Periodic Modules, Perverse Equivalences and Blocks of Symmetric Groups

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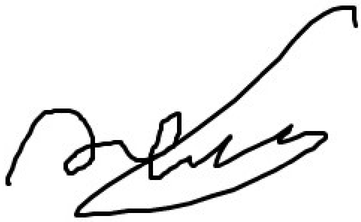
Declaration

I, Alfred George Dabson, confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the thesis.



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Abstract

This thesis focuses on perverse autoequivalences of finite-dimensional symmetric algebras A over an algebraically closed field k. These are autoequivalences, introduced by Chuang and Rouquier, of the bounded derived category $D^b(A)$ of A-modules filtered by shifted Morita equivalences. We pay special attention to what we call *two-step* perverse autoequivalences, for which the filtration is of length two.

In particular, we demonstrate that two-step perverse autoequivalences of a certain form give rise to distinguished modules in the endomorphism algebra E of some projective A-module exhibiting a property we term *strong periodicity*. Such modules are periodic, and the periodicity arises from an extension of the E-E-bimodule E by itself.

We then prove a converse: given strongly periodic E-modules, we construct endofunctors of $D^b(A)$, and prove that these endofunctors are two-step perverse autoequivalences. This is closely related to work of Grant on perverse autoequivalences arising from periodic endomorphism algebras, and our result encompasses his. Building on Grant's work, we show that these autoequivalences coincide with iterated combinatorial tilts, as defined by Rickard and Okuyama.

Finally, we survey some applications of our result to blocks of symmetric groups of weight two. This recovers equivalences of Craven and Rouquier, arising from the geometry of the underlying groups, while our methods are based only on the algebras themselves.

Contents

Acknowledgements

First and foremost, I would like to thank my supervisor, Professor Joseph Chuang. The completion of this thesis is a testament to his diligence, enthusiasm and encyclopaediac knowledge of mathematics. He is always willing to follow a tempting mathematical lead and has helped me pushed those found in this thesis to their end.

I would also like to thank all the support and administrative staff at City, who made this possible. I am indebted to anyone else with whom I have discussed mathematics over the last four years, including (but not limited to) Karin Erdmann, Jeremy Rickard, Zain Kapadia and Kamilla Rekvényi. I am additionally thankful for my comrades in E202 and latterly CG41, who made the lack of sunlight bearable.

To my parents, without whom I could never have even considered postgraduate research, and to Kit and to Dom, my eternal brothers in arms. Lastly, but not leastly, my thanks go to Laura, for whom I am always grateful, and who I imagine is very pleased this is finally all over.

Introduction

Representation theory was born out of the need to better understand abstract finite groups. Frobenius developed in the 1890s a theory of group representations and group characters, by which one can represent elements of a group by matrices. One can therefore tackle problems in group theory, a remarkably protean subject, with the methods of linear algebra, which is much better behaved. Frobenius and Burnside demonstrated the efficacy of this new approach by proving a number of results, most famously Burnside's $p^a q^b$ theorem.

In the early 20th century, Frobenius and his student Schur realised that the representation theory of finite groups had the potential to be a fascinating and exciting subject in its own right. In particular, Frobenius classified the irreducible representations of the symmetric group $\mathfrak{S}_n$ over the complex numbers, utilising combinatorial constructions of Young. Schur extended this work in his doctoral thesis to determine the polynomial representations of the general linear group $\mathrm{GL}_n(\mathbb{C})$. Many fundamental results in representation theory, including Frobenius reciprocity and Schur's lemma, were formulated by these two mathematicians at this time.

Later, Noether developed the theory of modules over rings and algebras. Combined with work of Artin, this forms the bedrock of modern abstract algebra. Noether's theory encompasses Frobenius's, as the study of the representations of a finite group G over $\mathbb{C}$ is equivalent to the study of modules over the group algebra $\mathbb{C}G$. Translated into this language, Maschke's Theorem informs us that $\mathbb{C}G$ is a semisimple algebra, and the Artin-Wedderburn Theorem informs us that any semisimple algebra, and hence $\mathbb{C}G$, is a direct product of some number of matrix algebras. In the language of block theory, the decomposition of $\mathbb{C}G$ into indecomposable $\mathbb{C}$-algebras is

$$\mathbb{C}G \cong M_{n_1}(\mathbb{C}) \times \ldots \times M_{n_r}(\mathbb{C})$$

for some integers $n_i \geq 1$, where $M_{n_i}(\mathbb{C})$ is the algebra of $n_i \times n_i$ matrices over $\mathbb{C}$.

A natural extension of Frobenius's representation theory of finite groups is to replace the complex numbers $\mathbb{C}$ by an arbitrary field k. Dickson had proposed doing exactly this around the time of Frobenius's first endeavours, coining the term *modular* representation theory. However, one immediately faces difficulty moving to a field of positive characteristic p dividing the order of the group G: the algebra kG is no longer semisimple, so the structure given by Artin-Wedderburn is lost. That is, in the language of block theory,

$$kG \cong B_1 \times \ldots \times B_r$$

as a direct product of indecomposable k-algebras, where not all of the B_i are matrix algebras. It was not until the 1930s and Brauer that modular representation theory would take off in earnest.

Arguably Brauer's most significant contributions are his Three Main Theorems, relating the blocks of a finite group G with the blocks of its p-local subgroups: normalizers and centralizers of the p-subgroups of G. Most striking is the apparent relationship between a block B_i of kG and algebras built from its *defect groups*, a conjugacy class of p-subgroups of G, measuring how far B_i is from a matrix algebra. Given a p-subgroup D of G, Brauer determined a one-to-one correspondence between the set of blocks of kG with D as a defect group and the set of blocks of $kN_G(D)$ with D as a defect group. The block C_i of $kN_G(D)$ corresponding to B_i is the *Brauer correspondent* of B_i.

This led inevitably to a trove of local-global counting conjectures, notably Brauer's Height Zero Conjecture, the Alperin-McKay Conjecture, and Alperin's Weight Conjecture, based on highly remarkable numerical equalities on either side of this correspondence. The proof of many of these conjectures remains a significant outstanding problem in the representation theory of finite groups[1]. A very simple form of Alperin's Weight Conjecture is the following.

Conjecture (Alperin). *If B is a block of kG with abelian defect group D and C is the Brauer correspondent block of $kN_G(D)$, then B and C have an equal number of simple modules up to isomorphism.*

A statement such as this is highly suggestive of a profound structural relationship between the two algebras, rather than a serendipitous equality of sizes of sets. What, then, is this relationship?

[1]The author notes, with an unearned sense of self-satisfaction, that he was present at the presentation of the proof of Brauer's Height Zero Conjecture by Gunter Malle at Oberwolfach in 2023, based on joint work with Navarro, Schaeffer-Fry and Tiep.

In 1958, Kiiti Morita determined criteria for two algebras A and B to have additively equivalent module categories A-mod and B-mod. This relationship is known as *Morita equivalence*. Two Morita equivalent algebras have essentially the same representation theory; it is thus an important problem to determine the Morita equivalence classes of blocks of group algebras, and of finite-dimensional algebras more generally. It is tempting to hope that Alperin's Conjecture is a result of a Morita equivalence between the algebras B and C. However, a famous example of Rickard for the principal block B of the alternating group $\mathfrak{A}_5$ in characteristic $p = 2$ demonstrated that this is too strong a requirement. While this block B and its Brauer correspondent C are not Morita equivalent, they are in this case *derived equivalent*.

In the 1950s, Grothendieck revolutionised the study of algebra with his introduction of the abelian category. This axiomatized homological algebra and unified aspects of algebraic topology, algebraic geometry and representation theory. The module category A-mod of a finite-dimensional algebra A is an abelian category, and therefore an appropriate setting in which to perform homological algebra. Later, Grothendieck and his student Verdier defined the derived category $D(\mathcal{A})$ of an abelian category $\mathcal{A}$. Two abelian categories $\mathcal{A}$ and $\mathcal{A}'$ are *derived equivalent* if their derived categories $D(\mathcal{A})$ and $D(\mathcal{A}')$ are equivalent, with this being an equivalence of *triangulated categories*. For two finite-dimensional algebras A and A', a derived equivalence $D(A\text{-mod}) \simeq D(A'\text{-mod})$ is a weaker notion of equivalence than a Morita equivalence, however many interesting properties of the algebras are preserved. Strikingly, derived equivalent algebras A and A' must have the same number of simple modules up to isomorphism. This observation, together with the example of Rickard, results in the following famous conjecture of Broué.

Conjecture (Broué). *If B is a block of kG with abelian defect group D and C is the Brauer correspondent block of $kN_G(D)$, then B and C have equivalent derived categories of modules.*

In the interceding years, the study of derived equivalence of blocks of finite groups and, more generally, finite-dimensional algebras has exploded. Rickard formulated a complete working Morita theory for derived categories, based partly on the concurrently emerging tilting theory of Brenner, Butler, Happel, Ringel and others. Subsequent developments include proving Broué's conjecture for a large number of blocks, as well as characterising derived equivalence for various interesting families of algebras, for example Brauer tree algebras by Rickard, and Brauer graph algebras by Antipov, Opper and Zvonareva. In the finite group setting, possibly the most exciting advancement came with Chuang and Rouquier's completion of the proof of Broué's Conjecture for blocks of the symmetric groups. In this, they identified a particularly combinatorially friendly subclass of derived equivalences, intimately related to constructions in sheaf cohomology, which they termed *perverse equivalences*.

Perverse equivalences are derived equivalences filtered by shifted Morita equivalences. They induce a one-to-one map between isomorphism classes of simple modules, so in this sense are more appealing than an arbitrary derived equivalence. They occur naturally in the representation theory of finite groups of Lie type, and have wider significance in areas such as tilting theory and cluster theory. It is an important problem to determine from where these equivalences arise. Grant has shown the following.

Theorem (Grant). *Let k be an algebraically closed field, A a finite-dimensional symmetric k-algebra, P a finitely generated projective A-module, and $E = \mathrm{End}_A(P)^{\mathrm{op}}$. If E is periodic as a k-algebra of period n, then there is a autoequivalence*

$$\Psi_P : D^b(A) \xrightarrow{\sim} D^b(A)$$

such that $\Psi_P(P) \cong P[n]$ and $\Psi_P(S) \cong S$ for every simple A-module S with $\mathrm{Hom}_A(P, S) = 0$. Moreover, the equivalence Ψ_P is perverse.

This opens up a link between the existence of perverse autoequivalences of an algebra A and the periodicity of its idempotent algebras. This thesis investigates this link further.

In Chapter 1, we describe the relevant background for our results. We give a basic overview of the representation theory of finite-dimensional algebras over an algebraically closed field, particularly symmetric algebras, and the related tools of homological algebra, with a focus on triangulated categories and derived categories. We then recount Chuang and Rouquier's definition of perverse equivalences, presenting some useful formulations and key results. We devote special attention to what we call *two-step* perverse equivalences of width n, for which the filtration has length two and the shift is by n, and *self-perverse* equivalences, autoequivalences in which the filtration matches on either side.

The following chapter, Chapter 2, is the mathematical core of this thesis. Firstly, in §2.1, we recall Grant's result in more detail, and see certain applications related to geometry. In §2.2, we demonstrate the relevance of twisted periodic modules. We have our first key result, Theorem 2.2.2.

Theorem. *With A as in Grant's Theorem, if $\Phi : D^b(A) \xrightarrow{\sim} D^b(A)$ is a two-step self-perverse equivalence of width n, then there are finitely generated projective A-modules P and Q such that, with $E = \mathrm{End}_A(P)^{\mathrm{op}}$, the E-module $M = \mathrm{Hom}_A(P, Q)$ is σ-periodic of period n, for σ an automorphism of E.*

This leads into an attempt to prove a converse, encompassing Grant's earlier result. Naive efforts to do so are discounted in §2.2.3.

We approach this problem in a novel way in §2.3. We introduce a notion of *strong periodicity* of a module, by which a σ-periodic E-module M, for σ an automorphism of E, is strongly σ-periodic if the periodicity of M arises from an element $\alpha \in \mathrm{Ext}^n_{E \otimes_k E^{\mathrm{op}}}(E, {}_\sigma E)$. We then strengthen the previous theorem into the following, Theorem 2.3.4.

Theorem. *With A as above, if $\Phi : D^b(A) \xrightarrow{\sim} D^b(A)$ is a two-step self-perverse equivalence of width n such that there is a natural transformation $\mathrm{Id}_{D^b(A)} \to \Phi$ satisfying a restriction condition, then there are projective A-modules P and Q such that, with $E = \mathrm{End}_A(P)^{\mathrm{op}}$, the E-module $M = \mathrm{Hom}_A(P, Q)$ is strongly σ-periodic and the E^{op}-module $M^\vee$ is strongly σ^{-1}-periodic, both of period n, relative to some $\alpha \in \mathrm{Ext}^n_{E \otimes_k E^{\mathrm{op}}}(E, {}_\sigma E)$.*

This point of view allows us to prove a converse in §2.3.3. Given a basic algebra A and projective A-modules P and Q such that $A \cong P \oplus Q$ as A-modules, set $E = \mathrm{End}_A(P)^{\mathrm{op}}$ and $M = \mathrm{Hom}_A(P, Q)$. Suppose that M is a strongly periodic E-module of period n and $M^\vee$ is a strongly periodic E^{op}-module of period n, with the periodicity arising from the same $\alpha \in \mathrm{Ext}^n_{E \otimes_k E^{\mathrm{op}}}(E, {}_\sigma E)$. Following Grant's method, we construct an endofunctor

$$\Phi_P : D^b(A) \longrightarrow D^b(A)$$

from α, called the *generalised periodic twist* at P. We have the following, Theorem 2.3.8.

Theorem. *The generalised periodic twist Φ_P is a two-step self-perverse equivalence of width n.*

These two results constitute the main part of the thesis. In §2.3.4, we show that, similarly to Grant's construction, our equivalence Φ_P can be realised as the inverse of an iterated combinatorial tilt, as defined by Rickard and Okuyama, producing a cycle of length n of perverse equivalences between different algebras. One can follow this cycle from any starting point to produce a two-step self-perverse equivalence of width n, corresponding to our construction.

Finally, in Chapter 3, we look at applications of our result to blocks of the symmetric groups. We first recall their representation theory and describe important definitions and results of Scopes, Chuang and Rouquier, and others. We then focus on the subclass of blocks of weight two in characteristic $p = 3$. These are in some sense the smallest symmetric group blocks of wild representation type, and therefore constitute an interesting and informative source of examples. In §3.3, we survey some applications of our main results to these blocks, most pertinently in §3.3.3 to a surprising example occurring in the principal block of $\mathfrak{S}_8$. Primarily, these examples show the intractability of by-hand applications of our result, and suggest a potentially favourable change of perspective for future work.

Chapter 1

Background

1.1 Basic Representation Theory

We collect the necessary representation theoretic tools for this thesis. Definitions and results are taken from [Alp93] or [Lin18], unless otherwise stated. A detailed primer on the tensor product can be found in [Lin18, §A.1].

Throughout, k is an algebraically closed field, and a k-algebra A is finite-dimensional, associative, and has an identity element 1_A. Unless otherwise stated, we will take an A-module to mean a left A-module. All A-modules are assumed to be finitely generated. Since A is finite-dimensional, an A-module M is finitely generated if and only if it is finite-dimensional. Given another k-algebra B, we will always assume that k acts centrally on A-B-bimodules. For an A-module M, we denote the n-fold direct sum of copies of M by $M^{\oplus n}$. We denote by A-mod the k-linear category of all finitely generated left A-modules.

1.1.1 Algebras and modules

Let A be a finite-dimensional k-algebra. Then A is a left and a right A-module, with left and right action given by left and right multiplication in A, respectively. These two actions commute, so A is an A-A-bimodule. We call the A-module A the *regular* A-module. The zero A-module is the abelian group $\{0\}$, which is both a left and a right A-module, with multiplication $a \cdot 0 = 0$ and $0 \cdot a = 0$ respectively.

The *opposite algebra* A^{op} of A is the k-module A, but with algebra multiplication reversed: $a \times b = ba$. A right A-module is a left A^{op}-module, and vice versa. In this way, we identify the k-linear category mod-A of all finitely generated right A-modules with A^{op}-mod.

Given two k-algebras A and B, under our assumption that k acts centrally on bimodules, an A-B-bimodule is a left $A \otimes_k B^{\mathrm{op}}$-module in the obvious way. We may identify the k-linear category A-mod-B of all A-B-bimodules with $A \otimes_k B^{\mathrm{op}}$-mod.

An A-module M is *simple* if it is non-zero and has no proper non-zero submodules. An A-module M is *semisimple* if it is a direct sum of simple modules. An A-module M is *indecomposable* if it cannot be written as a direct sum of two non-zero submodules. For a finite-dimensional k-algebra A, there are only finitely many simple A-modules up to isomorphism, and all are finitely generated. A central problem of representation theory is to identify the isomorphism classes of simple A-modules for a given A.

Given an A-module M and a submodule $N \subseteq M$, the usual abelian group quotient M/N is another A-module. A *composition series* of M is a chain of submodules

$$\{0\} = M_0 \subset M_1 \subset \ldots \subset M_r = M$$

such that each quotient M_{i+1}/M_i is a simple A-module. An A-module M has *finite length* if there exists a finite composition series of M. Since A is finite-dimensional, this happens if and only if M is finitely generated. The Jordan-Hölder Theorem holds for finite length modules: if the A-module M has a finite composition series, then the length of this composition series is uniquely determined, and the simple quotients M_{i+1}/M_i are defined up to isomorphism. The unique length of a compositions series is called the *composition length* of M and the simple quotients M_{i+1}/M_i are called the *composition factors* of M. Given A-modules M and S such that S is simple, we will denote by $[M : S]$ the multiplicity of S as a composition factor of M.

Given an A-module M, we denote by M-add the full additive subcategory of A-mod, whose objects are finite direct sums of direct summands of M. One can extend this definition to $\mathcal{X}$-add for $\mathcal{X}$ a collection of A-modules, in the obvious way.

Given an A-module M, we denote by M^* the *k-linear dual* of M,

$$M^* = \mathrm{Hom}_k(M, k),$$

and by $M^\vee$ the *A-linear dual* of M,

$$M^\vee = \mathrm{Hom}_A(M, A).$$

If M is a left A-module, then M^* and $M^\vee$ are right A-modules, and vice versa. The right A-module action of M^* is given by

$$\theta \cdot a \mapsto \big(x \mapsto \theta(ax)\big)$$

for $\theta : M \to k$, $a \in A$ and $x \in M$, and the right A-module action of $M^\vee$ by

$$\theta \cdot a \mapsto \big(x \mapsto \theta(x)a\big)$$

for $\theta : M \to A$, $a \in A$ and $x \in M$.

For example, A^*, the k-linear dual of the regular A-A-bimodule A, is an A-A-bimodule, with left action inherited from the right action of A on A, and right action inherited from the left action of A on A.

The k-module $\mathrm{Hom}_A(M, M) = \mathrm{End}_A(M)$ is a k-algebra, with algebra multiplication given by composition:

$$\psi_1 \cdot \psi_2 = \psi_1 \circ \psi_2.$$

We have an isomorphism of k-algebras

$$A \cong \mathrm{End}_A(A)^{\mathrm{op}},$$

given by

$$a \mapsto (x \mapsto xa)$$

for all $a, x \in A$.

While there are only finitely many simple A-modules up to isomorphism for a finite-dimensional k-algebra A, the same is not true of indecomposable A-modules in general. This determines the *representation type* of A. We have the following definitions.

- The k-algebra A is of *finite representation type* if and only if there are finitely many isomorphism classes of finitely generated indecomposable A-modules.
- The k-algebra A is of *tame representation type* if and only if all but finitely many isomorphism classes of finitely generated indecomposable A-modules fall into a finite number of one-parameter families.
- If neither of the above hold, then the k-algebra A is of *wild representation type*.

For more details on the definition of tame and wild representation type, see e.g. [Erd90]. Strictly, this is not the standard definition of wild representation type, but Drozd's Trichotomy Theorem [Dro80] informs us that a k-algebra A falls into exactly one of these three classes[1], so this definition suffices.

[1]Drozd's Trichotomy Theorem requires that the field k be algebraically closed.

1.1.2 Semisimple algebras

Let A be a finite-dimensional k-algebra. The *radical* of A, $\operatorname{rad}(A)$, is the two-sided ideal of A defined as the intersection of all maximal ideals of A.

The k-algebra A is a *semisimple algebra* if $\operatorname{rad}(A) = 0$. If A is a semisimple k-algebra, then A is a semisimple A-module. The classical Artin-Wedderburn Theorem tells us that A is semisimple if and only if A is a direct product of a finite number of matrix algebras over k. For any k-algebra A, the radical of A is the smallest (two-sided) ideal $\mathcal{I}$ of A such that $A/\mathcal{I}$ is a semisimple algebra.

For the sake of completeness, we note that a k-algebra A is a *simple algebra* if and only if A is isomorphic to a matrix algebra over k. Thus, the Artin-Wedderburn Theorem determines the analogous relationship between semisimple and simple algebras, as between semisimple and simple modules.

If $A/\operatorname{rad}(A) \cong k$, then A is a *local* algebra. This property characterises endomorphism rings of indecomposable modules: an A-module M is indecomposable if and only if $\operatorname{End}_A(M)$ is a local algebra.

Let M be an A-module. The *radical* of M, $\operatorname{rad}(M)$, is the submodule $\operatorname{rad}(A)M$ of M. Then $\operatorname{rad}(M)$ is the smallest submodule N of M such that the quotient M/N is a semisimple A-module. We call the quotient $M/\operatorname{rad}(M)$ the *head* of M. We define inductively the *radical series* of M by $\operatorname{rad}^{i+1}(M) = \operatorname{rad}(\operatorname{rad}^i(M))$ for $i \geq 0$, with the convention $\operatorname{rad}^0(M) = M$. The *radical length* of M is the smallest ℓ for which $\operatorname{rad}^{\ell-1}(M) \neq 0$ but $\operatorname{rad}^{\ell}(M) = 0$. If M is a semisimple module, then $\operatorname{rad}(M) = 0$, so M has radical length $\ell = 1$. The *ith radical layer* of M is the semisimple module

$$\operatorname{rad}^{i-1}(M)/\operatorname{rad}^i(M)$$

for $i \geq 1$. If every radical layer of M is simple, then the radical series of M is a composition series of M, in fact the unique composition series of M, and we say M is *uniserial*.

The *socle* of a finite length A-module M, $\operatorname{soc}(M)$, is the largest semisimple submodule of M. We define inductively the *socle series* of M as follows. The module $\operatorname{soc}^{i+1}(M)$ for $i \geq 1$ is the submodule of M containing $\operatorname{soc}^i(M)$ such that $\operatorname{soc}^{i+1}(M)/\operatorname{soc}^i(M)$ is the socle of the A-module $M/\operatorname{soc}^i(M)$. The semisimple module

$$\operatorname{soc}^i(M)/\operatorname{soc}^{i-1}(M)$$

is the *ith socle layer* of M, with the convention that $\operatorname{soc}^0(M) = 0$. The smallest ℓ such that $\operatorname{soc}^{\ell}(M) = M$ is the *socle length* of M. Every layer in the socle series of M is a semisimple A-module. If M is a semisimple module, then $\mathrm{soc}(M) = M$.

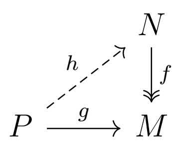
Given a finitely generated A-module M, the radical length and socle length of M are equal, and we call this common integer $\ell \geq 0$ the *Loewy length* of M. We also call the radical series of M the *Loewy* series of M. We note that M has Loewy length $\ell = 0$ if and only if M is the zero module.

We call an A-module M of Loewy length ℓ *stable* if for every i, we have $\mathrm{soc}^i(M) = \mathrm{rad}^{\ell-i}(M)$. In short, M is stable if the radical and socle series of M coincide.

1.1.3 Projective and injective modules

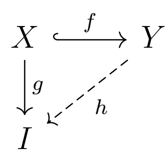
Let A be a finite-dimensional k-algebra.

An A-module P is *projective* if for every surjective A-module homomorphism $f : N \to M$ and every A-module homomorphism $g : P \to M$, there is an A-module homomorphism $h : P \to N$ such that $g = f \circ h$:



A finitely generated A-module M is *free* if $M \cong A^{\oplus n}$ for some $n \geq 1$. A finitely generated A-module P is projective if and only if it is a direct summand of a finitely generated free A-module.

The dual construction to projective modules is injective modules. An A-module I is *injective* if for every injective A-module homomorphism $f : X \to Y$ and every A-module homomorphism $g : X \to I$, there is an A-module homomorphism $h : Y \to I$ such that $g = h \circ f$.



A finitely generated A-module is injective if and only if it is a direct summand of a finitely generated *cofree* module[2].

[2] By a cofree module we mean a direct summand of copies of (the left A-module) A^*. We note that this dual notion to a free module exists only because A is a finite-dimensional algebra over a field k; in general, we do not cannot make such a nice dual definition.

Let P be a projective A-module. Then the head $P/\operatorname{rad}(P)$ is a semisimple A-module. Any projective A-module P is a direct sum of projective indecomposable A-modules; the projective module P has simple head exactly when P is indecomposable. Furthermore, the simple module in the head of a projective indecomposable A-module is unique up to isomorphism: if Q is another projective indecomposable A-module such that $Q/\operatorname{rad}(Q) \cong P/\operatorname{rad}(P)$, then $P \cong Q$. This gives a natural bijection between the (finite, since A is finite-dimensional) set of isomorphism classes of simple A-modules and the set of isomorphism classes of projective indecomposable A-modules. Suppose these sets are $\{S_1, \dots, S_n\}$ and $\{P_1, \dots, P_n\}$, such that $P_i/\operatorname{rad}(P_i) \cong S_i$. Then we have an isomorphism of A-modules

$$A \cong P_1^{\oplus d_1} \oplus \dots \oplus P_n^{\oplus d_n},$$

where $d_1 = \dim_k(S_i)$. If $d_i = 1$ for all i, then we say A is a *basic* algebra.

There is a full additive subcategory A-proj of A-mod, whose objects are the projective A-modules. Naturally, the subcategories A-proj and A-add coincide, where A-add is the full additive subcategory of A-mod generated by the regular A-module A, as in §1.1.1.

For $1 \leq i, j \leq n$, set $c_{ij} = [P_i : S_j]$, the multiplicity of S_j as a composition factor of P_i. The integers c_{ij} are the *Cartan numbers* of A, and the matrix C_A with (i, j) entry c_{ij} is the *Cartan matrix* of A. We have that $c_{ij} = \dim_k \operatorname{Hom}_A(P_j, P_i)$ for every pair (i, j).

Let M be an A-module. A *projective cover* of M is a projective A-module P, together with a surjective A-module homomorphism $p : P \to M$, such that $P/\operatorname{rad}(P) \cong M/\operatorname{rad}(M)$. Then (see [Alp93, Lemma 20.2]) the pair (P, p) is unique up to isomorphism, in that if (Q, q) is any other such pair, then there is an isomorphism $\varphi : Q \cong P$ such that $q = p \circ \varphi$. We may thus talk of *the* projective cover of M, meaning a choice of a projective A-module $P(M)$ and a surjective A-module homomorphism $\pi_M : P(M) \to M$ satisfying the above. Every A-module has a projective cover[3]. If S is a simple A-module, then the projective cover $\pi_S : P(S) \to S$ is such that $P(S)$ is the unique projective indecomposable A-module with $P(S)/\operatorname{rad}(P(S)) \cong S$.

Given an A-module M, we define the *Heller translate* $\Omega_A(M)$ of M to be the kernel of a projective cover π_M of M:

$$0 \longrightarrow \Omega_A(M) \hookrightarrow P \overset{\pi_M}{\twoheadrightarrow} M \longrightarrow 0.$$

[3]This is not true for a general ring, but is true for a finite-dimensional algebra over an algebraically closed field.

We will spend significantly more time with the Heller translate in §1.2.10.

Dually, an *injective hull* of M is an injective A-module I, together with an injective A-module homomorphism $\iota_M : M \to I$, such that $\operatorname{soc}(I) \cong \operatorname{soc}(M)$. As above, an injective hull of M is unique up to isomorphism. We may therefore speak of *the* injective hull of M, which we denote $I(M)$, with the injective homomorphism $\iota_M : M \to I(M)$. Every A-module has an injective hull[4].

Let S be a simple A-module and $\iota_S : S \to I(S)$ an injective hull of S. Then $I(S)$ is an indecomposable A-module such that $\operatorname{soc}(I(S)) \cong S$. Moreover, the A-module $I(S)$ is the unique indecomposable injective A-module with simple socle S. We thus have a complete set of indecomposable injective A-modules up to isomorphism, $\{I_1, \ldots, I_n\}$, in bijective correspondence with the set of simple A-modules up to isomorphism, labelled so that $\operatorname{soc}(I_i) \cong S_i$.

Given an A-module M, let $\iota_M : M \to I$ be an injective hull of M. We define $\Omega_A^{-1}(M)$ to be the cokernel of ι_M:

$$0 \longrightarrow M \stackrel{\iota_M}{\hookrightarrow} I \twoheadrightarrow \Omega_A^{-1}M \longrightarrow 0.$$

As the notation might indicate, we call $\Omega_A^{-1}(M)$ the *inverse Heller translate* of M.

Note that, as it is a free module, the regular A-module A is a projective A-module. If the regular A-module A is also an injective A-module, then the k-algebra A is *self-injective*. If A is self-injective, then projective A-modules are injective modules, and vice versa[5]. The following characterisation of self-injective algebras is due to Nakayama [Nak41].

Theorem 1.1.1. *Let A be a finite-dimensional k-algebra. Let $\{S_1, \ldots, S_n\}$ be a set of isomorphism classes of simple A-modules, and $\{P_1, \ldots, P_n\}$ the corresponding set of projective indecomposable A-modules. Then A is self-injective if and only if there is a permutation ν_A of the set $\{1, \ldots, n\}$ such that*

$$P_i / \operatorname{rad}(P_i) \cong S_i \cong \operatorname{soc}(P_{\nu_A(i)}).$$

The permutation ν_A in Theorem 1.1.1 is called the *Nakayama permutation* of A.

[4] This *is* true for a general ring.

[5] That is, the full additive subcategories A-proj and A-inj, the category of all injective A-modules, coincide.

As a final comment in this subsection, when A is self-injective and M is any A-module with no non-zero projective summands, we have

$$\Omega_A^{-1}(\Omega_A(M)) \cong M \cong \Omega_A(\Omega_A^{-1}(M)).$$

1.1.4 Functors between module categories

Let A and B be finite-dimensional k-algebras. Consider the abelian categories A-mod and B-mod. Let M be an A-B-bimodule. Then M gives rise to a succession of functors:

- $\mathrm{Hom}_A(M, -) : A\text{-mod} \to B\text{-mod}$ is a covariant left exact functor;
- $\mathrm{Hom}_{B^{\mathrm{op}}}(M, -) : B^{\mathrm{op}}\text{-mod} \to A^{\mathrm{op}}\text{-mod}$ is a covariant left exact functor;
- $\mathrm{Hom}_A(-, M) : A\text{-mod} \to B\text{-mod}$ is a contravariant left exact functor;
- $\mathrm{Hom}_{B^{\mathrm{op}}}(-, M) : B^{\mathrm{op}}\text{-mod} \to A^{\mathrm{op}}\text{-mod}$ is a contravariant left exact functor;
- $M \otimes_B - : B\text{-mod} \to A\text{-mod}$ is a covariant right exact functor;
- $- \otimes_A M : A^{\mathrm{op}}\text{-mod} \to B^{\mathrm{op}}\text{-mod}$ is a covariant right exact functor.

The functor $\mathrm{Hom}_A(M, -)$ is exact if and only if M is a projective left A-module (see §1.1.3), and the functor $\mathrm{Hom}_A(-, M)$ is exact if and only if M is an injective left A-module. Similarly, the functor $\mathrm{Hom}_{B^{\mathrm{op}}}(M, -)$ is exact if and only if M is a projective right B-module, and the functor $\mathrm{Hom}_{B^{\mathrm{op}}}(-, M)$ is exact if and only if M is an injective right B-module.

The functor $- \otimes_A M$ (resp. $M \otimes_B -$) is exact if and only if M is a *flat* left A-module (resp. right B-module). An A-module is flat if and only if it is projective[6].

If M is projective as a left A-module, we have an isomorphism of functors

$$\mathrm{Hom}_A(M, -) \cong M^\vee \otimes_A -.$$

If M is projective as a right B-module, we have an isomorphism of functors

$$\mathrm{Hom}_{B^{\mathrm{op}}}(M, -) \cong - \otimes_B M^\vee.$$

An exceedingly useful way to relate these functors is tensor-Hom adjunction. The functors

$$M \otimes_B - \dashv \mathrm{Hom}_A(M, -)$$

[6]This is not true in general, but is true for finitely generated modules over Noetherian rings. Note that a finite-dimensional k-algebra A is always Noetherian.

are left-right adjoint, as are the functors

$$- \otimes_A M \dashv \mathrm{Hom}_{B^{\mathrm{op}}}(M, -).$$

In particular, if A, B, C, D and E are k-algebras, M is an A-B-bimodule, N a B-C-bimodule, U a D-C-bimodule, and V an A-E-bimodule, then we have an isomorphism

$$\mathrm{Hom}_{C^{\mathrm{op}}}(M \otimes_B N, U) \cong \mathrm{Hom}_{B^{\mathrm{op}}}(M, \mathrm{Hom}_{C^{\mathrm{op}}}(N, U))$$

of A-D-bimodules, and an isomorphism

$$\mathrm{Hom}_A(M \otimes_B N, V) \cong \mathrm{Hom}_B(N, \mathrm{Hom}_A(M, V))$$

of C-E-bimodules.

The right derived functors of the left exact Hom functors are the Ext^i functors, admitting the following description, due to Yoneda. Given two A-modules M and N, $\mathrm{Ext}^i_A(M, N)$ is the set of *Yoneda i-extensions* of M by N, up to chain map equivalence. That is, the set of short exact sequences of A-modules

$$0 \longrightarrow N \longrightarrow X_{i-1} \longrightarrow \ldots \longrightarrow X_0 \longrightarrow M \longrightarrow 0,$$

up to the symmetric closure of the relation $\sim$, where for two i-extensions ξ and ξ', $\xi \sim \xi'$ if there is a chain map

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & N & \longrightarrow & X_{i-1} & \longrightarrow & \ldots & \longrightarrow & X_0 & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & N & \longrightarrow & X'_{i-1} & \longrightarrow & \ldots & \longrightarrow & X'_0 & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

which is the identity on M and N. Then $\mathrm{Ext}^i_A(M, N)$ is a group, under an operation known as the Baer sum (see e.g. [Wei95, Definition 3.4.4]), and forms a vector space over k. We have

$$\mathrm{Ext}^0_A(M, N) = \mathrm{Hom}_A(M, N),$$

and $\mathrm{Ext}^1_A(M, N)$ is the group of (homological) *extensions* of M by N.

There is a handy way to classify the extensions of a simple module by another simple module. Given two simple A-module S and T, let P_S and P_T be the indecomposable projective A-modules corresponding to S and T. Then

$$\mathrm{Ext}^1_A(S, T) \cong \mathrm{Hom}_A(P_T, \mathrm{rad}(P_S)/\mathrm{rad}^2(P_S)).$$

Thus, the dimension of this extension group is equal to the number of times T appears in the second radical layer of P_S.

When M is an A-A-bimodule, the Ext^i groups define the *Hochschild cohomology* of A with coefficients in M,

$$HH^*(A; M) = \mathrm{Ext}^*_{A\otimes A^{\mathrm{op}}}(A, M).$$

When we come to define the derived category in §1.2.7, we will have the following realisation of the Ext^i groups:

$$\mathrm{Ext}^i_A(M, N) \cong \mathrm{Hom}_{D^b(A)}(M, N[i]).$$

1.1.5 Morita equivalence

Let A and B be k-algebras. We say that A and B are *Morita equivalent* if there is an equivalence of k-linear categories

$$F : A\text{-mod} \xrightarrow{\sim} B\text{-mod}\,.$$

The functor F is called a *Morita equivalence*. It is clear that isomorphic algebras are Morita equivalent. The converse is not true in general.

Morita equivalence preserves a number of properties of an algebra. Let

$$F : A\text{-mod} \xrightarrow{\sim} B\text{-mod}$$

be a Morita equivalence. Then, for example, A is semisimple if and only if B is semisimple, and A is self-injective if and only if B is self-injective. Properties of modules are also preserved. An A-module M is simple if and only if the B-module $F(M)$ is simple, and M is projective (resp. injective) if and only if $F(M)$ is projective (resp. injective). In particular, Morita equivalent algebras have an equal number of simple modules, and of projective indecomposable modules, up to isomorphism. This list of preserved properties is far from exhaustive.

We can see the preservation of simple modules at the level of the Grothendieck groups of A-mod and B-mod. Let $\mathcal{A}$ be an abelian category. The Grothendieck group $K_0(\mathcal{A})$ of $\mathcal{A}$ is the free abelian group $K_0(\mathcal{A})$ on the symbols $[X]$ for objects X of $\mathcal{A}$, modulo the relation

$$[Y] = [X] + [Z]$$

for every short exact sequence

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

in $\mathcal{A}$.

For the abelian category A-mod, $K_0(A) = K_0(A\text{-mod})$ is the free abelian group on the symbols $[S]$, where S runs through a set of isomorphism classes of simple A-modules. Then a Morita equivalence

$$F : A\text{-mod} \xrightarrow{\sim} B\text{-mod}$$

induces an isomorphism of abelian groups

$$[F] : K_0(A) \xrightarrow{\sim} K_0(B),$$

that matches the classes of simple A-modules and simple B-modules[7].

A remarkable result, due to Morita, characterises Morita equivalence entirely. A *projective generator* or *progenerator* of an algebra A is a projective A-module V such that the regular A-module A is a summand of $V^{\oplus m}$ for some $m > 0$.

Theorem 1.1.2. *The following are equivalent.*

(i) The algebras A and B are Morita equivalent.

(ii) There exists a progenerator V of A such that $B \cong \operatorname{End}_A(V)^{\mathrm{op}}$.

(iii) There is an A-B-bimodule M and a B-A-bimodule N such that:

- *(a) M and N are projective as left and as right modules;*
- *(b) $M \otimes_B N \cong A$ as A-A-bimodules;*
- *(c) $N \otimes_A M \cong B$ as B-B-bimodules.*

This result is foundational in the representation theory of finite-dimensional algebras. We shall see thematic echoes of it in Rickard's Morita theory for derived categories, presented here as Theorem 1.2.17.

1.1.6 Blocks

Let A be a finite-dimensional k-algebra. The Krull-Schmidt theorem for algebras (see e.g. [Alp93, Theorem 13.1]) states that there is a well-defined collection of subsets A_i of A such that there is a direct product decomposition

$$A \cong A_1 \times A_2 \times \ldots \times A_r$$

[7] It is worth nothing that an arbitrary isomorphism $K_0(A) \xrightarrow{\sim} K_0(B)$ need not preserve classes of simple modules; this is a special property of a Morita equivalence.

of A into indecomposable subalgebras. In particular, this direct product decomposition is unique up to reordering. The k-algebras A_i are called the *blocks* of A. We have $A_i \cap A_j = \{0\}$ for $i \neq j$, so if $a_i \in A_i$ and $a_j \in A_j$, then $a_i a_j = 0$. When A is semisimple, every block of A is a matrix algebra[8]. When A is not semisimple, the blocks of A are not all matrix algebras, though some might be. Fix this decomposition for the remainder of this subsection.

The decomposition of A may be understood in terms of primitive decompositions of idempotents in A. An *idempotent* in A is an element[9] $0 \neq e \in A$ such that $e^2 = e$. Two idempotents $e, f \in A$ are *orthogonal* if $ef = 0 = fe$. An idempotent $e \in A$ is *primitive* if e cannot be written as the sum of two orthogonal idempotents. An idempotent $e \in A$ is *central* if $e \in Z(A)$. A *primitive central idempotent* of A is a central idempotent $e \in Z(A)$ primitive in $Z(A)$.

An algebra A is indecomposable if and only if the idempotent 1_A is a primitive central idempotent of A. The set of primitive central idempotents $\{e_1, \ldots, e_r\}$ corresponds to a *primitive decomposition* of the idempotent 1_A in A. In particular, the e_i are pairwise orthogonal, and $1_A = e_1 + \ldots + e_r$. Such a decomposition is unique in A. Further, if the k-algebra B is a direct factor of A, then there is some primitive central idempotent $b \in Z(A)$ such that $B = Ab$. The decomposition of A into blocks is given by

$$\begin{aligned} A &= A \cdot 1_A \\ &= A(e_1 + e_2 + \ldots + e_r) \\ &= Ae_1 \times Ae_2 \times \ldots \times Ae_r \\ &= A_1 \times A_2 \times \ldots \times A_r. \end{aligned}$$

For each i, the idempotent e_i is the identity element of $A_i = Ae_i$. We call the primitive central idempotents of A the *block idempotents*[10] of A. The uniqueness of the block decomposition of A up to reordering is a simple consequence of the uniqueness of the primitive decomposition of 1_A in A.

Let M be an A-module. If there is some bock idempotent e_i such that $e_i M = M$ and $e_j M = \{0\}$ for all $j \neq i$, then the module M *lies in the block* A_i. The set of modules lying in a block A_i is closed under taking submodules, quotients and direct sums. For an arbitrary A-module M, we

[8] This is the Artin-Wedderburn Theorem referenced in §1.1.2.

[9] Some authors allow $e = 0$, however this will complicate subsequent definitions.

[10] Many authors call the idempotents e_i the *blocks* of A, and the indecomposable direct factors $A_i = Ae_i$ the *block algebras* of A. Our choice of terminology is simply because we will be working more with the algebras directly.

have a decomposition

$$\begin{aligned} M &= 1_A M \\ &= (e_1 + \ldots + e_r)M \\ &= e_1 M \oplus \ldots \oplus e_r M \\ &= M_1 \oplus \ldots \oplus M_r, \end{aligned}$$

where the submodule $M_i = e_i M$ of M lies in the block A_i of A.

There is a Krull-Schmidt Theorem for modules. For an A-module M, there is decomposition of M into a direct sum of indecomposable submodules, unique up to isomorphism and reordering, say

$$M \cong M_{1,1} \oplus \ldots M_{1,n_1} \oplus \ldots \oplus M_{r,1} \oplus \ldots \oplus M_{r,n_r},$$

where for each $1 \leq i \leq r$, $1 \leq j \leq n_i$, the indecomposable submodule $M_{i,j}$ of M lies in the block A_i of A. In short, the decomposition of M into indecomposable submodules respects the decomposition of A into blocks.

To summarise the previous two paragraphs concisely, the k-linear category of A-modules decomposes as the direct sum of the k-linear categories of modules of its blocks:

$$A\text{-mod} \cong \bigoplus_{i=1}^{r} A_i\text{-mod}.$$

Further, the block decomposition

$$A = A_1 \times A_2 \times \ldots \times A_r$$

describes exactly the decomposition of the A-A-bimodule A. That is, the indecomposable A-A-bimodule summands of A are exactly the A-A-bimodules A_i, up to isomorphism and reordering.

Having introduced idempotents in this subsection, we make a couple of related notes. If P is a projective A-module, then there is a primitive idempotent (not necessarily central) $e \in A$ such that $P \cong Ae$ as a left A-module. Further, given two projective A-modules $P = Ae$ and $Q = Af$, there is an isomorphism of vector spaces

$$\mathrm{Hom}_A(P, Q) \cong eAf.$$

Moreover, there is an algebra isomorphism

$$\mathrm{End}_A(P)^{\mathrm{op}} \cong eAe,$$

where eAe is a non-unital subalgebra[11] of A, with multiplication inherited from A.

Recall that we may write the left A-module A as a direct sum of projective modules

$$A = P_1^{d_1} \oplus \ldots \oplus P_n^{d_n},$$

where $\{P_1, \ldots, P_n\}$ is a set of representatives of isomorphism classes of projective A-modules. We can phrase this in terms of idempotents. Take a complete set of primitive orthogonal idempotents $\{e_1, \ldots, e_m\}$, $m \geq 1$, in A. Then there is an isomorphism of left A-modules

$$A \cong \bigoplus_{i=1}^{m} Ae_i,$$

where for each i, Ae_i is a projective indecomposable A-module. Comparing the two decompositions, we have that, for each $l = 1, \ldots, n$, there are exactly d_l many $Ae_i \cong P_l$, so that $m = d_1 + \ldots + d_n$. In particular, A is a basic algebra precisely when $Ae_i \cong Ae_j$ if and only if $i = j$.

Suppose A is an arbitrary finite-dimensional k-algebra. Let $\{e_{a_1}, \ldots, e_{a_n}\}$ be a subset of $\{e_1, \ldots, e_m\}$ such that there is a bijection

$$\{Ae_{a_1}, \ldots, Ae_{a_n}\} \leftrightarrow \{P_1, \ldots, P_n\}$$

In particular, $Ae_{a_i} \cong Ae_{a_j}$ if and only if $i = j$. Set $e = e_{a_1} + \ldots + e_{a_n}$ and $\tilde{A} = eAe$. Then $\tilde{A}$ is a basic algebra by construction. Taking $P = Ae$, we have that P is a progenerator of A, since A is a direct summand of $P^{\oplus d}$, where $d = \max\{d_1, \ldots, d_n\}$, and

$$\tilde{A} = eAe \cong \mathrm{End}_A(P)^{\mathrm{op}},$$

so A and $\tilde{A}$ are Morita equivalent by Theorem 1.1.2. We call $\tilde{A}$ *the* basic algebra of A. It is the unique basic algebra Morita equivalent to A, up to isomorphism. Thus, up to Morita equivalence, we may always assume that A is a basic algebra.

1.1.7 Symmetric algebras

Let A be an k-algebra. We recall the following well-known result, due to Brauer, Nesbitt and Nakayama, here taken from [Ric02, Theorem 3.1].

[11] A *non-unital* subalgebra of A is a subalgebra whose unit element differs from that of A. Here, a non-unital subalgebra is still a unital algebra.

Theorem 1.1.3. *Let A be a finite-dimensional k-algebra. The following statements are equivalent.*

(i) There exists a symmetric, non-degenerate trace form on A. That is, a linear map $\mathrm{tr} : A \to k$ *such that:*

- $\mathrm{tr}(xy) = \mathrm{tr}(yx)$ *for all* $x, y \in A$*;*
- *for every* $0 \neq x \in A$*, there is a* $y \in A$ *such that* $\mathrm{tr}(xy) \neq 0$.

(ii) There is an isomorphism of A-A-bimodules

$$f : A \to A^*.$$

(iii) There is a natural isomorphism of contravariant functors

$$\mathrm{Hom}_k(-, k) \cong \mathrm{Hom}_A(-, A)$$

from A-mod *to* A^{op}-mod*.*

(iv) Given a finitely generated left A-module M and a finitely generated projective left A-module P, there is an isomorphism of k-vector spaces

$$\mathrm{Hom}_A(M, P) \cong \mathrm{Hom}_A(P, M)^*,$$

functorial in M and P.

An algebra A satisfying any of the above statements is *symmetric*. A symmetric, non-degenerate trace form $\mathrm{tr} : A \to k$ on A is called a *symmetrising form*[12]on A. Notice that condition (ii) is a special case of condition (iv).

We omit the full proof of this result, but it will be useful to show that the first two statements are equivalent. Suppose that A admits a symmetrising form $\mathrm{tr} : A \to k$. Define a map $f : A \to A^* = \mathrm{Hom}_k(A, k)$ by

$$f(x) = x \cdot \mathrm{tr} : y \mapsto \mathrm{tr}(yx).$$

Then f is an A-A-bimodule homomorphism, by the symmetry of tr, and it is injective by non-degeneracy. It is hence an isomorphism, since A and A^* have equal (finite) dimension over k. Conversely, if $f : A \to A^*$ is an A-A-bimodule isomorphism, we may take $\mathrm{tr} = f(1) : A \to k$.

We have an immediate corollary.

[12]A non-degenerate trace form is called a *Frobenius form* on A. An algebra admitting a Frobenius form is called a *Frobenius algebra*. As such, some authors use the terminology *symmetric Frobenius algebra* for what we are calling a symmetric algebra.

Corollary 1.1.4. *The k-algebra A is symmetric if and only if the k-algebra A^{op} is symmetric.*

Proof. The symmetrising form $\operatorname{tr} : A \to k$ on A can be applied to A^{op} and condition (i) still holds. □

Consider the decomposition

$$A = A_1 \times A_2 \times \ldots \times A_r$$

of A into blocks. The following is also well known.

Corollary 1.1.5. *The k-algebra A is symmetric if and only if every block A_i of A is symmetric.*

Proof. If A is symmetric and $\operatorname{tr} : A \to k$ is a symmetrising form on A, then the restriction of tr to A_i for any i defines a symmetrising form on A_i. Conversely, suppose for every i there is a symmetrising form $\operatorname{tr}_i : A_i \to k$. Then $\operatorname{tr}_1 + \operatorname{tr}_2 + \ldots + \operatorname{tr}_r$ is a symmetrising form on A. □

From condition (ii) of Theorem 1.1.3, one can immediately see that if A is a symmetric k-algebra, then A is self-injective. Moreover, from condition (iv), we also have that if A is symmetric, then the Nakayama permutation ν_A of A, as defined in Theorem 1.1.1, is the identity permutation. That is, if $\{S_1, \ldots, S_n\}$ is a complete set of simple A-modules up to isomorphism, and $\{P_1, \ldots, P_n\}$ are the corresponding projective indecomposable A-modules, then

$$P_i / \operatorname{rad}(P_i) \cong S_i \cong \operatorname{soc}(P_i).$$

In the symmetric algebra case, we have

$$\dim_k \operatorname{Hom}_A(P_i, P_j) = \dim_k \operatorname{Hom}_A(P_j, P_i),$$

so the Cartan matrix C_A of a symmetric algebra is a symmetric matrix.

1.1.8 Quivers and relations

Let k be an algebraically closed field and A a finite-dimensional k-algebra.

A *quiver* is a quadruple $Q = (Q_0, Q_1, s, t)$ such that:

- (Q_0, Q_1) is a graph, with finite vertex set Q_0 and finite edge set Q_1;
- $s : Q_1 \to Q_0$ is the map assigning to an arrow $\alpha \in Q_1$ its *source* $s(\alpha)$;

- $t : Q_1 \to Q_0$ is the map assigning to an arrow $\alpha \in Q_1$ its *target* $t(\alpha)$.

We say two arrows α and β in Q_1 are *compatible* if $t(\alpha) = s(\beta)$. In this case, we can define a *path* $\beta\alpha$ in Q.

$$\cdots \bullet \xrightarrow{\alpha} \bullet \xrightarrow{\beta} \bullet \cdots$$

In general, a *path* in Q is a sequence of arrows in Q such that any two successive arrows are compatible. The *path algebra* kQ of Q is the k-linear span of all paths in Q, with formal addition, and multiplication induced by concatenation of compatible arrows.

