Column 2) shows the considered portfolios. The remaining columns include some specific portfolio performance measures: (Column 3) Annualized Mean (Return); (Column 4) Annualized SD; (Column 5) Annualized Sharpe Ratio (SR) based on SD; (Column 6) Annualized skew-adjusted SR, which adjusts for negative skewness; (Column 7) Calmar Ratio; (Column 8) DI based on annualized SD estimates.

Table 4: Various performance measures in *DA500*

Structural Break	Portfolio	Mean	SD	SD-SR	skew-Adj SR	Calmar Ratio
Period 1	EW	-10.73%	0.2613	-0.4106	-0.4844	-0.3603
01/01/2002	RP-SD	**-7.83%**	**0.2178**	-0.3594	**-0.4300**	**-0.3155**
-	IWP-SD	-8.37%	0.2366	**-0.3539**	-0.4306	-0.3313
21/04/2003	S&P500	-16.24%	0.2538	-0.6398	-0.6520	-0.5236
Period 2	EW	20.32%	0.1255	1.6201	1.6769	2.5035
22/04/2003	RP-SD	**20.45%**	**0.1120**	**1.8248**	**1.9003**	**3.1634**
-	IWP-SD	19.77%	0.1154	1.7132	1.7852	2.9527
05/10/2006	S&P500	12.68%	0.1139	1.1132	1.1165	1.5657
Period 3	EW	-9.03%	0.2215	-0.4078	-0.4934	-0.3283
06/10/2006	RP-SD	**-7.09%**	**0.2040**	**-0.3475**	**-0.4362**	**-0.2995**
-	IWP-SD	-8.05%	0.2159	-0.3729	-0.4600	-0.3188
06/10/2008	S&P500	-10.26%	0.2052	-0.5002	-0.5628	-0.3587
Period 4	EW	**19.06%**	0.3225	**0.5911**	0.4393	**0.3734**
07/10/2008	RP-SD	16.96%	0.2912	0.5822	**0.4404**	0.3650
-	IWP-SD	17.09%	0.3032	0.5636	0.4181	0.3392
03/08/2011	S&P500	10.27%	**0.2846**	0.3609	0.2237	0.1786
Period 5	EW	**20.23%**	0.1865	1.0848	**0.7107**	1.6169
04/08/2011	RP-SD	19.90%	0.1698	**1.1718**	0.6978	**1.7967**
-	IWP-SD	19.87%	0.1745	1.1388	0.6963	1.7279
20/03/2014	S&P500	16.54%	**0.1692**	0.9776	0.6907	1.2757
Period 6	EW	6.03%	0.1373	0.4389	0.3694	0.4160
21/03/2014	RP-SD	**7.87%**	**0.1310**	**0.6009**	**0.5279**	**0.6352**
-	IWP-SD	7.30%	0.1320	0.5530	0.4826	0.5705
05/01/2016	S&P500	5.10%	0.1375	0.3707	0.3020	0.3431
Period 7	EW	**12.13%**	0.1220	0.9940	0.7160	1.1749
06/01/2016	RP-SD	11.62%	**0.1133**	1.0263	**0.7344**	1.1414
-	IWP-SD	11.88%	0.1148	**1.0344**	0.7275	1.1741
19/10/2018	S&P500	12.03%	0.1174	1.0243	0.6628	**1.1811**
Period 8	EW	14.45%	0.1474	0.9804	0.8186	0.8991
20/10/2018	RP-SD	14.69%	**0.1367**	1.0747	0.8618	0.9744
-	IWP-SD	**14.96%**	0.1384	**1.0808**	**0.8673**	**0.9982**
02/01/2020	S&P500	14.80%	0.1517	0.9754	0.7938	0.8892
Period 9	EW	**21.99%**	0.2855	0.7702	**0.4894**	0.5032
03/01/2020	RP-SD	19.75%	0.2674	0.7387	0.4696	0.4680
-	IWP-SD	20.65%	0.2759	0.7488	0.4801	0.4794
14/01/2022	S&P500	20.96%	**0.2588**	**0.8097**	0.4469	**0.5665**
Period 10	EW	2.91%	0.1934	0.1506	0.0544	0.0549
15/01/2022	RP-SD	2.40%	**0.1788**	0.1340	0.0449	0.0459
-	IWP-SD	2.67%	0.1823	0.1463	0.0556	**0.0565**
31/12/2023	S&P500	**3.08%**	0.1958	**0.1572**	**0.0597**	0.0503

