The Use of Asymptotic Expansions and Other Approximation Methods in Reliability

by

Abdel-Raouf Abdel-Rahman Abdel-Wahid

A Thesis submitted for the Degree of Doctor of Philosophy in Statistics

Department of Mathematics
The City University
London

November 1984

<u>CONTENTS</u>

Page

ACKNOWLEDGEMENTS

I would like to express my thanks and sincere gratitude to Dr Alan Winterbottom under whose supervision this work has been carried out, for offering valuable suggestions, helpful advice and constructive assistance throughout the study.

I record my thanks to the Egyptian Government for the financial support I received.

I am also indebted to my family for their support and encouragement which has helped me greatly.

I record my thanks to [redacted] the Departmental Secretary of the Mathematics Department for her skill in typing the manuscript.

ABSTRACT

In this work some asymptotic and other approximate methods for obtaining interval estimates for system reliability from binomial and/or exponential subsystem test data have been developed. The accuracy of these approximations has been assessed by comparison with exact values (sometimes obtained by simulation). Also, other applications of asymptotic techniques are made to the Weibull distribution and to tolerance limits.

In Chapter 2 and Chapter 5, we develop Edgeworth expansions, Cornish and Fisher expansions for percentage points and Saddlepoint expansions for application to the system reliability problem and to tolerance limits.

In Chapter 3, two simple approximate methods have been derived. The first method forms the basis of the Maximus report (1980) and depends on reducing the component test data to equivalent system test data. The second method depends on equating the system posterior mean and variance with the mean and variance of a single beta distribution. The parameters of this beta distribution, so determined, are used in the construction of interval estimates for system reliability.

In addition, in Chapter 4, approximations for the reliability function and for the hazard rate, posterior to censored Weibull life data, are obtained using an asymptotic expansion due to Lindley.

CHAPTER 1

INTRODUCTION

The purpose of this work is to develop and investigate the accuracy of some asymptotic and other approximate methods for obtaining lower confidence limits for the reliability of systems from binomial, and/or exponential subsystem test data. Other applications of asymptotic techniques are made to the Weibull distribution and to tolerance limits. We first discuss the system reliability problem.

The determination of confidence limits on series, parallel, or complex system reliability from time-to-fail, or pass-fail subsystem/component data is one of considerable interest and much practical importance. The computation of confidence limits by exact methods is intractable for all, but very specialised models, and for these there are problems such as the use of large amounts of computer time and loss of precision.

Various authors in recent years have investigated approximate techniques for estimating lower (and upper) confidence limits for some systems.

1.1 System Reliability Interval Estimates: A Review

There are many approximate methods for obtaining interval estimates on system reliability using complete or censored test data, where the independent subsystems form a series, parallel, or complex system.

For the case in which only pass-fail data are collected for each subsystem, many methods involving large or small sample approximations, or Bayesian techniques, have been derived for obtaining confidence limits on the probability of a successful operation of an independent series or parallel system. For example, Buehler (1957) considers exact solutions to the problem of determining upper confidence limits for series and parallel systems as a product of two binomial parameters, using Poisson subsystem test data only. The method of Madansky (1965), which is based on the asymptotic chi-squared distribution of -2 log likelihood ratio, can be used to construct confidence limits for series or parallel system reliability. Subsequently, Myhre and Saunders (1968a) derived a generalisation of Madansky's method. Myhre and Saunders (1968b) also yielded a method of the asymptotic normality of the maximum likelihood estimates to find confidence limits for the reliability of general systems for the binomial models. They concluded that the likelihood ratio gave better approximations than maximum likelihood.

Easterling (1972) gave a method for obtaining system confidence limits from component test data. The technique consists of estimating the asymptotic variance of the maximum likelihood estimate of system reliability and equating this to the estimate of the variance of a binomial proportion, thus obtaining pseudo numbers of trials and successes. Substituting these into the incomplete beta function yields the desired confidence limits.

Mann (1974a) **adapted** an approach similar to that of Mann and Grubbs (1972) for exponential-subsystem failure data to approximate confidence limits on both series and parallel system reliability in the case of the binomial model. In (1974b) Mann derived a method for obtaining approximately optimum lower confidence limits on system reliability for any independent series or parallel system with binomially distributed pass-fail subsystem data. It is assumed that failure data have been collected from life tests of the original of the subsystems making up the system using censored test data. Also Mann and Grubbs (1974) obtained confidence limits for a parallel-system in the case of type II censored test data, in some special cases.

In the case of the Bayesian approach Zimmer, et al (1965) derive confidence limits which are exact in the Bayesian sense using a uniform prior for each subsystem on the binomial model. Springer and Thompson (1966) use a Mellin transform technique for obtaining, in closed form, the distribution of series system reliability given binomial subsystem data. Also Parker (1972) uses a Bayesian approach to calculate confidence limits by assigning a prior to a system rather than to each of its subsystems in the case of the series system on binomial model. Again, for the Bayesian approach, Mann et al (1974) show that - log reliability function for series systems is well approximated by a non central chi-squared distribution. The corresponding central chi-squared variate with non integer degrees of freedom is transformed to normality yielding the lower confidence limits.

On the other hand for the case of exponential test data, there are also several approximate methods for obtaining confidence limits on system reliability. For example the method by Kraemer (1963) obtains confidence limits for series system reliability with exponential time-to-fail subsystems, using type II censored test data. This method is based upon the smallest of the observed times to failure and gives inaccurate confidence limits when subsystem times to failure are disparate. Lieberman and Ross (1971) derived a method for determining an exact lower confidence limit for series system reliability with two independent exponential subsystems using complete or type II censored test data. Their method depends on a combination of failure times selected from the various subsystems, rather than on the separate sufficient statistics. A similar method has been developed independently by Sarkar (1971) and is exact for equal numbers of failures for all subsystems. The method of Lieberman and Ross has been extended to complex systems by Saunders (1972). El-Mawaziny and Buehler (1967) derive confidence limits for the exponential model depending on the asymptotic normality of system failure rate for more than two subsystems, which depends upon large-sample theory and the fact that a function of the estimator for subsystem mean-time-to-failure has an approximate chi-squared distribution. Grubbs (1971) suggests a procedure for approximating the fiducial probability bounds on the true system reliability, which can be used to approximate confidence limits for the reliability of a series system for which each component has an exponential time-to-fail distribution. He uses the first two moments of the fiducial distribution of the system failure rate to fit a weighted chi-squared distribution. The method of Mann and Grubbs (1972) is a chi-squared technique, involving an extension and combination of the approximate fiducial approach suggested by Grubbs (1971) and the contributions of El-Mawaziny and Buehler (1967). After computations of the first two moments of system failure rate based on certain conditional variates, one constructs approximate chi-squared variates and, by using the Wilson-Hilferty transformation of chi-squared to an approximate standard normal variate, lower confidence limits on series system reliability for the exponential model can be obtained using standard normal percentiles. Mann and Grubbs (1974) made use of Patnaik's chi-squared approximation to the non central chi-squared distribution, and the Wilson-Hilferty transformation of chi-squared to approximate normality for censored test data. They obtain confidence limits for series and parallel system from time-to-fail, pass-fail subsystems or mixed systems. Myhre and Saunders (1971) extended Madansky's method to calculate approximate confidence limits for the reliability function of any coherent system of the exponential distribution using the asymptotic distribution of the log likelihood ratio.

In the case of Bayesian technique, Fertig (1972) investigated different prior functions to find confidence limits on the reliability of a series system composed of exponential subsystems. He showed that it is not possible to find prior distributions on subsystem failure rates so that the resulting confidence limits for system reliability are the same as exact limits. Springer and Thompson (1967) obtain Bayesian confidence intervals for the reliability of series system when all of the failure probability density functions are exponential. In (1968) they derived confidence limits for system reliability under the exponential-failure-time models using Bayesian approach for a parallel system with a single failure for each subsystem. In (1971) Springer and Byers obtain confidence intervals for the reliability of a mixed series system whose failure probability distributions are exponential for some subsystems and binomial for the rest. Springer and Thompson use the Mellin transform, together with test data obtained from the subsystems, to derive the posterior density function and cumulative distribution function for the system reliability, and to calculate confidence limits in their methods.

1.2 An Outline Of Methods Used In This Thesis

With the increasing importance of statistical inference, interest in approximations has appeared because of the increasing number and complexity of theoretical distributions, particularly for small sample sizes.

In subsequent Chapters, we are going to develop Edgeworth expansions, Cornish and Fisher expansions for percentage points and Saddlepoint expansions, for application to the system reliability problem and to tolerance limits. In addition an asymptotic approximation, due to Lindley, is applied to the Weibull reliability function and hazard rate.

(a) Edgeworth Expansion

Problems with distribution functions such as bias and skewness means that the crude approximation by a normal distribution is unsatisfactory. Then the Edgeworth asymptotic expansions enable successive adjustments to be made to the fit of a normal distribution.

Let T be an estimator of parameter θ based on a sample of size n. Define

$$x = \sqrt{n}(T-\theta)/\sigma ,$$

where

$$\sigma^2 = \lim_{n\to\infty} \mathrm{var}(\sqrt{n}T) ,$$

and suppose that x is asymptotically standard normal ($n\to\infty$). Then the Edgeworth approximation for F(x), the distribution function of x, is

$$F(x) \cong \Phi(x) - \phi(x)\{(\kappa_{11}+\frac{1}{6}\kappa_{32}H_2)/\sqrt{n} + [(\kappa_{11}^2+\kappa_{22})H_1/2 + \kappa_{11}\kappa_{32}H_3/6 + \kappa_{43}H_3/24 + \kappa_{32}H_5/72]/n + \ldots\} \qquad (1.2.1)$$

Where $\phi(x)$ and $\Phi(x)$ are the standard normal probability density function and distribution function respectively, κ's are the cumulants,

and

$H_0 = 1$, $H_1 = x$, $H_2 = x^2 - 1$, ... , and

$H_r = xH_{r-1} - (r-1)H_{r-2}$ for $r \geq 2$, are Hermite polynomials.

The Edgeworth expansion is valid to r+1 terms if κ_{r+1} exists and the distribution has an absolutely continuous part. (1.2.1) has been written explicitly in terms of the cumulants because, for some systems, the asymptotic variable n may be the number of components. In other cases it has an interpretation as the smallest sample size over component tests. Our interest is for the case when n is a sample size.

In practice, the Edgeworth approximation is simple to use, but on the other hand the approximation is often unsatisfactory in the tails and it can take on negative values or values greater than unity. See, for example, Kendall and Stuart (1958).

(b) Cornish and Fisher Expansions for Percentage Points

Let Φ be the standard normal distribution function and ξ be such that $\Phi(\xi) = 1 - \alpha$.

Then the $100(1-\alpha)\%$ point of the distribution of T is given by

$$T_\xi = \theta + (\kappa_{21}/n)^{\frac{1}{2}}\{\xi + (\ell_{11}+\ell_{32}(\xi^2-1)/6)/\sqrt{n} + (\ell_{22}\xi/2 +$$

$$\ell_{43}(\xi^3-3\xi)/24 - \ell_{32}^2(2\xi^3-5\xi)/36)/n + \dots\} \tag{1.2.2}$$

Where $\ell_{rs} = \kappa_{rs}/\kappa_{21}^{r/2}$ are the standardised cumulants. See Fisher and Cornish (1960) and Kendall and Stuart (1958).

(c) Saddlepoint Approximations

Saddlepoint methods can give especially good approximations in the tails of distribution functions; see, for example, Daniels (1954), Barndorff-Neilson and Cox (1979) and Robinson (1982).

Let $F_n(x)$ be the distribution function to be approximated. Define the exponentially shifted distribution function $P_n(x)$ by

$$P_n(x) = \int_{-\infty}^{x} e^{uy} d\, F_n(y)/M_n(u) \quad , \tag{1.2.3}$$

where

$$M_n(u) = \int_{-\infty}^{\infty} e^{uy} d\, F_n(y) \quad .$$

From (1.2.3) we can show that

$$1 - F_n(x) = M_n(u) \int_{x}^{\infty} e^{-uy} dP_n(y) \ . \tag{1.2.4}$$

Approximating $P_n(x)$ by an Edgeworth series, and using (1.2.4), $1 - F_n(x)$ can then be approximated indirectly. For this reason saddlepoint approximations are sometimes called indirect Edgeworth approximations. A judicious choice for u enables good approximations to be obtained for $1 - F_n(x)$.

The moment generating function of P is

$$\begin{aligned} M_p(\theta) &= M_F^{-1}(u) \int_{\infty}^{\infty} e^{\theta y} d\{\int_{-\infty}^{y} e^{ux} dF(x)\} \\ &= M_F^{-1}(u) \int_{-\infty}^{\infty} e^{(\theta+u)y} dF(y) \\ &= M_F(\theta+u)/M_F(u). \end{aligned}$$

Then the cumulant generating function of P is

$$K_p(\theta) = K_F(\theta+u) - K_F(u) \ .$$

Let $m(u) = \partial K_p(\theta+u)/\partial\theta \Big|_{\theta=0}$ and

$\sigma^2(u) = \partial^2 K_p(\theta+u)/\partial\theta^2 \Big|_{\theta=0}$ be the mean and variance of the distribution function P_n .

Consider the Edgeworth approximation $Q_n(x)$ of $P_n(x)$ given by

$$P_n(x) \cong Q_n(x) = \Phi(z) - \phi(z)\{\kappa_1/2 + (z^2-1)\kappa_3/6 + \dots\} ,$$

where ϕ and Φ are the standard normal probability density and distribution functions respectively and $z = (x-m(u))/\sigma(u)$.

We may express (1.2.4) in the form

$$1 - F_n(x) = M_n(u)\int_x^\infty e^{-uy} dQ_n(y) + M_n(u)\int_x^\infty e^{-uy} d(P_n(y)-Q_n(y)). \quad (1.2.5)$$

If we choose $m(u) = x$, then the first integral yields

$$M_n(u)\exp(-um+u^2\sigma^2/2)[1-\Phi(u\sigma)][1-\kappa_1 W_1(u\sigma)/2 + \kappa_3 W_2(u\sigma)/6] ,$$

where $W_1(u\sigma) = \dfrac{\phi(u\sigma)}{1-\Phi(u\sigma)} - u\sigma$, $W_3(u\sigma) = \dfrac{(u^2\sigma^2-1)\phi(u\sigma)}{1-\Phi(u\sigma)} - u^3\sigma^3$,

κ_1 , κ_3 are standardised cumulants of P_n, and in many applications $\kappa_1 = 0$. The second integral of (1.2.5) is of order n^{-1} and becomes very small for large values of x.

It is possible to develop further terms in the saddlepoint approximation but the above gives more than adequate approximations for practical purposes. Also, the theory of saddlepoint method can be developed using complex variable theory, as was the approach of Daniels, but using real variable method gives a clearer insight of the attendant statistical theory. Comparing the saddlepoint approximation with the Edgeworth series, we find the saddlepoint method does not suffer from the drawbacks of the Edgeworth approximation in the tail area.

(d) Lindley's Method

This method depends on evaluation of the ratio of integrals of the form

$$\int w(\theta)e^{L(\theta)}d\theta \Big/ \int \nu(\theta)e^{L(\theta)}d\theta , \quad (1.2.6)$$

the posterior expectation of $w(\theta)/\nu(\theta)$, where

$\theta = (\theta_1, \theta_2, \dots, \theta_m)$ is the parameter, $L(\theta)$ is the logarithm of the likelihood function for n observations and $w(\theta)$ and $\nu(\theta)$ are arbitrary, see Lindley (1980). $w(\theta)$ and $\nu(\theta)$ are $O(1)$.

The Taylor series expansion for $L(\theta)$ about the maximum likelihood value $\hat{\theta}$ may be written

$$L(\theta) = L(\hat{\theta}) + \sum_i L_i(\hat{\theta})(\theta_i-\hat{\theta}_i) + \frac{1}{2!}\sum_i\sum_j L_{ij}(\hat{\theta})(\theta_i-\hat{\theta}_i)(\theta_j-\hat{\theta}_j)$$

$$+ \frac{1}{3!}\sum_i\sum_j\sum_k L_{ijk}(\hat{\theta})(\theta_i-\hat{\theta}_i)(\theta_j-\hat{\theta}_j)(\theta_k-\hat{\theta}_k) + \dots$$

where $L_i(\hat{\theta}) = \dfrac{\partial L(\hat{\theta})}{\partial\hat{\theta}_i}$ and so on, and L and its derivatives are $O(n)$, whereas $(\theta_i-\hat{\theta}_i)$ is $O(n^{-\frac{1}{2}})$ for all i.

Also $w(\theta) = w(\hat{\theta}) + \sum_i w_i(\hat{\theta})(\theta_i-\hat{\theta}_i) + \frac{1}{2}\sum_i\sum_j w_{ij}(\hat{\theta})(\theta_i-\hat{\theta}_i)(\theta_j-\hat{\theta}_j) + \dots$.

For simplicity we put $w(\hat{\theta}) = w$, $L(\hat{\theta}) = L$ and $(\theta_i-\hat{\theta}_i) = \theta_i$. Then the numerator of (1.2.6) becomes

$$\int w(\theta)e^{L(\theta)}d\theta = \int (w + \sum_i w_i\theta_i + \frac{1}{2!}\sum_{ij}\sum w_{ij}\theta_i\theta_j + \dots) \,.$$

$$\exp(L + \sum_i L_i\theta_i + \frac{1}{2!}\sum_i\sum_j L_{ij}\theta_i\theta_j + \frac{1}{3!}\sum_i\sum_j\sum_k L_{ijk}\theta_i\theta_j\theta_k$$

$$+ \frac{1}{4!}\sum_i\sum_j\sum_k\sum_l L_{ijkl}\theta_i\theta_j\theta_k\theta_l + \dots)d\theta \,.$$

$$= we^L[\int (1 + \sum_i W_i\theta_i + \frac{1}{2}\sum_{ij}\sum W_{ij}\theta_i\theta_j + \dots) \exp(\frac{1}{2}\sum_{ij}\sum L_{ij}\theta_i\theta_j) \,.$$

$$\exp(\frac{1}{3!}\sum_{ijk}\sum\sum L_{ijk}\theta_i\theta_j\theta_k + \frac{1}{4!}\sum_{ijkl}\sum\sum\sum L_{ijkl}\theta_i\theta_j\theta_k\theta_l + \dots)d\theta]$$

$$= we^L[\int e^{\frac{1}{2}\sum_{ij}\sum L_{ij}\theta_i\theta_j}(1 + \sum_i W_i\theta_i + \frac{1}{2!}\sum_{ij}\sum W_{ij}\theta_i\theta_j + \dots) \,.$$

$$(1 + \frac{1}{3!}\sum_{ijk}\sum\sum L_{ijk}\theta_i\theta_j\theta_k + \frac{1}{4!}\sum_{ijkl}\sum\sum\sum L_{ijkl}\theta_i\theta_j\theta_k\theta_l$$

$$+ \frac{1}{2}(\frac{1}{3!}\sum_{ijk}\sum\sum L_{ijk}\theta_i\theta_j\theta_k)^2 + \dots)d\theta] \,. \tag{1.2.7}$$

where $W_i = w_i/w$ = etc, $L_i = 0$, since the expansion is about the maximum likelihood value, and all functions are evaluated at $\hat{\theta}$.

Collecting terms of like order together, the integral is

$$we^{L}\int \exp(\frac{1}{2}\sum_{i}\sum_{j}L_{ij}\theta_i\theta_j)[1 + \sum_{i}W_i\theta_i + \frac{1}{6}\sum_{i}\sum_{j}\sum_{k}L_{ijk}\theta_i\theta_j\theta_k$$

$$+ \frac{1}{2}\sum_{i}\sum_{j}W_{ij}\theta_i\theta_j + (\sum_{i}W_i\theta_i)(\frac{1}{6}\sum_{i}\sum_{j}\sum_{k}L_{ijk}\theta_i\theta_j\theta_k) + R]d\theta \quad (1.2.8)$$

up to **$O(n^{-1})$, and R ultimately** disappears because it does not involve w or its derivatives.

The integrations all involve the moments of the multivariate normal distribution with density proportional to $\exp(\frac{1}{2}\sum_{i}\sum_{j}L_{ij}\theta_i\theta_j)$.