The radical of kQ is the two-sided ideal $R_Q = \mathrm{rad}(kQ)$, generated by all arrows in Q, called the *arrow ideal*. For every integer $\ell \geq 1$, there is a subspace kQ_ℓ of kQ, the k-linear span of all paths in Q of length ℓ. We have

$$R_Q \cong \bigoplus_{\ell \geq 1} kQ_\ell$$

as k-vector spaces. For every $m \geq 1$, the mth radical is the two-sided ideal

$$\mathrm{rad}^m(kQ) = R_Q^m = \bigoplus_{\ell \geq m} kQ_\ell$$

of kQ, the k-linear span of all paths of length at least m. A two-sided ideal $\mathcal{I}$ of kQ is an *admissible ideal* if there is an $m \geq 2$ such that

$$R_Q^m \subset \mathcal{I} \subset R_Q^2.$$

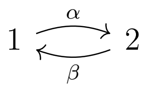
Typically, we define $\mathcal{I}$ by a finite set of relations in kQ generating $\mathcal{I}$. A *relation* in kQ is an identity in kQ of the form

$$\sum_{r=1}^{n} c_r \mu_r = 0,$$

where, for each r, $\mu_r = \alpha_1^{(r)} \ldots \alpha_{m_r}^{(r)}$ is a path in kQ of length $m_r \geq 2$, for $\alpha_i^{(r)} \in Q_1$, and $c_r \in k^*$.

Given an admissible ideal $\mathcal{I}$ of kQ, the quotient algebra $kQ/\mathcal{I}$ is a finite-dimensional k-algebra, called a *bound quiver algebra*. The algebra $A = kQ/\mathcal{I}$ is a basic algebra, and it is indecomposable if and only if Q is a connected graph. The simple A-modules are in one-to-one correspondence with the vertices in Q_0.

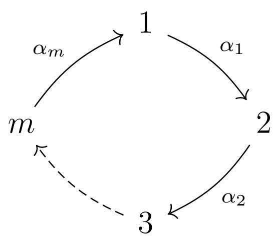
The following example is ubiquitous. Let Q be the following quiver.



Let $\mathcal{I}$ be the admissible ideal $\langle \alpha\beta\alpha, \beta\alpha\beta \rangle$ of kQ. Then $A = kQ/\mathcal{I}$ is a finite-dimensional k-algebra with two simple modules S_1 and S_2, and indecomposable projective A-modules P_1 and P_2 with Loewy structures

$$P_1 = \begin{matrix} 1 \\ 2 \\ 1 \end{matrix}, \quad P_2 = \begin{matrix} 2 \\ 1 \\ 2 \end{matrix}.$$

This algebra A falls into the wider class of Nakayama algebras, as introduced by Nakayama [Nak40a]. Given integers $m, n \in \mathbb{Z}_+$, the *Nakayama algebra* $N_{m,n}$ is the bound quiver algebra $kQ/\mathcal{I}$, where Q is the quiver



and $\mathcal{I} = \langle \alpha^n \rangle$. The projective indecomposable $N_{m,n}$-modules are uniserial of length n. The algebra A above is the Nakayama algebra $N_{2,3}$.

The Nakayama algebra $N_{m,n}$ is self-injective, with Nakayama permutation $\nu = \nu_{N_{m,n}}$ given by $\nu(i) = i - n + 1$, with indices modulo m. In particular, $N_{m,n}$ is symmetric when m divides $n - 1$.

Note that this definition allows the cases $m = 1$ and $n = 1$. However, for $n = 1$, the ideal $\mathcal{I}$ is not admissible, and the algebras $N_{m,1}$ are semisimple. For $m = 1$, we have $N_{1,n} \cong k[x]/\langle x^n \rangle$, the truncated polynomial ring of degree n.

Given a quiver Q and some ideal of relations $\mathcal{I}$, we have a finite-dimensional k algebra $kQ/\mathcal{I}$. It is natural to ask whether the converse is true: given a finite-dimensional k-algebra A, is there a quiver Q and an admissible ideal $\mathcal{I}$ of kQ such that $A \cong kQ/\mathcal{I}$? For finite-dimensional basic algebras over algebraically closed fields, the answer is yes, by Gabriel's Theorem.

Given a basic k-algebra A, we may construct the Ext^1-quiver[13] of A. This is the quiver $Q = (Q_0, Q_1)$, where Q_0 is in one-to-one correspondence with

[13] This is not the only quiver construction for A. The other most prominent construction is the Auslander-Reiten quiver of an Artinian algebra.

a set of representatives of isomorphism classes of simple A-modules, say $\{S_1, \ldots, S_r\}$, and the number of arrows $S_i \to S_j$ in Q_1 is equal to the dimension of $\mathrm{Ext}^1_A(S_i, S_j)$ over k. Recall that

$$\dim_k \mathrm{Ext}^1_A(S_i, S_j) = \dim_k \mathrm{Hom}_A(P_j, \mathrm{rad}(P_i)/\mathrm{rad}^2(P_i)),$$

where $P_i = Ae_i$ and $P_j = Ae_j$ are projective covers of S_i and S_j respectively, for some idempotents e_i, e_j in A. Then an arrow $\alpha : S_i \to S_j$ in Q corresponds to an element of $e_j \,\mathrm{rad}(A) e_i$. The structure of A then determines an admissible ideal $\mathcal{I} \subset \mathrm{rad}^2(kQ)$, of paths of length at least 2, such that $A \cong kQ/\mathcal{I}$. The paths in $\mathcal{I}$ define a set of *relations* in A.

For example, we will see in §3.1.5 that the quiver Q and relations $\mathcal{I}$ in the example above are the Ext^1-quiver and relations of the Brauer tree algebra of a star (or line) on two edges with exceptional multiplicity 1.

1.2 Homological Algebra

This section covers the homological underpinning of our work. We introduce the key players in §1.2.1 and §1.2.3: chain complexes, homology, and homotopy; before constructing the derived category of a finite-dimensional k-algebra A in §1.2.7 and focusing on equivalences at this level. To do this, we require an interjectory subsection on triangulated categories, §1.2.5, of which the homotopy and derived categories of A are the primary examples. We will recount Rickard's Morita theory for derived categories in §1.2.7, the analogue to Theorem 1.1.2 in this setting. Finally, we recall the definition of the stable category in §1.2.10, briefly discuss the existing Morita theory, and relate this to the derived category in a systematic way.

Throughout, k is an algebraically closed field and A a finite-dimensional k-algebra. We will always be working with the k-linear, abelian category A-mod of (finitely generated, left) A-modules, but one may just as well replace this with an arbitrary k-linear, abelian category $\mathcal{A}$. Unless otherwise stated, results come from Chapters 1, 2 and 10 of [Wei95].

1.2.1 Chain complexes

Let A be a finite-dimensional k-algebra. All A-modules are assumed to be finitely generated.

A $\mathbb{Z}$-*graded* (or simply *graded*) A-*module* is a sequence of A-modules $X = (X_n)_{n\in\mathbb{Z}}$. If $X = (X_n)$ and $Y = (Y_n)$ are graded A-modules, then a *graded*

A-module homomorphism $f : X \to Y$ of *degree* i is a family of A-module homomorphisms $f = (f_n : X_n \to Y_{n+i})$. There is a category A-grmod of all graded A-modules, whose Hom sets are graded A-module homomorphisms.

Graded A-modules occur naturally in many settings. The most familiar example is a chain complex of A-modules. A *chain complex* of A-modules is a pair (X, δ), where $X = (X_n)$ is a graded A-module and $\delta : X \to X$ is a graded A-module homomorphism of degree -1, called the *differential*, such that $\delta \circ \delta = 0$. We display this as

$$\ldots \longrightarrow X_{n+1} \overset{\delta_{n+1}}{\longrightarrow} X_n \overset{\delta_n}{\longrightarrow} X_{n-1} \overset{\delta_{n-1}}{\longrightarrow} \ldots$$

with $\delta_n \circ \delta_{n+1} = 0$ for all n. A chain complex X is *bounded above* or *to the left* if $X_n = 0$ for sufficiently large n, and *bounded below* or *to the right* if $X_n = 0$ for sufficiently small n. A chain complex X is *bounded* if it is both bounded above and bounded below.

Let (X, δ) and (Y, ϵ) be chain complexes of A-modules. A *chain map* $f : X \to Y$ is a graded A-module homomorphism of degree 0, commuting with the differentials on X and Y. That is, all squares in the following diagram commute:

$$\begin{array}{ccccccccc}
\ldots & \xrightarrow{\delta_{n+2}} & X_{n+1} & \xrightarrow{\delta_{n+1}} & X_n & \xrightarrow{\delta_n} & X_{n-1} & \xrightarrow{\delta_{n-1}} & \ldots \\
 & & \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \\
\ldots & \xrightarrow{\epsilon_{n+2}} & Y_{n+1} & \xrightarrow{\epsilon_{n+1}} & Y_n & \xrightarrow{\epsilon_n} & Y_{n-1} & \xrightarrow{\epsilon_{n-1}} & \ldots
\end{array}$$

The category of chain complexes of A-modules $\mathrm{Ch}(A)$ is the category with object class consisting of all chain complexes of A-modules. Given chain complexes (X, δ) and (Y, ϵ), the morphism set $\mathrm{Hom}_{\mathrm{Ch}(A)}(X, Y)$ is the set of all chain maps $f : X \to Y$. Each set $\mathrm{Hom}_{\mathrm{Ch}(A)}(X, Y)$ is a k-vector space, and the category $\mathrm{Ch}(A)$ is a k-linear abelian category. Direct sums of chain complexes are taken degree wise.

The full subcategories $\mathrm{Ch}^-(A)$, $\mathrm{Ch}^+(A)$ and $\mathrm{Ch}^b(A)$ of $\mathrm{Ch}(A)$ have object classes comprising respectively all bounded below, bounded above, and bounded chain complexes of A-modules. All three are abelian subcategories. By considering an A-module as a chain complex concentrated in degree 0, there is a nice embedding of A-mod into $\mathrm{Ch}^b(A)$. Full subcategories of A-mod also define full subcategories of $\mathrm{Ch}(A)$: for example, $\mathrm{Ch}(A\text{-proj})$ is the subcategory of $\mathrm{Ch}(A)$ whose objects are chain complexes of projective A-modules. If X is a chain complex, then we define X-add to be the full subcategory of $\mathrm{Ch}(A)$ whose objects are summands of direct sums of copies of X. We can extend this definition to $\mathcal{X}$-add, for $\mathcal{X}$ a collection of objects in $\mathrm{Ch}(A)$, in the obvious way.

For k-algebras A and B, we denote by $\mathrm{Ch}(A\text{-}B)$ the category of chain complexes of A-B-bimodules, so that $\mathrm{Ch}(A\text{-}B) \cong \mathrm{Ch}(A \otimes_k B^{\mathrm{op}})$.

For every $i \in \mathbb{Z}$, there is an equivalence of abelian categories $[i] : \mathrm{Ch}(A) \to \mathrm{Ch}(A)$, where, for a chain complex (X, δ), the chain complex $(X[i], \delta[i])$ is defined by $X[i]_n = X_{n-i}$ and $\delta[i]_n = (-1)^i \delta_{n-i}$. We think of $X[i]$ as the complex X shifted i positions to the left. For a chain map $f : X \to Y$, we have $f[i]_n = f_{n-i}$.

One may make the definition of a chain complex in an arbitrary abelian category $\mathcal{A}$. A chain complex in $\mathcal{A}$ is a sequence

$$\ldots \longrightarrow X_{n+1} \overset{\delta_{n+1}}{\longrightarrow} X_n \overset{\delta_n}{\longrightarrow} X_{n-1} \overset{\delta_{n-1}}{\longrightarrow} \ldots$$

of objects X_n in $\mathcal{A}$ and homomorphisms d_n in $\mathcal{A}$ such that $\delta_n \circ \delta_{n+1} = 0$ for all n. Chain maps are defined similarly, and we may define an abelian category $\mathrm{Ch}(\mathcal{A})$ of chain complexes in $\mathcal{A}$, so that the abelian subcategories $\mathrm{Ch}^-(\mathcal{A})$, $\mathrm{Ch}^+(\mathcal{A})$ and $\mathrm{Ch}^b(\mathcal{A})$ exist. Similarly, a full subcategory $\mathcal{B}$ of $\mathcal{A}$ gives rise to a full subcategory $\mathrm{Ch}(\mathcal{B})$ of $\mathrm{Ch}(\mathcal{A})$. In our notation, $\mathrm{Ch}(A) = \mathrm{Ch}(A\text{-mod})$. Notice that $\mathrm{Ch}(\mathcal{A})$ is an abelian category, so we have chain complexes of chain complexes, and chain complexes of chain complexes of chain complexes, and so on.

The notion of a chain complex of chain complexes is usually described in terms of double complexes. A *double complex* in $\mathcal{A}$ is a collection of objects $\{X_{i,j}\}$ of objects in $\mathcal{A}$, together with maps $\delta^h_{i,j} : X_{i,j} \to X_{i-1,j}$ and $\delta^v_{i,j} : X_{i,j} \to X_{i,j-1}$ for all i and j, such that $\delta^h_{i-1,j} \circ \delta^h_{i,j} = 0$, $\delta^v_{i,j-1} \circ \delta^v_{i,j} = 0$, and $\delta^v_{i-1,j}\delta^h_{i,j} + \delta^h_{i,j-1}\delta^v_{i,j} = 0$. One typically pictures a double complex as a lattice:

$$\begin{array}{ccccccccc}
 & & \ldots & & \ldots & & \ldots & & \\
 & & \downarrow{\scriptstyle\delta^v} & & \downarrow{\scriptstyle\delta^v} & & \downarrow{\scriptstyle\delta^v} & & \\
\ldots & \longrightarrow & X_{i+1,j+1} & \overset{\delta^h}{\longrightarrow} & X_{i,j+1} & \overset{\delta^h}{\longrightarrow} & X_{i-1,j+1} & \overset{\delta^h}{\longrightarrow} & \ldots \\
 & & \downarrow{\scriptstyle\delta^v} & & \downarrow{\scriptstyle\delta^v} & & \downarrow{\scriptstyle\delta^v} & & \\
\ldots & \longrightarrow & X_{i+1,j} & \overset{\delta^h}{\longrightarrow} & X_{i,j} & \overset{\delta^h}{\longrightarrow} & X_{i-1,j} & \overset{\delta^h}{\longrightarrow} & \ldots \\
 & & \downarrow{\scriptstyle\delta^v} & & \downarrow{\scriptstyle\delta^v} & & \downarrow{\scriptstyle\delta^v} & & \\
\ldots & \longrightarrow & X_{i+1,j-1} & \overset{\delta^h}{\longrightarrow} & X_{i,j-1} & \overset{\delta^h}{\longrightarrow} & X_{i-1,j-1} & \overset{\delta^h}{\longrightarrow} & \ldots \\
 & & \downarrow{\scriptstyle\delta^v} & & \downarrow{\scriptstyle\delta^v} & & \downarrow{\scriptstyle\delta^v} & & \\
 & & \ldots & & \ldots & & \ldots & &
\end{array}$$

A double complex is *bounded* if there are only finitely many non-zero terms $X_{i,j}$ with $i + j = n$, for every n. That is, if there are only finitely many non-zero terms on each diagonal line in the above lattice. For example, any double complex $\{X_{i,j}\}$ with non-zero terms concentrated in the upper left quadrant of the plane (that is, for which $X_{i,j}$ is zero whenever i is sufficiently small or j is sufficiently small) is a bounded double complex. Such bounded double complexes are called *upper left quadrant double complexes.*

From a bounded double complex $\{X_{i,j}\}$, one obtains complexes and chain maps in $\mathrm{Ch}(\mathcal{A})$, in the following way. Each column $X_{i,*}$ is a chain complex, and $f_{i,*} = (\delta^h_{i,j})$ is a chain map $X_{i,*} \to X_{i-1,*}$. Similarly, each row $X_{*,j}$ is a chain complex, and $f_{*,j} = ((-1)^i \delta^v_{i,j})$ is a chain map $X_{*,j} \to X_{*,j-1}$. Note that the anticommutativity condition $\delta^v_{i-1,j}\delta^h_{i,j} + \delta^h_{i,j-1}\delta^v_{i,j} = 0$ necessitates the changes of sign here.

One can define another chain complex in $\mathcal{A}$ from a bounded double complex $\{X_{i,j}\}$, the *total complex* $\mathrm{Tot}^{\oplus}(X)$, with

$$\mathrm{Tot}^{\oplus}(X)_n = \bigoplus_{i+j=n} X_{i,j}$$

and differential

$$\begin{aligned} &\delta_n : \mathrm{Tot}^{\oplus}(X)_n \to \mathrm{Tot}^{\oplus}(X)_{n-1} \\ &\delta_n = \sum_{i+j=n} \delta^h_{i,j} + \delta^v_{i,j}. \end{aligned}$$

The anticommutativity condition of the double complex guarantees that $\mathrm{Tot}^{\oplus}(X)$ is a well-defined complex in $\mathrm{Ch}(\mathcal{A})$. In particular, when $\{X_{i,j}\}$ is an upper left quadrant double complex, the total complex $\mathrm{Tot}^{\oplus}(X)_n$ is a bounded below chain complex in $\mathcal{A}$.

For now, we return to the case that $\mathcal{A}$ is the module category of a finite-dimensional k-algebra. One uses the total complex construction to produce the tensor product between two chain complexes.

Let A, B and C be k-algebras, where we allow any to be k. Recall that, given an A-B-bimodule M and a B-C-bimodule N, there are covariant right exact functors

$$M \otimes_B - : (B \otimes_k C^{\mathrm{op}})\text{-mod} \to (A \otimes_k C^{\mathrm{op}})\text{-mod}$$

and

$$- \otimes_B N : (A \otimes_k B^{\mathrm{op}})\text{-mod} \to (A \otimes_k C^{\mathrm{op}})\text{-mod}\,.$$

Let (X, δ) be a bounded below chain complex of A-B-bimodules and (Y, ϵ) a bounded below chain complex of B-C-modules. Then $X \otimes_B Y$ is the chain complex of A-C-bimodules, formed as the total complex of the double complex

$$
\begin{array}{ccccccccc}
 & & \dots & & \dots & & \dots & & \\
 & & \downarrow{\scriptstyle (-1)^{i+1}\epsilon} & & \downarrow{\scriptstyle (-1)^{i}\epsilon} & & \downarrow{\scriptstyle (-1)^{i-1}\epsilon} & & \\
\dots & \longrightarrow & X_{i+1}\otimes_B Y_{j+1} & \xrightarrow{\delta} & X_i\otimes_B Y_{j+1} & \xrightarrow{\delta} & X_{i-1}\otimes_B Y_{j+1} & \xrightarrow{\delta} & \dots \\
 & & \downarrow{\scriptstyle (-1)^{i+1}\epsilon} & & \downarrow{\scriptstyle (-1)^{i}\epsilon} & & \downarrow{\scriptstyle (-1)^{i-1}\epsilon} & & \\
\dots & \longrightarrow & X_{i+1}\otimes_B Y_{j} & \xrightarrow{\delta} & X_i\otimes_B Y_{j} & \xrightarrow{\delta} & X_{i-1}\otimes_B Y_{j} & \xrightarrow{\delta} & \dots \\
 & & \downarrow{\scriptstyle (-1)^{i+1}\epsilon} & & \downarrow{\scriptstyle (-1)^{i}\epsilon} & & \downarrow{\scriptstyle (-1)^{i-1}\epsilon} & & \\
\dots & \longrightarrow & X_{i+1}\otimes_B Y_{j-1} & \xrightarrow{\delta} & X_i\otimes_B Y_{j-1} & \xrightarrow{\delta} & X_{i-1}\otimes_B Y_{j-1} & \xrightarrow{\delta} & \dots \\
 & & \downarrow{\scriptstyle (-1)^{i+1}\epsilon} & & \downarrow{\scriptstyle (-1)^{i}\epsilon} & & \downarrow{\scriptstyle (-1)^{i-1}\epsilon} & & \\
 & & \dots & & \dots & & \dots & &
\end{array}
$$

Notice that X and Y being bounded below makes this double complex an upper left quadrant double complex, so that the total complex $X \otimes_B Y$ is a bounded below complex of A-C-bimodules.

This construction is functorial in both arguments, so we obtain functors

$$X \otimes_B - : \mathrm{Ch}^-(B\text{-}C) \to \mathrm{Ch}^-(A\text{-}C)$$

and

$$- \otimes_B Y : \mathrm{Ch}^-(A\text{-}B) \to \mathrm{Ch}^-(A\text{-}C).$$

As in the module category case, these are covariant right exact functors. The former is an exact functor when X is a bounded below complex of A-B-bimodules, each term of which is projective as a right B-module. The latter is exact when Y is a bounded complex of B-C-bimodules, each term of which is projective as a left B-module.

Similarly, recall that, given an A-B-module M and an A-C-bimodule N, there is a covariant left exact functor

$$\mathrm{Hom}_A(M,-) : (A \otimes_k C^{\mathrm{op}})\text{-mod} \to (B \otimes_k C^{\mathrm{op}})\text{-mod}$$

and a contravariant left exact functor

$$\mathrm{Hom}_A(-,N) : (A \otimes_k B^{\mathrm{op}})\text{-mod} \to (B \otimes_k C^{\mathrm{op}})\text{-mod}\,.$$

Given a bounded above chain complex X of A-B-bimodules and a bounded below chain complex Y of A-C-bimodules, then $\mathrm{Hom}_A(X,Y)$ is the chain complex of B-C-bimodules, formed as the total complex of the double complex

$$
\begin{array}{ccccccccc}
 & & \cdots & & \cdots & & \cdots & & \\
 & & \downarrow{\scriptstyle (-1)^{i+1}\epsilon_*} & & \downarrow{\scriptstyle (-1)^{i}\epsilon_*} & & \downarrow{\scriptstyle (-1)^{i-1}\epsilon_*} & & \\
\cdots & \longrightarrow & \operatorname{Hom}_A(X_{i-1}, Y_{j+1}) & \xrightarrow{\delta_*} & \operatorname{Hom}_A(X_i, Y_{j+1}) & \xrightarrow{\delta_*} & \operatorname{Hom}_A(X_{i+1}, Y_{j+1}) & \xrightarrow{\delta_*} & \cdots \\
 & & \downarrow{\scriptstyle (-1)^{i+1}\epsilon_*} & & \downarrow{\scriptstyle (-1)^{i}\epsilon_*} & & \downarrow{\scriptstyle (-1)^{i-1}\epsilon_*} & & \\
\cdots & \longrightarrow & \operatorname{Hom}_A(X_{i-1}, Y_{j}) & \xrightarrow{\delta_*} & \operatorname{Hom}_A(X_i, Y_{j}) & \xrightarrow{\delta_*} & \operatorname{Hom}_A(X_{i+1}, Y_{j}) & \xrightarrow{\delta_*} & \cdots \\
 & & \downarrow{\scriptstyle (-1)^{i+1}\epsilon_*} & & \downarrow{\scriptstyle (-1)^{i}\epsilon_*} & & \downarrow{\scriptstyle (-1)^{i-1}\epsilon_*} & & \\
\cdots & \longrightarrow & \operatorname{Hom}_A(X_{i-1}, Y_{j-1}) & \xrightarrow{\delta_*} & \operatorname{Hom}_A(X_i, Y_{j-1}) & \xrightarrow{\delta_*} & \operatorname{Hom}_A(X_{i+1}, Y_{j-1}) & \xrightarrow{\delta_*} & \cdots \\
 & & \downarrow{\scriptstyle (-1)^{i+1}\epsilon_*} & & \downarrow{\scriptstyle (-1)^{i}\epsilon_*} & & \downarrow{\scriptstyle (-1)^{i-1}\epsilon_*} & & \\
 & & \cdots & & \cdots & & \cdots & &
\end{array}
$$

Notice that X being bounded above and Y being bounded below makes this double complex an upper left quadrant double complex, so that the total complex $\operatorname{Hom}_A(X, Y)$ is a bounded below complex of B-C-bimodules.

Again, this construction is functorial in both arguments, and we obtain a covariant left exact functor

$$\operatorname{Hom}_A(X, -) : \operatorname{Ch}^-(A\text{-}C) \to \operatorname{Ch}^-(B\text{-}C)$$

and a contravariant left exact functor

$$\operatorname{Hom}_A(-, Y) : \operatorname{Ch}^+(A\text{-}B) \to \operatorname{Ch}^-(B\text{-}C).$$

These functors are exact functors respectively when each term of X is projective as a left A-module, and when each term of Y is injective as a left A-module.

Tensor-hom adjunction extends to chain complexes, too. If $X \in \operatorname{Ch}^+(A\text{-}B)$, $Y \in \operatorname{Ch}^+(B\text{-}C)$, and $Z \in \operatorname{Ch}^-(A\text{-}C)$, then

$$\operatorname{Hom}_{\operatorname{Ch}^-(A\text{-}C)}(X \otimes_B Y, Z) \cong \operatorname{Hom}_{\operatorname{Ch}^-(A\text{-}B)}(X, \operatorname{Hom}_{C^{\mathrm{op}}}(Y, Z))$$

and

$$\operatorname{Hom}_{\operatorname{Ch}^-(A\text{-}C)}(X \otimes_B Y, Z) \cong \operatorname{Hom}_{\operatorname{Ch}^+(B\text{-}C)}(Y, \operatorname{Hom}_A(X, Z)).$$

Both isomorphisms are given in each term by the usual tensor-Hom adjunction for modules.

1.2.2 Homology

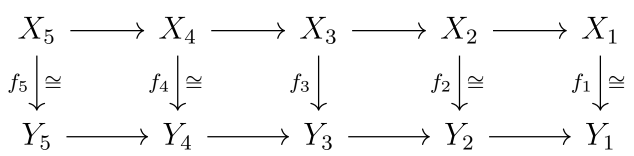
Let (X, δ) be a chain complex in $\operatorname{Ch}(A)$. The *homology* of X is the graded A-module $H_*(X) = (H_n(X))_{n \in \mathbb{Z}}$, where $H_n(X) = \ker(\delta_n)/\operatorname{im}(\delta_{n+1})$. This is well defined, as $\delta_n \circ \delta_{n+1} = 0$. A chain complex X is *acyclic* if $H_n(X) = 0$ for every n. That is, an acyclic chain complex is an exact sequence of A-modules.

Taking homology yields a functor $H_*(-) : \mathrm{Ch}(A) \to A\text{-grmod}$. If (X, δ) and (Y, ϵ) are chain complexes of A-modules and $f : X \to Y$ is a chain map, then the graded A-module homomorphism $H_*(f) : H_*(X) \to H_*(Y)$ is defined as $H_*(f) = (H_n(f))_{n\in\mathbb{Z}}$, where

$$\begin{aligned} H_n(f) : H_n(X) &\to H_n(Y), \\ x + \mathrm{im}(\delta_{n+1}) &\mapsto f_n(x) + \mathrm{im}(\epsilon_{n+1}). \end{aligned}$$

A chain map $f : X \to Y$ is a *quasi-isomorphism* if each $H_n(f) : H_n(X) \to H_n(Y)$ is an isomorphism.

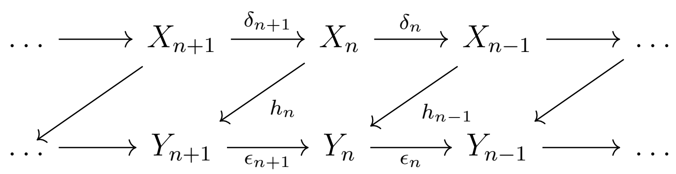
We make reference to a result that will be relevant later on, the 5-Lemma (see e.g. [Wei95, Exercise 1.3.3]). If there is a commutative diagram



with exact rows such that f_1, f_2, f_4 and f_5 are all isomorphisms, then f_3 is also an isomorphism.

1.2.3 Chain homotopy

Let A be a finite-dimensional k-algebra and let (X, δ) and (Y, ϵ) be chain complexes of A-modules. A *chain homotopy* between X and Y is a graded A-module homomorphism $h : X \to Y$ of degree 1.



Notice that $h \circ \delta + \epsilon \circ h$ is a chain map $X \to Y$. The homotopy h is a *homotopy between* two chain maps $f, f' : X \to Y$ if $f - f' = h \circ \delta + \epsilon \circ h$. In this case, the chain maps f, f' are *homotopic*, $f \sim f'$. This relation is an equivalence relation. We think of chain maps f and f' with $f \sim f'$ as being the same *up to homotopy*.

A *chain homotopy equivalence* is a chain map $f : X \to Y$ such that there is a chain map $g : Y \to X$ with $g \circ f \sim \mathrm{Id}_X$ and $f \circ g \sim \mathrm{Id}_Y$. If there is a chain homotopy equivalence $f : X \to Y$, we say the chain complexes X and Y are *chain homotopy equivalent*, $X \simeq Y$.

We have two useful results regarding chain homotopy.

Proposition 1.2.1. *Let X, Y be chain complexes, and $f, f' : X \to Y$ chain maps.*

1. *If $f \sim f'$, then $H_*(f) = H_*(f')$.*
2. *If f is a chain homotopy equivalence, then f is a quasi-isomorphism.*

The *homotopy category* of chain complexes of A-modules is the category $K(A\text{-mod}) = K(A)$ whose objects are chain complexes of A-modules, and whose morphism sets are the sets

$$\mathrm{Hom}_{K(A)}(X, Y) = \mathrm{Hom}_{\mathrm{Ch}(A)}(X, Y)/\sim$$

of chain maps up to homotopy. Isomorphisms in $K(A)$ are chain homotopy equivalences. The categories $K^+(A)$, $K^-(A)$ and $K^b(A)$ are the full subcategories of $K(A)$ whose objects are bounded above, bounded below and bounded chain complexes of A-modules, respectively. Again, A-mod embeds nicely into $K^b(A)$ by considering an A-module as a chain complex concentrated in degree 0. We can further define the category $K(A\text{-proj})$ as the subcategory of $K(A)$ whose objects are chain complexes of projective A-modules. Similarly, $K^b(A\text{-proj})$ is the subcategory of bounded chain complexes of projective A-modules. There is a k-linear quotient functor $\mathrm{Ch}(A) \to K(A)$, restricting to a k-linear functor $\mathrm{Ch}^b(A) \to K^b(A)$.

Given two k-algebras A and B, we denote by $K(A\text{-}B)$ the homotopy category of bounded chain complexes of A-B-bimodules. As before, under the assumption that k acts centrally on bimodules, $K(A\text{-}B) \cong K(A \otimes_k B^{\mathrm{op}})$.

A more general construction of the homotopy category $K(\mathcal{A})$ of an abelian category $\mathcal{A}$ exists, built in much the same way out of the category of chain complexes $\mathrm{Ch}(\mathcal{A})$ of objects of $\mathcal{A}$.

1.2.4 Projective resolutions and mapping cones

Let A be a finite-dimensional k algebra and let X be a bounded below chain complex of A-modules. A *projective resolution* of X is a pair $(P(X), \pi_X)$ comprising a bounded below chain complex of projective A-modules and a quasi-isomorphism $\pi_X : P(X) \to X$. Every bounded below complex of A-modules X has a projective resolution.

One useful result is the following, known as the Comparison Theorem, [Wei95, Theorem 2.2.6].

Theorem 1.2.2. *Let M, N be A-modules and*

$$\ldots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \xrightarrow{\epsilon} M \longrightarrow 0$$

a chain complex with each P_i a projective A-module. Let $P = (P_n)_{n \in \mathbb{Z}}$, with $P_i = 0$ for $i < 0$. If $f : M \to N$ is an A-module homomorphism, and $Q \xrightarrow{\eta} N$ is a projective resolution of N, then there is a chain map $f : P \to Q$, unique up to homotopy, such that the following diagram commutes:

$$\begin{array}{ccccccccccc} \ldots & \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & P_0 & \xrightarrow{\epsilon} & M & \longrightarrow & 0 \\ & & \downarrow f_2 & & \downarrow f_1 & & \downarrow f_0 & & \downarrow f & & \\ \ldots & \longrightarrow & Q_2 & \longrightarrow & Q_1 & \longrightarrow & Q_0 & \xrightarrow{\eta} & N & \longrightarrow & 0 \end{array}$$

In particular, $f \circ \epsilon = \eta \circ f_0$.

Note that we do not here require that P be a projective resolution of the A-module M, considered as a complex concentrated in degree 0. When P is a projective resolution of M, this says that a map between A-modules M and N lifts to a unique (up to homotopy) map between their projective resolutions.

We have one final definition in this section. If (X, δ) and (Y, ϵ) are chain complexes of A-modules and $f : X \to Y$ is a chain map, then the *mapping cone* of f is the chain complex $(\text{cone}(f), d)$ such that $\text{cone}(f)_n = X_{n-1} \oplus Y_n$ and $d_n(x, y) = (-\delta_{n-1}(x), \epsilon_n(y) - f_{n-1}(x))$.

The mapping cone of f fits into a short exact sequence of chain complexes

$$0 \longrightarrow Y \longrightarrow \text{cone}(f) \longrightarrow X[1] \longrightarrow 0,$$

where the left arrow sends elements $y \in Y_n$ to $(0, y)$ and the right arrow sends elements $(x, y) \in X_{n-1} \oplus Y_n$ to $-x$.

1.2.5 Triangulated categories

The notion of triangulated category was introduced by Verdier[14] in 1963, [Ver77]. He was primarily attempting to generalise his new definition of the derived category of an abelian category, which we come onto in the next section. In this sense, the exposition here is the reverse of Verdier's.

Definition 1.2.3. Let $\mathcal{T}$ be a category (assumed additive, k-linear).

[14] This is disputed by some, who believe the idea was originally Grothendieck's. The formal definition first appeared in print in the referenced paper of Verdier, however. Moreover, a less complete notion of triangulated category appeared a year before Verdier's, due to Puppe, described in [Pup67]. Puppe's definition is without the octahedral axiom, **(TR4)** below. Puppe's motivation for a general definition was the stable homotopy category.

- A *shift* or *translation functor* on $\mathcal{T}$ is an additive functor $T : \mathcal{T} \to \mathcal{T}$.
- A *triangle* in a category $\mathcal{T}$ equipped with a translation functor is a succession of morphisms

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T[X]$$

  between objects of $\mathcal{T}$. We also use the notation $X \longrightarrow Y \longrightarrow Z \rightsquigarrow$ to denote a triangle in $\mathcal{T}$.
- A *morphism of triangles* is a triple (u, v, w) of morphisms in $\mathcal{T}$ making the following diagram, in which both rows are triangles in $\mathcal{T}$, commute:

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & T(X) \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle T(u)} \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & T(X') \end{array}$$

  If all three of u, v and w are isomorphisms, then this is an *isomorphism of triangles.*

Given a translation functor T on a category $\mathcal{T}$, we will generally write $X[1]$ for $T(X)$, where X is an object of $\mathcal{T}$. We write iterated applications of T as $X[n] = T^n(X)$.

We can now define a triangulated category in general.

Definition 1.2.4. We say the category $\mathcal{T}$ is a *triangulated category* if there is a translation functor T on $\mathcal{T}$ and a family of *distinguished* or *exact triangles*

$$X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$$

such that the following four axioms hold.

(TR1) (i) Every morphism $f : X \to Y$ in $\mathcal{T}$ can be embedded in an exact triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$. The object Z of $\mathcal{T}$ is the mapping cone of f.

(ii) For any object X in $\mathcal{T}$, the triangle $X \xrightarrow{\text{id}} X \longrightarrow 0 \longrightarrow X[1]$ is exact.

(iii) If $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ is an exact triangle in $\mathcal{T}$ and we have an isomorphism of triangles

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & X[1] \\ \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & X'[1] \end{array}$$

in $\mathcal{T}$, then the triangle $X' \xrightarrow{f'} Y' \xrightarrow{g'} Z' \xrightarrow{h'} X'[1]$ is also exact.

(TR2) If $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ is an exact triangle in $\mathcal{T}$, then the rotated triangles

$$Y \xrightarrow{\quad g \quad} Z \xrightarrow{\quad h \quad} X[1] \xrightarrow{-f[1]} Y[1]$$

and

$$Z[-1] \xrightarrow{-h[-1]} X \xrightarrow{\quad f \quad} Y \xrightarrow{\quad g \quad} Z$$

are also exact in $\mathcal{T}$.

(TR3) Given two exact triangles

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1] \quad \text{and} \quad X' \xrightarrow{f'} Y' \xrightarrow{g'} Z' \xrightarrow{h'} X'[1]$$

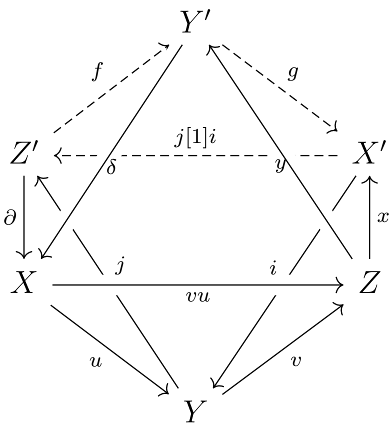
in $\mathcal{T}$ and morphisms $u : X \to X'$ and $v : Y \to Y'$ such that $f'u = vf$, there is a third morphism $w : Z \to Z'$ such that we have a morphism of triangles

$$\begin{array}{ccccccc} X & \xrightarrow{\quad f \quad} & Y & \xrightarrow{\quad g \quad} & Z & \xrightarrow{\quad h \quad} & X[1] \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle u[1]} \\ X' & \xrightarrow{\quad f' \quad} & Y' & \xrightarrow{\quad g' \quad} & Z' & \xrightarrow{\quad h' \quad} & X'[1]. \end{array}$$

This axiom is sometimes called the *completion* axiom for triangulated categories.

(TR4) **(The Octahedral Axiom)** Given exact triangles $X \xrightarrow{u} Y \xrightarrow{j} Z' \xrightarrow{\partial} X[1]$, $Y \xrightarrow{v} Z \xrightarrow{x} X' \xrightarrow{i} Y[1]$ and $X \xrightarrow{vu} Z \xrightarrow{y} Y' \xrightarrow{\delta} X[1]$ in $\mathcal{T}$, there is an exact triangle $Z' \xrightarrow{f} Y' \xrightarrow{g} X' \xrightarrow{h} Z'[1]$ such that $h = j[1]i$, $\partial = \delta f$, $x = gy$, $yv = fj$ and $u\delta = ig$. We can picture this[15] as follows:

[15] and justify the nomenclature



Notice that the cone of a morphism, as defined in **(TR1)**, is defined up to non-unique isomorphism, as per **(TR3)**. Some authors thus contend that the triangulated category framework is not the right one and could lead to errors. Alternative frameworks to avoid the non-uniqueness trap, such as the stable ∞-category, have been proposed as more suitable. However, for our purposes this is not important, as the derived category one recovers is still triangulated, and all relevant results hold in the usual framework.

Definition 1.2.5. Let $\mathcal{T}$ and $\mathcal{T}'$ be triangulated categories.

- An additive functor $F : \mathcal{T} \to \mathcal{T}'$ is a *triangulated functor* if it commutes with the translation functors T and T' and sends distinguished triangles to distinguished triangles.
- A triangulated functor $F : \mathcal{T} \to \mathcal{T}'$ is an *equivalence of triangulated categories* if F is an equivalence of categories.
- A full subcategory $\mathcal{C}$ of $\mathcal{T}$ is a *triangulated subcategory* if $\mathcal{C}$ is a triangulated category, and the inclusion functor $\iota : \mathcal{C} \to \mathcal{T}$ is a triangulated functor.

The 5-Lemma for abelian categories has its own triangulated version.

Lemma 1.2.6. *Let $\mathcal{T}$ be a triangulated category and (u, v, w) a morphism of distinguished triangles in $\mathcal{T}$:*

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle u[1]} \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1] \end{array} .$$

If u and v are isomorphisms, then so is w.

A useful application of Lemma 1.2.6 is the following.

Proposition 1.2.7. *Let $\mathcal{T}$ be a triangulated category, and*

$$X_1 \longrightarrow Y_1 \longrightarrow Z_1 \rightsquigarrow$$

$$X_2 \longrightarrow Y_2 \longrightarrow Z_2 \rightsquigarrow$$

two distinguished triangles in $\mathcal{T}$. Then the direct sum

$$X_1 \oplus X_2 \longrightarrow Y_1 \oplus Y_2 \longrightarrow Z_1 \oplus Z_2 \rightsquigarrow$$

is also a distinguished triangle in $\mathcal{T}$.

Proof. By the axiom **(TR1)**, there is a distinguished triangle

$$X_1 \oplus X_2 \longrightarrow Y_1 \oplus Y_2 \longrightarrow Z \rightsquigarrow$$

for some Z. Projecting $X_1 \oplus X_2$ and $Y_1 \oplus Y_2$ onto their factors, with the completion axiom **(TR3)** gives morphisms of triangles

$$\begin{array}{ccccc} X_1 \oplus X_2 & \longrightarrow & Y_1 \oplus Y_2 & \longrightarrow & Z \rightsquigarrow \\ \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow & & \downarrow \\ X_i & \longrightarrow & Y_i & \longrightarrow & Z_i \rightsquigarrow \end{array}$$

for $i = 1, 2$. Then one may take the direct sum to obtain a morphism of triangles

$$\begin{array}{ccccc} X_1 \oplus X_2 & \longrightarrow & Y_1 \oplus Y_2 & \longrightarrow & Z \rightsquigarrow \\ \downarrow\cong & & \downarrow\cong & & \downarrow \\ X_1 \oplus X_2 & \longrightarrow & Y_1 \oplus Y_2 & \longrightarrow & Z_1 \oplus Z_2 \rightsquigarrow \end{array}$$

and since the first two arrows are isomorphisms, by Lemma 1.2.6 the third is, too. □

Let A be a k-algebra. The homotopy category $K(A)$ of A is a triangulated category. The full subcategories $K^-(A)$, $K^+(A)$ and $K^b(A)$ are triangulated subcategories of the triangulated category $K(A)$, as is $K^b(A\text{-proj})$. The shift functor is $T(X) = X[1]$. A distinguished triangle $X \longrightarrow Y \longrightarrow Z \rightsquigarrow$ in $K^b(A)$ is any triple isomorphic to a *strict triangle*:

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\ \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong \\ V & \xrightarrow{u} & W & \longrightarrow & \mathrm{cone}(u) & \longrightarrow & V[1] \end{array}$$

Here, all arrows are chain maps, and the vertical arrows are isomorphisms in $K(A)$ (that is, chain homotopy equivalences).

Similarly, given an arbitrary abelian category $\mathcal{A}$, the homotopy category $K(\mathcal{A})$ is a triangulated category, with translation functor $-[1] : K(\mathcal{A}) \to K(\mathcal{A})$, and distinguished triangles formed in much the same way. Again, $K^-(\mathcal{A})$, $K^+(\mathcal{A})$ and $K^b(\mathcal{A})$ are triangulated subcategories of $K(\mathcal{A})$.

We introduce now an important class of subcategories of triangulated categories, that will play an significant role in the context of derived equivalence, notably in §1.2.9 and §1.3.1.

Definition 1.2.8. A full triangulated subcategory $\mathcal{C}$ of a triangulated category $\mathcal{T}$ is a *thick* or *épaisse subcategory* if, for $X, Y \in \mathcal{T}$, whenever $X \oplus Y$ is isomorphic in $\mathcal{T}$ to an object in $\mathcal{C}$, both X and Y are objects in $\mathcal{C}$.

This definition of a thick subcategory is equivalent to the original Verdier definition [Ver77], as proved by Rickard [Ric89a, Proposition 1.3]. Verdier's own formulation is that $\mathcal{C}$ is a thick subcategory if and only if there is a triangulated category $\mathcal{T}'$ and a triangulated functor $F : \mathcal{T} \to \mathcal{T}'$ such that $\mathcal{C}$ is the full subcategory of $\mathcal{T}$ consisting of objects X in $\mathcal{T}$ such that $F(X) \cong 0$ in $\mathcal{T}'$. In other words, $\mathcal{C}$ is the the kernel of the triangulated functor F.

Given a full triangulated subcategory $\mathcal{C}$ of $\mathcal{T}$, we denote by $\langle \mathcal{C} \rangle$ the intersection of all the thick subcategories of $\mathcal{T}$ containing $\mathcal{C}$; that is, $\langle \mathcal{C} \rangle$ is the smallest thick subcategory of $\mathcal{T}$ containing $\mathcal{C}$. The subcategory $\langle \mathcal{C} \rangle$ of $\mathcal{T}$ consists of all objects in $\mathcal{T}$ isomorphic to direct summands of objects in $\mathcal{C}$.

We may generalize this definition to any full subcategory of $\mathcal{T}$, or indeed any object or family of objects in $\mathcal{T}$. Let $\mathcal{X}$ be a collection of objects in $\mathcal{T}$. We denote by $\langle \mathcal{X} \rangle$ the smallest thick subcategory of $\mathcal{T}$ containing all the objects in $\mathcal{X}$. We say that $\mathcal{X}$ *generates* $\mathcal{T}$ as a triangulated category if $\langle \mathcal{X} \rangle = \mathcal{T}$.

Strictly, the construction of the thick subcategory $\langle \mathcal{X} \rangle$ of $\mathcal{T}$ involves taking iterative closures of certain full additive subcategories of $\mathcal{T}$ under shifts, direct sums and direct summands. For a precise rendering of this construction, the reader is encouraged to see [Stacks, Section 09SI].

1.2.6 Verdier localisation

Let $\mathcal{T}$ be a triangulated category and $\mathcal{D}$ a triangulated subcategory of $\mathcal{T}$. The *Verdier quotient* of $\mathcal{T}$ by $\mathcal{D}$ is the triangulated category specified by the following theorem of Verdier [Ver77]. In this subsection, we assume our categories are such that we avoid any set theoretic difficulties.

Theorem 1.2.9. *There is a triangulated category $\mathcal{T}/\mathcal{D}$ together with a triangulated functor $F : \mathcal{T} \longrightarrow \mathcal{T}/\mathcal{D}$ such that $\mathcal{D}$ is contained in the kernel of F, and F is universal with this property; that is, if $F' : \mathcal{T} \longrightarrow \mathcal{T}'$ is any triangulated functor whose kernel contains $\mathcal{D}$, then F' factors through F. Moreover, the kernel of F is the thick subcategory $\langle \mathcal{D} \rangle$.*

The object class of the category $\mathcal{T}/\mathcal{D}$ is the object class of $\mathcal{T}$. The morphism sets of $\mathcal{T}/\mathcal{D}$ are the equivalence classes of left $\mathcal{D}$-fractions or $\mathcal{D}$-roofs, as described below. We first need the following definition for an arbitrary category.

Definition 1.2.10. A collection of morphism $\mathcal{Q}$ in a category $\mathcal{C}$ is a *multiplicative system* if the following three conditions hold.

(i) The class $\mathcal{Q}$ is closed under composition and contains all identities and isomorphisms.

(ii) *(Ore condition)* For $t : Z \to Y$ in $\mathcal{Q}$ and $g : X \to Y$ in $\mathcal{C}$, there is a commutative diagram

$$\begin{array}{ccc} W & \xrightarrow{f} & Z \\ \big\downarrow{\scriptstyle s} & & \big\downarrow{\scriptstyle t} \\ X & \xrightarrow{g} & Y \end{array}$$

in $\mathcal{C}$ with $s \in \mathcal{Q}$. Similarly, given maps $s : W \to X$ in and $f : W \to Z$ with $s \in \mathcal{Q}$, there are maps $t : Z \to Y$ and $g : X \to Y$ with $t \in \mathcal{Q}$ making the above diagram commute.

(iii) For two morphisms $f, g : X \to Y$ in $\mathcal{C}$, there is an $s \in \mathcal{Q}$ such that $s \circ f = s \circ g$ if and only if there is a $t \in \mathcal{Q}$ such that $f \circ t = g \circ t$.

Let $\mathcal{T}$ be a triangulated category. Let $\mathcal{Q}_{\mathcal{D}}$ be the collection of morphisms s in $\mathcal{T}$ such that cone(s) is an object in $\mathcal{D}$. Then $\mathcal{Q}_{\mathcal{D}}$ is a multiplicative system in $\mathcal{T}$.

To avoid set theoretic difficulties, we assume that the multiplicative system $\mathcal{Q}_{\mathcal{D}}$ is *locally small.* That is, for every object X of $\mathcal{T}$, there is a set $Q_X = \{h : X' \to X\}$ of morphisms in $\mathcal{Q}_{\mathcal{D}}$ such that for every $X_1 \to X$ in $\mathcal{Q}_{\mathcal{D}}$, there is a morphism $X_2 \to X_1$ in $\mathcal{T}$ such that the composition $X_2 \to X_1 \to X$ is in Q_X. This ensures that, for every pair of objects X and Y in $\mathcal{T}$, the collection $\mathrm{Hom}_{\mathcal{Q}_{\mathcal{D}}}(X, Y)$ of morphisms $X \to Y$ in $\mathcal{Q}_{\mathcal{D}}$ is a set.

A *left $\mathcal{D}$-fraction* or *$\mathcal{D}$-roof* in $\mathcal{T}$ is a chain of morphisms

$$X \xleftarrow{s} W \xrightarrow{f} Y$$

with s in $\mathcal{Q}_{\mathcal{D}}$.

The composition of two $\mathcal{D}$-roofs $X \overset{s}{\longleftarrow} W_1 \overset{f}{\longrightarrow} Y$ and $Y \overset{t}{\longleftarrow} W_2 \overset{g}{\longrightarrow} Z$ is constructed using the Ore condition as below:

$$\begin{array}{ccccccccc} & & & & W & & & & \\ & & & \overset{t'}{\swarrow} & & \overset{f'}{\searrow} & & & \\ & & W_1 & & & & W_2 & & \\ & \overset{s}{\swarrow} & & \overset{f}{\searrow} & & \overset{t}{\swarrow} & & \overset{g}{\searrow} & \\ X & & & & Y & & & & Z \end{array}$$

We note that the composition of two $\mathcal{D}$-roofs is defined only up to isomorphism.

The fractions $X \overset{s}{\longleftarrow} W_1 \overset{f}{\longrightarrow} Y$ and $X \overset{t}{\longleftarrow} W_2 \overset{g}{\longrightarrow} Y$ are *equivalent* if there is a fraction $X \longleftarrow W_3 \longrightarrow Y$ such that the following diagram commutes in $\mathcal{T}$:

$$\begin{array}{ccccc} & & W_1 & & \\ & \overset{s}{\swarrow} & \uparrow & \overset{f}{\searrow} & \\ X & \longleftarrow & W_3 & \longrightarrow & Y. \\ & \underset{t}{\nwarrow} & \downarrow & \underset{g}{\nearrow} & \\ & & W_2 & & \end{array}$$

We denote by fs^{-1} the equivalence class of the roof $X \overset{s}{\longleftarrow} X_1 \overset{f}{\longrightarrow} Y$. Composition behaves nicely with respect to this equivalence relation.

Denote by $\mathrm{Hom}_{\mathcal{T}/\mathcal{D}}(X, Y)$ the equivalence classes of roofs between X and Y. Then the Verdier quotient $\mathcal{T}/\mathcal{D}$ exists, and has morphism sets $\mathrm{Hom}_{\mathcal{T}/\mathcal{D}}(X, Y)$. We may also think of $\mathcal{T}/\mathcal{C}$ as the *Verdier localisation* of $\mathcal{T}$ at the class $\mathcal{Q}_{\mathcal{D}}$, denoted $\mathcal{Q}_{\mathcal{D}}^{-1}\mathcal{T}$. From this point of view, the category $\mathcal{Q}_{\mathcal{D}}^{-1}\mathcal{T}$ is the category we obtain from $\mathcal{T}$ by formally inverting the morphisms in $\mathcal{Q}_{\mathcal{D}}$.

The functor $F : \mathcal{T} \longrightarrow \mathcal{T}/\mathcal{D}$ in Theorem 1.2.9 is such that $F(X) = X$, and if $X \overset{f}{\longrightarrow} Y$ is a morphism in $\mathcal{T}$, then $F(f)$ is the roof $X \overset{\mathrm{id}_X}{\longleftarrow} X \overset{f}{\longrightarrow} Y$. If $f \in \mathcal{Q}_{\mathcal{D}}$, then fs^{-1} is invertible in $\mathcal{T}/\mathcal{D}$ with inverse sf^{-1}. In particular, F maps morphisms in $\mathcal{Q}_{\mathcal{D}}$ to isomorphisms in $\mathcal{T}/\mathcal{C}$.

