A Dynamic Count Process

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Abstract

The current paper aims to complement the recent development of the observation-driven models of dynamic counts with a parametric-driven one for a general case, particularly discrete two parameters exponential family distributions. The current paper proposes a finite semiparametric exponential mixture of SETAR processes of the conditional mean of counts to capture the non-linearity and complexity. Because of the intrinsic latency of the conditional mean, the general additive state-space representation of dynamic counts is firstly proposed then stationarity and geometric ergodicity are established under a mild set of conditions. We also propose to estimate the unknown parameters by using quasi maximum likelihood estimation and establishes the asymptotic properties of the quasi maximum likelihood estimators (QMLEs), particularly $\sqrt{T}$-consistency and normality under the relatively mild set of conditions. Furthermore, the finite sample properties of the QMLEs are investigated via simulation exercises and an illustration of the proposed process is presented by applying the proposed method to the intraday transaction counts per minute of AstraZeneca stock.

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1. Introduction

The studies of univariate time series of counts have received extensive attention because of its applicabilities in many different disciplines (see Davis et al. (2021) for a comprehensive list of applications in different areas). In the preceding decades, one of the most well-accepted approaches to modelling the dynamic counts applied the generalised linear model framework of Nelder and Wedderburn (1972) because of its convenient interpretation of covariates on the observed counts and an easy extension of the Gaussian linear regression to an exponential family distribution. For instance, the approach is taken by Zeger (1988), Davis et al. (2000), Davis and Wu (2009), and Samia and Chan (2011), to just name a few. The first three studies are closely related to a parameter-driven model of the broad classification of Cox (1981) within the generalised state-space framework, that of Samia and Chan (2011) is related to an observation-driven one. Subsequently, observation-driven models have been actively developed in the past two decades, particularly by Fokianos et al. (2009), Neumann (2011), Fokianos and Tjøstheim (2011), Fokianos and Tjøstheim (2012), Wang et al. (2014), and Doukhan et al. (2021) for a Poisson process, and Davis and Liu (2016) for the general one parameter exponential family case, because of their convenient accessibility of estimating these models (see Davis et al. (2021) for an excellent review on the topic including more comprehensive references and a review of other methodological approaches). While the dynamic evolution of the stochastic conditional mean of counts is driven by the past observed counts in the case of an observation-driven model, for instance Poisson integer-valued ARCH (INARCH) or GARCH (INGARCH) processes (see Fokianos et al. (2009) for details and references therein), it is driven by its own dynamic evolution in the case of a parameter-driven one. The computation of the likelihoods of those parameter-driven models is, therefore, not straightforward, even for the simple AR(1) specification of the conditional mean (see Davis et al. (2021) for a more comprehensive discussion and references therein) because of its intrinsic latency. For instance, Harvey and Fernandes (1989) and Jørgensen et al. (1999) required the specific conjugate prior distributions to perform the linear filtering. Therefore, this paper aims to complement the recent development of the observation-driven models with a parameter-driven one for a general case in that the nonlinearity and complexity (see Doukhan et al. (2021) for details and references therein) of dynamic counts are described by modelling the latent stochastic conditional mean with the finite semiparametric mixture of self exciting threshold autoregressive (SETAR) processes.

The current paper firstly proposes to represent the discrete two parameters exponential family distributions within the structural state-space representation (see Harvey and Fernandes (1989) for details and references therein) by introducing a negligibly marginal tuning parameter. Jørgensen (1987) provided the extensive study on the exponential family distributions and named these processes as exponential dispersion processes (EDPs). Hence, the current study adopts his abbreviation of discrete EDPs for referring the discrete two parameters exponential family distributions. As a result of introducing the tuning parameter, a legitimately simple additive state-space representation of the proposed count process can be achieved via log-transformation. Although it seems to be quite similar to the case of the generalised linear regression model framework, the simple additive state-space representation of dynamic counts is nontrivial. Because it allows us straightforward implementation of the linear filtering, namely Kalman filter and, hence, accessible establishment of stationarity and geometric ergodicity for a mixture of nonlinear dynamic counts under a mild set of conditions. In that the explicit forms of up to the second moments are also presented. The unknown parameters in the proposed process are then estimated by using quasi maximum likelihood (QML) estimation and the asymptotic properties of quasi maximum likelihood estimators (QMLEs), particularly $\sqrt{T}$-consistency and normality, are also obtained under relatively primitive regularity conditions. Although one may advocate to apply other nonlinear filters such as the Extended Kalman, Unscented Kalman or Particle filters of the nonadditive form, it is not easy to establish the geometric convergence of these filters for the state estimation. A set of strict and meticulous stability conditions, particularly the number of inequalities and random tuning parameters, needs to be imposed for the stability of the Lyapunov functions of those filters (see Särkkä (2013) for a comprehensive treatment of the nonlinear filters) compared to their linear counterpart of relatively mild observability and controllability conditions of a system (see Chapter 3 of Caines (1987) for details).

The current paper is particularly related to those observation-driven ones, namely the work of Samia and Chan (2011), Wang et al. (2014), and Doukhan et al. (2021). The first two studies modelled the dynamic evolution of counts with SETAR of Chan (1993) for the generalised linear model framework of a discrete exponential family and Poisson INGARCH processes, respectively. The nonlinearity of the dynamic counts in their studies were driven by modelling the conditional mean of counts with the observed past counts following the discontinuous SETAR process. Unlike these two studies, our proposed process attempts to model the nonlinearity of dynamic counts by modelling the nonlinear dynamic mechanism of the stochastic latent conditional mean of counts with the continuous SETAR of Chan and Tsay (1998) (see Chan and Tsay (1998), and Xia et al. (2007) for details). On the other hands, Doukhan et al. (2021) studied a mixture of nonlinear INARCH and INGARCH Poisson processes with a time-homogenous hidden Markov switching model and also provided the criterion for selecting the correct number of regimes. More specifically, they proposed the mixture of the Poisson processes themselves, not the conditional means. For our proposed case, the exponential mixture of the conditional means of the discrete EDPs is proposed. Because of the simple additive state-space representation of the dynamic counts via log-transformation, the finite semiparametric exponential mixture of count processes through the conditional mean is easily deduced and the linear filtering is also easily implemented. Lindsay (1983a), Lindsay (1983b), and Van der Vaart (1996) studied the semiparametric mixture of distributions including exponential family distributions without performing Kalman filtering.

The rest of the paper is structured as follows. Section 2 proposes the discrete finite semiparametric exponential mixture of SETAR dynamic count processes and the QML estimation procedure, and establishes the asymptotic properties of the proposed QMLEs. The finite sample performances of the proposed QMLEs with simple but interesting Monte Carlo designs and the details of the proposed estimation procedure are presented in Section 3. In addition, Section 3 also illustrates our proposed process by applying to the intraday transaction counts per minute of AstraZeneca stock. The paper then concludes with the summary. The mathematical proofs of the main theoretical results of the paper are presented in the Appendix.

2. Exponential Mixture of SETAR Count Processes

2.1. Exponential Mixture of SETAR Count Processes

In this section, a discrete finite semiparametric exponential mixture of the discrete EDPs is introduced. In particular, the exponential mixture of the stochastic conditional means of the discrete EDPs is proposed as follows

$$\text{Prob}(I_t = i; \mu_t) = \prod_{k=1}^{K} ED^*\left(\mu_{k,t}, \beta_k, \pi_k\right), \; i = 0, 1, 2 \ldots, \tag{2.1.1}$$

where μ_t is a stochastic latent process specified in (2.1.2) below, $\pi_k \in (0, 1]$ is a mixing parameter such that $\sum_{k=1}^{K} \pi_k = 1$ with K being assumed to be finite and known, and $\beta_k \equiv \frac{1}{\lambda_k}$ with λ_k being a dispersion parameter that varies in a subset of positive real values. Furthermore, $ED^*(\cdot, \cdot, \cdot)$ denotes a discrete EDP with the mixture parameter, which is specified with the conditional mean and variance of counts such that $E(I_{k,t}; \mu_{k,t}, \pi_k) = \mu_{k,t}^{\pi_k} \equiv \tau(\theta_{k,t})$, where $I_{k,t}$ takes nonnegative integers and $\tau(\theta_{k,t}) = \frac{\partial \kappa_k(\theta_{k,t})}{\partial \theta_{k,t}}$ with $\kappa_k(\cdot)$ and $\theta_{k,t}$ being a cumulant function and a canonical parameter, respectively, and $\text{Var}(I_{k,t}; \mu_{k,t}, \pi_k) = \beta_k \text{V}(I_{k,t}; \mu_{k,t}, \pi_k)$ where $\text{V}(I_{k,t}; \mu_{k,t}, \pi_k) = \frac{\partial^2 \kappa_k(\theta_{k,t})}{\partial \theta_{k,t}^2}\Big|_{\theta_{k,t} = \tau^{-1}\left(\mu_{k,t}^{\pi_k}\right)}$. Importantly, the data generating processes of each clusters are assumed to be independent. Additionally, the conditional mean of $I_{k,t}$ is specified by the continuous SETAR process for a flexible dynamic evolution of counts (see (2.2.4) below). Hence, it is transpired that the conditional mean and variance of I_t in (2.1.1) are as follows. Firstly, the conditional mean is

$$\begin{aligned} E(I_t; \mu_t) &\equiv \mu_t \qquad (2.1.2) \\ &= \prod_{k=1}^{K} \mu_{k,t}^{\pi_k}, \qquad (2.1.3) \end{aligned}$$

where

$$\mu_{k,t} = \alpha_{k,0} \prod_{l=1,\neq d_k}^{p_k} \mu_{k,t-l}^{a_{k,l}} \left\{ \prod_{j_k=1}^{m_k} \left(\frac{\mu_{k,t-d_k}}{r_{k,j_k}} \right)^{a_{k,d_k,j_k}} \epsilon_{k,j_k,t} \right\}^{\mathbb{I}_k(r_{k,j_k-1}<\mu_{k,t-d_k}\leq r_{k,j_k})} \tag{2.1.4}$$

with $\alpha_{k,0} \geq 0$ ensuring the nonnegativeness of $\mu_{k,t}$, p_k being a nonnegative integer, d_k being a positive integer such that $d_k \leq p_k$, r_{k,j_k} being a positive real value and $r_{k,0} = 0$, and $\epsilon_{k,j_k,t}$ being independently, identically, absolutely and continuously distributed (i.i.a.c.d.) over the positive real values with $E(\epsilon_{k,j_k,t}) = 1$ and $\text{Var}(\epsilon_{k,j_k,t}) = \sigma^2_{\epsilon,k,j_k} < \infty$, and $\mathbb{I}_k(\cdot)$ denoting an indicator function. Note that the log-transformation of (2.1.4) is the standard continuous SETAR process (see (2.2.4) below). Then the conditional variance is

$$\text{Var}(I_t;\mu_t) \;=\; \prod_{k=1}^{K} \left\{ \sigma^2_{k,t} + \left(\mu_{k,t}^{\pi_k}\right)^2 \right\} - \prod_{k=1}^{K} \left(\mu_{k,t}^{\pi_k}\right)^2, \tag{2.1.5}$$

where $\sigma^2_{k,t}$ denotes $\text{Var}(I_{k,t};\mu_{k,t},\pi_k)$ for the sake of notational simplicity, respectively.