Then the result of the integration is that

$$\int w(\theta)e^{L(\theta)}d\theta \sim we^{L}(2\pi)^{m/2}|\Sigma|^{\frac{1}{2}}[1 + \frac{1}{2}\sum_{i}\sum_{j}W_{ij}\sigma_{ij}$$

$$+ \frac{1}{2}\sum_{i}\sum_{j}\sum_{k}\sum_{l}L_{ijk}W_l\sigma_{ij}\sigma_{kl} + R^*] .$$

where $|\Sigma|$ is the determinant of variance—covariance matrix, and $\sigma_{ij} = E_{\theta_i\theta_j}(\theta_i\theta_j)$ **, holding θ_i, θ_j fixed.**

In the same way we can calculate the integral of the denominator, so that finally we have

$$\int w(\theta)e^{L(\theta)}d\theta/\int \nu(\theta)e^{L(\theta)}d\theta \sim$$

$$\frac{w}{\nu}[1 + \frac{1}{2}\sum_{i}\sum_{j}(W_{ij}-V_{ij})\sigma_{ij} + \frac{1}{2}\sum_{i}\sum_{j}\sum_{k}\sum_{l}L_{ijk}(W_l-V_l)\sigma_{ij}\sigma_{kl} + \ldots] \quad (1.2.9)$$

for $\nu \neq 0$.

R^* and the terms with order higher than (n^{-1}) are cancelled. Indeed the first term to be cancelled is of order (n^{-2}) not $(n^{-3/2})$, because the odd moments for the normal distribution are zero.

CHAPTER 2

SYSTEM RELIABILITY

2.1 Introduction

Large sample normality is often invoked when exact distributions are intractable. However, if sample sizes are not large enough to justify normality, then approximations to exact results for tests, intervals, etc. may be poor. Asymptotic expansions can improve approximations by correcting for bias, skewness and higher cumulants all of which are zero for normal random variables. They can be regarded as extensions of the Central Limit Theorem. Successive corrections are of increasing powers, inverse in the sample size, or in a variable related to the sample size.

In this Chapter we investigate the use of asymptotic expansions to approximate system reliability posterior distributions with general structures. Winterbottom (1980) obtained the system reliability approximation in classical case. In some systems we assume that component posteriors are beta distributed and in other cases, they have negative log gamma distributions. We can allow the test data to be mixed, i.e. for some components, the test data is pass/fail and for others exponential times to failure are recorded.

Lampkin and Winterbottom (1983) investigate the case of mixed components in series systems. In the special case when the posterior distribution of a component is gamma with integer index we establish an equivalence with series systems and beta distributions. However, for non integer index, we can develop asymptotic expansions for the case of gamma posteriors for failure rate or, equivalently, negative log gamma distributions for reliability function.

Several approximate methods are described and their effectiveness compared. In particular, we introduce asymptotic expansions for percentage points, (Cornish and Fisher), and distribution function approximations, (Edgeworth series and Saddlepoint approximation). These methods are described and their effectiveness compared.

2.2 Cumulants of Systems

To explain the expansions consider a parameter θ and an estimator T of θ, based on a sample of size n. Suppose that the moments of $(T-\theta)$ have expansions in inverse powers of n, given to appropriate orders for two corrections, with θ fixed, as follows:

$$
\begin{aligned}
E_\theta(T-\theta) &= \frac{u}{n} + O(n^{-2}) , \\
E_\theta(T-\theta)^2 &= \frac{v}{n} + \frac{v^*}{n^2} + O(n^{-3}) , \\
E_\theta(T-\theta)^3 &= \frac{w}{n^2} + O(n^{-3}) , \\
E_\theta(T-\theta)^4 &= \frac{3v^2}{n^2} + \frac{z}{n^3} + O(n^{-4}) , \\
E_\theta(T-\theta)^5 &= (10vw-15vu^2)/n^3 + O(n^{-4}) , \\
E_\theta(T-\theta)^6 &= 15v^3/n^3 + O(n^{-4}) .
\end{aligned}
\tag{2.2.1}
$$

These expansions are typical of random variables with asymptotically normal distributions. u, v, v*, w and z are usually functions of the parameter θ. In Bayesian applications T is the posterior random variable and θ is a function of the prior parameters and the data.

Let θ_i be the parameter of component i, i = 1, 2, ..., m and let T_i be based on a sample of size n_i with $E(T_i) = \theta_i$. ψ is the system reliability function.

Consider the Taylor series expansion

$$T = \theta + \sum_i (T_i-\theta_i)\psi_i + \frac{1}{2!}\sum_{ij}\sum(T_i-\theta_i)(T_j-\theta_j)\psi_{ij} +$$

$$\frac{1}{3!}\sum_{i}\sum_{j}\sum_{k}(T_i-\theta_i)(T_j-\theta_j)(T_k-\theta_k)\psi_{ijk} + \dots \quad , \qquad (2.2.2)$$

where

$$T = \psi(T_1, T_2, \dots, T_m) \ , \quad \theta = \psi(\theta_1, \theta_2, \dots, \theta_m) \ ,$$

$\psi_i = \dfrac{\partial\psi}{\partial\theta_i}$, $\psi_{ij} = \dfrac{\partial^2\psi}{\partial\theta_i\partial\theta_j}$, etc. Taking the expectations of the expansion (2.2.2), the moments $E_\theta(T-\theta)^r$, $r = 1, 2, \dots$ can be obtained and take the same form as for the component expectations. For example, $E_\theta(T-\theta) = \frac{u}{n} + O(n^{-2})$, where $n = \min n_i$ and u is a function of $\theta_1, \theta_2, \dots, \theta_m$ and of $\lambda_1, \lambda_2, \dots, \lambda_m$, where $\lambda_i = n_i/n$.

Once the moment expansions are available they can be converted to cumulant expansions using the well known relationships between moments and cumulants, (see appendix 1). We get the cumulant coefficients for subsequent use in obtaining up to two corrections as:

$$\kappa_{1,1} = \frac{1}{2}\sum_i v_i\psi_{ii}/\lambda_i \ ,$$

$$\kappa_{2,1} = \sum_i v_i\psi_i^2/\lambda_i \ ,$$

$$\kappa_{3,2} = \sum_i w_i\psi_i^3/\lambda_i^2 + 3\sum_i\sum_j v_iv_j\psi_i\psi_j\psi_{ij}/\lambda_i\lambda_j \ ,$$

$$\kappa_{2,2} = \sum_i v_i^*\psi_i^2/\lambda_i^2 + \sum_i w_i\psi_i\psi_{ii}/\lambda_i^2 + \sum_i\sum_j v_iv_j\psi_i\psi_{ijj}/\lambda_i\lambda_j$$
$$+ \frac{1}{2}\sum_i\sum_j v_iv_j\psi_{ij}{}^2/\lambda_i\lambda_j \ ,$$

$$\kappa_{4,3} = \sum_i z_i\psi_i^4/\lambda_i^3 - 6\sum_i v_iv_i^*\psi_i^4/\lambda_i^3 + 12\sum_i\sum_j w_iv_j\psi_i^2\psi_j\psi_{ij}/\lambda_i^2\lambda_j$$
$$+ 4\sum_i\sum_j\sum_k v_iv_jv_k\psi_i\psi_j\psi_k\psi_{ijk}/\lambda_i\lambda_j\lambda_k$$

$$+ 12\sum_i\sum_j\sum_k v_i v_j v_k \psi_{ij}\psi_{ik}\psi_j\psi_k/\lambda_i\lambda_j\lambda_k \quad . \tag{2.2.3}$$

In the above all summations are unrestricted over the range, 1 through m.

Note that in (2.2.3) we have taken $E(T_i) = \theta_i$ so that $u_i = 0$, $i = 1, 2, \ldots, m$. Then we can represent the corresponding cumulant expansions for $n^{\frac{1}{2}}(T-\theta)$ by the following scheme.

1	$n^{-\frac{1}{2}}$	n^{-1}		$n^{-\frac{3}{2}}$	n^{-2}
	$\kappa_{1,1}$		¦	$\kappa_{1,2}$	
$\kappa_{2,1}$		$\kappa_{2,2}$	¦		$\kappa_{2,3}$
	$\kappa_{3,2}$		¦	$\kappa_{3,3}$	
		$\kappa_{4,3}$	¦		$\kappa_{4,4}$
			¦	$\kappa_{5,4}$	
			¦		$\kappa_{6,5}$

For example, $\kappa_2 = \kappa_{2,1} + \kappa_{2,2}/n + \kappa_{2,3}/n^2 + \ldots$

The vertical dotted line shows the extent of the corrections for the reliability problem with general system structures.

It is convenient to consider the partially standardised form $n^{\frac{1}{2}}(T-\theta)/\kappa_{2,1}^{\frac{1}{2}}$ where the scheme holds with $\kappa_{2,1}$ replaced by unity and all other cumulant coefficients $\kappa_{r,s}$ replaced by $\ell_{r,s} = \kappa_{r,s}/\kappa_{2,1}^{\frac{r}{2}}$. The first order correction **of $n^{\frac{1}{2}}(T-\theta)/\kappa_{2,1}^{\frac{1}{2}}$ in asymptotic** expansions reduces $\ell_{1,1}$ and $\ell_{3,2}$, both of order $n^{-\frac{1}{2}}$, to zero.

This correction is for the leading terms in the expansions for bias and skewness.

Moving progressively from left to right through the scheme the corrections provide successive approximations, in increasing powers of $n^{-\frac{1}{2}}$, to standard normal variate. Improved tests and intervals for θ can then be obtained. For certain systems there will be considerable simplification when some partial derivatives of the reliability function vanish.

(i) Beta Distributions

Suppose that for the i^{th} component the posterior reliability function R_i is beta distributed with probability density function

$$f(r_i) = r_i^{\alpha_i - 1}(1-r_i)^{\beta_i - 1}/B(\alpha_i,\beta_i), \qquad (2.2.4)$$

$$0 < r_i < 1 \quad , \quad \alpha_i > 0 \quad , \beta_i > 0.$$

We may write $\alpha_i = \alpha_{io} + s_i$ and $\beta_i = \beta_{io} + n_i - s_i$, where α_{io} and β_{io} are prior parameters and s_i the reliable components obtained out of n_i tested. **Taking expectations over the posterior distribution.**

Define $\rho_i = E(R_i/\alpha_i, \beta_i) = \dfrac{\alpha_i}{\alpha_i + \beta_i}$ **and** $n_i = \alpha_i + \beta_i$.

Then we have $u_i = 0$, $v_i = \rho_i(1-\rho_i)$, $v_i^* = -\rho_i(1-\rho_i)$, $w_i = 2\rho_i(1-\rho_i)(1-2\rho_i)$ and $z_i = 6\rho_i(1-\rho_i)(1-6\rho_i+6\rho_i^2)$.

(ii) Gamma Distributions

When component lives are exponentially distributed, we consider gamma posteriors for the failure rate. For the i^{th} component, let the failure rate Λ_i have the probability density function

$$g(\lambda_i) = \tau_i(\tau_i\lambda_i)^{\eta_i-1}e^{-\tau_i\lambda_i}/\Gamma(\eta_i) , \qquad (2.2.5)$$

$$\lambda_i > 0, \quad \tau_i > 0, \quad \eta_i > 0 .$$

We may write $\tau_i = \tau_{io} + t_i$ and $\eta_i = \eta_{io} + r_i$ where τ_{io} and η_{io} are prior parameters and r_i failures were observed in a total time on test t_i. Then we take $n_i = \tau_i$ and $\rho_i = e^{-\lambda_i t_m}$, where t_m is the mission time which, without loss of generality, may be set at unity. Noting that $E(\Lambda_i) = \eta_i/\tau_i = \phi_i$ (say) we can replace $(R_i - \rho_i)$ by $(\lambda_i - \phi_i)$ in the Taylor series expansions, suitably modifying the partial derivatives of the system reliability function.

For the moments of $(\lambda_i - \phi_i)$ we have $u_i = 0$, $v_i = \phi_i$, $v_i^* = 0$, $w_i = 2\phi_i$ and $z_i = 6\phi_i$.

2.3 Asymptotic Expansions for Percentage Points

Asymptotic expansions for percentage points provide successive corrections to large sample normal approximations which consist of a point estimate plus an appropriate multiple of its standard error. The expansions are due to Cornish and Fisher (1937) and can be considered as extensions of the Central Limit Theorem. In the reliability context the expansions take different forms depending on the asymptotic variable.

For series systems the asymptotic variable can be taken as the number of components and this case has been investigated by

Lampkin and Winterbottom (1983). In general, it is necessary for the asymptotic variable to be a quantity related to the smallest sample size over component tests. An analogous treatment for the corresponding classical problem was given by Winterbottom (1980).

For a system of m components, specialising the notations, the system reliability is given by $R = \psi(R_1, R_2, \ldots R_m)$, where R_i is the reliability of the i^{th} component and ψ is the system reliability function. Quantities n_i, ρ_i $(i = 1, 2, \ldots, m)$ can be constructed from the component reliability distributions, so that the cumulants of $(R_i - \rho_i)$ are $\kappa_1 \sim 0(n_i^{-1})$ and $\kappa_s \sim 0(n_i^{1-s})$, $s \geqq 2$. In fact we can take ρ_i to be the mean of R_i, whence $\kappa_1 = 0$. The n_i will be quantities involving the test sample sizes combined with corresponding prior quantities. The above cumulant properties enable Cornish and Fisher expansions to be obtained for percentage points of the component reliability distributions separately. However, using a result due to James and Mayne (1962), if we define $\rho = \psi(\rho_1, \rho_2, \ldots, \rho_m)$ and $n = \min n_i$, then the cumulants of $(R-\rho)$ are $\kappa_1 \sim 0(n^{-1})$ and $\kappa_s \sim 0(n^{1-s})$, $s \geqq 2$. Thus Cornish and Fisher expansions can be applied directly to the posterior distribution of system reliability.

More explicitly, we can write

$$\kappa_1 = n^{-1}\sum_{i=0}^{\infty}\kappa_{1,i+1}/n^i \quad ,$$

$$\kappa_s = n^{1-s}\sum_{i=0}^{\infty}\kappa_{s,s-1+i}/n^i \quad , \qquad s \geqq 2 \; .$$

For the standardised random variable $n^{\frac{1}{2}}(R-\rho)/\kappa_{2,1}^{\frac{1}{2}}$

the corresponding cumulants are

$$c_1 = n^{-\frac{1}{2}} \sum_{i=0}^{\infty} \ell_{1,1+i}/n^i \quad ,$$

$$c_s = n^{1-\frac{s}{2}} \sum_{i=0}^{\infty} \ell_{s,s-1+i}/n^i \quad , \quad s \geq 2 \; ,$$

where $\ell_{s,j} = \kappa_{s,j}/\kappa_{2,1}^{s/2}$.

The cumulant coefficients $\kappa_{s,j}$ will be obtained from a Taylor series expansion (2.2.2) after replacing T and θ by R and ρ respectively.

The large sample formula for system reliability posterior percentage points is

$$R_\xi \cong \rho + \xi(\kappa_{2,1}/n)^{\frac{1}{2}} \quad , \qquad (2.3.1)$$

whereas the Cornish and Fisher expansions, giving two corrections to (2.3.1), is

$$R_\xi \cong \rho + (\kappa_{2,1}/n)^{\frac{1}{2}}[\xi + \{\ell_{1,1} + \ell_{3,2}(\xi^2-1)/6\}/n^{\frac{1}{2}}$$

$$+ \{\ell_{3,2}\xi/2 + \ell_{4,3}(\xi^3-3\xi)/24 - \ell_{3,2}^2(2\xi^3-5\xi)/36\}/n]. \qquad (2.3.2)$$

For the 100 α percentage point, ξ is the standard normal value such that $\Phi(\xi)$ = 1 - α and Φ is the standard normal distribution function.

(i) Series Systems

Exact results can be obtained for series systems by inverting the Mellin integral transform (Springer and Byers, 1971), or by inverting the Laplace transform of $Y = -\ln R$ and then transforming to obtain the distribution of $R = e^{-Y}$ (Lampkin and Winterbottom, 1983). Even so, calculation of exact limits for system reliability becomes computationally very difficult as the n_i increase.

If it can be demonstrated that good approximations are obtained for reasonably small values of the n_i then, by the nature of asymptotic expansions, even greater accuracy will be achieved as these values increase.

Using the results of the previous section, expressions for the cumulant coefficients required for up to two corrections are now given. When all component posterior distributions are beta we have,

$$\kappa_{1,1} = 0 \ ,$$

$$\kappa_{2,1} = \rho^2 \sum_i (1-\rho_i)/\lambda_i \rho_i \ ,$$

$$\kappa_{2,2} = \frac{\rho^2}{2} \{ (\sum_i (1-\rho_i)/\lambda_i \rho_i)^2 - \sum_i (1-\rho_i^2)/\lambda_i^2 \rho_i^2 \} \ ,$$

$$\kappa_{3,2} = \rho^3 \{ 3(\sum_i (1-\rho_i)/\lambda_i \rho_i)^2 - \sum_i (1-\rho_i^2)/\lambda_i^2 \rho_i^2 \} \ ,$$

$$\kappa_{4,3} = \rho^4 \{ 2\sum_i (1-\rho_i)(1+\rho_i+\rho_i^2)/\lambda_i^3 \rho_i^3 + 16(\sum_i (1-\rho_i)/\lambda_i \rho_i)^3$$

$$-12(\sum_i (1-\rho_i)/\lambda_i \rho_i)(\sum_i (1-\rho_i^2)/\lambda_i^2 \rho_i^2) \} \ . \tag{2.3.3}$$

Let $m = 2$, $\alpha_1 = \alpha_2 = 19$, $\beta_1 = \beta_2 = 1$, so that $n_1 = n_2 = n = 20$,

$\lambda_1 = \lambda_2 = 1$, $\rho_1 = \rho_2 = 0.95$ and $\psi = \rho_1 \rho_2 = 0.9025$.

The exact distribution function of system reliability R in this simple case is

$$G(r) = r^{19}(1-19 \ln r) \ . \tag{2.3.4}$$

Then, for example, the exact lower 90% limit is the solution for r in the equation $G(r) = 0.1$ and this is 0.8149 .

The values for the cumulant coefficents $\kappa_{r,s}$ and corresponding ratios $\ell_{r,s}$ are

$$\kappa_{1,1} = 0 , \quad \kappa_{2,1} = 0.0857375 , \quad \kappa_{2,2} = -0.0848125 ,$$

$$\kappa_{3,2} = -0.13439353 , \quad \kappa_{4,3} = 0.2727578 , \quad \text{whence}$$

$$\ell_{1,1} = 0 , \quad \ell_{2,2} = -0.97368421 ,$$

$$\ell_{3,2} = -5.35330685 , \quad \ell_{4,3} = 37.105263 .$$

Using the Cornish and Fisher expansion (2.3.2), we get the value of $r = \underline{0.8153}$, when $\xi = -1.28159$.

For each of the following series system cases we give lower 90, 95 and 97.5 percentage points. Cornish and Fisher with asymptotic variable m, the number of components, is denoted CF(m) and uses the formula given by Lampkin and Winterbottom (1983) to two corrections. CF(n) uses formula (2.3.2) . Table 1 gives exact and approximate CF(n) and CF(m) lower 90, 95 and 97.5 percentage points.

Table (1) Cornish and Fisher Approximations

%	ξ	CF(m)	CF(n)	EXACT
Case (a) $m = 2, \alpha_1 = \alpha_2 = 19, \beta_1 = \beta_2 = 1$.				
90	-1.28159	0.8148	0.8153	0.8149
95	-1.6449	0.7742	0.7800	0.7791
97.5	-1.95996	0.7448	0.7433	0.7458
Case (b) $m = 2, \alpha_1 = \alpha_2 = 18, \beta_1 = \beta_2 = 2$.				
90	-1.28159	0.6966	0.6970	0.6967
95	-1.6449	0.6571	0.6577	0.6574
97.5	-1.95996	0.6218	0.6223	0.6223
Case (c) $m = 4, \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 20, \beta_1 = \beta_2 = \beta_3 = \beta_4 = 1$.				
90	-1.28159	0.7160	0.7163	0.7160
95	-1.6449	0.6784	0.6789	0.6786
97.5	-1.95996	0.6447	0.6451	0.6451

It is easy to extend CF(m) to four corrections enabling very good accuracy of approximation to be achieved.

When all component posteriors are gamma distributed then there are two cases, the first case being when the index η in the posterior density function

$$g(r) = \tau\ (\tau\lambda)^{\eta-1} \bar{e}^{\tau\lambda} / \Gamma(\eta) \qquad (2.3.5)$$

is an integer. This would be so if the prior index η_0 (the prior parameter) were chosen to be integer or, alternatively, if the invariant prior $\pi(\lambda)\propto\lambda^{-1}$ were to be used for which $\eta = r$ and $\tau = t$.