The triangulated structure of $\mathcal{T}/\mathcal{D}$ is inherited directly from $\mathcal{T}$. The translation functor $[1]$ on $\mathcal{T}/\mathcal{D}$ is exactly the translation functor $[1]$ on $\mathcal{T}$. A distinguished triangle in $\mathcal{T}/\mathcal{D}$ is any triangle isomorphic to the image under F of a distinguished triangle in $\mathcal{T}$.

1.2.7 Derived categories

Let A be a finite-dimensional k-algebra. Recall the homotopy category $K(A)$ from §1.2.3. There is a thick subcategory $\operatorname{Acy}(A)$ whose objects are the acyclic chain complexes of A-modules.

Definition 1.2.11. The *derived category* of A is the Verdier quotient

$$D(A) = K(A)/\operatorname{Acy}(A).$$

Recall from 1.2.3 that a morphism $f : X \to Y$ in $K(A)$ is a quasi-isomorphism if and only if $\operatorname{cone}(f)$ is an acyclic chain complex. Thus, $D(A) = \mathcal{Q}^{-1}K(A)$, where $\mathcal{Q}$ is the class of quasi-isomorphisms in $K(A)$.

Verdier's Theorem 1.2.9 has the following form for the derived category.

Proposition 1.2.12. *There is a triangulated functor $F : K(A) \to D(A)$ satisfying the following two properties.*

(i) Whenever $f : X \to Y$ is a homotopy class of quasi-isomorphisms, $F(f)$ is an isomorphism in $D(A)$.

(ii) If $F' : K(A) \to \mathcal{T}$ is any other functor sending quasi-isomorphisms to isomorphisms, then there is a unique functor $G : D(A) \to \mathcal{D}$ making the following diagram commute:

$$\begin{array}{ccc} K(A) & \xrightarrow{F} & D(A) \\ & \searrow^{F'} & \Big\downarrow{\scriptstyle \exists G} \\ & & \mathcal{T}. \end{array}$$

The triangulated subcategories $D^-(A)$, $D^+(A)$ and $D^b(A)$ of $D(A)$ of bounded below, bounded above, and bounded chain complexes of A-modules can be constructed identically from the triangulated subcategories $K^-(A)$, $K^+(A)$ and $K^b(A)$ of $K(A)$ respectively. We call $D^b(A)$ the *bounded derived category* of A, and it is this category that we will primarily be working with. We denote by $D(A\text{-}B)$ the derived category of chain complexes of A-B-bimodules, for B another k-algebra. We have $D(A\text{-}B) \simeq D(A \otimes_k B^{\mathrm{op}})$ as triangulated categories. There are triangulated functors $D(A\text{-}B) \longrightarrow D(A)$ and $D(A\text{-}B) \longrightarrow D(B^{\mathrm{op}})$, given by restricting to the action on either side.

Given an abelian category $\mathcal{A}$, under some conditions (see [Wei95, Remark 10.4.5]) one can construct the derived category $D(\mathcal{A})$, and its relevant subcategories[16], in much the same way. When $D(\mathcal{A})$ exists, the abelian category

[16] For $D^-(\mathcal{A})$ to exist, we require that $\mathcal{A}$ has enough projectives, while for $D^+(\mathcal{A})$ to

$\mathcal{A}$ embeds as a full a subcategory of $D(\mathcal{A})$, with objects of $\mathcal{A}$ viewed as chain complexes concentrated in degree 0.

Quasi-isomorphisms of chain complexes do not in general have inverses, hence the introduction of a method of *formal* inversion via roofs. Direct calculation in the derived category $D(\mathcal{A})$ can therefore be tricky. Fortunately, when $\mathcal{A} = A\text{-mod}$ for A a finite-dimensional algebra over a field k, for bounded complexes of projective A-modules, we have the following result (see [Wei95, Corollary 10.4.7]).

Proposition 1.2.13. *Let P be a bounded below chain complex of projective A-modules and X an arbitrary chain complex of A-modules. Then*

$$\mathrm{Hom}_{D(A)}(P, X) \cong \mathrm{Hom}_{K(A)}(P, X).$$

Further, we have (see [Wei95, Theorem 10.4.8]) an equivalence of triangulated categories

$$D^-(A) \simeq K^-(A\text{-proj}).$$

We may realise $D^b(A)$ as the subcategory $K^{-,b}(A\text{-proj})$ of $K^-(A\text{-proj})$ of bounded below chain complexes of projective A-modules with *bounded homology*; that is, objects X of $K^-(A\text{-proj})$ such that $H_n(X) = 0$ for all but finitely many n. We thus have a triangulated subcategory $K^b(A\text{-proj}) \subset D^b(A)$. In this way, we may consider a chain complex X which is unbounded above but with bounded homology to be an object in the bounded derived category $D^b(A)$.

We call an object X in $D(A)$ a *perfect* complex if it is isomorphic in $D(A)$ to a bounded chain complex of finitely generated projective A-modules; that is, when X is isomorphic to an object of $K^b(A\text{-proj})$. Perfect complexes are the *compact* objects[17] of $D(A)$. The perfect complexes form a triangulated subcategory $\mathrm{Perf}(A)$ of $D(A)$ (actually of $D^b(A)$).

Remark 1.2.14. This definition of the derived category of A is slightly non-standard. Typically, one defines the derived category of A to be $D(A\text{-Mod})$, where A-Mod is the abelian category of all A-modules, not just finitely generated ones. We note that the natural embedding $A\text{-mod} \hookrightarrow A\text{-Mod}$ induces a full embedding $D(A\text{-mod}) \hookrightarrow D(A\text{-Mod})$. Thus, although the statements in §1.2.8 and §1.2.9 are generally given on $D(A\text{-Mod})$, for example in

exist, we require that $\mathcal{A}$ have enough injectives, see [Wei95, Theorem 10.4.8]. This is always the case when $\mathcal{A} = A\text{-mod}$, for A a finite-dimensional algebra over a field k.

[17] The compact objects of a k-linear, triangulated category $\mathcal{T}$ are those objects X for which the functor $\mathrm{Hom}_{\mathcal{T}}(X, -) : \mathcal{T} \to k\text{-mod}$ commutes with coproducts (in our case, direct sums).

[Ric91], for finite-dimensional algebras over a field, they apply just as well to $D(A) = D(A\text{-mod})$.

1.2.8 Functors between derived categories

Recall that there is a right exact bifunctor

$$- \otimes_B - : \mathrm{Ch}^-(A\text{-}B) \times \mathrm{Ch}^-(B\text{-}C) \to \mathrm{Ch}^-(A\text{-}C).$$

This bifunctor respects chain homotopy, translation and mapping cones, so extends to a triangulated bifunctor

$$- \otimes_B - : K^-(A\text{-}B) \times K^-(B\text{-}C) \to K^-(A\text{-}C).$$

In general, however, the tensor product bifunctor does not extend nicely to the derived category. This is because the tensor product of chain complexes is not well defined up to quasi-isomorphism.

A functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ between abelian categories extends to a triangulated functor $F : D^-(\mathcal{A}) \longrightarrow D^-(\mathcal{B})$ between their derived categories if F is exact. If F is left exact, we may take the *total right derived functor*

$$\mathbf{R}F : D^-(\mathcal{A}) \longrightarrow D^-(\mathcal{B}),$$

and if F is right exact, we may take the *total left derived functor*

$$\mathbf{L}F : D^-(\mathcal{A}) \longrightarrow D^-(\mathcal{B}),$$

see [Wei95, Definition 10.5.1]. The functors $\mathbf{R}F$ and $\mathbf{L}F$ restrict to triangulated functors between the bounded derived categories $D^b(\mathcal{A})$ and $D^b(\mathcal{B})$.

Exactness of the tensor product functor between module categories is determined by projectivity of the modules. Let $X \in D^-(A\text{-}B)$ and $Y \in D^-(B\text{-}C)$. When X is isomorphic in $D^b(B^{\mathrm{op}})$ to a bounded below complex of projective B^{op}-modules, we have a triangulated functor

$$X \otimes_B - : D^-(B\text{-}C) \to D^-(A\text{-}C),$$

and when Y is isomorphic to a bounded below complex of projective B-modules, a triangulated functor

$$- \otimes_B Y : D^-(A\text{-}B) \to D^-(A\text{-}C).$$

Otherwise, we take the total left derived functors of the right exact tensor product functors, giving a bifunctor

$$- \otimes^{\mathbf{L}}_B - : D^-(A\text{-}B) \times D^-(B\text{-}C) \longrightarrow D^-(A\text{-}C)$$

called the *derived tensor product.*

Given objects X in $D^-(A\text{-}B)$ and Y in $D^-(B\text{-}C)$, to explicitly construct the object $X \otimes^{\mathbf{L}}_B Y$ of $D^-(A\text{-}C)$, one takes a projective resolution P of X, or Q of Y, and sets $X \otimes^{\mathbf{L}}_B Y$ to be $P \otimes_B Q$, $P \otimes_B Y$ or $X \otimes_B Q$. All three are isomorphic in $D(A\text{-}C)$. In this way, the object $X \otimes^{\mathbf{L}}_B Y$ of $D^-(A\text{-}C)$ is unambiguously defined.

By a similar process, the bifunctor

$$\mathrm{Hom}_A(-,-) : \mathrm{Ch}^+(A\text{-}B) \times \mathrm{Ch}^-(A\text{-}C) \to \mathrm{Ch}^-(B\text{-}C),$$

left exact in both arguments, covariant in the right argument and contravariant in the left, induces a bifunctor

$$\mathbf{R}\,\mathrm{Hom}_A(-,-) : D^+(A\text{-}B) \times D^-(A\text{-}C) \to D^-(B\text{-}C).$$

The usual tensor-Hom adjunction extends to the derived functors on the derived category. For objects X in $D^+(A\text{-}B)$, Y in $D^+(B\text{-}C)$ and Z in $D^-(A\text{-}C)$, we have

$$\mathbf{R}\,\mathrm{Hom}_A(X \otimes^{\mathbf{L}}_B Y, Z) \cong \mathbf{R}\,\mathrm{Hom}_B(Y, \mathbf{R}\,\mathrm{Hom}_A(X, Z))$$

and

$$\mathbf{R}\,\mathrm{Hom}_{C^{\mathrm{op}}}(X \otimes^{\mathbf{L}}_B Y, Z) \cong \mathbf{R}\,\mathrm{Hom}_{B^{\mathrm{op}}}(X, \mathbf{R}\,\mathrm{Hom}_{C^{\mathrm{op}}}(Y, Z)).$$

1.2.9 Derived equivalence

Let A and B be finite-dimensional k-algebras.

Definition 1.2.15. A *derived equivalence* between A and B is an equivalence of triangulated categories

$$F : D(A) \xrightarrow{\sim} D(B).$$

The algebras A and B are *derived equivalent* if such an equivalence exists.

A derived equivalence F as above descends to

$$F : D^-(A) \xrightarrow{\sim} D^-(B)$$

and

$$F : D^b(A) \xrightarrow{\sim} D^b(B).$$

In fact, the categories $D^-(A)$ and $D^-(B)$ are equivalent as triangulated categories if and only if the categories $D^b(A)$ and $D^b(B)$ are, [Ric89b, Theorem 6.4].

It is obvious that Morita equivalent algebras are derived equivalent.

Recall in §1.1.5 that a Morita equivalence preserves a number of properties of an algebra and its modules. A derived equivalence also preserves a number of important properties, albeit fewer. Let

$$\Phi : D(A) \xrightarrow{\sim} D(B)$$

be a derived equivalence. Then (see [Ric91, Corollary 5.3]) if A is symmetric, then B is also symmetric.

Further, the number of simple A-modules is preserved, although Φ need not send simple A-modules to simple B-modules. Again, one can see this at the level of the Grothendieck group. For a triangulated category $\mathcal{T}$, the Grothendieck group $K_0(\mathcal{T})$ of $\mathcal{T}$ is the free abelian group on symbols $[X]$ for objects X of $\mathcal{T}$, subject to the relation that

$$[Y] = [X] + [Z]$$

whenever

$$X \longrightarrow Y \longrightarrow Z \rightsquigarrow$$

is a distinguished triangle in $\mathcal{T}$. A triangulated functor

$$F : \mathcal{T} \to \mathcal{T}'$$

induces an abelian group homomorphism

$$[F] : K_0(\mathcal{T}) \to K_0(\mathcal{T}'),$$

and F is an equivalence of triangulated categories if and only if $[F]$ is an isomorphism of abelian groups.

Here, the full embedding

$$A\text{-mod} \hookrightarrow D(A)$$

induces an isomorphism of abelian groups[18]

$$K_0(A) \xrightarrow{\sim} K_0(D(A)).$$

[18] This is a highly non-trivial statement, but the proof is not too important for us.

In particular, $K_0(D(A))$ captures the number of isomorphism classes of simple A-modules. The derived equivalence Φ induces isomorphisms

$$K_0(A) \xrightarrow{\sim} K_0(D(A)) \xrightarrow{\sim} K_0(D(B)) \xrightarrow{\sim} K_0(B),$$

so that A and B must have the same number of simple modules. However, $[\Phi]$ does not preserve simple modules in the derived categories. It is an interesting problem, given a derived equivalence Φ, to determine the images of the simple A-modules in $D(B)$. Work of Rickard and Okuyama, among others, has determined in certain situations objects in $D(B)$ that behave like simple modules, and are thus candidate images of simple modules under a derived equivalence.

Rickard introduced a Morita theory for derived categories in [Ric89b]. Here, the role played by the progenerator in Theorem 1.1.2 is played by the tilting complex.

Definition 1.2.16. A *one-sided tilting complex* for A is an object X of $K^b(A\text{-proj})$ which satisfies:

- $\mathrm{Hom}_{D^-(A)}(X, X[i]) = 0$ whenever $i \neq 0$;
- X-add generates $K^b(A\text{-proj})$ as a triangulated category.

Rickard's result, [Ric89b, Theorem 6.4] is the following.

Theorem 1.2.17. *The following are equivalent.*

(i) There is a derived equivalence $D^-(A) \xrightarrow{\sim} D^-(B)$.

(ii) There is a one-sided tilting complex X for A such that

$$\mathrm{End}_{D^-(A)}(X)^{\mathrm{op}} \cong B$$

as k-algebras.

A ubiquitous class of one-sided tilting complexes for symmetric algebras is defined as follows.

Let A be a finite-dimensional symmetric k-algebra A. Let $\{S_1, \ldots, S_n\}$ be a set of representatives of isomorphism classes of simple A-modules, and $\{P_1, \ldots, P_n\}$ the projective indecomposable A-modules, such that $P_i/\operatorname{rad}(P_i) \cong S_i$. Let $I = \{1, \ldots, n\}$. Let $J \subset I$.

Let M be an A-module. Take a projective cover $P(M) \xrightarrow{\pi_M} M$ of M. Denote by M_J the largest quotient of $P(M)$ by a submodule of $\ker(\pi_M)$ such that all composition factors of the kernel of the induced map $M_J \longrightarrow M$ are in J. Let $Q_{M,J} \overset{\pi_{M,J}}{\longrightarrow} \ker(\phi_{M,J})$ be a projective cover of the kernel of the canonical map $P(M) \overset{\phi_{M,J}}{\longrightarrow} M_J$.

Definition 1.2.18. Given $J \subset I$, the *combinatorial tilting complex at* J is the complex

$$T = \bigoplus_{j \in J} T_j \oplus \bigoplus_{i \in I \setminus J} P_i[1],$$

where, for $j \in J$, T_j is the complex

$$0 \longrightarrow Q_{S_j,J} \longrightarrow P_j \longrightarrow 0,$$

concentrated in degrees 1 and 0.

We note that Grant [Gra13, Definition 5.4] allows $T = \bigoplus_{i \in I} T_i^{\ell_i}$, where $\ell_i \geq 1$ for all i, with $T_i = P_i[1]$ for $i \in I \setminus J$. The complex T with $\ell_i = 1$ is the *basic* combinatorial tilting complex at J.

Typically, one takes J to be a proper, non-empty subset of I. This definition is still valid when $J = \emptyset$ or $J = I$, though such cases are less interesting.

When J is such that $\operatorname{Ext}^1_A(S_i, S_j) = 0$ for every $i, j \in J$, then for $j \in J$, the complex T_j is

$$P(\operatorname{rad}(P_j)) \overset{\pi}{\longrightarrow} P_j,$$

where $P(\operatorname{rad}(P_j)) \overset{\pi}{\longrightarrow} \operatorname{rad}(P_j)$ is a projective cover of $\operatorname{rad}(P_i)$. That is, $P(\operatorname{rad}(P_j)) = \bigoplus_{i \in I \setminus J} P_i \otimes_k \operatorname{Ext}^1_A(S_i, S_j)$, where $\operatorname{Ext}^1_A(S_i, S_j)$ is the *multiplicity module.*

Combinatorial tilting complexes were introduced by Rickard [Ric88] for J a single index, generalised to arbitrary subsets J by Okuyama [Oku97]. They are also called *Okuyama-Rickard two-term tilting complexes*, among other names. Combinatorial tilting complexes have wide-reaching applications, for example in silting theory [AI12] and cluster tilting theory [Bua+04].

Given a subset J, the basic combinatorial tilting complex T at J exists and is unique up to isomorphism, [Gra13, Lemma 5.5, Corollary 5.7]. Further. T is a tilting complex, [Oku97, Proposition 1.1], [Gra13, Proposition 5.6]. Thus, by Theorem 1.2.17, given a combinatorial tilting complex T, there is an algebra $B = \operatorname{End}_{D^b(A)}(T)^{\mathrm{op}}$ such that there is a derived equivalence $F_J : D^b(A) \xrightarrow{\sim} D^b(B)$. We call F_J the *combinatorial tilt of* A *at* J.

It is often beneficial for a derived equivalence to be coming from a complex of bimodules, rather than one-sided modules.

A *standard derived equivalence* between A and B is one of the form

$$X \otimes^{\mathbf{L}}_A - : D^-(A) \xrightarrow{\sim} D^-(B),$$

where X is an object of $D^b(B\text{-}A)$. The object X of $D^b(B\text{-}A)$ is a *two-sided tilting complex* if X induces a standard derived equivalence as above. A standard derived equivalences descends to an equivalence

$$X \otimes^{\mathbf{L}}_A - : D^b(A) \xrightarrow{\sim} D^b(B),$$

where here we must treat $D^b(A)$ as the homotopy category of bounded above complexes of projective A-modules with bounded homology, $K^{-,b}(A)$.

The following theorem, [Ric91, Corollary 3.5], tells us that we can always replace a derived equivalence by a standard derived equivalence, and it will behave the same on objects.

Theorem 1.2.19. *Let*

$$F : D^-(A) \xrightarrow{\sim} D^-(B)$$

be a derived equivalence. Then there is a standard derived equivalence

$$X \otimes^{\mathbf{L}}_A - : D^-(A) \xrightarrow{\sim} D^-(B),$$

that agrees with F on A-proj*, and is such that $F(Y) \cong X \otimes^{\mathbf{L}}_A Y$ for every object Y of $D^-(A)$.*

We have that, if X is a two-sided tilting complex, then X is perfect in $D(A^{\mathrm{op}})$ and $D(B)$. Further [Ric91, Definition 4.2], X is a two-sided tilting complex if and only if there is an object $\tilde{X}$ of $D^b(A\text{-}B)$ such that

$$X \otimes^{\mathbf{L}}_A \tilde{X} \cong B \text{ in } D^b(B\text{-}B)$$

and

$$\tilde{X} \otimes^{\mathbf{L}}_B X \cong A \text{ in } D^b(A\text{-}A).$$

The object $\tilde{X}$ is called the *inverse* of X. It is itself a two-sided tilting complex, inducing a standard derived equivalence

$$\tilde{X} \otimes^{\mathbf{L}}_B - : D^-(B) \xrightarrow{\sim} D^-(A).$$

1.2.10 The stable category

Let A be a finite-dimensional k-algebra. The *stable module category*, or simply *stable category*, A-$\underline{\mathrm{mod}}$ of A is obtained from the module category A-mod by factoring out the projective modules. More formally, $A\text{-}\underline{\mathrm{mod}}$ is the k-linear category whose object class is the class of all A-modules, and with morphisms

$$\underline{\mathrm{Hom}}_A(M,N) = \mathrm{Hom}_A(M,N)/\,\mathrm{PHom}_A(M,N),$$

where $\mathrm{PHom}_A(M,N)$ is the space of A-module homomorphisms $M \to N$ which factor through a projective module.

Given an A-module M, recall that the Heller translate $\Omega_A(M)$ of M is the kernel

$$0 \longrightarrow \Omega_A(M) \hookrightarrow P \overset{\pi}{\twoheadrightarrow} M \longrightarrow 0,$$

where $P \overset{\pi}{\twoheadrightarrow} M$ is a projective cover of M. If M is an A-module with no projective direct summands, we have $\Omega_A(M) \cong \Omega_{A\otimes_k A^{\mathrm{op}}}(A) \otimes_A M$.

Similarly, the inverse Heller translate $\Omega_A^{-1}(M)$ is the cokernel

$$0 \longrightarrow M \overset{\iota}{\hookrightarrow} I \twoheadrightarrow \Omega_A^{-1}M \longrightarrow 0$$

where $M \overset{\iota}{\hookrightarrow} I$ is an injective hull of M. If M is an A-module with no injective summands, we have $\Omega_A^{-1}(M) \cong \Omega^{-1}_{A\otimes_k A^{\mathrm{op}}}(A) \otimes_A M$.

When A is self-injective, we thus obtain endofunctors

$$\Omega_A = \Omega_{A\otimes_k A^{\mathrm{op}}}(A) \otimes_A - : A\text{-}\underline{\mathrm{mod}} \overset{\sim}{\longrightarrow} A\text{-}\underline{\mathrm{mod}}$$

and

$$\Omega_A^{-1} = \Omega^{-1}_{A\otimes_k A^{\mathrm{op}}}(A) \otimes_A - : A\text{-}\underline{\mathrm{mod}} \overset{\sim}{\longrightarrow} A\text{-}\underline{\mathrm{mod}}\,.$$

Moreover, the functors Ω_A and Ω_A^{-1} are mutually inverse k-linear autoequivalences of $A\text{-}\underline{\mathrm{mod}}$.

Assume for the remainder of this section that A is a self-injective algebra. The stable category $A\text{-}\underline{\mathrm{mod}}$ is a triangulated category. The translation functor is the inverse Heller translate $T(M) = \Omega_A^{-1}(M)$. Exact triangles can be described as follows.

First, we have isomorphisms (see e.g. [Lin18, Propositions 4.13.7, 2.5.15])

$$\underline{\mathrm{Hom}}_A(\Omega_A(N), M) \cong \mathrm{Ext}^1_A(N,M) \cong \underline{\mathrm{Hom}}_A(N, \Omega_A^{-1}(M))$$

for any two A-modules M, N. Thus, an element of $\mathrm{Ext}^1_A(N,M)$, that is, an exact sequence

$$0 \to M \to E \to N \to 0$$

in A-mod, gives rise to an element of $\underline{\mathrm{Hom}}_A(N, \Omega_A^{-1}(M))$. Hence, we obtain a sequence

$$M \to E \to N \to \Omega_A^{-1}(M).$$

Sequences of this form form the class of exact triangles in $A\text{-}\underline{\mathrm{mod}}$.

An explicit relationship between the stable category and the derived category is due to Rickard, [Ric89a, Theorem 2.1]. Recall that there is a triangulated subcategory $\mathrm{Perf}(A)$ of $D^b(A)$ whose objects are the perfect objects in $D(A)$. One can therefore form the Verdier quotient

$$D^b(A)/\,\mathrm{Perf}(A).$$

The diagram

$$\begin{array}{ccc} D^b(A) & \longrightarrow & D^b(A)/\,\mathrm{Perf}(A) \\ \uparrow & & \uparrow \\ A\text{-mod} & \longrightarrow & A\text{-}\underline{\mathrm{mod}} \end{array}$$

induces an equivalence of triangulated categories

$$A\text{-}\underline{\mathrm{mod}} \xrightarrow{\sim} D^b(A)/\,\mathrm{Perf}(A).$$

Following this diagram, given an object $X \in D^b(A)$, if $\overline{X}$ is the image of X under the quotient map

$$D^b(A) \longrightarrow D^b(A)/\,\mathrm{Perf}(A),$$

then there is an A-module M such that $M \cong \overline{X}$ in $A\text{-}\underline{\mathrm{mod}}$, under the above equivalence. We will freely identify these two objects in $A\text{-}\underline{\mathrm{mod}}$ in Chapter 2.

If B is another k-algebra, a *stable equivalence* between A and B is an equivalence of triangulated categories

$$F : A\text{-}\underline{\mathrm{mod}} \xrightarrow{\sim} B\text{-}\underline{\mathrm{mod}}\,.$$

We say A and B are *stably equivalent.*

An immediate consequence of the above result of Rickard is that, if A and B are derived equivalent, then A and B are stably equivalent. In particular, [Ric91, Corollary 5.5], when A and B are derived equivalent, there is an A-B-bimodule M and a B-A-bimodule N such that the functors

$$N \otimes_A - : A\text{-mod} \to B\text{-mod}$$

and

$$M \otimes_B - : B\text{-mod} \to A\text{-mod}$$

induce mutually inverse equivalences of the stable categories $A\text{-}\underline{\text{mod}}$ and $B\text{-}\underline{\text{mod}}$.

This leads to the definition of a stable equivalence *of Morita type*, due to Broué, [Bro94]. Suppose the A-B-bimodule M and the B-A-bimodule N are such that the following all hold:

(i) M and N are projective as left and as right modules;

(ii) $M \otimes_B N \cong A \oplus P$ as A-A-bimodules, where P is a projective A-A-bimodule;

(iii) $N \otimes_A M \cong B \oplus Q$ as B-B-bimodules, where Q is a projective B-B-bimodule.

Then the mutually inverse equivalences of the stable categories induced by the functors $N \otimes_A -$ and $M \otimes_B -$ are *stable equivalences of Morita type.*

1.3 Perverse Equivalences

Perverse equivalences, introduced by Chuang and Rouquier (described in full in [CR17], earlier described in [Rou06, §2.6]), are a subclass of derived equivalences, that are filtered by shifted equivalences on successive quotients of the underlying abelian category. They thus come with desirable combinatorial properties. In the block theory of finite-dimensional symmetric algebras, perverse equivalences appear to be foundational in constructing derived equivalences. They seem to have particular significance in the context of finite groups, especially finite groups of Lie type, as we will discuss in §3.1.2.

1.3.1 Definitions

We shall not reproduce here the definition of Chuang and Rouquier in full generality, [CR17, Definition 4.1], but rather a definition for abelian categories.

We require some preliminary definitions, analogous to the notion of Verdier localisation in §1.2.6, for abelian categories. Details on the following can be found in [Stacks, Section 02MN].

Definition 1.3.1. Let $\mathcal{A}$ be an abelian category. A full abelian subcategory $\mathcal{B}$ of $\mathcal{A}$ is a *Serre subcategory* if whenever

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

is an exact sequence in $\mathcal{A}$, the object M belongs to $\mathcal{B}$ if and only if both M' and M'' belong to $\mathcal{B}$.

Kernels of exact functors between abelian categories are Serre subcategories. That is, given abelian categories $\mathcal{A}$ and $\mathcal{A}'$ and an exact functor $F : \mathcal{A} \to \mathcal{A}'$, the full subcategory of objects X of $\mathcal{A}$ such that $F(X) = 0$ is a Serre subcategory of $\mathcal{A}$.

In fact, all Serre subcategories occur as kernels of exact functors between abelian categories.

Lemma 1.3.2. *Let $\mathcal{A}$ be an abelian category and $\mathcal{B}$ a Serre subcategory of $\mathcal{A}$. There is an abelian category $\mathcal{A}/\mathcal{B}$ and an essentially surjective exact functor $F : \mathcal{A} \to \mathcal{A}/\mathcal{B}$ such that $\mathcal{B}$ is the kernel of F, and for any exact functor $G : \mathcal{A} \to \mathcal{A}'$ whose kernel contains $\mathcal{B}$, there is a unique exact functor $H : \mathcal{A}/\mathcal{B} \to \mathcal{A}'$ such that $G = H \circ F$.*

Lemma 1.3.2 is an abelian category analogue of Verdier's Theorem 1.2.9 for triangulated categories.

Definition 1.3.3. The abelian category $\mathcal{A}/\mathcal{B}$ in Lemma 1.3.2 is the *Serre quotient* of $\mathcal{A}$ by $\mathcal{B}$.

The Serre quotient $\mathcal{A}/\mathcal{B}$ can be characterised as follows. The objects of $\mathcal{A}/\mathcal{B}$ are the objects of $\mathcal{A}$. Morphism sets $\mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(X, Y)$ are the sets of equivalence classes of left $\mathcal{B}$-roofs or $\mathcal{B}$-fractions, as follows.

The collection $\mathcal{Q}_{\mathcal{B}}$ of morphisms s in $\mathcal{A}$ such that $\ker(s)$ and $\mathrm{coker}(s)$ are in $\mathcal{B}$ is a multiplicative system in $\mathcal{A}$, as per Definition 1.2.10. A left $\mathcal{B}$-roof or $\mathcal{B}$-fraction is a diagram

$$X \xleftarrow{s} W \xrightarrow{f} Y$$

with s in $\mathcal{Q}_{\mathcal{B}}$. Two roofs $X \xleftarrow{s} W \xrightarrow{f} Y$ and $X \xleftarrow{t} W' \xrightarrow{g} Y$ are equivalent if there is a map $W' \longrightarrow W$ making the following diagram commute:

$$\begin{array}{ccccc} & & W' & & \\ & \swarrow^{t} & \downarrow & \searrow^{g} & \\ X & \xleftarrow{s} & W & \xrightarrow{f} & Y \end{array}$$

Composition of roofs follows the Ore condition, as in §1.2.6. Again, we can think of $\mathcal{A}/\mathcal{B}$ as the localisation $\mathcal{Q}_{\mathcal{B}}^{-1}\mathcal{A}$ of $\mathcal{A}$ at the multiplicative system $\mathcal{Q}_{\mathcal{B}}$.

Now, let $D^b(\mathcal{A})$ be the bounded derived category of the abelian category $\mathcal{A}$. If $\mathcal{B}$ is a Serre subcategory of $\mathcal{A}$, then we define $D^b_{\mathcal{B}}(\mathcal{A})$ to be $\langle \mathcal{B} \rangle$, the thick subcategory of $D^b(\mathcal{A})$ generated by $\mathcal{B}$. Then, since $\mathcal{B}$ is a Serre subcategory, as in [CR17, §4.2.2], $D^b_{\mathcal{B}}(\mathcal{A})$ is precisely the thick subcategory of $D^b(\mathcal{A})$ whose objects are the complexes with homology contained entirely in $\mathcal{B}$.

Let $\mathcal{A}$ and $\mathcal{A}'$ be abelian categories. Suppose we have filtrations

$$0 = \mathcal{A}_0 \subset \mathcal{A}_1 \subset \ldots \subset \mathcal{A}_r = \mathcal{A},$$
$$0 = \mathcal{A}'_0 \subset \mathcal{A}'_1 \subset \ldots \subset \mathcal{A}'_r = \mathcal{A}'$$

by Serre subcategories, and let $p : \{1, \ldots, r\} \to \mathbb{Z}$ be a function.

Definition 1.3.4. A derived equivalence $\Phi : D^b(\mathcal{A}) \xrightarrow{\sim} D^b(\mathcal{A}')$ is *perverse relative to* $(\mathcal{A}_\bullet, \mathcal{A}'_\bullet, p)$ if

- the functor Φ restricts to an equivalence $D^b_{\mathcal{A}_i}(\mathcal{A}) \xrightarrow{\sim} D^b_{\mathcal{A}'_i}(\mathcal{A}')$ of triangulated categories for every i, and
- for every i, $\Phi[p(i)]$ induces an equivalence $\mathcal{A}_i/\mathcal{A}_{i-1} \xrightarrow{\sim} \mathcal{A}'_i/\mathcal{A}'_{i-1}$ of abelian categories.

We call p the *perversity function*.

Perverse equivalences occur naturally in the representation theory of finite dimensional algebras. Standard examples can be found in Brauer tree algebras (see §3.1.5) and blocks of finite groups of Lie type (see §3.1.2).

An immediate consequence of the definition is the following, [CR17, Lemma 4.16].

Lemma 1.3.5. *If $\Phi : D^b(\mathcal{A}) \xrightarrow{\sim} D^b(\mathcal{A}')$ is perverse relative to $(\mathcal{A}_\bullet, \mathcal{A}'_\bullet, p)$, with perversity function $p \equiv 0$, then we have an equivalence $\mathcal{A} \xrightarrow{\sim} \mathcal{A}'$.*

Another useful fact, clear from the definition, is the following, [CR17, Lemma 4.2].

Lemma 1.3.6. *If $\Phi : D^b(\mathcal{A}) \xrightarrow{\sim} D^b(\mathcal{A}')$ is perverse relative to $(\mathcal{A}_\bullet, \mathcal{A}'_\bullet, p)$, then the inverse $\Phi^{-1} : D^b(\mathcal{A}') \xrightarrow{\sim} D^b(\mathcal{A})$ is perverse relative to $(\mathcal{A}'_\bullet, \mathcal{A}_\bullet, -p)$.*

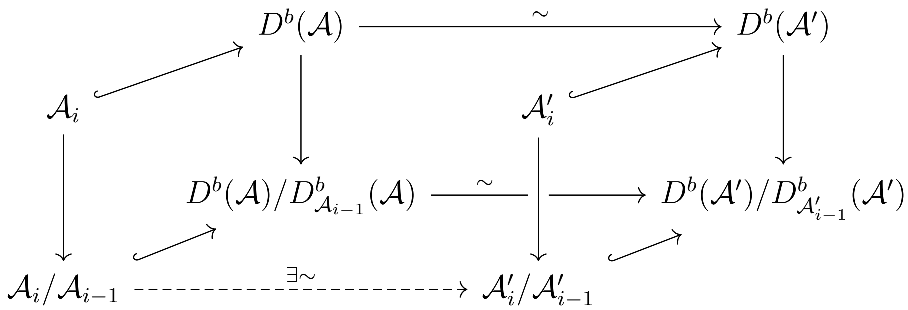
We can reframe the final clause of the definition diagrammatically, following Grant [Gra13, Remark 3.21]. For each i, restricting Φ gives an equivalence $D^b_{\mathcal{A}_i}(\mathcal{A}) \xrightarrow{\sim} D^b_{\mathcal{A}'_i}(\mathcal{A}')$, from which we obtain an equivalence

$$\Phi[p(i)] : D^b(\mathcal{A})/D^b_{\mathcal{A}_{i-1}}(\mathcal{A}) \xrightarrow{\sim} D^b(\mathcal{A}')/D^b_{\mathcal{A}'_{i-1}}(\mathcal{A}'),$$

which we require to fit into a commutative diagram:

$$\begin{array}{ccc} D^b(\mathcal{A})/D^b_{\mathcal{A}_{i-1}}(\mathcal{A}) & \xrightarrow{\sim} & D^b(\mathcal{A}')/D^b_{\mathcal{A}'_{i-1}}(\mathcal{A}') \\ \uparrow & & \uparrow \\ \mathcal{A}_i/\mathcal{A}_{i-1} & \overset{\exists\sim}{\dashrightarrow} & \mathcal{A}'_i/\mathcal{A}'_{i-1} \end{array}$$

To justify the vertical arrows, this commutative diagram can be seen to sit inside a larger one:



Here, the vertical arrows are quotients given by localisation, and the embedding $\mathcal{A}_i \hookrightarrow D^b(\mathcal{A})$ is via the usual embedding of $\mathcal{A}$ in $D^b(\mathcal{A})$.

The functor $\mathcal{A}_i/\mathcal{A}_{i-1} \hookrightarrow D^b(\mathcal{A})/D^b_{\mathcal{A}_{i-1}}(\mathcal{A})$ exists and is fully faithful by the universal property of the quotient $\mathcal{A}_i \longrightarrow \mathcal{A}_i/\mathcal{A}_{i-1}$. An identical argument works on the other side.

1.3.2 Composition of perverse equivalences

In general, the composition of two perverse equivalences need not remain perverse; we will see an example of this in §3.2. In certain circumstances, however, perversity upon composition is guaranteed.

Firstly, the composition of two perverse equivalences at a fixed middle filtration is perverse, [CR17, Lemma 4.4].

Lemma 1.3.7. *Let $\mathcal{A}$, $\mathcal{A}'$ and $\mathcal{A}''$ be abelian categories. Suppose we have an equivalence $F : D^b(\mathcal{A}) \xrightarrow{\sim} D^b(\mathcal{A}')$, perverse relative to $(\mathcal{A}_\bullet, \mathcal{A}'_\bullet, p)$, and an equivalence $F' : D^b(\mathcal{A}') \xrightarrow{\sim} D^b(\mathcal{A}'')$, perverse relative to $(\mathcal{A}'_\bullet, \mathcal{A}''_\bullet, p')$. Then the composition $F' \circ F$ is perverse relative to $(\mathcal{A}_\bullet, \mathcal{A}''_\bullet, p + p')$.*

Lemmas 1.3.7, 1.3.6 and 1.3.5 then produce the following result, [CR17, Proposition 4.17].

Proposition 1.3.8. *Suppose we have equivalences $F : D^b(\mathcal{A}) \xrightarrow{\sim} D^b(\mathcal{A}')$, perverse relative to $(\mathcal{A}_\bullet, \mathcal{A}'_\bullet, p)$, and $F' : D^b(\mathcal{A}) \xrightarrow{\sim} D^b(\mathcal{A}'')$, perverse relative to $(\mathcal{A}_\bullet, \mathcal{A}''_\bullet, p)$. Then the composition $F' \circ F^{-1}$ is perverse relative to $(\mathcal{A}_\bullet, \mathcal{A}_\bullet, \bar{p})$, with $\bar{p} \equiv 0$, and thus induces an equivalence $\mathcal{A}' \xrightarrow{\sim} \mathcal{A}''$.*

In other words, the filtration on the left hand side and the perversity function p completely determine the equivalence class of the abelian category $\mathcal{A}'$.

1.3.3 Simple modules

We confine our discussion now to when the abelian categories in question are module categories of finite-dimensional symmetric algebras over a field. For the remainder of this section, let k be a field, and let A and A' be finite-dimensional symmetric k-algebras.

Set $\mathcal{A} = A$-mod and $\mathcal{A}' = A'$-mod. Suppose there exists a derived equivalence $\Phi : D^b(A) \xrightarrow{\sim} D^b(A')$, and let $\mathcal{S} = \{S_1, \ldots, S_n\}$ and $\mathcal{S}' = \{S'_1, \ldots, S'_n\}$ denote complete sets of non-isomorphic simple A- and A'-modules, respectively. The Serre subcategories of $\mathcal{A}$ and $\mathcal{A}'$ are in one-to-one correspondence with the finite subsets of $\mathcal{S}$ and $\mathcal{S}'$, respectively. In particular, given a finite subset $\mathcal{S}_1 \subset \mathcal{S}$, the full subcategory $\mathcal{A}_1$ of $\mathcal{A}$, whose objects are the A-modules with composition factors are all in $\mathcal{S}_1$, is a Serre subcategory, called the Serre subcategory *generated by* $\mathcal{S}_1$.

Suppose we have filtrations

$$\emptyset = \mathcal{S}_0 \subset \mathcal{S}_1 \subset \ldots \subset \mathcal{S}_r = \mathcal{S},$$
$$\emptyset = \mathcal{S}'_0 \subset \mathcal{S}'_1 \subset \ldots \subset \mathcal{S}'_r = \mathcal{S}'$$

on the sets of simple modules. For each i, let $\mathcal{A}_i$ be the Serre subcategory of A-mod generated by $\mathcal{S}_i$ and $\mathcal{A}'_i$ the Serre subcategory of A'-mod generated by $\mathcal{S}'_i$.

Definition 1.3.9. We say the equivalence $\Phi : D^b(A) \xrightarrow{\sim} D^b(A')$ is *perverse relative to* $(\mathcal{S}_\bullet, \mathcal{S}'_\bullet, p)$ if it is perverse relative to $(\mathcal{A}_\bullet, \mathcal{A}'_\bullet, p)$.

We call the information $(\mathcal{S}_\bullet, \mathcal{S}'_\bullet, p)$ the *perversity data* for Φ.

For simplicity, we may take an indexing set $I = \{1, \ldots, n\}$ of $\mathcal{S}$ and set I_i to be the subset of I corresponding to the indices of the simple modules in $\mathcal{S}_i$, and I'_i the subset of I corresponding to the indices of the simple modules in $\mathcal{S}'_i$. Then the filtrations can be written as

$$\emptyset = I_0 \subset I_1 \subset \ldots \subset I_r = I,$$
$$\emptyset = I'_0 \subset I'_1 \subset \ldots \subset I'_r = I.$$

One may then say that Φ is perverse relative to $(I_\bullet, I'_\bullet, p)$.

We can rephrase the conditions in Definition 1.3.4 to conditions wholly on the simple modules themselves. The following is [CR17, Lemma 4.19].

Proposition 1.3.10. *A derived equivalence $\Phi : D^b(A) \xrightarrow{\sim} D^b(A')$ is perverse relative to $(\mathcal{S}_\bullet, \mathcal{S}'_\bullet, p)$ if the following hold:*

- *for every i and every $V \in \mathcal{S}_i \backslash \mathcal{S}_{i-1}$, the composition factors of $H_t(\Phi(V))$ for $t \neq p(i)$ are all in $\mathcal{S}'_{i-1}$, and there is a filtration $L_1 \subset L_2 \subset H_{p(i)}(\Phi(V))$ such that the composition factors of L_1 and the composition factors of $H_{p(i)}(\Phi(V))/L_2$ are all in $\mathcal{S}'_{i-1}$, and $L_2/L_1 \in \mathcal{S}'_i \setminus \mathcal{S}'_{i-1}$;*
- *the map $V \mapsto L_2/L_1$ described above is a bijection between $\mathcal{S}_i \setminus \mathcal{S}_{i-1}$ and $\mathcal{S}'_i \setminus \mathcal{S}'_{i-1}$.*

In fewer words, for every $V \in \mathcal{S}_i \setminus \mathcal{S}_{i-1}$, the composition factors of $H_t(\Phi(V))$ are all in $\mathcal{S}'_{i-1}$, except for a single composition factor of $H_{p(i)}(\Phi(V))$, which lies in $\mathcal{S}'_i \setminus \mathcal{S}'_{i-1}$. Notationally, to highlight the effect of Φ on the simple modules, we may write the filtration as

$$\emptyset = \mathcal{S}_0 \subset_{p(1)} \mathcal{S}_1 \subset_{p(2)} \ldots \subset_{p(r)} \mathcal{S}_r = \mathcal{S},$$

or similarly on the other side. For the filtration on indices, this is

$$\emptyset = I_0 \subset_{p(1)} I_1 \subset_{p(2)} \ldots \subset_{p(r)} I_r = I,$$

and similarly on the other side.

Recall that, for an arbitrary derived equivalence

$$F : D^b(A) \xrightarrow{\sim} D^b(A'),$$

the induced abelian group isomorphism

$$[F] : K_0(D^b(A)) \xrightarrow{\sim} K_0(D^b(A'))$$

does not preserve classes of simple modules. In particular, we know that the sets $\mathcal{S}$ and $\mathcal{S}'$ of isomorphism classes of simple A- and A'-modules have the same finite cardinality, but the equivalence F does not induce a canonical bijection between the two sets. However, when F is a perverse equivalence, by Proposition 1.3.10, F does in fact induce a bijection $\mathcal{S}_i \setminus \mathcal{S}_{i-1} \leftrightarrow \mathcal{S}'_i \setminus \mathcal{S}'_{i-1}$ between the layers of the two filtrations for every i. Gluing these stratified bijections together therefore defines a bijection between $\mathcal{S}$ and $\mathcal{S}'$. This is one sense in which a perverse equivalence gives us more information than an arbitrary equivalence.

1.3.4 Projective modules

One can also rephrase the conditions for perversity in terms of projective modules. The following is taken from [CR17, §4.2.4].

Suppose $\Phi : D^b(A) \xrightarrow{\sim} D^b(A')$ is a perverse equivalence, relative to $(\mathcal{S}_\bullet, \mathcal{S}'_\bullet, p)$, in the sense of Definition 1.3.9. For each i, let $\mathcal{P}_i$ be the set of projective indecomposable A-modules P_V corresponding to the simple modules $V \in \mathcal{S} \setminus \mathcal{S}_{r-i}$, and $\mathcal{P}'_i$ be the set of projective indecomposable A'-modules $P_{V'}$ corresponding to the simple modules $V' \in \mathcal{S}' \setminus \mathcal{S}'_{r-i}$. This gives filtrations

$$\emptyset = \mathcal{P}_0 \subset \mathcal{P}_1 \subset \ldots \subset \mathcal{P}_r = \mathcal{P}$$

and

$$\emptyset = \mathcal{P}'_0 \subset \mathcal{P}'_1 \subset \ldots \subset \mathcal{P}'_r = \mathcal{P}'$$

of the sets $\mathcal{P}$ and $\mathcal{P}'$ of projective A- and A'-modules respectively. Define a function $\bar{p} : \{1, \ldots, r\} \to \mathbb{Z}$ by $\bar{p}(i) = p(r - i + 1)$.

Definition 1.3.11. We say the equivalence Φ is perverse relative to $(\mathcal{P}_\bullet, \mathcal{P}'_\bullet, \bar{p})$.

We present this as a definition, but it is strictly speaking a consequence of Chuang and Rouquier's definition for additive categories [CR17, Definition 4.1, Lemma 4.7], that we do not reprint here. In particular, the equivalence Φ restricts to

$$\overline{\Phi} : K^b(A\text{-proj}) \xrightarrow{\sim} K^b(A'\text{-proj}).$$

A filtration of A-proj by full additive subcategories is given by $\mathcal{B}_\bullet$, where $\mathcal{B}_i = \mathcal{P}_i$-add, with a similar filtration for A'-proj. Then Φ is perverse relative to $(\mathcal{B}_\bullet, \mathcal{B}'_\bullet, \bar{p})$, in the sense of Chuang and Rouquier's more general definition.

The filtration on the indexing set I of isomorphism classes of simple A-modules is

$$\emptyset = \bar{I}_0 \subset_{\bar{p}(1)} \bar{I}_1 \subset_{\bar{p}(2)} \ldots \subset_{\bar{p}(r)} \bar{I}_r = I,$$

where $\bar{I}_i = I \setminus I_{r-i}$, and similar on the other side.

Given a perverse equivalence in terms of simple modules, we can thus reconfigure it in terms of projective modules by reversing the filtrations and the perversity, and vice versa. We have a natural analogue to Proposition 1.3.10, using [CR17, Lemma 4.21].

Proposition 1.3.12. *A derived equivalence $\Phi : D^b(A) \xrightarrow{\sim} D^b(A')$ is perverse relative to $(\mathcal{P}_\bullet, \mathcal{P}'_\bullet, \bar{p})$ if and only if the following hold:*

- *for every i and every indecomposable projective A-module $P \in \mathcal{P}_i \setminus \mathcal{P}_{i-1}$, the object $\Phi(P)$ is isomorphic in $D^b(A')$ to a complex X of projective A'-modules, such that every term of X is a direct sum of modules in $\mathcal{P}'_{i-1}$, except in degree $\bar{p}(i)$, which has exactly one indecomposable summand in $\mathcal{P}'_i \setminus \mathcal{P}'_{i-1}$, say P', with all others in $\mathcal{P}'_{i-1}$;*
- *the map $P \mapsto P'$ as above defines a bijection between $\mathcal{P}_i \setminus \mathcal{P}_{i-1}$ and $\mathcal{P}'_i \setminus \mathcal{P}'_{i-1}$.*

In particular, gluing these stratified bijections produces a bijection $\mathcal{P} \leftrightarrow \mathcal{P}'$, matching (in reverse) the bijection $\mathcal{S} \leftrightarrow \mathcal{S}'$. It will be useful to have these two concrete formulations of perversity.

For symmetric algebras, it is clear from Proposition 1.3.12 and the discussion following Definition 1.2.18 that derived equivalences arising from combinatorial tilting complexes are perverse equivalences. The following can be found in [CR17, Proposition 5.3].

Proposition 1.3.13. *Let A be a finite-dimensional symmetric k-algebra. Let I be an indexing set of the isomorphism classes of simple A-modules. Given $J \subset I$, the combinatorial tilt $F_J : D^b(A) \xrightarrow{\sim} D^b(B)$ at J is a perverse equivalence, with filtrations both given by*

$$\emptyset \subset_0 J \subset_{-1} I.$$

The filtration on projectives is

$$\emptyset \subset_{-1} I \setminus J \subset_0 I.$$

Chuang and Rouquier call these combinatorial tilts *elementary perverse equivalences*. Concrete examples of such tilts can be found as Kauer moves in Brauer graph algebras, [Kau98].

1.3.5 Self-perverse equivalences

A ubiquitous class of perverse equivalences is that of *self-perverse equivalences*, as in [CR17, §4.3]. These are derived autoequivalences

$$\Phi : D^b(A) \xrightarrow{\sim} D^b(A),$$

perverse relative to $(\mathcal{S}_\bullet, \mathcal{S}_\bullet, p)$ for which the filtration

$$\emptyset = \mathcal{S}_0 \subset \mathcal{S}_1 \subset \ldots \subset \mathcal{S}_r = \mathcal{S}$$

of simple modules is the same on both sides. We may in such circumstances say that Φ is perverse relative to $(\mathcal{S}_\bullet, p)$.

It is important to note that, given a self-perverse equivalence Φ, the induced permutation on the layers of the filtration of $\mathcal{S}$, given by Proposition 1.3.10, need not be the identity. Further, [CR17, Remark 4.27], a self-perverse equivalence Φ whose perversity function is $p \equiv 0$ need not be isomorphic to the identity functor on $D^b(A)$, even if the induced bijection on $\mathcal{S}$ is the identity map. Rather, by Lemma 1.3.5, Φ induces a self-Morita equivalence of A-mod.

1.3.6 Two-step perverse equivalences

We will encounter a number of perverse equivalences whose filtration on simple modules is of the form

$$\emptyset = \mathcal{S}_0 \subset_{n_1} \mathcal{S}_1 \subset_{n_2} \mathcal{S}_2 = \mathcal{S}.$$

When this occurs, one can always take a homological shift so that the perversity function satisfies

$$\emptyset = \mathcal{S}_0 \subset_0 \mathcal{S}_1 \subset_d \mathcal{S}_2 = \mathcal{S}$$

for some $d \in \mathbb{Z}$. We will call an equivalence with a filtration of this form a *two-step perverse equivalence of width d.*

Proposition 1.3.13 informs us that, with I the indexing set for the isomorphism classes of simple A-modules, and $J \subset I$, the combinatorial tilt (or elementary perverse equivalence) at J is a two-step perverse equivalence of width -1.

There is a natural dual construction to the combinatorial tilt at J, using injective hulls rather than projective covers, which also gives rise to a two-step perverse equivalence, this time of width 1. The inverse of the combinatorial tilt $F_J : D^b(A) \xrightarrow{\sim} D^b(B)$ is the dual of the elementary perverse equivalence at J for B:

$$F_J^{-1} : D^b(B) \xrightarrow{\sim} D^b(A).$$

For details of the dual construction, see [CR17, Proposition 5.4].

