Uniformly Moving Elastic Singularities in the Classical Theory of Elasticity

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Abstract

The linear, isotropic, elastodynamic displacement field may be broken down into two distinctive components by means of a Helmholtz resolution. Each component satisfies a characteristic equation which becomes a biharmonic equation in the static limit. This analysis opens the way towards quickly constructing the fields of uniformly moving singularities from those of the corresponding static singularities. Many known solutions are recovered and some new ones are discovered. Only problems of the full-space are considered.

List of material constants

E	Young's modulus
K	bulk modulus
λ, μ	Lamé constants
ν	Poisson's ratio
κ	defined as $\kappa = {}^{1}/_{4(1-\nu)}$
ρ	mass density
c_1	longitudinal wave velocity
c_2	transverse wave velocity

These constants are inter-related as follows :

$$K = \frac{E}{3(1-2\nu)}$$

$$\mu = \frac{E}{2(1+\nu)}$$

$$\nu = \frac{\lambda}{2(\lambda+\mu)} \quad ; \quad 0 < \nu \leq \frac{1}{2}$$

$$1-2\kappa = \frac{1-2\nu}{2(1-\nu)}$$

$$c_1^2 = \frac{\lambda+2\mu}{\rho}$$

$$c_2^2 = \frac{\mu}{\rho}$$

$$\frac{c_2^2}{c_1^2} = \frac{1-2\nu}{2(1-\nu)}$$

Three further parameters are as follows :

c uniform velocity of singularity

$\gamma_1 = (1 - \frac{c^2}{c_1^2})^{\frac{1}{2}}$; longitudinal wave contraction factor

$\gamma_2 = (1 - \frac{c^2}{c_2^2})^{\frac{1}{2}}$; transverse wave contraction factor

$$\frac{1-\gamma_1^2}{1-\gamma_2^2} = \frac{c_2^2}{c_1^2}$$

Introduction

The linear elastic continuum may include singularities or "nuclie of strain" of various types. The most important of these are dislocations, point forces, and centres of dilatation and rotation. These singularities have no mobility within the continuum and therefore produce no physical effects. However, the continuum provides a useful model for the crystalline medium. It was first shown by G. I. Taylor (1934) that an edge dislocation could propagate through a metal crystal under the action of applied shear stresses, so causing plastic deformation. However, his analysis was largely qualitative. F. C. Frank (1949) determined the elastic field of a screw dislocation moving uniformly through the linear elastic continuum. This was followed by the work of J. D. Eshelby (1949), who solved the more complicated problem of an edge dislocation moving uniformly through the continuum. The field of a uniformly moving point force, both 2-dimensional and 3-dimensional, was given by Eason _et al_ (1956) and this was extended from the full-space to the half-space by Papadopoulos (1963) and by Eason (1965). The solutions of Eshelby and Eason were obtained by extensive manipulations involving Fourier transforms; such methods afford little insight into the nature of the field, and they cannot readily be adapted to the other types of singularity. An entirely different approach is possible, depending upon a Helmholtz resolution of the elastodynamic field into irrotational and equivoluminal components (also termed dilatational and rotational, or longitudinal and transverse, or irrotational and solenoidal). Each component satisfies a characteristic equation which is uniquely related to the corresponding elastostatic equation by means of a Lorentz-type transformation. Accordingly, starting with a Helmholtz resolution of the elastostatic field, we may transform it by fairly straightforward procedures into a uniformly moving field. All the results of Frank, Eshelby and Eason are quickly recovered, as well as some new results of interest. The success of this approach raises the possibility that it might also work with accelerating singularities, e.g. a dislocation accelerated from rest by the action of applied stresses. However, such evolution of solutions falls outside our present scope.

This thesis divides naturally into three main parts. Part I examines the dynamic Cauchy-Navier equation in terms of a Helmholtz resolution of the displacement field into two distinctive component fields (Sommerfeld, 1964). Each component satisfies a characteristic fourth-order wave equation which becomes a biharmonic equation in the static limit. We show, however, that the complete field may be built up from the superposition of two components which satisfy second-order wave equations. By contrast, the corresponding elastostatic field can only be built up from components which satisfy biharmonic equations. This raises an interesting apparent paradox which does not seem to have been previously recognised (Sternberg, 1960). To fix ideas, the theory is first applied to simple singularities, which are either purely irrotational or purely equivoluminal.

In these cases, the elastodynamic field is expressed by means of a single wave function, which reduces directly to a harmonic function in the static limit.

Part II introduces the Papkovich-Neuber representation of an elastostatic field, which covers every possible type of static singularity. We show how to break down this representation into two distinctive biharmonic components. Each component may be readily transformed into a corresponding uniformly moving component by means of a Lorentz-type transformation. The superposition of these, with appropriate coefficients, provides the required elastodynamic solution. This theory is applied to determine the fields produced by uniformly moving point forces, covering both longitudinal and transverse configurations.

Part III is concerned with 2-dimensional singularities, in particular dislocations and point forces. A dislocation is characterised by its Burgers vector, i.e. the jumps in displacement on circuiting the dislocation line. The condition that this vector remains invariant through the motion determines a unique solution. We demonstrate the mathematical impossibility of a dislocation motion not previously considered. Two-dimensional point force fields generally include unwanted dislocation solutions unless special steps are taken to exclude them. This exclusion enables the solution to be uniquely determined, so providing fresh insight into its special features.

Our analysis refers only to an infinite continuum. Accordingly, the possibility of Rayleigh waves does not arise. This method could in principle apply to a half-space, assuming the static solution were known in the half-space. For instance, given the Boussinesq (1885) solution of a half-space, we could calculate the field of a point force moving in the half-space. However, this analysis would add nothing essentially new to the theory.

PART I

INTRODUCTORY ANALYSIS

Chapter 1

The Cauchy-Navier Equation

1.1 Introduction

The Cauchy-Navier equation of motion for the linear elastic isotropic continuum, free from body forces, may be written as

$$\mu\nabla^2 \underset{\sim}{U} + (\lambda+\mu)\nabla(\nabla.\underset{\sim}{U}) = \rho \frac{\partial^2 \underset{\sim}{u}}{\partial t^2} , \qquad (1.1.1)$$

where $\underset{\sim}{U}(\underset{\sim}{r},t)$ is the displacement vector ; λ,μ are Lamé elastic constants, and ρ is the mass density.

Operating with $\nabla.$ on both sides of equation (1.1.1), we get

$$(\lambda+2\mu)\nabla^2(\nabla.\underset{\sim}{U}) = \rho \frac{\partial^2(\nabla.\underset{\sim}{U})}{\partial t^2} \qquad (1.1.2)$$

where $\nabla.\underset{\sim}{U}$ is the local dilatation. Introducing the wave operator

$$\square_1^2 \equiv \nabla^2 - \frac{1}{c_1^2}\frac{\partial^2}{\partial t^2} \quad ; \quad c_1^2 = \frac{\lambda+2\mu}{\rho} , \qquad (1.1.3)$$

we may write (1.1.2) in the form

$$\square_1^2(\nabla.\underset{\sim}{U}) = 0, \qquad (1.1.4)$$

which signifies that $\nabla.\underset{\sim}{U}$ satisfies the wave equation with velocity c_1 . Also, operating with $\nabla\wedge$ on both sides of (1.1.1), we get

$$\mu\nabla^2\underset{\sim}{\omega} = \rho\frac{\partial^2\underset{\sim}{\omega}}{\partial t^2} \qquad (1.1.5)$$

where $\underset{\sim}{\omega}(= \frac{1}{2}\nabla\wedge\underset{\sim}{U})$ is the local rotation. Again, introducing the wave operator

$$\Box_2^2 \equiv \nabla^2 - \frac{1}{c_2^2}\frac{\partial^2}{\partial t^2} \quad ; \quad c_2^2 = \frac{\mu}{\rho} , \qquad (1.1.6)$$

we may write (1.1.5) in the form

$$\Box_2^2\underset{\sim}{\omega} = 0, \qquad (1.1.7)$$

which signifies that $\underset{\sim}{\omega}$ satisfies the wave equation with velocity c_2 . Equations (1.1.4), (1.1.7) were obtained by Stokes (1851).

Poisson (1829) proved that every solution of (1.1.1) has a representation

$$\underset{\sim}{U} = \underset{\sim}{U}_1 + \underset{\sim}{U}_2 , \qquad (1.1.8)$$

where

$$\nabla\wedge\underset{\sim}{U}_1 = 0, \qquad \nabla.\underset{\sim}{U}_2 = 0 \qquad (1.1.9)$$

provided that

$$\Box_1^2\underset{\sim}{U}_1 = 0 \quad , \quad \Box_2^2\underset{\sim}{U}_2 = 0. \qquad (1.1.10)$$

Poisson thus established that the complete solution of equation (1.1.1) may be expressed as a superposition of an irrotational and equivoluminal motion, associated with respective velocities c_1 and c_2 .

1.2 Lamé solution

Lamé (1852) proved that, if we write

$$\underset{\sim}{U} = \nabla\Phi + \nabla_{\wedge}\underset{\sim}{\Psi} , \qquad (1.2.1)$$

then $\underset{\sim}{U}$ satisfies (1.1.1) provided that

$$\Box_1^2 \Phi = 0, \quad \Box_2^2 \underset{\sim}{\Psi} = 0. \qquad (1.2.2)$$

Thus substituting from (1.2.1) into (1.1.1) gives

$$(\lambda+2\mu)\Box_1^2(\nabla\Phi) + \mu\Box_2^2(\nabla_{\wedge}\underset{\sim}{\Psi}) = 0, \qquad (1.2.3)$$

which is identically satisfied (noting that $\nabla, \nabla_{\wedge}$ commute with the wave operators $\Box_1^2, \Box_2^2$) if (1.2.2) holds. To prove the generality of the solution (1.2.1), we must show that (1.2.3) implies (1.2.2). However this requires an additional condition

$$\nabla.\underset{\sim}{\Psi} = 0 \qquad (1.2.4)$$

associated with (1.2.1). If so, the Stokes-Helmholtz resolution

(Sommerfeld, 1964)

$$\underset{\sim}{U} = \nabla\mathcal{S} + \nabla\wedge\underset{\sim}{\mathcal{A}} , \qquad \nabla.\underset{\sim}{\mathcal{A}} = 0, \tag{1.2.5}$$

provides a general solution of equation (1.1.1). The change in symbolism from (1.2.1) to (1.2.5) arises from the fact that the equations satisfied by $\mathcal{S}$, $\underset{\sim}{\mathcal{A}}$ remain to be determined.

1.3 Generality of Lamé solution

Substituting (1.2.5) into (1.1.1) we get

$$(\lambda+2\mu)\Box_1^2(\nabla\mathcal{S}) + \mu\Box_2^2(\nabla\wedge\underset{\sim}{\mathcal{A}}) = 0. \tag{1.3.1}$$

Taking the $\nabla.$ of this equation, we obtain

$$\nabla^2\Box_1^2\mathcal{S} = 0. \tag{1.3.2}$$

Hence, according to Boggio's theorem (1903), $\mathcal{S}$ has the general solution

$$\mathcal{S} = S_1 + S_0 \quad ; \quad \Box_1^2 S_1 = 0, \ \nabla^2 S_0 = 0. \tag{1.3.3}$$

Clearly, of course, S_0 is a harmonic function involving the time t as a parameter, since otherwise

$$\Box_1^2 S_0 = \nabla^2 S_0 - \frac{1}{c_1^2}\frac{\partial^2 S_0}{\partial t^2} = 0,$$

which means that S_0 could be included in S_1 . Also, operating with $\nabla\wedge$ on (1.3.1) and utilising the vector identity

$$\nabla\wedge\nabla\wedge\underset{\sim}{A} = \nabla(\nabla.\underset{\sim}{A}) - \nabla^2\underset{\sim}{A} \quad , \tag{1.3.4}$$

we obtain

$$\nabla^2\square_2^2\underset{\sim}{\mathcal{A}} = 0 \quad ; \quad \nabla.\underset{\sim}{\mathcal{A}} = 0. \tag{1.3.5}$$

Hence, as in (1.3.2), $\underset{\sim}{\mathcal{A}}$ has the general solution

$$\underset{\sim}{\mathcal{A}} = \underset{\sim}{A}_2 + \underset{\sim}{A}_0 \quad ; \quad \square_2^2\underset{\sim}{A}_2 = 0, \quad \nabla^2\underset{\sim}{A}_0 = 0 \tag{1.3.6}$$

where

$$\nabla.(\underset{\sim}{A}_2 + \underset{\sim}{A}_0) = \nabla.\underset{\sim}{\mathcal{A}} = 0. \tag{1.3.7}$$

Again $\underset{\sim}{A}_0$ is a harmonic vector field involving the time t as a parameter. Thus the representation (1.2.5) becomes

$$\underset{\sim}{U} = \nabla(S_1+S_0) + \nabla\wedge(\underset{\sim}{A}_2+\underset{\sim}{A}_0) \quad ; \quad \nabla.(\underset{\sim}{A}_2+\underset{\sim}{A}_0) = 0. \tag{1.3.8}$$