The unconditional first two moments of counts are then obtained by applying the law of iterated expectations to (2.1.3) and (2.1.5), and they are as follows

$$E(I_t) \;=\; E\left[E(I_t;\mu_t)\right] \tag{2.1.6}$$

and

$$\text{Var}(I_t) \;=\; E\left[\text{Var}(I_t;\mu_t)\right] + \text{Var}\left[E(I_t;\mu_t)\right]. \tag{2.1.7}$$

The unconditional autocovariance is also obtained by applying the law of iterated expectation, similar to Davis and Wu (2009), as follows

$$\text{Cov}(I_t, I_{t+\tau}) = 0 + \text{Cov}[E(I_t;\mu_t), E(I_{t+\tau};\mu_{t+\tau})]. \tag{2.1.8}$$

In addition, it is plausible to analyse the over or under-dispersion of the proposed process by applying Fisher's index to (2.1.6) and (2.1.7) as follows

$$\frac{E\left[\text{Var}(I_t;\mu_t)\right] + \text{Var}[E(I_t;\mu_t))]}{E[E(I_t;\mu_t)]} \;\lesseqgtr\; 1. \tag{2.1.9}$$

According to (2.1.9), $E\left[\text{Var}(I_t;\mu_t)\right] + \text{Var}[E(I_t;\mu_t)] < E[E(I_t;\mu_t)]$ indicates the under-dispersed case and otherwise it is over-dispersed.

In order to obtain the unconditional first two moments in (2.1.6) to (2.1.8) and thus (2.1.9), $\mu_{k,t}$ in (2.1.4) is first rewritten in terms of $\epsilon'_{k,j_k,t}s$ under the following condition below

$$\max_{1\leq k\leq K} \left(\max_{1\leq j\leq m_k} \left(\sum_{l=1,\neq d_k}^{p_k} a_{k,l} + a_{k,d_k,j_k} \right) \right) < 1. \tag{2.1.10}$$

The condition in (2.1.10) is the necessary condition for the stationarity and geometric ergodicity of the conditional mean of the log-transformed μ_t (see Assumption 2.1 (i) below). Therefore, the condition in (2.1.10) leads to the stationarity and geometric ergodicity of μ_t. Now the law of iterated expectations is applied to obtain the first two unconditional moments of I_t. Therefore, they are given below

$$\begin{aligned} E(I_t) &= \prod_{k=1}^{K} E\left(\mu_{k,t}^{\pi_k}\right) \\ &= \prod_{k=1}^{K} \left[\left(\alpha_{k,0}^{\frac{\pi_k}{1-\sum_l^{p_k} a_{k,j,l}}} \left(\frac{1}{r_{k,j_k}}\right)^{\frac{\pi_k a_{k,d_k,j_k}}{1-\sum_l^{p_k} a_{k,j,l}}} \right) \left\{ \prod_{l=1}^{\infty} E\left(\epsilon_{k,j_k,t}^{\pi_k} \epsilon_{k,j_k,t-l}^{\pi_k b_{k,l}}\right) \right\} \right]^{\mathbb{I}_{k,j_k}}, \end{aligned} \tag{2.1.11}$$

where $\sum_{l=1}^{\infty} |b_{k,l}(L)| < \infty$ with L being a lag-operator, $\mathbb{I}_{k,j_k} = \mathbb{I}(r_{k,j_k-1} < \mu_{k,t-d_k} \leq r_{k,j_k})$ and $\sum_l^{p_k} a_{k,j,l}$ denotes $\sum_{l=1,\neq d_k}^{p_k} a_{k,l} + a_{k,d_k,j_k}$ for the sake of notational simplicity, and

$$\begin{aligned} \mathrm{Var}(I_t) &= \prod_{k=1}^{K} E\left\{\sigma_{k,t}^2 + \mu_{k,t}^{2\pi_k}\right\} - \prod_{k=1}^{K} \left\{E\left(\mu_{k,t}^{\pi_k}\right)\right\}^2 \\ &= \prod_{k=1}^{K} \left[\left(E\left\{\sigma_{k,t}^2\right\} + \left\{ \alpha_{k,0}^{\frac{2\pi_k}{1-\sum_l^{p_k} a_{k,j,l}}} \left(\frac{1}{r_{k,j_k}}\right)^{\frac{2\pi_k a_{k,d_k,j_k}}{1-\sum_l^{p_k} a_{k,j,l}}} \prod_{l=1}^{\infty} E\left(\epsilon_{k,j_k,t}^{2\pi_k} \epsilon_{k,j_k,t-1}^{2\pi_k b_{k,l}}\right) \right\} \right) \right. \\ &\quad \left. - \left(\prod_{k=1}^{K} \alpha_{k,0}^{\frac{\pi_k}{1-\sum_l^{p_k} a_{k,j,l}}} \left(\frac{1}{r_{k,j_k}}\right)^{\frac{\pi_k a_{k,d_k,j_k}}{1-\sum_l^{p_k} a_{k,j,l}}} \left\{ \prod_{l=1}^{\infty} E\left(\epsilon_{k,j_k,l}^{\pi_k} \epsilon_{k,j_k,t-l}^{\pi_k b_{k,l}}\right) \right\} \right)^2 \right]^{\mathbb{I}_{k,j_k}}. \end{aligned} \tag{2.1.12}$$

In addition, the unconditional autocovariance between I_t and $I_{t+\tau}$ is given below

$$\begin{aligned} \mathrm{Cov}(I_t, I_{t+\tau}) &= \prod_{k=1}^{K} \left[\alpha_{k,0}^{\frac{2\pi_k}{1-\sum_l^{p_k} a_{k,j,l}}} \left(\frac{1}{r_{k,j_k}}\right)^{\frac{2\pi_k a_{k,d_k,j_k}}{1-\sum_l^{p_k} a_{k,j,l}}} \left\{ \prod_{n=0}^{\tau-1} \prod_{l=0}^{\infty} E\left(\epsilon_{k,j_k,t+\tau-n}^{\pi_k b_{k,j,n}} \epsilon_{k,j_k,t-l}^{2\pi_k b_{k,j,t+l}}\right) \right. \right. \\ &\quad \left. \left. - \prod_{l=0}^{\infty} E\left(\epsilon_{k,j_k,t+l}^{\pi_k b_{k,j,l}}\right) E\left(\epsilon_{k,j_k,t+\tau-l}^{\pi_k b_{k,j,l}}\right) \right\} \right]^{\mathbb{I}_{k,j_k}}. \end{aligned} \tag{2.1.13}$$

The first two moments of (2.1.11) to (2.1.13) show that our proposed count process is weakly stationary under the condition of (2.1.10). Furthermore, the under and over-dispersion of the dynamic counts can be evaluated by applying the law of iterated expectations to (2.1.9) as follows

$$
\begin{aligned}
&\prod_{k=1}^{K}\left[\left(E\left\{\sigma_{k,t}^{2}\right\}+\left\{\alpha_{k,0}^{\frac{2\pi_k}{1-\sum_{l}^{p_k}a_{k,j,l}}}\left(\frac{1}{r_{k,j_k}}\right)^{\frac{2\pi_k a_{k,d_k,j_k}}{1-\sum_{l}^{p_k}a_{k,j,l}}}\prod_{l=1}^{\infty}E\left(\epsilon_{k,j_k,t}^{2\pi_k}\epsilon_{k,j_k,t-1}^{2\pi_k b_{k,l}}\right)\right\}\right)\right]^{\mathbb{I}_{k,j_k}}\\
&\leqslant\left[\left(\prod_{k=1}^{K}\alpha_{k,0}^{\frac{\pi_k}{1-\sum_{l}^{p_k}a_{k,j,l}}}\left(\frac{1}{r_{k,j_k}}\right)^{\frac{\pi_k a_{k,d_k,j_k}}{1-\sum_{l}^{p_k}a_{k,j,l}}}\left\{\prod_{l=1}^{\infty}E\left(\epsilon_{k,j_k,t}^{\pi_k}\epsilon_{k,j_k,t-l}^{\pi_k b_{k,l}}\right)\right\}\right.\right.\\
&\quad\left.\left.+\prod_{k=1}^{K}\alpha_{k,0}^{\frac{2\pi_k}{1-\sum_{l}^{p_k}a_{k,j,l}}}\left(\frac{1}{r_{k,j_k}}\right)^{\frac{2\pi_k a_{k,d_k,j_k}}{1-\sum_{l}^{p_k}a_{k,j,l}}}\left\{\prod_{l=1}^{\infty}E\left(\epsilon_{k,j_k,t}^{\pi_k}\epsilon_{k,j_k,t-l}^{\pi_k b_{k,l}}\right)\right\}^{2}\right)\right]^{\mathbb{I}_{k,j_k}}.
\end{aligned}
$$

Moreover, the explicit forms of the above unconditional moments are obtained by considering the first few fractional moments of $\epsilon_{k,j_k,t}$. For example, consider a simple case where $K=1$, $m=1$, $p=1$, $\alpha_0=1$, $a_1=0.5$, and $\epsilon_{1,t}$ is independently, identically and exponentially distributed with $E(\epsilon_{1,t})=1$. The geometric approximation of $|a_1|<1$ produces the unconditional mean of I_t such that $E(I_t)\approx E\left(\epsilon^{\frac{1}{1-a_1}}\right)=2$; however, this is far from our expectations. In particular, the t-fold product of the fractional expectation of $\epsilon_{1,t}$ terms is expected to exponentially converge to 1 as the value of a_1 increases by the independence assumption. A fractional moment of $\epsilon_{1,t}$ can be obtained by applying the Riemann–Liouville fractional differ-integration technique to the moment generating function of $\epsilon_{1,t}$. Because the details of the Riemann–Liouville fractional differ-integration technique are easily found in Oldham and Jerome (1974), only the brief application of the technique to the moment generating function is given below

$$
\begin{aligned}
\frac{d^{q}M_{\epsilon}(s=0)}{ds^{q}} &= \int_{0}^{\infty}(-\epsilon)^{q}\exp(-s\epsilon)f(\epsilon)d\epsilon\Big|_{s=0}\\
&= (-)^{q}E(\epsilon^{q}),
\end{aligned}
$$

where q denotes a positive noninteger, $s\in\mathbb{R}$ is a neighbourhood value of 0, $M_{\epsilon}(\cdot)$ is a moment generating function of ϵ, and $f(\cdot)$ denotes a probability density function. The closed form of a fractional moment of ϵ can be obtained from the distributional assumption on ϵ.

