

The FitzHugh-Nagumo Equations and Quantum Noise

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Abstract

It is unclear whether quantum phenomena can be observed in brain recordings because of thermal noise causing decoherence, that is, quantum superpositions and entanglement quickly collapsing into classical, i.e. nonquantum states. This paper demonstrates that intrinsic neuronal noise naturally leads to a wave-particle description of neural activity of the sort predicted by a fundamental equation in quantum mechanics known as the Schrödinger equation. The predicted quantum-like fluctuations in firing rates are consistent with data. This approach could provide new insights into the underlying mechanisms of brain function, thus challenging existing paradigms in both quantum physics and neuroscience.

Part I

Basic Theory in 1D

1 Introduction

Quantum mechanics is widely regarded as the most fundamental theory of nature, suggesting its applicability to even brain functions. However, the prevailing view is that quantum coherence is lost at the neuronal level due to the large size and complex environment of the

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neurons, a process known as ‘decoherence’ [1, 2]. Each neuron is itself a massive and complex cell comprising a vast number of molecules, making it unlikely that quantum states could be maintained or measured across such a large scale. Neurobiological processes operate on much larger scales than those where quantum phenomena dominate. For example, action potentials are driven by ion exchanges across cell membranes that are well described by classical rather than quantum physics. Furthermore, cognitive functions, including memory, attention and sensory perception, typically involve neuronal populations interacting through electrochemical signals. While these might exhibit quantum effects, they might not scale up in a coherent way to affect brain activity at the population level. As a result, both physicists and neuroscientists have traditionally dismissed the relevance of standard quantum mechanics to neuronal processes. Still, classical models do consider stochastic processes in neurons [3, 4, 5, 7] as this is needed to explain brain data that show a large amount of variability. This can be due to fluctuations in synaptic transmission, stochastic opening and closing of ion channels or changes in incoming input. Stochastic effects are then modelled as stochastic terms added to differential equations describing synaptic transmission, ion channel activity or incoming inputs [8]. Brain recordings include examples of stochastic variations in neuronal dynamics such as membrane potential fluctuations [9], irregular spike trains [10], fluctuating synaptic currents [11], trial to trial variability in response to stimuli [12], fluctuations in spike-triggered average [13], spike amplitude variability [14], variability in LTP measurements [15] and calcium dynamics [16], etc.

Neuronal models used to study these phenomena are based on stochastic differential equations, and predict effects that can be studied using classical physics. Such modeling studies indicate that stochastic noise may affect various neural processes, including information processing, spike timing reliability, stochastic resonance, firing irregularity, subthreshold dynamics, as well as the initiation and propagation of action potentials. This can be due to ion channel, synaptic, cellular or even mechanical and thermal noise and other factors like background network activity [5, 6].

Motivated by a mathematical equivalence between stochastic and quantum mechanics shown by Nelson [17, 18, 19], we suggest that stochastic effects expressed as white noise type in neural data (Brownian motion) are equivalent to quantum effects. This link is established by describing stochastic effects in terms of a fundamental equation similar to one of the most fundamental equations, the Schrödinger equation in quantum mechanics. This can complement classical stochastic equations describing other types of stochastic effects, and can be used to come up with hypotheses that can be tested against real data. This is discussed in Section 7 below.

2 1D Random Walk Model with Drift for the Spike Activity of a Single Neuron

It would be helpful at this stage to introduce some technical terms which will be used in what follows. A stochastic process is a sequence of random variables whose values change over time in an uncertain way so that one knows the distribution of possible values at any point in time. Here we consider the neuron’s membrane potential as a random variable. A

Markov stochastic process is a particular type of stochastic process where only the current value of a variable is relevant for predicting the future movement, not past values. For neuronal data this is clearly a simplification as synaptic plasticity, refractory periods and mesoscale patterns of neuronal activity would need to be described by a correlated random walk. A random walk with and without drift is the stochastic process formed by successive summation of independent, identically (often normally) distributed random variables (Fig. 1). To describe neuronal dynamics resulting from the cumulative effect of several factors discussed above, like synaptic inputs, firing threshold, ion channel variability etc, we will assume a Wiener process. A Wiener process, also called Brownian motion, is a Markov process which is essentially a series of normally distributed random variables such that for later times the variances of these normally distributed random variables increase, a process called ‘diffusion’ (Fig 2).

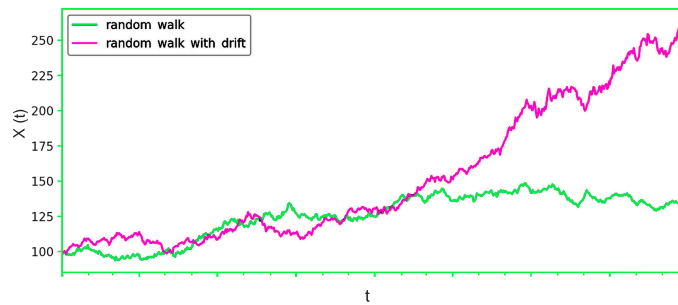


Figure 1: Random walk with and without drift. The value $x(t)$ of the random variable $X(t)$ at time t equals the last period’s value plus a constant (a drift) and a white noise. The drift can be in the direction of the threshold (forward) or away from it (backward).