Since S_0 is a harmonic scalar function, then ∇S_0 can be written as the curl of some divergence-free vector function $\underset{\sim}{f}$ (App. I), i.e.

$$\nabla S_0 = \nabla\wedge\underset{\sim}{f} \quad ; \quad \nabla.\underset{\sim}{f} = 0. \tag{1.3.9}$$

Hence the expression (1.3.8) becomes

$$\underset{\sim}{U} = \nabla S_1 + \nabla\wedge(\underset{\sim}{A}_2+\underset{\sim}{A}_0+\underset{\sim}{f}) \; ; \; \nabla.(\underset{\sim}{A}_2+\underset{\sim}{A}_0+\underset{\sim}{f}) = 0. \qquad (1.3.10)$$

Substituting from (1.3.10) into (1.1.1), we get

$$(\lambda+2\mu)\Box_1^2\nabla S_1 + \mu\Box_2^2\nabla\wedge(\underset{\sim}{A}_2+\underset{\sim}{A}_0+\underset{\sim}{f}) = 0, \qquad (1.3.11)$$

i.e. $\Box_2^2\nabla\wedge(\underset{\sim}{A}_2+\underset{\sim}{A}_0+\underset{\sim}{f}) = 0$ since $\Box_1^2 S_1 = 0$,

i.e. $\nabla\wedge\Box_2^2(\underset{\sim}{A}_2+\underset{\sim}{A}_0+\underset{\sim}{f}) = 0.$

Bearing in mind that $\nabla.$ commute with $\Box_2^2$ it follows from $\nabla.(\underset{\sim}{A}_2+\underset{\sim}{A}_0+\underset{\sim}{f}) = 0$ - See (1.3.10) - that $\Box_2^2\nabla.(\underset{\sim}{A}_2+\underset{\sim}{A}_0+\underset{\sim}{f}) = 0$ i.e. $\nabla.\Box_2^2(\underset{\sim}{A}_2+\underset{\sim}{A}_0+\underset{\sim}{f}) = 0.$ Since both the divergence and the curl of the vector $\Box_2^2(\underset{\sim}{A}_2+\underset{\sim}{A}_0+\underset{\sim}{f})$ vanish in all space, it follows (Landau and Lifshitz, 1959) that

$$\Box_2^2(\underset{\sim}{A}_2+\underset{\sim}{A}_0+\underset{\sim}{f}) = 0.$$

Accordingly, writing

$$\Phi = S_1 \quad , \quad \underset{\sim}{\Psi} = \underset{\sim}{A}_2+\underset{\sim}{A}_0+\underset{\sim}{f} \quad ,$$

we see that any solution $\underset{\sim}{U}$ of the equation (1.1.1) has the representation

$$\underset{\sim}{U} = \nabla\Phi + \nabla\wedge\underset{\sim}{\Psi} \quad ; \quad \nabla.\underset{\sim}{\Psi} = 0 \qquad (1.3.12)$$

characterized by

$$\Box_1^2\Phi = 0, \quad \Box_2^2\underset{\sim}{\Psi} = 0. \tag{1.3.13}$$

This completes the proof.

1.4 Iacovache solution

It is sometimes useful to write equation (1.1.1) in the form

$$\nabla^2\underset{\sim}{U} - \frac{\rho}{\mu}\frac{\partial^2\underset{\sim}{U}}{\partial t^2} + \frac{\lambda+\mu}{\mu}\nabla(\nabla.\underset{\sim}{U}) = 0$$

i.e.

$$\Box_2^2\underset{\sim}{U} + \frac{1}{1-2\nu}\nabla(\nabla.\underset{\sim}{U}) = 0 \; ; \quad \nu = \frac{\lambda}{2(\lambda+\mu)} \; , \tag{1.4.1}$$

where ν is Poisson's ratio. Iacovache (1949) found that the representation

$$\underset{\sim}{U}(\underset{\sim}{r},t) = \Box_1^2\underset{\sim}{\chi} - 2\kappa\nabla(\nabla.\underset{\sim}{\chi}) \quad ; \quad \kappa = \frac{1}{4(1-\nu)} \tag{1.4.2}$$

identically satisfies equation (1.4.1) , provided that $\underset{\sim}{\chi}$ satisfies the repeated wave equation

$$\Box_1^2\Box_2^2\underset{\sim}{\chi} = 0 \tag{1.4.3}$$

The representation (1.4.2) can be transformed into the representation (1.3.12). Using Boggio's theorem, equation (1.4.3) has a solution

$$\underset{\sim}{\chi} = \underset{\sim}{\chi}_1 + \underset{\sim}{\chi}_2 \tag{1.4.4}$$

such that

$$\square_1^2 \underset{\sim}{\chi}_1 = 0 \quad , \quad \square_2^2 \underset{\sim}{\chi}_2 = 0. \tag{1.4.5}$$

Now

$$\begin{aligned} \square_1^2 \underset{\sim}{\chi}_2 &= \nabla^2 \underset{\sim}{\chi}_2 - \frac{1}{c_1^2} \frac{\partial^2 \underset{\sim}{\chi}_2}{\partial t^2} \\ &= \left(1 - \frac{c_2^2}{c_1^2}\right) \nabla^2 \underset{\sim}{\chi}_2 \qquad \text{since } \frac{\partial^2 \underset{\sim}{\chi}_2}{\partial t^2} = c_2^2 \nabla^2 \underset{\sim}{\chi}_2 \quad . \end{aligned}$$

The velocities c_1, c_2 are related to Poisson's ratio ν by

$$\frac{c_2^2}{c_1^2} = \frac{\mu}{\lambda+2\mu} = \frac{1-2\nu}{2(1-\nu)} \quad ; \quad \nu = \frac{\lambda}{2(\lambda+\mu)} \quad , \tag{1.4.6}$$

i.e. $$1 - \frac{c_2^2}{c_1^2} = 1 - \frac{1-2\nu}{2(1-\nu)} = \frac{1}{2(1-\nu)} = 2\kappa \quad ; \kappa = \frac{1}{4(1-\nu)} \, , \tag{1.4.7}$$

so that

$$\square_1^2 \underset{\sim}{\chi}_2 = 2\kappa \nabla^2 \underset{\sim}{\chi}_2 \quad . \tag{1.4.8}$$

Utilising (1.4.4), (1.4.5) and (1.4.8), the representation (1.4.2) becomes

$$\begin{aligned} \underset{\sim}{U} &= \square_1^2 (\underset{\sim}{\chi}_1 + \underset{\sim}{\chi}_2) - 2\kappa \nabla\nabla . (\underset{\sim}{\chi}_1 + \underset{\sim}{\chi}_2) \\ &= \square_1^2 \underset{\sim}{\chi}_2 - 2\kappa \nabla(\nabla . \underset{\sim}{\chi}_1) - 2\kappa \nabla(\nabla . \underset{\sim}{\chi}_2) \end{aligned}$$

$$= 2\kappa\nabla^2 \underset{\sim}{X}_2 - 2\kappa\nabla(\nabla.\underset{\sim}{X}_2) - 2\kappa\nabla(\nabla.\underset{\sim}{X}_1)$$

$$= -2\kappa\nabla\wedge(\nabla\wedge \underset{\sim}{X}_2) - 2\kappa\nabla(\nabla.\underset{\sim}{X}_1)$$

on utilising the vector identity (1.3.4). Thus, we can identify the Lamé potentials as

$$\Phi = -2\kappa\nabla.\underset{\sim}{X}_1 \quad , \quad \underset{\sim}{\Psi} = -2\kappa\nabla\wedge \underset{\sim}{X}_2$$

characterised respectively by the properties

$$\Box_1^2 \Phi = -2\kappa\Box_1^2 (\nabla.\underset{\sim}{X}_1) = 0$$

$$\Box_2^2 \underset{\sim}{\Psi} = -2\kappa\Box_2^2 (\nabla\wedge \underset{\sim}{X}_2) = 0, \quad \nabla.\underset{\sim}{\Psi} \equiv 0$$

on noting that the operators $\nabla.,\nabla\wedge$ commute with the operators $\Box_1^2$, $\Box_2^2$. Therefore the representation (1.4.2) takes the form

$$\underset{\sim}{U} = \nabla\Phi + \nabla\wedge\underset{\sim}{\Psi} \quad ; \quad \nabla.\underset{\sim}{\Psi} = 0$$

$$\Box_1^2 \Phi = 0 \quad , \quad \Box_2^2 \underset{\sim}{\Psi} = 0$$

as in (1.3.12), (1.3.13). This reduction shows the completeness of Lamé solution on the basis of the generality of Iacovache's solution.

1.5 Limiting elastic field

We now examine the significance of Lamé solution in the limiting medium, i.e. when c_1, c_2 approach infinity and so define (see below) a rigid-continuum. First we note that ν remains fixed by the relation (1.4.6) as $c_1, c_2 \to \infty$. However, keeping ρ fixed, we see from the definitions of c_1, c_2 in (1.1.3), (1.1.6) respectively that $\mu \to \infty$ as $c_1, c_2 \to \infty$. Also, from the relation which holds between μ and Young's modulus E, i.e.

$$\frac{\mu}{E} = \frac{1}{2(1+\nu)} , \tag{1.5.1}$$

we see that $E \to \infty$ as $\mu \to \infty$ keeping ν fixed. Similarly from the relation which holds between μ and the bulk modulus K, i.e.

$$\frac{\mu}{K} = \frac{3}{2}\frac{1-2\nu}{1+\nu} , \tag{1.5.2}$$

we see that $K \to \infty$ as $\mu \to \infty$ keeping ν fixed. Accordingly, the medium becomes a rigid-body in the limiting case. This means physically that infinite stresses are required to maintain the displacements defined by the limiting equation (1.5.6) below.

Clearly, of course

$$\Box_1^2 \equiv \nabla^2 - \frac{1}{c_1^2}\frac{\partial^2}{\partial t^2} \to \nabla^2 \quad \text{as } c_1 \to \infty \tag{1.5.3}$$

$$\Box_2^2 \equiv \nabla^2 - \frac{1}{c_2^2}\frac{\partial^2}{\partial t^2} \to \nabla^2 \quad \text{as } c_2 \to \infty \quad . \tag{1.5.4}$$

If so, equation (1.3.13) apparently becomes

$$\nabla^2\Phi = 0 \quad , \quad \nabla^2\underset{\sim}{\Psi} = 0 \quad . \tag{1.5.5}$$

i.e. Φ now appears to be a harmonic function and $\underset{\sim}{\Psi}$ appears to be a harmonic vector, in which case $\underset{\sim}{U}$ as given by (1.3.12) apparently reduces to a harmonic vector. However, equation (1.4.1) becomes in this case

$$\nabla^2 \underset{\sim}{u} + \frac{1}{1-2\nu} \nabla(\nabla.\underset{\sim}{u}) = 0, \tag{1.5.6}$$

where $\underset{\sim}{u}$ is the limiting form of $\underset{\sim}{U}$. Now the general solution of (1.5.6) is a biharmonic vector (Jaswon and Symm, 1977) . So an apparent paradox arises, since the limiting form of the elastodynamic solution (1.3.12) is not a general solution of equation (1.5.6). We may resolve the paradox by noting that $\Phi, \underset{\sim}{\Psi}$ do not necessarily exist in the limiting case, since they include coefficients which become infinite as $c_1 \to \infty$, $c_2 \to \infty$. Examples will be given later. To construct acceptable alternatives to Φ , $\underset{\sim}{\Psi}$, we note that Boggio's theorem no longer applies when the two operators in (1.3.2) become identical. Thus the limiting form of (1.3.2) becomes

$$\nabla^2\nabla^2 S = 0 \tag{1.5.7}$$

where S is the limiting form of $\mathfrak{S}$. This is a biharmonic function which has a general solution (Almansi, 1897)

$$S = xh+f \quad ; \quad \nabla^2 h = 0 \ , \ \nabla^2 f = 0, \tag{1.5.8}$$

or equivalently

$$S = yh+f \ , \ S = zh+f \tag{1.5.9}$$

where h,f are harmonic functions. Similar remarks apply to $\underset{\sim}{A}$ (the limiting form of $\underset{\sim}{\mathfrak{A}}$), i.e. it becomes a biharmonic vector in the limiting case. If so, $\underset{\sim}{U}$ as defined by (1.3.12) becomes a biharmonic vector in the limiting case. An analysis of this vector will be given in Part II.

