Semiparametric Methods in Nonlinear Time Series Analysis: A Selective Review

PATRICK SAART†[1,2], JITI GAO‡ AND NAM HYUN KIM†
University of Canterbury† and *Monash University*‡

Abstract
Time series analysis is a tremendous research area in statistics and econometrics. In a previous review, the author was able break down up to fifteen key areas of research interest in time series analysis. Nonetheless, the aim of the review in this current paper is not to cover a wide range of somewhat unrelated topics on the subject, but the key strategy of the review in this paper is to begin with a core, the "curse of dimensionality" in nonparametric time series analysis, and explore further in a metaphorical domino-effect fashion into other closely related areas in semiparametric methods in nonlinear time series analysis.

JEL Classification: C12, C14, C22
Keywords: Autoregressive time series; nonparametric model; nonstationary process; partially linear structure, semiparametric method

1. Introduction

In time series regression, nonparametric methods have been quite popular both for prediction and for characterizing nonlinear dependence. Let $\{Y_t\}$ and $\{X_t\}$ be the one-dimensional and d-dimensional time series data, respectively. For a vector of time series data $\{Y_t, X_t\}$, the conditional mean function $E[Y_t|X_t = x]$ of Y_t on $X_t = x$ may be estimated nonparametrically by the Nadaraya–Watson (NW) estimator when the dimensionality d is less than or equal to three. When d is greater than three, the conditional mean can still be estimated using the NW estimator and asymptotic theory can be constructed. However, due to a well-known problem often referred to in the literature as the curse of dimensionality, this may not be recommended in practice unless the number of data points is extremely large. There are multiple phenomena in the literature which are referred to as the curse of dimensionality in various domains, e.g. numerical analysis, sampling, combinatorics, etc. For the sake of clarity, let us give a simple example of the curse of dimensionality in nonparametric regression.

Example 1. Let there be a set of data points (U, V), where

$$U = g(V) \ + \ \text{noise with a mean of zero.}$$

[1]Patrick Saart, Department of Mathematics and Statistics, University of Canterbury, New Zealand, Email: patrick.wsaart@canterbury.ac.nz, Website: https://sites.google.com/site/patrickwsaart/home
[2]The authors thank the three anonymous referees for their comments and suggestions.

Electronic copy available at: http://ssrn.com/abstract=2180598

The data $(\mathbf{V}, \mathbf{U}) \doteq \{(V_t, U_t)\}_{t=1}^n$ are assumed to be drawn identically and independently distributed (i.i.d.) from a distribution over a joint input-output space $\mathcal{V} \times \mathcal{U}$. The input space $\mathcal{V}$ is usually assumed to be a subset of $\mathbb{R}^d$, i.e. V is a vector of d features. The output space $\mathcal{U}$ is assumed to be a subset of $\mathbb{R}^{d\prime}$ and is a random vector satisfying $E[U|V = v] = g(v)$. An objective of the nonparametric regression is to approximate g with a nonparametric regressor, say, g_n. Under some smoothing condition of g; e.g. Lipschitz, a number of nonparametric estimators can be shown to satisfy $E_{\mathbf{V},\mathbf{U}}||g_n - g||^2 \leq O\left(n^{-2/(2+d)}\right)$. For instance, this is the rate for a kernel estimator. Such a rate implies that we need a sample size n exponential in d in order to approximate g. Hence, when d is high, as is often the case in modern applications, $n > 2^d$ is impractical. Furthermore, to get some intuition into the reason for such a rate, consider that nonparametric approaches, such as the NW estimator, operate by approximating the target function locally (on its domain $\mathcal{V}$) by simpler functions. There are necessarily some local errors and these errors aggregate globally. To approximate the entire function well, we need to do well in most local areas. Suppose, for instance, that the target function is well approximated by constants in regions with a radius of at most $0 < r < 1$. In how many ways can we divide up the domain $\mathcal{V}$ into smaller regions with a radius of at most r ? If $\mathcal{V}$ is d-dimensional then the smallest such partition is of size $O(r^{-d})$. We will need data points to fall into each such region if we hope to do well locally everywhere. In the other words, we will need a data set that is exponential in d in size. ■

Over recent years, a number of review papers on nonparametric and semiparametric methods have become available in the literature. Below, let us introduce a few of those that are the most relevant to the materials presented in the current paper. Firstly, there are two books by Härdle et al. (2000) and Gao (2007) that introduced some nonparametric and semiparametric nonlinear time series models as well as establishing various new results to enrich the literature. Meanwhile, a review by Fan (2005) focuses on nonparametric techniques used for estimating stochastic diffusion models, especially the drift and the diffusion functions, based on either discretely or continuously observed data. The paper begins with a brief review of some useful stochastic models for modeling stock prices and bond yields, which includes the Cox Ingersoll Ross model by Cox et al. (1985), Vasicek model by Vasicek (1977) and the Chan Karolyi Longstaff Sanders model of Chan et al. (2012). Furthermore, the paper reviews in the paper techniques for estimating state price densities and transition densities, and their applications in asset pricing and testing for parametric diffusion models. Some important references in that review are Aït-Sahalia (1996), Aït-Sahalia (2002), Aït-Sahalia and Lo (2002), and Fan et al. (1996).

Secondly, the review by Härdle et al. (2007) provides a fairly broad survey of many

Electronic copy available at: http://ssrn.com/abstract=2180598

nonparametric analysis techniques for time series. Specifically, the review discusses nonparametric methods for estimating the spectral density, the conditional mean, higher order conditional moments and conditional densities. Density estimation with correlated data, bootstrap methods for time series and nonparametric trend analysis were also reviewed.

Finally, the two review papers by Gao (2012) and Sun and Li (2012) summarize some recent theoretical developments in nonparametric and semiparametric techniques as applied to nonstationary or near-nonstationary variables. The first paper introduces a class of semi-linear time series models that incorporate both nonstationarity and endogeneity. The author also introduces and then discusses a class of the so–called "nearly integrated" time series models. The second paper begins with a review on various concepts of the integrated series of order zero and of order one, and cointegration for a linear model as they are available in the literature. It then discusses some popular nonlinear parametric models beginning with those for stationary data, such as the self-exciting threshold autoregressive models (Tong and Lim (1980), Chan (1993)) and the smooth transition autoregressive models (Chan and Tong (1986), and Van Dijk et al. (2002)), then some nonlinear error correction models and nonlinear cointegrating models (Teräsvirta et al. (2011), Dufrénot and Mignon (2002)). Thirdly, it discusses nonparametric models with nonstationary data in a similar fashion to the above parametric cases (nonparametric autoregressive then nonparametric cointegrating models). The review concentrates on existing works on the consistency of nonparametric estimators, with some key studies being Wang and Phillips (2009a), Wang and Phillips (2009b), Karlsen and Tjøstheim (2001), Karlsen et al. (2007), Karlsen et al. (2010). Finally, it presents a discussion on semiparametric models with nonstationary data. The focus of the review is on semiparametric varying coefficient cointegrating models and on semiparametric binary choice models.

The current paper complements these existing reviews by filling in some gaps, which are currently left unexplored. First, it discusses various issues (e.g. identification conditions, estimation procedures and asymptotic properties) involving semiparametric time series models, which are described as tools for circumventing the curse of dimensionality. Establishing ways to circumventing the curse of dimensionality is traditionally an important objective for a large number of studies in nonparametric statistics. There are essentially two approaches discussed in the literature. The first is largely concerned with dimension reduction; some well–known examples of studies that fall into this category are Li (1991), Cook (1998) and Xia et al. (2002). The review in the current paper focuses on studies in the second category, namely function approximation using semiparametric specifications. The paper first introduces three of the most well-known and successfully applied semiparametric time series regression specifications in the literature, namely the partially linear (PL), additive and the single-index models. It then reviews various specification tests for the semiparametric models in detail, including tests for linear regression. Since the focus is on time series, the current paper also presents a thorough discussion on re-specification of the above semiparametric models to form semiparametric autoregressive models and their specifications test. It is argued that although these semiparametric models are non-nested, they share some important similarities, especially their intolerance to the endogeneity of the error term in order to obtain consistent estimates of the models. Addressing such a problem, in practice, involves directly estimating semiparametric models with generated regressors. The current paper presents (in Section 5) a review of a recent method of addressing the endogeneity problem in semiparametric time series models and other semiparametric models with generated regressors, which share similar characteristics. One model, in particular, has a direct application to financial econometrics and explores a similar research area to those reviewed in Fan (2005). Finally, the current paper reviews semiparametric models with nonstationary data. However, the focus of this review is quite different to that of Sun and Li (2012). Since the issues of semiparametric models with nonstationary data is such a large area of research, which in itself warrants a separated review, the current review focuses mainly on (i) semiparametric models that have been established to help detect and estimate trend and seasonality, and (ii) semiparametric models involved both endogeneity and nonstationarity.

In summary, the logic of this paper can be described metaphorically as a domino-effect as follows. The first point of impact is on nonlinear time series analysis. The second, third and the fourth dominoes to fall are the curse of dimensionality, the semiparametric time series models and their specification testings, respectively. The fifth is the required conditions shared by these popular semiparametric models, namely the exogeneity of the error term and stationarity of the time series. The stationarity condition can then be linked to the respecification of the semiparametric time series models to construct nonlinear autoregressive time series models. Furthermore, addressing the breakdown in the exogeneity condition leads to the emergence of semiparametric models with generated regressors, while addressing the breakdown in the stationarity leads to semiparametric models of nonstationary data.

The remainder of this paper is structured as follows. Section 2 discusses semiparametric models for time series, while Section 3 considers some specification tests for these models. Section 4 discusses nonlinear autoregressive models and their specification testing. Section 5 reviews the endogeneity problem in semiparametric time series models and models with generated regressors. Section 6 discusses semiparametric models with nonstationary data. Section 7 concludes and presents a discussion on future research.

2. Semiparametric Models for Time Series

This section discusses various issues involving the estimation and identification of three of the most well known and successfully applied semiparametric time series regression models in the literature, namely the PL, additive and the single-index models. Below, let us begin with the semiparametric PL time series models.

Partially Linear Semiparametric Model for Time Series

Since their introduction to economic literature in the 1980s by Engle et al. (1986), the PL model has attracted much attention among econometricians and applied statisticians; see Heckman (1986), Robinson (1988), Fan et al. (1995), Härdle et al. (2000) and Gao (2007) for example. In some empirical studies, the PL model is able to help avoid the impact of the curse of dimensionality by allowing *a priori* information concerning the possible linearity of some of the components to be included in the model. More specifically, the PL models look at approximating the conditional mean function $m(X_t) = m(U_t, V_t) = E[Y_t|U_t, V_t]$ by a semiparametric function of the form:

$$m_1(U_t, V_t) = \mu + U_t^\tau \beta + g(V_t) \tag{2.1}$$

such that $E[Y_t - m_1(U_t, V_t)]^2$ is minimized over a class of semiparametric functions of the form $m_1(U_t, V_t)$ subject to $E[g(V_t)] = 0$ for the identifiability of $m_1(U_t, V_t)$, where μ is an unknown parameter, $\beta = (\beta_1, \ldots, \beta_q)^\tau$ is a vector of unknown parameters, $g(\cdot)$ is an unknown function over $\mathbb{R}^p$, and $U_t = (U_{t1}, \ldots, U_{tq})^\tau$ and $V_t = (V_{t1}, \ldots, V_{tp})^\tau$ may be vectors of time series variables. Such a minimization problem is equivalent to minimizing $E[Y_t - \mu - U_t^\tau \beta - g(V_t)]^2 = E\left[E\left\{(Y_t - \mu - U_t^\tau \beta - g(V_t))^2 | V_t\right\}\right]$ over some (μ, β, g). This implies that $g(V_t) = E[(Y_t - \mu - U_t^\tau \beta)|V_t]$ and $\mu = E[Y_t - U_t^\tau \beta]$, with β being given by:

$$\beta = \Sigma^{-1} E[(U_t - E[U_t|V_t])(Y_t - E[Y_t|V_t])], \tag{2.2}$$

provided that the inverse $\Sigma^{-1} = (E[U_t - E[U_t|V_t])(E[U_t - E[U_t|V_t])^\tau])^{-1}$ exists. This also shows that $m_1(U_t, V_t)$ is identifiable under the assumption of $E[g(V_t)] = 0$. Some important motivations for using the functional form in (2.1) for both independent and time series data analysis can be found in Härdle et al. (2000). Based on an i.i.d. random sample, it has been shown that the parameter vector β in various versions of (2.1) can be consistently estimated at $\sqrt{n}$-rate, see Heckman (1986), Robinson (1988) and Fan et al. (1995), for example. For dependent processes, traditionally such a result is established under a set of somewhat more stringent conditions, e.g. the independence between $\{U_t\}$ and $\{V_t\}$, as in Truong and Stone (1994). On the other hand, Fan and Li (1999b) extend the $\sqrt{n}$-consistency and asymptotic normality results of Robinson (1988) and Fan et al. (1995) for independent observations to a strictly stationary, absolutely regular β-mixing processes under a similar set of conditions. However, these results are not applicable to a weaker condition of strong mixing processes and the case where $p > 3$.