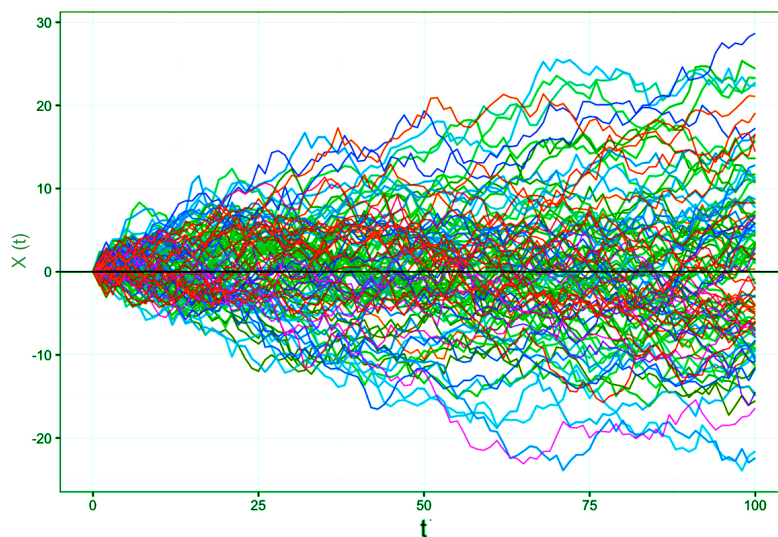


Figure 2: Wiener process or gaussian random walk (Brownian motion) showing diffusion, i.e. increase of variance with time.

Let us now consider a simple 1D random walk model to describe neuronal dynamics and spike activity. For instance, the electrical state of polarization in the somatic and dendritic membrane can be modelled as a random variable representing the coordinate $Q(t)$ and executing a random walk, influenced by excitatory and inhibitory post-synaptic potentials (EPSPs and IPSPs), and moving back and forth along a straight line between a resting potential and a threshold of neuron firing [3, 7]. Each incoming elemental EPSP (Excitatory Post Synaptic Potential) moves the random variable $Q(t)$ one unit toward the threshold, and each incoming elemental IPSP (Inhibitory Post Synaptic Potential) moves the random variable one unit away from the threshold (Fig.3). If the average rate of incoming elemental EPSP and elemental IPSP are the same, there is an equal probability at any time that the random variable moves either a unit toward or a unit away from the threshold, i.e. there is no “bias toward” either input. Immediately after the random variable has attained the threshold and caused the production of an action potential, it returns to the resting potential, only to begin its random walk again. Such a model would be a simple random walk model.

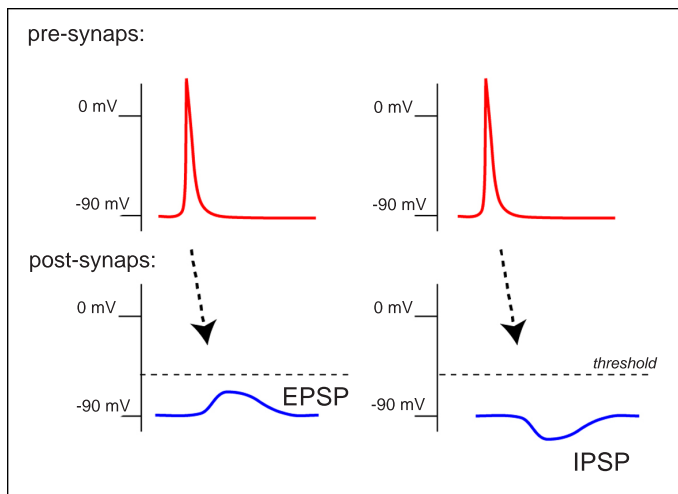


Figure 3: The figure illustrates the membrane potential’s forward and backward movements corresponding to Excitatory (EPSP) and Inhibitory (IPSP) Post-Synaptic Potentials, respectively.

In reality, however, these two rates may sometimes be different, and in realistic physiological models it would be far more reasonable to assume that there is some excess of either EPSP or IPSP inputs. In this case, the probability for the random variable $Q(t)$ to move one unit toward the threshold will be different from the probability for it to move away from the threshold. Considered as a diffusion process, the difference between these probabilities can be considered a “drift velocity”, either toward or away from the threshold. We will show that *this ‘drift velocity’ is responsible for quantum-like effects.*

Reasonably good agreement between this model with drift and interval histograms as well as the joint density of successive interspike intervals of spiking data has been observed [3].

3 Nelson's Method Applied to Single Neurons

Our aim is to demonstrate that a part of neuronal noise can exhibit coherence, stability and structure associated with quantum effects, a surprising example of ‘order from disorder’ [20]. The inherent stochastic nature of neural processes described by the ‘random walk with drift’ models that we discussed in the previous section, is equivalent to a Schrödinger-like dynamics of neurons. Such a perspective offers a new framework for understanding brain function in terms of quantum-like processes.

Following Nelson's approach and its further developments [17, 18, 19, 21, 22, 23, 24, 25], we start with some stochastic differential equations from classical physics and then transform them to a well known equation in quantum mechanics. The mathematical details are given in Appendix I.

Let the random variable $Q(t)$ representing the membrane potential of a neuron move back and forth along a straight line, executing Brownian motion without friction. Then $Q(t)$ follows the stochastic differential equation (SDE)

$$dQ(t) = b_f(Q(t), t)dt + \sigma dW_f(t). \quad (1)$$

Here, b_f represents the forward drift velocity (caused by EPSPs), σ is the square root of the diffusion coefficient, and $dW_f(t)$ is a forward Wiener process. The diffusion coefficient, σ^2 , describes the magnitude of the random fluctuation shown in Figure 2 over time, and is determined by the physiological characteristics of the membrane. Since these processes are conservative, backward processes (caused by IPSP) also exist. The SDE for such backward processes is

$$dQ(t) = b_b(Q(t), t)dt + \sigma dW_b(t) \quad (2)$$

where $dW_b(t)$ is the backward Wiener process. One can then define a *current velocity* as the sum of the forward and backward drift velocities,

$$v(q, t) = \frac{1}{2}(b_f(q, t) + b_b(q, t)) = \frac{1}{m} \frac{\partial}{\partial q} S(q, t) \quad (3)$$

where $S(q, t)$ is a scalar function which can be identified with the action, and an *osmotic velocity* as the difference of these two velocities,

$$u(q, t) = \frac{1}{2}(b_f(q, t) - b_b(q, t)) = \frac{\sigma}{2} \frac{\partial}{\partial q} \ln[\rho(q, t)]. \quad (4)$$

where $\rho(q, t)$ is the probability distribution of the random variable $Q(t)$. The right hand sides of these equations for v and u follow from the Fokker-Planck equations (see Appendix I).