Chapter 2

Some Specialized Fields

2.1 Centre of dilatation

We first consider an elastodynamic displacement field of the form $\underset{\sim}{U} = \nabla F$. In this case F must satisfy the wave equation $\Box_1^2 F = 0$ - as may be proved by direct substitution into (1.1.1) - except possibly at singularities. An interesting example is obtained by starting with a static "centre of dilatation" (Love, 1927), defined by

$$\underset{\sim}{u} = \nabla f \; ; \; f = \frac{1}{r} \, , \quad r^2 = x^2+y^2+z^2 \tag{2.1.1}$$

$$\nabla^2 f = -4\pi\delta(x,y,z).$$

Now introducing

$$\underset{\sim}{U} = k \nabla F \; ; \; F = \frac{1}{R_1} \, , \; R_1^2 = \left(\frac{x-ct}{\gamma_1}\right)^2 + y^2 + z^2 \, , \tag{2.1.2}$$

$$\gamma_1^2 = 1 - \frac{c^2}{c_1^2} \, , \quad k(\gamma_1) = 1 \text{ when } \gamma_1 = 1,$$

where c is parameter (uniform velocity), $k(\gamma_1)$ is a normalizing parameter to be determined (see later) so that the total strength (dilatation) of the singularity remains invariant, F has the following properties :

(i) It satisfies (see below)

$$\Box_1^2 F = -4\pi\delta\left(\frac{x-ct}{\gamma_1}, y,z\right) \qquad (2.1.3)$$

(ii) It reduces to f as $c \to 0$ ($\gamma_1 \to 1$)

(iii) It reduces to $\{(x-ct)^2 + y^2 + z^2\}^{-\frac{1}{2}}$ as $c_1 \to \infty$ ($\gamma_1 \to 1$).

This is simply the field (2.1.1) centred at $x = ct$, $y = 0$, $z = 0$ i.e. it defines a "centre of dilatation" moving with a uniform velocity c along the x-axis in a rigid continuum without change of form. This field satisfies the equation

$$\nabla^2\{(x-ct)^2 + y^2 + z^2\}^{-\frac{1}{2}} = -4\pi\delta(x-ct,y,z), \qquad (2.1.4)$$

which is seen to be the limiting form of the equation (2.1.3) as $c_1 \to \infty$ on utilising (1.5.3).

(iv) It becomes singular when $x = ct$, $y = 0$, $z = 0$.

Clearly, therefore, ∇F describes a "centre of dilatation" moving with a uniform velocity c along the x-axis in the elastic continuum.

Equation (2.1.3) may be easily proved by direct differentiation with respect to the variables x,y,z,t. More generally, we may argue as follows. If $\phi(x,y,z)$ is continuous and differentiable to the second order everywhere except possibly at a singularity (the origin), and it satisfies

$$\nabla^2 \phi = -4\pi\delta(x,y,z) \tag{2.1.5}$$

everywhere, then $\phi\left(\frac{x-ct}{\gamma_1}, y,z\right)$ is continuous and differentiable to the second order and satisfies

$$\Box_1^2 \phi = -4\pi\delta\left(\frac{x-ct}{\gamma_1}, y,z\right) \tag{2.1.6}$$

everywhere. To prove this theorem, we use the Lorentz-type transformation

$$x_1 = \frac{x-ct}{\gamma_1} \quad ; \quad \gamma_1^2 = 1 - \frac{c^2}{c_1^2} \tag{2.1.7}$$

and note that

$$\frac{\partial^2}{\partial x^2} - \frac{1}{c_1^2}\frac{\partial^2}{\partial t^2} = \frac{1}{\gamma_1^2}\frac{\partial^2}{\partial x_1^2} - \frac{c^2}{\gamma_1^2 c_1^2}\frac{\partial^2}{\partial x_1^2} \equiv \frac{\partial^2}{\partial x_1^2}$$

i.e.

$$\Box_1^2 \equiv \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \quad . \tag{2.1.8}$$

Accordingly, equation (2.1.6) transforms into

$$\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi = -4\pi\delta(x_1,y,z) \ ,$$

i.e. equation (2.1.5) with x replaced by x_1, and therefore (2.1.6) is true since (2.1.5) is true.

2.2 Centre of rotation

Another distinctive type of elastodynamic field is $\underset{\sim}{U} = \nabla\wedge \underset{\sim}{F}$. In this case $\underset{\sim}{F}$ must satisfy the wave equation $\Box_2^2 \underset{\sim}{F} = 0$ - as may be proved by direct substitution into (1.1.1) - except possibly at singularities. An example is obtained by starting with the static field (Love, 1927) defined by

$$\underset{\sim}{u} = \nabla\wedge \underset{\sim}{f} \quad ; \quad \underset{\sim}{f} = \langle 0,0,\frac{1}{r}\rangle \; , \qquad (2.2.1)$$

$$\nabla^2 \underset{\sim}{f} = \langle 0,0,-4\pi\delta(x,y,z)\rangle \; .$$

This field describes a singularity known as a "centre of rotation about the z-axis". Proceeding as before we introduce

$$\underset{\sim}{U} = k\, \nabla\wedge \underset{\sim}{F} \quad ; \quad \underset{\sim}{F} = \langle 0,0,\frac{1}{R_2}\rangle \qquad (2.2.2)$$

$$R_2^2 = \left(\frac{x-ct}{\gamma_2}\right)^2 + y^2 + z^2 \quad , \quad \gamma_2^2 = 1 - \frac{c^2}{c_2^2} \; ,$$

$$k(\gamma_2) = 1 \quad \text{when } \gamma_2 = 1,$$

where c is a parameter (uniform velocity), $k(\gamma_2)$ is a normalizing parameter to be determined (see later) so that the total strength (rotation about the z-axis) of the singularity remains invariant. The vector $\underset{\sim}{F}$ has the properties :

(i) It satisfies (see below)

$$\Box_2^2 \underset{\sim}{F} = <0,0,\ -4\pi\delta\left(\frac{x-ct}{\gamma_2}, y, z\right)> . \qquad (2.2.3)$$

(ii) It reduces to $\underset{\sim}{f} = <0,0,\frac{1}{r}>$ as $c \to 0$ $(\gamma_2 \to 1)$.

(iii) It reduces to $<0,0,\frac{1}{r'}>$; $(r')^2 = (x-ct)^2 + y^2 + z^2$

as $c_2 \to \infty(\gamma_2 \to 1)$. This is simply the field (2.2.1) centred about $x = ct$, $y = 0$, $z = 0$ i.e. it defines a "centre of rotation about the z-axis" moving with a uniform velocity c along the x-axis in a rigid continuum.

(iv) It becomes singular when $x = ct$, $y = 0$, $z = 0$.

Clearly, therefore, $\nabla\wedge\underset{\sim}{F}$ describes a "centre of rotation about the z-axis"moving with a uniform velocity c along the x-axis.

The proof of (2.2.3) follows by introducing the Lorentz-type transformation

$$x_2 = \frac{x-ct}{\gamma_2} \quad ; \quad \gamma_2^2 = 1 - \frac{c^2}{c_2^2} \qquad (2.2.4)$$

and following the same procedure as in section (2.1) we find

$$\Box_2^2 = \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} ,$$

in line with (2.1.8) ; which implies (2.2.3).

2.3 Strength of singularity

The dilatation singularity (2.1.1) has a local divergence $\nabla.\underset{\sim}{u} = \nabla.\nabla(\frac{1}{r}) = \nabla^2(\frac{1}{r})$, and therefore the total dilatation strength is given by

$$\int\nabla^2(\frac{1}{r})dx\,dy\,dz = -4\pi\int\delta(x,y,z)dx\,dy\,dz = -4\pi\ . \qquad (2.3.1)$$

In the uniformly moving case, we have

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \frac{1}{\gamma_1^2}\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$= \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) + \left(\frac{1}{\gamma_1^2}\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_1^2}\right)$$

$$= \square_1^2 + \frac{1-\gamma_1^2}{\gamma_1^2}\frac{\partial^2}{\partial x_1^2}\ . \qquad (2.3.2)$$

Also

$$dV \equiv dx\,dy\,dz = \gamma_1 dx_1 dy\,dz \equiv \gamma_1 d\,V_1 \qquad (2.3.3)$$

so that

$$\int\nabla^2\left(\frac{1}{R_1}\right)dV = \int\left\{\square_1^2\left(\frac{1}{R_1}\right) + \frac{1-\gamma_1^2}{\gamma_1^2}\frac{\partial^2}{\partial x_1^2}\left(\frac{1}{R_1}\right)\right\}\gamma_1 dV_1\ . \qquad (2.3.4)$$

By symmetry we have

$$\int \frac{\partial^2}{\partial x_1^2}\left(\frac{1}{R_1}\right) dV_1 \;=\; \frac{1}{3}\int \Box_1^2\left(\frac{1}{R_1}\right) dV_1 \quad , \tag{2.3.5}$$

and therefore

$$\int \nabla^2\left(\frac{1}{R_1}\right) dV \;=\; \left(1 + \frac{1-\gamma_1^2}{3\gamma_1^2}\right) \int \Box_1^2\left(\frac{1}{R_1}\right) \gamma_1\, dV_1$$

$$= -\left(\gamma_1 + \frac{1-\gamma_1^2}{3\gamma_1}\right)\int 4\pi\delta(x_1,y,z) = -\frac{2\gamma_1^2+1}{3\gamma_1} 4\pi \quad , \tag{2.3.6}$$

i.e.

$$\int \nabla.\underset{\sim}{U}\, dV \;=\; -k(\gamma_1)\,\frac{2\gamma_1^2+1}{3\gamma_1}\, 4\pi \to -4\pi \quad \text{as } \gamma_1 \to 1. \tag{2.3.7}$$

It is necessary to normalize the moving field so that its total dilatation remains invariant throughout the motion. Accordingly we choose

$$k(\gamma_1) \;=\; \frac{3\gamma_1}{1+2\gamma_1^2} \quad . \tag{2.3.8}$$

Clearly $k(\gamma_1) = 1$ when $\gamma_1 = 1$, i.e. (2.1.2) becomes

$$\underset{\sim}{U} \;=\; \frac{3\gamma_1}{1+2\gamma_1^2}\,\nabla\left(\frac{1}{R_1}\right) \;\to\; \nabla\left(\frac{1}{r}\right) \quad \text{as } c \to 0,$$

$$\underset{\sim}{U} \;=\; \frac{3\gamma_1}{1+2\gamma_1^2}\,\nabla\left(\frac{1}{R_1}\right) \;\to\; \nabla\left(\frac{1}{r'}\right) \; ; \quad (r')^2 = (x-ct)^2 + y^2 + z^2$$

$$\text{as } c_1 \to \infty .$$

Similarly, the rotation singularity (2.2.1) has a local rotation vector

$$\nabla \wedge \underset{\sim}{u} \;=\; \nabla \wedge \nabla \wedge \underset{\sim}{f} = \nabla(\nabla . \underset{\sim}{f}) - \nabla^2 \underset{\sim}{f} \;; \quad \underset{\sim}{f} \;=\; \langle 0,0,\tfrac{1}{r}\rangle$$

$$= \nabla \left(\frac{\partial r^{-1}}{\partial z}\right) \;-\; \nabla^2 \langle 0,0,\tfrac{1}{r}\rangle \quad ,$$

with a component in the z-direction

$$\frac{\partial^2 r^{-1}}{\partial z^2} \;-\; \nabla^2(\tfrac{1}{r}). \tag{2.3.9}$$

This provides a total rotation component of strength

$$\int \left\{ \frac{\partial^2 r^{-1}}{\partial z^2} \;-\; \nabla^2 (\tfrac{1}{r}) \right\} dV \;=\; -(\tfrac{1}{3} - 1) \int 4\pi\delta(x,y,z)$$

$$= \tfrac{2}{3} 4\pi . \tag{2.3.10}$$

In the uniformly moving case, we have

$$\nabla^2 \equiv \Box_2^2 + \frac{1-\gamma_2^2}{\gamma_2} \frac{\partial^2}{\partial x_2^2} \quad , \quad dV = \gamma_2 dV_2$$

and therefore the total rotation component has a strength

$$\int \frac{\partial^2}{\partial z^2}\left(\frac{1}{R_2}\right) dV \; - \int \nabla^2\left(\frac{1}{R_2}\right) dV \; = \frac{\gamma_2}{3}\int \Box_2^2\left(\frac{1}{R_2}\right) dV_2 \; - \int \nabla^2\left(\frac{1}{R_2}\right) dV$$

$$= \; -\frac{\gamma_2}{3}\; 4\pi + \frac{2\gamma_2^2+1}{3\gamma_2}\, 4\pi \; = \; \frac{1+\gamma_2^2}{3\gamma_2}\, 4\pi \to \frac{2}{3}\, 4\pi \quad \text{as } \gamma_2 \to 1$$

on utilising (2.3.5), (2.3.6).