Portfolio performance is illustrated for *DA500* (within each period and performance criterion), where the "**best"** portfolio is in bold and underlined. (Column 1) includes the ten periods identified by the Bai-Perron test. (Column 2) shows the considered portfolios. The remaining columns include some specific portfolio performance measures: (Column 3) Annualized Mean (Return); (Column 4) Annualized SD; (Column 5) Annualized Sharpe Ratio (SR) based on SD; (Column 6) Annualized skew-adjusted SR, which adjusts for negative skewness; (Column 7) Calmar Ratio.

EW, RB/RP, norm constrained and shortsale-constrained portfolios are elements of the GWMC set; these special cases had shown good out-of-sample performance, and our numerical analyses have confirmed that RP portfolios could better balance the trade-off between risk and return.

Our empirical evidence has shown that RP outperforms EW under favorable market conditions and it is no worse than EW otherwise; these comparisons are made in terms of SR, portfolio volatility and other performance measures. RP also reduces the portfolio losses under extremely unfavorable market conditions. This means that RP portfolios are not only risk conservative strategies, but also have a good trade-off between risk and return that pays off during adverse and booming market conditions. Statistical inferences for RB portfolios with time dependent data are also employed for CVaR and SD risk preferences, which is the first attempt in the existing literature; a by-product of our work is the introduction of a novel CVaR non-parametric estimator. Finally, we believe that further research is needed to understand the GWMC set of portfolios and its current mathematical characterization is briefly discussed as this is another by-product of the research questions raised in this paper.

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Appendix A. RB for elliptically distributed risks

Appendix A.1. Some theoretical results

Long-only and long-short RB portfolios are investigated by assuming a specific parametric distribution of portfolio loss (not returns) $\mathbf{X}$, namely the elliptical family due to its tractability of aggregating the risks (McNeil et al., 2015). The elliptical class includes multivariate Gaussian and multivariate t- families of distributions.

We work with a multivariate random vector $\mathbf{X}$ that is elliptically distributed. This is signified by $\mathbf{X} \sim E_d(\boldsymbol{\mu}, \Sigma, \psi)$, where $\boldsymbol{\mu}$ is the location vector, Σ is the covariance matrix, and ψ is its generator. This means that $\mathbf{X}$ and $\boldsymbol{\mu} + A\mathbf{Z}$ have the same joint distribution, where $A \in \Re^{d\times k}$ such that $\Sigma = AA^T$, and $\mathbf{Z}$ is a k-dimensional spherical random vector with generator ψ, i.e. $\mathrm{E}\big(\exp\{i\mathbf{t}^T\mathbf{Z}\}\big) = \psi(\mathbf{t}^T\mathbf{t})$ for all $\mathbf{t} \in \Re^k$ (McNeil et al., 2015). Without loss of generality, we assume that all variances are finite, and in turn, the elliptical distribution is precisely determined by the triplet $(\boldsymbol{\mu}, \Sigma, \psi)$.

Proposition Appendix A.1 provides an extension of Theorem 8.28 in McNeil et al. (2015) that determines closed-form risk measurements for elliptically distributed risks.