Although the PL specification can reduce the dimensionality of nonparametric time series regression significantly in some cases, it is also true that the PL time series model in (2.1) may still suffer from the curse of dimensionality when $g(\cdot)$ is not necessarily additive and $p \geq 3$. A method of addressing such an issue in the literature is to establish an effective model selection procedure to ensure that both the linear and the nonparametric components of the model are of the smallest possible dimension. Gao and Tong (2004), for example, propose using a semiparametric leave n_v out cross-validation function for the choice of both the parametric and nonparametric regressors, where $n_v > 1$ is a positive integer satisfying $n_v \to \infty$ as the number observations expands to infinity. Although the details of the test can be found in the paper (see also Gao (2007)), let us note an important advantage of such a method which is the fact that it provides a general model selection procedure in determining asymptotically whether both the linear time series component and the nonparametric time series component are of the smallest possible dimension. Hence, it can help to reduce the impact of the curse of dimensionality arising from using nonparametric techniques to estimate $g(\cdot)$ in (2.1).

Additive Semiparametric Model for Time Series

When $g(\cdot)$ is additive, i.e. $g(x) = \sum_{i=1}^{p} g_i(x_i)$, the form of $m_1(U_t, V_t)$ can be written as

$$m_1(U_t, V_t) = \mu + U_t^\tau \beta + \sum_{i=1}^{p} g_i(V_{ti}), \tag{2.3}$$

subject to $E[g_i(V_{ti})] = 0$, for all $1 \leq i \leq p$, for the identifiability of $m_1(U_t, V_t)$ in (2.3), where $g_i(\cdot)$ for $1 \leq i \leq p$ are all unknown one dimensional functions over $\mathbb{R}^1$. The main ideas of the discussion on the semiparametric additive model above can be taken from Gao et al. (2006), who established an estimation procedure for semiparametric spatial regression. The semiparametric kernel estimation approach, as discussed in Gao et al. (2006), involves a few important steps. The first step is to estimate μ and $g(\cdot)$ by assuming that β is known. Observe that under such an assumption, we have:

$$g(x) = g(x, \beta) = E[Y_t - \mu - U_t^\tau \beta)|V_t = x] = E[(Y_t - E[Y_t] - (U_t - E[U_t])^\tau \beta)|V_t = x], \tag{2.4}$$

using the fact that $\mu = E[Y_t] - E[U_t^\tau \beta]$, which can be estimated by the standard local linear estimation. (See, e.g. Fan and Gijbels (1996)) The second step is to apply the marginal integration technique of Linton and Nielsen (1995) to obtain $g_1, \ldots, g_p$ of (2.3) based on $g(V_t) = g(V_{t1}, \ldots, V_{tp}) = \sum_{t=1}^{p} g_l(V_{tl})$. Since $E[g_l(V_{tl})] = 0$ for $l = 1, \ldots, p$, we have, for a fixed value of k, $g_k(x_k) = E[g(V_{t1}, \ldots, x_k, \ldots, V_{tp})]$. Therefore, this method of estimating $g(\cdot)$ is based on an additive marginal integration projection on the set of additive functions, where the projection is taken with the product measure of V_{tl}, for $l = 1, \ldots p$, unlike in the backfitting case of Nielsen and Linton (1998), and Mammen et al. (1999). Although the marginal integration technique is inferior to backfitting in asymptotic efficiency for purely additive models, it seems well suited to the framework of PL estimation; see also Fan et al. (1998), and Fan and Li (2003) for details. The third and final step involves the estimation of β using the weighted least squares estimator $\widehat{\beta}$ of β derived in (2.2). The estimation procedure is completed by reintroducing $\widehat{\beta}$ into the previous steps. For the independent data case, orthogonal series estimation has been used as an alternative to some other nonparametric estimation method, such as the kernel method (see Eubank (1999), for example). By approximating each $g_i(\cdot)$ using an orthogonal series $\sum_{j=1}^{n_i} f_{ij}(\cdot)\theta_{ij}$ with $\{f_{ij}(\cdot)\}$ being a sequence of orthogonal functions and $\{n_i\}$ being a sequence of positive integers, we have an approximate model of the form:

$$Y_t = \mu + U_t^\tau \beta + \sum_{i=1}^{p} \sum_{j=1}^{n_i} f_{ij}(V_{ti})\theta_{ij} + e_t. \tag{2.5}$$

Model (2.5) covers some natural parametric time series models. For example, when $U_{tl} = U_{t-l}$ and $V_{ti} = Y_{t-i}$, model (2.5) becomes a parametric nonlinear additive time series model:

$$Y_t = \mu + \sum_{l=1}^{q} U_{t-l}\beta_l + \sum_{i=1}^{p} \sum_{j=1}^{n_i} f_{ij}(Y_{t-i})\theta_{ij} + e_t \tag{2.6}$$

The least squares estimators of (β, θ, μ) can be derived using (2.5):

$$\widehat{\beta} = \widehat{\beta}(n) = \left(\widehat{U}^\tau \widehat{U}\right)^+ \widehat{U}^\tau \widehat{Y},\ \widehat{\theta} = (F^\tau F)^+ F^\tau \left(\widetilde{Y} - \widetilde{U}\widehat{\beta}\right), \text{and } \widehat{\mu} = \bar{Y} - \bar{U}^\tau \widehat{\beta}, \tag{2.7}$$

where

$$\theta = (\theta_1^\tau, \ldots, \theta_p^\tau)^\tau, \quad \theta_i = (\theta_{i1}, \ldots, \theta_{in_i})^\tau,$$

$$F = (F_1, F_2, \ldots, F_p), \quad F_i = F_{in_i} = (F_i(V_{1i}), \ldots, F_i(V_{ni}))^\tau,$$

$$\bar{U} = \frac{1}{n}\sum_{t=1}^{n} U_t,\ \widetilde{U} = \left(U_1 - \bar{U}, \ldots, U_n - \bar{U}\right)^\tau,\ \bar{Y} = \frac{1}{n}\sum_{t=1}^{n} Y_t,\ \widetilde{Y} = \left(Y_1 - \bar{Y}, \ldots, Y_n - \bar{Y}\right)^\tau,$$

$$P = F(F^\tau F)^+ F^\tau,\ \widehat{U} = (I - P)\widetilde{U},\ \widehat{Y} = (I - P)\widetilde{Y},\ n = (n_1, \ldots, n_p)^\tau$$

and where A^+ deontes the Moore-Penrose inverse of A. A detailed discussion about the orthogonal series estimation method is available in Chapter 2 of Gao (2007).

(Extended Generalized) Single-Index Semiparametric Model

Alternatively, we may approximate the conditional mean function, $m(U_t, V_t)$, using a semiparametric function of the form:

$$m_2(U_t, V_t) = U_t^\tau \theta + \psi(V_t^\tau \eta), \tag{2.8}$$

where θ and η are unknown vector parameters and $\psi(\cdot)$ is an unknown function. A more general model, which has recently become available in the literature, is the semiparametric single-index model:

$$m_3(X_t) = X_t^\tau \theta + \psi(X_t^\tau \eta), \tag{2.9}$$

where $\{X_t^\tau\}$ is a stationary and α-mixing sequence with a mixing coefficient $\alpha(k) = O(c^k)$ for some large enough $0 < c < 1$. Xia et al. (1999) refer to the functional form in (2.9) as the extended generalized partially linear single–index (EG–PLSI) model.

Assumption 2.1. *Suppose that $X_t^\tau \theta + \psi(X_t^\tau \eta)$ can be written as $X_t^\tau(\theta + c\eta) + \psi(X_t^\tau \eta) - cX_t^\tau \theta$ such that all the roots of the function $x^d - (\theta_1 + c\eta_1)x^{d-1} - \ldots - (\theta_d + c\eta_d)$ are inside the unit circle. Moreover, suppose that $\lim_{|u|\to\infty} |\psi(u)/u| = 0$.*

In this case, the geometrical ergodicity of $\{Y_t\}$ is ensured under the conditions stated in Assumption 2.1. (See Theorem 3 of Xia et al. (1999) for details.) Furthermore, in order to ensure the estimatability of the model, it must also be the case that θ and η are perpendicular to each other with $\|\eta\| = 1$ and the first nonzero element must be positive. Now, let us define the following:

$$S(\theta, \eta) = E[Y_t - \varphi_\eta(X_t^\tau \eta) - \{X_t - \Gamma_\eta(X_t^\tau \eta)\}^\tau \theta]^2, \tag{2.10}$$

where $\varphi_\eta(u) = E[Y_t | X_t^\tau \eta = u]$ and $\Gamma_\eta(u) = E[X_t | X_t^\tau \eta = u]$, and $\mathcal{W}(\eta) = E[\{X - \Gamma_\eta(X_t^\tau \eta)\}\{X - \Gamma_\eta(X_t^\tau \eta)\}^\tau]$ and $\mathcal{V}(\eta) = E[\{X - \Gamma_\eta(X_t^\tau \eta)\}\{Y_t - \varphi_\eta(X_t^\tau \eta)\}]$. Xia et al. (1999) show that the minimum point of $S(\theta, \eta)$ with $\theta \perp \eta$ is unique at η and $\theta = \{\mathcal{W}(\eta)\}^+ \mathcal{V}(\eta)$, where $\{\mathcal{W}(\eta)\}^+$ is the Moore–Penrose inverse.

To estimate the model, Xia et al. (1999) introduce an estimation procedure, which is a semiparametric extension of the one introduced in Härdle et al. (1993) for a nonparametric single-index model. The procedure consists of four important steps as follows: (i) Compute the estimate $\widehat{\theta}_\eta$ of θ given η and the delete-one estimators of φ_η and Γ_η. Let Θ denote all the unit vectors in $\mathbb{R}^p$. (ii) Estimate $\eta \in \Theta$ and the bandwidth, h, using those values $\widehat{\eta}$ and $\widehat{h}$ that minimize $\widehat{S}(\eta, \mathrm{h})$ (an estimate of $S(\theta, \eta)$), where θ is replaced by $\widehat{\theta}_\eta$, and φ_η and $\Gamma_\eta(\cdot)$ are replaced by their nonparametric estimators; (iii) Re-estimate θ as in the first step, but with η being replaced by $\widehat{\eta}$; (iv) Estimate $\psi(\cdot)$ using the nonparametric kernel estimates and the fact that $\psi(x) = \varphi(x) - \theta\Gamma(x)$.

In order to illustrate the statistical validity of such an estimation procedure, the following asymptotic results are established:

(i) $\sqrt{n}$-consistency

$$\widetilde{n}\left(\widehat{\theta} - \theta\right) \to N(0, \mathbb{C}^{+}) \quad \text{and} \quad \widetilde{n}\left(\widehat{\eta} - \eta\right) \to N(0, \mathbb{D}^{+}) \tag{2.11}$$

in distribution, where $\widetilde{n}$ is the number of elements in $\mathcal{A} \subset \mathbb{R}$, i.e. the union of a number of open convex sets such that $f(x) > M$ for some constant $M > 0$, and $\mathbb{C}^{+}$ and $\mathbb{D}^{+}$ are some positive, finite constants; see also the corollary in page 836 of Xia et al. (1999);

(ii) Uniform convergence, where, almost surely:

$$\sup_{v \in \{x^{\tau}\eta \,:\, x \in \mathcal{A}\}} \left|\widehat{\psi}_{\widehat{\eta}}(v) - \psi(v)\right| = O\left\{(n^{-4/5}\log n)^{1/2}\right\}. \tag{2.12}$$

See also Theorem 5 of Xia et al. (1999).

These results warrant a few remarks. The asymptotic normality is a direct extension of the one presented in Härdle et al. (1993), but under α-mixing and a larger parameter cone Ω_n such that $\Omega_n = \{\eta : ||\eta - \eta|| \leq Mn^{-\delta}\}$ for some constant M, where $\frac{3}{10} < \delta < \frac{1}{2}$. The proof of such results is made possible using a decomposition of $\widehat{S}$ into various parts as follows.

Xia et al. (1999) derive the decomposition of $\widehat{S}(\eta, \mathrm{h})$ into a few important terms (see Theorem 4 of the paper). While one of these is shown to be $o(1)$, the remaining are:

$$\widetilde{S}(\eta) = \sum_{X_t \in \mathcal{A}} \{y_t - X_t^{\tau}\theta_{\eta} - \psi(X_t^{\tau}\eta)\}^2 \text{ and } T(\mathrm{h}) = \sum_{X_t \in \mathcal{A}} \{\widehat{\psi}_{\eta}(X_t^{\tau}\eta) - \psi(X_t^{\tau}\eta)\}^2, \tag{2.13}$$

where $\{y_t\}$ is a stationary and α-mixing process. Such a result suggests that estimating the EG-PLSI model can also be done in iterative steps, such as: (i) estimating h given an initial estimator of η, e.g. $\check{\eta}$; (ii) update $\check{\eta}$ using $\check{h}$ from the previous step; (iii) repeat the first two steps. In this setting, since it is clear that Step (i) is simply an estimation of a PL model for time series, such a result highlights a close connection between the PL and the EG-PLSI models.

Finally, some basic modifications to the formulation of the models bring about various special cases, which are well-known in the literature. For instance, if $\theta = 0$, (2.9) reduces to:

$$m_4(X_t) = \psi(X_t^{\tau}\eta), \tag{2.14}$$

which is the single index model discussed in Härdle et al. (1993). Furthermore, by partitioning $X_t = (U_t^{\tau}, V_t^{\tau})^{\tau}$ and by taking $\theta = (\beta^{\tau}, 0, \ldots, 0)^{\tau}$ and $\eta = (0, \ldots, 0, \alpha^{\tau})^{\tau}$, the

EG–PLSI model becomes the generalized partially linear single-index (G–PLSI) model introduced by Carroll et al. (1997) of the form

$$m_5(X_t) = U_t^{\tau}\theta + \psi(V_t^{\tau}\alpha), \tag{2.15}$$

which is a special case of the multiple-index model of Ichimura and Lee (1991).