In formula (2.3.5), if the mission time is one, then the reliability of the component is $R = e^{-\lambda}$. The probability density function of R is the negative log gamma distribution with p.d.f.

$$g(r) = \tau^{\eta}(-\ln r)^{\eta-1}r^{\tau-1}/\Gamma(\eta), \qquad 0 < r < 1. \tag{2.3.5a}$$

Let a series system have η components and let the posterior density for all components be Beta with parameters α, β. The Laplace transform of $Y = \sum_{i=1}^{\eta} Y_i$, where $Y = -\ln R$ and $Y_i = -\ln R_i$ is

$$L_y(s) = E(e^{-sy}) = E\prod_{i=1}^{\eta} R_i^s = \prod_{i=1}^{\eta} E(R_i^s)$$

$$= \prod_{i=1}^{\eta} \frac{B(\alpha+s,\beta)}{B(\alpha,\beta)} = \left(\frac{B(\alpha+s,\beta)}{B(\alpha,\beta)}\right)^{\eta} \tag{2.3.6}$$

Setting $\beta = 1$ and $\alpha = \tau$ gives

$$L_y(s) = \left(\frac{\tau}{\tau+s}\right)^{\eta} \tag{2.3.7}$$

Inverting this Laplace transform the p.d.f. of Y is

$$h(y) = \tau(\tau y)^{\eta-1}e^{-\tau y}/\Gamma(\eta), \tag{2.3.8}$$

and hence the p.d.f. of $R = e^{-Y}$ is

$$g(r) = \tau^{\eta}(-\ln r)^{\eta-1}r^{\tau-1}/\Gamma(\eta), \tag{2.3.9}$$

the negative log gamma as formula (2.3.5a). Thus, when η is an integer, we can replace a component which has a $\Gamma(\eta,\tau)$ posterior for failure rate by η independent series components each having a $B(\tau,1)$ posterior, to each of which the methods of expansions can be applied.

The second case is when the index η is non integer. In this case we can develop asymptotic expansions for the case of gamma posteriors for failure rate, or, equivalently, negative log gamma distributions for reliability.

Let $\rho = \exp(-\sum_i \phi_i)$, $\tau = \min \tau_i$ and $\lambda_i = \tau_i/\tau$

(i = 1, 2, ..., m). Then we have from (2.2.3)

$$\kappa_{1,1} = \frac{\rho}{2}\sum_i \phi_i/\lambda_i \quad ,$$

$$\kappa_{2,1} = \rho^2 \sum_i \phi_i/\lambda_i \quad ,$$

$$\kappa_{2,2} = \rho^2\{\frac{3}{2}(\sum_i \phi_i/\lambda_i)^2 - 2\sum_i \phi_i/\lambda_i^2\} \quad , \tag{2.3.10}$$

$$\kappa_{3,2} = \rho^3\{3(\sum \phi_i/\lambda_i)^2 - 2\sum_i \phi_i/\lambda_i^2\} \quad ,$$

$$\kappa_{4,3} = \rho^4\{6\sum_i \phi_i/\lambda_i^3 - 24(\sum \phi_i/\lambda_i)(\sum \phi_i/\lambda_i^2) + 16(\sum \phi_i/\lambda_i)^3\} \ .$$

Let $m = 3$, $\eta_1 = 1$, $\tau_1 = 11$; $\eta_2 = 1$, $\tau_2 = 20$; $\eta_3 = 2$, $\tau_3 = 30$.

Then $\phi_1 = \eta_1/\tau_1 = 0.0909 \ldots$, $\phi_2 = 0.05$,

$\phi_3 = =.066 \ldots$ and $\lambda_1 = 1$, $\lambda_2 = 1.8181 \ldots$,

$\lambda_3 = 2.7272 \ldots$, $\rho = 0.81255168$,

$\kappa_{2,1} = 0.09431765$, $\ell_{1,1} = 0.1889799$,

$\ell_{2,2} = -1.3957191$, $\ell_{3,2} = 3.1258327$,

$\ell_{4,3} = 13.106379$.

The Cornish and Fisher expansion for percentage points of the distribution of R, formula (2.3.2) (replacing n by τ), is <u>0.7003</u>. The exact 90% lower limit ($\xi = -1.28159$) is <u>0.6992</u> .

(ii) <u>More Complicated Structures and Mixed Test Data</u>

In this case we consider more complex systems. Also we can allow the test data to be mixed, i.e. for some components, the test data is pass/fail and for others exponential times to failure are recorded. Specifically we obtain lower limits for three systems:

a) A series-parallel system,

b) A parallel system with standby redundancy,

c) A K-out-of-N Quorum System.

For each system, and a given lower limit, 10,000 values of the posterior distribution were generated by Monte-Carlo methods and the proportion exceeding the lower limit determined. This proportion is then compared to the designated probability level.

Case (a) A Series-parallel System

Consider a series-parallel system with one component linked serially with two components in parallel. The two components in parallel are like components in that their times to failure are independent random values drawn from the same gamma posterior distribution. For the component linked in series to the parallel structure the reliability has probability density function

$$g(r) = \frac{r^{\alpha-1}(1-r)^{\beta-1}}{B(\alpha,\beta)} , \qquad 0 < r < 1. \tag{2.3.11}$$

For the two components in parallel, and for unit mission time, the corresponding distribution of reliability is

$$g(r) = \tau^{\eta}(-\ln r)^{\eta-1} r^{\tau-1}/\Gamma(\eta), \; 0 < r < 1 . \tag{2.3.12}$$

System reliability, as a random variable, is

$$\psi(R,\Lambda) = R(e^{-\Lambda_1} + e^{-\Lambda_2} - e^{-(\Lambda_1+\Lambda_2)}) . \tag{2.3.13}$$

Let $\rho = E(R) = \frac{\alpha}{\alpha+\beta}$ and $\phi = E(\lambda) = \eta/\tau$.

Then, by expanding $\psi(R,\lambda)$ about $\psi(\rho,\phi)$ in a Taylor's series, we have:-

For $(R-\rho)$,

$$u = 0 \ , \ v = \rho(1-\rho), \ v^* = -\rho(1-\rho), \ w = 2\,(1-\rho)(1-2\rho)$$

and $z = 6\rho(1-\rho)(1-6\rho+6\rho^2)$.

And for $(\lambda-\phi)$,

$$u = 0, \ v = \phi, \ v^* = 0, \ w = 2\phi \text{ and } z = 6\phi \ .$$

The structural derivatives required for the asymptotic expansions are

$$\psi_1 = \frac{\partial\psi}{\partial\rho} = e^{-\phi}(2-e^{-\phi}) \ ,$$

$$\psi_{11} = \frac{\partial^2\psi}{\partial\rho^2} = 0 \ ,$$

$$\psi_2 = \frac{\partial\psi}{\partial\phi} = -2\rho e^{-\phi}(1-e^{-\phi}) \ ,$$

$$\psi_{22} = \frac{\partial^2\psi}{\partial\phi^2} = 2\rho e^{-\phi}(1-2e^{-\phi}) \ ,$$

$$\psi_{12} = \frac{\partial^2\psi}{\partial\rho\partial\phi} = -2e^{-\phi}(1-e^{-\phi}) \ ,$$

$$\psi_{122} = \frac{\partial^3\psi}{\partial\rho\partial\phi^2} = 2e^{-\phi}(1-2e^{-\phi})$$

$$\psi_{222} = \frac{\partial^3\psi}{\partial\phi^3} = -2e^{-\phi}(1-4e^{-\phi}) \ .$$

All other required derivatives are zero.

Let the first component have a beta posterior with $\alpha = 19$ and $\beta = 1$ and the parallel components have a gamma posterior with $\tau = 20$ and $\eta = 2$. Then

$$\rho = \frac{\alpha}{\alpha+\beta} = 0.95 \quad \text{and} \quad \phi = \frac{\eta}{\tau} = 0.1 \ .$$

The moment coefficients and the above derivatives can be evaluated. For above values where $\lambda_1 = \lambda_2 = 1$ and from formula (2.2.3), we obtain

$$\kappa_{1,1} = -0.139198576 \quad , \quad \kappa_{2,1} = 0.051996749 \quad ,$$

$$\kappa_{2,2} = 0.041161491 \quad , \quad \kappa_{3,2} = -0.08782922,$$

$$\kappa_{4,3} = 0.184948466 \ .$$

The corresponding standardised cumulant coefficients are

$$\ell_{1,1} = -0.610445219 \quad , \quad \ell_{2,2} = 0.791616625 \quad ,$$

$$\ell_{3,2} = -7.40755252 \text{ and } \ell_{4,3} = 68.40665453 \ .$$

Finally, $\psi(\rho,\phi) = 0.9413969$. Using the Cornish and Fisher formula (2.3.2), where $n = \min((\alpha+\beta),\tau) = 20$ and $\psi(\rho,\phi)$ instead of ρ in the formula, the distribution function values of R are 0.86845 , 0.83582 and 0.80358 for the lower 90, 95 and 97.5 percentiles respectively .

Case (b) Parallel System with Standby Redundancy

Consider two components with identical gamma failure rate distributions and a switch which works with probability θ. We assume that pass/fail data is available from tests of the switch so that the posterior of the random variable θ is beta. The system works if the component set in operation at $t = 0$ survives for a time t_m or, if it fails at some time prior to t_m, the switch works and the second component survives for the remaining time to t_m. The system reliability for mission time t_m, given θ and λ is

$$P(T > t_m \ ; \ \theta,\lambda) = e^{-\lambda t_m} + \int_0^{t_m} \lambda e^{-\lambda x} \ \theta \ e^{-\lambda(t_m - x)} \, dx$$

$$= e^{-\lambda t_m}(1+\lambda\theta t_m) \ .$$

Without loss of generality let $t_m = 1$.

Then $$P(T > 1 \ ; \ \theta,\lambda) = e^{-\lambda}(1 + \lambda\theta). \qquad (2.3.14)$$

Thus system reliability is

$$\psi(\theta,\lambda) = \bar{e}^{\lambda}(1 + \lambda\theta) \tag{2.3.15}$$

Let $E(\theta) = \rho$ and $E(\lambda) = \phi$, expanding $\psi(\theta,\lambda)$ about $\psi(\rho,\phi)$ in a Taylor's series we have

$$\psi_1 = \frac{\partial\psi}{\partial\rho} = \phi\bar{e}^{\phi} , \qquad \psi_{11} = \frac{\partial^2\psi}{\partial\rho^2} = 0$$

$$\psi_2 = \frac{\partial\psi}{\partial\phi} = -\bar{e}^{\phi}(1 + (\phi-1)\rho) ,$$

$$\psi_{22} = \frac{\partial^2\psi}{\partial\phi^2} = \bar{e}^{\phi}(1 +(\phi-2)\rho) ,$$

$$\psi_{12} = \frac{\partial^2\psi}{\partial\rho\partial\phi} = \bar{e}^{\phi}(1-\phi) ,$$

$$\psi_{122} = \frac{\partial^3\psi}{\partial\rho\partial\phi^2} = (\phi-2)\bar{e}^{\phi} ,$$

$$\psi_{222} = \frac{\partial^3\psi}{\partial\phi^3} = -\bar{e}^{\phi}(1 + (\phi-3)\rho) .$$

All other required derivatives are zero.

The moment coefficients for the Beta and Gamma distributions are as given in Case (a).

Let $\alpha = 18$, $\beta = 2$ and $\tau = 20$, $\eta = 2$

Then

$$\kappa_{1,1} = -0.03212728 , \qquad \kappa_{2,1} = 0.003692476 ,$$

$$\kappa_{2,2} = -0.01869858 , \qquad \kappa_{3,2} = -0.002376641 ,$$

$$\kappa_{4,3} = 0.002379256 .$$

The corresponding standardised cumulants are:

$$\ell_{1,1} = -0.528615702 , \qquad \ell_{2,2} = -5.063967917 ,$$

$$\ell_{3,2} = -10.59222172 , \qquad \ell_{4,3} = 174.5041969 .$$

Also $\psi(\rho,\phi) = 0.986273$.

Using the Cornish and Fisher expansions (2.3.2), the lower .90, .95 and .975 limits for system reliability are 0.96392, 0.95317 and 0.94665 respectively.

Case (c) A K-out-of-N Quorum Structure

This system works if at least K of the N components work, $(K \leq N)$. For the case K = 2 and N = 3, the system reliability is

$$\psi(R_1, R_2, R_3) = \prod_{i=1}^{3} R_i + \sum_{i=1}^{3} \prod_{j \neq i} R_j (1-R_i) . \qquad (2.3.16)$$

Let R_i $(i = 1, 2, 3)$ be independent beta variates.

Then

$$\psi_1 = \frac{\partial\psi}{\partial\rho_1} = \rho_2(1-\rho_3) + \rho_3(1-\rho_2) \text{ and similarly for } \psi_2 \text{ and } \psi_3 .$$

$$\psi_{ii} = \frac{\partial^2\psi}{\partial\rho_i^2} = 0 ,$$

$$\psi_{12} = \frac{\partial^2\psi}{\partial\rho_1\partial\rho_2} = 1 - 2\rho_3 \text{ and similarly for } \psi_{13} \text{ and } \psi_{23} .$$

$$\psi_{123} = \frac{\partial^3\psi}{\partial\rho_1\partial\rho_2\partial\rho_3} = -2 .$$

All other derivatives required are zero.

Let $\alpha_1 = \alpha_2 = \alpha_3 = 18$; $\beta_1 = \beta_2 = \beta_3 = 2$.

Then $\rho_1 = \rho_2 = \rho_3 = 0.9$ and $\psi(\rho_1, \rho_2, \rho_3) = 0.972$.

Thus $\psi_i = 0.18$, $\psi_{ij} = -0.8$ and $\psi_{ijk} = -2$.

Further, for the common beta distribution, we have,

$u_i = 0$, $v_i = 0.09$, $v_i^* = -0.09$, $w_i = -0.144$ and $z_i = 0.297$.

Then

$\kappa_{1,1} = 0$, $\kappa_{2,1} = 0.008748$,

$\kappa_{2,2} = -0.008748$, $\kappa_{3,2} = -0.00629856$,

$\kappa_{4,3} = 0.006173219$, giving

$\ell_{1,1} = 0$, $\ell_{2,2} = -1$, $\ell_{3,2} = -7.69800359$,

$\ell_{4,3} = 80.66$.

Using Cornish and Fisher expansions (2.3.2), the lower .90, .95 and 0.975 limits, for system reliability are 0.9428 , 0.93109 and 0.91824 respectively.

Table (2) A Monte-Carlo Study

DESIGNATED LEVEL

SYSTEM	0.90	0.95	0.975
Case (a) (Series-Parallel)	0.905 (0.888)*	0.953 (0.920)*	0.978 (0.946)*
Case (b) (Standby Redundancy)	0.907 (0.882)*	0.95 (0.906)*	0.977 (0.934)*
Case (c) (2/3 Quorum)	0.902 (0.898)*	0.951 (0.928)*	0.975 (0.946)*

* Crude two term values

2.4 Distribution Function Approximations

An indirect way to obtain approximate percentage points is to approximate the posterior distribution function first. However, the distribution function itself may be of some interest. Here, we discuss and apply Edgeworth expansions and Saddlepoint approximations.

(i) Edgeworth Expansions (Section 1.2 (a))

Let the distribution function of system reliability be $F(r) = P(R \leq r)$. Then the Edgeworth expansion to two corrections is

$$F(r) \cong \Phi(x) - \phi(x)[\{\ell_{1,1} + \ell_{3,2}H_2/6\}/n^{\frac{1}{2}} + \{\ell_{1,1}^2 H_1/2 + \ell_{2,2}H_1/2$$

$$+ \ell_{1,1}\ell_{3,2}H_3/6 + \ell_{4,3}H_3/24 + \ell_{3,2}^2 H_5/72\}/n] \,, \qquad (2.4.1)$$

where $x = n^{\frac{1}{2}}(r-\rho)/\kappa_{2,1}^{\frac{1}{2}}$, H_r is the r^{th} Hermite polynomial defined by

$$H_0(x) = 1, \quad H_1(x) = x \,, \quad H_r(x) = xH_{r-1}(x) - (r-1)H_{r-2}(x) \,,$$

$$r \geq 2 \,.$$

Φ and ϕ denote the standard normal distribution and probability density functions respectively.

In order to illustrate the effectiveness of the Cornish and Fisher and Edgeworth expansions, let $m = 2$; $\alpha_1 = \alpha_2 = 19$; $\beta_1 = \beta_2 = 1$. The asymptotic expansion using Cornish and Fisher gives a lower 90% limit of 0·8153 as before compared to the exact value 0.8149. The corresponding Edgeworth expansion when $r = 0.8149$ is <u>0.092</u> compared to the exact value of <u>0.10</u> .

(ii) Saddlepoint Approximations (Section 1.2 (c))

Saddlepoint methods are useful for approximating the posterior distribution function of system reliability when, after a suitable transformation, it can be represented as a sum of independent, though not necessarily identically distributed, random variables. This is accomplished for either series or parallel systems by a logarithmic transformation of system reliability or unreliability respectively. The moment generating function of the transformed random variable is easily obtained in both cases and this facilitates the development of Saddlepoint approximations.

Let the distribution function of system reliability be $F(x) = P(x \leq x)$. Then Saddlepoint approximation is

$$1 - F(x) \cong M_F(u) \exp(-um + \sigma^2 u^2/2)\{1-\Phi(u\sigma)\}\{1+\ell_{3,2} w(u\sigma)/6\} , \tag{2.4.2}$$

where $w(u\sigma) = \dfrac{(u^2\sigma^2-1)\phi(u\sigma)}{(1-\Phi(u\sigma))} - u^3\sigma^3$, u is such that $m(u) = x$ and

$x = n^{\frac{1}{2}}(r-\rho)/\kappa_{2,1}^{\frac{1}{2}}$, see Robinson (1982) and (1.2(c)).

Consider a series system with beta distributed component posteriors. Then

$$x = \sum_i x_i \quad \text{where } x = -\ln R \text{ and } x_i = -\ln R_i \quad .$$

The moment generating function of x is

$$M_x(\theta) = E(R^{-\theta}) = \prod_{i=1}^{k} E(\bar{R}_i^{\theta}) = \prod_{i=1}^{k} \prod_{j=1}^{\beta_i} (\alpha_i+\beta_i-j)/(\alpha_i+\beta_i-\theta-j). \tag{2.4.3}$$

Then

$$\begin{aligned} K_x(\theta) &= \ln M_x(\theta) \\ &= \sum_{i=1}^{k} \sum_{j=1}^{\beta_i} [\ln(\alpha_i+\beta_i-j) - \ln(\alpha_i+\beta_i-\theta-j)] . \end{aligned} \tag{2.4.4}$$

Thus

$$\begin{aligned} \kappa_r &= \frac{d^r}{d\theta^r} \{K_x(\theta)\}_{\theta=0} \\ &= (r-1)! \sum_{i=1}^{k} \sum_{j=1}^{\beta_i} (\alpha_i+\beta_i-j)^{-r}. \end{aligned} \tag{2.4.5}$$

Let $K = 2$; $\alpha_1 = \alpha_2 = 19$; $\beta_1 = \beta_2 = 1$.

Then ,

$$M_X(u) = M_F(u) = \left(\frac{19}{19-u} \right)^2 ,$$

$$\kappa_{1,1} = m(u) = \frac{2}{19-u} ,$$

$$\kappa_{2,1} = \sigma^2(u) = \frac{2}{(19-u)^2} ,$$

$$\kappa_{r,s} = 2(r-1)!/(19-u)^r ,$$

from which $\ell_{r,s} = \kappa_{r,s}/\kappa_{2,1}^{r/2}$

$$= \frac{(r-1)!}{2^{r/2-1}} , \qquad r \geq 2 .$$

Specifically $\ell_{3,2} = \sqrt{2}$.

As we got before the distribution value for r = 0.81488 is 0.10, then x = -0.20471 .

Setting $-x = m(u)$, since $x < 0$, and solving gives u = 9.23, $M(u) = 3.782033$, $\sigma^2(u) = 0.02095231$. These values give

$$G(r) = 1 - F(x) \cong \underline{0.098} \text{ compared to } \underline{0.10} .$$

Table (3) gives comparisons with exact values for a series system with beta component posteriors, where m = 4 ; $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 20$; $\beta_1 = \beta_2 = \beta_3 = \beta_4 = 1$ so that n = 21 and $\rho = 0.81451$.