Accordingly we choose

$$k(\gamma_2) \quad = \quad \frac{2\gamma_2}{1+\gamma_2^2} \quad . \tag{2.3.11}$$

Clearly $k(\gamma_2) = 1$ when $\gamma_2 = 1$, i.e. (2.2.2) becomes

$$\underset{\sim}{U} \; = \; \frac{2\gamma_2}{1+\gamma_2^2}\, \nabla_\wedge \langle 0,0,\frac{1}{R_2}\rangle \; \to \; \nabla_\wedge \langle 0,0,\frac{1}{r}\rangle \quad \text{as} \quad c \to 0,$$

$$\underset{\sim}{U} \; = \; \frac{2\gamma_2}{1+\gamma_2^2}\, \nabla_\wedge \langle 0,0,\frac{1}{R_2}\rangle \; \to \; \nabla_\wedge \langle 0,0,\frac{1}{r'}\rangle \; \text{as } c_2 \to \infty \quad .$$

Chapter 3

The Screw Dislocation

3.1 Introduction

The displacement field

$$\underset{\sim}{u} = \langle 0,0,\theta \rangle \equiv \theta \underset{\sim}{k} \quad ; \quad \theta = \tan^{-1}\left(\frac{y}{x}\right) \qquad (3.1.1)$$

has a characteristic property, i.e. its z-component increases by $2\pi^{(*)}$ on making any circuit around the origin in the x,y plane. Expressed mathematically,

$$[\theta] \equiv \oint \left(\frac{\partial\theta}{\partial x}\,dx + \frac{\partial\theta}{\partial y}\,dy\right) = \oint d\theta = 2\pi \qquad (3.1.2)$$

where $\circlearrowright$ signifies any circuit enclosing the point $x = 0$, $y = 0$. This is a screw dislocation (Love, 1927 ; Cottrell,1964 ; Jaswon and Symm,1977). Strictly speaking, the dislocation is a line coinciding with the z-axis. In practice, we may think of it as a singularity located at the point $x = 0$, $y = 0$, which generates the field (3.1.1). To eliminate complications arising from the multi-valuedness of θ , we make a

---

(*) For a jump of amount p we replace 2π by $\frac{p}{2\pi}$ i.e. replacing θ by $\frac{p}{2\pi}\theta$.

cut along the positive x-axis ($0 \leqslant x < \infty$) so that θ varies only within the range $0 \leqslant \theta < 2\pi$. If so, θ satisfies the equation

$$\nabla^2\theta = 0 \; ; \quad y \neq 0 \text{ for } x \geqslant 0. \tag{3.1.3}$$

Also, it is continuous and differentiable everywhere except at the cut, and it is therefore a harmonic function everywhere except at the cut.

3.2 Uniformly moving screw dislocation

If the dislocation moves with a uniform velocity c along the x-axis, its displacement field (Frank, 1949) becomes (note $x_2 = \dfrac{x-ct}{\gamma_2}$)

$$\underset{\sim}{U} = \Theta_2 \underset{\sim}{k} \quad ; \quad \Theta_2 = \tan^{-1}\left(\frac{y}{x_2}\right) . \tag{3.2.1}$$

This has the following properties :

(i) $\square_2^2 \, \Theta_2 = 0 \; ; \; y \neq 0, \; x \geqslant ct.$

(ii) $\Theta_2 \to \theta$ as $c \to 0.$

(iii) $\Theta_2 \to \tan^{-1}\left(\dfrac{y}{x-ct}\right)$ as $c_2 \to \infty$.

(iv) Θ_2 has a branch point at $x = ct$, $y = 0$.

(v) $$[\Theta_2] \equiv \oint \left(\frac{\partial\Theta_2}{\partial x}\,dx + \frac{\partial\Theta_2}{\partial y}\,dy\right)$$

$$= \oint \left(\frac{\partial\Theta_2}{\partial x_2}\frac{dx_2}{dx}\,dx + \frac{\partial\Theta_2}{dy}\,dy\right)$$

$$= \oint \left(\frac{\partial\Theta_2}{\partial x_2}\,dx_2 + \frac{\partial\Theta_2}{\partial y}\,dy\right) = \oint d\Theta_2 = 2\pi\,. \qquad (3.2.2)$$

Here $\circlearrowleft$ indicates any circuit enclosing the point $x = ct$, $y = 0$.

3.3 Construction of a biharmonic vector potential

We now look for a vector $\underset{\sim}{A}$ such that

$$\nabla\wedge \underset{\sim}{A} = \theta \underset{\sim}{k} \quad ; \quad \nabla.\underset{\sim}{A} = 0. \qquad (3.3.1)$$

Operating with $\nabla\wedge$ upon equation (3.3.1) gives

$$-\nabla^2 \underset{\sim}{A} = < \frac{\partial\theta}{\partial y} , - \frac{\partial\theta}{\partial x} , 0 > , \qquad (3.3.2)$$

which is immediately seen to have the particular solution

$$\underset{\sim}{A}_0 = \frac{1}{2} < -y\theta , x\theta , 0 > . \qquad (3.3.3)$$

Since $\nabla.\underset{\sim}{A}_0 = \frac{1}{2}\left(-y\,\frac{\partial\theta}{\partial x} + x\,\frac{\partial\theta}{\partial y}\right) = \frac{1}{2} \neq 0$, it is necessary to superimpose upon this a harmonic vector $\nabla\psi$ defined by

$$\nabla.\underset{\sim}{A}_0 + \nabla^2\psi = 0\ , \quad \text{i.e. } \nabla^2\psi = -\frac{1}{2}\ , \qquad (3.3.4)$$

which equation has a particular solution

$$\psi = -\frac{1}{4}x^2 \text{ , if so } \nabla\psi = \langle -\frac{1}{2}x, 0, 0 \rangle . \tag{3.3.5}$$

Clearly $\nabla^2(\nabla\psi) = \nabla^2 \langle -\frac{1}{2}x, 0, 0 \rangle = 0$, showing that $\nabla\psi$ is a harmonic vector. Superimposing (3.3.5), (3.3.3) yields the vector

$$\underset{\sim}{A} = \frac{1}{2}\langle -(y\theta + x), x\theta, 0 \rangle \quad ; \quad \nabla . \underset{\sim}{A} = 0. \tag{3.3.6}$$

It will be noted that $\underset{\sim}{A}$ is a biharmonic vector which satisfies the equation

$$\nabla^4 \underset{\sim}{A} = \nabla^2(\nabla^2 \underset{\sim}{A}) = 0, \tag{3.3.7}$$

as follows from (3.3.2) since θ is a harmonic function. This also follows directly from (3.3.3) by Almansi's theorem (1897).

We now look for a vector $\underset{\sim}{\mathcal{A}}$ such that

$$\left.\begin{array}{ll} \text{(i)} & \nabla_\wedge \underset{\sim}{\mathcal{A}} = \Theta_2 \underset{\sim}{k} \\ \text{(ii)} & \nabla . \underset{\sim}{\mathcal{A}} = 0 \\ \text{(iii)} & \underset{\sim}{\mathcal{A}} \to \underset{\sim}{A}(x,y) \quad \text{as } c \to 0 \\ \text{(iv)} & \underset{\sim}{\mathcal{A}} \to \underset{\sim}{A}(x',y) \ ; \quad x' = x - ct \quad \text{as} \quad c_2 \to \infty \end{array}\right\} . \tag{3.3.8}$$

First we note that

$$x\theta = x\frac{\partial}{\partial x}(x\theta + y\log r) \ ; \ r^2 = x^2 + y^2 \tag{3.3.9}$$

$$y\theta = x \log r - (x \log r - y\theta)$$

$$= x \frac{\partial}{\partial x}(x \log r - y\theta - x) - (x \log r - y\theta) \qquad (3.3.10)$$

i.e.

$$y\theta + x = x \frac{\partial}{\partial x}(x \log r - y\theta) - (x \log r - y\theta).$$

If so, making use of later theory (Chap. 5), it follows that

$$\underset{\sim}{\mathcal{A}} = \frac{\gamma_2}{1-\gamma_2^2} < -(\gamma_2 G^* - g^*), (G^{**} - g^{**}), 0 > , \qquad (3.3.11)$$

where

$$G^* = x_2 \log R_2 - y\Theta_2 \quad , \quad g^* = x' \log r' - y\theta'$$

$$G^{**} = x_2\Theta_2 + y \log R_2 \quad , \quad g^{**} = x'\theta' + y \log r'$$

$$R_2^2 = x_2^2 + y^2 \quad , \quad (r')^2 = (x')^2 + y^2 \quad , \quad \theta' = \tan^{-1}\left(\frac{y}{x'}\right) .$$

This vector satisfies all the required conditions (3.3.8). Also, since $\nabla^2 g^* = 0$, $\nabla^2 g^{**} = 0$ everywhere except at the cut, it follows that

$$\left.\begin{aligned} &\left(\frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) G^* = 0 \quad \text{i.e. } \Box_2^2 G^* = 0 \\ &\left(\frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) G^{**} = 0 \quad \text{i.e. } \Box_2^2 G^{**} = 0 \end{aligned}\right\} y \neq 0, \ x \geqslant ct.$$

Accordingly,

$$\nabla^2 \Box_2^2 \underset{\sim}{\mathcal{A}} = 0 \; ; \quad y \neq 0, \; x \geqslant ct \tag{3.3.12}$$

which equation contrasts with the biharmonic equation (3.3.7) satisfied by the biharmonic vector $\underset{\sim}{A}$. This provides a good example of the construction of an elastodynamic vector $\underset{\sim}{\mathcal{A}}$ which reduces to a given biharmonic vector $\underset{\sim}{A}$ in the static limit.

PART II

THREE-DIMENSIONAL THEORY

Chapter 4

Three-Dimensional Fields

4.1 Papkovich-Neuber representation

The limiting equation (1.5.6) has a general solution given by (note $\kappa = \frac{1}{4(1-\nu)}$)

$$\underset{\sim}{u} = \underset{\sim}{h} - \kappa\nabla(\underset{\sim}{r}.\underset{\sim}{h} + f) \; ; \; \nabla^2\underset{\sim}{h}(\underset{\sim}{r}) = 0, \; \nabla^2 f(\underset{\sim}{r}) = 0, \qquad (4.1.1)$$

where $\underset{\sim}{h}$ is a harmonic vector function and f is a harmonic scalar function. This solution was first presented by Papkovich (1932) and then by Neuber (1934), and more recently adapted by Jaswon and Symm (1977). By a suitable choice of $\underset{\sim}{h}$, we may cover the field of any static elastic field, e.g. a point force. Generally speaking, the harmonic function f is not necessary, but in certain exceptional cases (e.g. $\nu = \frac{1}{2}$) it must be included (Eubanks & Sternberg, 1956; Jaswon & Symm, 1977). However, since the specialized harmonic field $\underset{\sim}{u} = \nabla f$ has already been considered, we may omit f without loss of generality.

4.2 Helmholtz resolution

We now attempt to generalize (4.1.1) so that it represents the field of uniformly moving singularities. An efficient approach is to introduce the Helmholtz resolution

$$\underset{\sim}{u} = \nabla S + \nabla_{\wedge}\underset{\sim}{A} \quad ; \quad \nabla.\underset{\sim}{A} = 0, \tag{4.2.1}$$

i.e. to determine $S,\underset{\sim}{A}$ so that

$$\underset{\sim}{h} - \kappa\nabla(\underset{\sim}{r}.\underset{\sim}{h} + f) = \nabla S + \nabla_{\wedge}\underset{\sim}{A}\ ;\ \nabla.\underset{\sim}{A} = 0, \tag{4.2.2}$$

and then undertake transformations

$$S \rightarrow \mathcal{S} \quad , \quad \underset{\sim}{A} \rightarrow \underset{\sim}{\mathcal{A}}$$

of the type already discussed. With a suitable choice of $\mathcal{S}$, $\underset{\sim}{\mathcal{A}}$ the reconstructed field

$$\underset{\sim}{U} = \nabla\mathcal{S} + \nabla_{\wedge}\underset{\sim}{\mathcal{A}} = 0 \quad ; \quad \nabla.\underset{\sim}{\mathcal{A}} = 0$$

defines a singularity moving with a uniform velocity c in a particular direction, and which reduces to the static singularity as $c \rightarrow 0$.

To determine $S,\underset{\sim}{A}$ in (4.2.2) we first operate with $\nabla.$ on both sides of (4.2.2) , which gives

$$\nabla^2 S = \nabla.\underset{\sim}{h} - \kappa\nabla^2(\underset{\sim}{r}.\underset{\sim}{h} + f). \tag{4.2.3}$$

This is a Poisson equation with the particular solution

$$S = \frac{1}{2}(1-2\kappa)(\underset{\sim}{r}.\underset{\sim}{h}) - \kappa f, \tag{4.2.4}$$

as may be readily verified (App. II). Also operating with $\nabla\wedge$ on both sides of (4.2.2), we obtain the vector Poisson equation

$$\nabla^2 \underset{\sim}{A} = -\nabla\wedge \underset{\sim}{h} \quad , \tag{4.2.5}$$

which has the particular solution

$$\underset{\sim}{A} = -\frac{1}{2}(\underset{\sim}{r} \wedge \underset{\sim}{h}), \tag{4.2.6}$$

as may be readily verified (App. II). It will be noted that

$$\nabla.\underset{\sim}{A} = -\frac{1}{2}\nabla.(\underset{\sim}{r} \wedge \underset{\sim}{h}) = -\frac{1}{2}(\underset{\sim}{h}.\nabla\wedge \underset{\sim}{r} - \underset{\sim}{r}.\nabla\wedge \underset{\sim}{h}) = \frac{1}{2}\underset{\sim}{r}.\nabla\wedge \underset{\sim}{h} \neq 0$$

in general. However we may re-define $\underset{\sim}{A}$ so that $\nabla.\underset{\sim}{A} = 0$ as shown in App. II.