1.3.7 Construction of perverse equivalences

Recall from §1.3.2 that the composition of perverse equivalences

$$D^b(A) \xrightarrow{\Phi_1} D^b(A^{(1)}) \xrightarrow{\Phi_2} D^b(A^{(2)})$$

need not remain perverse. However, in certain circumstances the composition will be perverse. For example, Lemma 1.3.7 tells us that iterated elementary perverse equivalences at a fixed subset I_0 of the common indexing set I will remain perverse. We can take this analysis further, and produce a construction of a canonical perverse equivalence for a given filtration and perversity function.

Now, let I be an indexing set of isomorphism classes of simple A-modules. Given a chain of subsets $\emptyset \subset I_1 \subset I_2 \subset I$, we have a diagram

$$\begin{array}{ccccc} & & D^b(A^{(1)}) & \xrightarrow{F_{I_2}} & D^b(B) \\ & \overset{F_{I_1}}{\nearrow} & & & \\ D^b(A) & & & & \\ & \underset{F_{I_2}}{\searrow} & & & \\ & & D^b(A^{(2)}) & \xrightarrow{F_{I_1}} & D^b(B') \end{array}$$

where F_{I_1} and F_{I_2} are the elementary perverse equivalences (or combinatorial tilts) at I_1 and I_2 respectively. By Proposition 1.3.13 and Lemma 1.3.7, both compositions are perverse relative to the filtration

$$\emptyset \subset_0 I_1 \subset_{-1} I_2 \subset_{-2} I.$$

By Proposition 1.3.8, the composition

$$F_{I_1} F_{I_2} F_{I_1}^{-1} F_{I_2}^{-1} : D^b(B) \xrightarrow{\sim} D^b(B')$$

is a perverse equivalence, with perversity function identically zero, so induces an equivalence

$$B\text{-mod} \xrightarrow{\sim} B'\text{-mod}.$$

That is, B and B' are Morita equivalent.

More generally, [CR17, Proposition 5.5], given a chain of subsets

$$\emptyset = I_0 \subset I_1 \subset \ldots \subset I_r = I,$$

and a function $p : \{1, \ldots, r\} \to \mathbb{Z}$, the composition of elementary equivalences

$$F_{I_{r-1}}^{p(r-1)-p(r)} \ldots F_{I_1}^{p(1)-p(2)} F_{I_0}^{-p(1)} : D^b(A) \xrightarrow{\sim} D^b(B),$$

for some finite-dimensional symmetric k-algebra B, is perverse, relative to the filtration

$$\emptyset = I_0 \subset_{p(1)} I_1 \subset_{p(2)} \ldots \subset_{p(r)} I_r = I,$$

and the filtration $I_\bullet$ and the perversity function p completely determine the Morita equivalence class of the k-algebra B. That is, given any equivalence

$$F : D^b(A) \xrightarrow{\sim} D^b(B')$$

perverse relative to the filtration

$$\emptyset = I_0 \subset_{p(1)} I_1 \subset_{p(2)} \ldots \subset_{p(r)} I_r = I,$$

the algebra B' is Morita equivalent to B.

Similar statements are true when one takes the dual elementary perverse equivalences, with the values of the function p negated.

In [CR17, Proposition 5.11], Chuang and Rouquier give conditions for the composition of elementary perverse equivalences at two arbitrary subsets $J, J' \subset I$ to remain perverse. Further, in [CR17, Proposition 5.12], Chuang and Rouquier give conditions for the braid relation $F_J F_{J'} F_J \cong F_{J'} F_J F_{J'}$ to hold and remain perverse, between elementary perverse equivalences at two arbitrary subsets $J, J' \subset I$.

1.3.8 Standard equivalences

Suppose $\Phi : D^b(A) \xrightarrow{\sim} D^b(B)$ is a derived equivalence. Let $\mathcal{S}$ and $\mathcal{S}'$ be sets of isomorphism classes of simple A- and B-modules respectively. Recall by Theorem 1.2.19 that there is a complex X of B-A-bimodules such that, for every $V \in D^b(A)$, $\Phi(V) \cong X \otimes^{\mathbf{L}}_A V$ in $D^b(B)$.

Proposition 1.3.14. *If the equivalence Φ is perverse relative to $(\mathcal{S}_\bullet, \mathcal{S}'_\bullet, p)$, then the equivalence*

$$X \otimes^{\mathbf{L}}_A - : D^b(A) \xrightarrow{\sim} D^b(B)$$

is perverse, with the same perversity data.

Proof. By Proposition 1.3.10, the perversity of the derived equivalence $X \otimes^{\mathbf{L}}_A -$ depends only on the images of simple A-modules. But for every simple A-module S_i, $X \otimes^{\mathbf{L}}_A S_i \cong \Phi(S_i)$. The result follows. □

Let X be as above. Set $X^\vee = \mathbf{R}\operatorname{Hom}_B(X, B)$, a complex of A-B-bimodules. Then by [Ric91, Proposition 4.1],

$$X^\vee \otimes_B^{\mathbf{L}} - : D^b(B) \xrightarrow{\sim} D^b(A)$$

is a derived equivalence, mutually inverse with the equivalence $X \otimes_A^{\mathbf{L}} -$, and is the standard derived equivalence agreeing with Φ^{-1} on objects of $D^b(B)$. By Lemma 1.3.6 and Proposition 1.3.12, $X^\vee \otimes_B^{\mathbf{L}} -$ and Φ^{-1} are both perverse relative to $(\mathcal{S}'_\bullet, \mathcal{S}_\bullet, -p)$.

The two-sided tilting complexes X and $X^\vee$ also induce perverse equivalences on the derived categories of right modules. The following combines [Ric91, Lemma 4.3] and [CR17, Lemma 4.20].

Proposition 1.3.15. *The functor*

$$- \otimes_B^{\mathbf{L}} X : D^b(B^{\mathrm{op}}) \longrightarrow D^b(A^{\mathrm{op}})$$

is an equivalence, and is perverse relative to $(\mathcal{S}_\bullet, \mathcal{S}'_\bullet, -p)$. *Similarly, the functor*

$$- \otimes_A^{\mathbf{L}} X^\vee : D^b(A^{\mathrm{op}}) \longrightarrow D^b(B^{\mathrm{op}})$$

is an equivalence, perverse relative to $(\mathcal{S}'_\bullet, \mathcal{S}_\bullet, p)$. *Moreover, these two equivalences are mutually inverse.*

Thus, the equivalences Φ and Φ^{-1} induce equivalences

$$\tilde{\Phi} : D^b(B^{\mathrm{op}}) \longrightarrow D^b(A^{\mathrm{op}}),$$

perverse relative to $(\mathcal{S}_\bullet, \mathcal{S}'_\bullet, -p)$, and

$$\tilde{\Phi}^{-1} : D^b(A^{\mathrm{op}}) \longrightarrow D^b(B^{\mathrm{op}}),$$

perverse relative to $(\mathcal{S}'_\bullet, \mathcal{S}_\bullet, p)$.

Chapter 2

Periodicity

In Chapter 3, we will encounter a number of perverse equivalences exhibiting a degree of periodicity. This phenomenon is closely linked to work of Grant [Gra13] on perverse equivalences arising from periodic endomorphism algebras. In our examples, however, the relevant endomorphism algebras are not periodic. This suggests we need a stronger result to capture the periodicity governing these equivalences in general.

In §2.1, we recount Grant's statements for perverse equivalences arising from periodic symmetric algebras, and demonstrate that, under his conditions, the periodicity of the relevant modules is guaranteed. This is, in itself, a natural consequence of the key result in §2.2, in which it is shown that any two-step self-perverse equivalence gives rise to periodic modules over an idempotent algebra. We strengthen this into an if-and-only-if statement in §2.3, where we impose fairly strong conditions to determine the origin of these periodic modules, and show that, if there are periodic modules originating in this way, then the algebra admits a self-perverse equivalence with the desired filtration.

2.1 Grant's Theorem

Throughout this section, let A be a finite-dimensional, symmetric k-algebra.

2.1.1 Endomorphism algebras

Let P be a projective A-module. Set $E = \mathrm{End}_A(P)^{\mathrm{op}}$. We collect a few easy facts about E.

Proposition 2.1.1. *The k-algebra E is a finite-dimensional, symmetric k-algebra.*

Proof. By Theorem 1.1.3 part (iv), since A is symmetric we have $\mathrm{End}_A(P) = \mathrm{Hom}_A(P, P) \cong \mathrm{Hom}_A(P, P)^*$, and this isomorphism is functorial in both arguments, so is an isomorphism of E-E-bimodules. Thus, $E^{\mathrm{op}} = \mathrm{End}_A(P)$ is symmetric by Theorem 1.1.3 part (ii), and so by Corollary 1.1.4, E is a symmetric k-algebra. That E is finite-dimensional is trivial. □

The A-module P is an A-E-bimodule. The right action of E is

$$x \cdot \varphi = \varphi(x)$$

for $x \in P$ and $\varphi \in E = \mathrm{End}_A(P)^{\mathrm{op}}$. We thus have a functor

$$\mathrm{Hom}_A(P, -) : A\text{-mod} \to E\text{-mod},$$

and since P is a projective A-module, this functor is exact.

The following is a special case of Auslander's projectivisation [Aus74], by which objects in an additive category are transformed into projective modules.

Proposition 2.1.2. *The functor* $\mathrm{Hom}_A(P, -)$ *restricts to an equivalence*

$$P\text{-add} \xrightarrow{\sim} E\text{-proj}$$

of additive categories.

In particular, suppose $P = \bigoplus_{i=1}^{r} P_i^{m_i}$, for pairwise non-isomorphic projective indecomposable A-modules P_i. Then

$$E = \mathrm{Hom}_A(P, P) \cong \bigoplus_{i=1}^{r} \mathrm{Hom}_A(P, P_i)^{m_i},$$

and the E-modules $\mathrm{Hom}_A(P, P_i)$ are precisely the projective indecomposable E-modules.

2.1.2 Twisted modules

Let E be a finite-dimensional k-algebra and M an E-module. If σ is an automorphism of E, then we denote by ${}_{\sigma}M$ the *twisted module*, with E-action

$$e \cdot m = \sigma(e)m$$

for $e \in E$ and $m \in M$. If M is an E-E-bimodule and τ is another automorphism of E, then the *twisted bimodule* ${}_\sigma M_\tau$ is the bimodule with left and right E-action

$$e \cdot m \cdot e' = \sigma(e) m \tau(e')$$

for $e, e' \in E$ and $m \in M$. As a twisted bimodule, we have ${}_\sigma M = {}_\sigma M_{\mathrm{id}}$, and we denote $M_\tau = {}_{\mathrm{id}} M_\tau$.

Given an automorphism σ of E, there is a natural isomorphism of functors

$${}_\sigma(-) \cong {}_\sigma E \otimes_E -,$$

and so for any E-module M, we have ${}_\sigma M \cong {}_\sigma E \otimes_E M$. Similarly, given an automorphism τ of E, there is a natural isomorphism of functors

$$(-)_\tau \cong - \otimes_E E_\tau,$$

so that, for every E^{op}-module M, $M_\tau \cong M \otimes_E E_\tau$.

There is an E-E-bimodule isomorphism $f : {}_\sigma E \to E_{\sigma^{-1}}$, given by $f(x) = \sigma^{-1}(x)$ for all $x \in E$. Indeed, we have

$$\begin{aligned} f(e \cdot x \cdot e') &= f(\sigma(e) x e') \\ &= \sigma^{-1}(\sigma(e) x e') \\ &= e \sigma^{-1}(x) \sigma^{-1}(e') \\ &= e \cdot f(x) \cdot e', \end{aligned}$$

for all $x, e, e' \in E$. Similarly, $E_\tau \cong {}_{\tau^{-1}} E$ as E-E-bimodules. Given an E-E-bimodule M, we therefore have ${}_\sigma M_\tau \cong {}_{\tau^{-1}} M_{\sigma^{-1}}$.

2.1.3 Periodic modules

Let E be a finite-dimensional k-algebra and M an E-module. Recall that the Heller translate of M is $\Omega_E(M) = \ker(\pi_M)$, where $P(M) \xrightarrow{\pi_M} M$ is a projective cover of M. One can iterate this construction: for $n \geq 1$, we set

$$\Omega_E^{n+1}(M) = \Omega_E(\Omega_E^n(M)).$$

Definition 2.1.3. The E-module M is *σ-periodic of period n* if there is an automorphism σ of E and an $n \geq 1$ such that $\Omega_E^n(M) \cong {}_\sigma M$.

That is, M is σ-periodic of period n if there is an exact sequence

$$0 \longrightarrow {}_\sigma M \longrightarrow P_{n-1} \longrightarrow \ldots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

of E-modules such that each P_i is projective. We call the complex

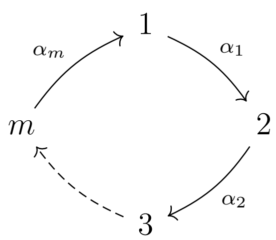
$$P_{n-1} \longrightarrow \ldots \longrightarrow P_1 \longrightarrow P_0$$

a *truncated projective resolution of M*. To avoid reference to a specific σ, we may say that M is *twisted periodic*. If $\sigma = \mathrm{id}$, then we say simply that M is *periodic*.

We note that, if there is some n such that M is σ-periodic of period n for some automorphism σ, then there must exist some minimal such n. We emphasise that we do not demand minimality, and in many of the examples that follow, it will be expedient to consider different periodicities of the same M at once. We also note that there may be different automorphisms σ of E for which M is σ-periodic.

There is an obvious dual definition for right modules. The E^{op}-module N is *τ-periodic of period n* for an automorphism τ of E and $n \in \mathbb{Z}_+$ if $\Omega^n_{E^{\mathrm{op}}}(N) \cong N_\tau$.

Periodic modules occur in a number of settings. Let $A = N_{m,n}$ be a Nakayama algebra. Recall that $A \cong kQ/\mathcal{I}$, where Q is the quiver



and $\mathcal{I} = \langle \alpha^n \rangle$. Given a simple A-module S_i, we have an exact sequence

$$0 \longrightarrow S_{i+n} \longrightarrow P_{i+1} \longrightarrow P_i \longrightarrow S_i \longrightarrow 0,$$

so that S_i is ν-periodic of period 2, where ν is the automorphism of A generating the Nakayama permutation $\nu_{N_{m,n}}$. In fact (see e.g. [ES08]), the Nakayama algebras coincide with the class of self-injective algebras A for which Ω^2_A permutes the set of isomorphism classes of simple A-modules.

More generally (see e.g. [GSS03]), if A is a finite-dimensional k-algebra for which every simple A-module is periodic, then A is a self-injective algebra. On the other hand (see e.g. [Dug10]), if A is a self-injective algebra of finite representation type, then every simple A-module (in fact, every non-projective indecomposable A-module) is periodic.

Periodic modules also exist in blocks of group algebras. Many examples of this can be found in [Ben].

2.1.4 Periodic algebras

Let E be a finite-dimensional k-algebra.

Definition 2.1.4. The k-algebra E is *σ-periodic of period n* if there is an automorphism σ of E and an $n \geq 1$ such that E is $\sigma \otimes \mathrm{id}_E$-periodic of period n as an $E \otimes E^{\mathrm{op}}$-module.

That is, E is σ-periodic of period n if there is an exact sequence of E-E-bimodules

$$0 \longrightarrow {}_\sigma E \longrightarrow Y_{n-1} \longrightarrow \ldots \longrightarrow Y_1 \longrightarrow Y_0 \longrightarrow E \longrightarrow 0$$

such that each Y_i is projective as an E-E-bimodule. Let Y be the object

$$Y_{n-1} \longrightarrow \ldots \longrightarrow Y_1 \longrightarrow Y_0$$

of $\mathrm{Ch}^b(E\text{-}E)$, the *truncated resolution of E*.

A survey of symmetric algebras with this property can be found in [ES08]. We will see a few examples in the next subsection. It remains an interesting open problem to classify the finite-dimensional periodic algebras.

If E is a finite-dimensional k-algebra and there exists an automorphism σ of E such that E is σ-periodic of period n, then every E-module M is σ-periodic of period n. Indeed, one can apply the functor $- \otimes_E M$ to the exact sequence of projective E-E-bimodules above, to obtain an exact sequence

$$0 \longrightarrow {}_\sigma M \longrightarrow Y_{n-1} \otimes_E M \longrightarrow \ldots \longrightarrow Y_1 \otimes_E M \longrightarrow Y_0 \otimes_E M \longrightarrow M \longrightarrow 0,$$

where every term $Y_i \otimes_E M$ is projective as an E-module. This is thus a truncated projective resolution of M, and we have $\Omega^n_E(M) \cong {}_\sigma M$.

Green, Snashall and Solberg [GSS03, Lemma 1.5] have shown that periodic algebras are necessarily self-injective. By way of a partial converse, Dugas [Dug10, Theorem 5.1] has shown that if A is a finite-dimensional self-injective k-algebra of finite representation type of which every block is not a matrix algebra, then A is periodic. Further, [GSS03, Theorem 1.4], if A is a finite-dimensional k-algebra such that every simple A-module is a periodic A-module, then there is an automorphism σ of A and an $n \geq 1$ such that A is a σ-periodic k-algebra of period n. For example, from the observations in the previous subsection, we have that the Nakayama algebras are σ-periodic of period 2. In fact (see e.g. [ES08]), if A is a Nakayama algebra, then there is some n such that A is a periodic k-algebra of period n.

2.1.5 Grant's result

We recall Grant's result on self-perverse equivalences arising from periodic algebras.

Let A be a finite-dimensional symmetric k-algebra, P a projective A-module, and $E = \mathrm{End}_A(P)^{\mathrm{op}}$. Suppose that E is σ-periodic of period n with truncated resolution Y. We have a map of chain complexes

$$f : Y \to E.$$

There is a chain of isomorphisms

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Ch}^b(E\text{-}E)}(Y, E) &= \mathrm{Hom}_{\mathrm{Ch}^b(E\text{-}E)}(Y, \mathrm{Hom}_A(P, P)) \\
&\cong \mathrm{Hom}_{\mathrm{Ch}^b(A\text{-}E)}(P \otimes_E Y, P) \\
&\cong \mathrm{Hom}_{\mathrm{Ch}^b(A\text{-}E)}(P \otimes_E Y, \mathrm{Hom}_{A^{\mathrm{op}}}(P^\vee, A)) \\
&\cong \mathrm{Hom}_{\mathrm{Ch}^b(A\text{-}E)}(P \otimes_E Y \otimes_E P^\vee, A)
\end{aligned}$$

given by tensor-Hom adjunction. Let

$$g : P \otimes_E Y \otimes_E P^\vee \to A$$

be the image of f under the above isomorphisms. Let $X = \mathrm{cone}(g)$ and set

$$\Psi_P = X \otimes_A - : D^b(A) \longrightarrow D^b(A).$$

Grant's result [Gra13, Theorem 3.9, Proposition 3.22] is the following.

Theorem 2.1.5. *The functor*

$$\Psi_P : D^b(A) \longrightarrow D^b(A)$$

is a derived equivalence. Moreover, Ψ_P is a two-step self-perverse equivalence with respect to the filtration

$$\emptyset \subset_0 J \subset_n I$$

on both sides, where I is an indexing set for the isomorphism classes of simple A-modules, and $J \subset I$ is the subset such that $I \setminus J$ is the subset of indices corresponding to the simple summands of $P/\operatorname{rad}(P)$.

We call the equivalence Ψ_P the *periodic twist at P*.

Grant's key examples are symmetric algebra analogues of geometric constructions of Seidel and Thomas and Huybrechts and Thomas.

Seidel and Thomas [ST00] defined *spherical objects* in $D^b(X)$, the bounded derived category of coherent sheaves on a smooth complex projective variety X. These are objects $\mathcal{E}$ for which

$$\bigoplus_{r\in\mathbb{Z}} \mathrm{Hom}_{D^b(X)}(\mathcal{E},\mathcal{E}[r]) \cong H^*(S^d) \cong k[x]/\langle x^2\rangle$$

as graded algebras, where S^d is the d-sphere, with d the dimension of X. Such objects give rise to twist functors $T_{\mathcal{E}}$, autoequivalences of $D^b(X)$ such that the twist functors arising from a collection of spherical objects are compatible in a precise way.

Grant [Gra13, §6.1] defines spherical objects in $D^b(A)$ to be projective A-modules P such that $E = \mathrm{End}_A(P)^{\mathrm{op}} \cong k[x]/\langle x^2\rangle$. Then there is a short exact sequence of (chain complexes of) E-E-bimodules

$$0 \longrightarrow {}_{\sigma}E \overset{\iota}{\hookrightarrow} E \otimes_k E \overset{m}{\twoheadrightarrow} E \longrightarrow 0,$$

where m is the multiplication map, σ is the automorphism of E generated by $\sigma(x) = -x$, and ι is the map $\iota(e) = e \otimes x - ex \otimes 1$. Thus, E is σ-periodic of period 1. The spherical twist at P is the periodic twist Ψ_P of A at P, given by a two-sided tilting complex

$$P \otimes_k P^\vee \xrightarrow{\mathrm{ev}} A,$$

where ev is the usual evaluation map. This construction specialises to Rouquier and Zimmerman's twists of Brauer tree algebras with exceptional multiplicity 1, [RZ03], which have a natural link to the representation theory of the symmetric groups.

Huybrechts and Thomas [HT05] generalised Seidel and Thomas's earlier construction, defining autoequivalences of the derived category $D^b(X)$ of coherent sheaves on a smooth complex projective variety arising from $\mathbb{P}^n$-*objects*. These are objects $\mathcal{E}$ in $D^b(X)$ for which

$$\bigoplus_{r\in\mathbb{Z}} \mathrm{Hom}_{D^b(X)}(\mathcal{E},\mathcal{E}[r]) \cong H^*(\mathbb{P}^n;\mathbb{C}) \cong k[x]/\langle x^{n+1}\rangle$$

as graded algebras, where $\mathbb{P}^n$ denotes complex projective n-space. Such objects give rise to autoequivalences $P_{\mathcal{E}}$ of $D^b(X)$ in such a way that, when $n = 1$, $P_{\mathcal{E}} = T_{\mathcal{E}}^2$.

Similarly, Grant [Gra13, §6.1] defines $\mathbb{P}^n$ objects in $D^b(A)$ to be projective A-modules P such that $E = \mathrm{End}_A(P)^{\mathrm{op}} \cong k[x]/\langle x^{n+1}\rangle$. In general, these algebras are periodic of period 2, with a short exact sequence of chain complexes of E-E-bimodules

$$0 \longrightarrow E[1] \hookrightarrow Y \twoheadrightarrow E \longrightarrow 0,$$

where Y is the complex

$$E \otimes_k E \stackrel{1\otimes x - x\otimes 1}{\longrightarrow} E \otimes_k E$$

in degrees 1 and 0. The $\mathbb{P}^n$-twist at P is the periodic twist Ψ_P of A at P, given by the two-sided tilting complex

$$P \otimes_k P^\vee \stackrel{1\otimes x - x\otimes 1}{\longrightarrow} P \otimes_k P^\vee \quad \stackrel{\text{ev}}{\longrightarrow} \quad A.$$

In the case $n = 1$, the $\mathbb{P}^1$-twist at P is the square of the spherical twist at P.

2.1.6 Relatively periodic algebras

Grant in fact proves a more general version of Theorem 2.1.5, based on the periodicity of the algebra E relative to some subalgebra.

Let E be a symmetric k-algebra and B a subalgebra of E. The algebra E is σ-periodic of period n *relative to* B if there is an automorphism σ of E and some $n \in \mathbb{Z}_+$ such that there is an exact sequence of E-E-bimodules

$$0 \longrightarrow {}_\sigma E \longrightarrow Y_{n-1} \longrightarrow \dots \longrightarrow Y_1 \longrightarrow Y_0 \longrightarrow E \longrightarrow 0$$

such that for every i, $Y_i \in E \otimes_B E$-add. Again, let Y be the object

$$Y_{n-1} \longrightarrow \dots \longrightarrow Y_1 \longrightarrow Y_0$$

of $\mathrm{Ch}^b(E\text{-}E)$, the truncated resolution of E *relative to* B.

Now, let A be a finite-dimensional symmetric k-algebra, P a projective A-module and $E = \mathrm{End}_A(P)^{\mathrm{op}}$. Let B be a subalgebra of E. Suppose E is periodic of period n relative to B with truncated resolution Y. Let X be the complex of A-A-bimodules constructed as in Theorem 2.1.5 and set

$$\Psi_P = X \otimes_A - : D^b(A) \longrightarrow D^b(A).$$

Grant's generalisation [Gra10, Theorem 4.3] is the following.

Theorem 2.1.6. *Suppose P is projective as a right B-module and that $P^\vee$ is projective as a left B-module. Then the functor*

$$\Psi_P : D^b(A) \longrightarrow D^b(A)$$

is a derived equivalence. Moreover, Ψ_P is a two-step self-perverse equivalence with respect to the filtration

$$\emptyset \subset_0 J \subset_n I$$

on both sides, where I is an indexing set for the isomorphism classes of simple A-modules, and $J \subset I$ is the subset such that $I \setminus J$ is the subset of indices corresponding to the simple summands of $P/\operatorname{rad}(P)$.

We call the equivalence Ψ_P the *relative* periodic twist at P. We note that the perversity clause in Theorem 2.1.6 is not explicitly proved by Grant, but his proof of Theorem 2.1.5 translates directly to this case. We note also that Grant assumes that the subalgebra B is symmetric, but the symmetrising form on B need not be that restricted from E. Dropping the symmetric constraint, we requite the additional assumption that $P^\vee$ be projective as a left B-module, but in the Grant setting this is a consequence of P being projective as a right B-module. Observe that, setting $B = k$, we recover Theorem 2.1.5 from Theorem 2.1.6.

Building on the idea of spherical twists, Grant [Gra10, §4.1.1] defines a *toric object* in $D^b(A)$ to be a projective A-module P for which $E = \operatorname{End}_A(P)^{\mathrm{op}} \cong k[x,y]/\langle x^2, y^2\rangle$, observing that this algebra with its natural grading is isomorphic to the coholomogy algebra of the torus, $H^*(S^1 \times S^1)$. The algebra E has a subalgebra $B = k[x]/\langle x^2\rangle$, and there is a short exact sequence of E-E-bimodules

$$0 \longrightarrow E \hookrightarrow E \otimes_B E \twoheadrightarrow E \longrightarrow 0,$$

so that E is periodic of period 1 relative to B. Thus, a toric object P in A gives rise to a relative periodic twist $\Psi_P : D^b(A) \xrightarrow{\sim} D^b(A)$ called the *toric twist* of A at P.

2.1.7 Cycle of equivalences

Let A be a finite-dimensional symmetric k-algebra and let I be an indexing set for the isomorphism classes of simple A-modules. For $i \in I$, let P_i be the projective indecomposable A-module with simple head S_i.

Let $J \subset I$. Recall the definition of a (basic) combinatorial tilting complex in A at J, Definition 1.2.18. Then there is an algebra $A^{(1)}$ and a derived equivalence

$$F_J : D^b(A) \xrightarrow{\sim} D^b(A^{(1)}),$$

the combinatorial tilt of A at J. Recall further that a combinatorial tilt is a two-step perverse equivalence of width 1, Proposition 1.3.13, or an elementary perverse equivalence at J, with filtration

$$\emptyset \subset_0 J \subset_{-1} I.$$

Given a subset J of I and a combinatorial tilt $F_J : D^b(A) \xrightarrow{\sim} D^b(A^{(1)})$ at J, the combinatorial tilt of $A^{(1)}$ at J again gives rise to an equivalence $F_J : D^b(A^{(1)}) \xrightarrow{\sim} D^b(A^{(2)})$, where $A^{(2)} \cong \mathrm{End}_{D^b(A^{(1)})}(T^{(1)})^{\mathrm{op}}$ for $T^{(1)}$ a combinatorial tilting complex of $A^{(1)}$ at J. One can iterate this procedure for arbitrary $n \geq 1$, so that the composition of n combinatorial tilts,

$$F_J^n : D^b(A) \xrightarrow{\sim} D^b(A^{(n)}),$$

the *nth iterated combinatorial tilt of A at J*, is a perverse equivalence, by Proposition 1.3.13 and the discussion in §1.3.2, with filtrations

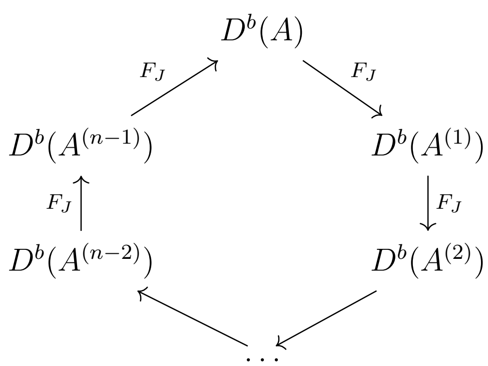
$$\emptyset \subset_0 J \subset_{-n} I.$$

The link between Grant's periodic twists and iterated combinatorial tilts is the following result, [Gra13, Theorem 5.11].

Theorem 2.1.7. *Let P be a projective A-module and suppose that the k-algebra $E = \mathrm{End}_A(P)^{\mathrm{op}}$ is σ-periodic of period n. Let $J \subset I$ be the subset of indices such that $P \cong \bigoplus_{i \in I \setminus J} P_i^{d_i}$ for some integers $d_i \geq 1$. Then the periodic twist Ψ_P at P agrees with the inverse of an nth iterated combinatorial twist F_J^{-n} at J.*

In particular, $A^{(n)} \cong A$. A very similar statement holds in the relatively periodic case.

One can picture this as a circle of n derived equivalences, starting and ending at $D^b(A)$.



If one can determine the algebras $A^{(i)}$, then we obtain for free a two-step perverse autoequivalence of each $A^{(i)}$ by following this circle.

Consider an algebra A with a $\mathbb{P}^n$-object P; that is, a projective A-module P such that $\mathrm{End}_A(P)^{\mathrm{op}} \cong k[x]/\langle x^{n+1}\rangle$. The $\mathbb{P}^n$-twist at P agrees with the iterated combinatorial tilts

$$\begin{array}{c} D^b(A) \\ {\scriptstyle\sim}\uparrow \quad \downarrow{\scriptstyle\sim} \\ D^b(A^{(1)}) \end{array}$$

at the appropriate set of indices J. In the case $n = 1$, the projective A-module P is both a $\mathbb{P}^1$-object and a spherical object. We have that $A^{(1)} \cong A$, and the combinatorial tilt at J is the inverse of the spherical twist at P. For an arbitrary $\mathbb{P}^n$-twist, $n > 1$, we do not expect to have $A^{(1)} \cong A$.

2.2 Periodic Modules

Let A be a finite-dimensional, symmetric k-algebra. Assume A is basic and indecomposable.

2.2.1 Periodic twists

Let P and Q be projective A-modules such that $A = P \oplus Q$ as an A-module. Set $E = \mathrm{End}_A(P)^{\mathrm{op}}$ and $M = \mathrm{Hom}_A(P, Q)$.

In our examples, we are interested less in the periodicity of the algebra E, and more in the periodicity of the E-module M. We know already that, when E is σ-periodic of period n, so is every E-module, and thus M is σ-periodic of period n. We can recover this same information in the relatively periodic case.

Proposition 2.2.1. *Suppose there is some automorphism σ of E and a subalgebra B of E such that E is σ-periodic of period n relative to B. Suppose that P is projective as a B^{op}-module and that $P^\vee$ is projective as a B-module. Then the E-module M is σ-periodic of period n and the E^{op}-module $M^\vee$ is σ^{-1}-periodic of period n.*

Proof. We have an exact sequence

$$0 \longrightarrow {}_{\sigma}E[n-1] \longrightarrow Y \longrightarrow E \longrightarrow 0$$

in $\mathrm{Ch}^b(E\text{-}E)$, where Y is a truncated resolution of E relative to B. Each term is a complex of bimodules, projective as E^{op}-modules, so we may apply the functor $- \otimes_E M$ to obtain another exact sequence

$$0 \longrightarrow {}_\sigma M[n-1] \longrightarrow Y \otimes_E M \longrightarrow M \longrightarrow 0$$

in $\mathrm{Ch}^b(E)$.

We thus need to show that each term of $Y \otimes_E M$ is a projective E-module. By construction, it suffices to show that $E \otimes_B E \otimes_E M \cong E \otimes_B M$ is a projective E-module.

We have

$$\begin{aligned} P^\vee &= \mathrm{Hom}_A(P, A) \\ &\cong \mathrm{Hom}_A(P, P \oplus Q) \\ &\cong \mathrm{Hom}_A(P, P) \oplus \mathrm{Hom}_A(P, Q) \\ &= E \oplus M, \end{aligned}$$

and since $P^\vee$ is a projective B-module, both E and M are projective B-modules. Since M is a projective B-module, $E \otimes_B M$ is a projective E-module. Thus, $Y \otimes_E M$ is a complex of projective E-modules, so we have

$$\Omega_E^n(M) \cong {}_\sigma M.$$

For the E^{op}-module $M^\vee$, we first recall that ${}_\sigma E \cong E_{\sigma^{-1}}$ as E-E-bimodules, and that by Theorem 1.1.3, since E is a symmetric algebra, we have

$$M^\vee = \mathrm{Hom}_A(P, Q)^\vee \cong \mathrm{Hom}_A(P, Q)^* \cong \mathrm{Hom}_A(Q, P).$$

We similarly need to show that $M^\vee \otimes_E Y$ is a complex of projective E^{op}-modules, for which it again suffices to show that

$$M^\vee \otimes_E E \otimes_B E \cong M^\vee \otimes_B E$$

is a projective E^{op}-module. We have

$$\begin{aligned} P &= \mathrm{Hom}_A(A, P) \\ &\cong \mathrm{Hom}_A(P \oplus Q, P) \\ &\cong \mathrm{Hom}_A(P, P) \oplus \mathrm{Hom}_A(Q, P) \\ &= E \oplus M^\vee, \end{aligned}$$

and since $P^\vee$ is a projective B-module, both E and M are projective B-modules. We can therefore similarly deduce that $M^\vee \otimes_E Y$ is a complex of projective E^{op}-modules, so that

$$\Omega^n_{E^{\mathrm{op}}}(M^\vee) \cong (M^\vee)_{\sigma^{-1}},$$

and we are done. $\square$

The method of this proof suggests that we may rephrase Grant's conditions in terms of the periodicity of M and $M^\vee$.

2.2.2 Two-step self-perverse equivalences

Proposition 2.2.1 tells us that a relative periodic twist of A at P gives rise to a periodic module M over the endomorphism ring E of P. This is emblematic of the following more general result for any two-step self-perverse equivalence.

Theorem 2.2.2. *Suppose $\Phi : D^b(A) \xrightarrow{\sim} D^b(A)$ is a two-step self-perverse equivalence relative to the filtration*

$$\emptyset \subset_0 J \subset_n I$$

on the indexing set I of isomorphism classes of simple A-modules, for some $n \geq 1$. Then there are projective A-modules P and Q such that $A \cong P \oplus Q$ as an A-module, and an automorphism σ of $E = \mathrm{End}_A(P)^{\mathrm{op}}$ such that, for the E-module $M = \mathrm{Hom}_A(P, Q)$, we have $\Omega^n_E(M) \cong {}_\sigma M$.

Proof. Let Q be the sum of the projective indecomposable A-modules corresponding to the simple A-modules in J and let P be the sum of the projective indecomposable A-modules corresponding to the simple A-modules in $I \setminus J$. Clearly, $A \cong P \oplus Q$. We have $\Phi(P) \cong P[n]$, while $\Phi(Q)$ is isomorphic to a complex

$$X_{n-1} \xrightarrow{d_{n-1}} \ldots \longrightarrow X_1 \xrightarrow{d_1} X_0 \xrightarrow{d_0} Q$$

with each $X \in P$-add such that applying the functor $\mathrm{Hom}_A(P, -)$ produces a minimal projective resolution of the E-module M.

The equivalence Φ induces an isomorphism

$$\mathrm{Hom}_{D^b(A)}(P, P)^{op} \xrightarrow{\sim} \mathrm{Hom}_{D^b(A)}(\Phi(P), \Phi(P))^{op}.$$

It is clear that

$$\mathrm{Hom}_{D^b(A)}(P,P)^{op} \cong \mathrm{Hom}_A(P,P)^{op} = E,$$

while $\Phi(P) = P[n]$, so

$$\begin{aligned}\mathrm{Hom}_{D^b(A)}(\Phi(P),\Phi(P))^{op} &= \mathrm{Hom}_{D^b(A)}(P[n],P[n])^{op}\\ &\cong \mathrm{Hom}_{D^b(A)}(P,P)^{op}\\ &\cong \mathrm{Hom}_A(P,P)^{op}\\ &= E.\end{aligned}$$

Thus, Φ induces an automorphism

$$\sigma^{-1} : E \xrightarrow{\sim} E.$$

Further, via this automorphism, Φ induces an isomorphism of E-modules

$$\mathrm{Hom}_{D^b(A)}(P,Q) \xrightarrow{\sim} {}_{\sigma^{-1}}\mathrm{Hom}_{D^b(A)}(P[n],\Phi(Q)).$$

Here, $\mathrm{Hom}_{D^b(A)}(P,Q) \cong \mathrm{Hom}_A(P,Q) = M$, so twisting by the action of σ we have

$${}_{\sigma}M \cong \mathrm{Hom}_{D^b(A)}(P[n],\Phi(Q)),$$

where σ is the inverse of the automorphism σ^{-1}. This right hand side is the space of chain maps (up to homotopy)

$$\begin{array}{l} P \\ \downarrow^{g} \\ X_{n-1} \xrightarrow{d_{n-1}} \dots \longrightarrow X_1 \xrightarrow{d_1} X_0 \xrightarrow{d_0} Q, \end{array}$$

that is, maps $P \xrightarrow{g} X_{n-1}$ such that the composition $P \xrightarrow{g} X_{n-1} \xrightarrow{d_{n-1}} Y$ is zero (here, Y is either X_{n-2} in P-add, or it is Q). Since P is projective, the functor $\mathrm{Hom}_A(P,-)$ is an exact functor, so we have an exact sequence

$$\mathrm{Hom}_A(P,X_{n-1}) \xrightarrow{(d_{n-1})_*} \dots \longrightarrow \mathrm{Hom}_A(P,X_0) \xrightarrow{(d_0)_*} M$$

of E-modules. Since $\Phi(Q)$ is a minimal resolution of Q and the functor $\mathrm{Hom}_A(P,-)$ is an equivalence P-add $\simeq E$-proj, this sequence is a minimal resolution of M of length n, so $\Omega^n_E(M) \cong \ker((d_{n-1})_*)$. But this kernel is the space of maps $g : P \to X_{n-1}$ such that $d_{n-1} \circ g : P \to X_{n-2}$ is zero, coinciding exactly with $\mathrm{Hom}_{D^b(A)}(P[n], \Phi(Q))$. Thus, in conclusion,

$$\Omega^n_E(M) \cong \ker((d_{n-1})_*) \cong \mathrm{Hom}_{D^b(A)}(P[n], \Phi(Q)) \cong {}_\sigma M,$$

so M is σ-periodic of period n. □

Recall Proposition 1.3.15. There is an induced equivalence

$$\tilde{\Phi} : D^b(A^{\mathrm{op}}) \xrightarrow{\sim} D^b(A^{\mathrm{op}})$$

such that the inverse $\tilde{\Phi}^{-1}$ is perverse relative to the filtration

$$\emptyset \subset_0 J \subset_n I.$$

A near-identical argument to that in Theorem 2.2.2 will then tell us that the E^{op}-module $M^\vee$ is σ^{-1}-periodic of period n.

2.2.3 Towards a converse

Theorem 2.2.2 tells us that the existence of a two-step self-perverse equivalence forces the existence of a periodic module M over an endomorphism algebra E. One might ask: is the converse true? That is, does the existence of a periodic module M over an endomorphism algebra $E = \mathrm{End}_A(P)^{\mathrm{op}}$ in A give rise to a two-step self-perverse equivalence

$$\Phi_P : D^b(A) \xrightarrow{\sim} D^b(A)$$

in a systematic way? That is, is the statement in Theorem 2.2.2 really an if-and-only-if? This prompts the following conjecture.

Conjecture 2.2.3. *Let A be a symmetric k-algebra and let P and Q be projective A-modules such that $A = P \oplus Q$. Set $E = \mathrm{End}_A(P)^{\mathrm{op}}$ and $M = \mathrm{Hom}_A(P, Q)$. If there is an automorphism σ of E and $n \in \mathbb{Z}$ such that the E-module M is σ-periodic of period n, then there is a two-step self-perverse equivalence*

$$\Phi_P : D^b(A) \xrightarrow{\sim} D^b(A)$$

with filtration

$$\emptyset \subset_0 J \subset_n I,$$

where I is an indexing set for the isomorphism classes of simple A-modules and J is the subset of indices of the simple summands of $Q/\operatorname{rad}(Q)$. Moreover, a one-sided tilting complex for Φ_P is

$$X = \bigoplus_{i \in I} X_i,$$

where for $i \in I \setminus J$, $X_i = P_i$, where P_i is the indecomposable summand of P with simple head corresponding to the index i, and for $i \in J$, X_i is lifted from the truncated periodic resolution of the summand $M_i = \operatorname{Hom}_A(P, Q_i)$, where Q_i is the summand of Q with simple head corresponding to the index i.

Consider the following example.

Example 2.2.4. Let A be the k-algebra with two simple modules S and T, with an Ext^1-quiver

$$\varepsilon \circlearrowright S \underset{\beta}{\overset{\alpha}{\rightleftarrows}} T \circlearrowleft \eta$$

and a set of relations

$$\mathcal{I} = \{\varepsilon^2 - \beta\alpha, \eta\alpha - \alpha\varepsilon, \beta\eta - \varepsilon\beta, \alpha\beta - \eta^2, \alpha\varepsilon^2, \beta\eta^2\}.$$

The projective indecomposable A-modules have Loewy series

$$\begin{matrix} & S & \\ S & & T \\ S & & T \\ & S & \end{matrix}, \quad \begin{matrix} & T & \\ T & & S \\ T & & S \\ & T & \end{matrix}.$$

The Cartan matrix of A is

$$\begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}.$$

The endomorphism algebra $E = \operatorname{End}_A(P_S)^{\mathrm{op}}$ is such that $E \cong k[x]/\langle x^4 \rangle$, so that P_S is a $\mathbb{P}^3$-object in A in the sense of Grant. In particular, E is a periodic algebra of period 2. Thus, by Theorem 2.1.5, there is a self-perverse equivalence $D^b(A) \to D^b(A)$ with filtration

$$\emptyset \subset_0 \{T\} \subset_2 \{S, T\},$$

the $\mathbb{P}^3$-twist at P. Grant's cycle of equivalences tells us that there is an algebra $A^{(1)}$ such that iterated combinatorial tilts at $J = \{T\}$ produce derived equivalences

$$\begin{array}{c} D^b(A) \\ {\scriptstyle\sim}\big\uparrow \quad \big\downarrow{\scriptstyle\sim} \\ D^b(A^{(1)}) \end{array}.$$

Unlike for $\mathbb{P}^1$-twists, we do not expect $A^{(1)}$ here to be isomorphic to A.

Let $M = \mathrm{Hom}_A(P_S, P_T)$. Then as an E-module,

$$M \cong \begin{smallmatrix} k \\ k \end{smallmatrix},$$

the unique non-split extension of the simple E-module k by itself. Thus, M is a periodic E-module of period 1. Conjecture 2.2.3 predicts a derived equivalence $D^b(A) \to D^b(A)$ given by a tilting complex X with summands

$$X_1 : \qquad P_S$$

$$X_2 : \qquad P_S \longrightarrow P_T.$$

However, we have that

$$\mathrm{End}_{D^b(A)}(P_T) \cong k[x]/\langle x^4 \rangle,$$

but

$$\mathrm{End}_{D^b(A)}(X_2) \cong k[x,y]/\langle x^2, y^2 \rangle,$$

so that $\mathrm{End}_{D^b(A)}(X)^{\mathrm{op}} \ncong A$.

To see this, we have non-zero irreducible maps in $\mathrm{End}_{D^b(A)}(X_2)$

$$\begin{array}{ccc} P_S & \xrightarrow{\alpha} & P_T \\ \big\downarrow{\scriptstyle\varepsilon} & & \big\downarrow{\scriptstyle\eta} \\ P_S & \xrightarrow{\alpha} & P_T \end{array} \qquad \text{and} \qquad \begin{array}{ccc} P_S & \xrightarrow{\alpha} & P_T \\ \big\downarrow{\scriptstyle 0} & & \big\downarrow{\scriptstyle\eta^2} \\ P_S & \xrightarrow{\alpha} & P_T \end{array}.$$

Label these $f_1, f_2 : X_2 \to X_2$. That $f_2^2 = 0$ is clear, while for f_1^2, we have a homotopy to zero

$$\begin{array}{ccc} P_S & \xrightarrow{\alpha} & P_T \\ \big\downarrow{\scriptstyle\varepsilon^2} & & \big\downarrow{\scriptstyle\eta^2} \\ P_S & \xrightarrow{\alpha} & P_T \end{array}$$

given by $P_T \xrightarrow{\beta} P_S$. That $f_2 f_1 = f_1 f_2$ is also clear:

$$\begin{array}{ccc} P_S & \xrightarrow{\alpha} & P_T \\ \big\downarrow{\scriptstyle 0} & & \big\downarrow{\scriptstyle \eta^3} \\ P_S & \xrightarrow{\alpha} & P_T \end{array}.$$

Thus, this equivalence Φ is between the derived category of A and that of another algebra A', with the same Cartan matrix as A, but such that A and A' are not Morita equivalent. In particular, A' has an Ext^1-quiver

$$\varepsilon \circlearrowleft S \underset{\beta}{\overset{\alpha}{\rightleftarrows}} T \circlearrowright \eta$$

and a set of relations

$$\mathcal{I} = \{\varepsilon^2 - \beta\alpha, \eta\alpha - \alpha\varepsilon, \beta\eta - \varepsilon\beta, \eta\alpha\beta - \alpha\beta\eta, \alpha\varepsilon^2, \eta^2\}.$$

The projective indecomposable A'-modules have Loewy series

$$\begin{array}{c} S \\ S\ \ T \\ S\ \ T \\ S \end{array}, \quad \begin{array}{c} T \\ S\ \ T \\ T\ \ S \\ S \end{array}.$$

One can see that the Loewy series of projective indecomposable A-modules and projective indecomposable A'-modules are the same. This equivalence is thus between two algebras with the same Cartan matrix, without being an autoequivalence.

Example 2.2.4 informs us that Conjecture 2.2.3 is not true in general. However, it is apparent from Theorem 2.2.2 that periodic modules are of great relevance to two-step self-perverse equivalences. We thus need to be more careful in determining the precise relationship.

2.3 Periodicity and Perversity

We come now to the main result of this chapter, Theorem 2.3.3. This theorem gives necessary and sufficient conditions for the existence of a two-step self-perverse equivalence

$$\Psi : D^b(A) \xrightarrow{\sim} D^b(A),$$

arising as the cone of a map $A \to X$ in $D^b(A\text{-}A)$.

We will fix some notation throughout this section. We will always denote by A a finite-dimensional symmetric k-algebra, $\{S_1, \ldots, S_r\}$ a complete set of simple A-modules up to isomorphism, $\{P_1, \ldots, P_r\}$ the set of projective indecomposable A-modules such that $P_i/\operatorname{rad}(P_i) \cong S_i \cong \operatorname{soc}(P_i)$, and $I = \{1, \ldots, r\}$ the indexing set.

2.3.1 The main theorem

Before stating our main theorem, it is necessary that we make the following definitions.

Definition 2.3.1. Let E be a finite-dimensional self-injective k-algebra, M an E-module, σ an automorphism of E, $n \in \mathbb{Z}_+$, and $\alpha \in \operatorname{Ext}^n_{E\otimes_k E^{\mathrm{op}}}(E, {}_\sigma E)$. We say that M is *strongly σ-periodic of period n relative to α* if $\alpha \otimes^{\mathbf{L}}_E M$ induces an isomorphism $\Omega^n_E(M) \cong {}_\sigma M$. Dually, we say that an E^{op}-module N is *strongly τ-periodic of period n relative to α*, for τ an automorphism of E, if there exists some $\alpha \in \operatorname{Ext}^n_{E\otimes_k E^{\mathrm{op}}}(E, E_\tau)$ such that $N \otimes^{\mathbf{L}}_E \alpha$ induces an isomorphism $\Omega^n_{E^{\mathrm{op}}}(N) \cong N_\tau$.

It is worth taking the time to unpack this definition. Recall that $\alpha \otimes^{\mathbf{L}}_E M \in \operatorname{Ext}^n_E(M, {}_\sigma M)$, and

$$\operatorname{Ext}^n_E(M, {}_\sigma M) \cong \operatorname{Hom}_{D^b(E)}(M, {}_\sigma M[n]) \cong \operatorname{Hom}_{E\text{-}\underline{\mathrm{mod}}}(M, \Omega^{-n}_E({}_\sigma M)).$$

The extension α therefore induces an E-module homomorphism $M \to \Omega^{-n}_E({}_\sigma M)$. Since E is self-injective, this in turn induces an E-module homomorphism $\Omega^n_E(M) \to {}_\sigma M$. The condition in Definition 2.3.1 is that this induced E-module homomorphism is an isomorphism.

Proposition 2.2.1 demonstrates that, with A a finite-dimensional symmetric k-algebra, P and Q projective A-modules, $E = \operatorname{End}_A(P)^{\mathrm{op}}$ and $M = \operatorname{Hom}_A(P, Q)$, if there is an automorphism σ of E, $n \in \mathbb{Z}$ and a subalgebra B of E such that E is σ-periodic of period n relative to B, then there is an $\alpha \in \operatorname{Ext}^n_{E\otimes_k E^{\mathrm{op}}}(E, {}_\sigma E)$ such that M is strongly σ-periodic relative to α. Indeed, with Y the truncated resolution of E relative to B, there is a triangle

$$Y \longrightarrow E \xrightarrow{\alpha} {}_\sigma E[n] \rightsquigarrow$$

in $D^b(E\text{-}E)$, and the proof of Proposition 2.2.1 shows that applying the functor $- \otimes^{\mathbf{L}}_E M$ induces an isomorphism $\Omega^n_E(M) \cong {}_\sigma M$.

One may wonder if other examples of strongly periodic modules exist. That is, does there exist a finite-dimensional self-injective k-algebra E and an E-module M such that M is strongly σ-periodic, but E is not σ-periodic, relative or otherwise? In §3.3, we will see examples of strongly periodic modules M over symmetric algebras E, which are not known to be (relatively)

σ-periodic. However, it remains open to find a strongly σ-periodic module over an algebra known to not be (relatively) σ-periodic.

Recall that a Serre subcategory of A-mod is generated by a set of simple A-modules.

Definition 2.3.2. Let P be a projective A-module. There is some subset $J \subset I$ such that $P = \bigoplus_{i \in I \setminus J} P_i^{m_i}$ for some integers $m_i \geq 1$. We call the Serre subcategory $\mathcal{A}_1$ of A-mod generated by the set $\{S_j\}_{j \in J}$ the Serre subcategory *prime to* P.

Clearly, the Serre subcategory $\mathcal{A}_1$ of A-mod prime to P depends only on the isomorphism classes of summands of P, and not their multiplicities.

The remainder of this section is dedicated to proving the following theorem.