Given these two velocities, one can introduce the Lagrangian field

$$\mathcal{L} = \frac{1}{2}m(v^2 - u^2)(q, t) - V(q) \quad (5)$$

where $V(q)$ is the electrostatic potential. The action $S(q, t)$ can be constructed from it. It can be shown using a stochastic variational principle [22] that such a system can be

mathematically mapped to the Schrödinger equation

$$i\hat{\hbar}\frac{\partial}{\partial t}\psi(q, t) = \left(-\frac{\hat{\hbar}^2}{2m}\partial_q^2 + V(q) \right) \psi(q, t) \quad (6)$$

where $\psi = \sqrt{\rho}\exp(iS/\hat{\hbar})$ and $\hat{\hbar} = m\sigma$. If $\hat{\hbar}$ is postulated to be a universal constant for neuronal membranes, one has a fundamental equation for a single neuron that is exactly the Schrödinger equation in quantum mechanics with $\hat{\hbar}$ replacing the reduced Planck constant \hbar and m standing for a neuron's self inductance. Then it follows from $\sigma = \hat{\hbar}/m$ that the diffusion tends to zero and the system tends to a classical one for large m , exactly as in standard quantum mechanics.

The wave function ψ describes the Markov process completely:

$$\rho = |\psi|^2, \quad (7)$$

$$u = \sigma\partial_q\Re\ln\psi, \quad (8)$$

$$v = \sigma\partial_q\Im\ln\psi. \quad (9)$$

This is the ‘Nelson map’. It maps the probability distribution function and the current and osmotic velocities in single neurons to a wave function. In other words, it associates a diffusion process in a neuron to every solution of the Schrödinger-like equation (6).

To sum up, we have seen that classical stochastic differential equations can be mapped to a fundamental equation similar to the Schrödinger equation in quantum mechanics. The mathematical steps are sketched in Appendix I. This relies upon a reformulation of a random walk in terms of a ‘quantum drift’ and a reinterpretation of the reduced Planck constant \hbar as the product of the coefficient σ associated with neuronal diffusion times the neuron self-inductance m . We will see below that under certain conditions, $\hat{\hbar}$ could be determined using real brain data.

This suggests that equations of neuronal dynamics like the Hodgkin-Huxley (H-H) [26, 27] and FitzHugh-Nagumo (FN) equations [28, 29] are connected to a Schrödinger-like equation, to which we turn below.

Part II

Random Walks and the Schrödinger Equation for Many Variables

4 Introduction

To link the Schrödinger with the FitzHugh-Nagumo (FN) equations, we need to consider two as opposed to one random variable, which is the number of random variables appearing in the FN equations. Above, we considered the basic stochastic mechanical theory of a neuron using a single random variable q . However, as shown by Gerstein and Mandelbrot

[3], that is not realistic enough to deal with the known geometric complexity of synapses and of the somatodendritic membrane. Hence, the electrical state of these structures cannot be adequately described by a single state variable. Several variables are needed, and one must allow the state to vary over the surface of the neuron. The random walk will then take place in the same number of dimensions as the number of random variables. We will see examples below.

There exist models of spiking neurons that use several random (also known as state) variables, like the 4D Hodgkin-Huxley (H-H) model [26] and its 2D variant, the FitzHugh-Nagumo model (FN) [28, 29]. Here, we will establish a link between a two-state generalization of the theory presented above and the FN model. To do that we first need to extend our formalism to N state variables, which is quite straightforward. Having done that, we will demonstrate that the 2-state wave function corresponding to the Hamiltonian of the FN model gives quantum corrections to the spiking rates calculated from the FN model. The wave function can then be mapped to the underlying 2D stochastic processes (ρ, u, v) using the Nelson map (eqns (7), (8), (9)) generalized to 2 state variables.

5 Generalization to N state Variables

Just as one can write a N -particle Schrödinger wave function $\psi(x_1, x_2, \dots, x_N, t)$, one can also write a N -state neuron wave function $\psi(q, t)$, $q = \{q_1, q_2, \dots, q_N\}$ where q is a set of generalized coordinates representing the state variables:

$$i\hat{\hbar}\frac{\partial}{\partial t}\psi(q, t) = \left(-\frac{\hat{\hbar}^2}{2m}\partial_q^2 + V(q)\right)\psi(q, t) \quad (10)$$

The Einstein summation convention has been used in writing this equation, i.e. $\partial_q^2\psi(q, t) = \sum_{i=q_1}^{i=q_N}\partial_{q_i}^2\psi(q_1, q_2, \dots, q_N)$. For notational simplicity, we have used the same m for all the state variables q_i .

By putting $\psi = \exp(R + iS/\hat{\hbar}) = \sqrt{\rho}\exp(iS/\hat{\hbar})$ into eqn (10) and separating the real and imaginary parts, the following coupled differential equations for the functions R and S can be derived [22, 30]:

$$\frac{\partial S}{\partial t} + \frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + V(q) + V_Q = 0, \quad V_Q = -\frac{\hat{\hbar}^2}{2m}\left[\left(\frac{\partial R}{\partial q}\right)^2 + \frac{\partial^2 R}{\partial q^2}\right], \quad (11)$$

$$\frac{\partial R}{\partial t} + \frac{1}{2m}\left(R\frac{\partial^2 S}{\partial q^2} + 2\frac{\partial R}{\partial q}\frac{\partial S}{\partial q}\right) = 0. \quad (12)$$

The first equation is the Hamilton-Jacobi-Bellman equation for N variables. It is the Hamilton-Jacobi equation for N variables with an additional stochastic term V_Q which takes the form

$$V_Q = -\frac{\hat{\hbar}^2}{4m}\left[\frac{\partial_q^2 \rho}{\rho} - \frac{(\partial_q \rho)^2}{2\rho^2}\right] \quad (13)$$

in terms of $\rho = e^{2R}$. The second equation can also be written in terms of ρ as

$$\frac{\partial \rho}{\partial t} + \partial_q \left[\rho \frac{\partial_q S}{m}\right] = 0 \quad (14)$$

which is the continuity equation for N variables. These two coupled partial differential equations determine the underlying stochastic processes with N variables.