The most important choice of $\underset{\sim}{h}$ is

$$\underset{\sim}{h} = \langle 0,0,\frac{1}{r}\rangle \; ; \; r^2 = x^2 + y^2 + z^2 ,$$

in which case

$$\nabla.\underset{\sim}{A} = -\frac{1}{2}\nabla.(\underset{\sim}{r} \wedge \underset{\sim}{h}) = -\frac{1}{2}\nabla.\langle\frac{y}{r}, -\frac{x}{r}, 0\rangle = 0.$$

Therefore, choosing $f = 0$ in (4.2.2) , we obtain the identity

$$<0,0,\frac{1}{r}> - \kappa\nabla\left(\frac{z}{r}\right) \equiv \frac{1}{2}(1-2\kappa)\nabla\left(\frac{z}{r}\right) - \frac{1}{2}\nabla\wedge<\frac{y}{r},-\frac{x}{r},0>. \qquad (4.2.7)$$

Symmetrical equivalents of (4.2.7) are

$$<0,\frac{1}{r},0> - \kappa\nabla\left(\frac{y}{r}\right) \equiv \frac{1}{2}(1-2\kappa)\nabla\left(\frac{y}{r}\right) - \frac{1}{2}\nabla\wedge<-\frac{z}{r},0,\frac{x}{r}>, \qquad (4.2.8)$$

$$<\frac{1}{r},0,0> - \kappa\nabla\left(\frac{x}{r}\right) \equiv \frac{1}{2}(1-2\kappa)\nabla\left(\frac{x}{r}\right) - \frac{1}{2}\nabla\wedge<0,\frac{z}{r},-\frac{y}{r}>. \qquad (4.2.9)$$

Chapter 5

Limiting Biharmonic Behaviour

5.1 Scalar limiting analysis

Given a biharmonic function

$$S = \frac{1}{2}(1-2\kappa)zh \; ; \quad \nabla^2 h(x,y,z) = 0, \qquad (5.1.1)$$

we show how to construct a function $\mathcal{S}$ with the properties :

(i) $\mathcal{S}$ is a function of the variables

$x,y,z_1 = \left(\frac{z-ct}{\gamma_1}\right) \; ; \quad \gamma_1^2 = 1 - \frac{c^2}{c_1^2}$, which satisfies

$$\nabla^2 \Box_1^2 \mathcal{S} = 0 \qquad (5.1.2)$$

everywhere except possibly at $x = 0$, $y = 0$, $z = ct$.

(ii) $\mathcal{S} \to S(x,y,z)$ as $c \to 0$. (5.1.3)

(iii) $\mathcal{S} \to S(x,y,z-ct)$ as $\frac{1}{c_1} \to 0$. (5.1.4)

Condition (iii) is equivalent to condition (ii) when $t = 0$.

We start with an arbitrary harmonic function $g_0(x,y,z)$ having continuous second-order partial derivatives except possibly at singularities. If so, there exists a corresponding harmonic function $g(x,y,z-ct)$ which satisfies $\nabla^2 g = 0$, and also a wave function $G(x,y,z_1)$ which satisfies

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z_1^2}\right) G = 0, \text{ i.e. } \Box_1^2 G = 0. \text{ Clearly } G \to g_0(x,y,z)$$

as $c \to 0$ and $G \to g(x,y,z-ct)$ as $\frac{1}{c_1} \to 0$, i.e. harmonic limits which have already been met (see Section 2.1). However, we may also obtain a biharmonic limit by observing that G has the Taylor expansion

$$G = g + \frac{1}{2}\frac{c^2}{c_1^2} z' \frac{\partial g}{\partial z'} + O\left(\frac{c^2}{c_1^2}\right)^2 \quad ; \quad z' = z-ct$$

$$= g + \frac{1}{2}(1-\gamma_1^2) z' \frac{\partial g}{\partial z'} + O(1-\gamma_1^2)^2 \tag{5.1.5}$$

in the neighbourhood of $\frac{1}{c_1} = 0$. If so, then

$$2\,\frac{G-g}{1-\gamma_1^2} = z' \frac{\partial g}{\partial z'} + O(1-\gamma_1^2)$$

$$\to z' \frac{\partial g}{\partial z'} \quad \text{as } \frac{1}{c_1} \to 0 . \tag{5.1.6}$$

This result also remains valid (App. III) as $c \to 0$ with $z\frac{\partial g_0}{\partial z}$ replacing $z'\frac{\partial g}{\partial z'}$. Now if we define g_0 so that $\frac{\partial g_0}{\partial z} = h$, then

$$\mathcal{J} = (1-2\kappa)\,\frac{G-g}{1-\gamma_1^2} . \tag{5.1.7}$$

Clearly $\mathcal{S}$ satisfies condition (ii), (iii) above. Also, it satisfies $\nabla^2\Box_1^2\mathcal{S} = 0$, since G satisfies $\Box_1^2 G = 0$ and g satisfies $\nabla^2 g = 0$.

Since c is a constant, there is no loss of generality in putting t = 0 in (5.1.5), so that

$$G(x,y,\frac{z}{\gamma_1}) = g_0 + \frac{1}{2}(1-\gamma_1^2)z\frac{\partial g_0}{\partial z} + O(1-\gamma_1^2)^2 \qquad (5.1.8)$$

in the neighbourhood of $\frac{1}{c_1} = 0$, bearing in mind that $g_0(x,y,z) = g(x,y,z-ct)$ when t = 0. This expression also holds in the neighbourhood of c = 0 (by interchanging c and $\frac{1}{c_1}$ in $\gamma_1^2 = 1 - \frac{c^2}{c_1^2}$).

In this case

$$2\,\frac{G(x,y,\frac{z}{\gamma_1}) - g_0(x,y,z)}{1-\gamma_1^2} \to z\frac{\partial g_0}{\partial z} \qquad (5.1.9)$$

either as $c \to 0$ or $\frac{1}{c_1} \to 0$.

5.2 Vector limiting analysis

Given a biharmonic vector

$$\underset{\sim}{A} = \frac{1}{2}\langle -yh, xh, 0\rangle \;;\quad \nabla^2 h = 0, \qquad (5.2.1)$$

we show how to construct a vector $\underset{\sim}{\mathcal{A}}$ with the properties

(i) $\underset{\sim}{\mathcal{A}}$ is a vector function of the variables $x, y, z_2 = \left(\frac{z-ct}{\gamma_2}\right)$;

$\gamma_2^2 = 1 - \frac{c^2}{c_2^2}$, which satisfies

$$\nabla^2 \Box_2^2 \underset{\sim}{\mathcal{A}} = 0 \qquad (5.2.2)$$

everywhere except possibly at x = 0, y = 0, z = ct.

(ii) $\underset{\sim}{\mathcal{A}} \to \underset{\sim}{A}(x,y,z)$ as $c \to 0$. (5.2.3)

(iii) $\underset{\sim}{\mathcal{A}} \to \underset{\sim}{A}(x,y,z')$ as $\frac{1}{c_2} \to 0$. (5.2.4)

Condition (iii) is equivalent to condition (ii) when t = 0.

First, using the Almansi representation of biharmonic functions, we write

$$\begin{aligned} yh &= zf^* + h^* \quad ; \quad \nabla^2 f^* = 0, \quad \nabla^2 h^* = 0, \\ &= z\frac{\partial g_0^*}{\partial z} + h^* \quad ; \quad \nabla^2 g_0^* = 0, \end{aligned} \qquad (5.2.5)$$

and

$$\begin{aligned} xh &= zf^{**} + h^{**} \quad ; \quad \nabla^2 f^{**} = 0, \quad \nabla^2 h^{**} = 0, \\ &= z\frac{\partial g_0^{**}}{\partial z} + h^{**} \quad ; \quad \nabla^2 g_0^{**} = 0, \end{aligned} \qquad (5.2.6)$$

where g_0^*, g_0^{**} are harmonic functions sufficiently defined by

$\frac{\partial g_0^*}{\partial z} = f^*$, $\frac{\partial g_0^{**}}{\partial z} = f^{**}$. Introducing the functions $G^*(x,y,z_2)$, $g^*(x,y,z')$ corresponding with $g_0^*(x,y,z)$, etc., and $H^*(x,y,z_2)$ corresponding with $h^*(x,y,z)$, etc., it follows that

$$\underset{\sim}{\mathcal{A}} = \left< -\frac{G^*-g^*}{1-\gamma_2^2}, \frac{G^{**}-g^{**}}{1-\gamma_2^2}, 0\right> + \frac{1}{2} <-H^*,H^*,0>$$

$$\to \frac{1}{2} <-zf^*,zf^{**},0> + \frac{1}{2} <-h^*,h^{**},0>$$

$$= \frac{1}{2} <-yh,xh,0> \quad \text{as} \quad c \to 0. \tag{5.2.7}$$

More generally, we write

$$\underset{\sim}{\mathcal{A}} = \left< -\frac{G^*-g^*}{1-\gamma_2^2}, \frac{G^{**}-g^{**}}{1-\gamma_2^2}, 0\right> + \beta(\gamma_2)<-H^*,H^{**},0>, \tag{5.2.8}$$

where $\beta(\gamma_2)$ is a parameter defined so that

$$\beta(\gamma_2) = \frac{1}{2} \quad \text{when} \quad \gamma_2 = 1. \tag{5.2.9}$$

This allows greater flexibility in the expression for $\underset{\sim}{\mathcal{A}}$ to suit specific problems.

If $h = \frac{1}{r}$; $r^2 = x^2+y^2+z^2$, then

$$S = \frac{1}{2} (1-2\kappa)\frac{z}{r} \quad ,$$

$$z\,\frac{1}{r} \;=\; z\,\frac{\partial}{\partial z}\left(\frac{1}{2}\log\frac{r+z}{r-z}\right) \quad , \qquad (5.2.10)$$

i.e. $\quad g_0 = \frac{1}{2}\log\frac{r+z}{r-z}$,

so that

$$\mathcal{S} = \frac{1-2\kappa}{1-\gamma_1^2}\left(\frac{1}{2}\log\frac{R_1+z_1}{R_1-z_1} - \frac{1}{2}\log\frac{r'+z'}{r'-z'}\right) \;;$$

$$R_1^2 = x^2+y^2+z_1^2 \quad , \quad (r')^2 = x^2+y^2+(z')^2 \quad , \quad z' = z-ct.$$

Also

$$\underset{\sim}{A} = \frac{1}{2} < -\frac{y}{r} \,,\, \frac{x}{r} \,,\, 0 >$$

where

$$-y\,\frac{1}{r} \;=\; z\left(\frac{yz}{r\xi^2}\right) - \frac{yr}{\xi^2} \quad ; \; \xi^2 = x^2+y^2$$

$$\equiv\; z\,\frac{\partial}{\partial z}\left(\frac{yr}{\xi^2}\right) - \frac{yr}{\xi^2} \quad , \qquad (5.2.11)$$

i.e. $\quad f^* = -\dfrac{yz}{r\xi^2}$, $\;g_0^* = -\dfrac{yr}{\xi^2}$, $\;h^* = -g_0^*$

and

$$-x\,\frac{1}{r} = z\left(\frac{xz}{r\xi^2}\right) - \frac{xr}{\xi^2}$$

$$\equiv z \frac{\partial}{\partial z}\left(\frac{xr}{\xi^2}\right) - \frac{xr}{\xi^2} \quad , \tag{5.2.12}$$

i.e. $f^{**} = -\dfrac{xz}{r\xi^2}$, $g_0^{**} = -\dfrac{xr}{\xi^2}$, $h^{**} = -g_0^{**}$.

Accordingly, we obtain

$$\underset{\sim}{\mathcal{A}} = \frac{1}{1-\gamma_2^2} \left< \left(\frac{yR_2}{\xi^2} - \frac{yr'}{\xi^2}\right) , -\left(\frac{xR_2}{\xi^2} - \frac{xr'}{\xi^2}\right) ,0 \right>$$

$$+ \beta(\gamma_2) \left< -\frac{yR_2}{\xi^2} , \frac{xR_2}{\xi^2} ,0 \right> ; \tag{5.2.13}$$

$$R_2^2 = x^2+y^2+z_2^2 \quad , (r')^2 = x^2+y^2+(z')^2 \quad , \quad z' = z-ct.$$

Expression (5.2.13) may be more compactly written as

$$\underset{\sim}{\mathcal{A}} = \frac{1}{1-\gamma_2^2} \left< \left(b \frac{yR_2}{\xi^2} - \frac{yr'}{\xi^2}\right) , -\left(b \frac{xR_2}{\xi^2} - \frac{xr'}{\xi^2}\right), 0 \right>$$

where

$$b(\gamma_2) = 1 - (1-\gamma_2^2)\beta \ . \tag{5.2.14}$$

This shows that

$$b(\gamma_2) = 1 \quad \text{when } \gamma_2 = 1 \tag{5.2.15}$$

for any choice of β . Also

$$\frac{db(\gamma_2)}{d\gamma_2} = -(1-\gamma_2^2)\frac{d\beta}{d\gamma_2} + 2\gamma_2\beta \quad ,$$

i.e.

$$\frac{db}{d\gamma_2} = 1 \quad \text{when } \gamma_2 = 1, \text{ by virtue of (5.2.9)}$$

which gives

$$b(\gamma_2) = 1 + (\gamma_2 - 1) + O(\gamma_2 - 1)^2 \equiv \gamma_2 + O(1-\gamma_2)^2 . \qquad (5.2.16)$$

5.3 Superposition of $\nabla\mathcal{S}$ and $\nabla_\wedge \underset{\sim}{\mathcal{A}}$

The superposition

$$\nabla\mathcal{S} + \nabla_\wedge \underset{\sim}{\mathcal{A}} \quad ; \quad \nabla . \underset{\sim}{\mathcal{A}} = 0, \qquad (5.3.1)$$

with $\mathcal{S}$, $\underset{\sim}{\mathcal{A}}$ defined by (5.1.2), (5.2.2) respectively, provides an elastodynamic displacement field. However, this would not be a physically acceptable field unless it reduces to a physically acceptable static field in the limit. By virtue of the theory of Chap. 1, we may always write (5.3.1) in the form

$$\nabla\Phi + \nabla \underset{\sim}{\Psi} \quad ; \quad \nabla . \underset{\sim}{\Psi} = 0 \qquad (5.3.2)$$

where Φ, $\underset{\sim}{\Psi}$ are wave functions. These become singular in the static limit as already noted (section 1.5), but the complete field (5.3.2) remains finite. We shall now apply this analysis to specific problems.