Theorem 2.3.3. *Let A be a finite-dimensional, symmetric k-algebra. Let P and Q be projective A-modules with no common direct summands up to isomorphism, and such that $P \oplus Q$ is a projective generator of A. Set $E = \mathrm{End}_A(P)^{\mathrm{op}}$ and $M = \mathrm{Hom}_A(P, Q)$. Let $\mathcal{A}_1$ be the Serre subcategory of A-mod prime to P. Then there exists a standard derived equivalence*

$$\Phi : D^b(A) \xrightarrow{\sim} D^b(A)$$

self-perverse relative to

$$0 \subset_0 \mathcal{A}_1 \subset_n A\text{-mod},$$

together with a natural transformation $\mathrm{Id}_{D^b(A)} \longrightarrow \Phi$ restricting to a natural isomorphism $\mathrm{Id}_{\mathcal{A}_1} \xrightarrow{\sim} \Phi|_{\mathcal{A}_1}$ if and only if there exists an automorphism σ of E and an extension $\alpha \in \mathrm{Ext}^n_{E \otimes_k E^{\mathrm{op}}}(E, {}_\sigma E)$ such that the E-module M is strongly σ-periodic and the E^{op}-module $M^\vee$ is strongly σ^{-1}-periodic, both of period n and relative to α.

The statement that $P \oplus Q$ is a projective generator of A means that there is some $m \in \mathbb{Z}_+$ such that the regular A-module A is a direct summand of $(P \oplus Q)^{\oplus m}$. Further, we are assuming that P and Q have no common direct summands. Then there is some $J \subset I$ and integers $m_i, m_j \geq 1$ such that $P = \bigoplus_{i \in I \setminus J} P_i^{m_i}$ and $Q = \bigoplus_{j \in J} P_j^{m_j}$.

We will prove Theorem 2.3.3 in two parts. Firstly, Theorem 2.3.4 tells us that the existence of such a derived autoequivalence Φ guarantees that M and $M^\vee$ are twisted strongly periodic. Theorem 2.3.8 demonstrates the converse: that twisted strong periodicity of the modules M and $M^\vee$ guarantee the existence of a derived autoequivalence Φ with the requisite properties.

2.3.2 Necessary conditions

Our first main result is the following.

Theorem 2.3.4. *Suppose*

$$\Phi : D^b(A) \xrightarrow{\sim} D^b(A)$$

is a standard derived autoequivalence, perverse relative to a filtration

$$0 \subset_0 \mathcal{A}_1 \subset_n A\text{-mod}$$

of Serre subcategories. Suppose also that there is a natural transformation of functors $\mathrm{Id}_{D^b(A)} \to \Phi$ *restricting to a natural isomorphism* $\mathrm{Id}_{\mathcal{A}_1} \xrightarrow{\sim} \Phi|_{\mathcal{A}_1}$. *Then there are projective* A*-modules* P *and* Q*, with no common direct summands, such that* $\mathcal{A}_1$ *is the Serre subcategory prime to* P*,* $P \oplus Q$ *is a projective generator of* A *and, with* $E = \mathrm{End}_A(P)^{\mathrm{op}}$*, there is an automorphism* σ *of* E *such that* E*-module* $M = \mathrm{Hom}_A(P, Q)$ *is strongly* σ*-periodic of period* n*, and the* E^{op}*-module* $M^\vee$ *is strongly* σ^{-1}*-periodic of period* n*, relative to some* $\alpha \in \mathrm{Ext}^n_{E \otimes_k E^{\mathrm{op}}}(E, {}_\sigma E)$.

We first note that the standard restriction on the derived equivalence Φ is not too strong. Indeed, if we were to drop this assumption, then by Theorem 1.2.19, there is a standard derived equivalence

$$X \otimes^{\mathbf{L}}_A - : D^b(A) \xrightarrow{\sim} D^b(A)$$

agreeing with Φ on objects of $D^b(A)$. Moreover, by Proposition 1.3.14, if Φ is perverse relative to the stated filtration, then $X \otimes^{\mathbf{L}}_A -$ is, too.

It will also be prudent to investigate the condition on the natural transformation $\mathrm{Id}_{D^b(A)} \to \Phi$. Since Φ is standard, there is some $X \in D^b(A\text{-}A)$ such that $\Phi = X \otimes^{\mathbf{L}}_A -$. Similarly, the identity functor is such that $\mathrm{Id}_{D^b(A)} = A \otimes^{\mathbf{L}}_A -$. Thus, we have a natural transformation

$$A \otimes^{\mathbf{L}}_A - \longrightarrow X \otimes^{\mathbf{L}}_A -,$$

which by Yoneda's Lemma must come from a morphism

$$A \xrightarrow{h} X$$

in $D^b(A\text{-}A)$. Since Φ is perverse relative to the given filtration, we have a commutative diagram

$$\begin{array}{ccc} D^b(A) & \xrightarrow{\Phi[n]} & D^b(A) \\ \uparrow & & \uparrow \\ D^b_{\mathcal{A}_1}(A) & \xrightarrow{\sim} & D^b_{\mathcal{A}_1}(A) \\ \uparrow & & \uparrow \\ \mathcal{A}_1 & \xrightarrow{\sim} & \mathcal{A}_1 \end{array}$$

and Φ restricts in this way to an autoequivalence $\Phi|_{\mathcal{A}_1}$ of $\mathcal{A}_1$. For every A-module $V \in \mathcal{A}_1$, the induced map

$$\begin{array}{ccc} A \otimes^{\mathbf{L}}_A V & \xrightarrow{h \otimes^{\mathbf{L}}_A V} & X \otimes^{\mathbf{L}}_A V \\ \| & & \| \\ V & \xrightarrow{\sim} & \Phi(V) \end{array}$$

is an isomorphism.

Now, in order to prove Theorem 2.3.4, we will need an alternative characterisation of strong σ-periodicity.

Let $\alpha \in \mathrm{Ext}^n_{E \otimes_k E^{\mathrm{op}}}(E, {}_\sigma E)$. Then, since

$$\mathrm{Ext}^n_{E \otimes_k E^{\mathrm{op}}}(E, {}_\sigma E) \cong \mathrm{Hom}_{D^b(E\text{-}E)}(E, {}_\sigma E[n]),$$

the element α gives rise to a triangle

$$Y \longrightarrow E \xrightarrow{\alpha} {}_\sigma E[n] \rightsquigarrow$$

in $D^b(E\text{-}E)$.

Recall that, for any finite-dimensional k-algebra A, we have a quotient functor

$$D^b(A) \longrightarrow D^b(A)/\operatorname{Perf}(A)$$

and this right hand side is canonically equivalent to the stable module category $A\text{-}\underline{\mathrm{mod}}$. Given an object X in $D^b(A)$, denote by $\overline{X}$ the image of X in $A\text{-}\underline{\mathrm{mod}}$ under this quotient, as described in §1.2.10.

The functor

$$Y \otimes^{\mathbf{L}}_E - : D^b(E) \longrightarrow D^b(E)$$

induces the functor

$$\overline{Y} \otimes_E - : E\text{-}\underline{\mathrm{mod}} \longrightarrow E\text{-}\underline{\mathrm{mod}}$$

so that, for any E-module U, we have $\overline{Y} \otimes_E U \cong \overline{Y \otimes^{\mathbf{L}}_E U}$ in $E\text{-}\underline{\mathrm{mod}}$.

By a similar construction, one may show that, for any E^{op}-module V, we have $V \otimes_E \overline{Y} \cong \overline{V \otimes^{\mathbf{L}}_E Y}$ in $E^{\mathrm{op}}\text{-}\underline{\mathrm{mod}}$.

This gives us the following characterisation of strongly periodic modules over E and E^{op}.

Lemma 2.3.5. *The E-module M is strongly σ-periodic relative to α if and only if $Y \otimes^{\mathbf{L}}_E M$ is a perfect complex of E-modules. Dually, the E^{op}-module N is strongly σ^{-1}-periodic relative to α if and only if $N \otimes^{\mathbf{L}}_E Y$ is a perfect complex of E^{op}-modules.*

Proof. The triangle

$$Y \otimes^{\mathbf{L}}_E M \xrightarrow{f \otimes^{\mathbf{L}}_E M} M \xrightarrow{\alpha \otimes^{\mathbf{L}}_E M} {}_\sigma M[n] \rightsquigarrow$$

in $D^b(E)$ induces a triangle

$$\overline{Y} \otimes_E M \longrightarrow M \longrightarrow \Omega^{-n}_E({}_\sigma M) \rightsquigarrow$$

in $E\text{-}\underline{\mathrm{mod}}$. If M is strongly σ-periodic relative to α, then this second arrow is an isomorphism, and thus $\overline{Y} \otimes_E M \cong \overline{Y \otimes^{\mathbf{L}}_E M} \cong 0$ in $E\text{-}\underline{\mathrm{mod}}$, so $Y \otimes^{\mathbf{L}}_E M$ is a perfect complex of E-modules. But on the other hand, if $Y \otimes^{\mathbf{L}}_E M$ is a perfect complex of E-modules, then $\overline{Y \otimes^{\mathbf{L}}_E M} \cong \overline{Y} \otimes_E M \cong 0$, giving an isomorphism $M \xrightarrow{\sim} \Omega^{-n}_E({}_\sigma M)$, so that M is strongly σ periodic relative to α. The proof of the dual statement is similar. □

Next, we note that we may assume that the projective A-modules P and Q are direct sums of projective indecomposable modules, no two of which are isomorphic.

Proposition 2.3.6. *Let P and Q be projective A-modules such that $P \oplus Q$ is a projective generator of A. Let $J \subset I$ be such that $P = \bigoplus_{i \in I \setminus J} P_i^{m_i}$ and $Q = \bigoplus_{j \in J} P_j^{m_j}$ for some integers $m_i, m_j \geq 1$. Set $P' = \bigoplus_{i \in I \setminus J} P_i$ and $Q' = \bigoplus_{j \in J} P_j$. With $E = \mathrm{End}_A(P)^{\mathrm{op}}$ and $M = \mathrm{Hom}_A(P, Q)$, and $E' = \mathrm{End}_A(P')^{\mathrm{op}}$ and $M' = \mathrm{Hom}_A(P', Q')$, there is an automorphism σ of E and $n \in \mathbb{Z}_+$ such that the E-module M is strongly σ-periodic of period n if and only if there is an automorphism σ' of E' such that the E'-module M' is strongly σ'-periodic of period n.*

Proof. Let $V = \mathrm{Hom}_A(P, P') \cong P^\vee \otimes_A P'$. Then V is an E-E'-bimodule, projective as a left E-module and as a right E'-module. As an E-module, V is a projective generator, so by Theorem 1.1.2 induces a Morita equivalence $E\text{-mod} \xrightarrow{\sim} E'\text{-mod}$. In particular, $V \otimes_{E'} V^\vee \cong E$, and $V^\vee \otimes_E V \cong E'$.

Let ε be the counit of the adjunction $P \otimes_E - \dashv P^\vee \otimes_A -$. Then, since $P \otimes_E P^\vee \otimes_A P \cong P \otimes_E E \cong P$, the map

$$\varepsilon_P : P \otimes_E P^\vee \otimes_A P \longrightarrow P$$

is an isomorphism, and hence so is

$$\varepsilon_{P'} : P \otimes_E P^\vee \otimes_A P' \longrightarrow P'$$

as $P' \in P\text{-add}$. Since $V = P^\vee \otimes_A P'$, we thus have $P \otimes_E V \cong P'$, and $P \cong P' \otimes_{E'} V^\vee$. Similarly, if η is the unit of the adjunction $- \otimes_{E'} (P')^\vee \dashv - \otimes_A P$, then the map

$$\eta_{P^\vee} : P^\vee \longrightarrow P^\vee \otimes_A P' \otimes_{E'} (P')^\vee$$

is an isomorphism, since $P^\vee \in (P')^\vee\text{-add}$. Again, $P^\vee \otimes_A P' \cong V$, so we have $P^\vee \cong V \otimes_{E'} (P')^\vee$, and $V^\vee \otimes_E P^\vee \cong (P')^\vee$.

Suppose first that there is some automorphism σ of E such that M is strongly σ-periodic of period n, relative to $\alpha \in \mathrm{Ext}^n_{E \otimes_k E^{\mathrm{op}}}(E, {}_\sigma E)$. There is a distinguished triangle

$$Y \longrightarrow E \xrightarrow{\alpha} {}_\sigma E[n] \rightsquigarrow$$

in $D^b(E\text{-}E)$, and by Lemma 2.3.5, the object $Y \in D^b(E\text{-}E)$ is such that $Y \otimes^{\mathbf{L}}_E M$ is a perfect complex of E-modules. Since $V = P^\vee \otimes_A P'$ is projective as a left E-module, the functor $V^\vee \otimes_E - \otimes_E V$ is exact. Thus, we obtain a triangle

$$Y' \longrightarrow E' \xrightarrow{\alpha'} {}_{\sigma'} E'[n] \rightsquigarrow,$$

where $Y' = V^\vee \otimes_E Y \otimes_E V$, $\alpha' = V^\vee \otimes_E \alpha \otimes_E V$, and σ' is the restriction of σ to E', noting that $V^\vee \otimes_E E \otimes_E V \cong V^\vee \otimes_E V \cong E'$. Then

$$\begin{aligned} Y' \otimes^{\mathbf{L}}_{E'} M' &\cong (V^\vee \otimes_E Y \otimes_E V) \otimes^{\mathbf{L}}_{E'} ((P')^\vee \otimes_A Q') \\ &\cong V^\vee \otimes_E Y \otimes^{\mathbf{L}}_E P^\vee \otimes_A Q', \end{aligned}$$

since $V \otimes_{E'} (P')^\vee \cong P^\vee$. Since $Q' \in Q$-add and $Y \otimes^{\mathbf{L}}_E M \cong Y \otimes^{\mathbf{L}}_E P^\vee \otimes_A Q$ is a perfect complex of E-modules by assumption, the same is true of $Y \otimes^{\mathbf{L}}_E P^\vee \otimes_A Q'$. Thus, since $V^\vee$ is projective as a right E-module, the complex of E'-modules $Y' \otimes^{\mathbf{L}}_{E'} M'$ is perfect. By Lemma 2.3.5, M' is strongly σ'-periodic as an E'-module, relative to α'.

Conversely, suppose that there is some automorphism σ' of E' and $\alpha' \in \mathrm{Ext}^n_{E' \otimes_k E'^{\mathrm{op}}}(E', {}_\sigma E')$ such that M' is strongly σ'-periodic relative to α'. Similarly to the above, we have a triangle

$$Y' \longrightarrow E' \xrightarrow{\alpha'} {}_{\sigma'}E'[n] \rightsquigarrow ,$$

and an exact functor $V \otimes_{E'} - \otimes_{E'} V : D^b(E'\text{-}E') \to D^b(E\text{-}E)$. Set $Y = V \otimes_{E'} Y' \otimes_{E'} V$. We have $E \cong V \otimes_{E'} E' \otimes_{E'} V$. We thus have a triangle

$$Y \longrightarrow E \xrightarrow{\alpha} W \rightsquigarrow ,$$

where $W = V \otimes_{E'} {}_{\sigma'}E'[n] \otimes_{E'} V$. It is clear that $W \cong E[n]$ in $D^b(E^{\mathrm{op}})$. Thus, as an object of $D^b(E\text{-}E)$, W is isomorphic to the E-E-bimodule E concentrated in degree n, with the regular right action of E. The left action of E then passes through an algebra homomorphism $\sigma : E \to E$. But it is clear that $W \cong E[n]$ in $D^b(E)$, too. Therefore, $W \cong {}_\sigma E[n]$ in $D^b(E\text{-}E)$, and we have a triangle

$$Y \longrightarrow E \xrightarrow{\alpha} {}_\sigma E[n] \rightsquigarrow ,$$

from which, by an analogous argument to the above, it follows that M is strongly σ-periodic relative to α. □

The obvious dual statement for right modules is also true, in a very similar way. In particular, we may assume that A is basic and that $A \cong P \oplus Q$ as A-modules.

Finally, we require a lemma.

Lemma 2.3.7. *Let P be a projective A-module. Let $\mathcal{A}_1$ be the Serre subcategory of A-*mod *prime to P. Then for a perfect complex $Z \in D^b(A^{\mathrm{op}})$, we have $Z \in \langle P^\vee \rangle$ if and only if $Z \otimes^{\mathbf{L}}_A V = 0$ for all $V \in D^b_{\mathcal{A}_1}(A)$.*

We comment that the V in Lemma 2.3.7 is different from the V in Proposition 2.3.6.

Proof. First, note that for $V \in D^b_{\mathcal{A}_1}(A)$, since $\mathcal{A}_1$ is prime to P, we have $P^\vee \otimes^{\mathbf{L}}_A V = \mathrm{Hom}_A(P, V) \cong 0$, so one direction is clear.

For the other, suppose that $Z \otimes^{\mathbf{L}}_A V = 0$ for all $V \in D^b_{\mathcal{A}_1}(A)$. It incurs no loss of generality to assume that Z is a bounded below complex of projective A^{op}-modules, with $Z_0 \neq 0$ and $Z_m = 0$ for all $m < 0$. Assume Z is such that the maximum non-zero degree

$$N_Z = \max\{m \geq 0 : Z_m \neq 0\}$$

is minimal among objects of $D^b(A^{\mathrm{op}})$ with these properties. Since Z is perfect, N_Z is finite.

Suppose for a contradiction that $Z \notin \langle P^\vee \rangle$. By assumption, we have

$$0 = Z \otimes^{\mathbf{L}}_A V \cong \mathbf{R}\,\mathrm{Hom}_A(Z^\vee, V),$$

for every $V \in D^b_{\mathcal{A}_1}(A)$, and for every $t \in \mathbb{Z}$,

$$H_t(\mathbf{R}\,\mathrm{Hom}_A(Z^\vee, V)) = \mathrm{Hom}_{D^b(A)}(Z^\vee, V[t]) = 0.$$

Let $X = Z^\vee \in D^b(A)$. Suppose that $X_0 \in P$-add. Then there is a commutative diagram

$$\begin{array}{ccccccccccc}
 & & & & & & 0 & \longrightarrow & X_0 & \longrightarrow & 0 \\
 & & & & & & & & \downarrow & & \\
0 & \longrightarrow & X_{N_Z} & \longrightarrow & \dots & \longrightarrow & X_1 & \longrightarrow & X_0 & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & & & \\
0 & \longrightarrow & X_{N_Z} & \longrightarrow & \dots & \longrightarrow & X_1 & \longrightarrow & 0 & &
\end{array}$$

giving rise to a triangle

$$X \longrightarrow X' \longrightarrow X_0[1] \rightsquigarrow$$

in $D^b(A)$, with X' the complex defined by the third row of the diagram. Since $X_0 \in P$-add and by the assumption on Z, we must have $\mathbf{R}\,\mathrm{Hom}_A(X', V) \cong 0$ for every $V \in D^b_{\mathcal{A}_1}(A)$. This contradicts the minimality of Z, since $(X')^\vee$ has strictly smaller maximum non-zero degree. Therefore $X_0 \notin P$-add.

Then, with $X_0 \notin P$-add, there is some simple module S in $D^b_{\mathcal{A}_1}(A)$ such that S is a summand of $X_0/\mathrm{rad}(X_0)$. But then the morphism

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & X_{N_Z} & \longrightarrow & \dots & \longrightarrow & X_1 & \longrightarrow & X_0 & \longrightarrow & 0 \\ & & & & & & & & \downarrow & & \\ & & & & & & 0 & \longrightarrow & S & \longrightarrow & 0 \end{array}$$

is a non-zero element of $\mathrm{Hom}_{D^b(A)}(X, S) = \mathrm{Hom}_{D^b(A)}(Z^\vee, S)$. This is a contradiction. Hence, $Z \in \langle P^\vee \rangle$. $\square$

We now have all the tools to prove Theorem 2.3.4.

Proof of Theorem 2.3.4. We will assume that A is basic. By Proposition 2.3.6, this is a legitimate assumption.

Let $J \subset I$ be such that set $\{S_j\}_{j \in J}$ generates $\mathcal{A}_1$. Set $P = \oplus_{i \in I \setminus J} P_i$ and $Q = \oplus_{j \in J} P_j$. Then by construction, $\mathcal{A}_1$ is the Serre subcategory prime to P, P and Q have no common direct summands, and $A \cong P \oplus Q$ as A-modules, so $P \oplus Q$ is a projective generator of A.

Since Φ is standard, there is some $X \in D^b(A\text{-}A)$ such that $\Phi = X \otimes^{\mathbf{L}}_A -$. By assumption, we have a map $A \xrightarrow{h} X$ in $D^b(A\text{-}A)$. This gives rise to a triangle

$$Z \longrightarrow A \xrightarrow{h} X \rightsquigarrow$$

in $D^b(A\text{-}A)$, say Δ. Applying the triangulated functor

$$\mathbf{R}\,\mathrm{Hom}_{A\text{-}A}(P \otimes_k P^\vee, -) : D^b(A\text{-}A) \longrightarrow D^b(E\text{-}E)$$

to Δ, we obtain a triangle

$$P^\vee \otimes^{\mathbf{L}}_A Z \otimes^{\mathbf{L}}_A P \longrightarrow P^\vee \otimes^{\mathbf{L}}_A A \otimes^{\mathbf{L}}_A P \longrightarrow P^\vee \otimes^{\mathbf{L}}_A X \otimes^{\mathbf{L}}_A P \rightsquigarrow$$

in $D^b(E\text{-}E)$. First, we have

$$P^\vee \otimes^{\mathbf{L}}_A A \otimes^{\mathbf{L}}_A P \cong E.$$

Next, we note that by Proposition 1.3.15, the equivalence

$$- \otimes^{\mathbf{L}}_A X : D^b(A^{\mathrm{op}}) \xrightarrow{\sim} D^b(A^{\mathrm{op}})$$

is also a perverse equivalence, relative to

$$0 \subset_0 \mathcal{A}'_1 \subset_n A^{\mathrm{op}}\text{-mod},$$

where $\mathcal{A}'_1$ is the Serre subcategory of A^{op}-mod prime to $P^\vee$. In particular, we have $P^\vee \otimes^{\mathbf{L}}_A X \cong P^\vee[n]$ as an object of $D^b(A^{\mathrm{op}})$. We thus have that

$$P^\vee \otimes^{\mathbf{L}}_A X \otimes^{\mathbf{L}}_A P \cong P^\vee[n] \otimes^{\mathbf{L}}_A P \cong E[n]$$

in $D^b(E^{\mathrm{op}})$. We are interested in this as an object of $D^b(E\text{-}E)$. That is, we have a complex of E-E-bimodules, isomorphic to the E-E-bimodule E concentrated in degree n, with the regular right action of E. The left action of E on this E-E-bimodule must therefore pass through some algebra homomorphism $\sigma : E \longrightarrow E$. But note that $X \otimes^{\mathbf{L}}_A P \cong P[n]$ in $D^b(A)$, so that

$$P^\vee \otimes^{\mathbf{L}}_A X \otimes^{\mathbf{L}}_A P \cong P^\vee \otimes^{\mathbf{L}}_A P[n] \cong E[n]$$

in $D^b(E)$, too. Thus, the homomorphism σ must be an isomorphism. In other words,

$$P^\vee \otimes^{\mathbf{L}}_A X \otimes^{\mathbf{L}}_A P \cong {}_\sigma E[n]$$

in $D^b(E\text{-}E)$.

Setting $Y = P^\vee \otimes^{\mathbf{L}}_A Z \otimes^{\mathbf{L}}_A P$, we therefore have a triangle

$$Y \longrightarrow E \longrightarrow {}_\sigma E[n] \rightsquigarrow$$

in $D^b(E\text{-}E)$, say ∇. This defines an element $\alpha \in \mathrm{Ext}^n_{E\text{-}E}(E, {}_\sigma E[n])$. By Lemma 2.3.5, to prove that M is strongly σ-periodic relative to α, it suffices to show that $Y \otimes^{\mathbf{L}}_E M$ is a perfect object in $D^b(E)$.

To this end, take the object $Y \otimes^{\mathbf{L}}_E M$ of $D^b(E)$. We have

$$Y \otimes^{\mathbf{L}}_E M \cong P^\vee \otimes^{\mathbf{L}}_A Z \otimes^{\mathbf{L}}_A P \otimes^{\mathbf{L}}_E P^\vee \otimes_A Q.$$

Consider the adjunction

$$D^b(A^{\mathrm{op}}) \underset{-\otimes^{\mathbf{L}}_E P^\vee}{\overset{-\otimes^{\mathbf{L}}_A P}{\rightleftarrows}} D^b(E^{\mathrm{op}}).$$

For any object W in the thick subcategory $\langle P^\vee \rangle$ of $D^b(A^{\mathrm{op}})$, we have $W \otimes^{\mathbf{L}}_A P \otimes^{\mathbf{L}}_E P^\vee \cong W$ in $D^b(A^{\mathrm{op}})$. Thus, if we can show that $P^\vee \otimes^{\mathbf{L}}_A Z \in \langle P^\vee \rangle$, then we will have $P^\vee \otimes^{\mathbf{L}}_A Z \otimes^{\mathbf{L}}_A P \otimes^{\mathbf{L}}_E P^\vee \cong P^\vee \otimes^{\mathbf{L}}_A Z$, so that

$$Y \otimes^{\mathbf{L}}_E M \cong P^\vee \otimes^{\mathbf{L}}_A Z \otimes^{\mathbf{L}}_A Q.$$

Since $P^\vee \otimes^{\mathbf{L}}_A Z$ is a summand of $A \otimes^{\mathbf{L}}_A Z$, we need only show that $Z \in \langle P^\vee \rangle$, considered as an object of $D^b(A^{\rm op})$. By Lemma 2.3.7, this is equivalent to showing that $Z \otimes^{\mathbf{L}}_A V = 0$ for all $V \in D^b_{\mathcal{A}_1}(A)$.

Given $V \in D^b_{\mathcal{A}_1}(A)$, we have a triangle $\Delta \otimes^{\mathbf{L}}_A V$,

$$Z \otimes^{\mathbf{L}}_A V \longrightarrow A \otimes^{\mathbf{L}}_A V \stackrel{h \otimes^{\mathbf{L}}_A V}{\longrightarrow} X \otimes^{\mathbf{L}}_A V \rightsquigarrow .$$

By assumption, $h \otimes^{\mathbf{L}}_A V$ is an isomorphism, so $Z \otimes^{\mathbf{L}}_A V = 0$. Thus, $Y \otimes^{\mathbf{L}}_E M \cong P^\vee \otimes^{\mathbf{L}}_A Z \otimes^{\mathbf{L}}_A Q$.

It thus suffices to show that this right hand side is a perfect complex of left E-modules. The complex $P^\vee \otimes^{\mathbf{L}}_A Z$ fits into a triangle

$$P^\vee \otimes^{\mathbf{L}}_A Z \longrightarrow P^\vee \otimes^{\mathbf{L}}_A A \longrightarrow P^\vee \otimes^{\mathbf{L}}_A X \rightsquigarrow .$$

Clearly, $P^\vee \otimes^{\mathbf{L}}_A A \cong P^\vee$ is a perfect complex of $A^{\rm op}$-modules. Since Φ is a standard equivalence induced by X by assumption, X is a perfect complex in $D^b(A)$ and in $D^b(A^{\rm op})$. Thus, $P^\vee \otimes^{\mathbf{L}}_A X$ is a perfect complex in $D^b(A^{\rm op})$, so $P^\vee \otimes^{\mathbf{L}}_A Z$ is, too. Then, since Q is a projective A-module, the object $P^\vee \otimes^{\mathbf{L}}_A Z \otimes^{\mathbf{L}}_A Q \cong Y \otimes^{\mathbf{L}}_E M$ is perfect in $D^b(E)$. This completes the proof of the claim for M.

For the claim on $M^\vee$, it suffices to show that

$$M^\vee \otimes^{\mathbf{L}}_E Y \cong Q^\vee \otimes_A P \otimes^{\mathbf{L}}_E P^\vee \otimes^{\mathbf{L}}_A Z \otimes^{\mathbf{L}}_A P$$

is perfect in $D^b(E^{\rm op})$. A similar argument to the above, using a dual statement to Lemma 2.3.7, will show that $Z \otimes^{\mathbf{L}}_A P \in \langle P \rangle$, so that by the adjunction

$$D^b(A) \underset{P \otimes^{\mathbf{L}}_E -}{\overset{P^\vee \otimes^{\mathbf{L}}_A -}{\rightleftarrows}} D^b(E)$$

we have

$$M^\vee \otimes^{\mathbf{L}}_E Y \cong Q^\vee \otimes_A P \otimes^{\mathbf{L}}_E P^\vee \otimes^{\mathbf{L}}_A Z \otimes^{\mathbf{L}}_A P \cong Q^\vee \otimes^{\mathbf{L}}_A Z \otimes^{\mathbf{L}}_A P.$$

Then, the triangle

$$Z \otimes^{\mathbf{L}}_A P \longrightarrow A \otimes^{\mathbf{L}}_A P \longrightarrow X \otimes^{\mathbf{L}}_A P \rightsquigarrow$$

guarantees that $Z \otimes_A^{\mathbf{L}} P$ is perfect in $D^b(A)$, so that, since Q is projective and A is symmetric, $M^\vee \otimes_E^{\mathbf{L}} Y \cong Q^\vee \otimes_A^{\mathbf{L}} Z \otimes_A^{\mathbf{L}} P$ is perfect in $D^b(E^{\mathrm{op}})$. $\square$

We thus have the first part of our main theorem.

2.3.3 Sufficient conditions

The next step is to show that the converse to Theorem 2.3.4 also holds true. That is, twisted strongly periodic E-modules of period n give rise to two-step self-perverse equivalences of width n of the appropriate form.

Theorem 2.3.8. *Let P and Q be projective A-modules with no common direct summands such that $P \oplus Q$ is a projective generator of A. Let $E = \mathrm{End}_A(P)^{\mathrm{op}}$ and $M = \mathrm{Hom}_A(P, Q)$. If there is an automorphism σ of E, an $\alpha \in \mathrm{Ext}^n_{E \otimes_k E^{\mathrm{op}}}(E, {}_\sigma E)$ and an $n \in \mathbb{Z}_+$ such that the E-module M is strongly σ-periodic and the E^{op}-module $M^\vee$ is strongly σ^{-1}-periodic of period n relative to α, then there is a standard derived equivalence*

$$\Phi_P : D^b(A) \xrightarrow{\sim} D^b(A),$$

perverse relative to

$$0 \subset \mathcal{A}_1 \subset A\text{-mod},$$

where $\mathcal{A}_1$ is the Serre subcategory of A-mod prime to P.

Our proof of Theorem 2.3.8 closely follows the method of Grant [Gra13], itself based on Ploog's simplified proof that (geometric) spherical twists are derived autoequivalences [Plo05]. We require the following definitions.

Definition 2.3.9. Let $\mathcal{S}$ be a collection of objects in a triangulated category $\mathcal{T}$. The *right orthogonal complement* of S is

$$\mathcal{S}^\perp = \{V \in \mathcal{T} : \mathrm{Hom}_{\mathcal{T}}(U, V[i]) = 0 \text{ for all } U \in \mathcal{S}, i \in \mathbb{Z}\}.$$

The *left orthogonal* complement ${}^\perp\mathcal{S}$ of $\mathcal{S}$ is defined similarly.

By a result of Rickard [Ric02, Corollary 3.2], if Z is a bounded complex of projective A-modules and V is any object of $D^b(A)$, since A is symmetric, $\mathrm{Hom}_{D^b(A)}(Z, V)$ and $\mathrm{Hom}_{D^b(A)}(V, Z)$ are naturally dual as k-vector spaces[1]. In such instances, the right and left orthogonal complements $Z^\perp$ and ${}^\perp Z$ coincide, and we may refer unambiguously to the *orthogonal complement* $Z^\perp$ of Z. In particular, if P is a projective A-module, the orthogonal complement $P^\perp$ is unambiguously defined. We comment that $P^\perp = D^b_{\mathcal{A}_1}(A)$, while the proof of Lemma 2.3.7 shows that ${}^\perp D^b_{\mathcal{A}_1}(A) = \mathrm{Perf}(A) \cap \langle P \rangle$.

[1] Such objects Z are *0-Calabi-Yau* objects.

Definition 2.3.10. A collection of objects $\mathcal{S}$ in a triangulated category $\mathcal{T}$ is a *spanning class* for $\mathcal{T}$ if for every $V \in \mathcal{T}$, if $\operatorname{Hom}_{\mathcal{T}}(U, V[i]) = 0$ for every $U \in \mathcal{S}$ and all $i \in \mathbb{Z}$, then $V \cong 0$, and if $\operatorname{Hom}_{\mathcal{T}}(V[i], U) = 0$ for every $U \in \mathcal{S}$ and all $i \in \mathbb{Z}$, then $V \cong 0$.

The following lemma is [Gra13, Lemma 3.14].

Lemma 2.3.11. *If P is a projective A-module, then the collection of objects $\mathcal{S} = \{P\} \cup P^{\perp}$ is a spanning class for $D^b(A)$.*

Suppose the conditions of Theorem 2.3.8 hold. We now identify our functor $\Phi : D^b(A) \longrightarrow D^b(A)$.

The extension $\alpha \in \operatorname{Ext}_{E \otimes_k E^{\mathrm{op}}}(E, {}_{\sigma}E)$ gives rise to a triangle

$$Y \xrightarrow{f} E \xrightarrow{\alpha} {}_{\sigma}E[n] \rightsquigarrow$$

in $D^b(E\text{-}E)$, say ∇. We have a chain of isomorphims

$$\begin{aligned}
\operatorname{Hom}_{D^b(E\text{-}E)}(Y, E) &\cong \operatorname{Hom}_{D^b(E\text{-}E)}(Y, \mathbf{R}\operatorname{Hom}_A(P, P)) \\
&\cong \operatorname{Hom}_{D^b(A\text{-}E)}(P \otimes^{\mathbf{L}}_E Y, P) \\
&\cong \operatorname{Hom}_{D^b(A\text{-}E)}(P \otimes^{\mathbf{L}}_E Y, \mathbf{R}\operatorname{Hom}_{A^{\mathrm{op}}}(P^{\vee}, A)) \\
&\cong \operatorname{Hom}_{D^b(A\text{-}A)}(P \otimes^{\mathbf{L}}_E Y \otimes^{\mathbf{L}}_E P^{\vee}, A)
\end{aligned}$$

given by tensor-Hom adjunction. Let

$$P \otimes^{\mathbf{L}}_E Y \otimes^{\mathbf{L}}_E P^{\vee} \xrightarrow{g} A$$

be the image of $Y \xrightarrow{f} E$ under this chain of isomorphisms. As in [Gra13, Lemma 3.4] we can characterise the map g as the resulting map in the commutative diagram

$$\begin{array}{ccc}
P \otimes^{\mathbf{L}}_E Y \otimes^{\mathbf{L}}_E P^{\vee} & \overset{g}{\dashrightarrow} & A \\
\Big\downarrow {\scriptstyle P \otimes^{\mathbf{L}}_E f \otimes^{\mathbf{L}}_E P^{\vee}} & & \Big\uparrow {\scriptstyle \varepsilon^R_A} \\
P \otimes^{\mathbf{L}}_E E \otimes^{\mathbf{L}}_E P^{\vee} & \xrightarrow{\sim} & P \otimes^{\mathbf{L}}_E P^{\vee}
\end{array}$$

where ε^R is the counit of the adjunction $- \otimes^{\mathbf{L}}_E P^{\vee} \dashv - \otimes^{\mathbf{L}}_A P$. We note that ε^R_A is the usual evaluation map $P \otimes^{\mathbf{L}}_E P^{\vee} \to A$. This in turn gives rise to a triangle

$$P \otimes^{\mathbf{L}}_E Y \otimes^{\mathbf{L}}_E P^{\vee} \xrightarrow{g} A \longrightarrow X \rightsquigarrow$$

in $D^b(A\text{-}A)$, say Δ.

Definition 2.3.12. The functor

$$\Phi_P = X \otimes^{\mathbf{L}}_A - : D^b(A) \longrightarrow D^b(A)$$

is the *generalised periodic twist* of A at P.

Our task is to show that the generalised periodic twist Φ_P is an equivalence. We first show that we may again reduce to the case that P and Q are direct sums of projective indecomposable modules, no two of which are isomorphic.

Proposition 2.3.13. *Let $J \subset I$ such that $P = \bigoplus_{i \in I \setminus J} P_i^{m_i}$ and $Q = \bigoplus_{j \in J} P_j^{m_j}$ for integers $m_j, m_i \geq 1$. Let $P' = \bigoplus_{i \in I \setminus J} P_i$ and $Q' = \bigoplus_{j \in J} P_j$, $E' = \mathrm{End}_A(P')^{\mathrm{op}}$ and $M' = \mathrm{Hom}_A(P', Q')$. Then the generalised periodic twists of A at P and at P' coincide, $\Phi_P \cong \Phi_{P'}$.*

Proof. By construction, P' and Q' have no common direct summands and $P' \oplus Q'$ is a projective generator of A. By Proposition 2.3.6, with $E' = \mathrm{End}_A(P')^{\mathrm{op}}$ and $M' = \mathrm{Hom}_A(P', Q')$, the E'-module M' and the $(E')^{\mathrm{op}}$-module $(M')^\vee$ are strongly σ'-periodic relative to α', where σ' and α' are the restrictions of σ and α respectively to E'. The generalised periodic twist $\Phi_{P'}$ therefore exists as constructed.

Recall from the proof of Proposition 2.3.6, there is an E-E'-bimodule V such that $P' \otimes_{E'} V^\vee \cong P$ and $V \otimes_{E'} (P')^\vee \cong P^\vee$, and applying the functor $V^\vee \otimes^{\mathbf{L}}_E - \otimes^{\mathbf{L}}_E V$ to ∇, we have a triangle

$$Y' \xrightarrow{f} E' \xrightarrow{\alpha'} {}_{\sigma'}E'[n] \rightsquigarrow$$

in $D^b(E'\text{-}E')$, with $Y' \cong V^\vee \otimes^{\mathbf{L}}_E Y \otimes^{\mathbf{L}}_E V$. Observe then that

$$\begin{aligned} P' \otimes^{\mathbf{L}}_{E'} Y' \otimes^{\mathbf{L}}_{E'} (P')^\vee &\cong P' \otimes^{\mathbf{L}}_{E'} (V^\vee \otimes_E Y \otimes_E V) \otimes^{\mathbf{L}}_{E'} (P')^\vee \\ &\cong P \otimes^{\mathbf{L}}_E Y \otimes^{\mathbf{L}}_E P^\vee. \end{aligned}$$

By the completion axiom for triangulated categories, we have a morphism of triangles

$$\begin{array}{ccccccc} P' \otimes^{\mathbf{L}}_{E'} Y' \otimes^{\mathbf{L}}_{E'} (P')^\vee & \longrightarrow & A & \longrightarrow & X' & \rightsquigarrow \\ \downarrow{\scriptstyle\sim} & & \| & & \vdots\downarrow & \\ P \otimes^{\mathbf{L}}_E Y \otimes^{\mathbf{L}}_E P^\vee & \longrightarrow & A & \longrightarrow & X & \rightsquigarrow \end{array}$$

in $D^b(A\text{-}A)$, and by the 5-Lemma for triangulated categories, this third arrow is an isomorphism. Thus,

$$\Phi_P = X \otimes^{\mathbf{L}}_A - \cong X' \otimes^{\mathbf{L}}_A - = \Phi_{P'},$$

and this completes the proof. □

In particular, we may assume that A is basic and that $A \cong P \oplus Q$ as A-modules.

We now work towards demonstrating that Φ_P is an equivalence. Recall that a functor is an equivalence if and only if it is fully faithful and essentially surjective. The following theorem of Bridgeland [Bri99, Theorem 2.3] will be useful.

Theorem 2.3.14. *Let $\mathcal{T}$, $\mathcal{T}'$ be triangulated categories and $F : \mathcal{T} \longrightarrow \mathcal{T}'$ a triangulated functor with a left and a right adjoint. Then F is fully faithful if and only if there is a spanning class $\mathcal{S}$ for $\mathcal{T}$ such that the homomorphisms*

$$\mathrm{Hom}_{\mathcal{T}}(U, V[i]) \longrightarrow \mathrm{Hom}_{\mathcal{T}'}(F(U), F(V[i]))$$

are bijective for every U, V in $\mathcal{S}$ and $i \in \mathbb{Z}$.

Our functor Φ_P satisfies the first clause of this theorem.

Lemma 2.3.15. *The object X is perfect in $D^b(A)$ and $D^b(A^{\mathrm{op}})$.*

Proof. Consider the triangle Δ,

$$P \otimes^{\mathbf{L}}_E Y \otimes^{\mathbf{L}}_E P^\vee \longrightarrow A \longrightarrow X \rightsquigarrow .$$

The A-A-bimodule A is projective as an A-module and as an A^{op}-module. We have that

$$P^\vee = \mathrm{Hom}_A(P, A) \cong \mathrm{Hom}_A(P, P \oplus Q) \cong E \oplus M$$

as an E-module, so that

$$Y \otimes^{\mathbf{L}}_E P^\vee \cong Y \oplus Y \otimes^{\mathbf{L}}_E M.$$

Similarly,

$$P \cong \mathrm{Hom}_A(A, P) \cong \mathrm{Hom}_A(P \oplus Q, P) \cong E \oplus M^\vee,$$

so that

$$P \otimes^{\mathbf{L}}_E Y \cong Y \oplus M^\vee \otimes^{\mathbf{L}}_E Y.$$

By assumption and Lemma 2.3.5, $Y \otimes_E^{\mathbf{L}} M$ is perfect in $D^b(E)$, and $M^\vee \otimes_E^{\mathbf{L}} Y$ is perfect in $D^b(E^{\mathrm{op}})$. The triangle ∇,

$$Y \longrightarrow E \longrightarrow {}_\sigma E[n] \rightsquigarrow$$

informs us that Y is a perfect object in $D^b(E)$ and $D^b(E^{\mathrm{op}})$. Then

$$P \otimes_E^{\mathbf{L}} Y \otimes_E^{\mathbf{L}} P^\vee \cong P \otimes_E^{\mathbf{L}} Y \oplus P \otimes_E^{\mathbf{L}} Y \otimes_E^{\mathbf{L}} M$$

in $D^b(A)$. Since P is a projective A-module, the functor

$$P \otimes_E^{\mathbf{L}} - : D^b(E) \longrightarrow D^b(A)$$

sends perfect objects to perfect objects, so since Y and $Y \otimes_E^{\mathbf{L}} M$ are perfect objects in $D^b(E)$, $P \otimes_E^{\mathbf{L}} Y \otimes_E^{\mathbf{L}} P^\vee$ is perfect in $D^b(A)$. Similarly, since $P^\vee$ is a projective A^{op} module,

$$P \otimes_E^{\mathbf{L}} Y \otimes_E^{\mathbf{L}} P^\vee \cong Y \otimes_E^{\mathbf{L}} P^\vee \oplus M^\vee \otimes_E^{\mathbf{L}} Y \otimes_E^{\mathbf{L}} P^\vee,$$

and Y and $M^\vee \otimes_E^{\mathbf{L}} Y$ are perfect objects of $D^b(E^{\mathrm{op}})$, $P \otimes_E^{\mathbf{L}} Y \otimes_E^{\mathbf{L}} P^\vee$ is perfect in $D^b(A^{\mathrm{op}})$. Thus, X fits into the triangle Δ with two objects perfect in $D^b(A)$ and in $D^b(A^{\mathrm{op}})$, so X must be, too. ∎

Thus, the functor

$$X^\vee \otimes_A^{\mathbf{L}} - : D^b(A) \longrightarrow D^b(A)$$

is both left and right adjoint to Φ_P. In order to apply Theorem 2.3.14, we now investigate how Φ_P acts on the spanning class $\mathcal{S} = \{P\} \cup P^\perp$.

Proposition 2.3.16. *For any V in $P^\perp$, $\Phi_P(V) \cong V$.*

Proof. Consider the triangle $\Delta \otimes_A^{\mathbf{L}} V$ in $D^b(A)$,

$$P \otimes_E^{\mathbf{L}} Y \otimes_E^{\mathbf{L}} P^\vee \otimes_A^{\mathbf{L}} V \longrightarrow A \otimes_A^{\mathbf{L}} V \longrightarrow X \otimes_A^{\mathbf{L}} V \rightsquigarrow .$$

Clearly, $A \otimes_A^{\mathbf{L}} V \cong V$. We have $P^\vee \otimes_A^{\mathbf{L}} V \cong \mathrm{Hom}_A(P, V)$; as P is a projective A-module, we need not derive these functors. Also since P is projective, we have $\mathrm{Hom}_{K^b(A)}(P, V) \cong \mathrm{Hom}_{D^b(A)}(P, V)$, so that the homology of the complex $\mathrm{Hom}_A(P, V)$ is given by

$$H_i(\mathrm{Hom}_A(P, V)) \cong \mathrm{Hom}_{K^b(A)}(P, V[i]) \cong \mathrm{Hom}_{D^b(A)}(P, V[i])$$

for every $i \in \mathbb{Z}$. But $V \in P^{\perp}$, so $H_i(\mathrm{Hom}_A(P, V)) = 0$ for every i. Thus, $P^{\vee} \otimes_A^{\mathbf{L}} V \cong \mathrm{Hom}_A(P, V) \cong 0$ in $D^b(A)$. The triangle $\Delta \otimes_A^{\mathbf{L}} V$ is therefore isomorphic to the triangle

$$0 \longrightarrow V \longrightarrow X \otimes_A^{\mathbf{L}} V \rightsquigarrow .$$

Thus, $\Phi_P(V) = X \otimes_A^{\mathbf{L}} V \cong V$. $\square$

Proposition 2.3.17. *We have* $\Phi_P(P) \cong P[n]$.

Proof. Consider the triangles $P \otimes_E^{\mathbf{L}} \nabla$ and $\Delta \otimes_A^{\mathbf{L}} P$ in $D^b(A)$. The former is

$$P \otimes_E^{\mathbf{L}} Y \longrightarrow P \otimes_E^{\mathbf{L}} E \longrightarrow P \otimes_E^{\mathbf{L}} E_{\sigma^{-1}}[n] \rightsquigarrow$$

and the latter

$$P \otimes_E^{\mathbf{L}} Y \otimes_E^{\mathbf{L}} P^{\vee} \otimes_A^{\mathbf{L}} P \longrightarrow A \otimes_A^{\mathbf{L}} P \longrightarrow X \otimes_A^{\mathbf{L}} P \rightsquigarrow .$$

Observe first that $P \otimes_E^{\mathbf{L}} E_{\sigma^{-1}}[n] \cong P[n]$ in $D^b(A)$. We wish to build a commutative diagram

$$\begin{array}{ccc} P \otimes_E^{\mathbf{L}} Y & \xrightarrow{P \otimes_E^{\mathbf{L}} f} & P \otimes_E^{\mathbf{L}} E \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\ P \otimes_E^{\mathbf{L}} Y \otimes_E^{\mathbf{L}} P^{\vee} \otimes_A^{\mathbf{L}} P & \xrightarrow{g \otimes_A^{\mathbf{L}} P} & A \otimes_A^{\mathbf{L}} P \end{array},$$

in which the vertical arrows are isomorphisms, from which the completion axiom and the 5-Lemma give an isomorphism of triangles

$$\begin{array}{ccccccc} P \otimes_E^{\mathbf{L}} Y & \longrightarrow & P \otimes_E^{\mathbf{L}} E & \longrightarrow & P \otimes_E^{\mathbf{L}} {}_{\sigma}E[n] & \rightsquigarrow \\ \downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle \sim} & \\ P \otimes_E^{\mathbf{L}} Y \otimes_E^{\mathbf{L}} P^{\vee} \otimes_A^{\mathbf{L}} P & \longrightarrow & A \otimes_A^{\mathbf{L}} P & \longrightarrow & X \otimes_A^{\mathbf{L}} P & \rightsquigarrow \end{array},$$

so that

$$\Phi_P(P) = X \otimes_A^{\mathbf{L}} P \cong P \otimes_E^{\mathbf{L}} {}_{\sigma}E[n] \cong P[n].$$

Let β be the obvious isomorphism induced by the isomorphisms

$$P \otimes_E^{\mathbf{L}} E \cong P \cong A \otimes_A^{\mathbf{L}} P.$$

Consider the adjunction $- \otimes^{\mathbf{L}}_E P^\vee \dashv - \otimes^{\mathbf{L}}_A P$. Let ε^R and η^R be the counit and unit of this adjunction. Define α by

$$P \otimes^{\mathbf{L}}_E Y \xrightarrow{\eta^R_{P \otimes^{\mathbf{L}}_E Y}} P \otimes^{\mathbf{L}}_E Y \otimes^{\mathbf{L}}_E P^\vee \otimes^{\mathbf{L}}_A P.$$

The triangle $P \otimes^{\mathbf{L}}_E \nabla$ informs us that $P \otimes^{\mathbf{L}}_E Y \in \langle P \rangle$ in $D^b(E^{\mathrm{op}})$, so α is an isomorphism.

It thus remains to show that $(g \otimes^{\mathbf{L}}_A P) \circ \alpha = \beta \circ (P \otimes^{\mathbf{L}}_E f)$. From the construction of g, we have a commutative diagram

$$\begin{array}{ccc} P \otimes^{\mathbf{L}}_E Y \otimes^{\mathbf{L}}_E P^\vee \otimes^{\mathbf{L}}_A P & \xrightarrow{g \otimes^{\mathbf{L}}_A P} & A \otimes P \\ \Big\downarrow{\scriptstyle P \otimes^{\mathbf{L}}_E f \otimes^{\mathbf{L}}_E P^\vee \otimes^{\mathbf{L}}_A P} & & \Big\uparrow{\scriptstyle \varepsilon^R_A \otimes^{\mathbf{L}}_A P} \\ P \otimes^{\mathbf{L}}_E E \otimes^{\mathbf{L}}_E P^\vee \otimes^{\mathbf{L}}_A P & \xrightarrow{\beta \otimes^{\mathbf{L}}_E P^\vee \otimes^{\mathbf{L}}_A P} & A \otimes^{\mathbf{L}}_A P \otimes^{\mathbf{L}}_E P^\vee \otimes^{\mathbf{L}}_A P \end{array}$$

so that

$$(g \otimes^{\mathbf{L}}_A P) \circ \alpha = (\varepsilon^R_A \otimes^{\mathbf{L}}_A P) \circ (\beta \otimes^{\mathbf{L}}_E P^\vee \otimes^{\mathbf{L}}_A P) \circ (P \otimes^{\mathbf{L}}_E f \otimes^{\mathbf{L}}_E P^\vee \otimes^{\mathbf{L}}_A P) \circ \eta^R_{P \otimes^{\mathbf{L}}_E Y}.$$

Since $P^\vee \otimes^{\mathbf{L}}_A P \cong E$, by the naturality of η^R we have that

$$(g \otimes^{\mathbf{L}}_A P) \circ \alpha = (\varepsilon^R_A \otimes^{\mathbf{L}}_A P) \circ \eta^R_{A \otimes^{\mathbf{L}}_A P} \circ \beta \circ (P \otimes^{\mathbf{L}}_E f),$$

and since

$$(\varepsilon^R_A \otimes^{\mathbf{L}}_A P) \circ \eta^R_{A \otimes^{\mathbf{L}}_A P} = \mathrm{id}_{A \otimes^{\mathbf{L}}_A P},$$

we have $(g \otimes^{\mathbf{L}}_A P) \circ \alpha = \beta \circ (P \otimes^{\mathbf{L}}_E f)$. The claimed commutative diagram therefore exists, and the result follows. □

Putting this all together, we conclude the following

Corollary 2.3.18. *The functor $\Phi_P : D^b(A) \longrightarrow D^b(A)$ is fully faithful.*

Proof. By Lemma 2.3.15, Φ_P has a left and a right adjoint. By Proposition 2.3.17,

$$\begin{aligned} \mathrm{Hom}_{D^b(A)}(\Phi_P(P), \Phi_P(P[i])) &\cong \mathrm{Hom}_{D^b(A)}(P[n], P[i][n]) \\ &\cong \mathrm{Hom}_{D^b(A)}(P, P[i]) \end{aligned}$$

By Proposition 2.3.16, if $U, V \in P^\perp$, then

$$\mathrm{Hom}_{D^b(A)}(\Phi(U), \Phi(V[i])) \cong \mathrm{Hom}_{D^b(A)}(U, V[i]),$$

and for $V \in P^{\perp}$,

$$\begin{aligned}\operatorname{Hom}_{D^b(A)}(\Phi(P), \Phi(V[i])) &\cong \operatorname{Hom}_{D^b(A)}(P[n], V[i]) \\ &\cong \operatorname{Hom}_{D^b(A)}(P, V[i]),\end{aligned}$$

and by [Ric02, Corollary 3.2], the spaces $\operatorname{Hom}_{D^b(A)}(P, V[i])$ and

$$\operatorname{Hom}_{D^b(A)}(V[i], P) \cong \operatorname{Hom}_{D^b(A)}(V, P[-i])$$

are naturally dual. Thus, for any U, V in the spanning class $\mathcal{S} = \{P\} \cup P^{\perp}$, the induced homomorphism

$$\operatorname{Hom}_{D^b(A)}(U, V[i]) \longrightarrow \operatorname{Hom}_{D^b(A)}(\Phi(U), \Phi(V[i]))$$

is an isomorphism. Therefore, by Theorem 2.3.14, the functor Φ_P is fully faithful. □

We can now show that Φ is an equivalence.