In the next section, let us review some useful procedures for testing the semiparametric specifications of these time series models.

3. Some Specification Tests for Semiparametric Models

In this section, we focus first on tests for a semiparametric (either PL or single-index) form against a nonparametric form. We then introduce the corresponding PL model to the EG-PLSI model in order to, finally, discuss testing a linear regression model against a semiparametric model.

Specification Tests for Semiparametric vs. Nonparametric Form

Most nonlinear time series specification tests, discussed in the literature, concentrate mainly on testing either:

$$\text{Nonparametric model: } y_t = m(X_t) + e_t \quad \text{or} \quad \text{Single-index model: } y_t = \psi(X_t^{\tau}\eta) + e_t, \tag{3.1}$$

where $\{X_t\}$ is a sequence of strictly stationary time series variables. As we will discuss below, these tests can be conveniently adopted to hypothesis testing in the semiparametric time series models discussed above.

Based on the nonparametric model in (3.1), Gao and Gijbels (2008) discuss a nonparametric testing procedure to test hypotheses of the form:

$$H_{01} : m(x) = m_{\theta_0}(x) \quad \text{versus} \quad H_{11} : m(x) = m_{\theta_1}(x) + C_n\Delta_n(x) \text{ for all } x \in \mathbb{R}^d,$$

where both θ_0 and $\theta_1 \in \Theta$ are unknown parameters, Θ is a parameter space of $\mathbb{R}^d$, C_n is a sequence of real numbers and $\Delta_n(x)$ is a sequence of nonparametrically unknown functions over $\mathbb{R}^d$, such that model (3.1) becomes a semiparametric time series model of the form:

$$y_t = m_{\theta_0}(X_t) + e_t \tag{3.2}$$

under H_{01}. Gao and Gijbels (2008) assume that $\{X_t\}$ is strictly stationary and α-mixing, with the mixing coefficient defined by

$$\alpha(t) = \sup\left\{|P(A \cap B) - P(A)P(B)| : A \in \Omega_1^s, B \in \Omega_{s+t}^{\infty}\right\} \leq C_\alpha \alpha^t$$

for all $s, t \geq 1$, where $0 < C_\alpha < \infty$ and $0 < \alpha < 1$ are constants and Ω_i^j denotes the σ-field generated by $\{X_k : i \leq k \leq j\}$.

Prior to Gao and Gijbels (2008), Härdle and Mammen (1993) suggest that one way of establishing the nonparametric kernel test statistic for such hypothesis is to do so based on the L_2-distance function:

$$M_{1n}(h) = nh^{\frac{d}{2}} \int \left\{\widehat{m}_h(x) - \widetilde{m}_{\widehat{\theta}}(x)\right\}^2 w(x)dx, \tag{3.3}$$

where $w(x)$ is some non-negative weight function, $\widehat{m}_h(x)$ is the nonparametric kernel estimator of $m(\cdot)$ defined by:

$$\widehat{m}_h(x) = \frac{\sum_{t=1}^n K_h(x - X_t)y_t}{\sum_{t=1}^n K_h(x - X_t)} \tag{3.4}$$

and $\widetilde{m}_{\widehat{\theta}}(x)$ is its parametric counterpart:

$$\widetilde{m}_{\widehat{\theta}}(x) = \frac{\sum_{t=1}^n K_h(x - X_t)m_{\widehat{\theta}}(X_t)}{\sum_{t=1}^n K_h(x - X_t)}, \tag{3.5}$$

where $\widehat{\theta}$ is a $\sqrt{n}$-consistent estimator of θ_0. Recently, a number of studies derived the nonparametric test statistics based on a modified version of the L_2-distance function in (3.3)). An example is the work by Horowitz and Spokoiny (2001), who used a discrete approximation to $M_{1n}(h)$ of the form

$$M_{2n}(h) = \sum_{t=1}^n \left(\widehat{m}_h(X_t) - \widetilde{m}_{\widehat{\theta}}(X_t)\right)^2, \tag{3.6}$$

where $\{X_t\}$ is only a sequence of fixed designs. They also considered a multiscale normalized version of the form:

$$M_{2n} = \max_{h \in H_n} \frac{M_{2n}(h) - \widehat{M}_n(h)}{\widehat{V}_n(h)}, \tag{3.7}$$

where H_n is a set of suitable bandwidth:

$$\widehat{M}_n(h) = \sum_{t=1}^n \left(\sum_{s=1}^n W_h(X_s, X_t)\right) \widehat{\sigma}_n^2(X_t)$$

and:

$$\widehat{V}_n^2(h) = 2\sum_{s=1}^n \sum_{t=1}^n \left(\sum_{\ell=1}^n W_h(X_\ell, X_t)\right)^2 \widehat{\sigma}_n^2(X_s)\widehat{\sigma}_n^2(X_t),$$

where $W_h(\cdot, X_t) = \frac{K_h(\cdot - X_t)}{\sum_{u=1}^{n} K_h(\cdot - S_u)}$ and $\widehat{\sigma}_n^2(X_s)$ is a consistent estimator of the variance function $\sigma_n^2(X_t) = E[e_t^2]$. They then show that M_{2n} is asymptotically consistent with an optimal rate of convergence for hypothesis testing.

An alternative approach employed by Gao and Gijbels (2008) is to consider a different type of distance function for the nonparametric kernel test statistic. In order to discuss this method, let us first rewrite the nonparametric model into a notational version so that, under the H_0, we have:

$$Y = m_{\theta_0}(X) + e, \tag{3.8}$$

where X is assumed to be random, θ_0 is the true value of θ under H_0 and $E[e|X] = 0$. In this case, the distance function employed can be written as follows:

$$E[eE(e|X)\pi(X)] = E[(E^2(e|X))\pi(X)], \tag{3.9}$$

where $\pi(\cdot)$ is the marginal density function of X. In order to establish the asymptotic distribution of their test statistic, Gao and Gijbels (2008) suggested studying asymptotic distribution and proposing an Edgeworth expansion for the quadratic form of the following type:

$$R_n(h) = \sum_{s=1}^{n} \sum_{t=1}^{n} e_s \phi_n(X_s, X_t) e_t, \tag{3.10}$$

where $\phi_n(\cdot, \cdot)$ may depend on n, the bandwidth h and the kernel function K. This is because, to derive the test statistic, they are able to use a normalized kernel-based sample analogue of (3.9) of the form

$$L_{1n}(h) = \frac{h^{\frac{d}{2}}}{n} \sum_{s=1}^{n} \sum_{t=1}^{n} \widehat{e}_s K_h\left(X_t - X_s\right) \widehat{e}_t, \tag{3.11}$$

where $\widehat{e}_t = y_t - m_{\widehat{\theta}}(X_t)$, which turns out to be simply the leading term of the quadratic form in (3.10). In order to proceed, let us now define the following:

$$\widehat{L}_{1n}(h) = \frac{L_{1n}(h) - E[L_{1n}(h)]}{\sqrt{\mathrm{var}[L_{1n}(h)]}}. \tag{3.12}$$

For each given h, we may also define a stochastically normalized version of the form

$$\bar{L}_{1n}(h) = \frac{\sum_{s=1}^{n} \sum_{t=1, \neq x}^{n} \widehat{e}_s K_h(X_s - X_t) \widehat{e}_t}{\sqrt{2 \sum_{s=1}^{n} \sum_{t=1}^{n} \widehat{e}_s^2 K_h(X_s - X_t) \widehat{e}_t^2}}. \tag{3.13}$$

Furthermore, it has been shown in Gao (2007), and Gao and Gijbels (2008) that we have:

$$\bar{L}_{1n}(h) = L_{1n}(h) + o_P(1) \tag{3.14}$$

for each given h. Hence, we may use the distribution of $\bar{L}_n(h)$ to approximate that of $\widehat{L}_n(h)$. Since the main objective of the research in Gao and Gijbels (2008) is to propose a suitable selection criterion for the choice of h (such that while the size function is appropriately controlled, the power function is maximized at this h), they also give Edgeworth expansions of both the size and power functions of the test. Nonetheless, instead of discussing these in detail here, we suggest that interested readers should consult Section 3 of Gao and Gijbels (2008).

In this review, let us proceed with hypothesis testing of the semiparametric time series specifications. We will begin with the corresponding hypothesis testing for the PL regression:

$$H_{02} : m(x) = u^\tau \beta + g(v) \quad \text{versus} \quad H_{12} : m(x) = u^\tau \beta + g(v) + C_n \Delta_n(x) \text{ for all } x \in \mathbb{R}^d,$$

where C_n and $\Delta_n(\cdot)$ are as defined previously, and u and v are subvectors of $x = (u^\tau, v^\tau)^\tau$. In this case, the test statistic can be written as:

$$L_{2n}(h) = \sum_{s=1}^{n} \sum_{t=1}^{n} \widehat{y}_s K\left(\frac{X_s - X_t}{h}\right) \widehat{y}_t, \tag{3.15}$$

where $\widehat{y}_s = y_s - U_s^\tau \widehat{\beta} - \widehat{g}(V_s)$, $\widehat{\beta} = (\widetilde{U}^\tau \widetilde{U})^+ \widetilde{U}^\tau \widetilde{y}$, $\widehat{g}(V_s) = \sum_{t=1}^n w_{2st}(y_t - U_t^\tau \widehat{\beta})$, $\widetilde{U} = (I - W_2)U$, $U = (U_1, \ldots, U_n)^\tau$, $\tilde{y} = (I - W_2)Y$ and $W_2 = \{w_{2st}\}$ is a $n \times n$ matrix such that $w_{2st} = \frac{K_2\left(\frac{V_s - V_t}{h}\right)}{\sum_{u=1}^n K_2\left(\frac{V_s - V_u}{h}\right)}$ with $K_2(\cdot)$ being a kernel function. Some existing results for a similar test statistic to $L_{2n}(h)$ as defined in (3.15) can be found in, for example, Fan and Li (1996) and Fan and Li (1997) (see also the detailed review of Fan and Li (1997) below).

Fan and Li (1996) consider a consistent test for a PL model where $\{U_t^\tau, V_t^\tau\}_{t=1}^n$ is a set of n i.i.d. observations on $\{U^\tau, V^\tau\}^\tau$ with U being $p \times 1$ and V being the $q \times 1$ regressors. Nonetheless, there are two useful results in the literature that may enable an extension of Fan and Li (1996) procedure to hypothesis testing in time series data, namely the Central Limit Theorem (CLT) established in Fan and Li (1999a) and the $\sqrt{n}$-consistent estimation of partially linear time series models in Fan and Li (1999b). Together, these two results can be used in the generalization of the consistent test of Fan and Li (1996) for testing a PL model versus a nonparametric regression model in the time series framework. This work is done by Li (1999). An important issue that should be noted is the dependence structure assumed. Fan and Li (1999a), Fan and Li (1999b), and Li (1999) considered absolutely regular (β-mixing) processes (though it is well known that such absolute regularity is stronger than strong mixing). This is because their method of mathematical proof relied on an inequality for β-mixing processes due to Yoshihara

(1976), which was not available for α-mixing. However, there are other recent works that studied PL models of α-mixing processes, such as Gao and Yee (2000), and Härdle et al. (2000).

With regard to the semiparametric single-index model, as a natural extension to the above tests, we may consider testing

$$H_{03} : m(x) = u^\tau \beta + \psi(x^\tau \eta) \ \text{ versus } \ H_{13} : m(x) = u^\tau \beta + \psi(x^\tau \eta) + C_n \Delta_n(x) \text{ for all } x \in R^d,$$

where both θ and η are vectors of unknown parameters, and $\psi(\cdot)$ is an unknown function. In this case, the test statistic can be written, similar to (3.15), as:

$$L_{3n}(h) = \sum_{s=1}^{n} \sum_{t=1}^{n} \widetilde{y}_s K\left(\frac{(X_s - X_t)^\tau \widehat{\eta}}{h}\right) \widetilde{y}_t, \tag{3.16}$$

where $\widehat{\theta}$, $\widehat{\eta}$ and $\widehat{\psi}(\cdot)$ are consistent estimators discussed previously, and we have:

$$\widetilde{y}_t = \left(y_t - U_t^\tau \widehat{\beta} - \widehat{\psi}(X_t^\tau \widehat{\eta})\right) \widehat{f}_3(X_t^\tau \widehat{\eta}),$$

in which $\widehat{f}_3(X_t^\tau \widehat{\eta}) = \frac{1}{h}\sum_{t=1}^{n} K\left(\frac{(X_s - X_t)^\tau \widehat{\eta}}{h}\right)$.