We now give expressions for all moments and cumulants required for use in the approximations.

$$E(R^s) = \prod_{i=1}^{m} E(R_i^s) = \left(\frac{20}{20 + s} \right)^m , \qquad s \geq 1 . \tag{2.4.6}$$

Moment generating function of x (distribution function F_m):

$$M_X(\theta) = E(e^{\theta X}) = E(e^{-\theta \ln R}) = \left(\frac{20}{20 - \theta} \right)^m . \tag{2.4.7}$$

Cumulant generating function of x :

$$K_x(\theta) = m[\ln 20 - \ln(20-\theta)].$$

Cumulants of x :

$$\kappa_s = m(s-1)!/20^s , \qquad s \geq 1 .$$

Cumulant generating function of the exponentially shifted distribution function P_m :

$$K_P(\theta) = K_{F_m}(\theta+u) - K_{F_m}(u)$$

$$= -m \ln [20 -(\theta+u)] + m \ln (20-u) . \qquad (2.4.8)$$

Then, the mean $m(u) = \dfrac{m}{20-u}$ and variance $\sigma^2(u) = \dfrac{m}{(20-u)^2}$.

Standardised cumulants of P_m :

$$\kappa_1 = 0 , \quad \kappa_s = (s-1)! m^{1-\frac{s}{2}} , \qquad s \geq 2 .$$

The exact posterior probability density function is

$$g(r) = 20^4(-\ln r)^3 r^{19}/3! \qquad (2.4.9)$$

and the distribution function can be conveniently calculated from its expansion as the cumulative Poisson sum

$$G(r) = \sum_{i=0}^{3} \exp(-\mu)\mu^i/i! , \qquad (2.4.10)$$

where $\mu = 20(-\ln r)$.

Table (3) Distribution Function Approximations

R	Edgeworth(m)	Edgeworth(n)	Saddlepoint	Exact
0.5	0.00007	0.00077	0.00052	0.00053
0.6	0.0113	0.00898	0.00881	0.00882
0.7	0.0717	0.0764	0.0742	0.0751
0.8	0.3487	0.3490	0.3470	0.3486
0.85	0.5914	0.5917	0.5939	0.5913
0.9	0.8325	0.8345	0.8394	0.8373
0.95	0.9739	0.9743	0.9797	0.9794
0.975	1.002	0.9995	0.9987	0.9982
0.99	1.009	1.0052	0.999941	0.999942

2.5 Remarks

The Saddlepoint and Edgeworth (m) approximations are only available for series (or parallel) system but Edgeworth (n) can be used for all coherent systems.

Saddlepoint is the best approximation for the distribution function especially in the tails but Edgeworth (m) and Edgeworth (n) are also good except in the extreme tails.

The (m) and (n) approximations for percentage points are both very good and can be used for general structures.

CHAPTER 3

SIMPLE APPROXIMATE METHODS

3.1 Introduction

This Chapter presents methods for calculating lower confidence bounds on system reliability for systems configured as series, parallel or complex arrangements of independent components. These methods depend upon the reduction of component test data to an equivalent system test data. One of these methods forms the basis of the Maximus report (1980). Maximus is based on classical statistics whereas we develop and investigate an analogous Bayesian approach. The other method depends on equating the system posterior mean and variance with the mean and variance of a single beta distribution. The parameters of this beta distribution, so determined, are used in the construction of interval estimates for system reliability. The test data will be taken as pass/fail and/or exponential life-times and the posterior component reliabilities will be distributed as beta and gamma with integer index. We can incorporate exponential time to failure data by the device described in Chapter 2 (Section 2.2).

3.2 Series Systems

The product of beta variables has a complicated form which causes difficulties in maintaining computational precision. In the first method, when the parameters of the component beta distributions are specially related, the product distribution is beta. In appendix II using independent beta distributions with inter-related parameters, we have derived the above result.

The result is as follows:

Let R_i have a beta distribution with probability density function

$$f_i(r_i) = \frac{r_i^{(l_0 + \sum_{j=i+1}^{k} l_j - 1)} (1-r_i)^{l_i - 1}}{B(l_0 + \sum_{j=i+1}^{k} l_j, l_i)} , \quad i = 1, 2, \ldots, k. \tag{3.2.1}$$

Then $R_0 = \prod_{i=1}^{k} R_i$ is beta distributed with p.d.f.

$$f_0(r_0) = \frac{r_0^{l_0-1}(1-r_0)^{l-l_0-1}}{B(l_0, l-l_0)} , \quad l = \sum_{j=0}^{k} l_j \, . \tag{3.2.2}$$

For example, let $k = 2$. Then $R_0 = R_1 R_2$ and from (3.2.1)

$$f_1(r_1) = \frac{r_1^{l_0+l_2-1}(1-r_1)^{l_1-1}}{B(l_0+l_2, l_1)} ,$$

$$f_2(r_2) = \frac{r_2^{l_0-1}(1-r_2)^{l_2-1}}{B(l_0, l_2)} .$$

Let $\alpha_1 = l_0 + l_2$, $\beta_1 = l_1$

$\alpha_2 = l_0$, $\beta_2 = l_2$.

Solving for α_1, α_2, β_1 and β_2 in terms of l_0, l_1 and l_2 we have four quantities (α_1, α_2, β_1, β_2) in three terms (l_0, l_1, l_2). There will be one constrainton the former. Thus we find that $\alpha_1 = \alpha_2 + \beta_2$. If this constraint can be obtained then R_0 has an exact beta distribution with p.d.f.

$$f_0(r_0) = \frac{r_0^{l_0-1}(1-r_0)^{l-l_0-1}}{B(l_0, l-l_0)} = \frac{r_0^{\alpha_2-1}(1-r_0)^{\beta_1+\beta_2-1}}{B(\alpha_2, \beta_1+\beta_2)} . \tag{3.2.3}$$

For example, suppose that $\alpha_1 = 18$, $\beta_1 = 2$; $\alpha_2 = 15$, $\beta_2 = 3$.

Then since $\alpha_1 = \alpha_2 + \beta_2$, R_0 is exactly beta with p.d.f.

$$f_0(r_0) = \frac{r_0^{14}(1-r_0)^4}{B(15,5)} .$$

For general k, where $\alpha_i = \alpha_{i+1} + \beta_{i+1}$, $i = 1, 2, \ldots, k-1$, it is easily seen that

$$f_0(r_0) = r_0^{\alpha_k - 1}(1-r_0)^{\sum_{i=1}^{k}\beta_i - 1} \Big/ B(\alpha_k, \sum_{i=1}^{k}\beta_i) .$$

When the parameters do not follow this pattern we shall show that inducing them to do so gives good approximations.

The second method sets the variance of the system equal to the variance of a single beta distribution. For the case of beta component posteriors the variance of series system reliability is

$$V_s = \frac{\rho^2}{n}\sum_i \frac{(1-\rho_i)}{\lambda_i \rho_i} + O(n^{-2}) , \quad \text{where } \rho_i = \alpha_i/(\alpha_i+\beta_i) ,$$

$$n_i = \alpha_i + \beta_i = \lambda_i n \quad \text{and} \quad n = \min n_i .$$

Let

$$V_c = \frac{\rho(1-\rho)}{\hat{n}} + O(\hat{n}^{-2}) ,$$

where $\rho = \alpha/(\alpha+\beta)$ and $\hat{n} = \alpha + \beta$, be the variance of a single beta distribution. Then to obtain the value of $\hat{n} = \alpha + \beta$ for the equivalent component, set

$$\frac{\rho(1-\rho)}{\hat{n}} + O(\hat{n}^{-2}) = \frac{\rho^2}{n}\sum_i \frac{(1-\rho_i)}{\lambda_i \rho_i} + O(n^{-2}) .$$

Thus

$$\hat{n} \cong \frac{n(1-\rho)}{\rho} \left(\sum_i \frac{(1-\rho_i)}{\lambda_i \rho_i} \right)^{-1},$$

$$\hat{\alpha} = \hat{n}\rho , \tag{3.2.4}$$

$$\hat{\beta} = \hat{n}(1-\rho)$$

and $$f_o(r_o) = \frac{r_o^{\alpha-1}(1-r_o)^{\beta-1}}{B(\alpha,\beta)} .$$

This method can be used for systems with general structure, (see Section 3.4).

(i) Series Systems Without Repetitions

If some or all of the constraints $\alpha_i = \alpha_{i+1} + \beta_{i+1}$ do not obtain, then a beta approximation for the posterior distribution of R_o can be obtained by inducing the appropriate constraints under certain rules which appear to give the best approximations.

The Rules

a) No α_i or $\alpha_i + \beta_i$ can be increased but they can be decreased.

b) If α_i is decreased to a value c, say, then $\alpha_i + \beta_i$ is reduced to the value $c(\alpha_i+\beta_i)/\alpha_i$ and β_i becomes $c\beta_i/\alpha_i$.

c) If $(\alpha_i+\beta_i)$ is decreased to a value d then α_i is decreased to $d\alpha_i/(\alpha_i+\beta_i)$ and β_i becomes

$d\beta_i/(\alpha_i+\beta_i)$.

Operations to induce the constraints can be carried out in any order and will give the same system result.

Some Examples:-

1. Suppose that

$\alpha_1 = 100,\ \beta_1 = 2,\ n_1 = 102$;

$\alpha_2 = 80,\ \beta_2 = 2,\ n_2 = 82$;

$\alpha_3 = 70,\ \beta_3 = 1,\ n_3 = 71$;

$\alpha_4 = 60,\ \beta_4 = 1,\ n_4 = 61$.

In this example $\alpha_i \geqq \alpha_{i+1} + \beta_{i+1}$, $i = 1, 2, 3, 4$.

In the first method, starting with components 3 and 4, reduce $\alpha_3 = 70$ to the value $\alpha_4 + \beta_4 = 61$. Using the rule (b), we get the new

$\alpha_3 + \beta_3$ as $\alpha_3' + \beta_3' = \dfrac{71 \times 61}{70}$ and $\alpha_3' = \alpha_4 = 60$.

In the same way with component 2 and the new component, $\alpha_2 > \alpha_3' + \beta_3'$, $\alpha_2 = 80$ is reduced to the value $\alpha_3' + \beta_3' = \dfrac{71 \times 61}{70}$. The new

$\alpha_2' + \beta_2' = \dfrac{82 \times 71 \times 61}{70 \times 80}$ and $\alpha_2' = 60$. Also using the new component

with component 1, we get $\alpha_s + \beta_s = \dfrac{(\alpha_1+\beta_1)(\alpha_2'+\beta_2')}{\alpha_1}$, $\alpha_s = \alpha_K$,

$$\alpha_s + \beta_s = \frac{102 \times 82 \times 71 \times 61}{100 \times 80 \times 70} = 64.6866 ,$$

$\alpha_s = 60$ then $\beta_s = 4.6866$.

Thus $$f_o(r_o) = \frac{r_o^{59}(1-r_o)^{3.6866}}{B(60,4.6866)} .$$

Using the second method, we get

$$\alpha_s = 77.549 ,$$

$$\beta_s = 6.057 ,$$

and

$$f_o(r_o) = \frac{r_o^{76.549}(1-r_o)^{5.057}}{B(77.549,6.057)} .$$

The lower limits for system reliability are

	0.90	0.95	0.975
1st*	0.885	0.8685	0.8536
2nd	0.8895	0.87497	0.8620
Exact	0.889	0.8748	0.8616

* 1st: Using first method ; 2nd: Using the second method .
The approximate values were obtained by interpolation in Table(16) of Biometrika tables vol. 1

2. Let $\alpha_1 = 4$, $\beta_1 = 2$

$\alpha_2 = 3$, $\beta_2 = 2$

$\alpha_3 = 2$, $\beta_3 = 2$.

In this example $\alpha_i \leqq \alpha_{i+1} + \beta_{i+1}$ $i = 1, 2, 3$.

In this case, using the first method we reduce n_{i+1} to α_i, as in rule (c). The equivalent values of α and β are

$$\alpha_s = 1.2 , \qquad \beta_s = 4.8 .$$

And the second method gives values

$$\alpha_s = 1.714 , \quad \beta_s = 6.856 .$$

The lower limits of the system reliability are

	0.90	0.95	0.975
1st	0.03565	0.02015	0.01485
2nd	0.0535	0.034602	0.02283
Exact	0.0535	0.0352	0.0237

3. Suppose that

$\alpha_1 = 11$, $\beta_1 = 1$

$\alpha_2 = 6$, $\beta_2 = 1$

$\alpha_3 = 6$, $\beta_3 = 1$.

In this example $\alpha_1 > \alpha_2 + \beta_2 = \alpha_3 + \beta_3$, but

$\alpha_1 \neq \alpha_2 + \beta_2$ and $\alpha_2 \neq \alpha_3 + \beta_3$.

In the first method, starting with components 2 and 3, reduce $\alpha_3 + \beta_3 = 7$ to the value $\alpha_2 = 6$. By using rule (c) the new α_3 is

$\alpha_3' = \dfrac{6 \times 6}{7} = 5.143$ and the new β_3 is $\beta_3' = \dfrac{6 \times 1}{7} = 0.857$.

The equivalent single test result for components 2 and 3 now

$$\alpha = \alpha_3' = 5.143 \text{ , } \beta = \beta_3 + \beta_3' = 1.857 \text{ and } n = \alpha + \beta = 7 \text{ .}$$

The new equivalent component with component 1, reduces $\alpha_1 = 11$ to the value $\alpha + \beta = 7$ then, from rule (b),

$$n_1' = \frac{7 \times 12}{11} = 7.636 \quad \text{and} \quad \beta_1' = \frac{7 \times 1}{11} = 0.636 \text{ .}$$

The equivalent system test result for component 1, 2 and 3 is

$$\alpha_s = \alpha_3' = 5.143 \ , \quad \beta_s = \beta_1' + \beta_2 + \beta_3' = 2.493 \ .$$

Using the second method we get

$$\alpha_s = 6.0343 \quad , \quad \beta_s = 2.9257 \quad .$$

Then the lower limits for system reliability are

	0.90	0.95	0.975
1st	0.452	0.385	0.3303
2nd	0.469	0.4021	0.3558
Exact	0.47	0.404	0.351

(ii) Series Systems With Repetitions

Let the k components of a series system have independent identical beta distributions with parameters α, β. Then the system reliability is R^k. If there are k such repetitions in series then each component is assigned the parameters α/k, β/k. The same rules as before are used to determine system equivalent parameters, whence limits for system reliability can be obtained. This is best illustrated by an example.

Example

Let $\alpha = 30 \quad , \quad \beta = 2$

If $k = 2$ (the probability limit for R^2)

$\frac{\alpha}{2} = 15 \quad , \quad \frac{\beta}{2} = 1 \quad ; \quad \frac{n}{2} = 16$ •

Following the rules of operation for the first method, given before, we obtain for the system

$$\alpha_s = 14.0625 \quad , \quad \beta_s = 1.9375 \quad .$$

Using the second method we get

$$\alpha_s = 14.53125 \quad , \quad \beta_s = 2.01415 \; .$$

The lower limits for the system reliability are

	0.90	0.95	0.975
1st	0.7701	0.72651	0.6866
2nd	0.7704	0.72806	0.6889
Exact	0.775	0.733	0.693

<u>If $k = 4$</u> (the probability limit for R^4)

$$\frac{\alpha}{4} = 7.5 \quad , \quad \frac{\beta}{4} = 0.5 \quad ; \quad \frac{n}{4} = 8 \; .$$

When each component has these parameters the equivalent system parameters are $\alpha_s = 6.1798$, $\beta_s = 1.8202$ and when we use the second method we get $\alpha_s = 6.8257$, $\beta_s = 2.01044$. Then the lower limits for system reliability are

	0.90	0.95	0.975
1st	0.5769	0.5083	0.4493
2nd	0.5852	0.5201	0.464
Exact	0.60	0.5367	0.4815

Both cases show reasonable accuracy and both are slightly conservative. The second method appears to be consistently better than the first method.

(iii) Series Systems with Repetitions in Some Components and Without in Others

Suppose that three sets of parameters are

$\alpha_1 = 30$, $\beta_1 = 2$

$\alpha_2 = 19$, $\beta_2 = 1$

$\alpha_3 = 18$, $\beta_3 = 2$,

with repetition in the first set. For R_1^2 the equivalent parameters are as for the case k = 2 with repetitions. The $\alpha_s = 14.0625$, $\beta_s = 4.651$. Using the second method we obtain $\alpha_s = 15.0441$, $\beta_s = 4.9756$.

The approximate lower limits are

	0.90	0.95	0.975
1st	0.625 (0.8968)	0.5842 (0.9498)	0.5488 (0.9741)
2nd	0.6241 (0.9047)	0.5826 (0.9500)	0.54604 (0.9750)

The values within brackets are the achieved probability levels obtained by simulating the posterior distribution of system reliability.

(iv) Series Systems With Mixed Test Data

If some components have gamma posteriors, with integer indices, we use the relation between gamma and beta variates, as in Chapter 2, to transform all the system posteriors into beta and use the same rules as before to obtain the equivalent system test result and then determine the lower limits for the system reliability. For example, suppose

$$\alpha_1 = 11 \quad , \beta_1 = 1$$

$$\alpha_2 = 20 \quad , \beta_2 = 2$$

$$\tau_1 = 40 \quad , \eta_1 = 2 \cdot$$

In this example we find the third component has a gamma posterior with parameters τ_1 and η_1. From Chapter 2 we can substitute this component by two components in series with a beta posterior with $\alpha = 40$, $\beta = 1$ for each component. Then we deal with a system of four components, each one has a beta posterior, as:

$$\alpha_1 = 11 \quad , \quad \beta_1 = 1$$

$$\alpha_2 = 20 \quad , \quad \beta_2 = 2$$

$$\alpha_3 = 40 \quad , \quad \beta_3 = 1$$

$$\alpha_4 = 40 \quad , \quad \beta_4 = 1 \cdot$$

Using the first method to obtain the equivalent system parameters, we get $\alpha_s = 11$, $\beta_s = 2.8683$. Also from the second method, we obtain $\alpha_s = 15.503$, $\beta_s = 4.04241$.

Then the lower limits for system reliability are

	0.90	0.95	0.975
1st	0.6491	0.5987	0.5540
2nd	0.6704	0.6288	0.5919
Exact	0.6727	0.6287	0.5887

3.3 Parallel Systems

We proceed in the same way as for series systems, but we deal with unreliability instead of reliability, i.e. we use the formula (3.2.1), replacing r_i by $Q_i = 1 - r_i$ to use the first method and ρ_i by $Q_i = 1 - \rho_i$ to use the second method.

In parallel structures we first investigate the case when there are no repetitions. In the case of repetitions, or mixed test data, the new system will be parallel and we use in this case the same idea as for series systems but changed into a complex system and this will be dealt with later.

In the same way there are some rules to give the best approximations when we use the first method, and these rules are:

(a) the equivalent system $\beta_i' = \min (\beta_i)$

(b) if $(\alpha_i + \beta_i)$ is decreased to a value d then $\alpha_i = d - \beta_i'$

(c) if $(\alpha_i + \beta_i)$ is increased to a value c then $\alpha_i = c - \beta_i'$

Examples

1. Let

$$\beta_1 = 7 , \quad \alpha_1 = 1 \quad ; \quad n_1 = 8$$

$$\beta_2 = 5 , \quad \alpha_2 = 1 \quad ; \quad n_2 = 6$$

$$\beta_3 = 3 , \quad \alpha_3 = 1 \quad ; \quad n_3 = 4$$

$$\beta_4 = 1 , \quad \alpha_4 = 1 \quad ; \quad n_4 = 2$$

In this example $\beta_i > \alpha_{i+1} + \beta_{i+1}$ and $\beta_i > \beta_{i+1}$.

Using the first method, we find $\beta_s = \min (\beta_i) = 1$ and from rule (b)

$n_s = 3.6572$ then $\alpha_s = 2.6572$ which is the equivalent system test result. The second method gives the values $\alpha_s = 3.0426$, $\beta_s = 1.1451$.

The lower limits for system reliability are

	0.90	0.95	0.975
1st	0.41887 (0.909)	0.32267 (0.964)	0.24869 (0.986)
2nd	0.44378 (0.899)	0.35212 (0.951)	0.2797 (0.973)

In all the examples, as above, the values within brackets are the achieved probability levels.

2. Suppose that

$\beta_1 = 6$, $\alpha_1 = 2$; $n_1 = 8$

$\beta_2 = 5$, $\alpha_2 = 2$; $n_2 = 7$

$\beta_3 = 4$, $\alpha_3 = 2$; $n_3 = 6$

$\beta_4 = 3$, $\alpha_4 = 2$; $n_4 = 5$

In this example $\beta_i > \beta_{i+1}$ and $\beta_i < \beta_{i+1} + \alpha_{i+1}$.