Proposition Appendix A.1. *Let $\mathbf{X} \sim E_d(\boldsymbol{\mu}, \Sigma, \psi)$. If φ is a homogeneous risk measure of order $\tau > 0$ that is shift invariant, then*

$$\varphi(\boldsymbol{\alpha}^T\mathbf{X} + c) = \big(\boldsymbol{\alpha}^T\Sigma\boldsymbol{\alpha}\big)^{\tau/2}\varphi(Z_1) \quad \textit{for any} \quad c \in \Re, \tag{A.1}$$

and if φ is a homogeneous risk measure of order $\tau > 0$ that is translation invariant, then $\tau = 1$ and

$$\varphi(\boldsymbol{\alpha}^T\mathbf{X} + c) = c + \boldsymbol{\alpha}^T\boldsymbol{\mu} + \left(\boldsymbol{\alpha}^T\Sigma\boldsymbol{\alpha}\right)^{1/2}\varphi(Z_1) \quad \textit{for any} \quad c \in \Re, \tag{A.2}$$

where Z_1 is a spherical random variable with generator ψ.

Proof. Theorem 8.28 (1) of McNeil et al. (2015) gives that $\boldsymbol{\alpha}^T\mathbf{X}$ and $||\boldsymbol{\alpha}^T\Sigma\boldsymbol{\alpha}||_2 Z_1 + \boldsymbol{\alpha}^T\boldsymbol{\mu}$ have the same distribution, which immediately justifies (A.1) if φ is shift invariant. The other case, when φ is translation invariant, is also true if $\tau = 1$. The latter follows from the fact that

$$\varphi(tY + tc) = t^\tau\varphi(Y + c) = t^\tau(\varphi(Y) + c) \text{ and } \varphi(tY + tc) = \varphi(tY) + tc = t^\tau\varphi(Y) + tc$$

hold for any $t > 0$ and $c \in \Re$, which in turn implies that $\tau = 1$. ∎

If $\mathbf{X} \sim E_d\left(\boldsymbol{\mu}, \Sigma, \psi\right)$ with φ being a shift invariant and homogeneous risk measure of order $\tau > 0$ such that $\varphi(Z_1) \neq 0$, then (A.1) implies that finding RB portfolio strategies relative to φ is equivalent to solving in $\boldsymbol{\alpha} \in \Delta_d$

$$\alpha_k \sum_{i=1}^{d} \alpha_i \Sigma_{ik} = b_k \boldsymbol{\alpha}^T\Sigma\boldsymbol{\alpha} \quad \text{for all} \quad k \in \{1, 2, \ldots, d\}, \tag{A.3}$$

for any given $\mathbf{b} \in \Delta_d^{++}$, where Σ_{ik} represents the $(i,k)^{th}$ entry of Σ. Equation (A.3) tells us that all shift invariant and homogeneous risk measures of order $\tau > 0$ lead to the same set of RB portfolio strategies for a fixed $\mathbf{b} \in \Delta_d^{++}$ if the assets in a short position are pre-specified. If φ is a translation invariant and homogeneous risk measure, then the latter conclusion holds under the condition that the aggregated risk position does not change. These are summarized in Corollary Appendix A.2 below.

Corollary Appendix A.2. *Let $\boldsymbol{X} \sim E_d\left(\boldsymbol{\mu}, \Sigma, \psi\right)$ and $\boldsymbol{b} \in \Delta_d^{++}$. Further, let φ and $\tilde{\varphi}$ be two homogeneous risk measures of order $\tau > 0$ and $\tilde{\tau} > 0$, respectively such that $\varphi(Z_1) \neq 0$ and $\tilde{\varphi}(Z_1) \neq 0$.*

- *a) Assume that φ and $\tilde{\varphi}$ are shift invariant risk measures. If $\boldsymbol{\alpha}^* \in \mathcal{RB}(\boldsymbol{b}, \varphi)$, then $\boldsymbol{\alpha}^* \in \mathcal{RB}(\boldsymbol{b}, \tilde{\varphi})$.*
- *b) Assume that φ and $\tilde{\varphi}$ are translation invariant risk measures such that $\tau = \tilde{\tau} = 1$ and $\varphi(Z_1) = \tilde{\varphi}(Z_1)$. If $\boldsymbol{\alpha}^* \in \mathcal{RB}(\boldsymbol{b}, \varphi)$, then $\boldsymbol{\alpha}^* \in \mathcal{RB}(\boldsymbol{b}, \tilde{\varphi})$.*