Specification Test for Linear Regression vs. a Semiparametric Form

For the case where $\{X_t\}$ is a vector time series regressor and $g(\cdot)$ is an unknown function defined on $\mathbb{R}^p$ (where $1 \le p \le 3$), an attempt is made in the work of Gao (2012) to extend the semiparametric PL models in (2.1) to the semi-linear (SL) model of the form

$$m_6(X_t) = \mu + X_t^\tau \beta + g(X_t), \tag{3.17}$$

which is a direct counterpart of the EG–PLSI model in (2.9), where the SL model has different motivations and applications from the conventional semiparametric time series model presented in (2.1) as follows: (i) In (3.17), the linear component in many cases plays the leading role, while the nonparametric component behaves like a type of unknown departure from such classic linear model. In order to establish the empirical support for such a condition, Gao (2012) uses the SL model to investigate time series properties of quarterly consumer price index numbers of 11 classes of commodities for eight Australian capital cities between 1994 and 2008. The author has found that linearity remains the leading component of the trending component of the consumer price index data. (ii) The SL model can be motivated as a model to address some endogenous problems involved in a class of linear models of the form $Y_t = X_t^\tau \beta + \varepsilon_t$, where $\{\varepsilon_t\}$ is a sequence of errors with $E[\varepsilon] = 0$ but $E[\varepsilon_t | X_t] \neq 0$, i.e. it might be the case that $\varepsilon_t = g(X_t) + e_t$, where e_t is an i.i.d. error. Unfortunately, in the process of estimating β and $g(\cdot)$, existing methods are not directly applicable, especially given the fact that $\Sigma = (E[U_t - E[U_t|U_t])(E[U_t - E[U_t|U_t])^\tau]) = 0$. To this end, Gao (2012) studies the estimation of the SL model and its asymptotic properties in two different contexts, namely (i) where $\{X_t\}$ is a vector of stationary time series regressors; (ii) where $\{X_t\}$ is stochastically nonstationary.

In the following, we focus first on the case of stationary time series regressors, while the case of nonstationary regressors will be considered later. In this case, essential assumptions are the identifiability and the smallness conditions of $g(\cdot)$.

Assumption 3.1. *(Assumption 2.1(i) of Gao (2012)) Let $g(\cdot)$ be an integrable function*

$$\int ||x||^i |g(x)|^i dF(x) \leq \infty$$

for $i = 1, 2$ and $\int x g(x) dF(x) = 0$, where $F(x)$ is the cumulative distribution function of $\{X_t\}$ and $||\cdot||$ denotes the conventional Euclidean norm.

Under such conditions, the parameter β is identifiable and chosen such that $E[Y_t - X_t^\tau \beta]^2$ is minimized over β, which implies $\beta = (E[X_1 X_1^\tau])^{-1} E[X_1 Y_1]$, provided that the inverse matrix exists. Such a definition of β suggests that $\int x g(x) dF(x) = 0$, so β can be estimated by the ordinary least squares estimator of the form:

$$\widehat{\beta} = \left(\sum_{t=1}^{n} X_t X_t^\tau \right)^{-1} \left(\sum_{t=1}^{n} X_t Y_t \right) \text{ such that } \widehat{g}(x) = \sum_{t=1}^{n} w_{nt}(x) \left(Y_t - X_t^\tau \widehat{\beta} \right), \quad (3.18)$$

where $w_{nt}(x)$ is a probability (kernel) weight function. Gao (2012) then establishes the asymptotic normality of such estimators. Nonetheless, the full proof of such results is not shown, since it is a straightforward result of the central limit theorems for partial sums of stationary and α-mixing time series; see Fan and Yao (2003), for example.

To this end, an existing hypothesis testing procedure that can be used to determine whether $g(\cdot)$ is small enough to be negligible, is that developed by Gao (1995). The null hypothesis in this case is $H_0 : g(\cdot) = 0$, while the asymptotic distribution of the test statistic is derived as

$$\widehat{L}_{1n} = \frac{\sqrt{n}}{\widehat{\sigma}_1} \left(\frac{1}{n} \sum_{t=1}^{n} \left(Y_t - X_t^\tau \widehat{\beta} \right)^2 - \widehat{\sigma}_0^2 \right) \xrightarrow{D} N(0,1), \quad (3.19)$$

where $\widehat{\sigma}_1^2$ and $\widehat{\sigma}_0^2$ are consistent estimators of $\sigma_1^2 = E[e_1^4] - \sigma_0^4$ and $\sigma_0^2 = E[e_1^2]$, respectively.

Finally, let us note that the EG–PLSI model discussed in the previous section can always be used for the case where $p \geq 4$. For the sake of convenience and clarity, we will leave the discussion on the case of a nonstationary time series to a later section.

In the next section, let us shift our attention to a number of nonlinear autoregressive models that can be derived based on the semiparametric models defined above and their specification testing in practice.

4. Nonlinear Autoregressive Models and Their Specification Testing

Since the focus of the current review is on time series, it is important that we also discuss the re-specification of the semiparametric models in the previous section to form semiparametric autoregressive models and their specification testing.

If the observations are allowed to be taken over time, then the above mentioned semiparametric models give rise to a number of well-known nonlinear autoregressive models discussed in the literature as follows:

(i) A similar partitioning of X_t to that of (2.1) such that $U_t = (Y_{t-c_1}, Y_{t-c_2}, \ldots, Y_{t-c_p})^\tau$ and $V_t = (Y_{t-d_1}, Y_{t-d_2}, \ldots, Y_{t-d_p})^\tau$, where $c_i \neq d_j$ for all $1 \leq i \leq p$ and $1 \leq j \leq q$, giving rises to the autoregressive semiparametric PL additive model discussed in Gao and Yee (2000). Gao and Yee (2000) found that the PL regression is more appropriate than a completely nonparametric autoregression for the Canadian lynx data, which comprises of the annual record of the number of lynx trapped in the MacKenzie River district in the Canadian Northwest Territories from 1821 to 1934.

(ii) The autoregressive single–index model discussed in Xia et al. (1999) is obtained simply by letting $X_t = (Y_{t-1}, Y_{t-2}, \ldots, Y_{t-p})^\tau$ in (2.9). Using the projection pursuit method to investigate the autoregressive process of sunspot numbers in a year, Xia et al. (1999) found some strong empirical evidence in support of such a model. Furthermore, a specification test of linearity can be developed based on the fact that statistical insignificance of the nonlinear component signals the superiority of a linear model. To see this, let us write an autoregressive EG-PLSI model in the form:

$$y_t = \beta^\tau X_t + \phi(\eta^\tau X_t) + \varepsilon_t, \tag{4.1}$$

where $X_t = (y_{t-1}, y_{t-2}, y_{t-3})^\tau$. Such a specification can be tested against a linear regressive model through testing $H_0 : \phi(u) \equiv 0$. Xia et al. (1999) suggested that the testing procedure can be developed based on the method discussed in Xia (1998). In their empirical analysis of the shape-invariant Engel curves in Australia, Kim et al. (2013) follow this suggestion and construct the Bonferroni-type variability bands in order to determine the statistical significance of, for example, $\phi(\cdot)$.

(iii) Another useful alternative is to establish an autoregressive SL model. In this case, the process $\{Y_t\}$ is stochastically stationary and α-mixing under the following conditions:

Assumption 4.1. *(Assumption 4.1 of Gao (2012)) (i)* $\beta = (\beta_1, \ldots, \beta_p)^\tau$ *satisfy* $Y^p - \beta_1 Y^{p-1} - \ldots - \beta_{p-1} Y - \beta_p \neq 0$ *for any* $|Y| \geq 1$*; (ii)* $g(X)$ *is bounded on any bounded Borel measurable set and satisfy* $g(X) = o(||X||)$ *as* $||X|| \to \infty$*, where* $||\cdot||$ *denotes the conventional Euclidean norm.*

In this case, the test statistic described in (3.19) above can be used to test a linear autoregressive model against a semiparametric alternative. Clearly, this is the corresponding test to that in Xia et al. (1999) above.

In order to provide a brief background and introduction into issues surrounding the specification testing of the autoregressive semiparametric models, let us first consider the following general autoregressive model of a finite order p:

$$Y_t = g(Y_{t-1}, \ldots, Y_{t-p}) + \epsilon_t, \tag{4.2}$$

where the autoregressive function g is unknown and $\{\epsilon_t\}$ is a sequence of martingale differences. The process $\{Y_t\}$ is absolutely regular with a coefficient $\phi_\tau = O(\rho^\tau)$, where ρ is a constant $0 < \rho < 1$. One of the first natural steps in the analysis of time series is to decide whether to use a nonlinear model. For convenience, we let $X_t = (Y_{t-1}, \ldots, Y_{t-p})^\tau$ so that we observe $X_1, \ldots, X_{n+1}$. To this end, Fan and Li (1997) establish a consistent nonparametric test for the linearity of AR(p) models. In terms of X_t, the hypotheses can be written as:

$$H_{03} : P(g(X_t) = \alpha^\tau X_t) = 1 \text{ and } H_{13} : P(g(X_t) = \alpha^\tau X_t) < 1 \tag{4.3}$$

for some $\alpha \in (-1, 1)^p$ and for all $\alpha \in (-1, 1)^p$, respectively. If the null hypothesis holds, then the ordinary least squares estimator $\widehat{\alpha}$, for example, provides a consistent estimator of α. Furthermore, by letting $\widehat{\epsilon}_t = Y_t - \widehat{\alpha}^\tau X_t$, the test statistic of Fan and Li (1997) is based on the kernel estimate of the sample analogue of $E[\epsilon_t E(\epsilon_t | X_t) f(X_t)]$, i.e.:

$$I_n = \frac{1}{n(n-1)h^p} \sum \sum_{s \neq t} \widehat{\epsilon}_t \widehat{\epsilon}_s K_{st}, \tag{4.4}$$

where $h \equiv h_n \to 0$ is a sequence of smoothing parameters, $K_{st} = K((X_s - X_t)/h)$, $K(\cdot)$ is a kernel function satisfying certain conditions and $\sum \sum_{s \neq t} = \sum_{s=1}^n \sum_{t \neq s, t=1}^n$. Under the null hypothesis, it is the case that $\widehat{\epsilon}_t = \epsilon_t - (\widehat{\alpha} - \alpha)^\tau X_t$ so that the asymptotic distribution of I_n is determined by that of $nh^{p/2} I_{n1}$, where

$$I_{n1} = \frac{1}{n(n-1)h^p} \sum \sum_{s \neq t} \epsilon_t \epsilon_s K_{st}. \tag{4.5}$$

To this end, Fan and Li (1997) derive the asymptotic normality of $nh^{p/2}I_{n1}$ by invoking on the CLT for degenerate U-statistics of absolutely regular processes of Khashimov (1993). In addition, Fan and Li (1999a), focus on one of the conditions in Khashimov (1993) which requires the error term ϵ_t to bounded and to provide a new CLT that can be used to relax such a boundedness. (Note that in the model specification testing introduced in Fan and Li (1999a) the error term is defined instead as $Y_t - g(X_t, \gamma)$ to reflect the null hypothesis which involves a specific parametric family.)

In the previous sections, we have noted results in the literature which suggested a close connection between the semiparametric models reviewed above. Another important feature that these models share is their intolerance of the endogeneity of the error term in order to obtain consistent estimates of the models. In the following section, we review the literature on the endogeneity problem in semiparametric models and a few methods of dealing with it. It will soon be clear that these can be directly related to studies of model estimation with generated regressors.

5. Endogeneity and Semiparametric Models with Generated Regressors

As noted previously, consistent estimation of the above mentioned PL and EG-PLSI models for time series relies on the exogeneity of the error term with respect to both the parametric and nonparametric regressors. The breakdown of such a condition is famously known in the literature as the endogeneity problem (see Blundell and Powell (2003), for example). Let e_t (for $t = 1, 2, \ldots, n$) form a sequence of i.i.d. random errors with a mean of zero and a finite variance of σ^2, so that the PL model for time series can be written as:

$$y_t = \mu + U_t^\tau \beta + g(V_t) + e_t. \tag{5.1}$$

An important assumption, which is required to ensure the consistent estimation of β and $g(\cdot)$, is the exogeneity of the error term with respect to both the parametric and nonparametric regressors, mathematically described as $E[e|U = u] = 0$ and $E[e|V = v] = 0$. Such an exogeneity condition is also needed for the EG-PLSI model:

$$y_t = X_t^\tau \theta + \psi(X_t^\tau \eta) + e_t, \tag{5.2}$$

where, in this case, it is necessary that $E[e|X^\tau \eta = v] = 0$. Kim and Saart (2013), and Kim et al. (2013) discuss in detail a set of simulation exercises to illustrate the seriousness of the impacts of endogeneity problem in semiparametric regression models.

While Kim and Saart (2013) attempted to address the endogeneity problem in the PL model, Kim et al. (2013) did so for the EG-PLSI model. In principle, the methods considered in Kim and Saart (2013) closely followed the logic of Robinson's (1988) two-step estimation procedure mentioned previously, i.e. first obtaining consistent estimators of the unknown parameters and then using them in order to identify an unknown structural function. If the parametric regressors are exogenous, then the least-squares estimators of the parametric parameters are consistent. Otherwise, if parametric endogeneity is present, then the parametric instrumental variable (PIV) estimation can be used. The consistency of the parametric estimators is important not only in its own right but also for identifying an unknown nonlinear function, $g(\cdot)$.

The presence of nonparametric endogeneity can induce further complication in the identification of the unknown function. There are two alternative methods in the literature which may be helpful in identifying the unknown function in such a case, namely the nonparametric instrumental variable (NpIV) estimation and the control function (CF) approach. Newey and Powell (2003), Hall and Horowitz (2005), and Darolles et al. (2011) developed the NpIV estimation for a pure nonparametric model, while Ai and Chen (2003) did so for semiparametric models, which included the PL model as a special case. One of the difficulties with using NpIV estimation resides in the well-known ill-posed inverse problem; see O'Sullivan (1986), for example. To overcome such an obstacle, Ai and Chen (2003) based their estimation on a complex sieve estimation under some regularity conditions on the inversion matrix and a constraint on the space of the reduced relation to keep it compact. On the other hand, Newey et al. (1999) and Pinkse (2000) considered the CF approach in a pure nonparametric model, while Blundell and Powell (2004) did so for a special case of a single index model, i.e. a case where only the discrete dependent variable was considered. With regard to the nonparametric estimation employed, Newey et al. (1999) and Pinkse (2000) relied on series approximation, while Su and Ullah (2008) used the local polynomial estimation of Fan and Gijbels (1996). Blundell and Powell (2004), on the other hand, relied on the local constant kernel estimation method.