Let $L(q, \dot{q})$ be the Lagrangian of a deterministic classical system with N variables from which one can construct the action

$$S(q) = \int L(q, \dot{q}) dt \quad (15)$$

and the Hamiltonian

$$H(S) = \sum_{i=1}^{i=N} \frac{1}{2m_i} \frac{\partial S}{\partial q_i} \frac{\partial S}{\partial q_i} + V(q) := \frac{1}{2m} \left(\frac{\partial S}{\partial q} \right)^2 + V_Q \quad (16)$$

in terms of the action. The classical Hamilton-Jacobi equation is of the form

$$\frac{\partial S}{\partial t} + H(S) = 0. \quad (17)$$

On the other hand, if one uses the form $\psi(q) = \sqrt{\rho(q)} \exp(iS(q)/\hbar)$ in the Schrödinger equation (10) for $\psi(q)$ and separates the real and imaginary parts, one gets, as we have just seen, the Hamilton-Jacobi-Bellman equation

$$\frac{\partial S}{\partial t} + H(S) + V_Q = 0 \quad (18)$$

with V_Q given by eqn (13), i.e. the classical and deterministic Hamilton-Jacobi equation with an additional stochastic term V_Q .

Now consider the momentum $p = mv$, v being the current velocity $\frac{1}{m} \partial_q S$. Then

$$\frac{dp}{dt} = \frac{d}{dt} \left(\frac{\partial S}{\partial q} \right) = \left(\frac{\partial^2 S}{\partial q^2} \right) \dot{q} + \frac{\partial}{\partial q} \left(\frac{\partial S}{\partial t} \right) \quad (19)$$

$$= \frac{\partial}{\partial q} \left[\frac{1}{2m} \left(\frac{\partial S}{\partial q} \right)^2 + \frac{\partial S}{\partial t} \right] \quad (20)$$

$$= -\partial_q [V + V_Q] \quad (21)$$

on using eqns (16, 18). Thus, V_Q gives the quantum corrections to the classical or deterministic equation of motion. This shows that the quantum corrections vanish when V_Q is independent of q . That is the case if ρ is independent of q . Now, notice that the osmotic velocity $u(q, t)$ (eqn (4)) vanishes if ρ is independent of q . Hence, the system behaves classically when the osmotic velocity vanishes, i.e. diffusion via Brownian motion vanishes. This is why the osmotic velocity is also called the ‘quantum drift’.

Deterministic neuron models such as the H-H, FN or the ‘integrate-and-fire’ model typically generate a regular sequence of spikes. Spike trains of typical neurons *in vivo*, however, show a much more irregular behaviour, indicating noise [31]. The stochastic mechanics of single neurons developed in this paper predicts the existence of such noise, at least in part, as intrinsic quantum fluctuations. Below, we will study neuronal noise using a key model of brain dynamics known as the FitzHuh-Nagumo model.

6 The FitzHugh-Nagumo Model

The FitzHugh-Nagumo model consists of two coupled differential equations including the membrane potential and a recovery variable [28, 29]. The model has been widely used to study neuronal dynamics. Its applicability however, extends into other domains like cell division and physiology as well as population dynamics. We here use it to consider neuronal dynamics. The first equation describes the rapid changes in a neuron's membrane potential after a spike is generated, while the second equation describes the dynamics of the recovery variable and the dynamics of slower processes that restore the neuron to its resting state after excitation, typically modeling ion channel dynamics that inhibit or modulate the firing. The FitzHugh-Nagumo equations are as follows:

$$\frac{dv}{dt} = v - \frac{v^3}{3} - w + I, \quad (22)$$

$$\frac{dw}{dt} = \epsilon(v + a - bw), \quad (23)$$

where v is the membrane potential, w is the recovery variable, I is an external stimulus current that leads to excitation, ϵ , a , and b are parameters that control the model's dynamics. The nonlinear term in the first equation above describes rapid activation and deactivation. The parameter ϵ controls the time-scale of recovery (usually much slower than the membrane potential dynamics), while the parameters a and b determine the model's behaviour, adjusting the sensitivity and response of the recovery variable to changes in v . They determine threshold activation, spike duration, and recovery.

By adding some stochastic noise terms, the FN equations describe variability in stimulus current, the opening and closing of sodium channels or slow dynamics associated with potassium channels. Thus we obtain the equations

$$\frac{dv}{dt} = v - \frac{v^3}{3} - w + I + \xi_v(t), \quad (24)$$

$$\frac{dw}{dt} = \epsilon(v + a - bw) + \xi_w(t) \quad (25)$$

where $\xi_v(t)$ and $\xi_w(t)$ are Gaussian white noise processes [8].

The FN equations are a simplification of the Hodgkin-Huxley equations [26] where the channel gating variables have been combined into one recovery variable, w , that represents the cumulative effects of ion channel inactivation and potassium increase that repolarizes the membrane.