2.2. Quasi Maximum Likelihood Estimation

The intrinsic difficulty of estimating our proposed discrete finite exponential mixture of the SETAR discrete EDPs is the presence of the latent stochastic conditional means and the unknown mixing parameters in the likelihood. Hence, a legitimate state-space representation of (2.1.1) is proposed. Before proceeding with the proposed state-space representation, it is imperative to discuss a few underlying remarks. Let us first introduce a stochastic process ζ_t, where ζ_t is defined below

(2.2.1), such that $E(I_t\zeta_t|\mu_t) = \mu_t$. Therefore, I_t can be rewritten by using ζ_t as follows

$$I_t\zeta_t = \mu_t, \tag{2.2.1}$$

where ζ_t is i.i.a.c.d. over the positive real values with $E(\zeta_t) = 1$ and $\text{Var}(\zeta_t) = \sigma_\zeta^2 < \infty$.

It seems attractive to make an immediate logarithmic transformation of (2.2.1); however, this is not plausible because counts take nonnegative integers. This paper therefore proposes to introduce a tuning parameter, the so-called negligible marginal, such that $\Delta_T = O\left(c^{\frac{T}{2}+\delta}\right)$, where $c \in (0,1)$ is unknown and $\delta > 0$ is arbitrarily small, in order to obtain the logarithmic transformation of (2.2.1). In fact, the tuning parameter in $\Delta_T = O\left(c^{\frac{T}{2}+\delta}\right)$ is $c \in (0,1)$. The regularity condition on the rate of the tuning parameter, particularly $\frac{T}{2} + \delta$ where δ is fixed with a very small value, produces a fast enough convergence rate of Δ_T (faster than $\sqrt{T}$). This ensures the asymptotic uniform equivalence between $\ln \mu_t$ and $\ln(\mu_t + \Delta_T\zeta_t)$ (see (2.2.2) below) with a faster rate than $\sqrt{T}$, and ultimately generalises the existing state-space representation of dynamic count processes and consolidates the mixture of the discrete EDPs within a legitimately simple but general enough framework.

The state-space representation of (2.2.1) is thus as follows

$$y_t = \tilde{X}_t + \xi_t \text{ and } X_t = \ln \mu_t, \tag{2.2.2}$$

where $y_t = \ln(I_t + \Delta_T)$, $\tilde{X}_t = \ln(\mu_t + \Delta_T\zeta_t)$ and $\xi_t = -\ln \zeta_t$. At first glance, it seems implausible to apply the filtering to (2.2.2). However, $\tilde{X}_t$ is shown to be asymptotically equivalent to X_t uniformly over the parameter space of $(a's, r's, \beta's, c, \pi's, d's)^\top \in D$, where $a's$ includes $\ln \alpha'_{k,0}s$ hereafter, and D is a compact parameter space of $\mathbb{R}^{K+\sum_{k=1}^K m_k+p_k} \times \mathbb{R}_+^{\sum_{k=1}^K 2m_k-K} \times \mathbb{R}_+^K \times \mathbb{R}_{(0,1)} \times \mathbb{R}_{(0,1]}^K \times \mathbb{Z}_{0,1,\ldots,p_K}^K$ with $\mathbb{R}_+$, $\mathbb{R}_{(0,1)}$ and $\mathbb{Z}_{0,1,\cdots,p_K}$ representing the positive real values, real values between 0 and 1 and integer values from 0 to p_K (p_K denotes $\max\limits_{1\le k\le K}(p_k)$), respectively, for the sake of notational simplicity. Additionally, the vectors of the parameters with a 0 subscript denote the vector of the true parameters hereafter. Before discussing the asymptotic equivalence between $\tilde{X}_t$ and X_t uniformly over D, the geometric ergodicity of X_t is presented in Remark 2.1 under the appropriate regularity conditions as follows.

Assumption 2.1. *(i) The logarithmic transformed conditional mean of $I_{k,t}$, $X_{k,t}$, requires that $\max\limits_{1\le j_k\le m_k} \left|\sum_{l=1,\neq d_k}^{p_k} a_{k,l} + a_{k,d_k,j_k}\right| < 1$ for all $k = 1,\ldots,K$. (ii) The stochastic process, $\epsilon_{k,j_k,t}$, is i.i.a.c.d. over positive real values with $E(\epsilon_{k,j_k,t}) = 1$, $Var(\epsilon_{k,j_k,t}) = \sigma^2_{\epsilon,k,j_k}$ and $E(\epsilon^{4+\delta}_{k,j_k,t}) < \infty$ for all $j_k = 1,\ldots,m_k$ and $k = 1,\ldots,K$. In addition, $\epsilon_{k,j_k,t}$ is independent to the initial state variable, $\mu_{k,1}$, for all $k = 1,\ldots,K$ and $t = 1,\ldots,T$.*

Remark 2.1. *Under Assumption 2.1, An and Huang (1996) showed that X_t is geometrically ergodic by using Tweedie's drift criterion (see Tweedie (1976)) and Tjøstheim's h-step criterion (see Tjøstheim (1990)).*

The additional regularity conditions of the asymptotic equivalence between $\tilde{X}_t$ and X_t are then as follows.

Assumption 2.2. *(i) The negligible marginal, $\triangle_T = O\left(c^{\frac{T}{2}+\delta}\right)$, where $c \in (0,1)$ is unknown and δ is arbitrarily small. (ii) The stochastic process, ζ_t, is i.i.a.c.d. over positive real values with $E(\zeta_t) = 1$, $Var(\zeta_t) = \sigma_\zeta^2$ and $E(\zeta_t^{4+\delta}) < \infty$. In addition, ζ_t is independent from the initial state variable, $\mu_{k,1}$, for all $k = 1, \ldots, K$ and $t = 1, \ldots, T$.*

Firstly, the regularity condition on the rate of the tuning parameter is necessary to ensure the faster rate of convergence of $\tilde{X}_t$ to X_t, particularly a faster rate than $\sqrt{T}$. Furthermore, the positive distributional assumption on ζ_t is necessary to ensure the positivity of μ_t and the finite moments of ζ_t up to fourth moments are necessary to apply the Cauchy–Schwartz inequality to establish the stochastic equi-continuity of the remainder term of $\tilde{X}_t - X_t$. The independence of ζ_t from the initial state variable $\mu_{1,k}$ for all $k = 1, \ldots, K$ is also a necessary condition for the stability of the state-space representation in (2.2.2) (see Chapter 3 of Caines (1987) for details). The asymptotic equivalence between $\tilde{X}_t$ and X_t uniformly over D is now ready to be presented.

Lemma 2.1. *Under Assumptions 2.1 and 2.2, and where $\vartheta_0 = (a_0's, r_0's, \beta_0's, c_0, \pi_0's, d_0's)^\top \in D$, it can be shown that*

$$\sup_{\vartheta \in D} \left| \tilde{X}_t - X_t \right| = o(T^{-1/2}) \ a.s., \ \ as \ T \to \infty,$$

where a.s. denotes almost surely.

As a result of Lemma 2.1, the current paper finally proposes to represent the discrete finite semiparametric exponential mixture of the nonlinear dynamic count processes in (2.1.1) by the state-space representation below

$$y_t = \sum_{k=1}^{K} \pi_k X_{k,t} + \xi_t \tag{2.2.3}$$

and

$$X_{k,t} = a_{k,0} + \sum_{l=1,\neq d_k}^{p_k} a_{k,l} X_{k,t-l} + \left\{ \sum_{j_k}^{m_k} a_{k,d_k,j_k} (X_{k,t-d_k} - \mathfrak{r}_{k,j_k}) + \eta_{k,j_k,t} \right\} \mathbb{I}_{k,j_k}, \tag{2.2.4}$$

where $X_{k,t} = \ln \mu_{k,t}$, $\mathfrak{r}_{k,j_k} = \ln r_{k,j_k}$, and $\eta_{k,j_k,t} = \ln \epsilon_{k,j_k,t}$.

Let us now briefly discuss the estimation procedure of our proposed discrete finite exponential mixture of the discrete EDPs in (2.1.1). The first step is to apply the Expectation Maximisation (EM) algorithm of Shumway and Stoffer (1982) to (2.2.3) and (2.2.4), given the tuning and threshold parameters. The estimation procedure of the algorithm is similar to that of Chan and Tsay (1998). In particular, we firstly estimate $(a's, \beta's, d's)^\top$ given $\mathfrak{r}'s$. $\mathfrak{r}'s$ is then estimated by maximising the log-likelihood in (2.2.5) but substituting with $(\hat{a}'s, \hat{\beta}'s, \hat{d}'s)^\top$. The EM algorithm is the iterative estimation procedure of alternating between Kalman filtering and recursive smoothing, and QML estimation (see Shumway and Stoffer (1982) for details). Note that linear piecewise Kalman filtering and smoothing are required for the proposed procedure in our case. The tuning parameter is then estimated by maximising the Gaussian likelihood of ξ_ts in (2.2.3). The second step is to perform the EM algorithm above with the estimated tuning parameter. The monotonicity of the sequence of the conditional log-likelihoods at each iterations of the EM algorithm ensures the convergence of the sequence of the conditional log-likelihoods to the one defined in (2.2.5) below (see Wu (1983) for details). Further details of the estimation procedures are presented in Section 3.1 below.

As a result of Lemma 2.1, $\hat{c} = c_0$ as $T \to \infty$, and the parameter space is modified accordingly such that $\psi_0 = (a_0's, \mathfrak{r}_0's, \beta_0's, d_0's)^\top \in D_\psi$, where $D_\psi \subset D$ is a compact parameter space. The feasible conditional log-likelihood of I_t is then as follows

$$\mathcal{L}(\psi|\mathcal{F}_{t-1}) = \frac{1}{T}\sum_{t=p_K+1}^{T+p_K}\left(\sum_{k=1}^{K}\ln\widehat{ED}^*_{k,t}\right), \qquad (2.2.5)$$

where $\widehat{ED}^*_{k,t}$ denotes $ED^*(\hat{\mu}_{k,t|t-1}, \beta_k, \hat{\pi}_k)$, and $\hat{\pi}_k$ and $\hat{\mu}_{k,t|t-1}$ are obtained by using the result of $\hat{X}_{k,t|t-1}$ which is the minimum conditional mean squared error estimate of $X_{k,t}$ given the sigma-field, $\mathcal{F}_{t-1}$, generated by $(I_1, I_2, \ldots, I_{t-1})$ (see Chapters 3 and 7 of Caines (1987) for details). The asymptotic properties of our proposed QMLEs are then established by showing the almost sure convergence of the feasible likelihood to the infeasible one uniformly over D_ψ with additional regularity conditions on EDPs. Hereafter, let us use $ED^*_{k,t}$ to denote $ED^*(\mu_{k,t}, \beta_k, \pi_k)$ for the sake of notational simplicity.