As in the first example $\beta_s = \min(\beta_i) = 3$ and, from rule (c), n_s increases to the value 14 then $\alpha_s = n_s - \beta_s = 11$. For the same example with the second method we get $\alpha_s = 9.1321$, $\beta_s = 2.4906$.

Then the lower limits for the system reliability are

	0.90	0.95	0.975
1st	0.64022 (0.890)	0.5899 (0.946)	0.54553 (0.962)
2nd	0.62582 (0.909)	0.5699 (0.956)	0.52088 (0.976)

3. Let

$\beta_1 = 2$, $\alpha_1 = 6$; $n_1 = 8$

$\beta_2 = 3$, $\alpha_2 = 4$; $n_2 = 7$

$\beta_3 = 4$, $\alpha_3 = 2$; $n_3 = 6$

$\beta_4 = 4$, $\alpha_4 = 1$; $n_4 = 5$

In this example $\beta_i < \beta_{i+1}$.

Then from the first method $\beta_s = \min(\beta_i) = 2$ and using rule (c) $n_s = 35$ then $\alpha_s = n_s - \beta_s = 33$. Also the value of α_s, β_s from the second method are $\alpha_s = 22.2624$, $\beta_s = 1.3492$.

The lower limits for system reliability are

	0.90	0.95	0.975
1st	0.88598 (0.876)	0.862777 (0.929)	0.840883 (0.960)
2nd	0.87675 (0.911)	0.84705 (0.953)	0.818748 (0.973)

3.4 Some Complex Systems

In this Section we study three cases, a Series-Parallel system, a Parallel-Series system and a Quorum Structure.

(i) Series-Parallel System

Consider a series-parallel system with two components linked serially to two components of the same kind in parallel. The posterior distributions of the reliability of components are beta with $\alpha_1 = 6$, $\beta_1 = 1$; $\alpha_2 = 4$, $\beta_2 = 1$ for the serially linked components and $\alpha_3 = 3$, $\beta_3 = 1$; $\alpha_4 = 2$, $\beta_4 = 1$ for the parallel linked components. Using the first method, the system is divided into two subsystems; one is series and the other is parallel. Then we follow the same methods as before for series and parallel systems. So we get

$\alpha_1' = 4 \quad \beta_1' = 1.8333$ for the series part and

$\alpha_2' = 11 \quad \beta_2' = 1$ for the parallel part.

Then we consider them as a series system with two components. The equivalent system test result is $\alpha_s = 4$, $\beta_s = 2.36364$. The equivalent system test result using the second method is $\alpha_s = 4.3435$, $\beta_s = 2.5669$.

The lower limits for system reliability are

	0.90	0.95	0.975
1st	0.38245 (0.909)	0.31351 (0.956)	0.258671 (0.982)
2nd	0.39112 (0.899)	0.32451 (0.951)	0.27107 (0.976)

(ii) Parallel-Series System

In the same way, suppose a system contains two components in series connected in parallel with another two components in series. The posterior distribution of the reliability of components are beta with $\alpha_1 = 5$, $\beta_1 = 2$; $\alpha_2 = 3$, $\beta_2 = 1$ for the first two in series and $\alpha_3 = 3$, $\beta_3 = 1$; $\alpha_4 = 1$, $\beta_4 = 1$ for the other two. As before we first get the equivalent single test result for the subsystems in series and then we treat them as a parallel system with two components.

Using the first method we get $\alpha_S = 3.975$, $\beta_S = 1.625$; from the second method we obtain $\alpha_S = 4.3734$, $\beta_S = 1.7879$.

Then the lower limits for system reliability are

	0.90	0.95	0.975
1st	0.459514 (0.912)	0.38039 (0.962)	0.316074 (0.980)
2nd	0.4712 (0.904)	0.39544 (0.953)	0.3333 (0.978)

(iii) Quorum Structure

In this system the first method is not applicable, so we use only the second method.

Let $\alpha_1 = \alpha_2 = \alpha_3 = 8$, $\beta_1 = \beta_2 = \beta_3 = 2$ be the parameters of a 2/3 Quorum system. As in Chapter 2 the system reliability function is

$$\psi(\rho) = \prod_{i=1}^{3} \rho_i + \sum_{i=1}^{3} \prod_{j\neq i} \rho_j (1-\rho_i) \cdot$$

Then we get $\alpha_s = 16.9867$, $\beta_s = 1.9717$.

The lower limits for system reliability are

0.90	0.95	0.975
0.7993 (0.910)	0.7608 (0.953)	0.7252 (0.978)

Let $\alpha_1 = \alpha_2 = \alpha_3 = 9$, $\beta_1 = \beta_2 = \beta_3 = 1$ be a 2/3 Quorum with high reliability. In this example we obtain the equivalent system test result as $\alpha_s = 30.24$, $\beta_s = 0.871$, then the lower limits for system reliability are

0.90	0.95	0.975
0.934 (0.907)	0.9137 (0.956)	0.8937 (0.971)

3.5 Remarks

As we see the second method is preferable to the first method for two reasons. The first reason is that results are more accurate than the first method and secondly, we can use it with any structure.

Although no great accuracy can be expected from such a procedure, it is quick and potentially useful for series, parallel and complex systems. One way of regarding the first procedure is that, by inducing the constraints, we are extracting information from the component posteriors which is relevant to the system reliability.

C H A P T E R 4

APPROXIMATE BAYESIAN ESTIMATES FOR THE WEIBULL RELIABILITY FUNCTION AND HAZARD RATE

4.1 Introduction

Suppose that censored life data is available from a Weibull distribution. Using a given prior for the parameters of this distribution it is of interest to obtain Bayes' estimates for the reliability function and for the hazard rate. Computation of these estimates can be difficult and we examine the accuracy of a general approximate method due to Lindley (1980). The development of this expansion requires the determination of maximum likelihood estimates and, in this connection, we considered a widely employed iterative procedure for censored Weibull data given by Cohen (1965). However, we have chosen to present a new, but equally effective, procedure which has the added advantage of reducing the number of terms in Lindley's expansion. This is achieved by first transforming to the log Weibull (extreme value) distribution and then making use of its centre of location.

In Section 2 Lindley's approximate method is described in a general context. In Section 3 Cohen's method of maximum likelihood is also described. Then, using results from Section 3, we specialise the general expansion to obtain approximate Bayes' estimates for the Weibull reliability function and for the hazard rate in Section 4.

The accuracy of the expansion is assessed by comparing results with exact values obtained using nine point extended Gauss-Laguerre quadrature.

4.2 Lindley's Expansion

From Section 1.2 (d) Lindley's expansion, to $O(n^{-1})$ is

$$\int w(\theta)\ \exp(L(\theta))d\theta \Big/ \int v(\theta)\ \exp(L(\theta))d\theta \ \sim$$

$$u\{1 + \frac{1}{2}\sum_{i}\sum_{j}(W_{ij} - V_{ij})\sigma_{ij} + \frac{1}{2}\sum_{i}\sum_{j}\sum_{k}\sum_{\ell}L_{ijk}(W_{\ell} - V_{\ell})\sigma_{ij}\sigma_{k\ell}\}\ . \qquad (4.2.1)$$

In the above formula $L_{ijk} = \dfrac{\partial^3 L}{\partial\theta_i\partial\theta_j\partial\theta_k}$, evaluated at $\hat{\theta}$, and θ_i stands for $(\theta_i - \hat{\theta}_i)$ and the element $E(\theta_i\theta_j)$ in the dispersion matrix is written as σ_{ij} . Also $u = \frac{w}{v}$ is evaluated at $\hat{\theta}$ and $W_i = \frac{w_i}{w}$, $V_i = \frac{v_i}{v}$.

Lindley gives an equivalent form for (4.2.1) by writing $v(\theta)$ as $\exp(\rho(\theta))$. In (4.2.1) the first term is $O(1)$ and is the estimate obtained on replacing parameters in $u(\theta)$ by maximum likelihood estimates. The second and third terms are both $O(n^{-1})$. In our applications $u(\theta)$ will be the reliability function and the hazard rate.

In the next Section we shall calculate the maximum likelihood estimates of the parameters of Weibull distribution.

4.3 Maximum Likelihood Estimation

The iterative procedure given in this Section is an alternative to Cohen's method and has the virtue that $\sigma_{ij} (i \neq j)$ is zero, thus reducing the number of terms in Lindley's expansion.

Consider the Weibull probability density function in the form

$$f(t) = (p/\theta)\, t^{P-1} e^{-t^P/\theta} \quad , \quad t > 0 \ , \ \theta > 0 \quad , \quad P > 0 . \qquad (4.3.1)$$

The simultaneous variate and parameter transformations to $y = \ell n t$, $\beta = P^{-1}$, $\alpha = P^{-1} \ell n \theta$ yield the log Weibull density

$$g(y) = \beta^{-1} \exp(z) \exp(-\exp(z)) \quad , \qquad (4.3.2)$$

where $z = (y-\alpha)/\beta$, $-\infty < y < \infty$, $-\infty < \alpha < \infty$, $\beta > 0$.

Following Fisher (1921a), to solve the problem of finding efficient estimators of location and scale parameters, let α, β change to α^*, β where $\alpha^* = \alpha + \beta c$ and c is a value to be determined. Then $z = (y-\alpha^*)/\beta + c$, $\frac{\partial z}{\partial \alpha^*} = \frac{\partial z}{\partial \alpha} = -\beta^{-1}$ and $\frac{\partial z}{\partial \beta} = -\beta^{-1}(z-c)$. When c is the origin which makes the estimators asymptotaclly uncorrelated , it is called the "Centre of Location" of the distribution. The development will be illustrated for type II censored data but the method generalises easily to any data set consisting of complete and incomplete lifetimes. Suppose that for type II (r-out-of-n) censoring $t_{(1)}, t_{(2)}, \ldots , t_{(r)}$ are the ordered complete lives so that there are (n-r) incomplete lives, all having the same value $t_{(r)}$. Under the log Weibull model, with

$y_i = \ell n t_i$ and $z_i = (y_i - \alpha_i^*)/\beta_i + c$, the log likelihood is

$$L(\alpha^*,\beta) = -r\ell n\beta + \sum_{i=1}^{r} z_i - \{\sum_{i=1}^{r} e^{z_i} + (n-r)e^{z_r}\} .$$

It is convenient to use the simplified notations

$$S_0 = \sum_{i=1}^{r} z_i \quad , \quad T_s = \sum_{i=1}^{r} z_i^s e^{z_i} + (n-r)z_r^s e^{z_r} \quad , \quad s = 0,1,2,3, \ldots .$$

Then

$$\frac{\partial L}{\partial \alpha^*} = \frac{\partial L}{\partial \alpha} = (T_0 - r)/\beta \quad ,$$

and

$$\frac{\partial L}{\partial \beta} = \{-r(1-c) - S_0 + T_1 - cT_0\}/\beta .$$

Maximum likelihood estimates of α^*, β(or α,β) are obtained as the simultaneous solution of $\frac{\partial L}{\partial \alpha} = 0$ and $\frac{\partial L}{\partial \beta} = 0$. For the iterative procedure we require the second partial derivatives of the log likelihood and, noting that partial differentiation with respect to α^* and α is the same, we have

$$\frac{\partial^2 L}{\partial \alpha^2} = -T_0/\beta^2 ,$$

$$\frac{\partial^2 L}{\partial \beta^2} = \{r(1-2c) + 2S_0 + c(2-c)T_0 \mp 2(1-c)T_1 - T_2\}/\beta^2 ,$$

$$\frac{\partial^2 L}{\partial \alpha \partial \beta} = \{r - (1-c)T_0 - T_1\}/\beta^2 .$$

Suppose that approximate values α_1, β_1 for the maximum likelihood estimates have been obtained by a graphical or other procedure. Then we proceed iteratively as follows:

Determine the value of $c = c_1$, the origin which makes $\frac{\partial^2 L}{\partial\alpha_1 \partial\beta_1} = 0$. Thus,

$$c_1 = (T_1 + T_0 - r)/T_0 \quad , \tag{4.3.3}$$

where T_0 and T_1 are evaluated using α_1, β_1 . Then, since

$$\frac{\partial^2 L}{\partial\alpha_1 \partial\beta_1} = 0, \text{ we have by Newton-Raphson}$$

$$\alpha_2 = \alpha_1 - \left(\frac{\partial L}{\partial\alpha_1} \right) / \left(\frac{\partial^2 L}{\partial\alpha_1{}^2} \right) , \tag{4.3.4}$$

and

$$\beta_2 = \beta_1 - \left(\frac{\partial L}{\partial\beta_1} \right) / \left(\frac{\partial^2 L}{\partial\beta_1{}^2} \right) . \tag{4.3.5}$$

In these equations one can conveniently approximate $\frac{\partial^2 L}{\partial\alpha_1{}^2}$ by $\frac{-r}{\beta_1{}^2}$, the expected value of $\frac{\partial^2 L}{\partial\alpha_1{}^2}$, and $\frac{\partial^2 L}{\partial\beta_1{}^2}$ by

$$-\{r(1-c_1^2) + T_2\}/\beta_1^2 \quad .$$

Equations (4.3.4) and (4.3.5) then become

$$\alpha_2 = \alpha_1 - \beta_1(1-T_0/r)$$

and

$$\beta_2 = \beta_1\{1 - \frac{r(1-c_1)+S_0+cT_0-T_1}{r(1-c_1^2) + T_2}\} .$$

Having calculated α_2, β_2, c_1 is recalculated to give a value of c_2 and the procedure is repeated until the desired accuracy is achieved.

A random sample of size 10, generated from a Weibull distribution (4.3.1), with $p = 2$ and $\theta = 4$, gave the ordered failure times:

0.2127 ,	0.3423 ,	0.4240 ,	0.6095 ,	1.0159 ,
1.1441 ,	1.3933 ,	1.4006 ,	1.6639 ,	1.7492 .

The first case studied treats the data as censored at the failure time 1.0159. Thus $r = 5$ and $n = 10$. The second case censors the data at the failure time 1.3933, i.e. $r = 7$ and $n = 10$. For the third case we use the complete sample so that $r = n = 10$. For the censored data the maximum likelihood estimates are $\hat{\alpha} = 0.251781$ $\hat{\beta} = 0.684401$, or $\hat{P} = 1.1848201$ and $\hat{\theta} = 1.3475865$, for $r = 5$. For $r = 7$, $\hat{\alpha} = 0.22833413$,$\hat{\beta} = 0.65612651$, or $\hat{P} = 1.5240963$ and $\hat{\theta} = 1.4162523$.

Also for the complete sample $\hat{\alpha} = 0.115197$ and $\hat{\beta} = 0.51995$, or $\hat{P} = 1.9232619$ and $\hat{\theta} = 1.2480146$. In these cases this accuracy was achieved after five iterations using starting values $\alpha_1 = 0.15$, $\beta_1 = 0.80$, which were obtained from a hazard plot.

Another random sample of size 20 generated from the same distribution with the same parameters, gave the first 10 ordered failure times:

$$\begin{array}{ccccc} 0.3428 , & 0.4495 , & 1.2237 , & 1.2386 , & 1.2851 , \\ 1.2898 , & 1.3088 , & 1.517 , & 1.6821 , & 1.6824 . \end{array}$$

In this case we study only the censored data when $r = 10$ and $n = 20$ to see the results when the sample size is increased. Now $\hat{\alpha} = 0.68367441$ and $\hat{\beta} = 0.38809799$, or $\hat{P} = 2.5766688$ and $\hat{\theta} = 5.8217594$.

In what follows we associate the subscript 1 with α and subscript 2 with β. The third partial derivatives evaluated at $\hat{\alpha},\hat{\beta}$ are required and these are

$$L_{111} = r/\hat{\beta}^3 , \quad L_{112} = 2r/\hat{\beta}^3 , \quad L_{122} = (T_2 - r\hat{c}^2)/\hat{\beta}^3$$

and

$$L_{222} = \{2r(1-\hat{c})(2+2\hat{c}-\hat{c}^2) + 3(1-\hat{c})T_2 + T_3\}/\hat{\beta}^3 .$$

4.4 Estimation of the Weibull Reliability Function and Hazard Rate

4.4.1 Prior Distribution

Working with formula (4.3.2) the Bayesian uses a prior which expresses his beliefs about the parameters α and β but here, largely for convenience in illustrating the development of the expansion and also for the computation of exact values, we use Jeffrey's (1961) prior. This prior is proportional to the square root of the determinant of I, the information matrix. Since z has a distribution which is parameter free we see by inspecting the second partial derivatives of the log likelihood, given in the previous Section, that

$$|I(\alpha,\beta)| = -E \begin{vmatrix} \dfrac{\partial^2 \log g(y)}{\partial\alpha^2} & \dfrac{\partial^2 \log g(y)}{\partial\alpha\partial\beta} \\ \dfrac{\partial^2 \log g(y)}{\partial\alpha\partial\beta} & \dfrac{\partial^2 \log g(y)}{\partial\beta^2} \end{vmatrix}$$

$$= \begin{vmatrix} 1/\beta^2 & 0 \\ 0 & \dfrac{(1-c_1^2) + E(T_2)}{\beta^2} \end{vmatrix}$$

$$= \frac{(1-c_1^2) + \psi'(2)}{\beta^4} \quad ,$$

where $\psi(2)$ and its derivatives are the values of di-gamma and related functions. Then we take $v(\alpha,\beta) \propto \sqrt{|I|} = \beta^{-2}$. Sinha (1983) has also considered this problem, but only for uncensored data, and he did not transform from the Weibull (4.3.1) to the log Weibull (4.3.2). He also used Jeffreys' prior for which $v(\theta,P) = P^{-1}\theta^{-1}$. Because of the invariance of Jeffreys' prior his numerical results would be exactly the same as ours for the same data sets. The disadvantage of using (4.3.1) directly is that there are several more terms in the resulting expansion due to the fact that $\sigma_{ij} \neq 0$, $i \neq j$.

4.4.2 The Reliability Function

At time t the reliability function for the Weibull distribution, as specified in (4.3.1), is $\overline{F}(t|P,\theta) = \exp(-t^P/\theta)$. The reliability function of the log Weibull distribution, from (4.3.2), is $\overline{G}(y|\alpha,\beta) = \exp[-\exp((y-\alpha)/\beta)]$ and, when $y = \ell nt$, $\beta = P^{-1}$, $\alpha = P^{-1}\ell n\theta$, we have $\overline{G}(y|\alpha,\beta) = \overline{F}(t|P,\theta)$.

If we choose u in (4.2.1) as the reliability function, then from (4.3.2) $u(\alpha,\beta) = \exp(-\exp(z))$ and $v(\alpha,\beta) = \beta^{-2}$, the prior function. Then $w(\alpha,\beta) = u(\alpha,\beta)v(\alpha,\beta) = \beta^{-2}\exp(-\exp(z))$. To use the formula (4.2.1) to evaluate the Bayes estimator of $\overline{F}(t/p,\theta)$, we require the derivatives of $w(\alpha,\beta)$ and $v(\alpha,\beta)$ as

$$w_1 = e^z/\hat{\beta} \quad , \quad w_2 = \{(z-\hat{c})e^z - 2\}/\hat{\beta} \quad ,$$

$$w_{11} = e^z(e^z-1)/\hat{\beta}^2 \quad , \quad w_{12} = -e^z(1+(z-\hat{c}))/\hat{\beta}^2 \quad ,$$

$$w_{22} = \{6(1-(z-\hat{c})e^z) + (z-\hat{c}^2)e^z(e^z-1)\}/\hat{\beta}^2 \quad ,$$

$$v_1 = v_{11} = 0 \quad , \quad v_2 = -2/\hat{\beta} \quad \text{and} \quad v_{22} = 6/\hat{\beta}^2 \quad .$$

In the determination of $\hat{\alpha},\hat{\beta}$ the final value of $\hat{c}$ (the Centre of Location) is chosen so that $L_{12}(\alpha,\beta)\Big|_{\hat{\alpha},\hat{\beta}} = 0$. It can also be chosen so that $E\{L_{12}(\alpha,\beta)\} = 0$. If it is so chosen then $\sigma_{11}^{-1} = -E\{L_{11}\}$ and $\sigma_{22}^{-1} = -E\{L_{22}\}$. σ_{11} is easily obtained and is

β^2/r, but σ_{22} requires the evaluation of di-gamma and its related functions. However, since the argument is asymptotic, expectations may be replaced by "observed" values in the terms of order n^{-1} in the expansion and the order of terms neglected will remain at $n^{-3/2}$. This means that we can write $\sigma_{11} \sim \hat{\sigma}_{11} = \hat{\beta}^2/r$ and

$\sigma_{22} \sim \hat{\sigma}_{22} = \hat{\beta}^2\{r(1-\hat{c}^2) + T_2\}^{-1}$, where T_2 is evaluated at $\hat{\alpha}$ and $\hat{\beta}$.