The above equations can be written as a second order equation

$$\ddot{v} = \left(1 - b\epsilon - \frac{2v^2}{3}\right) \dot{v} + v\epsilon \left(b - 1 - b\frac{v^2}{3}\right) - a\epsilon + b\epsilon + \dot{I}. \quad (26)$$

Consider the Lagrangian

$$L = \frac{1}{2}\dot{v}^2 + \frac{\epsilon}{2}(b-1)v^2 - \frac{\epsilon b}{12}v^4 - \frac{\epsilon b}{2}w^2 \quad (27)$$

where v and w are generalized coordinates. We also define the Rayleigh dissipation function [32]

$$R = \left(\frac{1}{2} - \frac{b\varepsilon}{2} - \frac{2v^2}{6} \right) \dot{v}^2 + \frac{1}{2} \dot{w}^2 \quad (28)$$

and the generalized force

$$Q = \left(-a\varepsilon + b\varepsilon + \dot{I} \right). \quad (29)$$

Then the Euler-Lagrange equations with Q on the right hand side

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{v}} \right) - \frac{\partial L}{\partial v} - \frac{\partial R}{\partial \dot{v}} = Q, \quad (30)$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{w}} \right) - \frac{\partial L}{\partial w} - \frac{\partial R}{\partial \dot{w}} = \varepsilon (v + a) \quad (31)$$

yield the FN equations.

Notice that the Lagrangian L (eqn 27) is a function of \dot{v} but not of \dot{w} which occurs only in the Rayleigh dissipation function R . Hence, using the Legendre transformation

$$H = \frac{\partial L}{\partial \dot{v}} \dot{v} + \frac{\partial L}{\partial \dot{w}} \dot{w} - L := p\dot{v} - L, \quad (32)$$

we get the Hamiltonian

$$H = \frac{1}{2} p^2 - \frac{\varepsilon}{2} (b-1) v^2 + \frac{\varepsilon b}{12} v^4 + \frac{\varepsilon b}{2} w^2 := \frac{1}{2} p^2 + V(v, w; a, \varepsilon, b). \quad (33)$$

Then, in a representation in which v is an independent variable and $|\psi(v, t)|^2 dv$ is the probability that the system has coordinate between v and $v + dv$, the time evolution of the state function is governed by the equation of motion

$$i\hat{\hbar}\partial_t|\psi\rangle = \hat{H}|\psi\rangle, \quad \hat{H} = \frac{1}{2}\hat{p}^2 + V(v, w; a, \varepsilon, b) \quad (34)$$

where $\hat{p} = -i\hat{\hbar}\partial_v$. Time evolution of the state function $\psi(v, t) = \langle v|\psi\rangle$ where $|\psi(v, t)|^2 dv$ is probability that the system has coordinates between v and $v + dv$ is governed by the Schrödinger equation

$$i\hat{\hbar}\frac{\partial}{\partial t}\psi(v, t) = \left(-\frac{\hat{\hbar}^2}{2}\partial_v^2 + V(v, w; a, \varepsilon, b) \right) \psi(v, t). \quad (35)$$

To sum up, the stochastic version of the FN equations can be rewritten as a Schrödinger equation. This suggests that predictions about variability in brain dynamics in the form of stochastic noise offered by these equations may also include noise due to quantum effects such as shot noise [33] and thermal fluctuations in ion channels [34].

7 Quantum Effects and the Planck constant \hat{h}

It is not clear at which spatial scale quantum effects might be apparent. Given the connection between classical equations of neuronal activity like the FN equations and the Schrödinger equation considered above, to find experimental evidence of quantum effects in brain data one could turn to subthreshold neural oscillations [35]. Besides the FN equations, there might be other equations that are linked to the Schrödinger equation in a similar fashion to the FN equations. These are equations that follow from the solution of the Euler-Lagrange equations (30) and (31) after defining an appropriate Lagrangian like the function given by equation (27). An important example of such equations that underlie sensory perception [35] are subthreshold oscillations [36].

These can be modelled as LC circuits [37, 38]. The total energy of such a circuit is

$$U = \frac{1}{2}(CV^2 + LI^2) = \frac{1}{2} \left(\frac{C^2V^2}{C} + \frac{L^2I^2}{L} \right) \quad (36)$$

where C is the capacitance, L is the inductance, V is the voltage and I the current. This can be written in the form

$$\begin{aligned} H &= \frac{1}{2} \left(\frac{L^2I^2}{L} + \frac{C^2V^2}{C} \right) \\ &:= \frac{p^2}{2m} + \frac{\omega^2 Lq^2}{2} = \frac{p^2}{2m} + \frac{m\omega^2 q^2}{2} \end{aligned} \quad (37)$$

with $U = H$ (the Hamiltonian), $L = m$ (the mass), $q = CV$ (the coordinate), $I = C dV/dt = dq/dt$ (the velocity), $p = LI$ (the momentum) and $\omega = 1/\sqrt{LC}$ (the angular frequency). The identical formal structure of a mechanical and LC harmonic oscillator prompts us to postulate that the quantum mechanical description of the LC oscillator is in the form of a state vector $|\psi\rangle$. Then, in a representation in which q is an independent variable and $|\psi(q, t)|^2 dq$ is the probability that the oscillator has coordinate between q and $q + dq$, the time evolution of the state function is governed by the equation of motion

$$i\hat{h}\partial_t|\psi\rangle = \hat{H}|\psi\rangle, \quad \hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{q}^2. \quad (38)$$

This corresponds exactly to the Schrödinger equation (6) with $\psi(q, t) = \langle q|\psi\rangle$, $\hat{p} = -i\hat{h}\partial_q$, $V = m\omega^2 q^2/2$. We therefore conclude that the Schrödinger equation (6) with $V = m\omega^2 q^2/2$ describes the subthreshold oscillations of neurons.