Corollary Appendix A.2 shows that the set of RB portfolios is invariant with respect to the choice of risk measure whenever the loss returns are jointly elliptically distributed. For example, the set of RB portfolios based on either variance, standard deviation, *skewness (skew)*, *kurtosis (kurt)*, excess risk (measured by either MS/VaR or CVaR) over the expected return, or any other risk measure that is a function of centred moments would lead to the same RB set, i.e., if

$$\varphi \in \{var, SD, skew, kurt, \mathrm{VaR}_p - \mathbb{E}, \mathrm{CVaR}_q - \mathbb{E}\} \quad \text{for any } 0 < p, q < 1. \tag{A.4}$$

This invariance result shows that RB/RP portfolios balance the risk across individual assets in the same way across all shift invariant and homogeneous risk preferences. The same result follows when the risk preferences are modelled by translation invariant and homogeneous risk measures of order $\tau = 1$ provided that the aggregate risk for these RB portfolios are equal. A similar result was shown in Asimit et al. (2019) in the context of capital allocation, where VaR and CVaR based capital allocations are found to be equivalent if the same total amount of capital ought to be allocated.

Appendix A.2. Some examples

The idealized elliptical assumption in Corollary Appendix A.2 is a good illustration of the inherit properties of RB/RP portfolios. The next example given as Example Appendix A.3 shows that Corollary Appendix A.2 does not hold if the elliptical assumption is removed.

Example Appendix A.3. *Assume that $d = 2$, $X_2 = X_1^3$ almost surely and $X_1 \sim N(0,1)$, i.e. X_1 and X_2 are* comonotonic*; by definition, X_1 and X_2 are* comonotonic *if there exists a non-decreasing function f such that $\Pr(X_2 = f(X_1)) = 1$. By construction, (X_1, X_2) is not elliptically distributed.*

We now discuss the long-only portfolios for Example Appendix A.3 if the budgeting target vector is $\mathbf{b} = (b, 1-b)$ with $0 < b < 1$, then the RB-SD long-only portfolio, denoted as $\boldsymbol{\alpha}^{*SD}(\mathbf{b})$, is the solution of

$$x_1^2 + 3x_1x_2 = b\big(x_1^2 + 6x_1x_2 + 15x_2^2\big) \text{ such that } \mathbf{x} \in \Delta_2^{++},$$

since $cov(X_1, X_2) = 3$ and $var(X_2) = 15$. Thus,

$$\alpha_1^{*SD}(\mathbf{b}) = \frac{24b + 3 - \sqrt{-24b^2 + 24b + 9}}{20b + 4} \quad \text{and} \quad \alpha_2^{*SD}(\mathbf{b}) = 1 - \alpha_1^{*SD}(\mathbf{b}),$$

for any $0 < b < 1$. Particularly, the unique RP-SD long-only portfolio is achieved with $\alpha_1^{*SD}\big(\frac{1}{2}\mathbf{1}\big) = 0.7948$.

Since $\varphi \in \{\mathrm{VaR}, \mathrm{CVaR}\}$ are comonotonic additive risk measures, we have that $\varphi(X_1 + X_2) = \varphi(X_1) + \varphi(X_2)$ for any comonotonic risks X_1 and X_2 (Föllmer and Schied, 2011). Thus, the RB-$\mathrm{CVaR}_{95\%}$ long-only portfolio, denoted as $\boldsymbol{\alpha}^{*\mathrm{CVaR}_{95\%}}(\mathbf{b})$, is the solution of

$$x_1\mathrm{CVaR}_{95\%}(X_1) = b\Big(x_1\mathrm{CVaR}_{95\%}(X_1) + x_2\mathrm{CVaR}_{95\%}(X_2)\Big) \text{ such that } \mathbf{x} \in \Delta_2^{++}.$$