Kim and Saart (2013) addressed nonparametric endogeneity in the estimation and inference of the PL model in a simple but widely-used framework of nonparametric simultaneous equations, specifically a nonparametric triangular model. Although the full details can be found in the paper, let us discuss this briefly here. They considered the following model:

$$y = x'\beta + g(v) + \epsilon, \tag{5.3}$$

where x may be either exogenous or endogenous, while v is endogenous. In addition, the following nonparametric reduced-form equation exists:

$$v = m_v(z) + \eta, \tag{5.4}$$

where z is a vector of the instrumental variables such that $E(\eta|z) = 0$ and $E(\epsilon|z, \eta) = E(\epsilon|\eta) \neq 0$. In order to identify and to estimate the structural function $g(\cdot)$, they take the CF approach, as in Newey et al. (1999), namely

$$E(y|v, \eta) = E(x|v, \eta)'\beta + g(v) + \iota(\eta), \tag{5.5}$$

where the endogeneity (i.e. $E(\epsilon|\eta) = \iota(\eta) \neq 0$) is controlled by introducing an additional unknown function. This structure enabled Kim and Saart (2013) to write the model as a simple nonparametric additive structure and, therefore, to employ the local constant kernel estimation and the marginal integration technique of Linton and Nielsen (1995), and Tjøstheim and Austad (1996) to identify the unknown function. As discussed in Kim et al. (2013), this procedure can also be used to address an endogeneity problem in the EG-PLSI model for the time series of Xia et al. (1999).

Nonetheless, this estimation procedure involves a generated regressor in the sense that an estimate of η must be used in estimating the conditional expectation in (5.5). In fact, there are many nonparametric and semiparametric models in econometrics that contain generated regressors. For example, Lewbel and Linton (2007) dealt with non-parametrically generated regressors when considering homothetically separable functions. Moreover, Newey et al. (1999) and Su and Ullah (2008) studied the nonparametric estimation of triangular simultaneous equation models. Li and Wooldridge (2002) considered the semiparametric estimation of PL models for dependent data with generated regressors. In a sense, Li and Wooldridge's (2002) model can be seen as a special case of the regression model in (5.5). Let $\mathcal{W}_t = \{Y_t, U_t^\tau, S_t, Z_t^\tau\}$ be a stationary and absolutely regular process, i.e., as $\tau \to \infty$:

$$\beta_\tau = \sup_{s \in \mathcal{N}} E\left[\sup_{A \in \mathcal{M}_{s+\tau}^{\infty}} \{|P(A|\mathcal{M}_{-\infty}^{s}(\mathcal{W})) - P(A)|\}\right] \to 0, \tag{5.6}$$

where $\mathcal{M}_s^t(\mathcal{W})$ denotes $\sigma(\mathcal{W}_s, \dots, \mathcal{W}_t)$, the sigma algebra generated by $(\mathcal{W}_s, \dots, \mathcal{W}_t)$, for $s \leq t$. Li and Wooldridge's (2002) model can be written as:

$$Y_t = U_t^\tau \beta + g(\eta_t) + \varepsilon_t \tag{5.7}$$

$$\eta_t = S_t - Z_t^\tau \alpha \tag{5.8}$$

such that $E(\varepsilon_t|U_t, Z_t, \eta_t) = 0$ and $E(\eta_t|Z_t) = 0$, where U_t is $p \times 1$, Z_t is $q \times 1$, Y_t and S_t are scalars, β and α are the vectors of unknown parameters, and $g(\cdot)$ is an unknown smooth function. The model can be modified so that nested within it are a nonlinear regression model and a Tobit–3 model. These modifications have been found to be very useful in practice. Bachmeier (2002), for example, applies a modified version of Li and Wooldridge's

(2002) model, which is written in the form of a semiparametric error correction model, to investigate nonlinearity in the term structure; see also Galego and Pereira (2010) for an application of the model to labour economics.

Overall, the model's estimation procedure is similar to that introduced in Robinson (1988), which we discussed earlier. The only exception in this case is the fact that the parametric estimation of η, as defined in (5.8), is now required in the first step. Hence, the mathematical proof of the $\sqrt{n}$-consistency of the unknown parameters must rely on an assumption that a $\sqrt{n}$-consistent estimator of α exists. Compared to those in Robinson (1988), Li and Wooldridge (2002) have to impose slightly stronger moment and smoothness conditions on the regression, density and kernel functions. This is mainly because they have to use Taylor expansions in their proof to deal with the regressor η_t, which was initially generated parametrically.

A similar generated regressor problem was also encountered by Saart et al. (2013) in order to develop their so–called semiparametric autoregressive conditional duration (SEMI–ACD) model. This is with an exception to the fact that, in their study, the unobservable regressor is computed semiparametrically based on an iterative estimation algorithm instead of using a linear regression as stated in (5.8). Saart et al. (2013) first derived the uniform consistency of the estimation algorithm, then used the Taylor expansions (together with the uniform convergence rates for kernel estimation with dependent data derived in Hansen (2008)) in the proof to deal with the generated regressor. Below, let us discuss the SEMI-ACD model in more detail. Let Y_t denotes financial duration, i.e. the waiting time between two consecutive financial events, associated with the t-th event. Engle and Russell (1998) develop the ACD model by assuming that

$$Y_t = \psi_t \varepsilon_t, \tag{5.9}$$

where $\{\varepsilon_t\}$ is an i.i.d. innovation series with non-negative support density $p(\varepsilon; \phi)$, in which ϕ is a vector of parameters and:

$$\psi_t \equiv \sum_{j=1}^{p} \alpha_j Y_{t-j} + \sum_{k=1}^{q} \beta_k \psi_{t-k}, \tag{5.10}$$

where $\{\psi_t\}$ denotes the process of conditional expectation, which summarizes the dynamics of the duration process. Suppose that the processes $\{Y_t\}$ and $\{\psi_t\}$ are both strictly stationary and α-mixing with the mixing coefficients $\alpha_x(n)$ and $\alpha_\psi(n)$ satisfying $\alpha_x(n) \leq C_x\ q_x^n$ and $\alpha_\psi(n) \leq C_\psi\ q_\psi^n$, respectively, where $0 < C_x,\ C_\psi < \infty$ and $0 < q_x, q_\psi < 1$.

The ACD model in (5.9) is considered by many to be too restrictive to take care of the dynamics of the duration process in practice. Furthermore, estimating the model requires the imposition of a distributional assumption on ε_t, a requirement that is not popular in the literature; see Pacurar (2008) for an excellent review of the ACD literature. Saart et al. (2013) attempt to minimize impacts of such issues by introducing the SEMI–ACD model such that

$$\psi_t \equiv \sum_{j=1}^{p} \gamma_j Y_{t-j} + \sum_{k=1}^{q} \mathrm{g}_k \left(\psi_{t-k}\right), \tag{5.11}$$

where γ_j is an unknown parameter and $\mathrm{g}_k(\cdot)$ is an unknown function on the real line. Even though the above mentioned distributional assumption is not required to estimate these semiparametric models, a latency problem arises because the conditional duration (ψ) is not observable in practice.

To estimate the model, the authors rely on an iterative estimation algorithm. For a special case of the model where $p = q = 1$, i.e. the so–called SEMI–ACD(1,1) model, the algorithm can be summarized as follows: *Step 1:* Choose the starting values for the vector of the n conditional durations. Index these values with a zero. Let $\{\widehat{\psi}_{t,0}; 1 \leq t \leq n\}$ satisfy $\widehat{\psi}_{t,0} = \psi_{t,0}$. Set $m = 1$. *Step 2:* Compute $\widehat{\gamma}_m$ and $\widehat{g}_{h,m}$, by regressing $\{Y_t; 2 \leq t \leq n\}$ against $\{Y_{t-1}; 2 \leq t \leq n\}$ and the estimates of ψ computed in the previous step, i.e. $\{\widehat{\psi}_{t-1,m-1}; 2 \leq t \leq n\}$. *Step 3:* Compute $\{\widehat{\psi}_{t,m}; 2 \leq t \leq n\}$. Furthermore, use the average of $\{\widehat{\psi}_{t,m}; 2 \leq t \leq n\}$ as a proxy for $\widehat{\psi}_{1,m}$, which cannot be computed recursively. *Step 4:* For $1 \leq m < m^*$, where $m^* = O(\log(n))$ is the (pre-specified) maximum number of iterations, increment m and return to Step 2. At $m = m^*$, perform the final estimation to obtain the final estimates of γ and g.

Saart et al. (2013) studied the asymptotic properties of such a procedure for the SEMI–ACD(1,1) model by first deriving the consistency of the estimation algorithm, i.e.

$$\left|\left|\widehat{\Psi}_m - \Psi\right|\right|_{1e} \leq \Delta_{1n}(\widehat{\psi})\ C_m(G) + G^m\ \Delta_{2n}(\psi), \tag{5.12}$$

where $0 < G < 1$, $\widehat{\Psi}_m = (\widehat{\psi}_{m+1,m}, \ldots, \widehat{\psi}_{n,m})^\tau$ and $\Psi = (\psi_{m+1}, \ldots, \psi_n)^\tau$. Although the details are shown in Theorem 3.1 of the paper, let us simply note that while the first term on the right side of (5.12) converges to zero uniformly over all the possible values of the bandwidth, h, at the same rate as the mean squared error of a usual PL time series model, the limit of the second term is zero as $m \to 0$. Hence, m^* in this case is selected so that the second term is bounded in probability by the first. Note that the proof of (5.12) requires the following contraction property on g.

Assumption 5.1. *(Assumption 3.1 of Saart et al. (2013)) Suppose that the function g on the real line satisfies the following Lipschitz type condition:*

$$|g(x+\delta) - g(x)| \leq \varphi(x)|\delta| \tag{5.13}$$

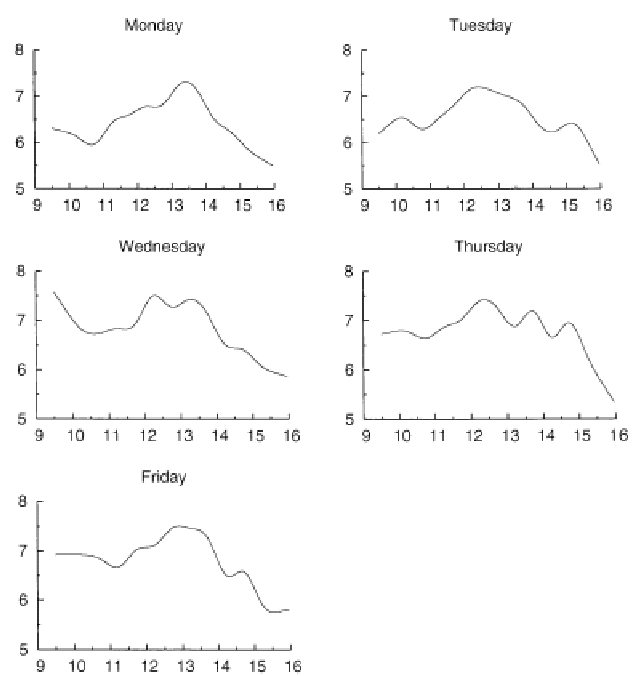
for each given $x \in S_\omega$, where S_ω is a compact support. Furthermore, $\varphi(\cdot)$ is a nonnegative measurable function such that with probability one and for some $0 < G < 1$:

$$\max_{t\geq 1} E\left[\varphi^2(\psi_t)|(\psi_{t-1},\cdots,\psi_1)\right] \leq G^2;\ \max_{t\geq 1} E\left[\varphi^2(\psi_{t,m})|(\psi_{t-1,m-1},\cdots,\psi_{1,1})\right] \leq G^2.$$

The asymptotic normality of the least squares estimator of the unknown parameter is then proved using the Taylor expansions together with the convergence rates for kernel estimation derived in Hansen (2008). Saart et al. (2013) also showed that an extension of the SEMI–ACD(1,1) model to, for example, a SEMI–ACD(p,q) model, where $q \leq 3$, is also possible without requiring an additional assumption. This is in the sense that the $\sqrt{n}$-asymptotic normality of the relevant estimators still holds, provided that $q \leq 3$. Such a claim is supported by the results found in Robinson (1988) and Fan and Li (1999b).