Assumption 2.3. *(i)* $\max_{1\le k\le K} E\left(\sup_{\psi_k\in D_\psi}\ln ED^*_{k,t}\right)^2 < \infty$ *and* $\max_{1\le k\le K} E\left(\sup_{\psi_k\in D_\psi}\frac{ED^{*(1)}_{k,t,\pi}}{ED^*_{k,t}}\right)^4 < \infty$ *for all* $t = 1,\ldots,T$*, where* $ED^{*(1)}_{k,t,\pi}$ *denotes the first derivative of* $ED^*_{k,t}$ *with respect to* π_k*. (ii)* $\max_{1\le k\le K} E\left(\sup_{\psi_k\in D_\psi}\frac{ED^*_{k,t}}{ED^{*(1)}_{k,t,\mu}}\right)^4 < \infty$ *and* $\max_{1\le k\le K} E\left(\sup_{\psi_k\in D_\psi}\frac{ED^{*(1)}_{k,t,\pi}}{ED^{*(1)}_{k,t,\mu}}\right)^4 < \infty$ *for all* $t = 1,\ldots,T$*, where* $ED^{*(1)}_{k,t,\mu}$ *denotes the first derivative of* $ED^*_{k,t}$ *with respect to* $\mu_{k,t}$*. (iii)* $\max_{1\le k\le K} E\left(\sup_{\psi_k\in D_\psi}\frac{ED^*_{k,t}}{ED^{*(2)}_{k,t,\mu,\pi}}\right)^4 < \infty$, $\max_{1\le k\le K} E\left(\sup_{\psi_k\in D_\psi}\frac{ED^{*(1)}_{k,t,\pi}}{ED^{*(2)}_{k,t,\mu,\pi}}\right)^4 < \infty$ *and*

$\max_{1\le k\le K} E\left(\sup_{\psi_k\in D_\psi} \frac{ED^{*(1)}_{k,t,\mu}}{ED^{*(2)}_{k,t,\mu,\pi}}\right)^4 < \infty$ *for all* $t = 1,\ldots,T$, *where* $ED^{*(2)}_{k,t,\mu,\pi}$ *denotes the second derivative of* $ED^*_{k,t}$ *with respect to* $\mu_{k,t}$ *and* π_k.

The above conditions are required to establish the almost sure convergence of the feasible log-likelihood to the infeasible one uniformly over the compact parameter space. The main strategy of showing the almost sure convergence is to show the almost sure negligibility of the remainder term of the difference between the two log-likelihoods over the parameter space by using the Taylor expansion arguments of a logarithmic function. This produces a number of first and second derivatives of $ED^*_{k,t}$ with respect to π_k and $\mu_{k,t}$ for all $k = 1,\ldots,K$. Therefore, the finite moments of the suprema of their derivatives up to fourth moment over the parameter space are required to the apply Cauchy–Schwartz inequality. With these regularity conditions on EDPs, the strong convergence of the feasible likelihood to the infeasible one uniformly over the parameter space is established as follows.

Lemma 2.2. *Under Assumptions 2.1 to 2.3, and with* $\psi_0 = (a_0's, \mathfrak{r}_0's, \beta_0's, d_0's)^\top \in D_\psi$, *it is shown that*

$$\sup_{\psi\in D_\psi} |\mathcal{L}(\psi|\mathcal{F}_{t-1}) - \mathcal{L}^*(\psi)| = O(T^{-1/2}) \text{ a.s., as } T\to\infty,$$

where

$$\mathcal{L}^*(\psi) = \frac{1}{T}\sum_{t=1}^{T}\left(\sum_{k=1}^{K} \ln ED^*_{k,t}\right).$$

Next, the almost sure convergence and asymptotic normality of the QMLEs are discussed below with additional regularity conditions as follows.

Assumption 2.4. *(i)* $\mu_{k,t}$ *is an* α*-mixing process such that* $\alpha(T) = O\left(T^{-\frac{(2+\delta)}{2}}\right)$ *for all* $k = 1,\ldots,K$. *(ii)* $\max_{1\le k\le K} E\left(\sup_{\psi_k^*\in D_{\psi^*}} \left|\frac{\partial \ln ED^*_{k,t}}{\partial\psi^*}\right|^{2+\delta}\right) < \infty$ *for all* $t = 1,\ldots,T$, *where* $\psi^* = (a's, \mathfrak{r}'s, \beta's)^\top \in D_{\psi^*}$ *and* $D_{\psi^*} \subset D_\psi$ *is the compact parameter space. (iii)* $\max_{1\le k\le K} E\left(\sup_{\psi_k^*\in D_{\psi^*}} \frac{\partial^2 \ln ED^*_{k,t}}{\partial\psi^*\partial\psi^{*\top}}\right)^2 < \infty$ *and* $\max_{1\le k\le K} E\left(\sup_{\psi_k^*\in D_{\psi^*}} \frac{\partial^3 \ln ED^*_{k,t}}{\partial\psi^*\partial\psi^{*\top}\partial\psi^*}\right)^2 < \infty$ *for all* $t = 1,\ldots,T$.

The first condition of Assumption 2.4, particularly the mixing condition, is the least restrictive serial dependence of the time series and the regularity condition of the rate on the mixing coefficient ensures the convergence rate $\sqrt{T}$ (see Chapter 2 of Fan and Yao (2008) for more details). The rest of the regularity conditions are necessary to establish the strong consistency uniformly over the parameter space and the asymptotic normality of our proposed QMLEs (see Chapter 4 of Amemiya (1985) for an example). In particular, these conditions are used to apply the Chebyshev inequality and the Borel–Cantelli lemma.

Theorem 2.1. *Under Assumptions 2.1 to 2.4, and with* $\limsup_{T\to\infty}\left(E\max_{\psi\in\bar{D}_\delta(\psi_0)\cap D_\psi}\mathcal{L}^*(\psi)\right)\neq\limsup_{T\to\infty}E\mathcal{L}^*(\psi_0)$ *for any* $\psi\in D_\psi$*, where* $\bar{D}_\delta(\psi_0)$ *is the complement of an open* δ*-neighbourhood of* ψ_0*,* ψ_0 *is uniquely identified and* $\hat{\psi}=\psi_0+O(T^{-1/2})$ *a.s., as* $T\to\infty$*.*

As a result of Theorem 2.1 and the discreteness of $d's$, $\hat{d}'s=d_0's$ as $T\to\infty$ (see Chan and Tsay (1998) for details), the parameter space is modified accordingly. The asymptotic normality of our proposed QMLEs can then be as follows.

Theorem 2.2. *Under Assumptions 2.1 to 2.4, and when* ψ_0^* *is an interior of* D_{ψ^*}*,*

$$\sqrt{T}(\hat{\psi}^*-\psi_0^*)\sim N(0,\Sigma),$$

where $\Sigma=B_0^{-1}(\psi_0^*)A_0(\psi_0^*)B_0^{-1}(\psi_0^*)$ *with* $B_0(\psi_0^*)=\lim_{T\to\infty}E\frac{\partial^2\mathcal{L}^*(\psi^*)}{\partial\psi^*\partial\psi^{*\top}}\Big|_{\psi^*=\psi_0^*}$ *and* $A_0(\psi_0^*)=\lim_{T\to\infty}E\left(\sqrt{T}\frac{\partial\mathcal{L}^*(\psi^*)}{\partial\psi^*}\right)\left(\sqrt{T}\frac{\partial\mathcal{L}^*(\psi^*)}{\partial\psi^*}\right)^\top\Big|_{\psi^*=\psi_0^*}$*, as* $T\to\infty$*.*

3. Simulation and Illustration

In this section, the finite sample performance of our proposed QMLEs is investigated with Monte Carlos simulation exercises. In addition, we illustrate the proposed process and estimation procedure by applying those to the intraday transaction counts of AstraZeneca stock.

3.1. Simulation Study

The finite sample performances of the QMLEs are investigated with the most fundamental data generating process of counts, namely a Poisson process, and its extensions. This simulation exercise also focuses on how to implement the proposed estimation procedure presented in Section 2.2. The number of replications in all the simulation exercises is 10,000.

Let us firstly define the Poisson process with the conditional mean specified by the rudimentary SETAR process below

$$\text{Prob}(I_t=i;\mu_t)=\frac{\mu_t^{I_t}\exp(-\mu_t)}{I_t!},\ i=0,1,2\ldots,\tag{3.1.1}$$

where $\mu_t=\begin{cases}\mu_{t-1}^{0.5}\epsilon_{1t}\text{ if }\mu_t\leq 1\\ \mu_{t-1}^{-0.5}\epsilon_{2t}\text{ if }\mu_t>1\end{cases}$ with ϵ_{1t} and $\epsilon_{2t}\sim\text{Lognormal}(0,1)$. The vector of parameters, namely $(\alpha_{0,1,1}=0.5,\alpha_{0,1,2}=-0.5,r_0=1,c_0)'$, is estimated by the proposed estimation procedure as follows. First, the EM algorithm with linear piecewise Kalman filtering and smoothing is applied, given the initial tuning and threshold parameters below

$$y_t = X_t + \xi_t \quad \text{and} \quad X_t = \begin{cases} 0.5(X_{t-1}-0)_- + \eta_{1t} \\ -0.5(X_{t-1}-0)_+ + \eta_{2t}, \end{cases}$$

where $y_t = \ln\left(I_t + c^{\frac{T}{2}+\delta}\right)$ with $0 < c < 1$ and $0 < \delta < 1$ being arbitrary, and $(X_{t-1}-0)_-$ and $(X_{t-1}-0)_+$ denote $X_{t-1} \le 0$ and $X_{t-1} > 0$, respectively, until the sequence of the conditional likelihoods converges to the below

$$\mathcal{L}(\alpha_{1,1}, \alpha_{1,2}|\mathcal{F}_{t-1}, r) = \frac{1}{T}\sum_{t=p_K+1}^{T+p_K} \frac{\hat{\mu}_{t|t-1}^{I_t}\exp(-\hat{\mu}_{t|t-1})}{I_t!},$$

where $\hat{\mu}_{t|t-1} = \exp(\hat{X}_{t|t-1})$ with $\hat{X}_{t|t-1}$ being obtained by implementing Kalman filtering with $(\hat{\alpha}_{1,1}, \hat{\alpha}_{1,2})'$ from the last iteration of the EM algorithm. We then estimate r by maximising the following

$$\mathcal{L}(r|\mathcal{F}_{t-1}, \hat{\alpha}_{1,1}, \hat{\alpha}_{1,2}) = \frac{1}{T}\sum_{t=p_K+1}^{T+p_K} \frac{\hat{\mu}_{t|t-1}^{I_t}\exp(-\hat{\mu}_{t|t-1})}{I_t!},$$

where $\hat{\mu}_{t|t-1} = \exp(\hat{X}_{t|t-1})$, and $\hat{X}_{t|t-1} = \begin{cases} \hat{\alpha}_{1,1}\hat{X}_{t-1|t-1} \text{ if } \hat{X}_{t-1|t-1} \le 0 \\ \hat{\alpha}_{2,1}\hat{X}_{t-1|t-1} \text{ otherwise} \end{cases}$. The tuning parameter is then estimated by maximising the Gaussian likelihood of $\hat{\xi}_t$s, where $\hat{\xi}_t = \log\left(I_t + c^{\frac{T}{2}+\delta}\right) - \hat{X}_{t|t-1}$. The next step is to apply the EM algorithm with the estimated tuning parameter to the below

$$\hat{y}_t = X_t + \xi_t \quad \text{and} \quad X_t = \begin{cases} 0.5(X_{t-1}-0)_- + \eta_{1t} \\ -0.5(X_{t-1}-0)_+ + \eta_{2t}, \end{cases}$$

where $\hat{y}_t = \ln\left(I_t + \hat{c}^{\frac{T}{2}+\delta}\right)$. The estimation procedure is similar to that of the two-step estimation procedure. First, estimate $(\alpha_{1,1}, \alpha_{1,2})'$ then r. The results for (3.1.1) are presented in Table 1.