Since we are interested in membrane oscillations, we need time-dependent solutions of (6) with $V = m\omega^2 q^2/2$. Such states are called ‘coherent states’ $|\alpha\rangle$ of the harmonic oscillator. They were first introduced by Schrödinger in 1926, and their theory was developed further by Glauber [39] and Sudarshan [40] in connection with laser optics. Their importance lies in the fact that they are ‘minimum uncertainty’ states and closely approximate classical oscillators. One defines the ‘displacement operator’ $D(\alpha)$ as $D(\alpha) = e^{\alpha a^\dagger - \alpha^* a}$ where $\alpha = |\alpha|e^{i\phi}$ is a complex number. A coherent state $|\alpha\rangle$ is then defined as a displaced vacuum state $|0\rangle$,

$$|\alpha\rangle = D(\alpha)|0\rangle, \quad (39)$$

$$a|0\rangle = 0. \quad (40)$$

Using $\alpha(t) = \alpha e^{-i\omega t} = |\alpha|e^{-i(\omega t - \phi)}$, one can verify (see the Appendix II for details) that the expectation value of x in such a state is

$$\langle q \rangle = \sqrt{2}q_0|\alpha| \cos(\omega t - \phi) \quad (41)$$

where $q_0 = \sqrt{\frac{\hat{h}}{m\omega}}$ is the width of the harmonic oscillator ground state which is a gaussian distribution. The uncertainties in position and momentum in such a state satisfy the relation $(\sigma_q)_0^2(\sigma_p)_0^2 = \hat{h}^2/4$, showing they are minimum uncertainty states, i.e. states with minimum quantum uncertainty and hence closest to classical states.

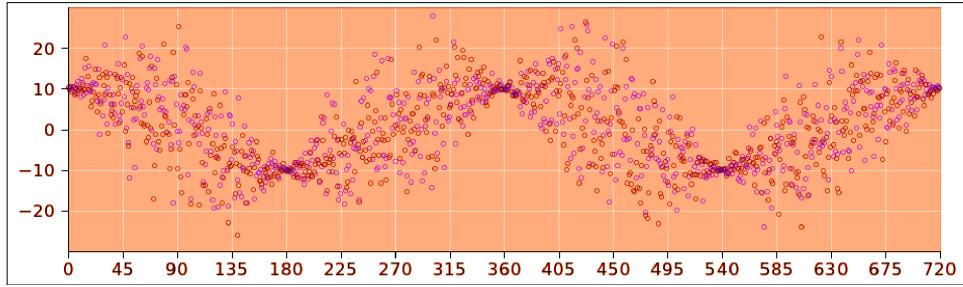


Figure 4: Coherent state representation of the harmonic oscillator: the expectation value $\langle q \rangle$ as a function of time in the ground state of the harmonic oscillator, showing minimum uncertainty scatter points around a classical cosine wave.

Hence, the unequivocal prediction is that subthreshold neural oscillations should be sinusoidal with a scatter of $\hat{h}^2/4$ around the classical values (Fig. 4). A measurement of the scatter will therefore determine the value of \hat{h} . Future work should therefore investigate quantum-like fluctuations in subthreshold neural oscillations using, for example, high-resolution electrophysiological recordings, such as patch-clamp techniques [41, 42], capable of detecting minute fluctuations in membrane potentials.

Another possibility is to search for the discrete stationary energy levels $E_n = (n + \frac{1}{2})\hat{h}\omega$ of neurons implied by the harmonic oscillator model (38) (see Appendix II). These are solutions of the time-independent Schrödinger equation and should be observable during the quiescent or refractory periods of the neurons. The Helmholtz free energy F and the average energy $\langle E \rangle$ of a quantum harmonic oscillator are given by (see Appendix II)

$$F = \frac{\hat{h}\omega}{2} + \frac{1}{\beta} \ln \left(1 - e^{-\beta\hat{h}\omega} \right), \quad (42)$$

$$\langle E \rangle = \frac{\hat{h}\omega}{2} + \frac{\hat{h}\omega e^{-\hat{h}\omega}}{1 - e^{-\beta\hat{h}\omega}} \quad (43)$$

where $\beta = 1/kT$, T being the absolute temperature. These expressions show that both F and $\langle E \rangle$ tend to the zero-point energy as the temperature T tends to zero. These results are experimentally verifiable in principle and can be used to determine the value of \hat{h} .

Interestingly, ‘entangled states’ also exist in stochastic mechanics, and stochastic mechanics and quantum mechanics agree in predicting all observed correlations at different times. The reader is referred to the papers by Faris [23] and Petroni and Morato [24] for details. Looking for entanglement in neural systems is therefore another important area for further research, not only for its intrinsic value but also because entanglement is a key resource in quantum information processing, and it is important to find out whether the brain makes use of it, as conjectured [43]. Hameroff and Penrose have suggested that entanglement might involve quantum states in microtubules, dendrites and other parts of a neuron [44]. These would occur at a smaller scale than the one we consider here, the scale of subthreshold oscillations. We predict that the presence of non-classical correlations in neuronal fluctuations, akin to quantum entanglement, could be detected through cross-correlation analyses of simultaneous recordings from neighbouring neurons.

8 Discussions

We have considered both a 1D random walk model with drift (Section 2) and the FN model of spiking neurons (Section 6), and, using Nelson’s method of stochastic mechanics (Section 3), shown that the FN equations imply a Schrödinger-like equation in the space of neuron state variables q , predicting corrections due to quantum fluctuation to the classical results. Since the noise processes are often modelled as white noise, the remaining noise, if any, should be small enough not to cause decoherence.

Given that neural plasticity is often linked to the probabilistic nature of synaptic changes, our results suggest that quantum-like effects might influence how plasticity occurs in specific neural circuits.

An alternative to the above hypothesis is that quantum entanglement happens within microtubules and is central to consciousness, allowing for non-local connections and unique information processing capabilities in the brain, as suggested by Hameroff and Penrose [46, 47]. They are shielded by Debye layers and actin gel among others [48]. This could protect microtubule quantum coherence even at warm temperatures. While experimental evidence is limited, these ideas have sparked interesting discussions at the intersection of quantum mechanics, consciousness studies, and theoretical neuroscience.