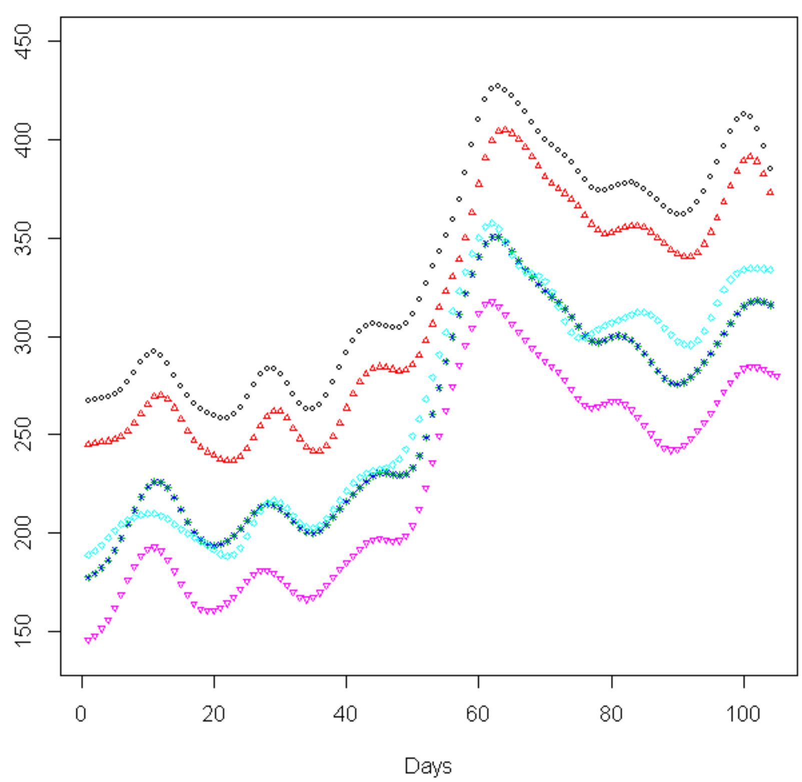
Let us finish this section with discussion of some important issues on the use of the SEMI-ACD model in practice. Firstly, the construction of the ACD models is done under the assumption that the duration process is stationary. However, it is well known that intraday financial data often involve some strongly diurnal patterns. Hence, the first step toward the econometric analysis of financial duration is always to perform a diurnal adjustment.

Figure 5.1. *Diurnal Patterns of Financial Trading for Monday to Friday*



Nonetheless, the remaining problems are other trading patterns, for example day of the week effects, that have not been taken care of. To illustrate the problem, let us consider Figure 5.1, which is a reproduction of Figure 7 of Meitz and Teräsvirta (2006). The figure presents the nonparametric kernel estimate of diurnal components of the price duration series for IBM for each of the trading days. The data are high-frequency data for IBM shares between July 2002 and December 2002. The results show a similar but not exactly identical inverted U-shaped pattern in the moving average of durations over the days. An idea, which is a work in progress, is to use the fact that the EG-PLSI model in (2.9) allows the nonparametric shape-invariant analysis (Härdle and Marron (1990)) and to jointly model the regular components of the duration process without unpooling the data. We have applied the idea to the total number of daily hospital admissions of circulatory and respiratory patients in Hong Kong between 1994 to 1996, i.e. the data set originally used in Xia et al. (2002). Figure 5.2 below presents the estimated time trend taking day-effects into account ($\circ$ Monday, $\bigtriangledown$ Tuesday, $\times$ Wednesday, $\diamond$ Thursday and $\triangle$ Friday). (Details of this work are available from the authors upon request.)

Figure 5.2. *Trend and Day-Effects for Hong Kong Patients*



Finally, the SEMI-ACD model can potentially be used in various empirical studies of financial market microstructure. Saart and Gao (2012), for example, applied the SEMI-ACD model to model the intertemporal dynamics of the price change duration process in stock exchange markets. The probability distribution of the resulting estimates of the so-called standardized durations were then hypothetically tested in order to obtain some information about that of the true duration processes. Although the details can be found in the paper, it is noted here that the outcomes of the above mentioned test are different when it is implemented based on the SEMI-ACD model compared to when it is based on the parametric ACD model. Furthermore, a work in progress is being conducted in the use of the resulting standardized duration from the SEMI-ACD model to study the exogeneity of trade arrivals in the financial market (details of this work are available from the authors upon request).

6. Semiparametric Models with Nonstationary Data

Firstly, in this section we will review a number of semiparametric models, which have been established to help detect and estimate trend and seasonality. Furthermore, since we have discussed the endogeneity problem in semiparametric models in detail in the previous section, it will also be of particular interest to also review the estimation of semiparametric models that involves both endogeneity and nonstationarity.

Semiparametric Detection and Estimation of Trend and Seasonality

Many important macroeconomic and financial data, such as income, unemployment and retail sale, are found to exhibit deterministic/stochastic trends. The closest semiparametric model to the PL model that explicitly allows for a trend detection is the PL time series error model introduced by Gao and Hawthorne (2006), of the form:

$$Y_t = U_t^\tau \beta + g\left(\frac{t}{n}\right) + \varepsilon_t,\ t = 1, 2, \ldots, n, \tag{6.1}$$

where $\{Y_t\}$ is a response variable (e.g. the mean temperature series), $U_t = (U_{t1}, \ldots, U_{tq})^\tau$ is a vector of q-explanatory variables (e.g. the southern oscillation index), t is the time in years, β is a vector of unknown coefficients for the explanatory variables, $g(\cdot)$ is an unknown smooth function of time representing the trend and $\{\varepsilon_t\}$ represents a sequence of stationary time series errors with $E[e_t] = 0$ and $0 < \text{var}[e_t] = \sigma^2 < \infty$. In order to estimate the model, Gao and Hawthorne (2006) introduce an estimation procedure, which is closely similar to that of the above mentioned PL time series model: (i) compute an estimate of $g(\cdot)$ for a given β, i.e. similar to the second term of (3.18); (ii) compute the least-squares estimate of β; (iii) compute the new estimate of $g(\cdot)$ based on that of β estimated in the previous step. Gao and Hawthorne (2006) also consider an alternative case where $\{\varepsilon_t\}$ is allowed to be $I(1)$. This is to say that $\{\varepsilon_t\}$ itself may be nonstationary, but its differences $\delta_t = \varepsilon_t - \varepsilon_{t-1}$ are assumed to be stationary. In this case, we need only to consider the first differenced version of (6.1) of the form

$$V_t = W_t^\tau \beta + m\left(\frac{t}{n}\right) + \delta_t,\ t = 1, 2, \ldots, n, \tag{6.2}$$

where $V_t = Y_t - Y_{t-1}$, $W_t = U_t - U_{t-1}$, and $m\left(\frac{t}{n}\right) = g\left(\frac{t}{n}\right) - g\left(\frac{t-1}{n}\right)$.

These models enable us to study an important issue in practice, which is to determine whether a linear trend is able to approximate the behavior of the series in question adequately. Using the model in (6.1), such a problem can be written as the hypotheses

$$H_0 : g\left(\frac{t}{n}\right) = \alpha_0 + \gamma_0 t \quad \text{versus} \quad H_1 : g\left(\frac{t}{n}\right) \neq \alpha + \gamma t \tag{6.3}$$

for some $\theta_0 = (\alpha_0, \gamma_0) \in \Theta$ and all $\theta = (\alpha, \gamma) \in \Theta$, where Θ is a parameter space in R^2. Hence, this issue is coherent with the general interest in statistics and econometrics, which involves testing the hypotheses of a parametric form against a nonparametric alternative. Inspired by Horowitz and Spokoiny (2001), Gao and Hawthorne (2006) propose a novel test for linearity in the trend function $g(\cdot)$ under such semiparametric settings such as (6.1) and (6.2). For each given value of bandwidth h, to test H_0, Gao and Hawthorne (2006) propose using the following:

$$L_{4n}(h) = \frac{\sum_{t=1}^n \sum_{s=1,\neq t}^n K\left(\frac{s-t}{nh}\right) \widetilde{\varepsilon}_s \widetilde{\varepsilon}_t}{\widetilde{S}_n}, \tag{6.4}$$

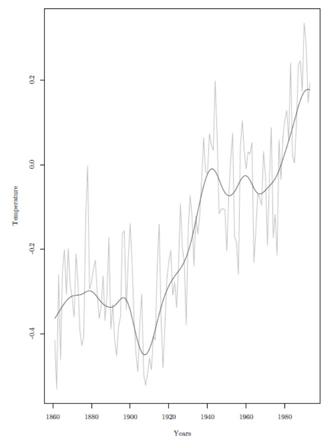
where $\widetilde{S}_n^2 = 2\sum_{t=1}^n \sum_{s=1}^n K^2\left(\frac{s-t}{nh}\right) \widetilde{\varepsilon}_s^2 \widetilde{\varepsilon}_t^2$, $\widetilde{\varepsilon}_t = Y_t - U_t^\tau \widetilde{\beta} - f(t, \widetilde{\theta})$ in which $f(t, \widetilde{\theta})$ is the least-squares estimate of $f(t, \theta_0)$.

By applying the above method, Gao and Hawthorne (2006) shows that the trend estimate of the global temperature series for 1867 to 1993 appears to be distinctly nonlinear. Figure 5.3 below is a reproduction Figure 4 of Gao and Hawthorne (2006), which shows the global temperature series for 1867 to 1993 and the estimated trend. Furthermore, Gao and Hawthorne (2006) also consider the possible nonstationarity of the residuals in the models by applying the first differenced version of the model defined in (6.2). The hypothesis testing described in (6.3) and (6.4) is then employed. They report that a similar conclusion – rejecting the linearity in the trend – can be drawn by using either the level or differenced version of the data. Note that in order to perform a nonparametric kernel testing such as this, bandwidth selection can be crucial and may significantly affect the outcome of the test. A novel idea about the testing procedure in Gao and Hawthorne (2006) is the use of a maximized version of the test such that:

$$L^* = \max_{h \in H_n} L_{4n}(h). \tag{6.5}$$

The main theoretical results of the paper show the consistency of such a test; see also Gao and King (2004), Gao and Gijbels (2008), and Saart and Gao (2012) for related works on nonparametric kernel testing and bandwidth selection.

Figure 5.3. *Global temperature series for 1867 to 1993 (light line) and the estimated trend (solid curve)*



More recently, there is a new semiparametric PL time series model has been developed by Chen et al. (2011). Although this model does not assist us with the dimension reduction problem, Chen et al. (2011) procedure provides a convenient estimation of the following extended version of the PL time series model:

$$Y_t = \beta(U_t, \theta_1) + g(U_t) + \varepsilon_t, \tag{6.6}$$

where $\beta(\cdot, \theta_1)$ is the known link function indexed by an unknown parameter vector $\theta_1 \in \Theta \subset \mathbb{R}^p$ ($p \geq 1$). An important point to note about the model in (6.6) is the fact that $\{U_t\}$ is allowed to be generated by

$$U_t = H\left(\frac{t}{n}\right) + u_t, \tag{6.7}$$

where $H(t)$ is unknown functions defined on $\mathbb{R}^d$ and $\{u_t\}$ is a sequence of i.i.d. random errors. In other words, it allows for the existence of deterministic trends in the regressors. Chen et al. (2011) studied a case where nonstationarity was allowed and was driven by a deterministic trending component. Regarding the model's estimation procedure, Chen et al. (2011) provided two alternative methods, namely the nonlinear least squares (see Gao (1995) and Gao (2012) for example) and the semiparametric weighted least squares estimations (see Härdle et al. (2000) for example). Among these methods, the former first estimates θ_1; such an estimate is then used in order to compute that of $g(\cdot)$, while the latter operates in just the reverse order. More important issues, however, are the identifiability and estimatability of the model. The following conditions are needed in Chen et al. (2011) in order to ensure that θ_1 in (6.6) is identifiable and estimable.

Assumption 6.1. *(Assumption A2 of Chen et al. (2011)) (i) $\beta(U_t, \theta)$ is twice differentiable with respect to θ, and both $g(\cdot)$ and $H(\cdot)$ are continuous. (ii) Denoting the partial derivative of $\beta(U_t, \theta)$ with respect to θ by $\dot{\beta}(U_t, \theta)$, then*

$$\Gamma(\theta) := \int_0^1 \left\{ \int g(v)\dot{\beta}(v,\theta)p_u(v - H(r))dv \right\} dr = 0$$

for all $\theta \in \Theta$ and $\int_0^1 \left\{ \int [\beta(v,\theta_1) - \beta(v,\theta)]\dot{\beta}(v,\theta)p_u(v - H(r))dv \right\} dr \neq 0$ uniformly in $\theta \in \Theta(\delta) = \{\theta : ||\theta - \theta_1|| \leq \delta\}$ for any $\delta > 0$.

In addition, there is an alternative model that is closely related to (6.6), which is discussed in Gao (2012). Unfortunately, due to the unavailability of the asymptotic results, the study focuses only on the case where $p = 1$. Gao's (2012) model can be obtained simply by replacing the parametric component with $x_t\beta$ and the nonparametric component with $g(x_t)$, whereby the regressor is defined as in the assumption below.

Assumption 6.2. *(Assumption 3.2(i) of Gao (2012)) Let $x_t = x_{t-1} + u_t$ with $x_0 = 0$ and $u_t = \sum_{i=0}^{\infty} \psi_i \eta_{t-i}$, where $\{\eta_t\}$ is a sequence of independent and identically distributed random errors, and $\{\psi_i : i \geq 0\}$ is a sequence of real numbers such that $\sum_{i=0}^{\infty} i^2 |\psi_i| < \infty$ and $\sum_{i=0}^{\infty} \psi_i \neq 0$.*

The required smallness conditions on $g(\cdot)$ are provided for two cases: stationary and nonstationary regressors. While the conditions of the stationary regressors are discussed in details in our discussion of the SL model for time series, those required for the nonstationary regressor case are the following:

Assumption 6.3. *(Assumption 3.1 of Gao (2012)) (i) Let $g(\cdot)$ be a real function on $\mathbb{R}^1 = (-\infty, \infty)$ such that $\int |x|^i |g(x)|^i dx < \infty$ for $i = 1, 2$, and $\int xg(x)dx \neq 0$; (ii) In addition, let $g(\cdot)$ satisfy $\int \left| \int e^{ixy} yg(y)dy \right| dx < \infty$ when $\int xg(x)dx = 0$.*

An important point to note about these assumptions is the fact that both exclude the case where $g(x)$ is a simple linear function of x. An interesting application of this smallness condition in practice is presented in Example 5.3 of Gao (2012). The author considers the logarithm of British pound/American dollar real exchange rate defined by:

$$y_t = \log(e_t) + \log(p_t^{UK}) - \log(p_t^{US}), \tag{6.8}$$

where $\{e_t\}$ is the monthly average of the nominal exchange rate, and $\{p_t^j\}$ denotes the consumer price index of country j. He finds that $\{y_t\}$ approximately follows a threshold model of the form

$$y_t = y_{t-1} - 1.1249 y_{t-1} I[|y_{t-1}| \leq 0.0134] + e_t. \tag{6.9}$$

This result suggests that, although $\{y_t\}$ does not necessarily follow an integrated time series model, e.g. $y_t = y_{t-1} + e_t$, it behaves like a nearly integrated time series, since the nonlinear component is a small departure function (see also the discussion on the semiparametric threshold models in Section 7).