Chapter 6

Three-Dimensional Displacement Fields

6.1 Longitudinal case

A static point-force of magnitude $4\pi\mu^{(*)}$ in an infinite elastic continuum, acting in the z-direction at the origin of co-ordinates, generates the displacement field (Kelvin solution in terms of the Papkovich-Neuber representation)

$$\underset{\sim}{u} = <0,0,\frac{1}{r}> - \kappa\nabla\left(\frac{z}{r}\right) \quad ; \quad \kappa = \frac{1}{4(1-\nu)} \quad . \qquad (6.1.1)$$

In components :

$$u_1 = \kappa\frac{xz}{r^3} \quad , \quad u_2 = \kappa\frac{yz}{r^3} \quad , \quad u_3 = \frac{1-\kappa}{r} + \kappa\frac{z^2}{r^3} \quad . \qquad (6.1.2)$$

As shown in Chap. 4, we may write

$$\underset{\sim}{u} = \nabla S + \nabla\wedge\underset{\sim}{A} \quad ; \quad \nabla.\underset{\sim}{A} = 0$$

where

$$S = \frac{1}{2}(1-2\kappa)\frac{z}{r} \quad , \quad \underset{\sim}{A} = \frac{1}{2}<-\frac{y}{r}, \frac{x}{r}, 0> . \qquad (6.1.3)$$

(*) For a point force of magnitude P we simply replace $4\pi\mu$ by $\frac{P}{4\pi\mu}$ i.e. $\frac{1}{r}$ by $\frac{P}{4\pi\mu r}$.

Following our previous analysis (Chap. 1) , we write

$$\underset{\sim}{U} = \nabla \mathcal{S} + \nabla_{\wedge} \underset{\sim}{\mathcal{A}} \; ; \quad \nabla . \underset{\sim}{\mathcal{A}} = 0, \tag{6.1.4}$$

for the field produced by a point force of magnitude $4\pi\mu$ moving with a uniform velocity c along the z-axis (the line of action of the force), where $\mathcal{S}$, $\underset{\sim}{\mathcal{A}}$ have been determined (Chap. 5) subject to the transformations :

$$z_1 = \frac{z-ct}{\gamma_1} \quad , \quad R_1^2 = x^2+y^2+z_1^2 \; ; \; \gamma_1^2 = 1 - \frac{c^2}{c_1^2} \quad , \tag{6.1.5}$$

$$z_2 = \frac{z-ct}{\gamma_2} \quad , \quad R_2^2 = x^2+y^2+z_2^2 \; ; \; \gamma_2^2 = 1 - \frac{c^2}{c_2^2} \quad . \tag{6.1.6}$$

Accordingly,

$$\mathcal{S} = \frac{1}{2} \frac{1-2\kappa}{1-\gamma_1^2} \left(\log \frac{R_1+z_1}{R_1-z_1} - \log \frac{r'+z'}{r'-z'} \right) \tag{6.1.7}$$

$$\underset{\sim}{\mathcal{A}} = \frac{1}{1-\gamma_2^2} \left\langle \left(b\, \frac{yR_2}{\xi^2} - \frac{yr'}{\xi^2} \right) , - \left(b\, \frac{xR_2}{\xi^2} - \frac{xr'}{\xi^2} \right) ,0 \right\rangle \tag{6.1.8}$$

with $\nabla . \underset{\sim}{\mathcal{A}} = 0$ for any choice of $b(\gamma_2) ; \gamma_2 \neq 1$, $(r')^2 = x^2+y^2+(z')^2$, $z' = z-ct$. Substituting (6.1.7), (6.1.8) into (6.1.4), we get

$$\underset{\sim}{U} = \frac{1}{2} \frac{1-2\kappa}{1-\gamma_1^2} \nabla \left(\log \frac{R_1+z_1}{R_1-z_1} - \log \frac{r'+z'}{r'-z'} \right) + \frac{1}{1-\gamma_2^2} \nabla_{\wedge} \left\langle \left(b\, \frac{yR_2}{\xi^2} - \frac{yr'}{\xi^2} \right) , \right.$$

$$\left. - \left(b\, \frac{xR_2}{\xi^2} - \frac{xr'}{\xi^2} \right) ,0 \right\rangle . \tag{6.1.9}$$

Now, from (1.4.7) i.e. $1 - \dfrac{c_2^2}{c_1^2} = 2\kappa$, we have

$$1-2\kappa = \frac{c_2^2}{c_1^2} . \tag{6.1.10}$$

But

$$\frac{1-\gamma_1^2}{1-\gamma_2^2} = \frac{c_2^2}{c_1^2} , \tag{6.1.11}$$

so that

$$\frac{1-2\kappa}{1-\gamma_1^2} = \frac{1}{1-\gamma_1^2}\,\frac{1-\gamma_1^2}{1-\gamma_2^2} = \frac{1}{1-\gamma_2^2} . \tag{6.1.12}$$

Since

$$\frac{1}{2}\nabla \log \frac{r'+z'}{r'-z'} = \nabla\wedge \left\langle -\frac{yr'}{\xi^2} , \frac{xr'}{\xi^2} ,0\right\rangle, \tag{6.1.13}$$

the expression (6.1.9) becomes

$$\underset{\sim}{U} = \frac{1}{2(1-\gamma_2^2)} \nabla \log \frac{R_1+z_1}{R_1-z_1} + \frac{1}{1-\gamma_2^2} \nabla\wedge \left\langle b\frac{yR_2}{\xi^2} , -b\frac{xR_2}{\xi^2} ,0\right\rangle; \tag{6.1.14}$$

$$\nabla . \left\langle b\frac{yR_2}{\xi^2} , -b\frac{xR_2}{\xi^2} ,0\right\rangle = 0.$$

It will be noted that

$$\left.\begin{aligned}\Box_1^2\left(\frac{1}{2(1-\gamma_2^2)}\log\frac{R_1+z_1}{R_1-z_1}\right) &= 0\ ,\\[2ex] \Box_2^2\left(\frac{1}{1-\gamma_2^2}\left\langle b\,\frac{yR_2}{\xi^2}\ ,\ -b\,\frac{xR_2}{\xi^2}\ ,0\right\rangle\right) &= 0\end{aligned}\right\}\text{for } \gamma_2 \neq 1,$$

showing that $\underset{\sim}{U}$ is given by

$$\underset{\sim}{U} = \nabla\Phi + \nabla\wedge\underset{\sim}{\Psi} \quad ; \quad \nabla.\underset{\sim}{\Psi} = 0 \tag{6.1.15}$$

with

$$\Phi = \frac{1}{2(1-\gamma_2^2)}\log\frac{R_1+z_1}{R_1-z_1}\ ,\ \underset{\sim}{\Psi} = \frac{1}{1-\gamma_2^2}\left\langle b\,\frac{yR_2}{\xi^2}\ ,-b\,\frac{xR_2}{\xi^2}\ ,0\right\rangle, \tag{6.1.16}$$

in accordance with (1.3.12), (1.3.13). Clearly Φ , $\underset{\sim}{\Psi} \to \infty$ as $\gamma_2 \to 1$, so providing a good example of the non-existence of Φ , $\underset{\sim}{\Psi}$ in the limiting case as mentioned in Section 1.5. This breakdown of Φ , $\underset{\sim}{\Psi}$ occurs in every subsequent problem.

In components (note $\frac{\partial}{\partial z} = \frac{1}{\gamma_1}\frac{\partial}{\partial z_1}$ or $\frac{1}{\gamma_2}\frac{\partial}{\partial z_2}$ as appropriate) :

$$\left.\begin{aligned} U_1 &= \frac{1}{1-\gamma_2^2}\left(\frac{b}{\gamma_2}\frac{xz_2}{R_2\xi^2} - \frac{xz_1}{R_1\xi^2}\right)\\[2ex] U_2 &= \frac{1}{1-\gamma_2^2}\left(\frac{b}{\gamma_2}\frac{yz_2}{R_2\xi^2} - \frac{yz_2}{R_2\xi^2}\right)\\[2ex] U_3 &= -\frac{1}{1-\gamma_2^2}\left(\frac{b}{R_2} - \frac{\gamma_1^{-1}}{R_1}\right)\end{aligned}\right\} . \tag{6.1.17}$$

These expressions reduce to the corresponding static expressions (6.1.2) as $c \to 0$, bearing in mind that $b = \gamma_2 + \mathcal{O}(1-\gamma_2)^2$.
To calculate b more precisely, we write

$$b = \gamma_2 + e(1-\gamma_2)^2 \qquad (6.1.18)$$

where e is a constant to be determined. If so,

$$U_1 = \frac{1}{1-\gamma_2^2}\left(\frac{xz_2}{R_2\xi^2} - \frac{xz_1}{R_1\xi^2}\right) + \frac{e(1-\gamma_2)}{\gamma_2(1+\gamma_2)}\,\frac{xz_2}{R_2\xi^2} \quad . \qquad (6.1.19)$$

Now

$$\frac{e(1-\gamma_2)}{\gamma_2(1+\gamma_2)}\,\frac{xz_2}{R_2\xi^2} \to 0 \quad \text{as } c \to 0 \quad (\gamma_2 \to 1).$$

Also

$$\frac{e(1-\gamma_2)}{1+\gamma_2}\,\frac{xz_2}{R_2\xi^2} \to \frac{ex}{\xi^2} \qquad \text{as} \quad c \to c_2 \;(\gamma_2 \to 0)$$

since $\dfrac{z_2}{R_2} = \mathcal{O}(1)$ as $\gamma_2 \to 0$, showing that

$$\frac{e(1-\gamma_2)}{\gamma_2(1+\gamma_2)}\,\frac{xz_2}{R_2\xi^2} = \mathcal{O}\left(\frac{1}{\gamma_2}\right) \qquad \text{as } c \to c_2 \quad .$$

Therefore, the second term in (6.1.19) makes no contribution to the static limit $c = 0$, and it becomes unstable as $c \to c_2$.
This instability would preclude the appearance of a transonic

regime $c_2 < c < c_1$, which always exists on physical grounds. We therefore eliminate the instability by choosing e = 0, so allowing us to connect the subsonic solution ($c < c_2$) with the transonic solution ($c_2 < c < c_1$). This gives $b = \gamma_2$. Inserting this value of b into (6.1.17), we obtain

$$\left.\begin{aligned} U_1 &= \frac{1}{1-\gamma_2^2}\left(\frac{xz_2}{R_2\xi^2} - \frac{xz_1}{R_1\xi^2}\right) \\ U_2 &= \frac{1}{1-\gamma_2^2}\left(\frac{yz_2}{R_2\xi^2} - \frac{yR_1}{R_1\xi^2}\right) \\ U_3 &= -\frac{1}{1-\gamma_2^2}\left(\frac{\gamma_2}{R_2} - \frac{\gamma_1^{-1}}{R_1}\right) \end{aligned}\right\} , \qquad (6.1.20)$$

in agreement with results already obtained by Eason et al (1956).