Spatial patterns of the sort predicted by the FN equations have been observed during actin polymerization and depolymerization in neuronal axons and dendrites. Thus it might be that quantum noise at the neuronal level is due to protein function.

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10 Appendix I

The solutions $Q(t)$ of the stochastic differential equations (1), (2) are known to be continuous at all state points but nowhere differentiable. Hence, Nelson suggested the following average forward and backward differentials which we adopt:

$$D_f Q(t) = \lim_{\Delta t \rightarrow 0} E_t \left[\frac{Q(t + \Delta t) - Q(t)}{\Delta t} \right], \quad (44)$$

$$D_b Q(t) = \lim_{\Delta t \rightarrow 0} E_t \left[\frac{Q(t) - Q(t - \Delta t)}{\Delta t} \right] \quad (45)$$

where E_t denotes the expectation conditional on $Q(t) = q$. For differentiable curves $D_Q X(t) = D_b Q(t) = \dot{q} = v(t)$, the ‘velocity’ of the random variable. It follows that the drift coefficients in the forward and backward equations are given by

$$D_f(Q(t)) = b_f(Q(t), t), \quad D_b(Q(t)) = b_b(Q(t), t) \quad (46)$$

This amounts to a complete description of the motion, as in classical mechanics.

As is well known, the forward and backward SDEs lead to two Fokker-Planck equations for the probability density $\rho(x, t)$ of the random variable executing Brownian motion:

$$\frac{\partial}{\partial t} \rho(q, t) = -\frac{\partial}{\partial q} [b_f(q, t) \rho(q, t)] + \frac{\sigma^2}{2} \frac{\partial^2}{\partial q^2} \rho(q, t), \quad (47)$$

$$\frac{\partial}{\partial t} \rho(q, t) = -\frac{\partial}{\partial q} [b_b(q, t) \rho(q, t)] - \frac{\sigma^2}{2} \frac{\partial^2}{\partial q^2} \rho(q, t) \quad (48)$$

Adding these equations results in the continuity equation

$$\frac{\partial}{\partial t} \rho(q, t) + \frac{\partial}{\partial q} [v(q, t) \rho(q, t)] = 0 \quad (49)$$

with the *current velocity* defined by $v(q, t) = (b_f(q, t) + b_b(q, t))/2$. This displays the role of the current velocity in maintaining the probability distribution:

The difference of the forward and backward drifts $u(q, t) = (b_f(q, t) - b_b(q, t))/2$ is defined as the *osmotic velocity*. Subtracting the two Fokker-Planck equations results in

$$u(q, t) = \frac{\sigma}{2} \frac{\partial}{\partial q} \ln[\rho(q, t)] = \frac{\sigma}{2} \frac{\partial_q \rho}{\rho} = \sigma \frac{\partial R}{\partial q} \quad (50)$$

where $\ln \rho(q, t) = 2R(q, t)$. The coupled forward-backward stochastic differential equations for the position process can thus be written as

$$dQ(t) = (v(Q(t), t) + u(Q(t), t)) + \sigma dW_f(t), \quad (51)$$

$$dQ(t) = (v(Q(t), t) - u(Q(t), t)) + \sigma dW_b(t). \quad (52)$$

It follows from this that the current velocity is curl-free and can be written as

$$v(q, t) = \frac{1}{m} \frac{\partial}{\partial q} S(q, t) \quad (53)$$

where $S(q, t)$ is a scalar function which can be identified with the action.

Now, following Guerra and Morato [22], let us introduce the Lagrangian field

$$\mathcal{L} = \frac{1}{2}m(v^2 - u^2)(q, t) - V(q) \quad (54)$$

where $V(q)$ is the electrostatic potential, from which the action $S(q, t)$ can be constructed. It can then be shown, using the variational principle, that the main features of Nelson's stochastic mechanics including eqns (4) and (53) can be derived from such an action.

Using stochastic control theory and the current velocity as the control, Guerra and Morato showed that the following differential equations for the functions R and S extremize the action:

$$\frac{\partial S}{\partial t} + \frac{1}{2m} \left(\frac{\partial S}{\partial q} \right)^2 + V + V_Q = 0, \quad V_Q = -\frac{m\sigma^2}{2} \left[\left(\frac{\partial R}{\partial q} \right)^2 + \frac{\partial^2 R}{\partial q^2} \right], \quad (55)$$

$$\frac{\partial R}{\partial t} + \frac{1}{2m} \left(R \frac{\partial^2 S}{\partial q^2} + 2 \frac{\partial R}{\partial q} \frac{\partial S}{\partial q} \right) = 0. \quad (56)$$

The first equation is the Hamilton-Jacobi-Bellman equation, i.e., the Hamilton-Jacobi equation with an additional stochastic term V_Q which takes the form

$$V_Q = -\frac{m\sigma^2}{4} \left[\frac{\partial_q^2 \rho}{\rho} - \frac{(\partial_q \rho)^2}{2\rho^2} \right] \quad (57)$$

in terms of $\rho = e^{2R}$. It is the analog of the Bohm quantum potential [30]. The second equation can also be written in terms of ρ as

$$\frac{\partial \rho}{\partial t} + \partial_q \left[\rho \frac{\partial_q S}{m} \right] = 0 \quad (58)$$

which, using eqn (53) for the current velocity, is a continuity equation. These two coupled partial differential equations determine the stochastic process. These equations can be derived from the Schrödinger-like equation

$$im\sigma \frac{\partial}{\partial t} \psi(q, t) = \left(-\frac{m\sigma^2}{2} \partial_q^2 + V(q) \right) \psi(q, t) \quad (59)$$

by putting $\psi = \exp(R + iS/m\sigma) = \sqrt{\rho} \exp(iS/m\sigma)$ and separating the real and imaginary parts [22, 30]. The argument q in the wave function $\psi(q, t)$ representing the value of the random variable $Q(t)$ of the neuron membrane plays the role of the spatial coordinate x and the coefficient σ plays the role of the factor \hbar/m in standard quantum mechanics.