Next, the Type II Negative Binomial (NB) process is considered. Although it is a simple extension of a Poisson process, it is one of the most popular count processes in practice because of its over-dispersion property. The Type II NB process is commonly obtained by mixing a Poisson process with Gamma distribution as follows

$$\text{Prob}(I_t = i; \mu_t) = \frac{\gamma_t^{I_t}\exp(-\gamma_t)}{I_t!}, \; i = 0, 1, 2\ldots, \tag{3.1.2}$$

where $\gamma_t = \mu_t\nu_t$ with $\mu_t = \begin{cases} \mu_{t-1}^{0.2}\epsilon_{1t} \text{ if } \mu_t \le 1 \\ \mu_{t-1}^{-0.2}\epsilon_{2t} \text{ if } \mu_t > 1 \end{cases}$ with ϵ_{1t} and $\epsilon_{2t} \sim \text{Exp}(1)$, and $\nu_t \sim \text{Gamma}(\beta, 1/\beta)$ with $\beta = 1$. The count process in (3.1.2) can be rewritten as

Table 1: *Simulation results for (3.1.1)*

T	$\hat{\alpha}_{1,1}$	$s.e.(\hat{\beta}_{1,1})$	$mse(\hat{\beta}_{1,1})$	$\hat{\alpha}_{1,2}$	$s.e.(\hat{\beta}_{1,2})$	$mse(\hat{\beta}_{1,2})$
100	0.4925	0.0665	0.0046	-0.4870	0.1007	0.0111
250	0.4976	0.0378	0.0015	-0.4951	0.0570	0.0033
500	0.4984	0.0258	0.0006	-0.4977	0.0383	0.0015
700	0.4989	0.0214	0.0005	-0.4986	0.0317	0.0010
1000	0.4996	0.0177	0.0003	-0.4990	0.0263	0.0007
T	$\hat{r}$	$s.e.(\hat{r})$	$mse(\hat{r})$	$\hat{c}$	$s.e.(\hat{c})$	
100	1.0096	0.1155	0.0125	0.9963	0.0027	·
250	1.0051	0.0670	0.0044	0.9985	0.0006	·
500	1.0012	0.0464	0.0021	0.9992	0.0002	·
700	1.0013	0.0389	0.0014	0.9994	0.0001	·
1000	1.0012	0.0323	0.0010	0.9996	0.0001	·

Table 2: *Simulation results for (3.1.3)*

T	$\hat{\alpha}_{1,1}$	$s.e.(\hat{\alpha}_{1,1})$	$mse(\hat{\alpha}_{1,1,})$	$\hat{\alpha}_{1,2}$	$s.e.(\hat{\alpha}_{1,2})$	$mse(\hat{\alpha}_{1,2})$	$\hat{c}$	$s.e.(\hat{c})$
100	0.2208	0.1200	0.0105	-0.2276	0.2510	0.0196	0.9923	0.0024
250	0.2070	0.0754	0.0049	-0.2120	0.1579	0.0114	0.9969	0.0006
500	0.2024	0.0529	0.0026	-0.2014	0.1030	0.0067	0.9984	0.0002
700	0.2010	0.0447	0.0020	-0.1989	0.0849	0.0052	0.9989	0.0001
1000	0.2011	0.0376	0.0014	-0.1976	0.0698	0.0039	0.9992	0.0001
T	$\hat{r}$	$s.e.(\hat{r})$	$mse(\hat{r})$	$\hat{\beta}$	$s.e.(\hat{\beta})$	$mse(\hat{\beta})$		
100	0.9360	0.8529	0.4956	0.9467	0.3660	0.0378	·	·
250	0.9774	0.5135	0.1751	0.9723	0.2355	0.0209	·	·
500	0.9966	0.3500	0.0770	0.9869	0.1704	0.0134	·	·
700	0.9963	0.2917	0.0539	0.9900	0.1446	0.0110	·	·
1000	1.0008	0.2410	0.0365	0.9579	0.1136	0.0054	·	·

follows

$$\text{Prob}(I_t = i; \mu_t) = \frac{\Gamma(\beta + I_t)}{\Gamma(\beta)\Gamma(I_t + 1)} \left(\frac{\beta}{\beta + \mu_t}\right)^{\beta} \left(\frac{\mu_t}{\beta + \mu_t}\right)^{I_t}, \; i = 0, 1, 2 \ldots, \qquad (3.1.3)$$

where $\Gamma(\cdot)$ denotes a gamma function. The vector of the parameters, $(\alpha_{0,1,1} = 0.2, \alpha_{0,1,2} = -0.2, r_0 = 1, \beta_0 = 1, c_0)'$, are estimated by the similar procedure as above, and the results are presented in Table 2.

The last exercise involves a mixture of a Poisson process with the conditional means specified by a SETAR and a simple autoregressive (AR(1)) processes as follows

$$\text{Prob}(I_t = i; \mu_t) = \frac{\mu_t^{I_t} \exp(-\mu_t)}{I_t!}, \; i = 0, 1, 2 \ldots, \qquad (3.1.4)$$

where $\mu_t = \prod_{k=1}^{2} \mu_{k,t}^{\pi_k}$ with $\pi_1 = \pi_2 = 0.5$,

$$\mu_{1,t} \begin{cases} \mu_{1,t-1}^{0.5} \epsilon_{11,t} \text{ if } \mu_{1,t} \leq 1 \\ \mu_{1,t-1}^{-0.5} \epsilon_{12,t} \text{ if } \mu_{1,t} > 1 \end{cases} \quad \text{with } \epsilon_{11,t} \text{ and } \epsilon_{12,t} \sim \text{Lognormal}(0, 1)$$

Table 3: *Simulation results for (3.1.4)*

T	$\hat{\alpha}_{1,1,1}$	$s.e.(\hat{\alpha}_{1,1,1})$	$mse(\hat{\alpha}_{1,1,1})$	$\hat{\alpha}_{1,1,2}$	$s.e.(\hat{\alpha}_{1,1,2})$	$mse(\hat{\alpha}_{1,1,2})$	$\hat{c}$	$s.e.(\hat{c})$
100	0.4689	0.1672	0.0333	-0.4511	0.2143	0.0643	0.9977	0.0028
250	0.4864	0.0931	0.0096	-0.4826	0.1319	0.0187	0.9992	0.0007
500	0.4946	0.0621	0.0042	-0.4934	0.0878	0.0080	0.9996	0.0002
700	0.4969	0.0516	0.0029	-0.4956	0.0727	0.0056	0.9997	0.0001
1000	0.4979	0.0428	0.0020	-0.4967	0.0602	0.0038	0.9998	0.0001
T	$\hat{r}_1$	$s.e.(\hat{r}_1)$	$mse(\hat{r}_1)$	$\hat{\alpha}_{2,1}$	$s.e.(\hat{\alpha}_{2,1})$	$mse(\hat{\alpha}_{2,1})$		
100	0.9270	0.3858	0.0148	0.1847	0.1870	0.0375	·	·
250	0.9073	0.1749	0.0106	0.1950	0.1123	0.0130	·	·
500	0.9050	0.1098	0.0100	0.1984	0.0774	0.0064	·	·
700	0.9048	0.0900	0.0100	0.1989	0.0648	0.0044	·	·
1000	0.9048	0.0741	0.0100	0.1997	0.0540	0.0030	·	·
T	$\hat{\pi}_1$	$s.e.(\hat{\pi}_1)$	$mse(\hat{\pi}_1)$	$\hat{\pi}_2$	$s.e.(\hat{\pi}_2)$	$mse(\hat{\pi}_2)$		
100	0.490	0.080	0.005	0.481	0.085	0.007	·	·
250	0.496	0.049	0.002	0.489	0.052	0.003	·	·
500	0.498	0.034	0.001	0.493	0.037	0.001	·	·
700	0.498	0.029	0.001	0.495	0.031	0.001	·	·
1000	0.498	0.024	0.001	0.496	0.026	0.001	·	·

and

$$\mu_{2,t} = \mu_{2,t-1}^{0.2}\epsilon_{2,t} \text{ with } \epsilon_{2,t} \sim \text{ Lognormal}(0,1).$$

The vector of the parameters, $(\alpha_{0,1,1,1} = 0.5, \alpha_{0,1,1,2} = -0.5, r_{0,1} = 1, \alpha_{0,2,1} = 0.2, \pi_{0,1} = 0.5, \pi_{0,2} = 0.5, c_0)'$, are estimated by a similar procedure to (3.1.1) and (3.1.3). The results for (3.1.4) are presented in Table 3.

For all these cases, the simulation exercise shows the satisfactory finite sample performance of our proposed QML estimation procedure. The estimates of the tuning parameters are close to the value of 1, as we expected. Additionally, notice that our proposed process is the special cases of those parametric-driven specifications within the generalised linear regression framework of Nelder and Wedderburn (1972), particularly Zeger (1988) for a Poisson process and Davis and Wu (2009) for a negative binomial process, where there is no covariate. The simulation results for the simple Poisson and negative binomial processes of Zeger (1988) and Davis and Wu (2009) without a covariate can be obtained by requesting the authors. In the following, we apply the proposed process and estimation procedure to the intraday transaction counts of AstraZeneca stock.

3.2. Illustration of Real Data Analysis

We now illustrate our proposed count process by applying it to analyse the number of transactions per minute for AstraZeneca stock, closely following Fokianos et al. (2009). The randomly selected trading day is 30 July 2019. There are about 500 observations after eliminating the first and last minutes of transactions for about

Table 4: *Estimation results of (3.2.1)*

	$\hat{\alpha}_0$	$\hat{\alpha}_1$	$\hat{\beta}$	$\hat{c}$
estimates	3.2034	-0.1511	0.7650	0.9503
s.e.	(0.1664)	(0.0559)	(0.0459)	(0.0028)

8 trading hours. The autocorrelation function of this data (see Figure 1 (b)) shows the moderate dependence between transactions not as strong as the case of Ericsson B stock in Fokianos et al. (2009). Furthermore, it is an over-dispersed case. The value of the sample mean is 10.0266 with the variance of 65.3661. The over-dispersion in this data might be caused by the frequent zero transactions and a few large number of transactions (see Figure 1(a)). Therefore, the Type II NB process discussed in Section 3.1 is considered in this analysis.