For the data given in Section (4.3), for the first case, when $r = 5$ and $n = 10$, we have $\hat{\alpha} = 0.251781$, $\hat{\beta} = 0.684401$ and $\hat{c} = -0.524308$. The second case, when $r = 7$ and $n = 10$, we get $\hat{\alpha} = 0.22833413$, $\hat{\beta} = 0.65612651$ and $\hat{c} = -0.10809403$. For $r = n = 10$, $\hat{\alpha} = 0.115197$, $\hat{\beta} = 0.51995$ and $\hat{c} = 0.37729563$. Finally, when $r = 10$ and $n = 20$, $\hat{\alpha} = 0.68367441$, $\hat{\beta} = 0.38809799$ and $\hat{c} = -0.5613782$. Tables (4.1) - (4.4) give exact and approximate (Lindley) results for several values of t together with the leading term in the expansion (M.L.) for the above four cases. Approximate and (M.L.) results are obtained from formula (4.2.1) and the exact result is obtained using the formula

$$\frac{\int_0^\infty (\beta^{-r-1}\exp(\sum_{i=1}^{r} y_i/\beta)/(\sum_{i=1}^{r}\exp(y_i/\beta) + (n-r)\exp(y_r/\beta) + \exp(t/\beta))^r)d\beta}{\int_0^\infty (\beta^{-r-1}\exp(\sum_{i=1}^{r} y_i/\beta)/(\sum_{i=1}^{r}\exp(y_i/\beta) + (n-r)\exp(y_r/\beta))^r)d\beta},$$

using nine Point extended Gauss- Laguerre quadrature.

Table 4.1 Censored (5,10)

t	M. L.	Lindley	Exact
0.01	0.9992	0.9962	0.9930
0.05	0.9913	0.9798	0.9765
0.10	0.9763	0.9567	0.9570
0.50	0.7777	0.7754	0.7762
1.00	0.5005	0.5310	0.5287
1.50	0.2860	0.3443	0.3386
2.00	0.1487	0.2402	0.2256
2.50	0.0713	0.1842	0.1607
3.00	0.0319	0.1422	0.1214

Table 4.2 Censored (7,10)

t	M. L.	Lindley	Exact
0.01	0.99937	0.99752	0.99647
0.05	0.99268	0.98337	0.98499
0.10	0.9791	0.96327	0.96965
0.50	0.78231	0.79098	0.79351
1.00	0.49357	0.51596	0.56805
1.50	0.26984	0.31049	0.36044
2.00	0.13123	0.18584	0.22318
2.50	0.05765	0.11763	0.14345
3.00	0.02311	0.07612	0.09746

Table 4.3 Uncensored (10,10)

t	M. L.	Lindley	Exact
0.01	0.9999	o.9995	0.9989
0.05	0.9975	0.9938	0.9927
0.10	0.9905	0.9820	0.9804
0.50	0.8096	0.7940	0.7955
1.00	0.4488	0.4666	0.4655
1.50	0.1749	0.2056	0.2050
2.00	0.0479	0.0782	0.0767
2.50	0.0093	0.0282	0.0316
3.00	0.0013	0.0084	0.0142

Table 4.4 Censored (10,20)

t	M. L.	Lindley	Exact
0.01	0.999999	0.999989	0.999953
0.05	0.999924	0.999649	0.999581
0.10	0.999545	0.998499	0.998745
0.50	0.971617	0.960731	0.973052
1.00	0.842173	0.836033	0.832931
1.50	0.613677	0.622234	0.657098
2.00	0.358901	0.380147	0.394998
2.50	0.161867	0.205521	0.198639
3.00	0.054317	0.114866	0.097522

4.4.3 The Hazard Rate

The hazard rate for the Weibull distribution (4.3.1) is $h(t|p,\theta) = pt^{P-1}/\theta$ and for the log Weibull distribution (4.3.2) it is $h(y|\alpha,\beta) = \beta^{-1}\exp((y-\alpha)/\beta)$. Under the variate and parameter transformations given in Section (4.3) the relationship between the two functions is

$$h(t|P,\theta) = e^{-y}h(y|\alpha,\beta) = t^{-1}h(y|\alpha,\beta) .$$

We approximate the log Weibull hazard rate and then convert to the Weibull hazard rate using the above relationship. As before we take the Jeffreys' prior and use $v(\alpha,\beta) = \beta^{-2}$. Then $w(\alpha,\beta) = \beta^{-3}e^{z}$ from which we obtain $w_1 = -\hat{\beta}^{-1}$, $w_{11} = \hat{\beta}^{-2}$, $w_2 = -\hat{\beta}^{-1}\{3+(z-\hat{c})\}$ and $w_{22} = \hat{\beta}^{-2}\{12+8(z-\hat{c})+(z-\hat{c})^2\}$.

Tables (4.5) - (4.8) give the exact and approximate (Lindley) values of the hazard rate for the same values of t used in tables (4.1) - (4.5) and for the same data for the type II censored data and complete sample. As before we use (4.2.1) to get approximate and (M. L.) results and the formula

$$\frac{r\int_0^\infty (\beta^{-r-2}\exp[(\sum_{i=1}^r y_i+y)/\beta]/A^r)d\beta}{t\int_0^\infty (\beta^{-r-2}\exp(\sum_{i=1}^r y_i/\beta)/A^r)d\beta} , \quad y = \ell nt \quad \text{and}$$

$A = \sum_{i=1}^r \exp(y_i/\beta) + (n-r)\exp(y_r/\beta)$, to get the exact results.

Table 4.5 Censored (5,10)

t	M. L.	Lindley	Exact
0.01	0.1210	0.4246	0.4936
0.05	0.2541	0.4539	0.4155
0.10	0.3498	0.4632	0.4296
0.50	0.7347	0.6613	0.6682
1.00	1.0114	1.0086	1.0082
1.50	1.2193	1.3805	1.3994
2.00	1.3923	1.7583	1.8537
2.50	1.5432	2.1360	2.3787
3.00	1.6785	2.5116	2.9923

Table 4.6 Censored (7,10)

t	M. L.	Lindley	Exact
0.01	0.09631	0.30121	0.32929
0.05	0.22388	0.39221	0.37929
0.10	0.32194	0.43897	0.41873
0.50	0.74836	0.70005	0.70276
1.00	1.07616	1.02787	1.02947
1.50	1.33096	1.36234	1.36192
2.00	1.54755	1.69865	1.71016
2.50	1.73954	2.03467	2.07844
3.00	1.91397	2.36939	2.46692

Table 4.7 Uncensored (10,10)

t	M. L.	Lindley	Exact
0.01	0.0219	0.0776	0.1241
0.05	0.0970	0.1936	0.2041
0.10	0.1839	0.2840	0.2805
0.50	0.8126	0.8001	0.7984
1.00	1.5411	1.4705	1.4733
1.50	2.2408	2.2303	2.2285
2.00	2.9225	3.0710	3.0728
2.50	3.5910	3.9822	4.0106
3.00	4.2494	4.9555	5.0451

Table 4.8 Censored (10,20)

t	M. L.	Lindley	Exact
0.01	0.000311	0.002339	0.012113
0.05	0.003933	0.014896	0.02345
0.10	0.011731	0.031141	0.036251
0.50	0.148382	0.167881	0.164154
1.00	0.442593	0.421436	0.422521
1.50	0.838768	0.822357	0.822465
2.00	1.320167	1.399429	1.40246
2.50	1.876826	2.169305	2.208216
3.00	2.501881	3.14285	3.293772

4.5 Remarks

The accuracy of approximation, in general, is quite good for the reliability function and for the hazard rate. It must be remembered that the method is asymptotic and that the sample sizes are small.

As we see from the results in the above tables, we find that when the sample size increases the accuracy of approximation is increased too. Also the accuracy of approximation is increased as the level of censoring is decreased.

C H A P T E R 5

TOLERANCE LIMITS FOR TRUNCATED NORMAL DISTRIBUTIONS

5.1 Introduction

This Chapter deals with certain aspects of the general problem of errors and tolerance in the design and testing of equipment. Suppose that components have been assembled in series to made a piece of equipment. It is assumed that this piece of equipment is required to operate within certain well defined limits (tolerances). Suppose that each component error is normally distributed with known tolerance. It follows that the distribution for the component errors is a truncated normal distribution.

In the following sections we shall describe the general procedure for obtaining the Saddlepoint expansion, Edgeworth expansion and Cornish and Fisher expansion for percentage point for the problem of a series system with truncated normal components.

5.2 Cumulants of the Truncated Normal Distribution

Let x be a truncated normal variate with probability density function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \Big/ \int_{\mu-\lambda\sigma}^{\mu+\lambda\sigma} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2} dy$$

$$= \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \Big/ A(\lambda) \quad , \quad \mu-\lambda\sigma \leqq x \leqq \mu+\lambda\sigma, \qquad (5.2.1)$$

where

$$A(\lambda) = \Phi(\lambda) - \Phi(-\lambda).$$

Then the moment generating function is

$$M_X(\theta) = \int_{\mu-\lambda\sigma}^{\mu+\lambda\sigma} \frac{1}{\sqrt{2\pi}\sigma} e^{\theta y} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2} dy/A(\lambda)$$

$$= e^{\mu\theta+\theta^2\sigma^2/2} \int_{\mu-\lambda\sigma}^{\mu+\lambda\sigma} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{y-\mu-\theta\sigma^2}{\sigma}\right)^2} dx/A(\lambda).$$

Let

$$z = \frac{x-\mu-\theta\sigma^2}{\sigma},$$

then

$$M_X(\theta) = \frac{e^{\mu\theta+\theta^2\sigma^2/2}}{A(\lambda)} \int_{-\lambda-\theta\sigma}^{\lambda-\theta\sigma} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$$

$$= e^{\mu\theta+\theta^2\sigma^2/2} A(\lambda,\theta)/A(\lambda), \tag{5.2.2}$$

where

$$A(\lambda,\theta) = \Phi(\lambda-\theta\sigma) - \Phi(-\lambda-\theta\sigma) .$$

The cumulant generating function is

$$K_X(\theta) = \mu\theta + \theta^2\sigma^2/2 + \log A(\lambda,\theta) - \log A(\lambda) , \qquad (5.2.3)$$

and the first two derivatives are

$$\left.\frac{\partial K_X(\theta)}{\partial\theta}\right|_{\theta=0} = \mu + \frac{A'(\lambda)}{A(\lambda)}$$

and

$$\left.\frac{\partial^2 K_X(\theta)}{\partial\theta^2}\right|_{\theta=0} = \sigma^2 + \frac{A''(\lambda)}{A(\lambda)} - \frac{A'^2(\lambda)}{A^2(\lambda)} . \qquad (5.2.4)$$

As we see from before,

$$A(\lambda,\theta) = \Phi(\lambda-\theta\sigma) - \Phi(-\lambda-\theta\sigma) .$$

Then

$$A'(\lambda,\theta) = -\sigma\{\phi(\lambda-\theta\sigma) - \phi(-\lambda-\theta\sigma)\} ,$$

$$A''(\lambda,\theta) = \sigma^2\{\phi'(\lambda-\theta\sigma) - \phi'(-\lambda-\theta\sigma)\} ,$$

$$\vdots$$

$$A^{(r)}(\lambda,\theta) = (-1)^r\sigma^r\{\phi^{(r-1)}(\lambda-\theta\sigma) - \phi^{(r-1)}(-\lambda-\theta\sigma)\} ,$$

and

$$\phi^{(r-1)}(\lambda-\theta\sigma) = (-1)^{r-1}H_{r-1}(\lambda-\theta\sigma)\phi(\lambda-\theta\sigma) \quad ,$$

$$\phi^{(r-1)}(-\lambda-\theta\sigma) = (-1)^{r-1}H_{r-1}(-\lambda-\theta\sigma)\phi(-\lambda-\theta\sigma) \quad ,$$

where $H_r(x)$ is Hermite Polynomial of degree r.

$$\therefore \quad A^{(r)}(\lambda,\theta) = -\sigma^r[H_{r-1}(\lambda-\theta\sigma)\phi(\lambda-\theta\sigma)-H_{r-1}(-\lambda-\theta\sigma)\phi(-\lambda-\theta\sigma)] \; .$$

When $\theta = o$, we get

$$A^{(r)}(\lambda) = -\sigma^r[H_{r-1}(\lambda)\phi(\lambda)-H_{r-1}(-\lambda)\phi(-\lambda)]$$

$$= \begin{cases} -2\sigma^r H_{r-1}(\lambda)\phi(\lambda) \quad , & \text{when r is even .} \\ 0 \quad , & \text{when r is odd .} \end{cases} \tag{5.2.5}$$

From (5.2.4) and (5.2.5) we get

$$\kappa_1(0) = \mu \quad ,$$

$$\kappa_2(0) = \sigma^2 - \frac{2\lambda\sigma^2\phi(\lambda)}{A(\lambda)}$$

$$= \sigma^2(1 - \frac{2\lambda \quad \phi(\lambda)}{A(\lambda)})$$

$$= \sigma^2\eta(\lambda), \text{ say .} \tag{5.2.6}$$

Interest is in

$$\sum_{i=1}^{n} x_i \quad \text{where } x_1, x_2, \ldots, x_n \text{ are independent truncated normals.}$$

Consider the standardised sum

$$x = \frac{\sum_{i=1}^{n}(x_i-\mu_i)}{\sqrt{\sum_{i=1}^{n}\sigma_i^2\eta(\lambda_i)}} .$$

The cumulant generating function of x is

$$K_x(\theta+u) = \frac{(\theta+u)^2 \sum_{i=1}^{n}\sigma_i^2}{2\sum_{i=1}^{n}\sigma_i^2\eta(\lambda_i)} + \sum_{i=1}^{n} \log A\left(\lambda_i, \frac{\sigma_i(\theta+u)}{\sqrt{\sum_{i=1}^{n}\sigma_i^2\eta(\lambda_i)}}\right)$$

$$- \sum_{i=1}^{n} \log A(\lambda_i) . \qquad (5.2.7)$$

Let

$$A_i = A\left(\lambda_i, \frac{\sigma_i(\theta+u)}{\sqrt{\sum_{i=1}^{n}\sigma_i^2\eta(\lambda_i)}}\right) \text{ and } \kappa_i = \left.\frac{\partial^i K_x(\theta+u)}{\partial\theta^i}\right|_{\theta=0} .$$

Then, the first four derivatives of (5.2.7) are

$$\kappa_1 = \frac{u\sum_{i=1}^{n}\sigma_i^2}{\sum_{i=1}^{n}\sigma_i^2\eta(\lambda_i)} + \sum_{i=1}^{n}\frac{A_i'}{A_i} ,$$

$$\kappa_2 = \sum_{i=1}^{n}\sigma_i^2 \Big/ \sum_{i=1}\sigma_i^2\eta(\lambda_i) + \sum_{i=1}^{n}\left[\frac{A''_i}{A_i} - \frac{A'^2_i}{A_i^{\,2}}\right] \quad ,$$

$$\kappa_3 = \sum_{i=1}^{n}\left[\frac{A_i'''}{A_i} - \frac{3A_i'A_i''}{A_i^{\,2}} + \frac{2A_i'^{\,3}}{A_i^{\,3}}\right] \quad ,$$

and finally

$$\kappa_4 = \sum_{i=1}^{n}\left[\frac{A_i^{(4)}}{A_i} - \frac{4A_i'A_i'''}{A_i^2} - \frac{3A_i''^{\,2}}{A_i^2} + \frac{12A_i'^{\,2}A_i''}{A_i^{\,3}} - \frac{6A_i'^{\,4}}{A_i^{\,4}}\right].$$

(5.2.8)

In the above

$$A_i = A(\lambda_i, \frac{\sigma_i(u+\theta)}{\sqrt{\Sigma\sigma_i^2\eta(\lambda_i)}})$$

$$= \Phi(\lambda_i - \frac{\sigma_i(u+\theta)}{\sqrt{\Sigma\sigma_i^2\eta(\lambda_i)}}) - \Phi(-\lambda_i - \frac{\sigma_i(u+\theta)}{\sqrt{\Sigma\sigma_i^2\eta(\lambda_i)}}) \quad ,$$

$$A_i' = \frac{-\sigma_i}{\sqrt{\sum_{i=1}^{n}\sigma_i^2\eta(\lambda_i)}}(\phi(\lambda_i') - \phi(-\lambda_i')) \quad ,$$

$$A_i'' = \frac{\sigma_i^2}{\sum_{i=1}^{n}\sigma^2\eta(\lambda_i)}(H_1(\lambda_i')\phi(\lambda_i') - H_1(-\lambda_i')\phi(-\lambda_i')) \quad ,$$

.

.

.

$$A_i^{(r)} = \frac{(-1)^r \sigma_i^r}{(\sum_{i=1}^{n} \sigma_i^2 \eta(\lambda_i))^{r/2}} \quad (H_{r-1}(\lambda_i')\phi(\lambda_i') - H_{r-1}(-\lambda_i')\phi(-\lambda_i')) \quad , \tag{5.2.9}$$

where

$$H_r(\lambda_i') = H_r(\lambda_i - \frac{u\sigma_i}{\sqrt{\sum_{i=1}^{n} \sigma_i^2 \eta(\lambda_i)}}) \quad ,$$

$$H_r(-\lambda_i') = H_r(-\lambda_i - \frac{u\sigma_i}{\sqrt{\sum_{i=1}^{n} \sigma_i^2 \eta(\lambda_i)}}) \quad ,$$

$$\phi(\lambda_i') = \phi(\lambda_i - \frac{u\sigma_i}{\sqrt{\sum_{i=1}^{n} \sigma_i^2 \eta(\lambda_i)}}) \quad ,$$

and $$\phi(-\lambda_i') = \phi(-\lambda_i - \frac{u\sigma_i}{\sqrt{\sum_{i=1}^{n} \sigma_i^2 \eta(\lambda_i)}}) \quad .$$

From (5.2.8) and (5.2.9) we get

$$\kappa_1 = \frac{u\sum_{i=1}^{n}\sigma_i^2}{\sum_{i=1}^{n}\sigma_i^2\eta(\lambda_i)} - \sum_{i=1}^{n} \frac{\sigma_i}{\sqrt{\Sigma\sigma_i^2\eta(\lambda_i)}} \frac{(\phi(\lambda_i') - \phi(-\lambda_i'))}{(\Phi(\lambda_i') - \Phi(-\lambda'))} \quad ,$$

$$\kappa_2 = \frac{\sum_{i=1}^{n}\sigma_i^2}{\sum_{i=1}^{n}\sigma_i^2\eta(\lambda_i)} + \sum_{i=1}^{n} \frac{\sigma_i^2}{\Sigma\sigma_i^2\eta(\lambda_i)} \{ \frac{(H_1(\lambda_i')\phi(\lambda_i') - H_1(-\lambda_i')\phi(-\lambda_i'))}{\Phi(\lambda_i') - \Phi(-\lambda_i')}$$

$$- (\frac{\phi(\lambda_i') - \phi(-\lambda_i')}{\Phi(\lambda_i') - \Phi(-\lambda_i')})^2 \} \quad ,$$

$$\kappa_3 = -\sum_{i=1}^{n} \frac{\sigma_i^3}{(\Sigma\sigma_i^2\eta(\lambda_i))^{3/2}} \left\{ \left(\frac{H_2(\lambda_i')\phi(\lambda_i') - H_2(-\lambda_i')\phi(-\lambda_i')}{\Phi(\lambda_i') - \Phi(-\lambda_i')} \right) \right.$$

$$+ 3\left(\frac{(\phi(\lambda_i') - \phi(-\lambda_i'))(H_1(\lambda_i')\phi(\lambda_i') - H_1(-\lambda_i')\phi(-\lambda_i'))}{(\Phi(\lambda_i') - \Phi(-\lambda_i'))^2} \right)$$

$$\left. - 2\left(\frac{\phi(\lambda_i') - \phi(-\lambda_i')}{\Phi(\lambda_i') - \Phi(-\lambda_i')} \right)^3 \right\} ,$$

$$\kappa_4 = \sum_{i=1}^{n} \frac{\sigma_i^4}{(\sum_{i=1}^{n}\sigma_i^2\eta(\lambda_i))^2} \left\{ \left(\frac{H_3(\lambda_i')\phi(\lambda_i') - H_3(-\lambda_i')\phi(-\lambda_i')}{\Phi(\lambda_i') - \Phi(-\lambda_i')} \right) \right.$$

$$- 4 \frac{(\phi(\lambda_i') - \phi(-\lambda_i'))(H_2(\lambda_i')\phi(\lambda_i') - H_2(-\lambda_i')\phi(-\lambda_i'))}{(\Phi(\lambda_i') - \Phi(\lambda_i'))^2}$$

$$- 3\left(\frac{H_1(\lambda_i')\phi(\lambda_i') - H_1(-\lambda_i')\phi(-\lambda_i')}{\Phi(\lambda_i') - \Phi(-\lambda_i')} \right)^2$$

$$+ 12 \frac{(\phi(\lambda_i') - \phi(-\lambda_i'))^2(H_1(\lambda_i')\phi(\lambda_i') - H_1(-\lambda_i')\phi(-\lambda_i'))}{(\Phi(\lambda_i') - \Phi(-\lambda_i'))^3}$$

$$\left. - 6\left(\frac{\phi(\lambda_i') - \phi(-\lambda_i')}{\Phi(\lambda_i') - \Phi(-\lambda_i')} \right)^4 \right\} . \qquad (5.2.10)$$