Proposition 2.3.19. *The functor $\Phi_P : D^b(A) \longrightarrow D^b(A)$ is an equivalence.*

Proof. First, we note that, since X is perfect in $D^b(A)$ and in $D^b(A^{\mathrm{op}})$ by Lemma 2.3.15, Φ_P restricts to a functor

$$\hat{\Phi}_P : \operatorname{Perf}(A) \longrightarrow \operatorname{Perf}(A).$$

Moreover, since Φ_P is fully faithful by Corollary 2.3.18, this restriction is fully faithful, too. The image $\Phi_P(\operatorname{Perf}(A))$ is therefore a thick subcategory of $\operatorname{Perf}(A)$.

By Proposition 2.3.17, $\Phi_P(P) \cong P[n]$. Thus, $\langle P \rangle$ is contained in the image $\Phi_P(\operatorname{Perf}(A))$. Applying the functor $- \otimes_A^{\mathbf{L}} Q$ to the triangle Δ, we obtain a triangle

$$P \otimes_E^{\mathbf{L}} Y \otimes_E^{\mathbf{L}} P^{\vee} \otimes_A^{\mathbf{L}} Q \longrightarrow A \otimes_A^{\mathbf{L}} Q \longrightarrow X \otimes_A^{\mathbf{L}} Q \rightsquigarrow$$

in $D^b(A)$. We have $P^{\vee} \otimes_A^{\mathbf{L}} Q \cong M$, and since $Y \otimes_E^{\mathbf{L}} M$ is isomorphic to a perfect complex of E-modules,

$$P \otimes_E^{\mathbf{L}} Y \otimes_E^{\mathbf{L}} P^{\vee} \otimes_A^{\mathbf{L}} Q \cong P \otimes_E^{\mathbf{L}} Y \otimes_E^{\mathbf{L}} M$$

is isomorphic to an object in $\langle P \rangle$, and is therefore in $\Phi_P(\operatorname{Perf}(A))$. Since Q is projective, $X \otimes_A^{\mathbf{L}} Q \cong \Phi_P(Q) \in \Phi_P(\operatorname{Perf}(A))$. But $\Phi_P(\operatorname{Perf}(A))$ is closed under triangles, so $A \otimes^{\mathbf{L}}_A Q \cong Q \in \Phi_P(\mathrm{Perf}(A))$. But then $A \cong P \oplus Q \in \Phi_P(\mathrm{Perf}(A))$, so $\Phi_P(\mathrm{Perf}(A))$ must contain all of $\mathrm{Perf}(A)$. In particular, the restriction of Φ_P to $\mathrm{Perf}(A)$ is essentially surjective, so is an equivalence

$$\Phi_P : \mathrm{Perf}(A) \xrightarrow{\sim} \mathrm{Perf}(A).$$

By [Ric89b, Theorem 6.4], Φ_P is therefore an equivalence

$$\Phi_P : D^b(A) \xrightarrow{\sim} D^b(A),$$

and we are done. □

The final step is to show that Φ_P is a perverse equivalence with the expected perversity.

Proposition 2.3.20. *The generalised periodic twist $\Phi_P : D^b(A) \xrightarrow{\sim} D^b(A)$ is perverse relative to the filtration*

$$0 \subset_0 \mathcal{A}_1 \subset_n A\text{-mod}$$

on both sides.

Proof. We appeal to Proposition 1.3.12. Firstly, $\Phi_P(P) \cong P[n]$ by Proposition 2.3.17. Next, the triangle $\Delta \otimes^{\mathbf{L}}_A Q$ in the proof of Proposition 2.3.19,

$$P \otimes^{\mathbf{L}}_E Y \otimes^{\mathbf{L}}_E P^\vee \otimes^{\mathbf{L}}_A Q \longrightarrow A \otimes^{\mathbf{L}}_A Q \longrightarrow X \otimes^{\mathbf{L}}_A Q \rightsquigarrow$$

informs us that $\Phi_P(Q) \cong X \otimes^{\mathbf{L}}_A Q$ is isomorphic in $D^b(A)$ to a complex with Q in degree 0, and all other terms contained in $\langle P \rangle$. Thus, if $\mathcal{P}$ is the set of all projective indecomposable A-modules and $\mathcal{P}_1$ is the subset consisting of summands of P, we have demonstrated that Φ_P is perverse relative to the filtration

$$\emptyset \subset_n \mathcal{P}_1 \subset_0 \mathcal{P}.$$

The result then follows from Definition 1.3.11. □

Combining Proposition 2.3.19 with Proposition 2.3.20 completes the proof of Theorem 2.3.8. Combining Theorem 2.3.8 with Theorem 2.3.4 completes the proof of Theorem 2.3.3.

2.3.4 Cycle of equivalences

Grant's cycle of equivalences carries over to this generalised setting, with a slight adaptation.

Assume the finite-dimensional symmetric k-algebra A is basic. Again by Proposition 2.3.6 and 2.3.13, this restriction incurs no loss of generality. Let P and Q be projective A-modules such that $A \cong P \oplus Q$ as A-modules. Let $J \subset I$ be the subset such that $P \cong \bigoplus_{i \in I \setminus J} P_i$ and $Q \cong \bigoplus_{j \in J} P_j$.

Let $F_J^{(0)}$ be the elementary perverse equivalence for A at J,

$$F_J^{(0)} : D^b(A) \xrightarrow{\sim} D^b(A^{(1)}).$$

This is induced by a combinatorial tilting complex $T = \bigoplus_{i \in I} T_i$, as in Definition 1.2.18. For each $i \in I$, let $P_i^{(1)} = F_J^{(0)}(T_i)$. Then the $P_i^{(1)}$ form a complete set of projective indecomposable $A^{(1)}$-modules up to isomorphism. If $P^{(1)} = \bigoplus_{i \in I \setminus J} P_i^{(1)}$ and $\mathcal{A}_1^{(1)}$ is the Serre subcategory of $A^{(1)}$-mod prime to $P^{(1)}$, then $F_J^{(0)}$ is perverse relative to the filtrations

$$0 \subset_0 \mathcal{A}_1 \subset_{-1} A\text{-mod}$$

and

$$0 \subset_0 \mathcal{A}_1^{(1)} \subset_{-1} A^{(1)}\text{-mod}\,.$$

As before, we may iterate this construction. For each i, with $A^{(0)} = A$, let

$$F_J^{(i)} : D^b(A^{(i)}) \xrightarrow{\sim} D^b(A^{(i+1)})$$

be the elementary perverse equivalence for $A^{(i)}$ at J. Set $F == F_J^{(n-1)} \circ \ldots \circ F_J^{(0)}$, so that

$$F : D^b(A) \xrightarrow{\sim} D^b(A^{(n)})$$

is the nth iterated combinatorial tilt at J. Let $P_i^{(n)}$ be the projective indecomposable $A^{(n)}$-modules, obtained as the image point of the ith summand of iterative combinatorial tilting complexes. Set $P^{(n)} = \bigoplus_{i \in I \setminus J} P_i^{(n)}$, and $\mathcal{A}_1^{(n)}$ the Serre subcategory of $A^{(n)}$-mod prime to $P^{(n)}$. Then the equivalence F is perverse relative to the filtrations

$$0 \subset_0 \mathcal{A}_1 \subset_{-n} A\text{-mod}$$

and

$$0 \subset_0 \mathcal{A}_1^{(n)} \subset_{-n} A^{(n)}\text{-mod}\,.$$

Now, set $E = \mathrm{End}_A(P)^{\mathrm{op}}$ and $M = \mathrm{Hom}_A(P, Q)$. Suppose for some automorphism σ of E and some $\alpha \in \mathrm{Ext}^n_{E \otimes_k E^{\mathrm{op}}}(E, {}_\sigma E)$ that M is strongly σ-periodic and $M^\vee$ is strongly σ^{-1}-periodic of period n relative to α. Then by Theorem 2.3.8, the generalised periodic twist

$$\Phi_P : D^b(A) \xrightarrow{\sim} D^b(A)$$

exists and is an equivalence. Moreover, Φ_P is self-perverse relative to

$$0 \subset_0 \mathcal{A}_1 \subset_n A\text{-mod}\,.$$

By Lemma 1.3.7,

$$G = F \circ \Phi_P : D^b(A) \xrightarrow{\sim} D^b(A^{(n)})$$

is a perverse equivalence with perversity function identically zero. Thus, this induces a Morita equivalence

$$G : A\text{-mod} \xrightarrow{\sim} A^{(n)}\text{-mod}\,.$$

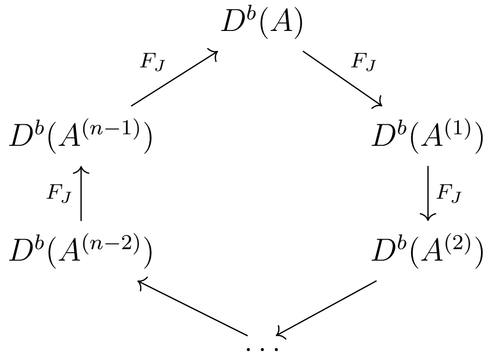
By Theorem 1.1.2, $G(A)$ is a progenerator of $A^{(n)}$, but since A is basic by assumption, and $A^{(n)}$ is basic by construction, taking opposites of endomorphism rings produces an isomorphism, $A \cong A^{(n)}$. Identifying A and $A^{(n)}$ via this isomorphism, we have a commutative diagram

$$\begin{array}{ccc} D^b(A) & \xrightarrow{F^{-1}} & D^b(A) \\ {\scriptstyle G}\uparrow & \nearrow{\scriptstyle \Phi_P} & \\ D^b(A) & & \end{array}$$

in which all the arrows are equivalences, and the two functors F^{-1} and Φ_P are naturally isomorphic. We have therefore shown the following.

Theorem 2.3.21. *The generalised periodic twist $\Phi_P : D^b(A) \xrightarrow{\sim} D^b(A)$ at P coincides with the inverse F^{-1} of the nth iterated combinatorial tilt F at J. That is, for every $V \in D^b(A)$, $\Phi_P(V) = X \otimes^{\mathbf{L}}_A V \cong F^{-1}(V)$.*

Thus, as in Grant's case, we obtain a cycle of derived equivalences



such that the complete cycle, starting and ending at $D^b(A)$, agrees with the inverse of the generalised periodic twist Φ_P. As in Grant's case, we obtain for free a two-step self-perverse equivalence

$$D^b(A^{(i)}) \xrightarrow{\sim} D^b(A^{(i)})$$

for every i, agreeing with the inverse of the generalised periodic twist $\Phi_{P^{(i)}}$. Determination of the algebras $A^{(i)}$ is far from trivial, but it will play an important role in our examples in Chapter 3.

2.3.5 Self-duality

It is natural to wonder if, in Theorem 2.3.3, the strong periodicity conditions on M and $M^\vee$ are both necessary. This question appears to be related to a question of the self-duality of the class α.

Let A, P, Q, E and M be as in the statement of Theorem 2.3.3. Let $\alpha \in \mathrm{Ext}^n_{E\otimes_k E^{\mathrm{op}}}(E, {}_\sigma E)$. Then α gives rise to a triangle

$$Y \xrightarrow{f} E \xrightarrow{\alpha} {}_\sigma E[n] \xrightarrow{h} Y[1]$$

in $D^b(E\text{-}E)$. Recall that M is strongly σ-periodic relative to α if $Y \otimes^{\mathbf{L}}_E M$ is a perfect object in $D^b(E)$. If Y is a self-dual object of $D^b(E\text{-}E)$, that is $Y^* \cong Y$, then, by Theorem 1.1.3, since A is a symmetric algebra and $M = \mathrm{Hom}_A(P, Q)$, we have

$$(Y \otimes^{\mathbf{L}}_E M)^* \cong M^* \otimes^{\mathbf{L}}_E Y^* \cong M^\vee \otimes^{\mathbf{L}}_E Y$$

is a perfect object of $D^b(E^{\mathrm{op}})$. Similarly, assuming the perfectness of $M^\vee \otimes^{\mathbf{L}}_E Y$ would give the perfectness of $Y \otimes^{\mathbf{L}}_E M$. In this case, then, we need only assume the strong periodicity of one of M or $M^\vee$.

Returning to the general setting, taking duals of the above triangle produces a triangle

$$Y^* \xleftarrow{f^*} E \xleftarrow{\alpha^*} {}_{\sigma^{-1}}E[n] \xleftarrow{h^*} Y^*[-1],$$

since by Proposition 2.1.1 and Theorem 1.1.3, $E \cong E^*$ as E-E-bimodules. Applying the shift functor $[n]$ and twisting the left action of E by σ produces a further triangle

$${}_{\sigma}Y^*[n-1] \xrightarrow{{}_{\sigma}h^*[n]} E \xrightarrow{{}_{\sigma}\alpha^*[n]} {}_{\sigma}E[n] \xrightarrow{{}_{\sigma}f^*[n]} {}_{\sigma}Y^*[n]$$

in $D^b(E\text{-}E)$. The morphism ${}_{\sigma}\alpha^*[n]$ corresponds to an element

$$\beta = {}_{\sigma}\alpha^*[n] \in \operatorname{Ext}^n_{E\otimes_k E^{\mathrm{op}}}(E, {}_{\sigma}E).$$

Let $W = {}_{\sigma}Y^*[n-1]$.

Proposition 2.3.22. *The E-module M is strongly σ-periodic of period n relative to α if and only if it is strongly σ-periodic of period n relative to β. Similarly, the E^{op}-module $M^{\vee}$ is strongly σ^{-1}-periodic of period n relative to α if and only if it is strongly σ^{-1}-periodic of period n relative to β.*

Proof. Firstly, suppose M is strongly σ-periodic of period n relative to α. Applying $- \otimes^{\mathbf{L}}_E M$ to the triangle above, we have

$$W \otimes^{\mathbf{L}}_E M \longrightarrow M \xrightarrow{\beta\otimes^{\mathbf{L}}_E M} {}_{\sigma}M[n] \rightsquigarrow$$

in $D^b(E)$. With the notation of §1.2.10 and the discussion preceding Lemma 2.3.5, passing to the stable category $E\text{-}\underline{\mathrm{mod}}$, we have a triangle

$$\overline{W} \otimes_E M \longrightarrow M \longrightarrow \Omega_E^{-n}({}_{\sigma}M) \rightsquigarrow .$$

By assumption and the discussion following Definition 2.3.1, $M \cong \Omega_E^{-n}({}_{\sigma}M)$ in $E\text{-}\underline{\mathrm{mod}}$. Thus, this second arrow is an isomorphism, and we have

$$\overline{W \otimes^{\mathbf{L}}_E M} \cong \overline{W} \otimes_E M \cong 0$$

in $E\text{-}\underline{\mathrm{mod}}$. In other words, $W \otimes^{\mathbf{L}}_E M$ is a perfect object of $D^b(E)$, so by Lemma 2.3.5, M is strongly periodic of period n relative to β. The other direction is analogous, and the dual statement is entirely similar. □

Now, Theorem 2.3.8 produces from the strong periodicity of M relative to α a two-step self-perverse equivalence

$$\Phi_{P,\alpha} : D^b(A) \xrightarrow{\sim} D^b(A).$$

However, the strong periodicity of M relative to β also produces such an equivalence

$$\Phi_{P,\beta} : D^b(A) \xrightarrow{\sim} D^b(A).$$

Following a similar argument to that in §2.3.4, the equivalences $\Phi_{P,\alpha}$ and $\Phi_{P,\beta}$ differ only by a self-Morita equivalence of A, which in turn induces some automorphism of A, so that via this automorphism, the functors $\Phi_{P,\alpha}$ and $\Phi_{P,\beta}$ are naturally isomorphic.

What, then, does this tell us about the classes α and β? Does the coincidence of $\Phi_{P,\alpha}$ and $\Phi_{P,\beta}$ guarantee that α and β are, in fact, representatives of the same class in $\mathrm{Ext}^n_{E\otimes_k E^{\mathrm{op}}}(E, {}_\sigma E)$?

If α and β are representatives of the same class, then the uniqueness of mapping cones in Definition 1.2.4 tells us that $Y \cong {}_\sigma Y^*[n-1]$ in $D^b(E\text{-}E)$, in which case the perfectness of $Y \otimes^{\mathbf{L}}_E M$ begets the perfectness of $M^\vee \otimes^{\mathbf{L}}_E Y$, as in the case where Y is self-dual, and we need only assume the strong periodicity of only one of M or $M^\vee$.

Unfortunately, to show that α and β are representatives of the same class appears difficult, if it is even the case at all. As such, we leave this for future work, possibly building on the work in this subsection, or that in [Gra13, §4.2].

Chapter 3

Application to Symmetric Groups

An interesting, and surprising, application of Theorem 2.3.3 occurs in the block theory of the symmetric groups. In this chapter, we briefly recount the representation theory of the symmetric groups, via some general group representation theory, before focusing on the class of blocks of symmetric group algebras of weight two in characteristic 3. It is in this context that we will see our examples, and some proposed further examples. We will then discuss the relevance of this to Broué's famous conjecture on blocks with abelian defect groups.

3.1 Representation Theory of the Symmetric Groups

We denote by $\mathfrak{S}_n$ the symmetric group on n letters. We will always take this to mean the group of permutations of the numbers $\{1, 2, \ldots, n\}$. Elements of $\mathfrak{S}_n$ can always be written uniquely as a product of disjoint cycles of weakly decreasing length.

The representation theory of the symmetric group is a broad and fascinating subject in its own right. We illustrate in §3.1.3 the relevant combinatorics of partitions and the p-abacus, before recounting in §3.1.4 the block theory of $k\mathfrak{S}_n$. The class of Brauer tree algebras play a key role in this story, as discussed in §3.1.5, and in §3.1.6, in which we describe an important class of blocks, named for Rouquier, Chuang and Kessar, as well as a combinatorial method of Scopes by which we may relate blocks of symmetric groups of different degrees.

In this section, classical results on the representation theory of $\mathfrak{S}_n$ are taken from [Jam06], unless otherwise stated. General definitions and results from representation theory are taken from [Alp93].

3.1.1 Finite group representation theory

Let G be a finite group and p a prime. Let K be a field of characteristic 0 and k a field of characteristic p.

For $R \in \{K, k\}$, the R-algebra RG is symmetric. The canonical symmetrising form on RG is

$$\mathrm{tr} : RG \to R, \quad \sum_{g \in G} \lambda_g g \mapsto \lambda_{1_G},$$

where $1_G \in G$ is the identity element.

By Maschke's Theorem, the K-algebra KG is semisimple. In particular, by Wedderburn's Theorem, if $\{U_1, \ldots, U_m\}$ is a complete set of non-isomorphic simple KG-modules, corresponding to the *ordinary irreducible representations* of G, then

$$KG \cong \bigoplus_{i=1}^{m} \mathrm{End}_K(U_i),$$

and each $\mathrm{End}_K(U_i)$ is a matrix algebra over K of degree equal to the dimension of the KG-module U_i. We note that the number of simple KG-modules is equal to the number of conjugacy classes of the group G. In general, there is no canonical bijection between these two sets.

The k-algebra kG has a block decomposition

$$kG = B_1 \times \ldots \times B_r.$$

The blocks B_i are symmetric k-algebras, because kG is a symmetric k-algebra. There is a *trivial* kG-module: a kG-module V such that $g \cdot v = v$ for every $g \in G$, $v \in V$. Then $V \cong k$ as k-vector spaces. The trivial module $V \cong k$ is simple and one-dimensional, and we call the block B_i such that $V \cong k$ lies in B_i the *principal block* of kG.

If $G \leq H$, then kG is a kH-kH-bimodule. The *restriction functor*

$$\mathrm{Res}^G_H : kG\text{-mod} \to kH\text{-mod}$$

sends a kG-module U to the kH-module $\mathrm{Res}^G_H(U)$ by restricting scalars. A kG-module homomorphism $\varphi : V \to V'$ is sent to the kH-module homomorphism

$$\mathrm{Res}^G_H(\varphi) : \mathrm{Res}^G_H(V) \to \mathrm{Res}^G_H(V')$$

by restricting scalars. The *induction functor*

$$\mathrm{Ind}_H^G : kH\text{-mod} \to kG\text{-mod}$$

sends a kH-module U to the kG-module

$$\mathrm{Ind}_H^G(U) = kG \otimes_{kH} U$$

by extension of scalars. Tensor-Hom adjunction specialises in this case to *Frobenius reciprocity*: if V is a kG-module and U is a kH-module, then we have an isomorphism

$$\mathrm{Hom}_{kG}(\mathrm{Ind}_H^G(U), V) \cong \mathrm{Hom}_{kH}(U, \mathrm{Res}_H^G(V))$$

of vector spaces. In other words, the functors

$$\mathrm{Ind}_H^G \dashv \mathrm{Res}_H^G$$

are left-right adjoint.

A kG-module M isomorphic to $\mathrm{Ind}_H^G(U)$ for some kH-module U is a *relatively H-free* kG-module. A kG-module Q is *relatively H-projective* if it is isomorphic to a summand of a relatively H-free module M. Free and projective kG-modules correspond respectively to relatively $\{1\}$-free and relatively $\{1\}$-projective kG-modules.

We do not expect kG to be a semisimple algebra. That is, the blocks B_i will not all be matrix algebras, although some might be. We would like to have a way of measuring *how close* a block is to a matrix algebra. Whence the notion of the *defect* of a block.

To each block $B = B_i$ of kG one assigns an important invariant: the *defect group* D of B. The subgroup D of G is a *defect group* of B if D is minimal with the property that, for every B-module M, M is relatively D-projective as a kG-module.

For a given block B of kG, the defect groups of B are all conjugate in G, and hence D is defined up to isomorphism. Furthermore, if D is a defect group of B, then D is a p-subgroup of G. Thus, there is some $d \geq 0$ such that $|D| = p^d$. We call the integer d the *defect* of the block B. When B is a principal block of kG, the defect groups of B and the Sylow p-subgroups of G coincide. At the other extreme, blocks of defect $d = 0$ are matrix algebras. In this sense, the defect measures how far a block is from being a matrix algebra.

Brauer's First Main Theorem, [Bra44, Theorem 4], here rephrased following [Alp93, Theorem 14.2], relates blocks of kG and blocks of kH, for certain subgroups $H \leq G$, that have common defect groups.

Theorem 3.1.1. *Let D be a p-subgroup of G and let H be a subgroup of G containing $N_G(D)$. There is a one-to-one correspondence between the set of blocks of kG with defect group D and the set of blocks of kH with defect group D.*

In the case $H = N_G(D)$, we call the block C of $kN_G(D)$ corresponding to the block B of kG the *Brauer correspondent* of B. By Brauer's Second Main Theorem [Bra44, Theorem 5], when B is a principal block, so that D is a Sylow p-subgroup of G, the Brauer correspondent C is the principal block of $kN_G(D)$.

3.1.2 Broué's conjecture

The p-local representation theory of finite groups is replete with so-called local-global counting conjectures, predicting many striking relationships between the representation theory of a finite group and that of its p-local subgroups. For example, Alperin's Weight Conjecture [Alp87] predicts that, if B is a block of kG with an abelian defect group D and C the Brauer correspondent block of $kN_G(D)$, then there are an equal number of simple B-modules as C-modules. One might expect that this equality comes from a deeper structural result; for example, that B and C are Morita equivalent. A famous example of Rickard [Ric88] regarding a block of the alternating group $\mathfrak{A}_5$ in characteristic 3 discounts this possibility, but prompts the following profound conjecture of Broué.

Conjecture 3.1.2. *Let B be a block of kG and D a defect group of B such that D is abelian. Let b be the Brauer correspondent of B. Then there is a derived equivalence*

$$D^b(B) \xrightarrow{\sim} D^b(b).$$

For obvious reasons, this conjecture is widely known as Broué's Abelian Defect Group Conjecture. The derived equivalence class of a block encodes the information of the number of isomorphism classes of simple modules lying in that block. Consequently, Conjecture 3.1.2 implies the stated conjecture of Alperin.

Conjecture 3.1.2 is known to hold in a number of cases, including for the symmetric groups, as we discuss in §3.1.7. A proof in full generality remains elusive. We point out a glaring insufficiency: what happens when the defect group D of B is non-abelian? In such cases, the block B may have more simple modules than the Brauer correspondent, immediately ruling out the possibility of a derived equivalence. One can take as an example the principal block of $\mathfrak{S}_4$ in characteristic 2, with Brauer correspondent the principal block of D_8. Conjecture 3.1.2 is then patently false. Unfortunately, a refinement allowing for non-abelian defect groups, or an alternative conjecture in these cases, is as of yet unforthcoming. For a broader discussion, see [CR07, §10].

3.1.3 Symmetric group representation theory

In characteristic 0, the isomorphism classes of simple $K\mathfrak{S}_n$-modules are in one-to-one correspondence with the set of *partitions* of n. That is, sequences of non-negative integers $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$ such that $\lambda_i \geq \lambda_{i+1}$ for every i, and $\sum_i \lambda_i = n$. The *length* of λ, $\ell(\lambda)$, is the largest integer $l \geq 1$ such that λ_l is non-zero.

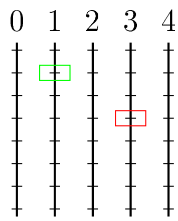
The classification of the simple modules in positive characteristic is due to James [Jam76]. To every partition λ of n, we assign a $k\mathfrak{S}_n$ module S^λ, called a *Specht module*. The Specht modules are not in general simple. A partition λ is *p-regular* if λ does not have p non-zero parts of the same size. When λ is p-regular, S^λ has a unique simple quotient D^λ. The D^λ for p-regular partitions λ form a complete set of simple $k\mathfrak{S}_n$-modules up to isomorphism.

One can therefore naturally describe the mathematics of the representation theory of the symmetric groups through the combinatorics of partitions. To utilise the power of this, James pioneered the use of the p-abacus. The presentation here is slightly non-standard, based on [CMT08].

Suppose λ is a partition, and take $r \in \mathbb{Z}$ with $r \geq \ell(\lambda)$. For $i = 1, \ldots, r$, let $\beta_i = \lambda_i + r - i$. The set $\mathfrak{B}_r(\lambda) = \{\beta_1, \ldots, \beta_r\}$ is called the *r-beta set* for λ. Distinct integers r give distinct r-beta sets for λ.

Let p be prime. The (empty) *p-abacus* is an abacus with p vertical runners of arbitrary (theoretically infinite) length, labelled $0, 1, \ldots, p-1$ from left-to-right, with positions indicated by dashes on the runners. The positions on runner i have values $i, i+p, i+2p, \ldots$ from top-to-bottom. If $a, b \in \mathbb{Z}_{\geq 0}$, then we say a comes *before* b in the p-abacus if $a < b$, and a comes *after* b in the p-abacus if $a > b$.

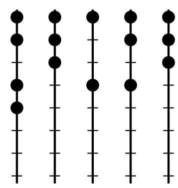
For example, for $p = 5$, the p-abacus is:



From top-to-bottom, the positions on runner 0 are $0, 5, 10, 15, \ldots$, on runner 1 are $1, 6, 11, 16, \ldots$, and so on. The position indicated in red is *after* the position indicated in green.

Let λ be a partition and $\mathfrak{B}_r(\lambda)$ its r-beta set. The r *bead p-abacus* display for λ is the p-abacus with a bead placed in position β_i for $i = 1, \ldots, r$. If p is understood, we call this the r bead abacus display for λ, or simply *an* abacus display for λ. When an abacus display is fixed, we may refer to it as *the* abacus display for λ.

For example, suppose $p = 5$, $\lambda = (6, 5^2, 4^2, 2, 1^2)$, and $r = 15$. Then $\mathfrak{B}_r(\lambda) = \{20, 18, 17, 15, 14, 11, 9, 8, 6, 5, 4, 3, 2, 1, 0\}$, and the r-bead p-abacus display for λ is



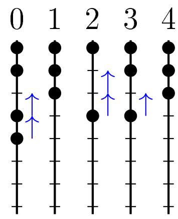
By counting the gaps in a left-to-right, top-to-bottom manner, one can easily read a partition off its abacus display. The bead corresponding to $\beta_1 = 20$ occurs after six gaps, so $\lambda_1 = 6$; the bead corresponding to $\beta_2 = 18$ appears after five gaps, so $\lambda_2 = 5$; and so on.

A position on an abacus display for λ is *occupied* if it has a bead in it, and *empty* if not. In other words, position b is occupied if and only if $b \in \mathfrak{B}_r(\lambda)$, for our chosen r. We say the bead at position $\beta_1 \in \mathfrak{B}_r(\lambda)$ in this display is in the *last occupied position*.

We call a bead in position b in an abacus display for λ *moveable* if $b - p \notin \mathfrak{B}_r(\lambda)$. Pictorially, the bead at position b is moveable if the position immediately above it is empty. In the previous example, the bead at position 18 is moveable, but the bead at position 8 is not.

Given a moveable bead in position b, we may draw an arrow from position b to $b - p$ on the abacus to indicate the movement of the bead up the runner. One can repeat this process, simulating the movement of each bead and recording the movement of a moveable bead on the resulting abacus display with an arrow, until no more beads can be moved. One is then left with an abacus display with no moveable beads, decorated with arrows. We call these arrows the *bead movements* of λ; note that the number of bead movements does not depend on the choice of r. In absolute terms, the number of bead movements on each runner is also independent of r, but the choice of r dictates the order in which the runners with each number of bead movements appears.

In our previous example:



The total number of bead movements is the *p-weight* of λ. The abacus display resulting from the completion of all bead movements (that is, following the movements indicated by the arrows) is the abacus display of a partition τ, the *p-core* of λ. A partition τ is an *p-core partition* if it has p-weight 0. If λ is a partition with p-weight w and p-core τ, then $|\lambda| = |\tau| + wp$.

In our example, the partition $\lambda = (6, 5^2, 4^2, 2, 1^2)$ has 5-weight 5 and 5-core (1^3); see Figure 3.1.3.

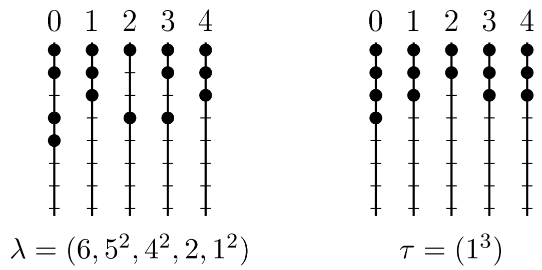


Figure 3.1: Our example partition and its 5-core.

3.1.4 Blocks of $k\mathfrak{S}_n$

The p-weight and p-core of a partition determine the block in which it lies. This is due to the following result, still known, several decades after its proof by Brauer and Robinson in [Bra47], as Nakayama's conjecture, as originally conjectured by Nakayama in [Nak40b].

Proposition 3.1.3. *Let λ, μ be partitions of n. Then S^λ and S^μ lie in the same block of $k\mathfrak{S}_n$ if and only if λ and μ have the same p-core.*

Note that two partitions λ and μ of the same n with the same p-core are necessarily of the same p-weight. Consequently, a block of $k\mathfrak{S}_n$ is completely determined by an integer $w \geq 0$ and a p-core partition τ of $n - pw$. We call w the weight and τ the p-core *of the block*. We write $B_{\tau,w}$ to indicate the block of $k\mathfrak{S}_n$ with p-core τ and of p-weight w. When the weight w is understood, we may write this as B_τ.

As an example, the Specht module $S^{(6,5^2,4^2,2,1^2)}$ lies in the block $B_{(1^3),5}$ of $k\mathfrak{S}_{28}$.

Suppose $B = B_{\tau,w}$ is a block of $k\mathfrak{S}_n$, with p-core τ and of p-weight w. Necessarily, $n \geq wp$. To construct the defect groups of B, we first do so for the principal block $B_{\emptyset,w}$ of $k\mathfrak{S}_{wp}$. In this case, the defect groups are the Sylow p-subgroups of $\mathfrak{S}_{wp}$.

Let P be a Sylow p-subgroup of $\mathfrak{S}_{wp}$. By a result of Kaloujnine [Kal48], P is a direct product of iterated wreath products of the cyclic group of order p. Write $wp = a_0 + a_1 p + \ldots + a_r p^r$, with $0 \leq a_i < p$. Then

$$P \cong (W_{p,1})^{a_1} \times (W_{p,2})^{a_2} \times \ldots \times (W_{p,r})^{a_r},$$

where $W_{p,i}$ is an iterated wreath product of i copies of C_p.

As in Conjecture 3.1.2, we focus our attention to the blocks whose defect groups are abelian. The group P is abelian exactly when it is a direct product of cyclic groups. This occurs if and only if $r = 1$; that is, when $w < p$. Assume then that $w < p$ and $P = C_p \times \ldots \times C_p = (C_p)^w$ is a Sylow p-subgroup of $\mathfrak{S}_{wp}$. The defect of $B_{\emptyset,w}$ is equal to the weight w. The normalizer is $N = N_{\mathfrak{S}_{wp}}(P) \cong (C_p \rtimes C_{p-1}) \wr \mathfrak{S}_w$. The Brauer correspondent of $B_{\emptyset,w}$ is the principal block b_0 of kN. By general group representation theory, $k(C_p \rtimes C_{p-1})$ is an indecomposable, non-semisimple k-algebra. The k-algebra $kN = k((C_p \rtimes C_{p-1}) \wr \mathfrak{S}_w)$ is indecomposable and non-semisimple, too (see e.g. [CT03]). Thus, $b_0 = kN$.

With this w fixed and τ a p-core partition, the group P is also a defect group of $B = B_{\tau,w}$ (see e.g. [JK84, §6.2.39]). Hence, for all blocks of symmetric groups, the defect groups are abelian if and only if $w < p$. In this case, B also has defect w. Thus, for blocks of symmetric groups with abelian defect, we may use the terms *weight* and *defect* synonymously.

We assume henceforth that $w < p$. Suppose $B = B_{\tau,w}$ is a block of $k\mathfrak{S}_n$ and P, N are as above. Then

$$N_{\mathfrak{S}_n}(P) \cong N \times \mathfrak{S}_{n-wp}.$$

By a result of Robinson [Rob51], the Brauer correspondent b of B is the block $b_0 \otimes B_{\tau,0}$ of $k(N \times \mathfrak{S}_{n-wp})$. Here, $B_{\tau,0}$ is a block of $k\mathfrak{S}_{n-wp}$ of defect zero, and is hence a matrix algebra. The block b is thus Morita equivalent to b_0, and we shall treat b_0 as the Brauer correspondent for our block B.

Observe that we have a chain of subgroups

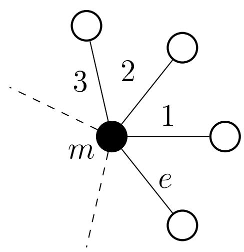
$$N = (C_p \rtimes C_{p-1}) \wr \mathfrak{S}_w \leq \mathfrak{S}_p \wr \mathfrak{S}_w \leq \mathfrak{S}_n.$$

Let $\tilde{b}_0$ be the principal block of $k(\mathfrak{S}_p \wr \mathfrak{S}_w)$. We will call $\tilde{b}_0$ the *intermediate block* and b_0 the *local block*.

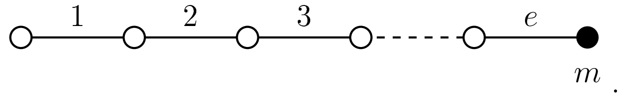
3.1.5 The local and intermediate blocks

The only non-trivial cyclic blocks of $k\mathfrak{S}_n$ are blocks of weight $w = 1$. If $B = B_{\tau,1}$ is a block of some $k\mathfrak{S}_n$ of weight 1, then abacus combinatorics tells us that there are p partitions of n with p-core τ and p-weight w, of which only one is not p-regular, and thus there are $p-1$ simple $k\mathfrak{S}_n$-modules lying in B. By a result of Dade [Dad66], B is Morita equivalent to a *Brauer tree algebra* on $p-1$ edges with exceptional multiplicity $m = 1$.

A *Brauer tree* is a quadruple $(\Gamma, \mathfrak{o}, v, m)$, where $\Gamma = (\Gamma_0, \Gamma_1)$ is a connected, acyclic graph (i.e. a tree) with vertex set Γ_0 and edge set Γ_1, $\mathfrak{o}$ is a cyclic ordering of the edges incident with each vertex $u \in \Gamma_0$, $v \in \Gamma_0$ is an *exceptional vertex*, and $m \in \mathbb{Z}_+$ is the *exceptional multiplicity* of the vertex v. To a Brauer tree Γ we may associate a k-algebra A_Γ, called a *Brauer tree algebra*; see [Alp93, §17] for details. The Brauer tree algebras are finite-dimensional, symmetric, basic and indecomposable. Two important Brauer tree and associated Brauer tree algebras are the *star* $\Gamma_{e,m}$ on e edges with exceptional multiplicity m



and the *line* $\tilde{\Gamma}_{e,m}$ on e edges with exceptional multiplicity m



We denote by $A_{e,m}$ and $\tilde{A}_{e,m}$ the Brauer tree algebras associated to $\Gamma_{e,m}$ and $\tilde{\Gamma}_{e,m}$, respectively.

The intermediate block $\tilde{b}_0$ in weight 1, the principal block of $k\mathfrak{S}_p$, is Morita equivalent to the Brauer tree algebra $\tilde{A}_{p-1,1}$ of a line on $p-1$ edges with exceptional multiplicity $m = 1$. Further, the weight 1 local block b_0 is the indecomposable algebra $k(C_p \rtimes C_{p-1})$. The algebra $k(C_p \rtimes C_{p-1})$ is Morita equivalent to the Brauer tree algebra $A_{p-1,1}$ of a star on $p-1$ edges with exceptional multiplicity $m = 1$ (see e.g. [Alp93, Theorem 17.2]).

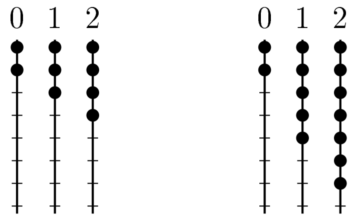
For higher weights $w \geq 2$, the block $\tilde{b}_0$ is Morita equivalent to the k-algebra $\tilde{A}_{p-1,1} \wr \mathfrak{S}_w$, while b_0 is Morita equivalent to the k-algebra $A_{p-1,1} \wr \mathfrak{S}_w$; see [CT03] for the notion of a wreath product of algebras.

3.1.6 Rouquier blocks and Scopes pairs

We define an important family of blocks, making use of the p-abacus.

Let $B = B_{\tau,w}$ be a block of $k\mathfrak{S}_n$. The block B is a *weight w Rouquier block* or *RoCK block* (for **Ro**uquier, **C**huang, **K**essar) if there is an abacus display for τ in which, for every i with $1 \leq i \leq e-1$, there are at least $w-1$ more beads on runner i than on runner $i-1$.

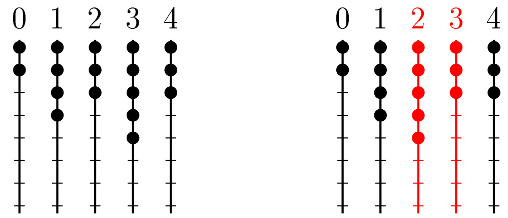
For example, with $p = 3$, the blocks $B_{(3,1^2),2}$ and $B_{(7,5,3^2,2^2,1^2),3}$ are weight 2 and weight 3 Rouquier blocks respectively.



We can relate all the blocks of $k\mathfrak{S}_n$ for all n of a given weight w to the Rouquier blocks, by the following construction.

Let $B = B_{\tau,w}$ be a block of $k\mathfrak{S}_n$. Suppose in some abacus display of the p-core partition τ, there are $m \geq 0$ more beads on runner i than runner $i-1$. Swap the runners i and $i-1$ to obtain an abacus display for another p-core partition partition, $\bar{\tau}$. Let $\overline{B} = B_{w,\bar{\tau}}$, a block of $\mathfrak{S}_{n-m}$. Then we say the blocks B and $\overline{B}$ form a *Scopes* $[w : m]$ *pair*. Scopes defined such pairs first in [Sco91] for $m \geq w$, and in general in [Sco95, Definition 2.1].

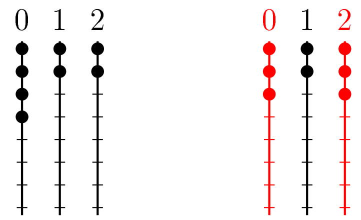
For example, with $p = 5$, the blocks $B = B_{(7,3,2,1^4),w}$ and $\overline{B} = B_{(6,2^2,1^4),w}$ form a Scopes $[w : 2]$ pair, for any w:



As an immediate remark, we are allowing the case $m = 0$. In this case, the blocks B and $\overline{B}$ are the same block, however the induced derived equivalence, described in §3.1.9, is not the identity functor.

A second important remark: we are taking the labels of the runners modulo p. That is, we allow $i = 0$, with a slight alteration: for $i = 0$, we require $m+1$ more beads on runner 0 than runner $p-1$, rather than m more beads. A clunky rewording of the condition will recover all cases: B and $\overline{B}$ form a $[w : m]$ pair if, in the abacus display for the p-core τ of B, the last occupied positions on runner i and $i-1$ respectively are $i + cp$ and $(i-1)+(c-m)p$. Here, the labels of runners are taken modulo p, but the positions are not.

For example, with $p = 3$, the following abacus display for the block $B_{(2),w}$ has two more beads in runner 0 than runner 2. The last occupied position in runner 0 is $0 + 3(3) = 9$, while in runner 2 the last occupied position is $-1+2(3) = 5$. By interchanging runners 0 and 2 as below, we have a Scopes $[w : 1]$ pair between $B_{(2),w}$ and $B_{(1),w}$, for any $w > 0$.



Note also the manner of runner interchange for $i = 0$. The bead at position $i + cp$ is moved to position $(i-1)+cp$, while the bead at position $(i-1)+(c-m)p$ is moved to position $i+(c-m)p$. For all practical purposes, the claim that runner i has more beads than runner $i-1$ includes this $i = 0$ case without further mention.

The motivation for Scopes's definition is the following observation, [Sco91, Theorem 5.1]. Given blocks A and A' of $k\mathfrak{S}_n$ and $k\mathfrak{S}_{n'}$ of the same weight w, there is a sequence of blocks

$$A = B_1, B_2, \ldots, B_t = A'$$

such that B_i and B_{i+1}, in either order, form a $[w : m]$ pair. Thus, all the blocks of a given weight communicate through a chain of Scopes pairs. This communication will be key §3.1.7.

The following result of Scopes, [Sco91, Theorem 4.2], is foundational.

Theorem 3.1.4. *If the blocks B and $\overline{B}$ form a $[w : m]$ pair and $m \geq w$, then there is a Morita equivalence*

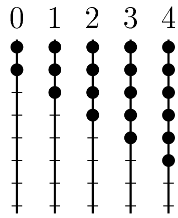
$$B\text{-mod} \xrightarrow{\sim} \overline{B}\text{-mod}\,.$$

Other Morita equivalences exist between blocks of symmetric groups, so we will call an equivalence arising from a Scopes $[w : m]$ pair with $m \geq w$ a *Scopes-Morita equivalence*. This naturally gives rise to an equivalence relation on the set of all blocks of $k\mathfrak{S}_n$ of weight w as n varies, whose equivalences classes we call the *Scopes-Morita equivalence classes*. For a given w and p, there are finitely many Scopes-Morita equivalence classes[1] of blocks of symmetric groups of weight w, by [Sco91, Theorem 5.1].

For the weight $w = 1$ blocks, we have the following consequence.

Corollary 3.1.5. *All blocks of symmetric groups of weight $w = 1$ are Morita equivalent.*

From their definition, it is clear that all of the Rouquier blocks for a given w and p fall into the same Scopes-Morita equivalence class. We typically choose as representative of this equivalence class the block $B_{\tau,w}$ such that τ has an abacus display for which, for every i with $1 \leq i \leq p-1$, there are exactly $w-1$ more beads on runner i than on runner $i-1$. For example, with $p = 5$ and $w = 2$, we would choose the block $B_{(10,6^2,3^3,1^4),2}$.



We then call this block *the* (weight w) Rouquier block.

[1] The alert reader may recall here Donovan's conjecture that, for a given finite p-group P, there are only finitely many Morita equivalence classes of blocks of kG for finite groups G with defect group isomorphic to P, see [Alp80, Conjecture M]. Scopes's result explicitly proves Donovan's conjecture for blocks of symmetric groups, [Sco91, Corollary 5.2]. For more on Donovan's conjecture, the interested reader is encouraged to visit the webpage [Eat19] dedicated to the status of the conjecture.

We note that the principal block of $k\mathfrak{S}_{wp}$ lies in a Scopes-Morita equivalence class on its own. We will call this block *the* weight w principal block. For a given w, the Rouquier block and the principal block are the blocks separated by the largest number of non-Morita equivalence Scopes pairs in a shortest chain of Scopes pairs between the two blocks. In this sense, they are the two extreme blocks on the scale.

3.1.7 Broué's conjecture for $\mathfrak{S}_n$

Broué's Conjecture 3.1.2 is known to hold for the symmetric groups. Reaching this point took a lot of effort and the best part of 20 years, so it is worth recounting the story of the proof here. The first point of call is the following result of Rickard [Ric89a, Theorem 4.2].

Theorem 3.1.6. *Let Γ be a Brauer tree on e edges with exceptional multiplicity m. Then there is a derived equivalence*

$$D^b(A_\Gamma) \xrightarrow{\sim} D^b(A_{e,m}).$$

Thus, all Brauer tree algebras of Brauer trees on a given number of edges with a given exceptional multiplicity are derived equivalent. Moreover, Rickard's proof is constructive. Let v be the exceptional vertex of Γ. For every edge i in Γ, there is a unique shortest path in Γ from v to the vertex at the furthest end of i. This defines a sequence of edges $i_0, i_1, \dots i_r = i$ in Γ. For each i, let X_i be the complex of projective A_Γ-modules

$$0 \longrightarrow P(i_0) \longrightarrow P(i_1) \longrightarrow \dots \longrightarrow P(i_r) \longrightarrow 0,$$

where by $P(j)$ we denote the projective cover of the simple A_Γ-module corresponding to the edge j. Set

$$X = \bigoplus_{i \in \Gamma_1} X_i \in K^b(A\text{-proj}).$$

Then Rickard proves that X is a one-sided tilting complex for A, so that Theorem 1.2.17 applies, and

$$\mathrm{End}_{D^b(A_\Gamma)}(X)^{\mathrm{op}} \cong A_{e,m}.$$

By construction, one can see from Proposition 1.3.12 that Rickard's equivalences are perverse, with the layers of the filtration and the perversity function determined by the distance of the furthest end of each edge from the exceptional vertex.

To prove Conjecture 3.1.2 for blocks of weight 0 is trivial, as every algebra in question is a matrix algebra. For blocks of weight 1, every algebra is a Brauer tree algebra, so by Theorem 3.1.6, Broué's Conjecture holds. One can say something stronger: all weight 1 blocks of all $k\mathfrak{S}_n$ in characteristic p are Morita equivalent, by Corollary 3.1.5 to Scopes's Theorem 3.1.4. In particular, all weight 1 blocks are Morita equivalent to the principal block of $k\mathfrak{S}_p$; that is, the Brauer tree algebra $\tilde{A}_{p-1,1}$. Thus, with $\Gamma = \tilde{\Gamma}_{p-1,1}$, a single derived equivalence

$$F_\Gamma : D^b(\tilde{A}_{p-1,1}) \xrightarrow{\sim} D^b(A_{p-1,1}),$$

induced by the tilting complex constructed by Rickard in the proof of Theorem 3.1.6, suffices.

The next step is to consider larger weights $w \geq 2$. This is immediately a harder problem, as the local block b_0 has a wreath product factor of $\mathfrak{S}_w$. Further, Scopes's Morita equivalences are only for blocks in a $[w : m]$ pair with $m \geq w$. For $1 \leq m < w$, such blocks are not Morita equivalent. It was conjectured, initially by Rickard, that blocks in a $[w : m]$ pair with $m < w$ are rather derived equivalent. In an unpublished theorem of 1990, Rickard proved this for $w \leq 5$ and all relevant p, so that, for a fixed p and a fixed $w \leq 5$, by this result and Scopes's, all blocks of weight w are derived equivalent.

This did not, however, tell us anything about how a given block B of $k\mathfrak{S}_n$ of weight $w \geq 2$ relates to either the intermediate weight w block $\tilde{b}_0$ or the local weight w block b_0. It was not until Okuyama [Oku97, Examples 4.3, 4.4] that concrete examples of derived equivalences between blocks of weight $w \geq 2$ and the local block b_0 appeared, here for the principal blocks of $k\mathfrak{S}_8$ and $k\mathfrak{S}_6$, both of weight $w = 2$ in characteristic $p = 3$.

After this, a systematic proof began to build. Rouquier had conjectured that the Rouquier blocks[2] should be Morita equivalent to the intermediate block $\tilde{b}_0$. By the following result of Marcus [Mar96, Example 5.7], this would prove Conjecture 3.1.2 for the Rouquier blocks.

Proposition 3.1.7. *There is a derived equivalence*

$$F_{\Gamma,w} : D^b(\tilde{A}_{p-1,1} \wr \mathfrak{S}_w) \xrightarrow{\sim} D^b(A_{p-1,1} \wr \mathfrak{S}_w).$$

This derived equivalence comes from an explicit lifting of the derived equivalence of Rickard in Theorem 3.1.6.

[2] Whence the name.

Chuang proves the conjecture on Rouquier blocks of weight 2 for any $p \geq 3$ in [Chu99, Theorem 3.1]. He further identifies tilting complexes inducing derived equivalences between blocks in a $[2:1]$ pair, giving a proof of Rickard's unpublished result in the case $w = 2$. Chuang's results combined with Proposition 3.1.7 thus prove Conjecture 3.1.2 for all blocks of weight $w = 2$.