In the literature, there is a number of mathematical approaches which have been established as tools for deriving an asymptotic theory for the nonparametric estimation of univariate models of nonstationary data. Below, let us mention a couple (see also the review in Sun and Li (2012) for details). Firstly, we have the Markov splitting technique used in; for example, Karlsen and Tjøstheim (2001) and Karlsen et al. (2007) that is used to model univariate time series with a null recurrent structure. Secondly, we have the local time methods developed by Phillips (2009) and Wang and Phillips (2011) used to derive an asymptotic theory for the nonparametric estimation of univariate models with an integrated time series. In Gao's (2012) model, since $\{x_t\}$ is nonstationary, the parameter β is identifiable and chosen such that $\frac{1}{n}\sum_{t=1}^{n}[y_t - x_t\beta]^2$ is minimized over β leading to

$$\widehat{\beta} = \left(\sum_{t=1}^{n} x_t^2\right)^{-1} \left(\sum_{t=1}^{n} x_t y_t\right), \tag{6.10}$$

which is closely related to the results of (3.18). Although the details are discussed in the paper, we note here that in order to establish an asymptotic distribution for $\widehat{\beta}$, it is necessary that, as $n \to \infty$, we have:

$$\frac{1}{n}\sum_{t=1}^{n} x_t g(x_t) \to_P 0.$$

Regarding the case of a nonstationary regressor, $\int x g(x) dx$ may or may not be zero. The asymptotic distribution of the estimators, namely the above ordinary least squares estimator of the unknown parameter β and the nonparametric estimator of $g(\cdot)$, are based very much on Theorem 2.1 of the studies by Wang and Phillips (2009a), and Wang and Phillips (2011).

Semiparametric Estimation in Multivariate Nonstationary Time Series Models

In the case of independent and stationary time series data, semiparametric methods have been shown to be particularly useful in modelling economic data in a way that retains generality where it is most needed while reducing dimensionality problems. Gao and Phillips (2013) sought to pursue these advantages in a wider context that allows for nonstationarities and endogeneities within a vector semiparametric regression model. In their study, the time series $\{(Y_t, X_t, V_t) : 1 \le t \le n\}$ were assumed to be modeled in a system of multivariate nonstationary time series models of the form:

$$
\begin{aligned}
Y_t &= AX_t + g(V_t) + e_t; \qquad (6.11)\\
X_t &= H(V_t) + U_t \quad t = 1, 2, \ldots, n;\\
E[e_t|V_t] &= E[e_t] = 0; \text{ and} \qquad (6.12)\\
E[U_t|V_t] &= 0, \qquad (6.13)
\end{aligned}
$$

where n is the sample size, A is a $p \times d$-matrix of unknown parameters, $Y_t = (y_{t1}, \ldots, y_{tp})^\tau$, $X_t = (x_{t1}, \ldots, x_{td})^\tau$ and V_t is a sequence of univariate integrated time series regressors, $g(\cdot) = (g_1(\cdot), \ldots, g_p(\cdot))^\tau$ and $H(\cdot) = (h_1(\cdot), \ldots, h_d(\cdot))^\tau$ are all unknown functions, and both e_t and U_t are vectors of stationary time series. Note that $\{X_t\}$ can be stationary only when $\{X_t\}$ and $\{V_t\}$ are independent. The identification condition $E[e_t|V_t] = E[e_t] = 0$ in (6.12) eliminates endogeneity between e_t and V_t while retaining endogeneity between e_t and X_t and potential nonstationarity in both X_t and V_t. In this setting, such a condition corresponds to the condition $E[e_t|V_t, U_t] = E[e_t|U_t]$ that is assumed in Newey et al. (1999), for example. The rational behind (6.12) is the fact that

$$E[e_t|V_t] = E(E[e_t|U_t, V_t]|V_t) = E(E[e_t, |U_t]|V_t) = E(E[e_t|U_t]) = E[e_t]$$

when U_t is independent of V_t and $E[e_t] = 0$. These conditions are less restrictive than the exogeneity condition between e_t and (X_t, V_t) that is common in the literature for the stationary case. In the study by Gao and Phillips (2013), the model is treated as a vector semiparametric structural model, and considers the case where X_t and V_t may be vectors of nonstationary regressors and X_t may be endogenous. The main contribution of the study resides in the derivation of a semiparametric instrumental variable least squares estimate of A to deal with endogeneity in X_t and a nonparametric estimator for the function $g(\cdot)$. Let us assume that there exists a vector of stationary variables η_t for which we have:

$$E[U_t \eta_t^\tau] \ne 0 \text{ and } E[e_t|\eta_t] = 0.$$

The derivation of the semiparametric instrumental variable least squares estimate of A can now be done based on the following expanded version of the system (6.11):

$$\begin{aligned} Y_t &= AX_t + g(V_t) + e_t \quad t = 1, 2, \ldots, n & (6.14) \\ X_t &= H(V_t) + U_t & \\ Q_t &= J(V_t) + \eta_t & \\ E[e_t|V_t] &= E[e_t] = 0, \; E[U_t|V_t] = 0 \text{ and } E[\eta_t|V_t] = 0, & (6.15) \end{aligned}$$

where $Q_t = (q_{t1}, \ldots, q_{td})^\tau$ is a vector of possible instrumental variables for X_t generated by a reduced form equation involving V_t, and $I(\cdot) = (J_1(\cdot), \ldots, J_d(\cdot))^\tau$ is a vector of unknown functions. The limiting theory in this kind of nonstationary semiparametric model depends on the probabilistic structure of the regressors and errors e_t, U_t, η_t and V_t, as well as the functional forms of $g(\cdot)$, $H(\cdot)$ and $J(\cdot)$. Gao and Phillips (2013) provide a list of the conditions required including, their detailed explanation in Appendix A of the paper.

7. Conclusions and Discussion

We have seen in the literature that theoretical and empirical research in time series analysis may be conducted on a large number of topics. Among these, we personally believe that perhaps nonlinear time series models are the most studied over recent years. In order to take the nonlinearity in time series regression in to account, nonparametric methods have been very popular both for predicting and characterizing nonlinear dependence. However, their developments has been significantly dampened by the so-called curse of dimensionality. Firstly, we reviewed a number of semiparametric time series models offered in the literature as the methods for combating the curse of dimensionality and their specification testings. In order to proceed along a linked sequence of materials, we identified two links between these semiparametric models, namely exogeneity and stationarity conditions. Addressing the breakdown in the former led to the emergence of semiparametric models with generated regressors, while addressing the breakdown in stationarity led to semiparametric models of nonstationary data. We presented a detailed review of recent developments of these models. Nonetheless, since time series models for nonstationary data provide a large field of research, the review in this paper focused on semiparametric models established to help detect and estimate trend and seasonality, and semiparametric models that involve both endogeneity and nonstationarity.

In many places throughout the previous sections, we have provided our views on future research. In the following, let us discuss some additional open questions about this area of research. In our view, the problems of endogeneity and nonstationarity are both important issues that future research in the area of semiparametric time series should be based on. As reviewed previously, Kim and Saart (2013) and Kim et al. (2013) successfully addressed the endogeneity problem in the PL and the EG-PLSI models. However, a number of questions are left unanswered, especially the importance of weak/strong instruments and the characteristics of the control function on the performance of the CF approach. Furthermore, a more detailed comparison between the CF and the NpIV approaches than what was done in Kim et al. (2013) is required, especially on the characteristics of the endogeneity for which these methods would be more advantageous. An advantage of the CF approach is its ability to disentangle the structural nonparametric relationship and the effect of endogeneity. Hence, a simple test of exogeneity can be developed by testing the statistical significance of the above mentioned effect of endogeneity. The first attempt by Kim et al. (2013) was to use bias-corrected confidence bands in nonparametric regression of Xia (1998). However, we believe that a more formal test can be developed based on this idea.

More recently, there has also been an attempt by Gao et al. (2013) to detect and to estimate a structural change from a nonlinear stationary regime to a linear nonstationary regime using a semiparametric threshold autoregressive model, which can be conveniently expressed as

$$\begin{aligned} Y_t &= g(Y_{t-1})I[Y_{t-1} \in C_\tau] + \alpha Y_{t-1} I[Y_{t-1} \in D_\tau] + \varepsilon_t \\ &= \begin{cases} g(Y_{t-1}) + \varepsilon_t & \text{if } Y_{t-1} \in C_\tau \\ \alpha Y_{t-1} + \varepsilon_t & \text{if } Y_{t-1} \in D_\tau, \end{cases} \end{aligned} \tag{7.1}$$

where C_τ is either a compact subset of R^1 or a set of the type $(-\infty, \tau]$ or $[\tau, \infty)$, D_τ is the complement of C_τ, $g(x)$ is an unknown and bounded function when $x \in C_\tau$ and $\alpha = 1$. Lemma 3.1 of the paper shows a special case of the model where $\alpha = 1$ is a β-null recurrent Markov Chain process; see also a detailed discussion on a null recurrent process in Karlsen et al. (2007). The existing asymptotic results for the stationary nonlinear time series models (for instance in Fan and Yao (2003), and Gao (2007)) are not directly applicable. While Gao et al. (2013) studied the asymptotic behavior of both a nonparametric estimator of $g(\cdot)$ and the least square estimator of α, their mathematical proof relied heavily on a number of general results of the β-null recurrent Markov chains discussed in Karlsen and Tjøstheim (2001).

As an alternative, we could establish a new threshold autoregressive model such that the response variable Y depends on the vector of stochastic explanatory variables or stochastic covariates $X = (X_1, \ldots, X_p)^T$, where $(p \geq 2)$ as follows:

$$Y = \beta_0^T X + \phi(\theta_0^T X) I[\theta_0^T X \in C_\tau] + \varepsilon = \begin{cases} \beta_0^T X + \phi(\theta_0^T X) + \varepsilon, & \text{if } \theta_0^T X \in C_\tau \\ \beta_0^T X + \varepsilon, & \text{otherwise,} \end{cases} \quad (7.2)$$

where the conditions stated in Assumption 2.1 hold and C_τ is either a compact subset of $\mathbb{R}^1$ or a set of the type $(-\infty, \tau]$ or $[\infty, \tau)$. We refer to the model as the partially-linear single-index threshold autoregressive (PlSi–TAR) model. Let us state a few remarks regarding the PlSi–TAR model: (i) Compared to the SEMI–TAR model of Gao et al. (2013), the PlSi–TAR model offers alternative types of flexibility, which can be quite useful when attempting to perform dimension reduction in modeling time series data. (ii) The model can be used to detect the structural change from a nonlinear stationary regime to a linear stationary regime. However, by relaxing some of the conditions in Assumption 2.1 the model can also be used to detect the structural change from a nonlinear stationary regime to a linear nonstationary regime. As another alternative, we may introduce an extended PlSi–TAR model of the form:

$$Y_t = g(X_t, \theta_1) + \phi(\theta_0^T X_t) I[\theta_0^T X_t \in C_\tau] + \varepsilon_t = \begin{cases} g(X_t, \theta_1) + \phi(\theta_0^T X_t) + \varepsilon_t & \text{if } \theta_0^T X_t \in C_\tau, \\ g(X_t, \theta_1) + \varepsilon_t & \text{otherwise,} \end{cases} \quad (7.3)$$

where $g(\cdot, \theta_1)$ is a known link function indexed by an unknown parameter vector $\theta_1 \in R^p$ $(p \geq 1)$.

Finally, let us give a remark on another important research direction, which focuses instead on improving parametric time series modeling. This line of development may be worth further exploration in parallel with those methods discussed previously. Clearly, an important benefit of the above semiparametric models resides in the additional flexibility that they provide given a constraint in the form of the curse of dimensionality. However, if we take a different point of view, for example, that all models are wrong, but some are useful (Box (1976)), then the usual arguments in favor of non/semi parametric models are substantially weakened (especially in time series analysis). As suggested by an anonymous referee, in this case, it is perhaps particularly relevant to explore ways to fit mis-specified parametric models more earnestly. An example of the studies in this area is that of Xia and Tong (2011).

References

Ai, C., Chen, X., 2003. Efficient estimation of models with conditional moment restrictions containing unknown functions. Econometrica 71 (6), 1795–1843.

Aït-Sahalia, Y., 1996. Testing continuous-time models of the spot interest rate. Review of Financial Studies 9 (2), 385–426.

Aït-Sahalia, Y., 2002. Transition densities for interest rate and other nonlinear diffusions. The Journal of Finance 54 (4), 1361–1395.

Aït-Sahalia, Y., Lo, A., 2002. Nonparametric estimation of state-price densities implicit in financial asset prices. The Journal of Finance 53 (2), 499–547.