Particularly, the unique RP-$\mathrm{CVaR}_{95\%}$ long-only portfolio is achieved with $\alpha_1^{*\mathrm{CVaR}_{95\%}}\left(\frac{1}{2}\mathbf{1}\right) = 0.8247$, since $\mathrm{CVaR}_{95\%}(X_1) = \psi\big(\mathrm{VaR}_{95\%}(X_1)\big)$ and

$$\mathrm{CVaR}_{95\%}(X_2) = \frac{1}{0.05}\int_{0.95}^{1}\big(\mathrm{VaR}_s(X_1)\big)^3\,ds = \psi\big(\mathrm{VaR}_{95\%}(X_1)\big)\Big(2 + \big(\mathrm{VaR}_{95\%}(X_1)\big)^2\Big),$$

where $\psi(\cdot) := \frac{1}{0.05\sqrt{2\pi}}e^{-\frac{\cdot^2}{2}}$ on $\Re$. Similarly, one may find that the unique RP-$\mathrm{VaR}_{95\%}$ long-only portfolio, denoted as $\boldsymbol{\alpha}^{*\mathrm{VaR}_{95\%}}\left(\frac{1}{2}\mathbf{1}\right)$, is achieved with $\alpha_1^{*\mathrm{VaR}_{95\%}}\left(\frac{1}{2}\mathbf{1}\right) = 0.7301$. Since $\mathbb{E}[X_1] = \mathbb{E}[X_2] = 0$, then $\boldsymbol{\alpha}^{*\mathrm{CVaR}_{95\%}}\left(\frac{1}{2}\mathbf{1}\right) \in \mathcal{RB}\big(\frac{1}{2}\mathbf{1}, \mathrm{CVaR}_{95\%} - \mathbb{E}\big) \cap \Delta_2^{++}$ and $\boldsymbol{\alpha}^{*\mathrm{VaR}_{95\%}}\left(\frac{1}{2}\mathbf{1}\right) \in \mathcal{RB}\big(\frac{1}{2}\mathbf{1}, \mathrm{VaR}_{95\%} - \mathbb{E}\big) \cap \Delta_2^{++}$. Note that X_2 is a non-linear function of X_1, and for this reason, SD does not capture the perfect association between the two assets[4] though the RP-SD weights are closer to those of RP-$\mathrm{CVaR}_{95\%}$ than RP-$\mathrm{VaR}_{95\%}$. Further, $\mathrm{CVaR}_{95\%}$ is tail sensitive, and the strong association between the assets is better captured by $\mathrm{CVaR}_{95\%}$ (than $\mathrm{VaR}_{95\%}$) that invests 82.47% (instead of 73.01%) in the less risky asset X_1; similarly, RP-SD balances the risk better than RP-$\mathrm{VaR}_{95\%}$ by partially capturing the strong association.

These show how different the portfolios in $\mathcal{RB}\big(\frac{1}{2}\mathbf{1}, SD\big) \cap \Delta_2^{++}$ are from those in $\mathcal{RB}\big(\frac{1}{2}\mathbf{1}, \mathrm{CVaR}_{95\%} - \mathbb{E}\big) \cap \Delta_2^{++}$ and $\mathcal{RB}\big(\frac{1}{2}\mathbf{1}, \mathrm{VaR}_{95\%} - \mathbb{E}\big) \cap \Delta_2^{++}$, even though all three risk preferences are based on shift invariant risk measures. This confirms that Corollary Appendix A.2 a) may not hold if the elliptical assumption is invalid.

Appendix B. Determination of structural breaks

The US equity market data from 01/01/2000 to 31/12/2023 is used in our data analyses, and its evolution could be captured by the S&P500 index dynamic.

We determine the ten periods in the S&P500 index that were delineated by various important events. The structural break points are identified with the test from Bai and Perron (2003) by using the S&P500 daily returns. Their timestamps are provided in the top of Figure B.2, while the bottom of Figure B.2 illustrates the accumulated S&P500 return over the same period in order to show the effectiveness of that test.