Applying first the simple AR(1) process to the conditional mean of the transaction counts, the Type II NB process is

$$\text{Prob}(I_t = i; \mu_t) = \frac{\Gamma(\beta + I_t)}{\Gamma(\beta)\Gamma(I_t + 1)} \left(\frac{\beta}{\beta + \mu_t}\right)^{\beta} \left(\frac{\mu_t}{\beta + \mu_t}\right)^{I_t}, \; i = 0, 1, 2 \ldots, \quad (3.2.1)$$

where $\mu_t = \alpha_0 \mu_{t-1}^{\alpha_1} \epsilon_t$. The results of this estimation are reported in Table 4. For examining the adequacy of the fit, an analysis of the Pearson residuals is performed. The Pearson residuals are defined as $e_t = \frac{I_t - \hat{\mu}_t}{\sqrt{\hat{\mu}_t \left(1 + \frac{\hat{\mu}_t}{\hat{\beta}}\right)}}$ in the case of Type II NB process, and e_t is an white noise process under a correct specification of I_t. The cumulative periodogram plot of e_ts (see Figure 1(d)) indicates that (3.2.1) is not adequate enough to model the intraday transactions of this data. The prediction of I_t of (3.2.1) is also shown in Figure 1(c). Furthermore, the mean squared error of (2.2.3) for the case of (3.2.1) is 6.6976.

Now let us apply the continuous two-regime SETAR with $p = d = 1$ to the conditional mean of the transaction counts, the Type II NB process is

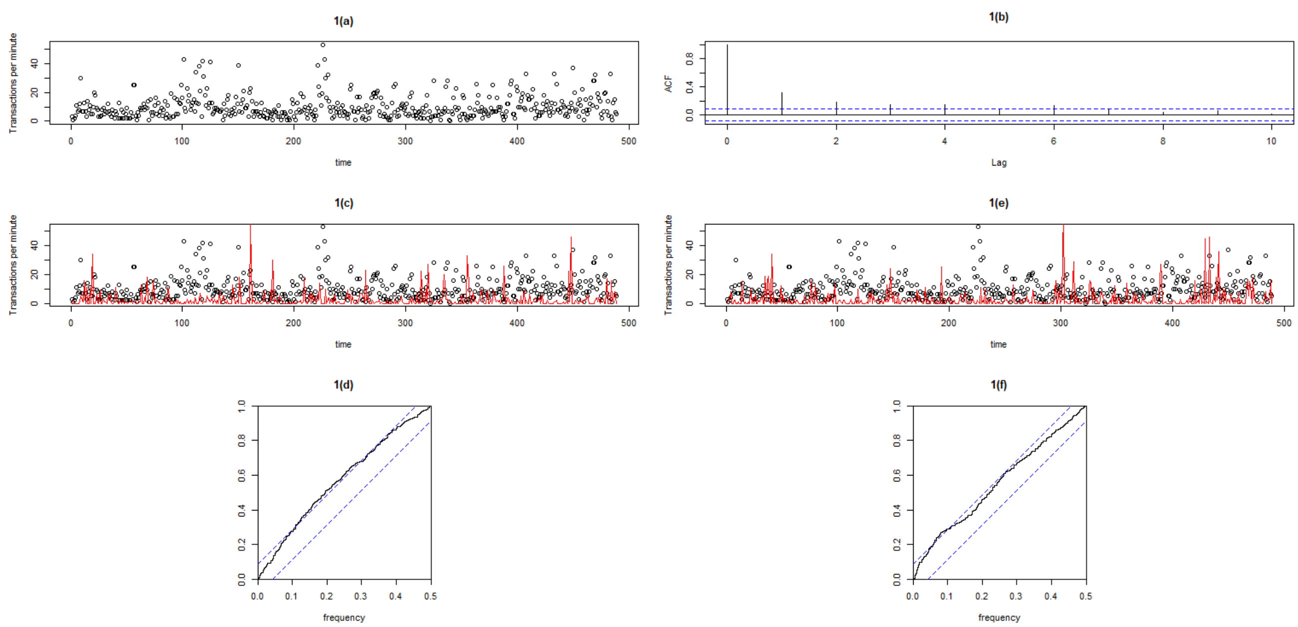
$$\text{Prob}(I_t = i; \mu_t) = \frac{\Gamma(\beta + I_t)}{\Gamma(\beta)\Gamma(I_t + 1)} \left(\frac{\beta}{\beta + \mu_t}\right)^{\beta} \left(\frac{\mu_t}{\beta + \mu_t}\right)^{I_t}, \; i = 0, 1, 2 \ldots, \quad (3.2.2)$$

where $\mu_t = \alpha_0 \prod_{k=1}^{2} \left\{ \left(\frac{\mu_{t-1}}{r}\right)^{\alpha_{1,k}} \epsilon_{k,t} \right\}^{\mathbb{I}_k}$. The results of this estimation are reported in Table 5. The cumulative periodogram plot of e_ts and the prediction of I_t of (3.2.2) are shown in Figure 1(e) and (f), respectively. The improvement on the Pearson's residuals (see Figure 1(f)) supports the use of (3.2.2), namely the nonlinear Type II NB process, instead of the linear one. There is also improvement on the mean squared error for the case of (3.2.2), it is 6.3317.

Table 5: *Estimation results of (3.2.2)*

	$\hat{\alpha}_0$	$\hat{\alpha}_{1,1}$	$\hat{\alpha}_{1,2}$	$\hat{r}$	$\hat{\beta}$	$\hat{c}$
estimates	3.3202	0.2700	-0.2023	1.2825	0.7493	0.9486
s.e.	(0.1498)	(0.0977)	(0.0490)	(0.3375)	(0.0455)	(0.0027)

Figure 1: Intraday transaction counts of AstraZeneca stock on 30 July, 2019



(a) Number of transactions per minute for AstraZeneca stock on 30 July, 2019. (b) Autocorrelation function of the transaction data. (c) Observed and predicted (red) number of transactions per minute calculated by using (3.2.1). (d) Cumulative periodogram plot of the Pearson residuals calculated by using (3.2.1). (e) Observed and predicted (red) number of transactions per minute calculated by using (3.2.2). (f) Cumulative periodogram plot of the Pearson residuals calculated by using (3.2.2)

4. Summary

This paper aims to complement the recent development of the observation-driven models of dynamic counts with a parameter-driven one for the general case, specifically the discrete two parameters exponential family distributions. In particular, we propose to model the mixture of nonlinear dynamic counts by representing a dynamic count process with a simple additive state-space representation. As a result of this, a more flexible dynamic evolution than a stationary AR(p) process of the conditional mean, particularly continuous SETAR process, and the discrete finite semiparametric exponential mixture of dynamic count processes are analysed with the well established linear filtering in that stationarity and geometric ergodicity of the process are obtained under a mild set of conditions. Furthermore, the unknown parameters are proposed to be estimated with quasi maximum likelihood estimation and the asymptotic properties of the QMLEs, particularly $\sqrt{T}$-consistency and normality, are established under a relatively primitive set of conditions.

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Appendix

In this section, the mathematical proofs of the main theoretical results of the paper, particularly Lemmas 2.1 and 2.2, and Theorems 2.1 and 2.2, are presented. The proofs of these are mainly shown within the conventional QML estimation literature, particularly the two main steps. The first step is to show the almost sure pointwise convergence, then to establish the almost sure stochastic equi-continuity. Hereafter, we omit $\mathbb{I}_{j_k}$ for notational simplicity.

Proof of Lemma 2.1

This proof shows the almost sure equivalence between X_t and $\tilde{X}_t$ uniformly over $\vartheta \in D$ in the two steps mentioned above, under Assumptions 2.1 and 2.2. Firstly, let us approximate $\tilde{X}_t$ by using the Taylor expansion of a logarithmic function such that $\tilde{X}_t = X_t + R_t$, where $R_t = \frac{\Delta_T \zeta_t}{\mu_t} + \sum_{l=2} \frac{(-1)^{(l+1)}}{l} \left(\frac{\Delta_T \zeta_t}{\mu_t}\right)^l$. The first main term in R_t, particularly $\frac{\Delta_T \zeta_t}{\mu_t} \equiv R_{1,t}$, is then easily shown to be $o(T^{-1/2})$ $a.s.$ as follows. By using the Cauchy–Schwartz inequality,

$$\begin{aligned} E\left(\frac{\Delta_T \zeta_t}{\mu_t}\right) &\leq \Delta_T \left\{ E\left(\frac{1}{\mu_t^2}\right)^{1/2} E(\zeta_t^2)^{1/2} \right\} \\ &= o(c^{T/2}) \end{aligned}$$

then apply the Markov inequality and Borel–Cantelli Lemma.

The proof is completed by showing the almost sure stochastic equi-continuity of $R_{1,t}$ as follows

$$\begin{aligned} \sup_{||\vartheta-\tilde{\vartheta}||<\delta} \left|R_{1,t}(\vartheta) - R_{1,t}(\tilde{\vartheta})\right| &\leq \sup_{||\vartheta-\tilde{\vartheta}||<\delta} \left\{ \left|\left|R_{1,t}^{(1)}(\bar{\vartheta}_{-d})\right|\right| + \left|\left|R_{1,t}(d) - R_{1,t}(\tilde{d})\right|\right| \right\} \cdot \left|\left|\vartheta - \tilde{\vartheta}\right|\right| \\ &= o(1)\ a.s., \end{aligned} \qquad \text{(A.1.1)}$$

where $\tilde{\vartheta}$ is an δ-neighbourhood of ϑ such that $\lim_{\delta\to 0} \sup_{||\vartheta-\tilde{\vartheta}||<\delta} ||\vartheta - \tilde{\vartheta}|| \to 0$, $\bar{\vartheta}$ lies on the line segment of $\{\rho\vartheta + (1-\rho)\tilde{\vartheta}; \rho \in (0,1)\}$, $R_{1,t}^{(1)}(\bar{\vartheta}_{-d})$ denotes the gradients of $R_{1,t}$ with respect to the vector of the parameters, $\bar{\vartheta}_{-d} = (\bar{a}'s, \bar{\mathfrak{r}}'s, \bar{\beta}'s, \bar{c}, \bar{\pi}'s)^\top$. (A.1.1) can be established by showing that $\left|\left|R_{1,t}^{(1)}(\bar{\vartheta}_{-d})\right|\right| = O(1)$ $a.s.$ because $||R_{1,t}(d) - R_{1,t}(\tilde{d})|| = o(1)$ $a.s.$, given the discreteness of $d's$. Now let us consider $R_{1,t}^{(1)}(\bar{\vartheta}_{-d})$ as follows

$$\begin{aligned} E\left(\left|\left|R_{1,t}^{(1)}(\bar{\vartheta}_{-d})\right|\right|\right) &\leq E||R_{a_{k,0},1,t}^{(1)}|| + E||R_{a_{j_k},1,t}^{(1)}|| + E||R_{\mathfrak{r}_{j_k},1,t}^{(1)}|| + E||R_{\pi_k,1,t}^{(1)}|| \\ &\quad + E||R_{\beta_k}^{(1)}|| + E|R_{c,t}^{(1)}| \\ &= O\left(\left(\frac{T}{2} + \delta\right) c^{\frac{T-2}{2}+\delta}\right) \end{aligned}$$

with the similar arguments to those. The each components of $R_{1,t}^{(1)}(\bar{\vartheta}_{-d})$ are $R_{a_{k,0},1,t}^{(1)} = -\frac{\Delta_T \zeta_t \pi_k}{\mu_t}$, $R_{a_{j_k},1,t}^{(1)} = -\frac{\Delta_T \zeta_t \left\{\pi_k X_{k,t-l}\right\}}{\mu_t}$, $R_{\mathfrak{r}_{j_k},1,t}^{(1)} = -\frac{\Delta_T \zeta_t}{\mu_t} \pi_k a_{k,d_k,j_k}$, $R_{\pi_k,t}^{(1)} = -\frac{\Delta_T \zeta_t \ln \mu_{k,t}}{\mu_t}$,