Then, the standard cumulants are

$$\ell_1(x) = 0 \quad , \qquad \ell_2(x) = 1 \quad ,$$

$$\ell_3(x) = \kappa_3 / \kappa_2^{3/2} ,$$

$$\ell_4(x) = \kappa_4/\kappa_2^2 \cdot$$

Suppose that all the sample variates have the same parameters, i.e. $\mu_1 = \mu_2 = \ldots = \mu$, $\sigma_1 = \sigma_2 = \ldots = \sigma$, and $\lambda_1 = \lambda_2 = \ldots = \lambda$. Then

$$\ell_3(x) = -\frac{1}{\sqrt{n}} \Big[\frac{(H_2(\lambda')\phi(\lambda') - H_2(-\lambda')\phi(-\lambda'))}{\Phi(\lambda') - \Phi(-\lambda')}$$

$$- 3 \frac{(\phi(\lambda') - \phi(-\lambda'))(H_1(\lambda')\phi(\lambda') - H_1(-\lambda')\phi(-\lambda'))}{(\Phi(\lambda') - \Phi(-\lambda'))^2}$$

$$+ 2\Big(\frac{\phi(\lambda') - \phi(-\lambda')}{\Phi(\lambda') - \Phi(-\lambda')} \Big)^2 \Big]$$

$$\Big/ \Big[1 + \frac{H_1(\lambda')\phi(\lambda') - H_1(-\lambda')\phi(-\lambda')}{\Phi(\lambda') - \Phi(-\lambda')} - \Big(\frac{\phi(\lambda') - \phi(-\lambda')}{\Phi(\lambda') - \Phi(-\lambda')}\Big)^2\Big]^{3/2} ,$$

$$\ell_4(x) = \frac{1}{n} \Big[\frac{H_3(\lambda')\phi(\lambda') - H_3(-\lambda')\phi(-\lambda'))}{\Phi(\lambda') - \Phi(-\lambda')}$$

$$- 4 \frac{((\phi(\lambda') - \phi(-\lambda'))(H_2(\lambda')\phi(\lambda') - H_2(-\lambda')\phi(-\lambda'))}{(\Phi(\lambda') - \Phi(-\lambda'))^2}$$

$$- 3\Big(\frac{H_1(\lambda')\phi(\lambda') - H_1(-\lambda')\phi(-\lambda')}{\Phi(\lambda') - \Phi(-\lambda')} \Big)^2$$

$$+ 12\Big(\frac{(\phi(\lambda') - \phi(-\lambda'))^2(H_1(\lambda')\phi(\lambda') - H_1(-\lambda')\phi(-\lambda'))}{(\Phi(\lambda') - \Phi(-\lambda'))^3} \Big)$$

$$- 6\left(\frac{\phi(\lambda') - \phi(-\lambda')}{\phi(\lambda') - \phi(-\lambda')} \right)^4]/[1 + \frac{H_1(\lambda')\phi(\lambda') - H_1(-\lambda')\phi(-\lambda')}{\phi(\lambda') - \phi(-\lambda')}$$

$$- \left(\frac{\phi(\lambda') - \phi(-\lambda')}{\phi(\lambda') - \phi(-\lambda')} \right)^2]^2 \; . \quad (5.2.11)$$

We can now substitute into the general results of Section 1.2 (c), we can get the approximation values to any degree of accuracy.

5.3 Numerical Example

Suppose that we have a system of two components connected together in series. Without loss of generality, let each component have a normal distribution with zero mean and unit variance. Then $x = \sqrt{n}\,\overline{x}/\sqrt{n}$, and for $\overline{x} = 0.5$, we calculate the Saddlepoint, Edgeworth and Cornish-Fisher expansions for some values of λ.

Table 5.1 Distribution Function Approximations

λ	CFI*	Edgeworth	Saddlepoint	Exact**
0.674498	0.02983	0.02982	0.0014	0.00356
1.2815602	0.14714	0.14709	0.13474	0.11648
1.6448745	0.18965	0.18962	0.18472	0.18509
1.9599645	0.21222	0.21219	0.21136	0.20912
2.5779564	0.2331	0.23306	0.23253	0.23216

* The values under CFI have been obtained by inverting the Cornish and Fisher expansion.

** The exact values have been obtained by generating 10,000 values of the distribution of x by Monte Carlo methods.

5.4 Remarks

We find from the results in the above table that Saddlepoint expansions are more accurate than inverted Cornish and Fisher and Edgeworth expansions especially in the tails.

CHAPTER 6

SUMMARY AND CONCLUSIONS

6.1 Conclusions

Quite generally, asymptotic expansions provide good approximations both for percentage points and for the distribution function. It will be noted that, in Chapter 2 and Chapter 5, the saddlepoint expansion (m) gives best results for the distribution function, especially in the tails, but it is only available for series and parallel systems.

Edgeworth (m) and (n) are also good except in the extreme tails and Edgeworth (n) can be used for all coherent systems. Also Cornish and Fisher (m) and (n) approximations for percentage points are very good. CF(n) can be used for general structures.

Although the sample is small and the degree of censoring is high, we find in Chapter 4 that the accuracy of Lindley's approximation is quite good for the reliability function and for the hazard rate. We see that as the sample size is increased and/or the degree of censoring is decreased the accuracy of approximation is increased.

In Chapter 3, simple approximate methods give better results than we expected. As we see the methods are quick and potentially useful for many structures, especially the second method which gives more accurate results and can be used for general structures.

Finally we are testing the accuracy of approximations against the exact results and results generated by Monte Carlo simulation when the exact are intractable.

6.2 Extended Research

As we see before all approximations, including Lindley's expansion, are calculated to $O(n^{-1})$ and $O(m^{-1})$ only. These approximations could usefully be extended to inverse powers more than $O(n^{-1})$ and $O(m^{-1})$.

The saddlepoint (n) method can in principle be derived as the cumulant generating function can be found. This would be worthwhile because saddlepoint (n) is more general than saddlepoint (m). Also saddlepoint (m) can be generalised to the multivariate case.

The simplification obtained by using the Centre of Location in Chapter 4 is apparent. It would be worthwhile investigating both Bayesian and classical uses for other location scale distributions.

APPENDICES

APPENDIX I

MOMENTS AND CUMULANTS OF SYSTEM RELIABILITY IN THE GENERAL CASE

For each component we consider an estimator T_i of some parameter θ_i which is related to the posterior distribution of the reliability. The moments take the following forms where n_i is the asymptotic variable. The asymptotic variable has a role akin to a sample size.

$$E_\theta(T_i - \theta_i) = u_i/n_i + 0(n_i^{-2}),$$

$$E_\theta(T_i - \theta_2)^2 = v_i/n_i + v_i^*/n_i^2 + 0(n_i^{-3}),$$

$$E_\theta(T_i - \theta_i)^3 = w_i/n_i^2 + 0(n_i^{-3}),$$

$$E_\theta(T_i - \theta_i)^4 = 3v_i^2/n_i^2 + z_i/n_i^3 + 0(n_i^{-4}),$$

$$E_\theta(T_i - \theta_i)^5 = (10v_iw_i - 15v_iu_i^2)/n_i^3 + 0(n_i^{-4}),$$

$$E_\theta(T_i - \theta_i)^6 = 15v_i^3/n_i^3 + 0(n_i^{-4}).$$

There are two points here. The first is that, if we can frame our parameters and estimators so that $u_i = 0$, the algebra is much simplified. The second is that T_i may not be directly the reliability of the i^{th} component. In general, therefore, we write system reliability as

$$R = \psi(R_1, R_2, \ldots, R_m) = \psi\{h_1(T_1), h_2(T_2), \ldots, h_m(T_m)\}.$$

Corresponding to this we may write parameters

$$\rho = \psi(\rho_1,\rho_2,. . .,\rho_m) = \psi\{h_1(\theta_1),\ h_2(\theta_2),. . .,\ h_m(\theta_m)\}\ .$$

Restricting attention to beta posteriors for component reliabilities and to gamma posteriors for component exponential failure rates we have the following.

Beta Posteriors

Let the reliability of the i^{th} component have a $\beta(\alpha_i,\ \beta_i)$ posterior. Thus the p.d.f. is

$$f(r_i) = r_i^{\alpha_i-1}\ (1 - r_i)^{\beta_i-1}/B(\alpha_i,\beta_i)$$

and we may take $f(T_i) = T_i = R_i$, $f(\theta_i) = \theta_i = \rho_i$,

where $\rho_i = E(R_i) = \alpha_i/(\alpha_i + \beta_i)$. With asymptotic variable $n_i = \alpha_i + \beta_i$ we then have $u_i = 0$, $v_i = \rho_i(1-\rho_i)$, $v_i^* = -\rho_i(1-\rho_i)$, $w_i = 2\rho_i(1-\rho_i)(1-2\rho_i)$ and $z_i = 6\rho_i(1-\rho_i)(1-6\rho_i + 6\rho_i^2)$.

Exponential Failure Rates

Let $T_i = \Lambda_i$, the exponential failure rate of the i^{th} component, have a $\Gamma(\eta_i,\ \tau_i)$ posterior.

$$\text{ie,}\quad f(\lambda_i) = \tau_i(\tau_i\lambda_i)^{\eta_i-1}\ e^{-\tau_i\lambda_i}/\Gamma(\eta_i)\ .$$

Let $\theta_i = \phi_i = \eta_i/\tau_i$ be the parameter and let $n_i = \tau_i$. Then, as shown in Chapter 2, we have

$$u_i = 0,\ v_i = \phi_i,\ v_i^* = 0,\ w_i = 2\phi_i,\ z_i = 6\phi_i.$$

also $h_i(T_i) = h_i(\Lambda_i) = e^{-\Lambda_i}$ and $h_i(\theta_i) = h_i(\phi_i) = e^{-\phi_i}$. Note that, in this case, the transformation $R_i = e^{-\Lambda_i}$ would lead to the negative log gamma distribution for component reliability but, for the moment, we prefer to work the general case in terms of the above.

In the following development of moments and cumulants we use the general notation (T,θ) for systems and (T_i,θ_i), $i = 1,2,\ldots, m$, for components, with the appropriate interpretations for beta and gamma variates.

Consider the multivariate Taylor's series expansion of T about θ given by

$$T = \theta + \sum_i (T_i-\theta_i)\psi_i + \frac{1}{2!}\sum_i\sum_j (T_i-\theta_i)(T_j-\theta_j)\psi_{ij}$$

$$+ \frac{1}{3!}\sum_i\sum_j\sum_k (T_i-\theta_i)(T_j-\theta_j)(T_k-\theta_k)\psi_{ijk} + \cdots ,$$

where $\psi_i = \dfrac{\partial\psi}{\partial\theta_i}$, $\psi_{ij} = \dfrac{\partial^2\psi}{\partial\theta_i\partial\theta_j}$, etc.

Let $n = \min n_i$ so that $n_i = n\lambda_i$, $\lambda_i \geq 1$, $i = 1,2, \ldots ,m$. Then to the required orders in inverse powers of n we have

$$\kappa_1 = M_1 = E(T-\theta) = \frac{1}{2n}\sum_i v_i\psi_{ii}/\lambda_i + O(n^{-2})$$

$$= \kappa_{11}/n + O(n^{-2}).$$

$$M_3 = E(T-\theta)^3 = n^{-2}[\textstyle\sum_i w_i \psi_i^3/\lambda_i^2$$

$$+ \tfrac{3}{2}\{3\textstyle\sum_i v_i^2\psi_i^2\psi_{ii}/\lambda_i^2 + \sum_{i\neq j}\sum v_i v_j \psi_i^2 \psi_{jj}/\lambda_i\lambda_j$$

$$+ 2\textstyle\sum_{i\neq j}\sum v_i v_j \psi_i \psi_j \psi_{ij}/\lambda_i\lambda_j\}] + O(n^{-3})$$

$$= n^{-2}\{\textstyle\sum_i w_i\psi_i^3/\lambda_i^2 + \tfrac{3}{2}(\sum_i v_i\psi_i^2/\lambda_i)(\sum_i v_i\psi_{ii}/\lambda_i)$$

$$+ 3\textstyle\sum_i\sum_j v_i v_j \psi_i \psi_j \psi_{ij}/\lambda_i\lambda_j\}$$

$$+ O(n^{-3}),$$

using the result $\sum_{i\neq j}\sum (ij) = \sum_i\sum_j (ij) - \sum_i (ii)$.

$$\therefore \quad \kappa_3 = M_3 - 3M_2M_1 + 2M_1^3$$

$$= n^{-2}\{\textstyle\sum_i w_i\psi_i^3/\lambda_i^2 + 3\sum_i\sum_j v_i v_j \psi_i\psi_j\psi_{ij}/\lambda_i\lambda_j\}$$

$$+ O(n^{-3})$$

$$= \kappa_{32}/n^2 + O(n^{-3}).$$

The algebra required for the first two terms in the expansions of κ_2 and the leading term in the expansion of κ_4 is extensive and some of the intermediate stages are reported following.

$$
\begin{aligned}
M_2 &= E(T-\theta)^2 \\
&= n^{-1}\textstyle\sum_i v_i\psi_i^2/\lambda_i \;+\; n^{-2}[\sum_i v_i^*\psi_i^2/\lambda_i^2 \\
&\quad + \textstyle\sum_i w_i\psi_i\psi_{ii}/\lambda_i^2 \\
&\quad + \tfrac{1}{3}\{3\textstyle\sum_i v_i^2\,\psi_i\psi_{iii}/\lambda_i^2 \;+\; 3\sum\sum_{i\neq j} v_iv_j\psi_i\psi_{ijj}/\lambda_i\lambda_j\} \\
&\quad + \tfrac{1}{4}\{3\textstyle\sum_i v_i^2\psi_{ii}^2/\lambda_i^2 \;+\; \sum\sum_{i\neq j} v_iv_j\psi_{ii}\psi_{jj}/\lambda_i\lambda_j \\
&\qquad + 2\textstyle\sum\sum_{i\neq j} v_iv_j\psi_{ij}^2/\lambda_i\lambda_j\}] \;+\; O(n^{-3}) \\
&= n^{-1}\textstyle\sum_i v_i\psi_i^2/\lambda_i + n^{-2}[\sum_i v_i^*\psi_i^2/\lambda_i^2 \\
&\qquad + \textstyle\sum_i w_i\psi_i\psi_{ii}/\lambda_i^2 \\
&\qquad + \textstyle\sum_i\sum_j v_iv_j\psi_i\psi_{ijj}/\lambda_i\lambda_j \\
&\qquad + \tfrac{1}{4}(\textstyle\sum_i v_i\psi_{ii}/\lambda_i)^2 \;+\; \tfrac{1}{2}\sum_i\sum_j v_iv_j\psi_{ij}^2/\lambda_i\lambda_j] \,. \\
&\qquad + O(n^{-3}).
\end{aligned}
$$

$$
\begin{aligned}
\kappa_2 &= M_2 - M_1^2 \\
&= n^{-1}\textstyle\sum_i v_i\psi_i^2/\lambda_i + n^{-2}\{\sum_i v_i^*\psi_i^2/\lambda_i^2 \\
&\qquad + \textstyle\sum_i w_i\psi_i\psi_{ii}/\lambda_i^2 \;+\; \sum_i\sum_j v_iv_j\psi_i\psi_{ijj}/\lambda_i\lambda_j
\end{aligned}
$$

$$+ \frac{1}{2} \sum_i \sum_j v_i v_j \psi_{ij}^2 / \lambda_i \lambda_j \} \quad + O(n^{-3})$$

$$= \kappa_{21}/n + \kappa_{22}/n^2 \quad + O(n^{-3})$$

$$M_4 = E(T-\theta)^4$$

$$= n^{-2}\, 3 \sum_i v_i^2 \psi_i^4 / \lambda_i^2 + n^{-3} \sum_i z_i \psi_i^4 / \lambda_i^3$$

$$+ 3 \sum_{i \neq j}\sum (v_i + v_i^*/n_i) \psi_i^2 / n_i \cdot (v_j + v_j^*/n_j) \psi_j^2 / n_j$$

$$+ n^{-3}[2\{10 \sum_i w_i v_i \psi_i^3 \psi_{ii} / \lambda_i^3 + \sum_{i \neq j}\sum w_i v_j \psi_i^3 \psi_{jj} / \lambda_i^2 \lambda_j$$

$$+ 3 \sum_{i \neq j}\sum v_i w_j \psi_i^2 \psi_j \psi_{jj} / \lambda_i \lambda_j^2$$

$$+ 6 \sum_{i \neq j}\sum w_i v_j \psi_i^2 \psi_j \psi_{ij} / \lambda_i^2 \lambda_j \}$$

$$+ \frac{2}{3}\{15 \sum_i v_i^3 \psi_i^3 \psi_{iii} / \lambda_i^3 + 9 \sum_{i \neq j}\sum v_i^2 v_j \psi_i^3 \psi_{ijj} / \lambda_i^2 \lambda_j$$

$$+ 9 \sum_{i \neq j}\sum v_i v_j^2 \psi_i^2 \psi_j \psi_{jjj} / \lambda_i \lambda_j^2$$

$$+ 27 \sum_{i \neq j}\sum v_i^2 v_j \psi_i^2 \psi_j \psi_{iij} / \lambda_i^2 \lambda_j$$

$$+ 6 \sum_{i \neq j \neq k}\sum\sum v_i v_j v_k \psi_i \psi_j \psi_k \psi_{ijk} / \lambda_i \lambda_j \lambda_k$$

$$+ 9 \sum_{i \neq j \neq k}\sum\sum v_i v_j v_k \psi_i^2 \psi_j \psi_{jkk} / \lambda_i \lambda_j \lambda_k \}$$

$$+ \frac{3}{2}\{15 \sum_i v_i^3\psi_i^2\psi_{ii}^2/\lambda_i^3 + 3 \sum_{i\neq j}\sum v_i v_j^2\psi_i^2\psi_{jj}^2/\lambda_i\lambda_j^2$$

$$+ 6 \sum_{i\neq j}\sum v_i^2 v_j\psi_i^2\psi_{ii}\psi_{jj}/\lambda_i^2\lambda_j$$

$$+ \sum_{i\neq j\neq k}\sum\sum v_i v_j v_k\psi_i^2\psi_{jj}\psi_{kk}/\lambda_i\lambda_j\lambda_k$$

$$+ 12 \sum_{i\neq j}\sum v_i^2 v_j\psi_i^2\psi_{ij}^2/\lambda_i^2\lambda_j$$

$$+ 2 \sum_{i\neq j\neq k}\sum\sum v_i v_j v_k\psi_i^2\psi_{jk}^2/\lambda_i\lambda_j\lambda_k$$

$$+ 24 \sum_{i\neq j}\sum v_i^2 v_j\psi_i\psi_j\psi_{ii}\psi_{ij}/\lambda_i^2\lambda_j$$

$$+ 4 \sum_{i\neq j\neq k}\sum\sum v_i v_j v_k\psi_i\psi_j\psi_{ij}\psi_{kk}/\lambda_i\lambda_j\lambda_k$$

$$+ 8 \sum_{i\neq j\neq k}\sum\sum v_i v_j v_k\psi_i\psi_j\psi_{ik}\psi_{jk}/\lambda_i\lambda_j\lambda_k\}]$$

$$+ O(n^{-4}).$$

On using the results $\sum_{i\neq j}\sum (i,j) = \sum_i\sum_j(i,j) - \sum_i(i,i)$

and $\sum_{i\neq j\neq k}\sum\sum (i,j,k) = \sum_i\sum_j\sum_k(i,j,k) + 2\sum_i(i,i,i)$