The next major breakthrough came from Chuang and Kessar [CK02, Theorem 2], who proved that Rouquier's conjecture was true[3] for the Rouquier blocks of any weight w. Thus, by Proposition 3.1.7, Conjecture 3.1.2 holds for all Rouquier blocks, and by the unpublished result of Rickard, Conjecture 3.1.2 holds for all blocks of weight $w \leq 5$.

The final piece in the puzzle was then to improve Rickard's result to arbitrary weights. The resolution finally came from Chuang and Rouquier [CR08, Theorem 7.2], who proved that there is a derived equivalence between any two blocks of weight w for a fixed p. In particular, suppose the blocks B and $\overline{B}$ occur in a $[w:m]$ pair. Then there is a derived equivalence

$$D^b(B) \xrightarrow{\sim} D^b(\overline{B}),$$

given by what they termed $\mathfrak{sl}_2$*-categorification*. Consequently, Broué's Conjecture 3.1.2 holds for all blocks of the symmetric groups with abelian defect groups.

To summarise, given a block B of weight w and C the weight w Rouquier block, there is a sequence of blocks of weight w

$$B = B_0, B_1, \ldots, B_r = C,$$

such that the blocks B_i and B_{i-1} form a $[w:m]$ pair. Each of these pairs of blocks are derived equivalent, by Chuang and Rouquier, and by Chuang and Kessar, C is Morita equivalent to the intermediate block $\tilde{b}_0$, which is in turn derived equivalent to the local block b_0 by Proposition 3.1.7. We may picture this as

$$\begin{array}{ccccccccccc} & & & & & & B_r\text{-mod} & \xrightarrow{\sim} & \tilde{b}_0\text{-mod} & & \\ & & & & & & \downarrow & & \downarrow & & \\ D^b(B_0) & \xrightarrow{\sim} & D^b(B_1) & \xrightarrow{\sim} & \ldots & \xrightarrow{\sim} & D^b(B_r) & \dashrightarrow^{\sim} & D^b(\tilde{b}_0) & \xrightarrow{\sim} & D^b(b_0). \end{array}$$

The equivalences for $[w:m]$ pairs constructed by Chuang and Rouquier are perverse, as discussed in the next subsection. However, in general their composition fails to remain perverse. One may thus ask if there exists a

[3] Whence the *other* name.

single perverse equivalence, realising Conjecture 3.1.2 for any given block B of some $k\mathfrak{S}_n$.

Conjecture 3.1.8. *When B is a block of $k\mathfrak{S}_n$, the equivalence realising Conjecture 3.1.2 can be chosen to be perverse.*

The motivation for this conjecture comes partly from the representation theory of finite groups of Lie type. For such groups G, Conjecture 3.1.2 has a particular geometric form: it is predicted that there is a derived equivalence coming from the cohomology of a Deligne-Lusztig variety for G. A great deal of progress has occurred on this problem in the last two decades, primarily due to work of Craven, Rouquier and Dudas, [Cra12], [CR13], [CDR20]. Craven [Cra12, Conjecture 1.4] has conjectured that these derived equivalences should be perverse. This is proved for many examples of blocks of defect 2 and 3 in [Cra12] and [CR13], some of which occur as blocks of symmetric groups.

3.1.8 Derived equivalences from $\mathfrak{sl}_2$-categorification

The concept of $\mathfrak{sl}_2$-categorification is based on the representation theory of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$. The standard basis of the vector space $\mathfrak{sl}_2(\mathbb{C})$ consists of the matrices

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Given a finite-dimensional $\mathfrak{sl}_2(\mathbb{C})$-module V, we have a decomposition

$$V = \bigoplus_{\lambda \in \mathbb{Z}} V_\lambda$$

of V into weight spaces, where the V_λ are the (integral) eigenspaces of the action of h on V. The action of e on V maps a weight space V_λ to the weight space $V_{\lambda+2}$, while the action of f maps V_λ to $V_{\lambda-2}$:

$$\dots\ V_{\lambda-2} \underset{f}{\overset{e}{\rightleftarrows}} V_\lambda \underset{f}{\overset{e}{\rightleftarrows}} V_{\lambda+2}\ \dots$$

Since V is finite-dimensional, we may integrate to an action of the Lie group $\mathrm{SL}_2(\mathbb{C})$, for which the element

$$\theta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$$

acts on the weight spaces as

$$\theta : V_\lambda \xrightarrow{\sim} V_{-\lambda}.$$

The action of θ on V is given by

$$\theta = \exp(-f)\exp(e)\exp(-f) = \sum_{a,b,c\geq 0} (-1)^{a+c} f^{(a)} e^{(b)} f^{(c)},$$

where by $f^{(a)}$ we denote the divided power

$$f^{(a)} = \frac{f^a}{a!}.$$

Chuang and Rouquier's idea was to replace the vector space V with the complexified Grothendieck group

$$V = K_{\mathbb{C},0}(\mathcal{A}) = \mathbb{C} \otimes_k K_0(\mathcal{A})$$

of a k-linear abelian category $\mathcal{A}$, for which we assume that objects have finite composition series. Their result [CR08, Theorem 6.4] is as follows.

Suppose we have the following.

- There exist exact functors $E, F : \mathcal{A} \longrightarrow \mathcal{A}$ such that E is both left and right adjoint to F and the action of $[E]$ and $[F]$ on $V = K_{\mathbb{C},0}(\mathcal{A})$ induces a locally finite action of $\mathfrak{sl}_2(\mathbb{C})$ corresponding to the action of e and f.
- There is a decomposition $\mathcal{A} = \oplus_{\lambda\in\mathbb{Z}}\mathcal{A}_\lambda$ corresponding to the weight space decomposition of V as an $\mathfrak{sl}_2(\mathbb{C})$-module; that is, $K_{\mathbb{C},0}(\mathcal{A}_\lambda) \cong V_\lambda$.
- There are natural transformations $X : E \to E$ and $T : EE \to EE$ satisfying some technical conditions (see statement in [CR08, Theorem 6.4]).

Then there is a complex of functors Θ inducing a derived equivalence

$$\Theta : D^b(\mathcal{A}) \longrightarrow D^b(\mathcal{A})$$

lifting the action of θ on V, restricting to equivalences

$$\Theta(\lambda) : D^b(\mathcal{A}_\lambda) \longrightarrow D^b(\mathcal{A}_{-\lambda}).$$

Moreover [CR17, Proposition 8.4], when every object of $\mathcal{A}$ has finite composition series and $\mathcal{S}$ is the set of simple objects in $\mathcal{A}$, the equivalence thus constructed is perverse, relative to $(\mathcal{S}_\bullet, \mathcal{S}'_\bullet, p)$, where the filtrations are given by

$$\mathcal{S}_i = \{V \in \mathcal{S} : F^{i+1}(V) = 0\}$$

and

$$\mathcal{S}'_i = \{V \in \mathcal{S} : E^{i+1}(V) = 0\},$$

and $p(i) = i$.

The ith degree terms of the complexes $\Theta(\lambda)$ are $E^{-(\lambda+i)}F^{(i)}$, where the divided powers are $F^{(n)} = c_n F^n$ and $E^{-(n)} = c_{-n}E^n$, where $c_n = \sum_{\sigma \in \mathfrak{S}_n} \sigma \in k\mathfrak{S}_n$ and $c_{-n} = \sum_{\sigma \in \mathfrak{S}_n} \mathrm{sgn}(\sigma)\sigma \in k\mathfrak{S}_n$.

The construction of the complex of functors Θ generalises that of Rickard, in his unpublished theorem on blocks of the symmetric groups of weight $w \leq 5$. The applicability of Chuang and Rouquier's result to the symmetric groups is due to work of Lascoux, Leclerc and Thibon [LLT96], who utilised a method by which one can work with all blocks of all symmetric groups over a fixed field k of characteristic $p > 0$ simultaneously.

For every n, we have $\mathfrak{S}_{n-1} \leq \mathfrak{S}_n$ in the obvious way. Thus, given any module U in $k\mathfrak{S}_n$-mod, the restriction $\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}}(U)$ is an object in $k\mathfrak{S}_{n-1}$-mod. The group $(k\mathfrak{S}_n)^{\mathfrak{S}_{n-1}}$ of elements of the group algebra $k\mathfrak{S}_n$ that commute with every element of $\mathfrak{S}_{n-1}$ acts naturally on the restriction $\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}}(U)$. In particular, the action of the Jucys-Murphy element

$$L_n = (1, n) + (2, n) + \ldots + (n-1, n)$$

on U has eigenvalues all lying in the prime subfield $\mathbb{F}_p$, and thus we have a decomposition

$$\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}}(U) = \bigoplus_{i=0}^{p-1} E_i(U).$$

The functor Ind is both left and right adjoint to Res, and we have a similar decomposition

$$\mathrm{Ind}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}}(V) = \bigoplus_{i=0}^{p-1} F_i(U)$$

for every module V in $k\mathfrak{S}_{n-1}$-mod, where the functor F_i is both left and right adjoint to E_i.

Consider the abelian category

$$\mathcal{F} = \bigoplus_{n \geq 0} k\mathfrak{S}_n\text{-mod},$$

called the *Fock space*. The functors Res and Ind are endofunctors of $\mathcal{F}$, and they decompose as

$$\mathrm{Res} = \bigoplus_{i=0}^{p-1} E_i$$

and

$$\mathrm{Ind} = \bigoplus_{i=0}^{p-1} F_i.$$

The functors E_i and F_i are exact, and E_i is both left and right adjoint to F_i. On blocks, they act as Robinson's i-restriction and i-induction functors, respectively. By a result of Puig, for every n we have isomorphisms of functors

$$E_i^n \cong (E_i^{(n)})^{\oplus n!} \cong (E_i^{-(n)})^{\oplus n!}$$

and

$$F_i^n \cong (F_i^{(n)})^{\oplus n!} \cong (F_i^{-(n)})^{\oplus n!}.$$

Lascoux, Leclerc and Thibon's result is the following.

Theorem 3.1.9. *The action of $e_i = [E_i]$ and $f_i = [F_i]$ on the Grothendieck group $K_0(\mathcal{F})$ extends to an action of the affine Kac-Moody algebra $\hat{\mathfrak{sl}}_p(\mathbb{C})$, with weight space decomposition*

$$K_0(\mathcal{F}) = \bigoplus_{B \text{ blocks of } k\mathfrak{S}_n} K_0(B).$$

Moreover, the action of the affine Weyl group W is transitive on blocks of a fixed weight.

The action of W is generated by reflections, which correspond to pairs (e_i, f_i), and thus we need only consider the action of $\mathfrak{sl}_2(\mathbb{C})$. Thus, by Chuang and Rouquier's method of $\mathfrak{sl}_2$-categorification, the functors E_i and F_i give rise to a derived autoequivalence of the Fock space $\mathcal{F}$, restricting to derived equivalences between blocks of the symmetric groups of the same weight. In particular, these give the derived equivalences between blocks in a $[w : m]$ pair.

3.1.9 Scopes pair derived equivalences

Suppose the blocks $B = B_{\tau,w}$ of $k\mathfrak{S}_n$ and $\overline{B} = B_{\bar{\tau},w}$ of $k\mathfrak{S}_{n-m}$ form a $[w : m]$ pair. Then for some $0 \leq i \leq p-1$, the derived equivalence

$$D^b(B) \xrightarrow{\sim} D^b(\overline{B})$$

is induced by a complex of functors of the form

$$\ldots \longrightarrow E_i^{(m+2)} F_i^{-(2)} \longrightarrow E_i^{(m+1)} F_i^{-(1)} \longrightarrow E_i^{(m)}.$$

Similarly, the equivalence

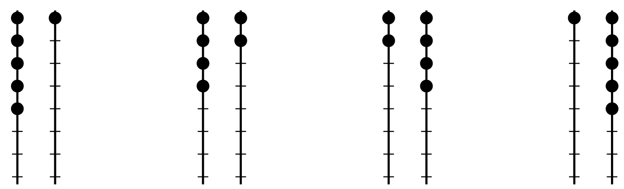
$$D^b(\overline{B}) \xrightarrow{\sim} D^b(B)$$

is induced by a complex of functors of the form

$$\ldots \longrightarrow F_i^{(m+2)}E_i^{-(2)} \longrightarrow F_i^{(m+1)}E_i^{-(1)} \longrightarrow F_i^{(m)}.$$

The index i here can be recovered from the combinatorics of the partitions lying in the blocks B and $\overline{B}$, as per the usual definition of i-restriction and i-induction.

One can describe this complex of functors using the p-abacus. Fixing an abacus display for τ, for some j there are m more beads on runner $j+1$ than on runner j. Without loss of generality, we may assume that there is at least one bead on runner j. The i-induction functor F_i maps the block B to the block B' of $k\mathfrak{S}_{n+1}$ of weight $w-m-1$, whose p-core has an abacus display identical to τ, but with one additional bead on runner $j+1$ and one fewer bead on runner j. The i-restriction functor E_i maps the block $\overline{B}$ to the block $\overline{B}'$ of $k\mathfrak{S}_{n-m-1}$ of weight $w-m-1$, whose p-core has an abacus display identical to $\bar{\tau}$, but with one additional bead on runner j and one fewer bead on runner $j+1$. Runners j and $j+1$ appear as the following for the blocks $\overline{B}'$, $\overline{B}$, B and B' from left to right, in the case $m=2$.



Consider the blocks $B_{(2),2}$ of $k\mathfrak{S}_8$ and $B_{(1),2}$ of $k\mathfrak{S}_7$ in characteristic 3. These blocks form a $[2:1]$ pair. The derived equivalence

$$\Phi : D^b(B_{(2),2}) \xrightarrow{\sim} D^b(B_{(1),2})$$

is induced by a complex of functors of the form

$$E_1^{(2)}F_1^{-(1)} \longrightarrow E_1,$$

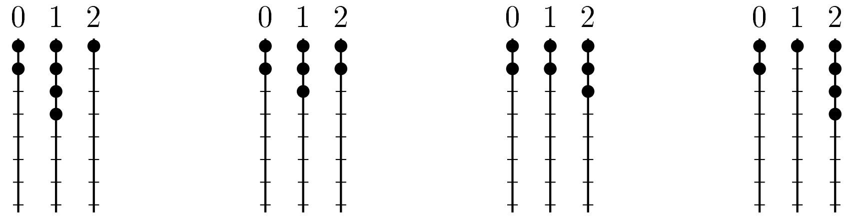
where F_1 is the induction functor to the block $B_{(5,3,1),0}$ of $k\mathfrak{S}_9$. The equivalence

$$\overline{\Phi} : D^b(B_{(1),2}) \xrightarrow{\sim} D^b(B_{(2),2})$$

is induced by the complex of functors

$$F_1^{(2)} E_1^{-(1)} \longrightarrow F_1,$$

where E_1 is the restriction functor to the block $B_{(4,2),0}$ of $k\mathfrak{S}_6$. Abacus displays for the 3-cores of the relevant blocks are below.



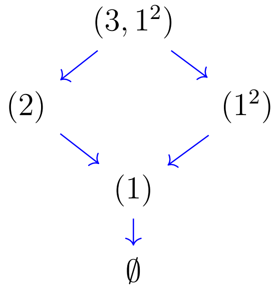
3.2 Blocks of Weight Two

Blocks of weight $w = 0$ are matrix algebras and blocks of weight $w = 1$ are Brauer tree algebras. These blocks are well understood: they are all Morita equivalent for w fixed, and are all of finite representation type. Blocks of weight $w \geq 2$ in characteristic $p \geq 3$ are of wild representation type, and are thus harder to pin down. In some sense, however, this makes them more interesting objects of study.

We will survey the landscape for blocks of weight $w = 2$ in characteristic $p = 3$. It is for these blocks that we will find examples of applications of Theorem 2.3.3. Fix an algebraically closed field k of characteristic $p = 3$. In what follows, by a block B, we will always mean the basic algebra Morita equivalent to B.

We start by collecting known information about these blocks. We present Ext^1-quivers of the blocks as k-algebras and the Loewy series of the projective indecomposable modules lying in each block.

Representatives of Scopes classes of the blocks (that is, choices of 3-core partitions) can be chosen as in the following diagram, with blue arrows denoting $[2:1]$-pairs between Scopes classes. For each class, we choose as representative the block of the symmetric group of smallest degree.



Projective indecomposable modules are known to have fixed Loewy length 5 and be stable (that is, the radical and socle layers coincide) [Sco95, Property 5.7], and methods for calculating composition multiplicities of simple modules can be found in [Ric96, Theorem 4.4, Conjecture 4.7] and [Fay12, Proposition 3.1]. The Scopes pairs are an easy application of [Sco95, Corollary 3.7, Lemma 4.3].

For a kG-module V, the k-linear dual $V^* = \mathrm{Hom}_k(V, k)$ is also a left kG-module, under the action

$$g \cdot \varphi : v \mapsto \varphi(g^{-1} \cdot v).$$

We note that, for all blocks B in question, every simple module S is self-dual, $S \cong S^*$ as kG-modules (see [Gre06, (3.5a), (3.3e)]). Hence, $\mathrm{Ext}^1_B(S, T) \cong \mathrm{Ext}^1_B(T, S)$ for all simple B-modules S and T. Further, $\dim \mathrm{Ext}^1_B(S, T) \leq 1$ (see [CT01, Theorem 3.1]). Additionally, the Ext^1-quivers of these blocks are bipartite (see [CT01, Corollary 3.2]), which greatly simplifies computation. Specific Ext^1-quivers come from [EM94, Theorem 7.1] and [Oku97, Examples 4.3, 4.4], with the general shape determined by [DE20, Theorem 4.1].

Fayers [Fay12, §2.3] has introduced a square bracket labelling convention for the p-regular partitions lying in a block B: if μ is such a partition, it is assigned a label $[i]$ for $1 \leq i < p$, or $[i, j]$ for $1 \leq i \leq j < p$, based on the more familiar angle bracket notation for a (not necessarily p-regular) partition and Richards's pyramid numbers for B. We will not cover the full details of this labelling here. For $p = 3$, we fix the following ordering on the simple modules in any block:

$$[2] > [2, 2] > [1, 2] > [1] > [1, 1].$$

We label the simple modules S_1 to S_5 from left to right. We will typically represent the simple module S_i as i. We denote by P_i the projective indecomposable module with simple head (and socle) S_i. In particular, $P_i \xrightarrow{\pi_i} S_i$ is a projective cover of S_i.

This labelling and ordering is non-standard. In particular, we may have two simple modules S and T such that $S > T$ in this ordering, but $T > S$ or

$T \rhd S$ in the usual lexicographic and dominance orderings[4]. The main benefit, however, is that the labels are fixed in a Scopes $[2:k]$-pair, regardless of whether $k \geq w$ or $k < w$, and thus the positions of a label in the quiver also remain fixed. A further benefit is that it allows us to continue without any reference to the partitions themselves. As all the combinatorial considerations here are at the level of the block (that is, the abacus combinatorics of the p-core), a detailed study of the partitions would be superfluous. Additionally, this means that, while we give a concrete realisation of each Scopes class, one can easily choose another, and we need only change the p-core and its abacus realisation; the ordering of the simple modules and the structure of the projective indecomposable modules remain exactly the same.

As a final comment before detailing the blocks in question, the composition of $[2:1]$ pairs

$$D^b(B_{(3,1^2)}) \xrightarrow{\sim} D^b(B_{(2)}) \xrightarrow{\sim} D^b(B_{(1)})$$

remains perverse; one way to see this is via [CR17, Proposition 5.11]. However, the composition

$$D^b(B_{(2)}) \xrightarrow{\sim} D^b(B_{(1)}) \xrightarrow{\sim} D^b(B_{\emptyset})$$

is not perverse. In order to resolve Conjecture 3.1.8 for the block $B_{\emptyset}$, one must thus utilise other methods.

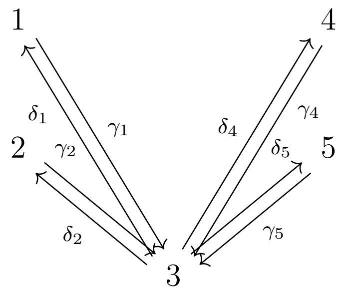
3.2.1 The block $B_{(3,1^2)}$

Consider the block $B_{(3,1^2)}$ of $k\mathfrak{S}_{11}$, the Rouquier block for $p = 3$. Uniquely among the Rouquier blocks for different values of p, this block is Morita equivalent to the local block b, not just the intermediate block b_0, due to the exceptional coincidence that the Brauer tree algebras $A_{2,1}$ and $\tilde{A}_{2,1}$ coincide:

.

The algebra $B_{(3,1^2)}$ is isomorphic to $kQ/\mathcal{I}$, where Q is the quiver

[4] For a description of these orderings, see [Jam06, Definitions 3.2, 3.4]



and $\mathcal{I}$ the admissible ideal of kQ generated by the relations (see [Oku97, §4 Case 2]):

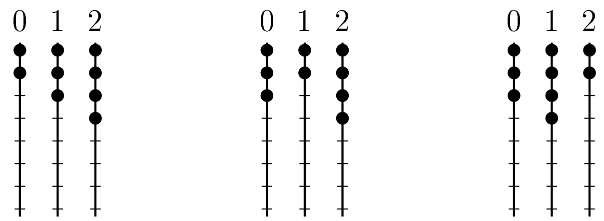
- $\delta_4\gamma_1 = 0$, $\delta_1\gamma_4 = 0$, $\delta_5\gamma_2 = 0$, $\delta_2\gamma_5 = 0$;
- $\gamma_1\delta_1 + \gamma_4\delta_4 = \gamma_2\delta_2 + \gamma_5\delta_5$;
- $\gamma_2\delta_2\gamma_1 = \gamma_5\delta_5\gamma_1$, $\gamma_1\delta_1\gamma_2 = \gamma_4\delta_4\gamma_2$, $\gamma_2\delta_2\gamma_4 = \gamma_5\delta_5\gamma_4$, $\gamma_1\delta_1\gamma_5 = \gamma_4\delta_4\gamma_5$;
- $\delta_1\gamma_2\delta_2 = \delta_1\gamma_5\delta_5$, $\delta_2\gamma_1\delta_1 = \delta_2\gamma_4\delta_4$, $\delta_4\gamma_2\delta_2 = \delta_4\gamma_5\delta_5$, $\delta_5\gamma_1\delta_1 = \delta_5\gamma_4\delta_4$;
- all paths of length four starting and ending at distinct vertices are 0.

The projective indecomposable modules, P_i corresponding to the simple i, have Loewy series

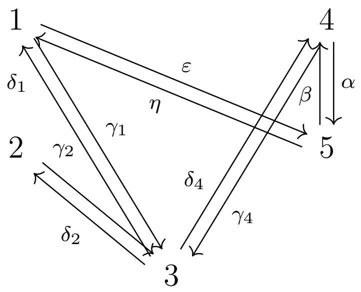
$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 3 & 3 & 1\ 2\ 4\ 5 & 3 & 3 \\ 1\ 2\ 5 & 1\ 2\ 4 & 3\ 3\ 3 & 2\ 4\ 5 & 1\ 4\ 5 \\ 3 & 3 & 4\ 5\ 1\ 2 & 3 & 3 \\ 1 & 2 & 3 & 4 & 5 \end{array}.$$

3.2.2 The block $B_{(2)}$

Consider the block $B_{(2)}$ of $k\mathfrak{S}_8$. There is a $[2:1]$ pair $D^b(B_{(3,1^2)}) \xrightarrow{\sim} D^b(B_{(3,1)})$. By composing this with the $[2:2]$ pair $D^b(B_{(3,1)}) \xrightarrow{\sim} D^b(B_{(2)})$, actually a Morita equivalence, we interpret this as an equivalence $D^b(B_{(3,1^2)}) \xrightarrow{\sim} D^b(B_{(2)})$.



The algebra $B_{(2)}$ is isomorphic to $kQ/\mathcal{I}$, where Q is the quiver



and $\mathcal{I}$ the admissible ideal of kQ generated by the relations (see [Oku97, Example 4.3]):

- $\beta\varepsilon = \delta_4\gamma_1$, $\eta\alpha = \delta_1\gamma_4$, $\varepsilon\delta_1 = \alpha\delta_4$, $\gamma_1\eta = \gamma_4\beta$;
- $\alpha\beta = \varepsilon\eta$, $\gamma_1\delta_1 + \gamma_4\delta_4 = \gamma_2\delta_2$;
- $\alpha\delta_4\gamma_2 = 0$, $\delta_2\gamma_4\beta = 0$;
- $\gamma_1\delta_1\gamma_1 = 0$, $\delta_1\gamma_1\delta_1 = 0$, $\gamma_4\delta_4\gamma_4 = 0$, $\delta_4\gamma_4\delta_4 = 0$;
- $\delta_2\gamma_1\delta_1 = \delta_2\gamma_4\delta_4$, $\gamma_1\delta_1\gamma_2 = \gamma_4\delta_4\gamma_2$;
- all paths of length four starting and ending at distinct vertices are 0.

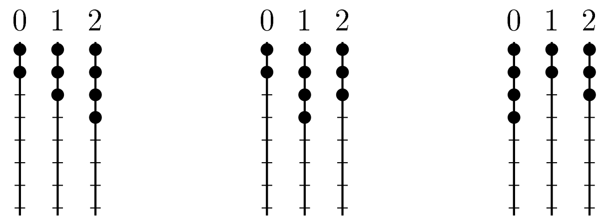
Note that this list of relations is not minimal.

The projective indecomposable modules have Loewy series

$$\begin{matrix} 1 \\ 3\ 5 \\ 1\ 2\ 4\ 1 \\ 5\ 3 \\ 1 \end{matrix} \qquad \begin{matrix} 2 \\ 3 \\ 1\ 2\ 4 \\ 3 \\ 2 \end{matrix} \qquad \begin{matrix} 3 \\ 1\ 2\ 4 \\ 3\ 5\ 3 \\ 4\ 2\ 1 \\ 3 \end{matrix} \qquad \begin{matrix} 4 \\ 3\ 5 \\ 4\ 1\ 2\ 4 \\ 5\ 3 \\ 4 \end{matrix} \qquad \begin{matrix} 5 \\ 1\ 4 \\ 3\ 5 \\ 4\ 1 \\ 5 \end{matrix}.$$

3.2.3 The block $B_{(1^2)}$

Consider the block $B_{(1^2)}$ of $k\mathfrak{S}_8$. There is a [2 : 1] pair $D^b(B_{(3,1^2)}) \xrightarrow{\sim} D^b(B_{(2,1^2)})$. By composing this with the [2 : 2] pair $D^b(B_{(2,1^2)}) \xrightarrow{\sim} D^b(B_{(1^2)})$, actually a Morita equivalence, we interpret this as an equivalence $D^b(B_{(3,1^2)}) \xrightarrow{\sim} D^b(B_{(2)})$.

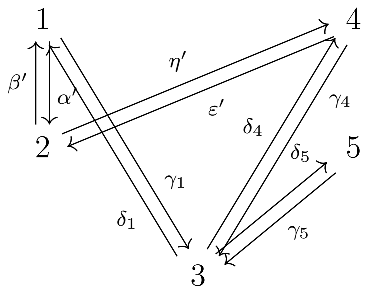


The blocks $B_{(1^2)}$ and $B_{(2)}$ of $k\mathfrak{S}_8$ are Morita equivalent, via the map

$$- \otimes_{k\mathfrak{S}_8} \mathrm{sgn} : B_{(1^2)}\text{-mod} \longrightarrow B_{(2)}\text{-mod},$$

with sgn the one-dimensional sign $k\mathfrak{S}_8$-module, though the two blocks are in strictly different Scopes-Morita equivalence classes. One can visualise this map as a reflection in the central vertical axis of the quiver. The 3-cores (1^2) and (2) are mutually conjugate as partitions; the blocks $B_{(1^2)}$ and $B_{(2)}$ are *conjugate blocks*.

The algebra $B_{(1^2)}$ is isomorphic to $kQ/\mathcal{I}$, where Q is the quiver



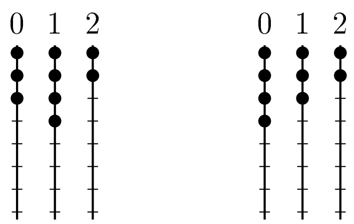
and $\mathcal{I}$ the admissible ideal in kQ generated by the relations as in §3.2.2, but with the obvious reflections in the central vertical axis of the quiver.

The projective indecomposable modules have Loewy series

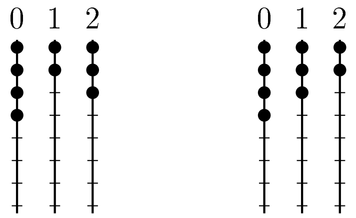
1	2	3	4	5
2 3	1 4	1 4 5	2 3	3
1 4 5 1	2 3	3 2 3	4 1 5 4	1 5 4 .
3 2	4 1	5 4 1	3 2	3
1	2	3	4	5

3.2.4 The block $B_{(1),2}$

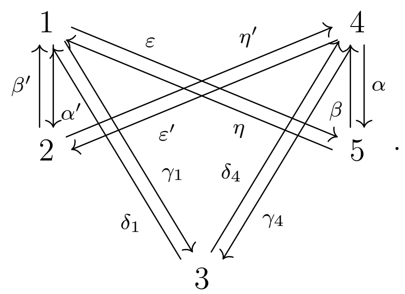
Consider the block $B_{(1)}$ of $k\mathfrak{S}_7$. There is a $[2:1]$ pair $D^b(B_{(2)}) \xrightarrow{\sim} D^b(B_{(1)})$.



There is also $[2:1]$ pair $D^b(B_{(1^2)}) \xrightarrow{\sim} D^b(B_{(1)})$.



The algebra $B_{(1)}$ is isomorphic to $kQ/\mathcal{I}$, with Q a quiver of the form (see [DE20, Appendix C])

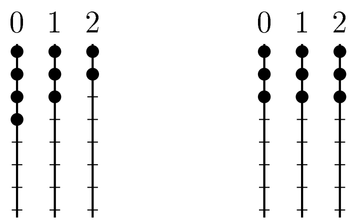


The projective indecomposable modules have Loewy series

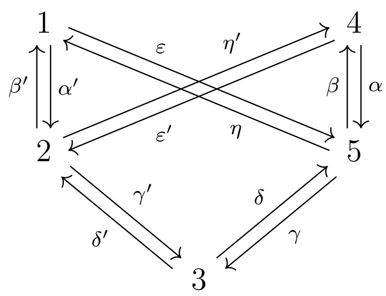
1	2	3	4	5
2 3 5	1 4	1 4	2 3 5	1 4
1 4 1 4 1	2 3	2 3 5	4 1 4 1 4	3 5 .
5 3 2	4 1	4 1	5 3 2	4 1
1	2	3	4	5

3.2.5 The block $B_{\emptyset,2}$

Consider the principal block $B_\emptyset$ of $k\mathfrak{S}_6$. There is a $[2:1]$ pair $D^b(B_{(1)}) \xrightarrow{\sim} D^b(B_\emptyset)$.



The algebra $B_\emptyset$ is isomorphic to $kQ/\mathcal{I}$, where Q is the quiver



with $\mathcal{I}$ the admissible ideal in kQ generated by the relations (see [Oku97, Example 4.4] or [EM94, Theorem 7.1]):

- $\alpha\eta' = \varepsilon\beta' = \delta\gamma'$, $\alpha'\eta = \varepsilon'\beta = \delta'\gamma$;
- $\beta\varepsilon = \eta'\alpha'$, $\beta'\varepsilon' = \eta\alpha$, $\gamma\delta + \gamma'\delta' = 0$;
- $\gamma'\alpha' = \gamma\varepsilon$, $\gamma\alpha = \gamma'\varepsilon'$, $\beta'\delta' = \eta\delta$, $\beta\delta = \eta'\delta'$;
- $\delta'\gamma' = \alpha'\beta' + \varepsilon'\eta'$, $\delta\gamma = \alpha\beta + \varepsilon\eta$;
- all paths of length four starting and ending at distinct vertices are 0.

The projective indecomposable modules have Loewy series

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 2\ 5 & 1\ 3\ 4 & 2\ 5 & 2\ 5 & 1\ 3\ 4 \\ 1\ 3\ 4\ 1 & 2\ 5\ 2 & 1\ 3\ 4 & 4\ 1\ 3\ 4 & 5\ 2\ 5 \\ 5\ 2 & 4\ 3\ 1 & 5\ 2 & 5\ 2 & 4\ 3\ 1 \\ 1 & 2 & 3 & 4 & 5 \end{array}.$$

3.3 Periodic Equivalences of Blocks of Symmetric Groups

We now explore some possible applications of Theorem 2.3.3 to the weight two blocks of symmetric groups in characteristic 3.

3.3.1 $[2:1]$ pairs

Recall from §3.1.9 that a $[w:m]$ pair derived equivalence

$$D^b(B) \longrightarrow D^b(\overline{B})$$

is induced by a complex of functors of the form

$$\ldots \longrightarrow E_i^{-(m+2)} F_i^{(2)} \longrightarrow E_i^{(m+1)} F_i \longrightarrow E_i^{(m)}.$$

We can give a more precise description of the action on simple modules for $[2:1]$ pairs.

Let k be an algebraically closed field of arbitrary odd prime characteristic $p \geq 3$. Let B be a block of $k\mathfrak{S}_n$ of p-weight $w = 2$ and $\overline{B}$ a block of $k\mathfrak{S}_{n-1}$ of p-weight $w = 2$ such that B and $\overline{B}$ are linked by a $[2:1]$ pair. With the square bracket notation of Fayers [Fay12], we fix an ordering of the simple modules lying in B and $\overline{B}$,

$$[p-1] > [p-1, p-1] > [p-2, p-1] > \ldots > [1] > [1, 1],$$

extending the ordering described in §3.2. Let $S_1, \ldots, S_r$ be the simple B-modules corresponding to the above list from left to right, and $\overline{S}_1, \ldots, \overline{S}_r$ the simple $\overline{B}$-modules corresponding to the above list from left to right. Let $I = \{1, \ldots, r\}$ be an indexing set for both sets of isomorphism classes of simple modules.

In [Sco95], Scopes identifies an exceptional p-regular partition α such that the simple $k\mathfrak{S}_n$-module D^α lies in B. Let i_α be the index in I such that $S_{i_\alpha} = D^\alpha$. Then [Sco95, Corollary 3.7, Remark 4.4], for $i \neq i_\alpha$, we have $\mathrm{Res}^B_{\overline{B}} S_i \cong \overline{S}_i$, and $\mathrm{Ind}^B_{\overline{B}} \overline{S}_i \cong S_i$.

There is a $\overline{B}$-B-bimodule V, projective as a left $\overline{B}$-module and as a right B-module, such that

$$\mathrm{Res}^B_{\overline{B}} \cong V \otimes_B -.$$

Let $P_V \xrightarrow{\delta} V$ be a projective cover of V as a $\overline{B}$-B-bimodule. By [Sco95, Corollary 3.7, Remark 4.4, Lemma 5.7] and [Rou95, Lemma 2],

$$P_V \cong \bigoplus_{i \in I} \overline{P}_i \otimes_k P_i^\vee,$$

where $P_i \xrightarrow{\pi_i} S_i$ is a projective cover of the B-module S_i and $\overline{P}_i \xrightarrow{\overline{\pi}_i} \overline{S}_i$ is a projective cover of the $\overline{B}$-module $\overline{S}_i$. Let P be the direct summand $\overline{P}_{i_\alpha} \otimes P_{i_\alpha}^\vee$ of P_V, and δ_P the restriction of δ to P. Let X be the complex of $\overline{B}$-B-bimodules

$$0 \longrightarrow P \xrightarrow{\delta_P} V \longrightarrow 0,$$

with V in degree 0. The first part of the following is taken from [Chu99, §2], based on the aforementioned unpublished result of Rickard, and the second part follows from [CR17, Remark 8.5], though it is also an immediate consequence of the construction of X.

Theorem 3.3.1. *The complex X of $\overline{B}$-B-bimodules is a two-sided tilting complex, inducing a derived equivalence*

$$F : D^b(B) \xrightarrow{\sim} D^b(\overline{B}).$$

Moreover, the equivalence F is perverse relative to the filtration

$$\emptyset \subset_0 I \setminus \{i_\alpha\} \subset_1 I$$

on both sides.

One can similarly construct a derived equivalence

$$\overline{F} : D^b(\overline{B}) \xrightarrow{\sim} D^b(B)$$

from the induction functor $\mathrm{Ind}_{\overline{B}}^{B} \cong \overline{V} \otimes_{\overline{B}} -$, perverse relative to the filtration

$$\emptyset \subset_0 I \setminus \{i_\alpha\} \subset_1 I$$

on both sides. By construction, the equivalence F is the inverse of a combinatorial tilt or elementary perverse equivalence of B at $\{i_\alpha\}$ and the equivalence $\overline{F}$ is the inverse of a combinatorial tilt of $\overline{B}$ at $\{i_\alpha\}$.

The compositions $\overline{F} \circ F$ and $F \circ \overline{F}$ are derived autoequivalences of the blocks B and $\overline{B}$ respectively, both self-perverse relative to the filtration

$$\emptyset \subset_0 I \setminus \{i_\alpha\} \subset_2 I.$$

By Theorem 2.3.4, we therefore expect projective B-modules P and Q, with no common direct summands and such that $P \oplus Q$ is a projective generator of B, and an autoequivalence σ of $E = \mathrm{End}_A(P)^{\mathrm{op}}$ and $\alpha \in \mathrm{Ext}^n_{E \otimes_k E^{\mathrm{op}}}(E, {}_\sigma)$ such that $M = \mathrm{Hom}_A(P, Q)$ is a strongly σ-periodic E-module of period 2 relative to α.

For the blocks in characteristic $p = 3$, as in §3.2, the $[2 : 1]$ pairs and exceptional simple modules are as follows.

- $F : D^b(B_{(3,1^2)}) \xrightarrow{\sim} D^b(B_{(2)})$, $S_{i_\alpha} = S_5$;
- $F : D^b(B_{(3,1^2)}) \xrightarrow{\sim} D^b(B_{(1^2)})$, $S_{i_\alpha} = S_2$;
- $F : D^b(B_{(2)}) \xrightarrow{\sim} D^b(B_{(1)})$, $S_{i_\alpha} = S_2$;
- $F : D^b(B_{(1^2)}) \xrightarrow{\sim} D^b(B_{(1)})$, $S_{i_\alpha} = S_5$;
- $F : D^b(B_{(1)}) \xrightarrow{\sim} D^b(B_{\emptyset})$, $S_{i_\alpha} = S_3$.

Observe that, in these examples, if the blocks B and $\overline{B}$ are linked by a $[2:1]$ pair, and S_{i_α} is the exceptional simple module in B, then $\mathrm{End}_A(P_{i_\alpha})^{\mathrm{op}} \cong k[x]/\langle x^3 \rangle$. In other words, the projective module P_{i_α} is a $\mathbb{P}^2$-object in B. Thus, there is a Grantian $\mathbb{P}^2$-twist

$$\Psi_{P_{i_\alpha}} : D^b(B) \xrightarrow{\sim} D^b(B),$$

perverse relative to the filtration

$$\emptyset \subset_0 I \setminus \{i_\alpha\} \subset_0 I.$$

By Theorem 2.1.7, there is a cycle of equivalences

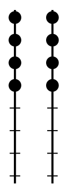
$$\begin{array}{c} D^b(B) \\ {\scriptstyle F_J^{-1}} \uparrow \quad \downarrow {\scriptstyle F_J^{-1}} \\ D^b(B^{(1)}) \end{array} .$$

However, the inverse of the combinatorial tilt at J is isomorphic to the $[2:1]$ pair derived equivalence $D^b(B) \xrightarrow{\sim} D^b(\overline{B})$. In this way, we recover the $[2:1]$ pair from the periodic twist, and vice versa.

This is not an especially groundbreaking observation, but it is interesting, and natural from the point of view of this thesis, to reframe the more classical understanding of derived equivalence for blocks of symmetric groups in terms of periodicity.

3.3.2 $[2:0]$ pairs

Let k be an algebraically closed field of characteristic p. Let B be a block of $k\mathfrak{S}_n$ of p-weight w, such that there is an abacus display of the p-core τ of B in which two consecutive runners have an equal number of beads.



This defines a $[w:0]$ pair of B, corresponding to a derived autoequivalence

$$\Phi : D^b(B) \xrightarrow{\sim} D^b(B).$$

induced by a complex of functors of the form

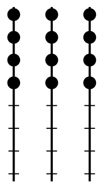
$$\ldots \longrightarrow E_i^{-(2)} F_i^{(2)} \longrightarrow E_i F_i \longrightarrow \mathrm{Id},$$

or dually with the functors E_i and F_i switched. By [CR17, Remark 8.5], this equivalence is self-perverse relative to a two-step filtration

$$\emptyset \subset_0 I_1 \subset_1 I$$

where I is an indexing set for the simple B-modules and $I_1 \subset I$ is a non-empty, proper subset.

Now, suppose the abacus display for τ has three consecutive runners with an equal number of beads.



We then have two derived autoequivalences

$$\Phi_1, \Phi_2 : D^b(B) \xrightarrow{\sim} D^b(B),$$

self-perverse relative to some two-step filtrations

$$\emptyset \subset_0 I \setminus I_1 \subset_1 I$$

and

$$\emptyset \subset_0 I \setminus I_2 \subset_1 I.$$

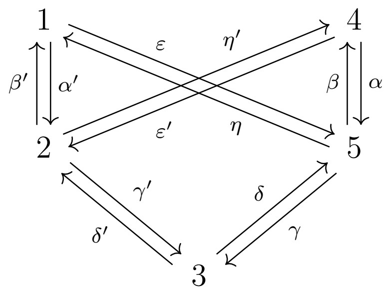
By a result of Cautis and Kamnitzer [CK12, Theorem 2.10], the braid relation $\Phi_1\Phi_2\Phi_1 \cong \Phi_2\Phi_1\Phi_2$ holds for these equivalences. Moreover, by a result of Halacheva, Licata, Losev and Yacobi [Hal+23, Theorem 6.8, Remark 6.9], the equivalence

$$\Phi_1\Phi_2\Phi_1 : D^b(B) \xrightarrow{\sim} D^b(B)$$

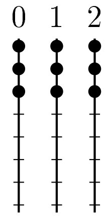
is perverse relative to the *isotypic filtration*

$$\emptyset \subset_0 I \setminus (I_1 \cup I_2) \subset_2 I.$$

Now, let k be an algebraically closed field of characteristic 3, and let A be the basic algebra of the block $B_\emptyset$, as in §3.2.5.



An abacus display for the p-core $\emptyset$ is



Clearly, we have two derived autoequivalences arising from $[2:0]$ pairs

$$\Phi_1, \Phi_2 : D^b(A) \xrightarrow{\sim} D^b(A)$$

as above. We claim that the perversity of the braid $\Phi_1\Phi_2\Phi_1 \cong \Phi_2\Phi_1\Phi_2$ can be understood at the level of periodicity, in a manner we outline below.

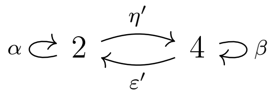
The equivalence Φ_1 is induced by a complex of functors

$$F_1E_1 \longrightarrow \mathrm{Id},$$

where E_1 is the 1-restriction functor to the weight 1 block $B_{(1^2)}$ of $k\mathfrak{S}_5$. The functor E_1 is zero on all simple A-modules except S_2 and S_4. Thus, Φ_1 is perverse relative to the filtration

$$\emptyset \subset_0 \{1,3,5\} \subset_2 I.$$

Let $P = P_2 \oplus P_4$, $Q = P_1 \oplus P_3 \oplus P_4$ and $E = \mathrm{End}_A(P)^{\mathrm{op}}$. Then E has an Ext^1-quiver



with relations

$$\{\eta'\varepsilon'\eta',\ \varepsilon'\eta'\varepsilon',\ \eta'\alpha - \beta\eta',\ \varepsilon'\beta - \alpha\varepsilon',\ \alpha^2,\ \beta^2\}.$$

The projective indecomposable E-modules, $\overline{P}_2$ and $\overline{P}_4$, have Loewy series

$$\begin{array}{c} 2 \\ 2\ \ 4 \\ 4\ \ 2 \\ 2 \end{array} \quad \text{and} \quad \begin{array}{c} 4 \\ 2\ \ 4 \\ 4\ \ 2 \\ 4 \end{array}.$$

We point out that neither of these is biserial, and

$$\mathrm{End}_E(\overline{P}_2) \cong k[x,y]/\langle x^2, y^2\rangle \cong \mathrm{End}_E(\overline{P}_4),$$

so that $\overline{P}_2$ and $\overline{P}_4$ are both toric objects, as described in §2.1.6, in E.

Let $M = \mathrm{Hom}_A(P, Q)$. Then as an E-module, M has three indecomposable summands with Loewy and socle series

$$\begin{array}{c} 2 \\ 4 \\ 2 \end{array} \oplus \begin{array}{c} 2 \\ 4 \\ 2 \end{array} \oplus \begin{array}{c} 4 \\ 2 \\ 4 \end{array}.$$

It is clear that $\Omega_E(M) \cong M$, so that M is periodic of period 1. We claim that M is in fact strongly periodic of period 1.

Let B be the subalgebra of E generated by the arrows η' and ε'. Then B has a quiver

$$2 \underset{\varepsilon'}{\overset{\eta'}{\rightleftarrows}} 4$$

with relations

$$\{\eta'\varepsilon'\eta', \varepsilon'\eta'\varepsilon'\}.$$

In particular, B is isomorphic to the Brauer tree algebra $A_{2,1}$ of a star on 2 edges with exceptional multiplicity $m = 1$. The projective indecomposable B-modules, say Q_2 and Q_4, have Loewy series

$$\begin{array}{c} 2 \\ 4 \\ 2 \end{array} \quad \text{and} \quad \begin{array}{c} 4 \\ 2 \\ 4 \end{array}.$$

One can see that M is a projective B-module. We have $E \cong \overline{P}_2 \oplus \overline{P}_4$ as an E-module, and $B \otimes_B \overline{P}_j^{\vee} \cong Q_j \oplus Q_j$, so that $E \otimes_B E \cong \overline{P}_2 \oplus \overline{P}_2 \oplus \overline{P}_4 \oplus \overline{P}_4$. We claim that there is an exact sequence of E-E-bimodules

$$0 \longrightarrow E \longrightarrow E \otimes_B E \longrightarrow E \longrightarrow 0.$$

Then E is periodic of period 1 relative to B, and this sequence defines a self-extension $\alpha \in \operatorname{Ext}^1_{E\otimes_k E^{\mathrm{op}}}(E,E)$. Then $E \otimes_B E$ is perfect in $D^b(E)$ and in $D^b(E^{\mathrm{op}})$, and, since E is projective as a B-module and an B^{op}-module,

$$E \otimes_B E \otimes^{\mathbf{L}}_E M \cong E \otimes_B M$$

is perfect in $D^b(E)$, and

$$M^\vee \otimes^{\mathbf{L}}_E E \otimes_B E \cong M^\vee \otimes_B E$$

is perfect in $D^b(E^{\mathrm{op}})$.

In conclusion, the $[2:0]$ pair Φ_1 corresponds to a generalised periodic twist, in this instance a Grantian relative periodic twist at P,

$$\Phi_P : D^b(A) \xrightarrow{\sim} D^b(A),$$

by Theorem 2.1.6.

A very similar description applies for the $[2:0]$ pair Φ_2, if we instead take P to be the projective A-module $P_1 \oplus P_5$.

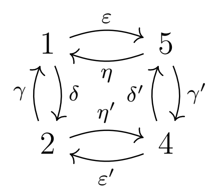
By the result of Halacheva, Licata, Losev and Yacobi, the braid

$$\Phi_1\Phi_2\Phi_1 : D^b(A) \xrightarrow{\sim} D^b(A)$$

is perverse, relative to the filtration

$$\emptyset \subset_0 \{3\} \subset_2 I.$$

Let $P = P_1 \oplus P_2 \oplus P_4 \oplus P_5$, $Q = P_3$ and $E = \operatorname{End}_A(P)^{\mathrm{op}}$. Then $E \cong kQ/\mathcal{I}$, where Q is the quiver



and $\mathcal{I}$ is the ideal of kQ generated by the relations

$$\{\varepsilon\eta\varepsilon,\ \eta\varepsilon\eta,\ \varepsilon'\eta'\varepsilon',\ \eta'\varepsilon'\eta',\ \gamma'\varepsilon - \eta'\delta,\ \gamma\varepsilon' - \eta\delta'\}.$$

The projective indecomposable E-modules, $\overline{P}_1, \overline{P}_2, \overline{P}_4, \overline{P}_5$, have Loewy series

$$\begin{matrix} & 1 & \\ 2 & & 5 \\ 1 & 4 & 1 \\ 5 & & 2 \\ & 1 & \end{matrix}\ ,\qquad \begin{matrix} & 2 & \\ 1 & & 4 \\ 2 & 5 & 2 \\ 4 & & 1 \\ & 2 & \end{matrix}\ ,\qquad \begin{matrix} & 4 & \\ 2 & & 5 \\ 4 & 1 & 4 \\ 5 & & 2 \\ & 4 & \end{matrix}\ ,\qquad \begin{matrix} & 5 & \\ 1 & & 4 \\ 5 & 2 & 5 \\ 4 & & 1 \\ & 5 & \end{matrix}\ .$$

There is an automorphism σ of E, induced by the graph automorphism of the quiver given by reflecting through the horizontal line of symmetry. The automorphism σ acts on I as the permutation $(1,2)(4,5)$.

Let $M = \mathrm{Hom}_A(P,Q)$. Then as an E-module, M has coinciding Loewy and socle series

$$M = \begin{array}{cc} 2 & 5 \\ 1 & 4 \\ 2 & 5 \end{array}$$

and a truncated projective resolution

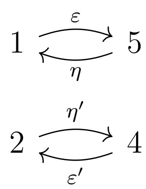
$$\overline{P}_1 \oplus \overline{P}_4 \longrightarrow \overline{P}_2 \oplus \overline{P}_5 \longrightarrow M.$$

One can calculate that

$$\Omega^2_E(M) = \begin{array}{cc} 1 & 4 \\ 2 & 5 \\ 1 & 4 \end{array} \cong {}_{\sigma}M,$$

so that M is σ-periodic of period 2. We claim that M is strongly σ-periodic of period 2, and, noting that $\sigma^{-1} = \sigma$, so is $M^{\vee}$.