$R^{(1)}_{\beta_k} = \frac{\Delta_T \zeta_t}{2\mu_t}\pi_k V(I_{k,t};\mu_{k,t})$ and $R^{(1)}_{c,1,t} = \frac{\zeta_t}{\mu_t} O\left(\left(\frac{T}{2}+\delta\right) c^{\frac{T-2}{2}+\delta}\right)$. Note that the derivative of $X_{k,t}$ with respect to $\mathfrak{r}_{j_k}$ is not well defined, so we set $\frac{\partial X_{k,t}}{\partial \mathfrak{r}_{j_k}} = a_{k,d_k,j_k}$, following the arguments in Chan and Tsay (1998). By applying the Markov inequality and Borel–Cantelli Lemma, $||R^{(1)}_{1,t}(\bar{\vartheta}_{-d})|| = o(1)\ a.s.$ □

Proof of Lemma 2.2

This proof establishes (A.2.1) below, under Assumptions 2.1 to 2.3 and the independence assumption of the data generating processes of each cluster.

$$\begin{aligned}\sup_{\psi\in D_\psi} |\mathcal{L}(\psi|\mathcal{F}_{t-1}) - \mathcal{L}^*(\psi)| &= \sup_{\psi\in D_\psi}\left|\frac{1}{T}\sum_{t=p_K+1}^{T+p_K}\left\{\sum_{k=1}^{K}\left(\ln \widehat{ED}^*_{k,t} - \ln ED^*_{k,t}\right)\right\}\right| \\ &= O(T^{-1/2})\ a.s.. \qquad \text{(A.2.1)}\end{aligned}$$

Let us firstly consider the strong pointwise convergence of $\mathcal{L}(\psi|\mathcal{F}_{t-1}) - \mathcal{L}^*(\psi)$. By using the Taylor expansion argument of a logarithmic function and the results in Chapters 3 and 7 of Caines (1987), particularly $\hat{\mu}_{k,t|t-1} = \mu_{k,t} + o(c^{T/2})\ a.s.$ and $\hat{\pi}_k = \pi_k + O(T^{-1/2})\ a.s.$, and hence

$$\ln \widehat{ED}^*_{k,t} = \ln ED^*_{k,t} + \frac{ED^{*(1)}_{k,t,\pi}(\hat{\pi}-\pi_k)}{ED^*_{k,t}} + \frac{ED^*_{k,t,\mu}(\hat{\mu}_{k,t|t-1}-\mu_{k,t})}{ED^*_{k,t} + ED^{*(1)}_{k,t,\pi}(\hat{\pi}-\pi)} + o(c^{T/2})\ a.s..$$

We can then rewrite $\mathcal{L}(\psi|\mathcal{F}_{t-t}) - \mathcal{L}^*(\psi)$ as follows

$$\begin{aligned}&\mathcal{L}(\psi|\mathcal{F}_{t-1}) - \mathcal{L}^*(\psi) \\ &= \frac{1}{T}\sum_{t=p_K+1}^{T+p_K}\sum_{k=1}^{K}\left\{\frac{ED^{*(1)}_{k,t,\pi}(\hat{\pi}-\pi_k)}{ED^*_{k,t}} + \frac{ED^{*(1)}_{k,t,\mu}(\hat{\mu}_{k,t|t-1}-\mu_{k,t})}{ED^*_{k,t} + ED^{*(1)}_{k,t,\pi}(\hat{\pi}-\pi)}\right\} + o(T^{-1/2})\ a.s..\end{aligned}$$

By using the Cauchy–Schwartz inequality,

$$\begin{aligned}&E\left(\mathcal{L}(\psi|\mathcal{F}_{t-1}) - \mathcal{L}^*(\psi)\right)^2 \\ &= \sum_{k=1}^{K}\left(\frac{1}{T^2}\sum_{t=p_K+1}^{T+p_K} E\left\{\left((\hat{\pi}_k-\pi_k)^2\left(\frac{ED^{*(1)}_{k,t,\pi}}{ED^*_{k,t}}\right)^2 + \left(\frac{ED^{*(1)}_{k,t,\mu}(\hat{\mu}_{k,t|t-1}-\mu_t)}{ED^*_{k,t} + ED^{*(1)}_{k,t,\pi}(\hat{\pi}_k-\pi_k)}\right)^2\right.\right.\right. \\ &\qquad\left.\left. +2(\hat{\pi}_k-\pi_k)(\hat{\mu}_{k,t|t-1}-\mu_{k,t})\frac{ED^{*(1)}_{k,t,\pi}}{ED^*_{k,t}}\frac{ED^{*(1)}_{k,t,\mu}}{ED^*_{k,t} + ED^{*(1)}_{k,t,\pi}(\hat{\pi}_k-\pi_k)}\right)\right\} \\ &\quad +2\frac{1}{T^2}\sum_{t=p_K+1}^{T+p_K}\sum_{\iota\neq t}^{T=p_K} E\left\{(\hat{\pi}_k-\pi_k)^2\left(\frac{ED^{*(1)}_{k,t,\pi}}{ED^*_{k,t}}\right)\left(\frac{ED^{*(1)}_{k,\iota,\pi}}{ED^*_{k,\iota}}\right)\right. \\ &\qquad + \left(\frac{ED^{*(1)}_{k,t,\mu}(\hat{\mu}_{k,t|t-1}-\mu_{k,t})}{ED^*_{k,t} + ED^{*(1)}_{k,t,\pi}(\hat{\pi}_k-\pi_k)}\right)\left(\frac{ED^{*(1)}_{k,\iota,\mu}(\hat{\mu}_{k,\iota|\iota-1}-\mu_{k,\iota})}{ED^*_{k,\iota} + ED^{*(1)}_{k,\iota,\pi}(\hat{\pi}_k-\pi_k)}\right) \\ &\qquad\left.\left. +2(\hat{\pi}_k-\pi)(\hat{\mu}_{k,\iota|\iota-1}-\mu_{k,\iota})\left(\frac{ED^{*(1)}_{k,t,\pi}}{ED^*_{k,t}}\right)\left(\frac{ED^{*(1)}_{k,\iota,\mu}}{ED^*_{k,\iota} + ED^{*(1)}_{k,\iota,\pi}(\hat{\pi}_k-\pi_k)}\right)\right\}\right) \\ &= O(T^{-1}),\end{aligned}$$

particularly under Assumption 2.3. Then, by applying the Chebyshev inequality and Borel–Cantelli Lemma,

$$\mathcal{L}(\psi|\mathcal{F}_{t-1}) - \mathcal{L}^*(\psi) = O(T^{-1/2}) \ a.s..$$

The next step is then to show the stochastic equi-continuity of $\mathcal{L}(\psi|\mathcal{F}_{t-1}) - \mathcal{L}^*(\psi)$. Hereafter, let us denote $\mathcal{L}(\psi|\mathcal{F}_{t-1}) - \mathcal{L}^*(\psi)$ as $\mathcal{L}_1(\psi)$ for the sake of notational simplicity.

$$\begin{aligned} \sup_{||\psi-\tilde{\psi}||<\delta} \left|\mathcal{L}_1(\psi) - \mathcal{L}_1(\tilde{\psi})\right| &\leq \sup_{||\psi-\tilde{\psi}||<\delta} \left\{||\mathcal{L}_1^{(1)}(\bar{\psi}_{-d})|| + |\mathcal{L}_1(d) - \mathcal{L}_1(\tilde{d})|\right\} \cdot ||\psi - \tilde{\psi}|| \\ &= o(1) \ a.s., \end{aligned}$$

where $\mathcal{L}_1^{(1)}(\bar{\psi}_{-d})$ denotes the first gradients of $\mathcal{L}_1(\bar{\psi})$ with respect to $\bar{\psi}_{-d} = (\bar{a}'s, \bar{\mathfrak{r}}'s, \bar{\beta}'s)^\top$. Hence, it is

$$\frac{\partial \mathcal{L}_1(\bar{\psi}_{-d})}{\partial \bar{\psi}_{-d}} = \frac{1}{T} \sum_{t=p_K+1}^{T+p_K} \left\{ \sum_{k=1}^{K} \left(\frac{\widehat{ED}^{*(1)}_{k,t,\mu}(\hat{\mu}_{k,t|t-1})'_{\bar{\psi}_{-d}}}{\widehat{ED}^*_{k,t}} - \frac{ED^{*(1)}_{k,t,\mu}(\mu_{k,t|t-1})'_{\bar{\psi}_{-d}}}{ED^*_{k,t}} \right) \right\}, \tag{A.2.2}$$

where $\widehat{ED}^{*(1)}_{k,t,\mu}$ is the first derivative of $\widehat{ED}^*_{k,t}$ with respect to $\hat{\mu}_{k,t|t-1}$, and $(\mu_{k,t})'_{\bar{\psi}_{-d}}$ and $(\hat{\mu}_{k,t|t-1})'_{\bar{\psi}_{-d}}$ denote the first gradients of $\mu_{k,t}$ and $\hat{\mu}_{k,t|t-1}$ with respect to $\bar{\psi}_{-d}$, respectively, and which are as follows

$$(\mu_{k,t})'_{\bar{\psi}_{-d}} = \begin{pmatrix} \mu_{k,t} & \mu_{k,t}X_{k,t-l} & \mu_{k,t}a_{k,j,d-} & \mu_{k,t}(X_{k,t})'_{\bar{\beta}_k} \end{pmatrix}^\top$$

and

$$(\hat{\mu}_{k,t|t-1})'_{\bar{\psi}_{-d}} = \begin{pmatrix} \hat{\mu}_{k,t|t-1} \\ \hat{\mu}_{k,t|t-1}\hat{X}_{k,t-l|t-1} \\ \hat{\mu}_{k,t|t-1}a_{k,j,d-} \\ \hat{\mu}_{k,t|t-1}\{(X_{k,t})'_{\bar{\beta}_k} + (X_{k,t})''_{\bar{\beta}_k,\bar{\mu}_{k,t}}(\hat{X}_{k,t|t-1} - X_{k,t})\} \end{pmatrix}$$

with $\hat{X}_{k,t-l|t-1}$ denoting the minimum conditional mean-squared error estimate of $X_{k,t-l}$ given $\mathcal{F}_{t-1}$, and $(X_{k,t})'_{\bar{\beta}_k} = \frac{V(I_{k,t};\mu_{k,t})}{2\mu^2_{k,t}}$ and $(X_{k,t})''_{\bar{\beta}_k,\bar{\mu}_{k,t}} = -\frac{V(I_{k,t};\mu_{k,t})}{\mu^3_{k,t}}$. We next use the Taylor expansion argument below

$$\widehat{ED}^{*(1)}_{k,t,\mu} = ED^{*(1)}_{k,t,\mu} + ED^{*(2)}_{k,t,\mu}(\hat{\mu}_{k,t|t-1} - \mu_{k,t}) + o(T^{-1/2}) \ a.s.,$$