Bachmeier, L., 2002. Is the term structure nonlinear? A semiparametric investigation. Applied Economics Letters 9 (3), 151–153.

Blundell, R., Powell, J., 2003. Endogeneity in nonparametric and semiparametric regression models. Econometric Society Monographs 36, 312–357.

Blundell, R., Powell, J., 2004. Endogeneity in semiparametric binary response models. Review of Economics Studies 71, 655–679.

Box, G. E., 1976. Science and statistics. Journal of the American Statistical Association 71 (356), 791–799.

Carroll, R., Fan, J., Wand, M., 1997. Generalized partially linear single-index models. Journal of the American Statistical Association 92 (438), 477–489.

Chan, K., 1993. Consistency and limiting distribution of the least squares estimator of a threshold autoregressive model. The Annals of Statistics 21 (1), 520–533.

Chan, K., Karolyi, G., Longstaff, F., Sanders, A., 2012. An empirical comparison of alternative models of the short-term interest rate. The Journal of Finance 47 (3), 1209–1227.

Chan, K. S., Tong, H., 1986. On estimating thresholds in autoregressive models. Journal of time series analysis 7 (3), 179–190.

Chen, J., Gao, J., Li, D., 2011. Estimation in semiparametric time series regression. Statistics and Its Interface 4 (4), 243–252.

Cook, R., 1998. Regression Graphics. Wiley: New York.

Cox, J., Ingersoll, E., Ross, S. A., 1985. A theory of the term structure of interest rates1. Econometrica 53 (2), 385–407.

Darolles, S., Fan, Y., Florens, J.-P., Renault, E., 2011. Nonparametric instrumental regression. Econometrica 79 (5), 1541–1565.

Dufrénot, G., Mignon, V., 2002. Recent Developments in Nonlinear Cointegration with Applications to Macroeconomics and Finance. Springer.

Engle, R., Russell, J., 1998. Autoregressive conditional duration: a new model for irregularly spaced transaction data. Econometrica 66, 1127–1162.

Engle, R. F., Granger, C., Rice, J., Weiss, A., 1986. Semiparametric estimates of the relation between weather and electricity sales. Journal of the American Statistical Association 81 (394), 310–320.

Eubank, R., 1999. Nonparametric Regression and Spline Smoothing. Vol. 157. CRC Press: Boca Raton.

Fan, J., 2005. A selective overview of nonparametric methods in financial econometrics. Statistical Science 20 (4), 317–337.

Fan, J., Gijbels, I., 1996. Local Polynomial Modeling and its Applications. Chapman & Hall, London.

Fan, J., Härdle, W., Mammen, E., 1998. Direct estimation of low-dimensional components in additive models. The Annals of Statistics 26 (3), 943–971.

Fan, J., Yao, Q., 2003. Nonlinear Time Series: Nonparametric and Parametric Methods. Springer Verlag: New York.

Fan, J., Yao, Q., Tong, H., 1996. Estimation of conditional densities and sensitivity measures in nonlinear dynamical systems. Biometrika 83 (1), 189–206.

Fan, Y., Li, Q., 1996. Consistent model specification tests: omitted variables and semiparametric functional forms. Econometrica: Journal of the Econometric Society 64 (4), 865–890.

Fan, Y., Li, Q., 1997. A consistent nonparametric test for linearity of ar (p) models. Economics Letters 55 (1), 53–59.

Fan, Y., Li, Q., 1999a. Central limit theorem for degenerate U-statistics of absolutely regular processes with applications to model specification testing. Journal of Nonparametric Statistics 10 (3), 245–271.

Fan, Y., Li, Q., 1999b. Root-N-consistent estimation of partially linear time series models. Journal of Nonparametric Statistics 11 (1-3), 251–269.

Fan, Y., Li, Q., 2003. A kernel-based method for estimating additive partially linear models. Statistica Sinica 13 (3), 739–762.

Fan, Y., Li, Q., Stengos, T., 1995. Root-N-consistent semiparametric regression with conditional heteroscedastic disturbances. Journal of Quantitative Economics 11, 229–140.

Galego, A., Pereira, J., 2010. Evidence on gender wage discrimination in Portugal: parametric and semi-parametric approaches. Review of Income and Wealth 56 (4), 651–666.

Gao, J., 1995. Parametric test in a partially linear model. Acta Mathematica Scientia (English Edition) 15, 1–10.

Gao, J., 2007. Nonlinear Time Series: Semiparametric and Nonparametric Methods. Chapman & Hall, London.

Gao, J., 2012. Handbook on Non- and Semi-Parametric Econometrics and Statistics. Oxford University Press, Oxford, Ch. Identification, estimation and specification in a class of semi-linear time series models.

Gao, J., Gijbels, I., 2008. Bandwidth selection in nonparametric kernel testing. Journal of the American Statistical Association 103 (484), 1584–1594.

Gao, J., Hawthorne, K., 2006. Semiparametric estimation and testing of the trend of temperature series. Econometrics Journal 9, 332–355.

Gao, J., King, M. L., 2004. Adaptive testing in continuous-time diffusion models. Econometric Theory 20 (05), 844–882.

Gao, J., Lu, Z., Tjøstheim, D., 2006. Estimation in semiparametric spatial regression. The Annals of Statistics 34 (3), 1395–1435.

Gao, J., Phillips, P. C., 2013. Semiparametric estimation in triangular system equations with nonstationarity. Journal of Econometrics.

Gao, J., Tjøstheim, D., Yin, J., 2013. Estimation in threshold autoregressive models with nonstationarity. Journal of Econometrics 172, 1 − 13.

Gao, J., Tong, H., 2004. Semiparametric non-linear time series model selection. Journal of the Royal Statistical Society: Series B (Statistical Methodology) 66 (2), 321–336.

Gao, J., Yee, T., 2000. Adaptive estimation in partially linear autoregressive models. Canadian Journal of Statistics 28, 571–586.

Hall, P., Horowitz, J., 2005. Non-parametric method for inference in the presence of instrumental variables. The Annals of Statistics 33, 2904–2929.

Hansen, B., 2008. Uniform convergence rates for kernel estimation with dependent data. Econometric Theory 24, 726–748.

Härdle, W., Hall, P., Ichimura, H., 1993. Optimal smoothing in single-index models. The Annals of Statistics 21 (1), 157–178.

Härdle, W., Liang, H., Gao, J., 2000. Partially Linear Models. Springer-Verlag, Berlin.

Härdle, W., Lütkepohl, H., Chen, R., 2007. A review of nonparametric time series analysis. International Statistical Review 65 (1), 49–72.

Härdle, W., Mammen, E., 1993. Comparing nonparametric versus parametric regression fits. The Annals of Statistics 21 (4), 1926–1947.

Härdle, W., Marron, J., 1990. Semiparametric comparison of regression curves. The Annals of Statistics 18 (1), 63–89.

Heckman, N., 1986. Spline smoothing in a partly linear model. Journal of the Royal Statistical Society, Series B 48, 244–248.

Horowitz, J. L., Spokoiny, V. G., 2001. An adaptive, rate-optimal test of a parametric mean-regression model against a nonparametric alternative. Econometrica 69 (3), 599–631.

Ichimura, H., Lee, L., 1991. Semiparametric least squares estimation of multiple index models: single equation estimation. In: Nonparametric and Semiparametric Methods in Econometrics and Statistics: Proceedings of the Fifth International Symposium in Economic Theory and Econometrics. Cambridge. pp. 3–49.

Karlsen, H., Myklebust, T., Tjøstheim, D., 2007. Nonparametric estimation in a nonlinear cointegration type model. The Annals of Statistics 35 (1), 252–299.

Karlsen, H., Tjøstheim, D., 2001. Nonparametric estimation in null recurrent time series. The Annals of Statistics, 372–416.

Karlsen, H. A., Myklebust, T., Tjøstheim, D., 2010. Nonparametric regression estimation in a null recurrent time series. Journal of Statistical Planning and Inference 140 (12), 3619–3626.

Khashimov, S., 1993. Limiting behavior of generalized U-statistics of weakly dependent stationary processes. Theory of Probability & Its Applications 37 (1), 148–150.

Kim, N., Saart, P., 2013. Estimation in semiparametric partially linear models with parametric and/or nonparametric endogeneity. SSRN Working Paper Series. Available at SSRN: http://ssrn.com/abstract=2213071 (accessed 18 June 2013).

Kim, N., Saart, P., Gao, J., 2013. Semi-parametric analysis of shape-invariant Engel curves with control function approach. SSRN Working Paper Series. Available at SSRN: http://ssrn.com/abstract=2213067 (accessed 18 June 2013).

Lewbel, A., Linton, O., 2007. Nonparametric matching and efficient estimators of homothetically separable functions. Econometrica 75 (4), 1209–1227.

Li, K., 1991. Sliced inverse regression for dimension reduction. Journal of the American Statistical Association 86 (414), 316–327.

Li, Q., 1999. Consistent model specification tests for time series econometric models. Journal of Econometrics 92 (1), 101–147.

Li, Q., Wooldridge, J., 2002. Semiparametric estimation of partially linear models for dependent data with generated regressors. Econometric Theory 18, 625–645.

Linton, O., Nielsen, J., 1995. A kernel method of estimating structural nonparametric regression based on marginal integration. Biometrika 82, 93–100.

Mammen, E., Linton, O., Nielsen, J., 1999. The existence and asymptotic properties of a backfitting projection algorithm under weak conditions. The Annals of Statistics 27 (5), 1443–1490.

Meitz, M., Teräsvirta, T., 2006. Evaluating models of autoregressive conditional duration. Journal of Business & Economic Statistics 24 (1).

Newey, W., Powell, J., 2003. Instrumental variable estimation of nonparametric models. Econometrica 71 (5), 1565–1578.

Newey, W., Powell, J., Vella, F., 1999. Nonparametric estimation of triangular simultaneous equations models. Econometrica 67 (3), 565–603.

Nielsen, J., Linton, O., 1998. An optimization interpretation of integration and back-fitting estimators for separable nonparametric models. Journal of the Royal Statistical Society: Series B (Statistical Methodology) 60 (1), 217–222.

O'Sullivan, F., 1986. A statistical perspective on ill-posed inverse problems. Statistical Science 1 (4), 502–518.

Pacurar, M., 2008. Autoregressive conditional duration models in finance: a survey of the theoretical and empirical literature. Journal of Economic Surveys 22, 711–751.

Phillips, P., 2009. Local limit theory and spurious nonparametric regression. Econometric Theory 25 (06), 1466–1497.

Pinkse, J., 2000. Nonparametric two-step regression estimation when regressors and error are dependent. The Canadian Journal of Statistics 28 (2), 289–300.

Robinson, P., 1988. Root-N-consistent semiparametric regression. Econometrica 56, 931–954.

Saart, P. W., Gao, J., 2012. An alternative nonparametric specification test in nonlinear autoregressive conditional duration model. Working Paper Series. Available at SSRN: `http://papers.ssrn.com/sol3/papers.cfm?abstract_id=2130454` (accessed on 18 June 2013).

Saart, P. W., Gao, J., Allen, D., 2013. Semiparametric autoregressive conditional duration model: theory and practice. Econometric Reviews (in press) 33.

Su, L., Ullah, A., 2008. Local polynomial estimation of nonparametric simultaneous equations models. Journal of Econometrics 144 (1), 193–218.

Sun, Y., Li, Q., 2012. Nonparametric and semiparametric estimation and hypothesis testing with nonstationary time series. Working Paper.

Teräsvirta, T., Tjøstheim, D., Granger, C. W., 2011. Modelling nonlinear economic time series. Oxford University Press, New York.

Tjøstheim, D., Austad, R., 1996. Nonparametric identification of nonlinear time series: projection. Journal of the American Statistical Association 89, 1398–1409.

Tong, H., Lim, K., 1980. Threshold autoregression, limit cycles and cyclical data. Journal of the Royal Statistical Society. Series B (Methodological) 42 (3), 245–292.

Truong, Y., Stone, C., 1994. Semiparametric time series regression. Journal of Time Series Analysis 15 (4), 405–428.

Van Dijk, D., Teräsvirta, T., Franses, P., 2002. Smooth transition autoregressive modelsa survey of recent developments. Econometric Reviews 21 (1), 1–47.

Vasicek, O., 1977. An equilibrium characterization of the term structure. Journal of Financial Economics 5 (2), 177–188.

Wang, Q., Phillips, P., 2009a. Asymptotic theory for local time density estimation and nonparametric cointegrating regression. Econometric Theory 25 (3), 710.

Wang, Q., Phillips, P., 2009b. Structural nonparametric cointegrating regression. Econometrica 77 (6), 1901–1948.

Wang, Q., Phillips, P., 2011. Asymptotic theory for zero energy functionals with nonparametric regression applications. Econometric Theory 27 (2), 235.

Xia, Y., 1998. Bias-corrected confidence bands in nonparametric regression. Journal of the Royal Statistical Society, Series B 60, 797–811.

Xia, Y., Tong, H., 2011. Feature matching in time series modeling. Statistical science 26 (1), 21–46.

Xia, Y., Tong, H., Li, W., 1999. On extended partially linear single-index models. Biometrika 86 (4), 831–842.

Xia, Y., Tong, H., Li, W., Zhu, L., 2002. An adaptive estimation of dimension reduction space. Journal of the Royal Statistical Society: Series B (Statistical Methodology) 64 (3), 363–410.

Yoshihara, K.-I., 1976. Limiting behavior of U-statistics for stationary, absolutely regular processes. Probability Theory and Related Fields 35 (3), 237–252.