Let B be the subalgebra of E generated by the horizontal arrows:



Then $B \cong A_{2,1} \times A_{2,1}$ as k-algebras. The automorphism σ restricted to B swaps the two factors. The projective indecomposable B-modules, say Q_1, Q_2, Q_4, Q_5, have Loewy series

$$\begin{array}{c} 1 \\ 5 \\ 1 \end{array}, \quad \begin{array}{c} 2 \\ 4 \\ 2 \end{array}, \quad \begin{array}{c} 4 \\ 2 \\ 4 \end{array}, \quad \begin{array}{c} 5 \\ 1 \\ 5 \end{array}.$$

The relatively B-projective E-modules are U_1, U_2, U_4 and U_4, with Loewy series

$$U_1 = \begin{array}{c} 1 \\ 2 \\ 1 \end{array}, \; U_2 = \begin{array}{c} 2 \\ 1 \\ 2 \end{array}, \; U_4 = \begin{array}{c} 4 \\ 5 \\ 4 \end{array}, \; U_5 = \begin{array}{c} 5 \\ 4 \\ 5 \end{array}.$$

We claim that there is an exact sequence of E-E-bimodules of the form

$$0 \longrightarrow {}_\sigma E \longrightarrow {}_\sigma E \otimes_B E \xrightarrow{d} E \otimes_B E \longrightarrow E \longrightarrow 0.$$

Indeed, applying the functors $- \otimes^{\mathbf{L}}_E S_i$, where S_i are the simple E-modules, gives complexes of the form

$$0 \longrightarrow S_2 \longrightarrow U_2 \longrightarrow U_1 \longrightarrow S_1 \longrightarrow 0,$$

$$0 \longrightarrow S_1 \longrightarrow U_1 \longrightarrow U_2 \longrightarrow S_2 \longrightarrow 0,$$

$$0 \longrightarrow S_5 \longrightarrow U_5 \longrightarrow U_4 \longrightarrow S_4 \longrightarrow 0,$$

$$0 \longrightarrow S_4 \longrightarrow U_4 \longrightarrow U_5 \longrightarrow S_5 \longrightarrow 0.$$

We have a triangle

$$Y \longrightarrow E \xrightarrow{\alpha} {}_\sigma E[2] \rightsquigarrow$$

defining

$$\alpha \in \mathrm{Hom}_{D^b(A)}(E, {}_\sigma E[2]) \cong \mathrm{Ext}^2_{E \otimes_k E^{\mathrm{op}}}(E, {}_\sigma E).$$

Similarly to before, the complex Y is perfect in $D^b(E)$ and in $D^b(E^{\mathrm{op}})$, and since E is a projective B-module and a projective B^{op}-module, $Y \otimes^{\mathbf{L}}_E M$ is perfect in $D^b(E)$, and $M^\vee \otimes^{\mathbf{L}}_E Y$ is perfect in $D^b(E^{\mathrm{op}})$. Noting that $\sigma^{-1} = \sigma$, this shows that both M and $M^\vee$ are strongly σ-periodic of period 2, relative to α.

The resulting generalised periodic twist, given by Theorem 2.3.8, is the Grantian relative periodic twist at P,

$$\Psi_P : D^b(A) \xrightarrow{\sim} D^b(A),$$

perverse relative to the filtration

$$\emptyset \subset_0 \{3\} \subset_2 I,$$

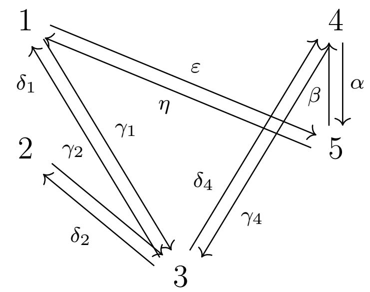
coinciding with the braid of the $[2:0]$ pairs $\Phi_1\Phi_2\Phi_1$.

A crucial aspect of this proposed construction is the role of the subalgebra B. Since $B \cong A_{2,1} \times A_{2,1}$, the formulation here appears to be masking the restriction and induction to the two weight one blocks in the $\mathfrak{sl}_2$-categorification picture, as both of these blocks are Morita equivalent to the Brauer tree algebra $A_{2,1}$. In this sense, this example is not really saying anything new, but reinterpreting the usual setting in terms of periodicity.

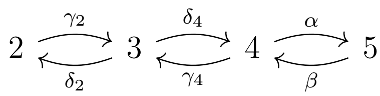
3.3.3 An autoequivalence of $B_{(2),2}$

In the previous two subsections, we saw examples of perverse equivalences in blocks of the symmetric groups arising from periodic and relatively periodic modules, in the sense of Grant. The example in this section is not known to fall into the Grantian setting, but rather offers an interesting application directly of Theorem 2.3.3.

Let A be the basic algebra of the block $B_{(2)}$, as in §3.2.2.



Let $P = P_2 \oplus P_3 \oplus P_4 \oplus P_5$, $Q = P_1$, $E = \mathrm{End}_A(P)^{\mathrm{op}}$ and $M = \mathrm{Hom}_A(P, Q)$. Then E has an Ext^1-quiver



with relations:

- $\alpha\delta_4\gamma_2 = 0 = \delta_2\gamma_4\beta$;
- $\delta_4\gamma_4\delta_4 = 0 = \gamma_4\delta_4\gamma_4$;
- $\gamma_2\delta_2\gamma_2 + \gamma_4\delta_4\gamma_2 = 0 = \delta_2\gamma_2\delta_2 + \delta_2\gamma_4\delta_4$;
- $\delta_4\gamma_2\delta_2 + \beta\alpha\delta_4 = 0 = \gamma_2\delta_2\gamma_4 + \gamma_4\beta\alpha$;
- $\alpha\delta_4\gamma_4 + \alpha\beta\alpha = 0 = \delta_4\gamma_4\beta + \beta\alpha\beta$;
- all paths of length four between distinct vertices are zero.

We note that all paths involving arrows in A with source or target 1 between the remaining vertices are in the ideal of A generated by the arrows in E.

The projective indecomposable E-modules have coinciding Loewy and socle series

$$\begin{array}{ccc} & 2 & \\ & 3 & \\ 2 & & 4 \\ & 3 & \\ & 2 & \end{array} \qquad \begin{array}{ccc} & 3 & \\ 2 & & 4 \\ 3 & 5 & 3 \\ 4 & & 2 \\ & 3 & \end{array} \qquad \begin{array}{ccc} & 4 & \\ 3 & & 5 \\ 4 & 2 & 4 \\ 5 & & 3 \\ & 4 & \end{array} \qquad \begin{array}{ccc} & 5 & \\ & 4 & \\ 3 & & 5 \\ & 4 & \\ & 5 & \end{array}.$$

There is an automorphism σ of E, induced by the graph automorphism of the above quiver given by rotating the quiver 180° about the centre. Then σ acts on the set $\{2, 3, 4, 5\}$ as the permutation $(2, 5)(3, 4)$.

The E-module M has Loewy and socle series given by

$$M = \begin{array}{cc} 3 & 5 \\ 2 & 4 \\ 3 & 5 \end{array}$$

and a truncated projective resolution

$$\overline{P}_2 \oplus \overline{P}_4 \longrightarrow \overline{P}_4 \oplus \overline{P}_3 \longrightarrow \overline{P}_3 \oplus \overline{P}_5 \longrightarrow M$$

One can then see that

$$\Omega^3_E(M) = \begin{array}{cc} 2 & 4 \\ 3 & 5 \\ 2 & 4 \end{array} \cong {}_\sigma M.$$

Thus, M is a σ-periodic E-module of period 3. We claim that there is an $\alpha \in \mathrm{Ext}^3_{E \otimes_k E^{\mathrm{op}}}(E, {}_\sigma E)$ such that M is strongly σ-periodic relative to α.

We describe a construction, Grantian in nature, with a complex of E-E-bimodules constructed from terms projective relative to some subalgebras of E. We first identify these subalgebras.

For $i \in \{2, 3, 4, 5\}$, let e_i be the primitive idempotent of E such that $\overline{P}_i = Ee_i$. Let B be the subalgebra of E generated by the idempotents $e = e_2 + e_4$ and $f = e_3 + e_5$ and the arrows $\zeta = \gamma_2 + \gamma_4 + \alpha$ and $\xi = \delta_2 + \delta_4 + \beta$. Then B has an Ext^1-quiver

$$e \underset{\xi}{\overset{\zeta}{\rightleftarrows}} f$$

with relations $\zeta\xi\zeta = 0 = \xi\zeta\xi$. To see this, we have

$$\begin{aligned} \zeta\xi &= (\gamma_2 + \gamma_4 + \alpha)(\delta_2 + \delta_4 + \beta) \\ &= \gamma_2\delta_2 + \gamma_4\delta_4 + \gamma_4\beta + \alpha\delta_4 + \alpha\beta \end{aligned}$$

so that

$$\begin{aligned}
\zeta\xi\zeta &= (\gamma_2\delta_2 + \gamma_4\delta_4 + \gamma_4\beta + \alpha\delta_4 + \alpha\beta)(\gamma_2 + \gamma_4 + \alpha) \\
&= \gamma_2\delta_2\gamma_2 + \gamma_2\delta_2\gamma_4 + \gamma_4\delta_4\gamma_2 + \gamma_4\delta_4\gamma_4 + \gamma_4\beta\alpha + \alpha\delta_4\gamma_2 + \alpha\delta_4\gamma_4 + \alpha\beta\alpha \\
&= (\gamma_2\delta_2\gamma_2 + \gamma_4\delta_4\gamma_2) + (\gamma_2\delta_2\gamma_4 + \gamma_4\beta\alpha) + \gamma_4\delta_4\gamma_4 + (\alpha\delta_4\gamma_4 + \alpha\beta\alpha) + \alpha\delta 4\gamma_2 \\
&= 0 + 0 + 0 + 0 + 0 \\
&= 0,
\end{aligned}$$

and

$$\begin{aligned}
\xi\zeta\xi &= (\delta_2 + \delta_4 + \beta)(\gamma_2\delta_2 + \gamma_4\delta_4 + \gamma_4\beta + \alpha\delta_4 + \alpha\beta) \\
&= \delta_2\gamma_2\delta_2 + \delta_2\gamma_4\delta_4 + \delta_2\gamma_4\beta + \delta_4\gamma_2\delta_2 + \delta_4\gamma_4\delta_4 + \delta_4\gamma_4\beta + \beta\alpha\delta + \beta\alpha\beta \\
&= (\delta_2\gamma_2\delta_2 + \delta_2\gamma_4\delta_4) + \delta_2\gamma_4\beta + (\delta_4\gamma_2\delta_2 + \beta\alpha\delta_4) + \delta_4\gamma_4\delta_4 + (\delta_4\gamma_4\beta + \beta\alpha\beta) \\
&= 0 + 0 + 0 + 0 + 0 \\
&= 0.
\end{aligned}$$

We comment that $B \cong A_{2,1}$ as k-algebras, where $A_{2,1}$ is the Brauer tree algebra of a star on two edges with exceptional multiplicity $m = 1$.

Next, let C be the subalgebra of E generated by the idempotents e_2, e_3, e_4, e_5 and the arrows γ_4, δ_4. Then C has an Ext^1-quiver

$$2 \qquad\quad 3 \underset{\gamma_4}{\overset{\delta_4}{\rightleftarrows}} 4 \qquad\quad 5$$

with relations $\gamma_4\delta_4\gamma_4 = 0 = \delta_4\gamma_4\delta_4$. Then $C \cong k \times A_{2,1} \times k$ as k-algebras. Note that C is not an indecomposable algebra.

Finally, let D be the subalgebra generated by the four idempotents e_2, e_3, e_4, e_5. Then D has an Ext^1-quiver

$$2 \qquad\quad 3 \qquad\quad 4 \qquad\quad 5$$

and, clearly, $D \cong k \times k \times k \times k$ as k-algebras. Again, D is not an indecomposable algebra.

Consider the sequence of E-E-bimodules

$$0 \longrightarrow {}_{\sigma}E \xrightarrow{d_3} {}_{\sigma}E \otimes_C E \xrightarrow{d_2} E \otimes_{{}_{\tau}D} E \xrightarrow{d_1} E \otimes_B E \xrightarrow{d_0} E \longrightarrow 0$$

where $E \otimes_{{}_{\tau}D} E$ denotes the E-E-bimodule $E \otimes_D {}_{\tau}D \otimes_D E$, where τ is the automorphism of D acting as the permutation $(2,4)(3,5)$ on labels of simple

D-modules. We comment that

$$E \otimes_{\tau D} E = \bigoplus_{i=2}^{5} \overline{P}_{i\tau(i)},$$

where $\overline{P}_{i\tau(i)} = \overline{P}_i \otimes_k \overline{P}^{\vee}_{\tau(i)}$, so this term is projective as an E-E-bimodule. We define the maps d_i below and claim that this sequence is exact.

The map $d_0 : E \otimes_B E \to E$ is the multiplication map, $d_0(x \otimes y) = xy$. The map $d_1 : E \otimes_{\tau D} E \to E \otimes_B E$ is given by $d_1(e_i \otimes e_{\tau(i)}) = e_i \otimes e_{\tau(i)}$. These elements generate $E \otimes_{\tau D} E$, so this completely defines d_1. Further, d_1 well-defined, as we need only consider the action of idempotents in E on either side. Then $d_0 \circ d_1 = 0$, since

$$d_0(d_1(e_i \otimes e_{\tau(i)})) = d_0(e_i \otimes e_{\tau(i)}) = e_i e_{\tau(i)} = 0.$$

Next, we define $d_2 : {}_\sigma E \otimes_C E \to E \otimes_{\tau D} E$ as follows. We first set

$$\begin{aligned} d_2(e_2 \otimes e_2) &= \alpha \otimes e_2 - e_5 \otimes \gamma_2, \\ d_2(e_5 \otimes e_5) &= \delta_2 \otimes e_5 - e_2 \otimes \beta. \end{aligned}$$

Next, set

$$\begin{aligned} d_2(e_3 \otimes e_3) =& \delta_4\gamma_4 \otimes \delta_2 + \beta \otimes \gamma_4\delta_4 + \delta_4 \otimes \alpha\delta_4 + \delta_4\gamma_2 \otimes \delta_4 \\ &- \beta\alpha\beta \otimes e_3 - e_4 \otimes \delta_2\gamma_2\delta_2 \end{aligned}$$

and

$$\begin{aligned} d_2(e_4 \otimes e_4) =& \gamma_4\delta_4 \otimes \alpha + \gamma_2 \otimes \delta_4\gamma_4 + \gamma_4 \otimes \delta_2\gamma_4 + \gamma_4\beta \otimes \gamma_4 \\ &- \gamma_2\delta_2\gamma_2 \otimes e_4 - e_3 \otimes \alpha\beta\alpha. \end{aligned}$$

Then one can show that $d_2(\delta_4 \otimes e_3) = d_2(e_4 \otimes \delta_4)$ and $d_2(e_3 \otimes \gamma_4) = d_2(\gamma_4 \otimes e_4)$, so that d_2 is well-defined.

We have

$$\begin{aligned} d_2(e_2 \otimes e_2) &= e_5\alpha e_4 \otimes e_2 - e_5 \otimes e_3\gamma_2 e_2 \\ &= e_5\zeta e_4 \otimes e_2 - e_5 \otimes e_3\zeta e_2 \\ &= e_5(\zeta \otimes 1_E - 1_E \otimes \zeta)e_2 \end{aligned}$$

in $E \otimes_{\tau D} E$, so that

$$\begin{aligned} d_1(d_2(e_2 \otimes e_2)) &= d_1(e_5(\zeta \otimes 1_E - 1_E \otimes \zeta)e_2) \\ &= e_5(\zeta \otimes 1_E - 1_E \otimes \zeta)e_2 \\ &= 0 \end{aligned}$$

in $E \otimes_B E$, while

$$\begin{aligned} d_2(e_5 \otimes e_5) &= e_2\delta_2 e_3 \otimes e_5 - e_2 \otimes e_4\beta e_5 \\ &= e_2\xi e_3 \otimes e_5 - e_2 \otimes e_4\xi e_5 \\ &= e_2(\xi \otimes 1_E - 1_E \otimes \xi)e_5 \end{aligned}$$

in $E \otimes_{\tau D} E$, so that

$$\begin{aligned} d_1(d_2(e_5 \otimes e_5)) &= d_1(e_2(\xi \otimes 1_E - 1_E \otimes \xi)e_5) \\ &= e_2(\xi \otimes 1_E - 1_E \otimes \xi)e_5 \\ &= 0 \end{aligned}$$

in $E \otimes_B E$.

Next, we have that

$$d_2(e_3 \otimes e_3) = e_4(\xi\zeta \otimes \xi + \xi \otimes \zeta\xi - 1_E \otimes \xi\zeta\xi - \xi\zeta\xi \otimes 1_E)e_3$$

in $E \otimes_{\tau D} E$, so that

$$\begin{aligned} d_1(d_2(e_3 \otimes e_3)) &= e_4(\xi\zeta \otimes \xi + \xi \otimes \zeta\xi - 1_E \otimes \xi\zeta\xi - \xi\zeta\xi \otimes 1_E)e_3 \\ &= 0 \end{aligned}$$

in $E \otimes_B E$, while

$$d_2(e_4 \otimes e_4) = e_3(\zeta\xi \otimes \zeta + \zeta \otimes \xi\zeta - 1_E \otimes \zeta\xi\zeta - \zeta\xi\zeta \otimes 1_E)e_4$$

in $E \otimes_{\tau D} E$, so that

$$\begin{aligned} d_1(d_2(e_4 \otimes e_4)) &= e_3(\zeta\xi \otimes \zeta + \zeta \otimes \xi\zeta - 1_E \otimes \zeta\xi\zeta - \zeta\xi\zeta \otimes 1_E)e_4 \\ &= 0 \end{aligned}$$

in $E \otimes_B E$, since $\zeta\xi\zeta = 0 = \xi\zeta\xi$ in B. Thus, $d_2 \circ d_1 = 0$.

Finally, we define $d_3 : {}_\sigma E \to {}_\sigma E \otimes_C E$ by

$$d_3(1_E) = y_2 + y_{34} + y_5,$$

where

$$\begin{aligned} y_2 &= e_2 \otimes \delta_2\gamma_2\delta_2\gamma_2 + \delta_2\gamma_2 \otimes \delta_2\gamma_2 + \delta_2\gamma_2\delta_2\gamma_2 \otimes e_2 \\ &\quad + \gamma_2\delta_2\gamma_2 \otimes \delta_2 + \gamma_2 \otimes \delta_2\gamma_2\delta_2 - \delta_4\gamma_2 \otimes \delta_2\gamma_4 \ \in Ee_2 \otimes_C e_2E, \\ y_{34} &= \delta_2 \otimes \gamma_2 - \alpha \otimes \beta + \gamma_2\delta_2 \otimes e_3 - \beta\alpha \otimes e_4 + e_3 \otimes \gamma_2\delta_2 \\ &\quad - e_4 \otimes \beta\alpha - \gamma_4 \otimes \delta_4 + \delta_4 \otimes \gamma_4 \ \in E(e_3 + e_4) \otimes_C (e_3 + e_4)E, \\ y_5 &= e_5 \otimes \alpha\beta\alpha\beta + \alpha\beta \otimes \alpha\beta + \alpha\beta\alpha\beta \otimes e_5 \\ &\quad + \beta\alpha\beta \otimes \alpha + \beta \otimes \alpha\beta\alpha - \gamma_4\beta \otimes \alpha\delta_4 \ \in Ee_5 \otimes_C e_5E. \end{aligned}$$

Observe that $y_5 = \sigma(y_2)$. The element $d_3(1_E)$ is central in $E \otimes_C E$, so d_3 is a well-defined homomorphism of bimodules. A rather painstaking calculation will then show that $d_2(d_3(1_E)) = 0$ in $E \otimes_{\tau D} E$, and our sequence is a complex of E-E-bimodules.

To show that this complex is an exact sequence, one considers the complexes obtained by applying the functors $- \otimes_E S_i$, for S_i the simple E-modules. There are relatively B-projective E-modules U_{24} and U_{35}, with Loewy series

$$U_{24} = \begin{matrix} 2 \ \ 4 \\ 3 \\ 2 \ \ 4 \end{matrix} \quad \text{and} \quad U_{35} = \begin{matrix} 3 \ \ 5 \\ 4 \\ 3 \ \ 5 \end{matrix} .$$

The projective E-modules $\overline{P}_2$ and $\overline{P}_5$ are relatively C-projective, as are the modules V_3 and V_4, with Loewy series

$$V_3 = \begin{matrix} 3 \\ 2 \\ 3 \end{matrix} \quad \text{and} \quad V_4 = \begin{matrix} 4 \\ 5 \\ 4 \end{matrix} .$$

Applying $- \otimes_E S_2$, we obtain a sequence

$$0 \longrightarrow S_5 \longrightarrow P_5 \longrightarrow P_4 \longrightarrow U_{24} \longrightarrow S_2 \longrightarrow 0 .$$

Applying $- \otimes_E S_3$, we obtain a sequence

$$0 \longrightarrow S_4 \longrightarrow V_4 \longrightarrow P_5 \longrightarrow U_{35} \longrightarrow S_3 \longrightarrow 0 .$$

Applying $- \otimes_E S_4$, we obtain a sequence

$$0 \longrightarrow S_3 \longrightarrow V_3 \longrightarrow P_2 \longrightarrow U_{24} \longrightarrow S_4 \longrightarrow 0 .$$

Applying $- \otimes_E S_5$, we obtain a sequence

$$0 \longrightarrow S_2 \longrightarrow P_2 \longrightarrow P_3 \longrightarrow U_{35} \longrightarrow S_5 \longrightarrow 0 .$$

The exactness of these sequences shows that the above sequence of E-E-bimodules is exact.

Now, given this exact sequence of E-E-bimodules, we have a triangle

$$Y \longrightarrow E \xrightarrow{\alpha} {}_{\sigma}E[3] \rightsquigarrow$$

in $D^b(E\text{-}E)$, with Y perfect in $D^b(E)$ and $D^b(E^{\mathrm{op}})$. This defines an element

$$\alpha \in \mathrm{Hom}_{D^b(E\text{-}E)}(E, {}_{\sigma}E[3]) \cong \mathrm{Ext}^3_{E \otimes_k E^{\mathrm{op}}}(E, {}_{\sigma}E).$$

Similarly to in §3.3.2, since E is projective as a left and a right B-module, C-module and D-module, we have that $Y \otimes_E^{\mathbf{L}} M$ is perfect in $D^b(E)$ and $M^\vee \otimes_E^{\mathbf{L}} Y$ is perfect in $D^b(E^{\mathrm{op}})$. Noting that $\sigma^{-1} = \sigma$, both M and $M^\vee$ are thus strongly σ-periodic of period 3, relative to α. Thus, Theorem 2.3.8 then gives a generalised periodic twist

$$\Phi_P : D^b(A) \xrightarrow{\sim} D^b(A),$$

self-perverse relative to the filtration

$$\emptyset \subset_0 \{1\} \subset_3 \{1, 2, 3, 4, 5\}.$$

We comment that an autoequivalence of $D^b(B_{(2)})$, perverse with respect to this filtration, does indeed exist by other means. To produce it, one composes the equivalence

$$\Phi_1 : D^b(B_{(2)}) \xrightarrow{\sim} D^b(B_{(3,1^2)})$$

constructed by Craven and Rouquier in [CR13, §5.5.3], perverse with respect to the filtrations

$$\emptyset \subset_0 \{1\} \subset_3 \{1, 5\} \subset_4 \{1, 5, 4\} \subset_5 \{1, 5, 4, 2\} \subset_6 \{1, 5, 4, 2, 3\}$$

and

$$\emptyset \subset_0 \{1\} \subset_3 \{1, 2\} \subset_4 \{1, 2, 3\} \subset_5 \{1, 2, 3, 4\} \subset_6 \{1, 2, 3, 4, 5\}$$

with the inverse of the equivalence

$$\Phi_2 : D^b(B_{(3,1^2)}) \xrightarrow{\sim} D^b(A_{2,1} \wr \mathfrak{S}_2)$$

of Rickard and Marcus, noting that $A_{2,1} \wr \mathfrak{S}_2$ and $B_{(3,1^2)}$ are Morita equivalent, perverse with respect to the filtration

$$\emptyset \subset_0 \{1, 2\} \subset_{-1} \{1, 2, 3\} \subset_{-2} \{1, 2, 3, 4, 5\}$$

on both sides, and the equivalence

$$\Phi_3 : D^b(B_{(3,1^2)}) \xrightarrow{\sim} D^b(B_{(2)})$$

perverse with respect to the filtration

$$\emptyset \subset_0 \{1, 2, 3, 4\} \subset_{-1} \{1, 2, 3, 4, 5\}$$

on both sides, arising from the $[2 : 1]$ pair between $B_{(3,1^2)}$ and $B_{(2)}$. The equivalence

$$\Phi_3 \circ \Phi_2^{-1} \circ \Phi_1 : D^b(B_{(2)}) \xrightarrow{\sim} D^b(B_{(2)})$$

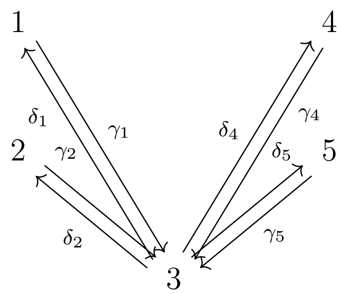
thus obtained is a self-perverse equivalence, relative to the filtration

$$\emptyset \subset_0 \{1\} \subset_3 \{1,2,3,4,5\}.$$

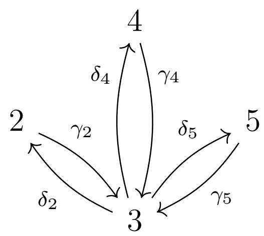
Note here again that the algebra B is isomorphic to the Brauer tree algebra $A_{2,1}$ and that C has an irreducible constituent isomorphic to $A_{2,1}$. This is highly suggestive of the functor Φ_P masking some restriction and induction functors to some weight one blocks. We conjecture that this is related to the self-stable equivalences of the Brauer correspondent block described by Craven and Rouquier in [CR13, §5.5], though it is not entirely clear precisely how.

3.3.4 Autoequivalences of $B_{(1),2}$ and $B_{(3,1^2),2}$

Let $A = B_{(3,1^2)}$ and recall the description of this block in §3.2.1.



Let $P = P_2 \oplus P_3 \oplus P_4 \oplus P_5$, $Q = P_1$, $E = \mathrm{End}_A(P)^{\mathrm{op}}$ and $M = \mathrm{Hom}_A(P, Q)$. Then E has an Ext^1-quiver



with relations:

- $\delta_5\gamma_2 = 0$, $\delta_2\gamma_5 = 0$;
- $\gamma_2\delta_2\gamma_2 + \gamma_4\delta_4\gamma_2 = 0$;
- $\gamma_2\delta_2\gamma_4 = \gamma_5\delta_5\gamma_4$, $\gamma_4\delta_4\gamma_4 + \gamma_2\delta_2\gamma_4 = 0$;

- $\gamma_5\delta_5\gamma_5 + \gamma_4\delta_4\gamma_5 = 0$;
- $\delta_2\gamma_2\delta_2 + \delta_2\gamma_4\delta_4 = 0$;
- $\delta_4\gamma_2\delta_2 = \delta_4\gamma_5\delta_5$, $\delta_4\gamma_2\delta_2 + \delta_4\gamma_4\delta_4 = 0$;
- $\delta_5\gamma_4\delta_4 + \delta_5\gamma_5\delta_5 = 0$.

We comment that the loop $\gamma_1\delta_1$ in A falls into the ideal generated by the remaining arrows in E.

The projective indecomposable E-modules, $\overline{P}_2, \overline{P}_3, \overline{P}_4, \overline{P}_5$, have Loewy series

$$\begin{array}{c} 2\\ 3\\ 2\ \ 4\\ 3\\ 2 \end{array} \qquad \begin{array}{c} 3\\ 2\ \ 4\ \ 5\\ 3\ \ 3\ \ 3\\ 5\ \ 4\ \ 2\\ 3 \end{array} \qquad \begin{array}{c} 4\\ 3\\ 2\ \ 4\ \ 5\\ 3\\ 4 \end{array} \qquad \begin{array}{c} 5\\ 3\\ 4\ \ 5\\ 3\\ 5 \end{array}.$$

The E-module M has Loewy series

$$M = \begin{array}{c} 3\\ 2\ \ 5\\ 3 \end{array}$$

and a truncated projective resolution

$$\overline{P}_3 \to \overline{P}_3 \oplus \overline{P}_4 \to \overline{P}_2 \oplus \overline{P}_4 \oplus \overline{P}_5 \to \overline{P}_2 \oplus \overline{P}_4 \oplus \overline{P}_5 \to \overline{P}_4 \oplus \overline{P}_3 \to \overline{P}_3 \to M$$

from which one can calculate that $\Omega^6_E(M) \cong M$. Thus, M is a periodic E-module of period 6. Let $N = \Omega^3_E(M)$. Then

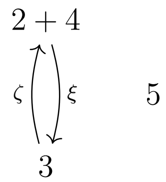
$$N = \begin{array}{c} 2\ \ 4\ \ 5\\ 3\ \ 3\\ 2\ \ 4\ \ 5 \end{array}$$

and N is an E-module not isomorphic to M, also periodic of period 6. A truncated projective resolution of N is

$$\overline{P}_2 \oplus \overline{P}_4 \oplus \overline{P}_5 \to \overline{P}_4 \oplus \overline{P}_3 \to \overline{P}_3 \to \overline{P}_3 \to \overline{P}_3 \oplus \overline{P}_4 \to \overline{P}_2 \oplus \overline{P}_4 \oplus \overline{P}_5 \to N.$$

We claim that M and N are strongly periodic of period 6, relative to some $\alpha \in \mathrm{Ext}^6_{E\otimes_k E^{\mathrm{op}}}(E, E)$. We outline a method of demonstrating this.

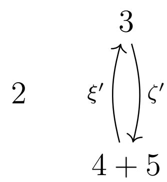
Let B be the subalgebra of E generated by the idempotents $e_2 + e_4$, e_3 and e_5 and the arrows $\xi = \gamma_2 + \gamma_4$ and $\zeta = \delta_2 + \delta_4$.



Then $\xi\zeta\xi = 0 = \zeta\xi\zeta$, so $B \cong A_{2,1} \times k$, where $A_{2,1}$ is the Brauer tree algebra of a star on 2 edges with exceptional multiplicity 1. The relatively B-projective E-modules are

$$U_{24} = \begin{array}{c} 2\ 4 \\ 3 \\ 2\ 4 \end{array}, \quad U_3 = \begin{array}{c} 3 \\ 5 \\ 3 \end{array}, \quad U_5 = \overline{P}_5.$$

Let C be the subalgebra of E generated by the idempotents e_2, e_3 and $e_4 + e_5$ and the arrows $\xi' = \gamma_4 + \gamma_5$ and $\zeta' = \delta_4 + \delta_5$.



Then $\xi'\zeta'\xi' = 0 = \zeta'\xi'\zeta'$, so $C \cong A_{2,1} \times k$, too. The relatively C-projective E-modules are

$$V_2 = \overline{P}_2, \quad V_3 = \begin{array}{c} 3 \\ 2 \\ 3 \end{array}, \quad V_{45} = \begin{array}{c} 4\ 5 \\ 3 \\ 4\ 5 \end{array}.$$

Let D be the subalgebra of E generated by the idempotents e_2, e_3, e_4 and e_5.

$$2 \qquad 3 \qquad 4 \qquad 5$$

Then $D \cong k \times k \times k \times k$. Let τ_1, τ_2, τ_3 be the automorphisms of D which act on the simple labels via the following permutations:

- $\tau_1 = (2, 5, 4, 3)$;
- $\tau_2 = (2, 4, 3, 5)$;
- $\tau_3 = (2, 5)(3, 4)$.

We claim that there is an exact sequence of E-E-bimodules of the form

$$0 \longrightarrow E \longrightarrow Y_5 \longrightarrow Y_4 \longrightarrow Y_3 \longrightarrow Y_2 \longrightarrow Y_1 \longrightarrow Y_0 \longrightarrow E \longrightarrow 0,$$

where:

- $Y_0 = (E \otimes_B E) \oplus (E \otimes_C E)$;
- $Y_1 = (E \otimes_{\tau_1 D} E) \oplus (E \otimes_D E) \oplus (E \otimes_{\tau_2 D} E)$;
- $Y_2 = (E \otimes_{\tau_1 D} E) \oplus (E \otimes_{\tau_3 D} E) \oplus (E \otimes_{\tau_2 D} E)$;
- $Y_3 = (E \otimes_{\tau_1 D} E) \oplus (E \otimes_{\tau_3 D} E) \oplus (E \otimes_{\tau_2 D} E)$;
- $Y_4 = (E \otimes_{\tau_1 D} E) \oplus (E \otimes_D E) \oplus (E \otimes_{\tau_2 D} E)$;
- $Y_5 = (E \otimes_B E) \oplus (E \otimes_C E)$.

For exactness, one checks the complexes obtained by applying the functors $- \otimes^{\mathbf{L}}_E S_i$ for the simple E-modules S_i. These complexes are

$$0 \longrightarrow S_2 \longrightarrow U_{24} \oplus \overline{P}_2 \longrightarrow \overline{P}_{234} \longrightarrow \overline{P}_{455} \longrightarrow \overline{P}_{455} \longrightarrow \overline{P}_{234} \longrightarrow U_{24} \oplus \overline{P}_2 \longrightarrow S_2 \longrightarrow 0,$$

$$0 \longrightarrow S_3 \longrightarrow U_3 \oplus V_3 \longrightarrow \overline{P}_{235} \longrightarrow \overline{P}_{245} \longrightarrow \overline{P}_{245} \longrightarrow \overline{P}_{235} \longrightarrow U_3 \oplus V_3 \longrightarrow S_3 \longrightarrow 0,$$

$$0 \longrightarrow S_4 \longrightarrow U_{24} \oplus V_{45} \longrightarrow \overline{P}_{245} \longrightarrow \overline{P}_{333} \longrightarrow \overline{P}_{333} \longrightarrow \overline{P}_{245} \longrightarrow U_{24} \oplus V_{45} \longrightarrow S_4 \longrightarrow 0,$$

$$0 \longrightarrow S_5 \longrightarrow \overline{P}_5 \oplus V_{45} \longrightarrow \overline{P}_{345} \longrightarrow \overline{P}_{224} \longrightarrow \overline{P}_{224} \longrightarrow \overline{P}_{345} \longrightarrow \overline{P}_5 \oplus V_{45} \longrightarrow S_5 \longrightarrow 0,$$

where, given a sequence $\{i_j\}_{j \in J'}$ of elements of $\{2,3,4,5\}$, we denote by $\overline{P}_{\{i_j\}}$ the projective E-module $\bigoplus_{j \in J'} \overline{P}_{i_j}$.

The complex of E-E-bimodules above then defines a triangle

$$Y \longrightarrow E \xrightarrow{\alpha} E[6] \rightsquigarrow$$

in $D^b(E\text{-}E)$, and thus an element

$$\alpha \in \operatorname{Hom}_{D^b(E\text{-}E)}(E, E[6]) \cong \operatorname{Ext}^6_{E \otimes_k E^{\mathrm{op}}}(E, E)$$

By construction, the complex Y is perfect in $D^b(E)$ and $D^b(E^{\mathrm{op}})$. We also note that, with reference to the proof of Proposition 2.2.1, E is projective as a left and a right B-module, C-module and D-module, so that, for each i, $Y_i \otimes^{\mathbf{L}}_E M$ is a projective E-module and $M^\vee \otimes^{\mathbf{L}}_E Y_i$ is a projective E^{op}-module. Therefore $Y \otimes^{\mathbf{L}}_E M$ is perfect in $D^b(E)$ and $M^\vee \otimes^{\mathbf{L}}_E Y$ is perfect in $D^b(E^{\mathrm{op}})$, so that M and $M^\vee$ are both strongly periodic of period 6, relative to α.

By Theorem 2.3.8, the generalised periodic twist

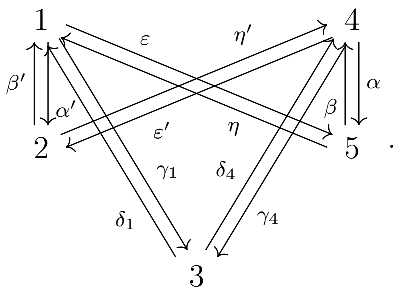
$$\Phi_P : D^b(A) \xrightarrow{\sim} D^b(A),$$

is a perverse equivalence, relative to the filtration

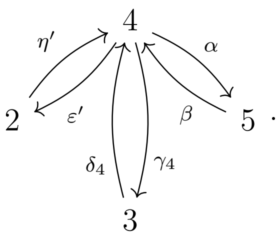
$$\emptyset \subset_0 \{1\} \subset_6 \{1,2,3,4,5\}.$$

Now, for $i \in \{2,3,4,5\}$, let X_i be the image of the simple module S_i. We observe that, for every i, $H_3(X_i) \cong S_{\sigma(i)}$, where here we identify the automorphism σ with the permutation $(2,5)(3,4)$. This is related to the following.

Take $A' = B_{(1)}$, and recall the description in §3.2.4.



Let $P' = P'_2 \oplus P'_3 \oplus P'_4 \oplus P'_5$ and $E' = \mathrm{End}_A(P)^{\mathrm{op}}$. Then E' has an Ext^1-quiver of the form



We claim that, by calculating relations in A', one can see that $E \cong E'$, with a change of simple labels given by the permutation $(2,3)(4,5)$. The E'-module M' has Loewy and socle series

$$M' = \begin{array}{ccc} 2 & 3 & 5 \\ & 4 \; 4 & \\ 2 & 3 & 5 \end{array}.$$

Then under the change of simple labels, as an E-module $M' \cong N = \Omega^3_E(M)$. Thus, the periodicity analysis above should apply to M', too. We claim that this then induces a generalised periodic twist

$$\Phi_{P'} : D^b(A') \xrightarrow{\sim} D^b(A'),$$

perverse relative to the filtration

$$\emptyset \subset_0 \{1\} \subset_6 \{1,2,3,4,5\}.$$

We believe the significance of the modules M and N appearing as each other's third Heller translate is related to certain perverse equivalences of Craven and Rouquier. In [CR13, §5.5.1], Craven and Rouquier construct an equivalence

$$D^b(B_\emptyset) \xrightarrow{\sim} D^b(B_{(3,1^2)}),$$

perverse relative to the filtrations

$$\emptyset \subset_0 \{1\} \subset_3 \{1,2,4,5\} \subset_4 \{1,2,4,5,3\}$$

and

$$\emptyset \subset_0 \{1\} \subset_3 \{1,5,3,2\} \subset_4 \{1,5,3,2,4\}$$

The inverse of the derived equivalence arising from the [2 : 1] pair between $B_{(1)}$ and $B_\emptyset$ is an equivalence

$$D^b(B_{(1)}) \xrightarrow{\sim} D^b(B_\emptyset),$$

perverse relative to the filtration

$$\emptyset \subset_0 \{1,2,4,5\} \subset_{-1} \{1,2,4,5,3\}$$

on both sides. Composing the two thus induces an equivalence

$$D^b(B_{(1)}) \xrightarrow{\sim} D^b(B_{(3,1^2)}),$$

perverse relative to the filtrations

$$\emptyset \subset_0 \{1\} \subset_3 \{1,2,3,4,5\}$$

and

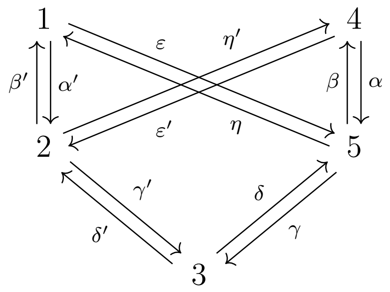
$$\emptyset \subset_0 \{1\} \subset_3 \{1,5,4,3,2\}.$$

Composing this equivalence with its inverse in either direction therefore gives perverse autoequivalences of $D^b(B_{(1)})$ and $D^b(B_{(3,1^2)})$, with perversity functions and filtrations matching our proposed generalised periodic twists Φ_P and $\Phi_{P'}$. This also offers an explanation for the degree three homology of the images X_i of simple modules S_i above being simple, and isomorphic to $S_{\sigma(i)}$ for every i.

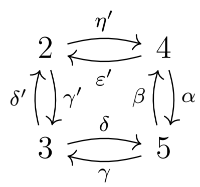
We again observe that the subalgebras B and C of E have irreducible constituents isomorphic to the Brauer tree algebra $A_{2,1}$, suggesting the involvement of some restriction and induction functors to weight one blocks. It is likely that this is related to the self-stable equivalences of the Brauer correspondent in [CR13, §5.5], in a way that is slightly easier to see in this example, by comparing images of simple modules above to those in [CR13, §5.5.1].

3.3.5 Autoequivalences of $B_{\emptyset,2}$ and $B_{(2),2}$

Let $A = B_\emptyset$ and recall the description in §3.2.5.



Let $P = P_2 \oplus P_3 \oplus P_4 \oplus P_5$, $Q = P_1$, $E = \mathrm{End}_A(P)^{\mathrm{op}}$ and $M = \mathrm{Hom}_A(P,Q)$. Then E has an Ext^1-quiver



with relations:

- $\alpha\eta' = \delta\gamma'$, $\varepsilon'\beta = \delta'\gamma$;
- $\gamma\delta + \gamma'\delta' = 0$;
- $\gamma\alpha = \gamma'\varepsilon'$, $\beta\delta = \eta'\delta'$;
- all paths of length four between distinct vertices are zero.

The projective indecomposable E-modules, $\overline{P}_2, \overline{P}_3, \overline{P}_4, \overline{P}_5$, have Loewy series

$$\begin{array}{c}2\\3\ 4\\2\ 5\ 2\\4\ 3\\2\end{array} \qquad \begin{array}{c}3\\2\ 5\\3\ 4\\5\ 2\\3\end{array} \qquad \begin{array}{c}4\\2\ 5\\4\ 3\ 4\\5\ 2\\4\end{array} \qquad \begin{array}{c}5\\3\ 4\\5\ 2\ 5\\4\ 3\\5\end{array}.$$

The E-module M has Loewy series

$$M = \begin{array}{c}2\ 5\\3\ 4\\2\ 5\end{array},$$

and a truncated projective resolution

$$\overline{P}_2 \oplus \overline{P}_5 \to \overline{P}_3 \oplus \overline{P}_4 \to \overline{P}_4 \to \overline{P}_4 \to \overline{P}_3 \oplus \overline{P}_4 \to \overline{P}_2 \oplus \overline{P}_5 \to M$$

One can calculate that

$$\Omega_E^6(M) = \begin{array}{cc} 2 & 5 \\ 3 & 4 \\ 2 & 5 \end{array} \cong M,$$

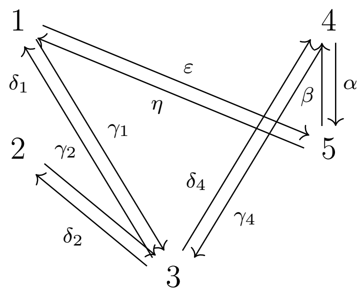
so M is a periodic E-module of period 6. Let $N = \Omega_E^3(M)$. Then

$$N = \begin{array}{ccc} & 4 & \\ 2 & & 5 \\ & 4 & \end{array}$$

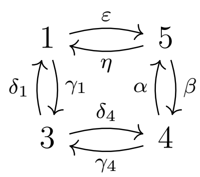
and N is another E-module, not isomorphic to M. A truncated projective resolution of N is

$$\overline{P}_4 \to \overline{P}_3 \oplus \overline{P}_4 \to \overline{P}_2 \oplus \overline{P}_5 \to \overline{P}_2 \oplus \overline{P}_5 \to \overline{P}_3 \oplus \overline{P}_4 \to \overline{P}_4 \to N$$

Now let $A' = B_{(2)}$ and recall the description in §3.2.2.



Let $P' = P'_1 \oplus P'_3 \oplus P'_4 \oplus P'_5$, $Q' = P'_2$, $E' = \mathrm{End}_A(P)^{\mathrm{op}}$, $M' = \mathrm{Hom}_{A'}(P', Q')$. Then E' has an Ext^1-quiver



with relations:

- $\beta\varepsilon = \delta_4\gamma_1$, $\eta\alpha = \delta_1\gamma_4$, $\varepsilon\delta_1 = \alpha\delta_4$, $\gamma_1\eta = \gamma_4\beta$;
- $\alpha\beta = \varepsilon\eta$;
- $\gamma_1\delta_1\gamma_1 = 0$, $\delta_1\gamma_1\delta_1 = 0$, $\gamma_4\delta_4\gamma_4 = 0$, $\delta_4\gamma_4\delta_4 = 0$;
- all paths of length four between distinct vertices are 0.

We claim that $E \cong E'$, with a change of simple labels given by

$$2 \mapsto 1, \ 3 \mapsto 5, \ 4 \mapsto 3, \ 5 \mapsto 4.$$

The E'-module M' has Loewy series

$$M' = \begin{matrix} & 3 & \\ 1 & & 4. \\ & 3 & \end{matrix}$$

Then, as an E-module, $M' \cong N = \Omega_E^3(M)$. So again, M' is periodic of period 6.

We believe that there is an $\alpha \in \mathrm{Ext}^6_{E \otimes_k E^{\mathrm{op}}}$ such that M, $M^\vee$, N and $N^\vee$ are all strongly periodic of period 6 relative to α. This being the case, we would obtain an autoequivalence

$$D^b(B_\emptyset) \xrightarrow{\sim} D^b(B_\emptyset)$$

self-perverse relative to the filtration

$$\emptyset \subset_0 \{1\} \subset_6 \{1, 2, 3, 4, 5\},$$

and an autoequivalence

$$D^b(B_{(2)}) \xrightarrow{\sim} D^b(B_{(2)})$$

self-perverse relative to the filtration

$$\emptyset \subset_0 \{2\} \subset_6 \{2, 1, 3, 4, 5\}.$$

We also hope that, as in §3.3.4, this produces an equivalence

$$D^b(B_\emptyset) \xrightarrow{\sim} D^b(B_{(2)})$$

perverse relative to the filtrations

$$\emptyset \subset_0 \{1\} \subset_3 \{1, 2, 3, 4, 5\}$$

and

$$\emptyset \subset_0 \{2\} \subset_3 \{2, 1, 5, 3, 4\}.$$

Unfortunately, standard methods to produce this extension α have proved fruitless. Moreover, it does not appear obvious from the equivalences of Craven and Rouquier [CR13] and Chuang and Rouquier [CR08] that there are two-step self-perverse equivalences

$$D^b(B_\emptyset) \xrightarrow{\sim} D^b(B_\emptyset)$$

and

$$D^b(B_\emptyset) \xrightarrow{\sim} D^b(B_\emptyset)$$

of width 6, relative to the appropriate filtrations.

This suggests at least two possibilities. Firstly, the periodicity of these modules is a pure coincidence, and there is no such α relative to which they are strongly periodic. However, this would be a remarkable coincidence, especially considering our other examples that have at least some reasoning behind their existence. Secondly, it may be that one will have to appeal to the structure of $\mathrm{Ext}^6_{E\otimes_k E^{\mathrm{op}}}(E, E)$ itself to produce the correct α abstractly. This would rely on a good working knowledge of the sixth Hochschild cohomology class

$$HH^6(E) \cong \mathrm{Ext}^6_{E\otimes_k E^{\mathrm{op}}}(E, E).$$

Because the algebra E has arisen in a rather ad hoc manner and does not obviously fall into a well-studied class of algebras, its Hochschild cohomology is not immediately apparent. It is suggestive, though, that studying the Hochschild cohomology classes

$$HH^*(E) \cong \mathrm{Ext}^*_{E\otimes_k E^{\mathrm{op}}}(E, E)$$

and the *twisted* Hochschild cohomology classes

$$HH^*(E; {}_\sigma E) \cong \mathrm{Ext}^*_{E\otimes_k E^{\mathrm{op}}}(E, {}_\sigma E)$$

will greatly benefit potential future applications of Theorem 2.3.3. Moreover, this would largely sidestep the arduous task of producing explicit exact sequences, like those presented in §3.3.3 and §3.3.4, to demonstrate the wider practicality of our main result.

3.3.6 Relevance to Broué's Conjecture

We note that Conjecture 3.1.8 is known to be true for the symmetric group blocks of weight $w = 2$ in characteristic $p = 3$, due to the examples of Craven and Rouquier [CR13, §5.5] and Chuang and Rouquier [CR08, Theorem 7.2, Remark 7.5]. However, this is reliant on the exceptional coincidence that the Rouquier block $B_{(3,1^2)}$ and the Brauer correspondent $A_{2,1} \wr \mathfrak{S}_2$ are Morita equivalent, rather than just derived equivalent. It offers greater evidence in favour of Conjecture 3.1.8 in generality if the equivalences realising Broué's Conjecture 3.1.2 remain perverse on composition with the equivalence

$$D^b(B_{(3,1^2)}) \xrightarrow{\sim} D^b(A_{2,1} \wr \mathfrak{S}_2)$$

of Rickard and Marcus.

We note that there are two such equivalences, one, say Φ_{RoCK}, perverse relative to the filtration

$$\emptyset \subset_0 \{1,2\} \subset_1 \{1,2,3\} \subset_2 \{1,2,3,4,5\}$$

on the left hand side, and one, say Φ'_{RoCK}, perverse relative to the filtration

$$\emptyset \subset_0 \{4,5\} \subset_1 \{4,5,3\} \subset_2 \{4,5,3,1,2\}$$

on the left hand side.

In particular, the previously known perverse equivalence

$$D^b(B_{(1)}) \xrightarrow{\sim} D^b(A_{2,1} \wr \mathfrak{S}_2)$$

is the composition of two $[2:1]$ pair derived equivalences, perverse relative to the filtration

$$\emptyset \subset_0 \{2,5\} \subset_1 \{4,5,3,1,2\}$$

This does not remain perverse on composition with either Φ_{RoCK} or Φ'_{RoCK}. Instead, consider the equivalence

$$\hat{\Phi}_{P'} : D^b(B_{(1)}) \xrightarrow{\sim} D^b(B_{(3,1^2)})$$

given as the half-way point of the conjectured generalised periodic twist

$$\Phi_{P'} : D^b(B_{(1)}) \xrightarrow{\sim} D^b(B_{(1)})$$

in §3.3.4. This is perverse relative to the filtrations

$$\emptyset \subset_0 \{1\} \subset_3 \{1,2,3,4,5\}$$

and

$$\emptyset \subset_0 \{1\} \subset_3 \{1,5,4,3,2\}.$$

The composition

$$\Phi_{\text{RoCK}}\hat{\Phi}_{P'} : D^b(B_{(1)}) \xrightarrow{\sim} D^b(A_{2,1} \wr \mathfrak{S}_2)$$

is a perverse equivalence, relative to the filtration

$$\emptyset \subset_0 \{1\} \subset_3 \{1,5\} \subset_4 \{1,5,4\} \subset_5 \{1,5,4,2,3\}$$

on the left hand side. This therefore gives stronger evidence supporting Conjecture 3.1.8 in general.

Further Work

The central result of this thesis, Theorem 2.3.3, provides necessary and sufficient conditions for the existence of two-step perverse autoequivalences of the derived category of a finite-dimensional symmetric algebra of a particular form. We have seen that attempts to apply this result can be stifled by unwieldy sequences of bimodules. The role of the extension

$$\alpha \in \operatorname{Ext}^n_{E\otimes_k E^{\mathrm{op}}}(E, {}_\sigma E) = HH^n(E; {}_\sigma E)$$

indicates that it is perhaps better to attempt to apply our result more abstractly, using knowledge of the Hochschild cohomology of the idempotent algebras E.

One may then wonder how widely applicable our result is. In the context of the symmetric groups, for example, can we find strongly periodic modules in idempotent algebras of blocks of larger weight, or in higher characteristic? Moreover, one might ask the same question for blocks of the Iwahori-Hecke algebras, whose representation theory is intimately related to that of the symmetric groups.

Attempts to produce any such examples have thus far proven futile. This suggests attempting to relax the conditions in Theorems 2.3.4 and 2.3.8. For example, what, if anything, can we say about periodicity for a self-perverse equivalence

$$\Phi : D^b(A) \xrightarrow{\sim} D^b(A)$$

with a filtration

$$\emptyset = \mathcal{S}_0 \subset \mathcal{S}_1 \subset \ldots \subset \mathcal{S}_r = \mathcal{S}$$

of arbitrary length? It is not immediately obvious, though examples suggest that some periodicity statement should hold.

Finally, we observe that the projectivity of P is not really the important aspect, but rather the Calabi-Yau property. Thus, we believe we should be able to replace P with some perfect object, say Z, in $D^b(A)$, and E with the endomorphism differential graded algebra E_Z of Z. Seidel and Thomas's geometric spherical twists are highly suggestive that such a result is plausible. However, this would require a rephrasing of our statements in the language of dg-categories, and is therefore left as work for a future date.

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