(A.2.2) is then rewritten as follows

$$\mathcal{L}_1^{(1)}(\bar{\psi}_{-d}) = \mathcal{L}_{11}^{(1)}(\bar{\psi}_{-d}) + \mathcal{L}_{12}^{(1)}(\bar{\psi}_{-d}) + O(T^{-1}) \ a.s.,$$

where

$$\mathcal{L}_{11}^{(1)}(\bar{\psi}_{-d})$$
$$= \frac{1}{T}\sum_{t=p_K+1}^{T+p_K}\left\{\sum_{k=1}^{K}\frac{\left(ED_{k,t}^{*}ED_{k,t\mu,\pi}^{*(2)}(\hat{\mu}_{k,t|t-1})'_{\bar{\psi}_{-d}} - ED_{k,t,\mu}^{*(1)}ED_{k,t,\pi}^{*(1)}(\mu_{k,t|t-1})'_{\bar{\psi}_{-d}}\right)(\hat{\pi}_k - \pi_k)}{ED_{k,t}^{*}\left(ED_{k,t}^{*} + ED_{k,t,\pi}^{*(1)}(\hat{\pi}_k - \pi_k) + ED_{k,t,\mu}^{*(1)}(\hat{\mu}_{k,t|t-1} - \mu_{k,t})\right) + o(T^{-1})}\right\}$$

and

$$\mathcal{L}_{12}^{(1)}(\bar{\psi}_{-d})$$
$$= \frac{1}{T}\sum_{t=p_K+1}^{T=p_K}\sum_{k=1}^{K}\left\{\frac{ED_{k,t}^{*}ED_{k,t\mu}^{*(1)}\left\{(\hat{\mu}_{k,t|t-1})'_{\bar{\psi}_{-d}} - (\mu_{k,t|t-1})'_{\bar{\psi}_{-d}}\right\}}{ED_{k,t}^{*}\left(ED_{k,t}^{*} + ED_{k,t,\pi}^{*(1)}(\hat{\pi}_k - \pi_k) + ED_{k,t,\mu}^{*(1)}(\hat{\mu}_{k,t|t-1} - \mu_{k,t})\right) + o(T^{-1})}\right.$$
$$\left. + \frac{\left(ED_{k,t}^{*}ED_{k,t,\mu,\pi}^{*(2)}(\hat{\mu}_{k,t|t-1})'_{\hat{\psi}_{-d}} - \left(ED_{k,t,\mu}^{*(1)}\right)^2(\mu_{k,t|t-1})'_{\bar{\psi}_{-d}}\right)(\hat{\mu}_{k,t|t-1} - \mu_{k,t})}{ED_{k,t}^{*}\left(ED_{k,t}^{*} + ED_{k,t,\pi}^{*(1)}(\hat{\pi}_k - \pi_k) + ED_{k,t,\mu}^{*(1)}(\hat{\mu}_{k,t|t-1} - \mu_{k,t})\right) + o(T^{-1})}\right\}.$$

By applying the Cauchy–Schwartz and Chebyshev inequalities, and the Borel–Cantelli lemma to $\mathcal{L}_{11}^{(1)}(\bar{\psi}_{-d})$, and the Markov inequality and Borel–Cantelli lemma to $\mathcal{L}_{12}^{(1)}(\bar{\psi}_{-d})$, respectively, we obtain $\mathcal{L}_{1}^{(1)}(\bar{\psi}_{-d}) = o(1)$ *a.s.*. □

Proof of Theorem 2.1

This proof can be shown in the two steps under Assumptions 2.1 to 2.4, with the independence assumption on the data generating process of each cluster. The first step is to show the almost sure convergence of $\hat{\psi}$ to ψ uniformly over D_ψ by using similar arguments to those in Lemma 2.2. We can then verify the identification condition of ψ_0.

The first step can be shown by establishing (A.3.1) below

$$\sup_{\psi\in D_\psi}|\mathcal{L}^*(\psi) - E\mathcal{L}^*(\psi)| = O(T^{-1/2})\ a.s.. \tag{A.3.1}$$

Firstly, it is

$$E\left(\mathcal{L}^*(\psi) - E\mathcal{L}^*(\psi)\right)^2 = \frac{1}{T^2}\sum_{t=p_K+1}^{p_K+T}\left\{\sum_{k=1}^{K}E\left(\ln ED_{k,t}^{*} - E\ln ED_{k,t}^{*}\right)^2\right\}$$
$$+2\frac{1}{T^2}\sum_{t=p_K+1}^{T+p_K}\sum_{\iota\neq t}^{T+p_K}\left\{\sum_{k=1}^{K}E\left(\left\{\ln ED_{k,t}^{*} - E\ln ED_{k,t}^{*}\right\}\left\{\ln ED_{k,\iota}^{*} - E\ln ED_{k,\iota}^{*}\right\}\right)\right\}$$

then apply the Chebyshev inequality and Borel–Cantelli lemma. We thus obtain that $\mathcal{L}^*(\psi) = E\mathcal{L}^*(\psi) + O(T^{-1/2})$ *a.s.*. The next step is to show the stochastic equi-continuity of $\mathcal{L}^*(\psi) - E\mathcal{L}^*(\psi)$. This can be established by showing that

$E\left(\frac{\partial \mathcal{L}^*(\psi_{-d})}{\partial \psi_{-d}}\right)^2 = O(T^{-1})$ as follows

$$\begin{aligned} & E\left(\frac{\partial \mathcal{L}^*(\psi_{-d})}{\partial \psi_{-d}}\right)^2 \\ & = \sum_{k=1}^{K} \frac{1}{T^2}\left(\sum_{t=1}^{T}\left\{E\left(\frac{ED^{*(1)}_{k,t,\psi_{-d}}}{ED^*_{k,t}}\right)^2 + \sum_{t=1}^{T}\sum_{\iota\neq t}^{T} E\left(\frac{ED^{*(1)}_{k,t,\psi_{-d}}}{ED^*_{k,t}}\frac{ED^{*(1)}_{k,\iota,\psi_{-d}}}{ED^*_{k,\iota}}\right)\right\}\right) \\ & = O(T^{-1}) \end{aligned} \tag{A.3.2}$$

particularly under Assumptions 2.4 (i) and (ii). By applying the Chebyshev inequality and Borel–Cantelli lemma to (A.3.2), (A.3.1) is shown.

The identification condition of ψ_0 is then verified by using the counter argument as follows. Consider the Jensen's inequality in (A.3.3), taking the expectation with ψ_0 as follows

$$\begin{aligned} E\mathcal{L}^*(\psi) - E\mathcal{L}^*(\psi_0) & \leq \frac{1}{T}\sum_{t=p_K+1}^{T+p_K}\left\{\sum_{k=1}^{K} \ln E\left(\frac{ED^*(\mu_{k,t},\beta_k,\pi_k)}{ED^*(\mu_{0,k,t},\beta_{0,k},\pi_k)}\right)\right\} \\ & \leq 0. \end{aligned} \tag{A.3.3}$$

The equality of (A.3.3) holds when $\psi \to \psi_0$. Hence, ψ_0 is not uniquely identified when there is a sequence such that $\psi_T \in D_\delta(\psi^*)$ converges to $\psi^* \in \bar{D}_\delta(\psi_0) \cap D_\psi$, where $D_\delta(\cdot)$ and $\bar{D}_\delta(\cdot)$ represent an open δ-neighbourhood and its complement, respectively, and $\lim_{T\to\infty} E\mathcal{L}^*(\psi^*) \to \lim_{T\to\infty} E\mathcal{L}^*(\psi_0)$. Hence, the unique identification condition requires that $\limsup_{T\to\infty}\left(\max_{\psi\in\bar{D}_\delta(\psi_0)\cap D_\psi} E\mathcal{L}^*(\psi)\right) \neq E\mathcal{L}^*(\psi_0)$ for any ψ. □

Proof of Theorem 2.2

The asymptotic normality of our proposed QMLEs can be obtained by considering the extension of the mean value theorem of $\frac{\partial \mathcal{L}^*(\hat{\psi}^*)}{\partial \psi^*}$ as follows

$$\frac{\partial \mathcal{L}^*(\hat{\psi}^*)}{\partial \psi^*} = \frac{\partial \mathcal{L}^*(\psi_0^*)}{\partial \psi^*} + \frac{\partial^2 \mathcal{L}^*(\bar{\psi}^*)}{\partial \psi^* \partial \psi^{*\top}}(\hat{\psi}^* - \psi_0^*),$$

where $\bar{\psi}^*$ is between $\hat{\psi}^*$ and ψ_0^*. Therefore, the asymptotic normality of $\hat{\psi}^*$ can be obtained by showing that $\sqrt{T}\frac{\partial \mathcal{L}^*(\psi^*)}{\partial \psi^*}\Big|_{\psi^*=\psi_0^*} \sim N(0, A_0(\psi_0^*))$ and $\frac{\partial^2 \mathcal{L}^*(\psi^*)}{\partial \psi^* \partial \psi^{*\top}} = \lim_{T\to\infty} E\frac{\partial^2 \mathcal{L}^*(\psi^*)}{\partial \psi^* \psi^{*\top}} + O_P(T^{-1/2})$ uniformly over D_{ψ^*}. Particularly under Assumptions 2.4 (i) and (iii), $\sqrt{T}\frac{\partial \mathcal{L}^*(\psi^*)}{\partial \psi^*} \sim N(0, A_0(\psi_0^*))$ can be easily shown by using the small and large blocks arguments (see Chapter 2 of Fan and Yao (2008) for details).

The last step of this proof is to establish below

$$\sup_{\psi^*\in D_{\psi^*}} ||B_T(\psi^*) - B_0(\psi^*)||_F = O_P(T^{-1/2}), \tag{A.4.1}$$

where $||\cdot||_F$ denotes the Frobenius norm, $B_T(\psi^*) = \frac{\partial^2 \mathcal{L}^*(\psi^*)}{\partial \psi^* \partial \psi^{*\top}}$ and $B_0(\psi^*) = \lim_{T\to\infty} E \frac{\partial^2 \mathcal{L}^*(\psi^*)}{\partial \psi^* \partial \psi^{*\top}}$. The result of (A.4.1) is obtained by applying the Chebyshev inequality. Next, the stochastic equi-continuity of $B_T(\psi^*)$ can be established by showing that

$$||C_T(\bar{\psi}^*) - C_0(\bar{\psi}^*)||_F \leq ||C_T(\bar{\psi}^*)||_F = O_p(T^{-1/2}), \tag{A.4.2}$$

where $C_T(\bar{\psi}^*) = \frac{\partial B_T(\bar{\psi}^*)}{\partial \bar{\psi}^*}$ and $C_0(\bar{\psi}^*) = \lim_{T\to\infty} E C_T(\bar{\psi}^*)$. The result of (A.4.2) is then obtained by applying the Chebyshev inequality. $\square$