Bearing in mind that the stochastic processes which occur in neural membranes are entirely different from those in configuration space that give rise to standard quantum mechanics, it would be useful to introduce a new universal constant $\hat{\hbar} = m\sigma$ for neuronal media and rewrite the above equation in the form

$$i\hat{\hbar} \frac{\partial}{\partial t} \psi(q, t) = \left(-\frac{\hat{\hbar}^2}{2m} \partial_q^2 + V(q) \right) \psi(q, t) \quad (60)$$

and treat it as the Schrödinger equation for single neurons. Like the Schrödinger equation in standard quantum mechanics, this equation will also ensure a novel level of stability and structure in the stochastic world of neurons.

Fig. 5 is a flow chart that summarizes the various mathematical steps used in deriving the Schrödinger equation in Nelson's stochastic mechanics.

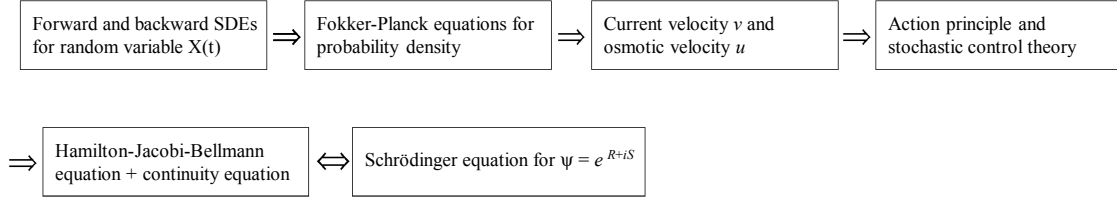


Figure 5: Flow chart depicting the main mathematical steps used in deriving the Schrödinger equation from stochastic mechanics.

11 Appendix II

When considering harmonic oscillators, it is convenient to introduce the ladder operators

$$a = \frac{1}{\sqrt{2m\omega\hat{\hbar}}}(m\omega q + i\hat{p}), \quad (61)$$

$$a^\dagger = \frac{1}{\sqrt{2m\omega\hat{\hbar}}}(m\omega q - i\hat{p}). \quad (62)$$

Using the commutation rule $[\hat{p}, q] = -i\hat{\hbar}$, one gets the commutation relation $[a, a^\dagger] = 1$, and

$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle, \quad (63)$$

$$a|n\rangle = \sqrt{n}|n-1\rangle, \quad (64)$$

$$a^\dagger a|n\rangle = n|n\rangle \quad (65)$$

where $n = 0, 1, 2, \dots$ is an integer. The Hamiltonian operator can be written in the form

$$\hat{H} = \hat{\hbar}\omega \left(a^\dagger a + \frac{1}{2} \right) = \hat{\hbar}\omega \left(N + \frac{1}{2} \right) \quad (66)$$

where $N = a^\dagger a$ is the number operator. It then follows from the time-independent Schrödinger equation $\hat{H}\psi_n = E_n\psi_n$ that the energy eigenvalues are

$$E_n = \hat{\hbar}\omega \left(n + \frac{1}{2} \right). \quad (67)$$

The factor $\hat{\hbar}\omega/2$ is the ‘zero-point energy’ of neurons. The corresponding energy eigenfunctions are given by

$$\psi_n(q) = \frac{1}{\sqrt{n!}}(a^\dagger)^n\psi_0(q) \quad (68)$$

with the ground state

$$\psi_0(q) = \left(\frac{m\omega}{\pi\hat{h}} \right) e^{-\frac{m\omega q^2}{2\hat{h}}} \quad (69)$$

which is a Gaussian distribution with width $q_0 = \sqrt{\frac{\hat{h}}{m\omega}}$.

Now note that the coherent state can be written as

$$a|\alpha\rangle = \alpha|\alpha\rangle, \quad (70)$$

$$\langle\alpha|a^\dagger = \langle\alpha|\alpha^*, \quad (71)$$

$$\langle\alpha|a^\dagger a|\alpha\rangle = |\alpha|^2. \quad (72)$$

Using these results, one can compute the uncertainties in q and p :

$$(\sigma_q)^2 = \langle q^2 \rangle - \langle q \rangle^2 = q_0^2 \left(n + \frac{1}{2} \right), \quad (73)$$

$$(\sigma_p)^2 = \langle p^2 \rangle - \langle p \rangle^2 = \frac{\hat{h}^2}{q_0^2} \left(n + \frac{1}{2} \right), \quad (74)$$

and hence for the ground state ($n = 0$) $(\sigma_q)_0^2(\sigma_p)_0^2 = \hat{h}^2/4$. For further details of coherent states the reader is referred to Ref [49].

The thermodynamic properties of the quantum harmonic oscillator can be calculated using the standard techniques of statistical mechanics. The partition function

$$Z = \text{Tr} e^{-\beta\hat{H}} = \sum_{n=0}^{\infty} e^{-\beta E_n} = \sum_{n=0}^{\infty} e^{-\beta(n+\frac{1}{2})\hat{h}\omega} = \frac{1}{2} \text{csch} \left(\frac{\beta\hat{h}\omega}{2} \right) \quad (75)$$

where \hat{H} is the Hamiltonian operator. Hence

$$F = -\frac{1}{\beta} \ln Z = \frac{\hat{h}\omega}{2} + \frac{1}{\beta} \ln \left(1 - e^{-\beta\hat{h}\omega} \right), \quad (76)$$

$$\langle E \rangle = -\frac{\partial \ln Z}{\partial \beta} = \frac{\hat{h}\omega}{2} + \frac{\hat{h}\omega e^{-\hat{h}\omega}}{1 - e^{-\beta\hat{h}\omega}}. \quad (77)$$

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