6.2 Transverse case

As before, we start with a static point force of magnitude $4\pi\mu$ acting in the z-direction at the origin of co-ordinates. But it now moves with a uniform velocity c along the x-axis , i.e. transverse to its line of action, and we therefore use the transformations

$$x_1 = \frac{x-ct}{\gamma_1} \;, \quad R_1^2 = x_1^2+y^2+z^2 \;; \quad \gamma_1^2 = 1 - \frac{c^2}{c_1^2} \qquad (6.2.1)$$

$$x_2 = \frac{x-ct}{\gamma_2} \quad , \quad R_2^2 = x_2^2+y^2+z^2 \quad ; \; \gamma_2^2 = 1 - \frac{c^2}{c_2^2} \tag{6.2.2}$$

in place of (6.1.5), (6.1.6). The static potential (6.1.3) still holds, and we have

$$+\, x\,\frac{1}{r} = x\,\frac{\partial}{\partial x}\left(\frac{1}{2}\log\frac{r+x}{r-x}\right) \quad , \tag{6.2.3}$$

$$-\, y\,\frac{1}{r} = x\left(\frac{yx}{r\eta^2}\right) - \frac{yr}{\eta^2} \quad ; \; \eta^2 = y^2+z^2$$

$$= x\,\frac{\partial}{\partial x}\left(\frac{yr}{\eta^2}\right) - \frac{yr}{\eta^2} \quad , \tag{6.2.4}$$

$$-\, z\,\frac{1}{r} = x\left(\frac{zx}{r\eta^2}\right) - \frac{zr}{\eta^2}$$

$$= x\,\frac{\partial}{\partial x}\left(\frac{zr}{\eta^2}\right) - \frac{zr}{\eta^2} \quad . \tag{6.2.5}$$

Adapting the previous analysis, we write

$$\underset{\sim}{U} = \nabla\mathcal{S} + \nabla\wedge\underset{\sim}{\mathcal{A}} \quad ; \; \nabla.\underset{\sim}{\mathcal{A}} = 0, \tag{6.2.6}$$

where

$$\mathcal{S} = -\frac{1-2\kappa}{1-\gamma_1^2}\left(a\,\frac{zR_1}{\eta^2} - \frac{zr'}{\eta^2}\right) \quad , \tag{6.2.7}$$

$$\underset{\sim}{\mathcal{A}} = \frac{1}{1-\gamma_2^2}\left\langle\left(b\,\frac{yR_2}{\eta^2} - \frac{yr'}{\eta^2}\right), \left(\frac{1}{2}\log\frac{R_2+x_2}{R_2-x_2} - \frac{1}{2}\log\frac{r'+x'}{r'-x'}\right), 0\right\rangle, \tag{6.2.8}$$

$$(r')^2 = (x')^2+y^2+z^2 \quad , \quad x' = x-ct,$$

and $a(\gamma_1)$, $b(\gamma_2)$ are parameters to be determined. By contrast with (6.1.8), the requirement $\nabla.\underset{\sim}{\mathcal{A}} = 0$ implies $b = \gamma_2$ as may be verified, but nothing can be said about $a(\gamma_1)$ at this stage. Therefore, inserting (6.2.7), (6.2.8) into (6.2.6), we get

$$\underset{\sim}{U} = -\frac{1-2\kappa}{1-\gamma_1^2}\nabla\left(a\frac{zR_1}{\eta^2} - \frac{zr'}{\eta^2}\right) + \frac{1}{1-\gamma_2^2}\nabla\wedge\left<\left(\gamma_2\frac{yR_2}{\eta^2} - \frac{yr'}{\eta^2}\right),\right.$$

$$\left.\left(\frac{1}{2}\log\frac{R_2+x_2}{R_2-x_2} - \frac{1}{2}\log\frac{r'+x'}{r'-x'}\right), 0\right> . \quad (6.2.9)$$

Bearing in mind (6.1.12), and observing that

$$\nabla\left(\frac{zr'}{\eta^2}\right) = \nabla\wedge\left<\frac{yr'}{\eta^2}, \frac{1}{2}\log\frac{r'+x'}{r'-x'}, 0\right>, \quad (6.2.10)$$

the expression (6.2.9) becomes

$$\underset{\sim}{U} = -\frac{1}{1-\gamma_2^2}\nabla\left(a\frac{zR_1}{\eta^2}\right) + \frac{1}{1-\gamma_2^2}\nabla\wedge\left<\gamma_2\frac{yR_2}{\eta^2}, \frac{1}{2}\log\frac{R_2+x_2}{R_2-x_2}, 0\right>; \quad (6.2.11)$$

$$\nabla.\left<\gamma_2\frac{yR_2}{\eta^2}, \frac{1}{2}\log\frac{R_2+x_2}{R_2-x_2}, 0\right> = 0.$$

In components (note $\frac{\partial}{\partial x} = \frac{1}{\gamma_1}\frac{\partial}{\partial x_1}$ or $\frac{1}{\gamma_2}\frac{\partial}{\partial x_2}$ as appropriate) :

$$\left.\begin{aligned}
U_1 &= \frac{1}{1-\gamma_2^2}\frac{z}{\eta^2}\left\{\frac{x_2}{R_2} - \frac{a}{\gamma_1}\frac{x_1}{R_1}\right\} \\
U_2 &= -\frac{1}{1-\gamma_2^2}\frac{yz}{\eta^4}\left\{(\gamma_2 R_2 - aR_1) + \left(\frac{\gamma_2 x_2^2}{R_2} - \frac{ax_1^2}{R_1}\right)\right\} \\
U_3 &= \frac{1}{1-\gamma_2^2}\left\{\frac{1-\gamma_2^2}{\gamma_2 R_2} + \frac{y^2}{\eta^4}(\gamma_2 R_2 - aR_1) - \frac{z^2}{\eta^4}\left(\frac{\gamma_2 x_2^2}{R_2} - \frac{ax_1^2}{R_1}\right)\right\}
\end{aligned}\right\} \quad (6.2.12)$$

We may prove that $a = \gamma_1$ by the same line of argument as used to prove $b = \gamma_2$ from (6.1.18). Of course $c_1 > c_2$ and therefore $c \to c_1$ implies $c > c_2$. This is the transonic regime $c_2 < c < c_1$, which connects up with the subsonic regime $c < c_2$ at $c = c_2$ ($\gamma_2 = 0$). Inserting $a = \gamma_1$ into (6.2.12) we get

$$\left.\begin{aligned}
U_1 &= \frac{1}{1-\gamma_2^2}\frac{z}{\eta^2}\left\{\frac{x_2}{R_2} - \frac{x_1}{R_1}\right\} \\
U_2 &= -\frac{1}{1-\gamma_2^2}\frac{yz}{\eta^4}\left\{(\gamma_2 R_2 - \gamma_1 R_1) + \left(\frac{\gamma_2 x_2^2}{R_2} - \frac{\gamma_1 x_1^2}{R_1}\right)\right\} \\
U_3 &= \frac{1}{1-\gamma_2^2}\left\{\frac{1-\gamma_2^2}{\gamma_2 R_2} + \frac{y^2}{\eta^4}(\gamma_2 R_2 - \gamma_1 R_1) - \frac{z^2}{\eta^4}\left(\frac{\gamma_2 x_2^2}{R_2} - \frac{\gamma_1 x_1^2}{R_1}\right)\right\}
\end{aligned}\right\} \quad (6.2.13)$$

in agreement with Eason et al (1956). These expressions reduce to the static expressions (6.1.2) as $c \to 0$.

PART III

TWO-DIMENSIONAL THEORY

Chapter 7

Two-Dimensional Dislocations

7.1 Displacement formulae

Problems of two-dimensional isotropic elasticity, in the absence of body forces, are most efficiently solved by introducing the Airy stress function χ (Airy, 1863). This is a biharmonic function i.e. $\nabla^4\chi = \nabla^2(\nabla^2\chi) = 0$, with the properties :

$$p_{11} = \frac{\partial^2\chi}{\partial x^2}, \quad p_{22} = \frac{\partial^2\chi}{\partial y^2}, \quad p_{12} = -\frac{\partial^2\chi}{\partial x\partial y}, \qquad (7.1.1)$$

where p_{11}, p_{22}, p_{12} $(= p_{21})$ are the 2-dimensional stress components. It is convenient to work with the Almansi representation (Almansi, 1897)

$$\chi = x\phi+\psi \;;\quad \nabla^2\phi = 0,\ \nabla^2\psi = 0, \qquad (7.1.2)$$

or equivalently

$$\chi = y\phi+\psi \;;\quad \nabla^2\phi = 0,\ \nabla^2\psi = 0, \qquad (7.1.3)$$

where $\phi = \phi(x,y)$, $\psi = \psi(x,y)$. In terms of χ, the displacement components in plane strain may be expressed

(Coker and Filon, 1975) by

$$2\mu u_1 = (1-\nu)H - \frac{\partial\chi}{\partial x}, \qquad (7.1.4)$$

$$2\mu u_2 = (1-\nu)\overline{H} - \frac{\partial\chi}{\partial y}, \qquad (7.1.5)$$

where μ, ν denote the shear modulus and Poisson's ratio respectively and $H,\overline{H}$ are conjugate harmonic functions sufficiently defined by

$$\frac{\partial H}{\partial x} = \frac{\partial \overline{H}}{\partial y} = \nabla^2\chi. \qquad (7.1.6)$$

Accordingly, utilising the representation (7.1.2), we find

$$\nabla^2\chi = 2\,\frac{\partial\phi}{\partial x}, \quad \text{i.e.} \quad H = 2\phi \;, \; \overline{H} = 2\overline{\phi}, \qquad (7.1.7)$$

so that

$$2\mu\underset{\sim}{u} = 2(1-\nu)\langle\phi,\overline{\phi}\rangle - \nabla\chi \quad ; \; \chi = x\phi+\psi \;. \qquad (7.1.8)$$

Alternatively, utilising the representation (7.1.3), we find

$$\nabla^2\chi = 2\,\frac{\partial\phi}{\partial y} \quad \text{i.e.} \quad H = -2\overline{\phi} \;, \; \overline{H} = 2\phi, \qquad (7.1.9)$$

so that

$$2\mu\underset{\sim}{u} = 2(1-\nu)\langle-\overline{\phi},\phi\rangle - \nabla\chi \quad ; \; \chi = y\phi+\psi \;. \qquad (7.1.10)$$

These formulae may be proved directly from the Papkovich-Neuber representation (4.1.1), See App. IV, and they may be adapted to plane stress on replacing ν by $\frac{\nu}{1+\nu}$. It may be remarked that formula (7.1.8) covers the case of a rigid-body rotation $\langle d_1 y, -d_1 x\rangle$ and a rigid-body translation $\langle d_2, d_3\rangle$ by writing

$$\phi = \frac{\mu}{1-\nu}(d_1 y + d_2) \quad \text{i.e.} \quad \overline{\phi} = \frac{\mu}{1-\nu}(-d_1 x + d_3),$$

$$\psi = -\frac{\mu}{1-\nu}(d_1 xy + d_2 x) \ ,$$

where d_1, d_2, d_3 are arbitrary constants. Alternative descriptions involving ϕ , $\overline{\phi}$ are given by Bhattacharyya and Symm (1980). Similar remarks apply to formula (7.1.10).

Following the theory of Part II, we write

$$\underset{\sim}{u} = \nabla S + \nabla_\wedge \underset{\sim}{A} \ ; \ \nabla . \underset{\sim}{A} = 0. \tag{7.1.11}$$

If so, $\nabla . \underset{\sim}{u} = \nabla^2 S$ which gives from (7.1.8) :

$$\nabla^2 S = \frac{2(1-\nu)}{2\mu}\left(\frac{\partial\phi}{\partial x} + \frac{\partial\overline{\phi}}{\partial y}\right) - \frac{1}{2\mu}\nabla^2(x\phi+\psi)$$

$$= \frac{1-\nu}{\mu}\left(2\frac{\partial\phi}{\partial x}\right) - \frac{1}{\mu}\frac{\partial\phi}{\partial x} - \frac{1}{2\mu}\nabla^2\psi,$$

i.e.

$$\nabla^2 S = \frac{1-2\nu}{\mu}\frac{\partial\phi}{\partial x} - \frac{1}{2\mu}\nabla^2\psi \text{ or } \frac{1-2\nu}{\mu}\frac{\partial\bar{\phi}}{\partial y} - \frac{1}{2\mu}\nabla^2\psi\,, \qquad (7.1.12)$$

noting that $\frac{\partial\phi}{\partial x} = \frac{\partial\bar{\phi}}{\partial y}$. This is a Poisson equation for S with the particular solutions

$$S = \frac{1-2\nu}{2\mu}x\phi - \frac{1}{2\mu}\psi \quad \text{or} \quad \frac{1-2\nu}{2\mu}y\bar{\phi} - \frac{1}{2\mu}\psi\,. \qquad (7.1.13)$$

Also,

$$\nabla\wedge\underset{\sim}{u} = \nabla(\nabla.\underset{\sim}{A}) - \nabla^2\underset{\sim}{A}, \text{ which gives}$$

$$-\nabla^2\underset{\sim}{A} = \frac{2(1-\nu)}{2\mu}\left(\frac{\partial\bar{\phi}}{\partial x} - \frac{\partial\phi}{\partial y}\right)\underset{\sim}{k}\,,$$

i.e.

$$\nabla^2\underset{\sim}{A} = -\frac{2(1-\nu)}{\mu}\frac{\partial\bar{\phi}}{\partial x}\underset{\sim}{k} \quad \text{or} \quad \frac{2(1-\nu)}{\mu}\frac{\partial\phi}{\partial y}\underset{\sim}{k}\,, \qquad (7.1.14)$$

noting that $\frac{\partial\phi}{\partial y} = -\frac{\partial\bar{\phi}}{\partial x}$. This is a vector Poisson equation with the particular solutions

$$\underset{\sim}{A} = -\frac{1-\nu}{\mu}x\bar{\phi}\underset{\sim}{k} \quad \text{or} \quad \underset{\sim}{A} = \frac{1-\nu}{\mu}y\,\phi\,\underset{\sim}{k}\,. \qquad (7.1.15)$$

Accordingly, choosing

$$S = \frac{1-2\nu}{2\mu}x\phi - \frac{1}{2\mu}\psi,\ \underset{\sim}{A} = -\frac{1-\nu}{\mu}x\bar{\phi}\underset{\sim}{k}\,, \qquad (7.1.16)$$

we obtain

$$\underset{\sim}{u} = \frac{1}{2\mu}\nabla\{(1-2\nu)x\phi-\psi\} - \frac{1}{\mu}\nabla_\wedge\{(1-\nu)x\bar{\phi}\}\underset{\sim}{k}$$

i.e.

$$2\mu\underset{\sim}{u} = \nabla\{(1-2\nu)x\phi-\psi\} - \nabla_\wedge\{2(1-\nu)x\bar{\phi}\}\underset{\sim}{k} \quad . \tag{7.1.17}$$

It may be verified that (7.1.17) is identically equivalent to (7.1.8), and that no other choice of $S,\underset{\sim}{A}$ has this property.