These ten structural breaks could be explained by the following events: April 21, 2003 is the end of the dot-com aftermath period; October 5, 2006 is shortly after the housing prices fell by more than 6% in 20 large metropolitan areas, according to Standard & Poor's/Case-Shiller indices; October 6, 2008 is about two weeks after the collapse of Lehman Brothers; August 3, 2011 indicates the time when US and global stock markets crashed upon Standard & Poor's credit rating downgrade of the US sovereign debt from AAA to AA+, the first time in history the United States was downgraded; March

[4] Note that (X_1, X_2) are comonotonic, which is the strongest positive association (Nelsen, 2006), but such non-linear perfect dependence is not captured by the linear correlation measure of association as $corr(X_1, X_2) = 0.7746$.

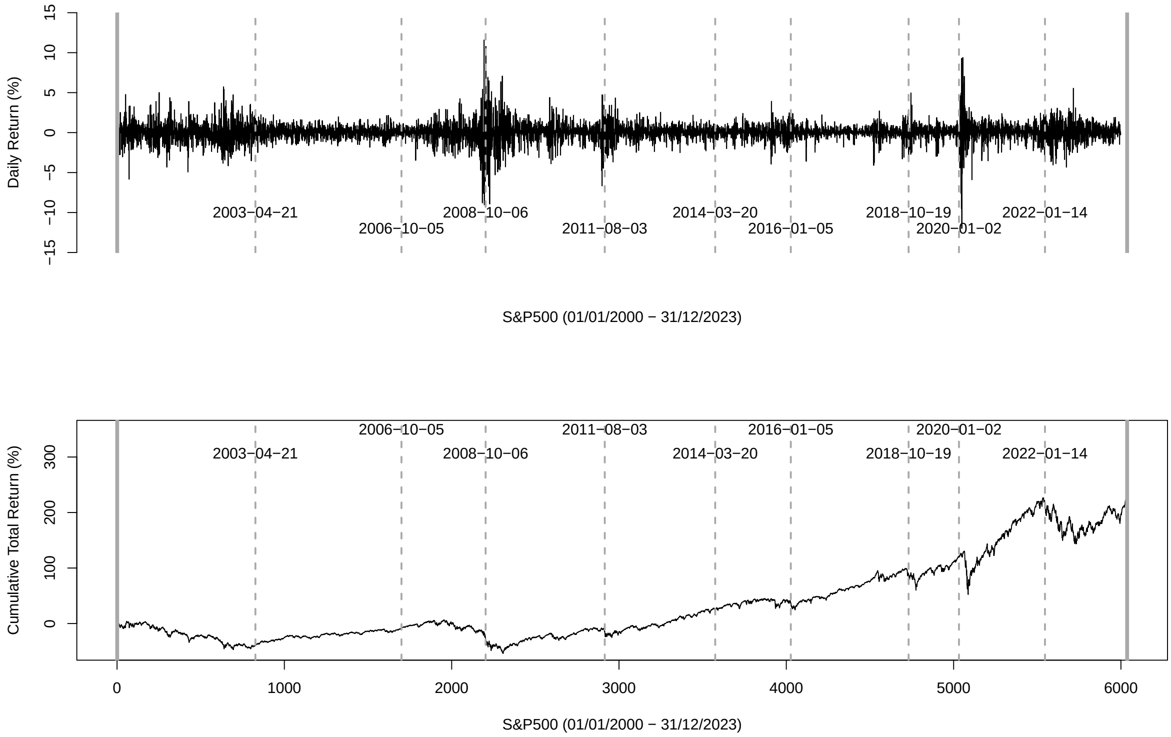

Figure B.2: Structural break points for the S&P500 time series based on a Bai-Perron test.

20, 2014 marks the Maidan revolution in Ukraine; January 5, 2016 relates to the period August 2015-2016 stock market sell off when S&P500 and Dow Jones dropped twice by more than 10%; October 19, 2018 is associated with the loss of nearly 2 trillion dollars in the US stock markets leading to S&P500 losing about 20% by the end of that year; January 2, 2020 marks the beginning of the COVID-19 period; finally, January 14, 2022 is the ending of a two-year COVID-19 period with signals of high inflation.