$$- \sum_i\sum_j (i,i,j) - \sum_i\sum_j(i,j,i) - \sum_i\sum_j (i,j,j)$$

we obtain

$$M_4 = n^{-2}\, 3(\textstyle\sum_i v_i\psi_i^2/\lambda_i)^2$$

$$+ \; n^{-3}\{\textstyle\sum_i z_i\psi_i^4/\lambda_i^3 \; + \; 6(\sum_i v_i\psi_i^2/\lambda_i)(\sum_i v_i^*\psi_i^2/\lambda_i^2)$$

$$- \; 6 \textstyle\sum_i v_i v_i^*\psi_i^4/\lambda_i^3$$

$$+ \; 2(\textstyle\sum_i w_i\psi_i^3/\lambda_i^2)(\sum_i v_i\psi_{ii}/\lambda_i) + 6(\sum_i v_i\psi_i^2/\lambda_i)(\sum_i w_i\psi_i\psi_{ii}/\lambda_i^2)$$

$$+ \; 12 \textstyle\sum_i\sum_j w_i v_j\psi_i^2\psi_j\psi_{ij}/\lambda_i^2\lambda_j$$

$$+ \; 4 \textstyle\sum_i\sum_j\sum_k v_i v_j v_k\psi_i\psi_j\psi_k\psi_{ijk}/\lambda_i\lambda_j\lambda_k$$

$$+ \; 6 \; (\textstyle\sum_i v_i\psi_i^2/\lambda_i) \; (\sum_i\sum_j v_i v_j\psi_i\psi_{ijj}/\lambda_i\lambda_j)$$

$$+ \; {}^3/_2 \; (\textstyle\sum_i v_i\psi_i^2/\lambda_i)(\sum_i v_i\psi_{ii}/\lambda_i)^2$$

$$+ \; 3(\textstyle\sum_i v_i\psi_i^2/\lambda_i)(\sum_i\sum_j v_i v_j\psi_{ij}^2/\lambda_i\lambda_j)$$

$$+ \; 6(\textstyle\sum_i v_i\psi_{ii}/\lambda_i)(\sum_i\sum_j v_i v_j\psi_i\psi_j\psi_{ij}/\lambda_i\lambda_j)$$

$$+ \; 12 \textstyle\sum_i\sum_j\sum_k v_i v_j v_k\psi_i\psi_j\psi_{ik}\psi_{jk}/\lambda_i\lambda_j\lambda_k\}$$

$$+ \; O(n^{-4}).$$

Finally $\kappa_4 = M_4 - 3M_2^2 - 4M_3M_1 + 12M_2M_1^2 - 6M_1^4$

$$= n^{-3}\{ \sum_i z_i\psi_i^4/\lambda_i^3 - 6\sum_i v_iv_i^*\psi_i^4/\lambda_i^3$$

$$+ 12\sum_i\sum_j w_iv_j\psi_i^2\psi_j\psi_{ij}/\lambda_i^2\lambda_j$$

$$+ 4\sum_i\sum_j\sum_k v_iv_j v_k\psi_i\psi_j\psi_k\psi_{ijk}/\lambda_i\lambda_j\lambda_k$$

$$+ 12\sum_i\sum_j\sum_k v_iv_jv_k\psi_i\psi_j \psi_{ik}\psi_{jk}/\lambda_i\lambda_j\lambda_k \}$$

$$+ O(n^{-4})$$

$$= \kappa_{43}/n^3 + O(n^{-4}).$$

We now have the cumulant coefficients κ_{21}; κ_{11}, κ_{32}; κ_{22}, κ_{43} for use in the asymptotic expansions.

Appendix II

The Dirichlet distribution and the product of Independent Beta Variates

The probability density function (p.d.f.) of the Dirichlet distribution is

$$\frac{\Gamma(\ell)}{\Gamma(\ell_0)\Gamma(\ell_1)\dots\Gamma(\ell_k)} r_0^{\ell_0-1} u_1^{\ell_1-1} \dots u_k^{\ell_k-1}, \; 0 \leqslant u_i \leqslant 1, \; 0 \leqslant r_0 \leqslant 1,$$

where $r_0 + \sum_{i=1}^{k} u_i = 1$ and $\ell = \ell_0 + \sum_{i=1}^{k} \ell_i$.

The marginal distributions of the U_i are beta distributions with p.d.f.'s

$$f_i(u_i) = \frac{\Gamma(\ell)}{\Gamma(\ell_i)\Gamma(\ell-\ell_i)} u_i^{\ell_i-1} (1-u_i)^{\ell-\ell_i-1}, \; 0 \leqslant u_i \leqslant 1.$$

$$= u_i^{\ell_i-1}(1-u_i)^{\ell-\ell_i-1}/B(\ell_i, \ell-\ell_i)$$

The marginal distribution for R_0 has p.d.f.

$$f_0(r_0) = \tau_0^{\ell_0-1}(1-r_0)^{\ell-\ell_0-1}/B(\ell_0, \ell-\ell_0), \; 0 \leqslant r_0 \leqslant 1.$$

Consider the transformations

$$U_1 = 1-R_1, \; U_2 = R_1(1-R_2), \; \dots, \; U_i = R_1R_2 \dots R_{i-1}(1-R_i), \; \dots$$

Then $R_0 = 1 - \sum_{i=1}^{k} U_i = \prod_{i=1}^{k} R_i$

The Jacobian for these transformations is

$J = (-1)^k \, r_1^{k-1} \, r_2^{k-2} \ldots r_{k-1}^1$ and thus the joint distribution of

$\underset{\sim}{R} = (R_1, R_2 \ldots, R_k)$ has p.d.f.

$$f(\underset{\sim}{r}) = \frac{\Gamma(\ell)}{\prod_{i=0}^{k} \Gamma(\ell_i)} \left\{ \prod_{i=1}^{k} r_i \right\}^{\ell_0 - 1} \prod_{i=1}^{k} \left\{ r_1 r_2 \ldots r_{i-1} (1-r_i) \right\}^{\ell_i - 1} |J|$$

$$= \frac{\Gamma(\ell)}{\prod_{i=0}^{k} (\ell_i)} \prod_{i=1}^{k} r_i^{\ell_0 + \sum_{j=i+1}^{k} \ell_j - 1} (1-r_i)^{\ell_i - 1}, \quad \sum_{j=k+1}^{k} \ell_j = 0.$$

$$= \prod_{i=1}^{k} \frac{r_i^{\ell_0 + \sum_{j=i+1}^{k} \ell_j - 1} (1-r_i)^{\ell_i - 1}}{B(\ell_0 + \sum_{j=i+1}^{k}, \ell_j, \ell_i)} .$$

Since $f(\underset{\sim}{r}) = \prod_{i=1}^{k} f_i(r_i)$, where

$$f_i(r_i) = \frac{r_i^{\ell_0 + \sum_{j=i+1}^{k} \ell_j - 1} (1-r_i)^{\ell_i - 1}}{B\ell_0 + \sum_{j=i+1}^{k} \ell_j, \ell_i)}, \quad i = 1, 2, \ldots, k,$$

the random variables $R_1, R_2, \ldots, R_k$ have <u>independent beta distributions with inter-related parameters</u>.

The distribution of $R_0 = \prod_{i=1}^{k} R_i$, given previously, is beta with p.d.f.

$$f_0(r_0) = \frac{r_0^{\ell_0 - 1}(1 - r_0)^{\ell - \ell_0 - 1}}{B(\ell_0, \ell - \ell_0)} \quad , \quad 0 \leqq r_0 \leqq 1 \ .$$

This is the result called upon in Chapter 3.

REFERENCES

Abdel-Wahid, A. R. and Winterbottom, A. (1983) "Approximating Posterior Distributions of System Reliability," *The Statistician*, Vol. 32, pp. 224-228.

Ad-Hoc Methodology Working Group on Nuclear Weapons Reliability Assessment (1980) "Handbook for the Calculation of Lower Statistical Confidence Bounds on System Reliability." *Maximus*, Report No. 1.

Aitchison, J. and Dunsmore, I. R. (1975) *Statistical Prediction Analysis*, Cambridge University Press.

Bache, N. (1979) "Approximate Percentage Points for the Distribution of a Product of Independent Positive Random Variables," *J. Roy. Statist. Soc. C*, Vol. 28, No. 2, pp. 158-162.

Bagelanoff, D. A. and Pierce, D. A. (1973) "Bayes-Fiducial Inference for the Weibull Distribution," *J. A. S. A.*, Vol. 68, No. 343, pp. 659-664.

Bain, L. J. and Engelhardt, M. (1981) "Simple Approximate Distributional Results for Confidence and Tolerance Limits for the Weibull Distribution Based on Maximum Likelihood Estimators," *Technometrics*, vol. 23, No. 1, pp. 15-20.

Barlow, R. and Proschan, F. (1975) *Statistical Theory of Reliability and Life Testing*, Rinehard and Winston, New York.

Barndorff-Nielson, O. and Cox, D. R. (1979) "Edgeworth and Saddlepoint Approximations with Statistical Applications," *J. Roy. Statist. Soc. B.*, vol. 41, No. 3, pp. 279-312.

Bhattacharya, R. N. and Ghosh, J. K. (1978) "On the Validity of the Formal Edgeworth Expansion," *The Annals of Statistics*, vol. 6, No. 2, pp. 434-451.

Bhattacharya, R. N. and Rao, R. R. (1976) *Normal Approximation and Asymptotic Expansions*, John Wiley and Sons, New York.

Billmann, B. R., Antle, C. E. and Bain, L. J. (1972) "Statistical Inference from Censored Weibull Samples," *Technometrics*, vol. 14, No. 4, pp. 831-840.

Buehler, R. J. (1957) "Confidence Intervals for the Product of Two Binomial Parameters," *J. A. S. A.*, vol. 52, No. 280, pp. 482-493.

Butcher, A. C., Lampkin, H. and Winterbottom, A. (1978) "TRansformations Improving Maximum Likelihood Confidence Intervals for System Reliability," *Technometrics*, vol. 20, No. 4, pp. 467-473.

Chang, E. Y. and Thompson, W. E. (1976) "Bayes Analysis of Reliability for Complex Systems," *Operations Research*, vol. 24, No. 1, pp. 156-168.

Cohen, A. C. (1965) "Maximum Likelihood Estimation in the Weibull Distribution Based on Complete and on Censored Samples," *Technometrics*, vol. 7, No. 4, pp. 579-588.

Cornish, E. A. and Fisher, R. A. (1937) "Moments and Cumulants in the Specification of Distributions." *Rev. Inst. Int. Statist.*, vol. 5, pp. 307-322.

Daniels, H. E. (1954) "Saddlepoint Approximations in Statistics," *Ann. Math. Statist.*, vol. 25, pp. 631-650.

Daniels, H. E. (1980) "Exact Saddlepoint Approximations," *Biometrika*, vol. 67, No. 1, pp. 59-63.

Daniels, H. E. (1983) "Saddlepoint Approximations for Estimating Equations," *Biometrika*, vol. 70, No. 1, pp. 89-96.

Dunsmore, I. R. (1976) "Asymptotic Predition Analysis," *Biometrika*, vol. 63, No. 3, pp. 627-630.

Easterling, R. G. (1972) "Approximate Confidence Limits for System Reliability," *J. A. S. A.*, vol. 67, No. 337, pp. 220-222.

Efrom, B. and Hinkley, D. V. (1978) "Assessing the Accuracy of the Maximum Likelihood Estimator: Observed Versus Expected Fisher Information," *Biometrika*, vol. 65, No. 3, pp. 457-487.

El-Mawaziny, A. H. and Buehler, R. J. (1967) "Confidence Limits for the Reliability of a Series System," *J. A. S. A.*, vol. 62, No. 320, pp. 1452-1459.

Erdelyi, A. (1962) *Asymptotic Expansions*, Dover Publications, Inc., U.S.A.

Fertig, K. W. (1972) "Bayesian Prior Distribution for Systems with Exponential Failure-Time Data," *Annals of Math. Statist.*, vol. 43, No. 5, pp. 1441-1448.

Fisher, R. A. (1921) "On the Mathematical Foundations of Theoretical Statistics," *Phil. Trans. Roy. Soc. A*, vol. 222, pp. 309-368.

Fisher, R. A. and Cornish, E. A. (1960) "The Percentile Points of Distributions Having Known Cumulants," *Technometrics*, vol. 2, No. 2, pp. 209-225.

Fraser, D. A. S. (1979) *Inference and Linear Models*, McGraw-Hill International Book Company, London.

Grubbs, F. E. (1971) "Approximate Fiducial Bounds for the Reliability of a Series System for which Each Component Has an Exponential Time-to-Fail Distribution," *Technometrics*, vol. 13, No. 4, pp. 865-871.

Hinkley, D. V. (1978) "Likelihood Inference about Location and Scale Parameters," *Biometrika*, vol. 65, No. 2, pp. 253-261.

Jayne, L. W. and Ross, E. W. (1972) "Probability of Equipment Failure When Tolerances are Incorrect." *Technometrics*, vol. 14, No. 4, pp. 899-909.

Johns, M. V. and Lieberman, G. J. (1966) "An Exact Asymptotically Efficient Confidence Bound for Reliability in the Case of the Weibull Distribution," *Technometrics*, vol. 8, No. 1. pp. 135-175.

Kendall, M. G. and Stuart, A. (1958) *The Advanced Theory of Statistics*, vol. 1, Charles Griffin and Company Limited, London.

Kendall, M. G. and Stuart, A. (1961) *The Acvanced Theory of Statistics*, vol. 2, Charles Griffin and Company Limited, London.

Kraemer, H. C. (1963) "One-Side Confidence Intervals for the Quality Indices of a Complex Item," *Technometrics*, vol. 5, No. 3, pp. 400-403.

Lampkin, H. and Winterbottom, A. (1983) "Approximate Bayesian Intervals for the Reliability of Series Systems from Mixed Subsystem Test Data," *Naval Research Logistics Quarterly*, vol. 30, pp. 313-317.

Lawless, J. F. (1973) "Conditional Versus Unconditional Confidence Intervals for the Parameters of the Weibull Distribution," *J. A. S. A.*, vol. 68, No. 343, pp. 665-669.

Lawless, J. F. (1978) "Confidence Interval Estimation for the Weibull and Extreme Value Distributions," *Technometrics*, vol. 20, No. 4, pp. 355-368.

Lawless, J. F. (1983) "statistical Methods in Reliability," *Technometrics*, vol. 25, No. 4, pp. 305-335.

Lieberman, G. J. and Ross, S. M. (1971) "Confidence Intervals for Independent Exponential Series Systems," *J. A. S. A.*, vol. 66, No. 336, pp. 837-840.

Lindley, D. V. (1961) "The Use of Prior Probability Distributions in Statistical Inference and Decisions," *Proc. 4th Berkeley Symp. 1*, pp. 453-468.

Lindley, D. V. (1971) *Bayesian Statistics, A Review*, Philadelphia: S.I.A.M.

Lindley, D. V. (1980) "Approximate Bayesian Methods," *Trabajos Estadistica*, vol. 31, pp. 223-237.

Madansky, A. (1965) "Approximate Confidence Limits for the Reliability of Series and Parallel Systems," *Technometrics*, vol. 7, No. 4, pp. 495-503.

Mann, N. R. (1974a) "Simplified Expressions for Obtaining Approximately Optimum System Reliability Confidence Bounds from Exponential Subsystem Data," *J. A. S. A.*, vol. 69, No. 346, pp. 492-495.

Mann, N. R. (1974b) "Approximately Optimum (Randomised and Nonrandomised) Confidence Bounds on Series and Parallel-System Reliability for Systems with Binomial Subsystem Data," *I.E.E.E. Trans. on Reliability*, vol. R. 23, No. 5, pp. 295-303.

Mann, N. R. and Grubbs, F. E. (1972) "Approximately Optimum Confidence Bounds on Series-System Reliability for Exponential Time to Fail Data," *Biometrika*, vol. 59, No. 1, pp. 191-204.

Mann, N. R. and Grubbs, F. E. (1974) "Approximately Optimum Confidence Bounds for System Reliability Based on Component Test Data," *Technometrics*, vol. 16, No. 3, pp. 335-347.

Mann, N. R., Schafer, R. E. and Singpurwalla, N. D. (1974) *Methods for Statistical Analysis of Reliability and Life Data*, John Wiley and Sons, New York.

Myhre, J. M. and Saunders, S. C. (1968a) "On Confidence Limits for the Reliability of Systems," *Annals of Meth. Statist.*, vol. 39, No. 5, pp. 1463-1472.

Myhre, J. M. and Saunders, S. C. (1968b) "Comparison of Two Methods of Obtaining Approximate Confidence Intervals for System Reliability," *Technometrics*, vol. 10, No. 1, pp. 37-49.

Myhre, J. M. and Saunders, S. C. (1971) "Approximate Confidence Limits for Complex Systems with Exponential Component Lives," *Annals of Math. Statist.*, vol. 42, No. 1, pp. 342-348 .

Ord, J. K. (1972) *Families of Frequency Distributions*, Charles Griffin and Company Limited, London.

Parker, J. B. (1972) "Bayesian Prior Distributions for Multi-Component Systems," *Naval Research Logistics Quarterly*, vol. 19, No. 3, pp. 509-515.

Patnaik, P. B. (1949) "The Non-Central X^2 and F Distributions and their Applications," *Biometrika* , vol. 36, No. 2, pp. 202-232.

Petrov, V. V. (1975) *Sums of Independent Random Variables*, Springer-Verlag Berlin Heidelberg, New York.

Robinson, J. (1982) "Saddlepoint Approximation for Permutation Tests and Confidence Intervals," *J. Roy. Statist. Soc. B*, vol. 44, No. 1, pp. 91-101.

Sarker, T. K. (1971) "An Exact Confidence Bound for the Reliability of a Series System Where Each Component Has an Exponential Time to Failure Distribution," *Technometrics*, vol. 13, No. 3, pp. 535-546.

Saxena, K. M. L. (1981) "Distribution-Free Tolerance Bounds for the Lifetime of an m-out-of-K System of Components," *Technometrics* vol. 23, No. 1, pp. 97-103.

Pearson, E. S. and Hartley, H. O. (1958) *Biometrika Tables for Statistics*, vol. 1, Cambridge University Press.

Shenton, L. R. and Bowman, K. O. (1977) *Maximum Likelihood Estimation in Small Samples*, Charles Griffin and Company Limited, London.

Sinha, S. K. (1983) "Bayes' Estimation of the Reliability Function and Hazard Rate of a Weibull Failure Time Distribution," *Technical Report*, No. 127, University of Manitoba.

Springer, M. D. and Byers, J. K. (1971) "Bayesian Confidence Limits for the Reliability of Mixed Exponential and Distribution-Free Cascade Subsystems," *I.E.E.E. Trans. on Reliability*, vol. R-20 No. 1, pp. 24-28.

Springer, M. D. and Thompson, W. E. (1966) "Bayesian Confidence Limits for the Product of N Binomial Parameters," *Biometrika*, vol. 53 No. 3, pp. 611-613.

Springer, M. D. and Thompson, W. E. (1968) "Bayesian Confidence Limits for Reliability of Redundant Systems when Tests are Terminated at First Failure," *Technometrics*, vol. 10, No. 1, pp. 29-36.

Thoman, D. R., Bain, L. J. and Antle, C. E. (1969) "Inferences on the Parameters of the Weibull Distribution," *Technometrics*, vol. 11, No. 3, pp. 445-460.

Wallace, D. L. (1958) "Asymptotic Approximations to Distributions," *Annals of Math. Statist.*, vol. 29, pp. 635-654.

Wilks, S. S. (1962) *Mathematical Statistics*, John Wiley and Sons, Inc., New York.

Winterbottom, A. (1974) "Lower Confidence Limits for Series System Reliability from Binomial Subsystem Data," *J. A. S. A.*, vol. 69, No. 347, pp. 782-788.

Winterbottom, A. (1979) "Cornish-Fisher Expansions for Confidence Limits," *J. Roy. Statist. Soc. B.*, vol. 41, No. 1, pp. 69-75.

Winterbottom, A. (1980) "Asymptotic Expansions to Improve Large Sample Confidence Intervals for System Reliability," *Biometrika*, vol. 67, No. 2, pp. 351-357.

Winterbottom, A. (1984) "The Interval Estimation of System Reliability from Component Test Data," *Operations Research*, vol. 32, No. 3, pp. 628-640.

Zimmer, W. J., Prairie, R. R. and Breipohl, A. M. (1965) "A Consideration of the Bayesian Approach in Reliability Evaluation," *I.E.E.E. Trans. on Reliability*, vol. R-14, No. 2, pp. 107-113.