Similarly, choosing

$$S = \frac{1-2\nu}{2\mu}y\phi - \frac{1}{2\mu}\psi \quad , \quad \underset{\sim}{A} = -\frac{1-\nu}{\mu}\, y\bar{\phi}\underset{\sim}{k} \ , \tag{7.1.18}$$

we obtain

$$2\mu\underset{\sim}{u} = \nabla\{(1-2\nu)y\phi-\psi\} - \nabla_\wedge\{2(1-\nu)y\bar{\phi}\}\underset{\sim}{k} \quad , \tag{7.1.19}$$

corresponding with formula (7.1.10).

7.2 Edge dislocation (null case)

Corresponding with $\chi = x\phi+\psi$ where

$$\left.\begin{aligned} \phi &= \frac{\mu}{1-\nu}\log r \quad , \quad \bar{\phi} = \frac{\mu}{1-\nu}\theta \ , \ \psi = 0 \ ; \\ r^2 &= x^2+y^2 \quad , \quad \theta = \tan^{-1}\!\left(\frac{y}{x}\right) \end{aligned}\right\} \ , \tag{7.2.1}$$

formula (7.1.8) provides the displacement field

$$2\mu \underset{\sim}{u} = 2(1-\nu)\left\langle \frac{\mu}{1-\nu} \log r \, , \frac{\mu}{1-\nu} \theta \right\rangle - \frac{\mu}{1-\nu} \nabla(x \log r),$$

i.e. $$\underset{\sim}{u} = \langle \log r \, , \theta \rangle - 2\kappa \nabla(x \log r) \; ; \; \kappa = \frac{1}{4(1-\nu)} \, . \qquad (7.2.2)$$

In components :

$$u_1 = \log r - 2\kappa \frac{\partial}{\partial x}(x \log r) = (1-2\kappa)\log r - 2\kappa \frac{x^2}{r^2} \, , \qquad (7.2.3)$$

$$u_2 = \theta - 2\kappa \frac{\partial}{\partial y}(x \log r) = \theta - 2\kappa \frac{xy}{r^2} \, . \qquad (7.2.4)$$

This shows that u_2 jumps by an amount 2π(*) for any anti-clockwise circuit around the origin, while the component u_1 remain unaltered. More precisely, introducing the symbol [ ] - see (3.1.2) - to indicate the jump in a quantity on circuiting the origin :

$$[\log r\,] = 0, \quad \left[\frac{\partial}{\partial x}(x \log r)\right] = 0 \qquad (7.2.5)$$

$$[\theta] = 2\pi, \quad \left[\frac{\partial}{\partial y}(x \log r)\right] = 0 \qquad (7.2.6)$$

(*) For a jump of amount p we replace 2π by $\frac{p}{2\pi}$
i.e. choosing $\phi = \frac{p}{2\pi}\frac{\mu}{1-\nu}\log r$, etc.

so that

$$[u_1] = 0, \quad [u_2] = 2\pi \quad . \tag{7.2.7}$$

This property defines an edge dislocation of Burgers' vector (Cottrell, 1964 ; Nabarro, 1967) $[u_2]\underset{\sim}{j}$. It is convenient to think of the edge dislocation as a line coinciding with the z-axis, i.e. at right angles to the Burgers' vector. If this dislocation moves with a uniform velocity c along the x-axis, in the positive direction, then a jump of amount $[u_2]$ appears behind it, separating the quadrant $y > 0$ from $y < 0$ as indicated in fig.1. Such a motion would not be physically possible, since it separates the continuum into two non-interacting parts. We may also prove this below by a mathematical analysis following the method of Chap. 5.

Utilising (7.2.1), formulae (7.1.16) give

$$\left.\begin{aligned} S &= \frac{1-2\nu}{2\mu}\,\frac{\mu}{1-\nu}\, x \log r \;=\; (1-2\kappa)x \log r \\ \underset{\sim}{A} &= -\,\frac{1-\nu}{\mu}\,\frac{\mu}{1-\nu}\, x\,\theta\,\underset{\sim}{k} \;=\; -x\,\theta\,\underset{\sim}{k} \end{aligned}\right\} . \tag{7.2.8}$$

If so, the Helmholtz expression (7.1.17) becomes

$$\underset{\sim}{u} = (1-2\kappa)\nabla(x \log r) - \nabla_\wedge(x\theta)\underset{\sim}{k} \quad . \tag{7.2.9}$$

It can be verified that (7.2.9) is identically equivalent to (7.2.2). Now

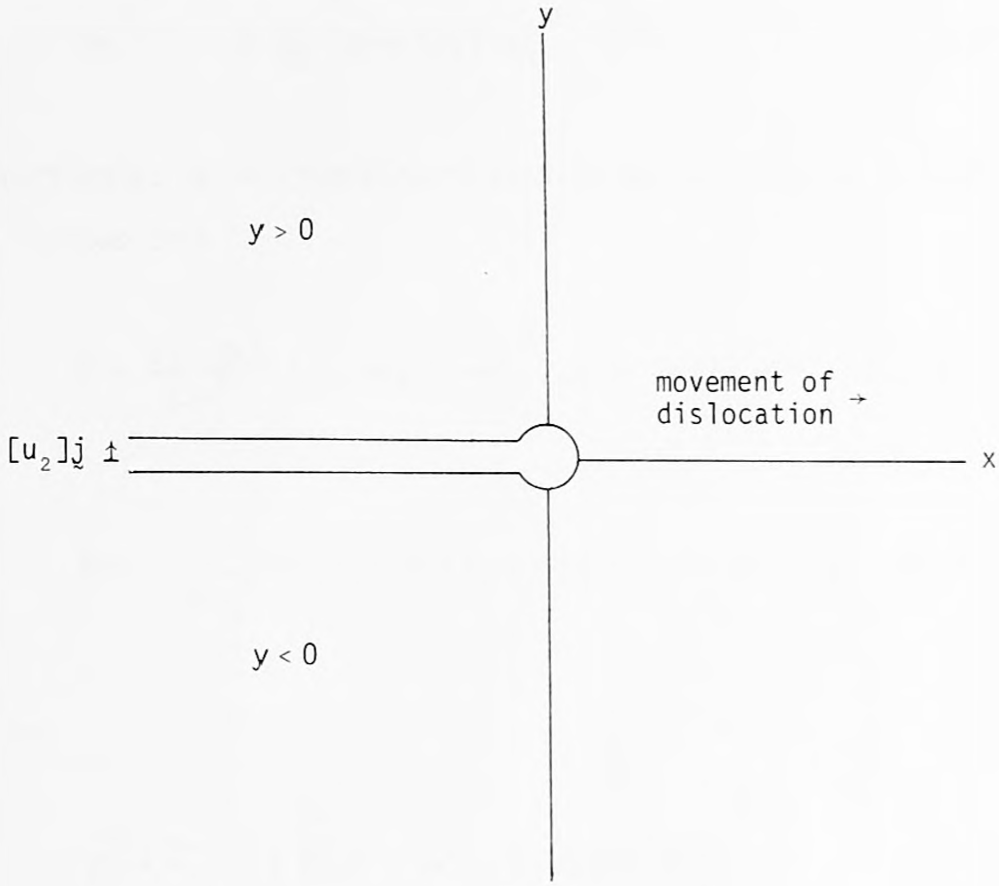

Fig. 1

Edge dislocation (coinciding with z-axis-perpendicular to the plane of paper at the origin) of Burgers' vector $[u_2]\underset{\sim}{j}$ separating the quadrant $y > 0$ from $y < 0$.
As the dislocation moves forward along the positive x-axis it separates the material into two distinct halves, being therefore physically impossible.

$$x \log r = x \frac{\partial}{\partial x}(x \log r - y\theta - x), \qquad (7.2.10)$$

$$x\theta = x \frac{\partial}{\partial x}(x\theta + y \log r - y). \qquad (7.2.11)$$

Accordingly, by straightforward adaptations of (5.1.7), (5.2.8), it follows that

$$\mathfrak{S} = \frac{2(1-2\kappa)}{1-\gamma_1^2}\{(x_1 \log R_1 - y\Theta_1 - x_1) - (x' \log r' - y\theta' - x')\}, \qquad (7.2.12)$$

$$\underset{\sim}{\mathfrak{A}} = -\frac{2}{1-\gamma_2^2}\{(x_2\Theta_2 + y \log R_2 - y) - (x'\theta' + y \log r' - y)\}\underset{\sim}{k}, \qquad (7.2.13)$$

where

$$x_1 = \frac{x-ct}{\gamma_1} \quad ; \; R_1^2 = x_1^2 + y^2 \; , \; \Theta_1 = \tan^{-1}\left(\frac{y}{x_1}\right) \; , \qquad (7.2.14)$$

$$x_2 = \frac{x-ct}{\gamma_2} \quad ; \; R_2^2 = x_2^2 + y^2 \; , \; \Theta_2 = \tan^{-1}\left(\frac{y}{x_2}\right) \; , \qquad (7.2.15)$$

$$x' = x - ct \quad ; \; (r')^2 = (x')^2 + y^2 \quad , \; \theta' = \tan^{-1}\left(\frac{y}{x'}\right) \; .$$

Now, following the general theory of Part II,

$$\underset{\sim}{U} = \nabla\mathfrak{S} + \nabla_\wedge \underset{\sim}{\mathfrak{A}} \qquad ; \; \nabla . \underset{\sim}{\mathfrak{A}} = 0,$$

for the uniformly moving field. Substituting for $\mathfrak{S}$, $\underset{\sim}{\mathfrak{A}}$ from (7.2.12), (7.2.13), we find

$$\underset{\sim}{U} = \frac{2(1-2\kappa)}{1-\gamma_1^2} \nabla\{(x_1 \log R_1 - y\Theta_1 - x_1) - (x' \log r' - y\theta' - x')\}$$

$$- \frac{2}{1-\gamma_2^2} \nabla\wedge \{(x_2\Theta_2 + y \log R_2 - y) - (x'\theta' + y \log r' - y)\}\underset{\sim}{k}. \quad (7.2.16)$$

Bearing in mind that

$$\frac{2(1-2\kappa)}{1-\gamma_1^2} = \frac{2}{1-\gamma_2^2}, \quad \text{see (6.1.12)},$$

and that

$$\nabla(x' \log r' - y\theta' - x') = \nabla\wedge(x'\theta' + y \log r' - y)\underset{\sim}{k}, \quad (7.2.17)$$

the expression (7.2.16) becomes

$$\underset{\sim}{U} = \frac{2}{1-\gamma_2^2} \nabla(x_1 \log R_1 - y\Theta_1 - x_1) - \frac{2}{1-\gamma_2^2} \nabla\wedge(x_2\Theta_2 + y \log R_2 - y)\underset{\sim}{k}. \quad (7.2.18)$$

In components (note $\frac{\partial}{\partial x} = \frac{1}{\gamma_1}\frac{\partial}{\partial x_1}$ or $\frac{1}{\gamma_2}\frac{\partial}{\partial x_2}$ as appropriate):

$$U_1 = \frac{2}{1-\gamma_2^2}\left(\frac{1}{\gamma_1} \log R_1 - \log R_2\right), \quad (7.2.19)$$

$$U_2 = -\frac{2}{1-\gamma_2^2}\left(\Theta_1 - \frac{1}{\gamma_2}\Theta_2\right). \quad (7.2.20)$$

As $c \to 0$, the expressions (7.2.19), (7.2.20) reduce to the corresponding static expressions (7.2.3), (7.2.4) respectively. Also

$$[U_1] = \frac{2}{1-\gamma_2^2}\left\{\frac{1}{\gamma_1}\ [\log R_1] - [\log R_2]\right\} = 0, \qquad (7.2.21)$$

$$[U_2] = -\frac{2}{1-\gamma_2^2}\left\{[\Theta_1] - \frac{1}{\gamma_2}\ [\Theta_2]\right\} = -\frac{2}{1-\gamma_2^2}\left(1-\frac{1}{\gamma_2}\right)2\pi$$

$$= \frac{2}{1-\gamma_2^2}\ \frac{1-\gamma_2}{\gamma_2}\ 2\pi = \frac{4\pi}{\gamma_2(1+\gamma_2)}\ , \qquad (7.2.22)$$

by virtue of (3.2.2). In the static case $\gamma_2 = 1$, and therefore $[U_2] = 2\pi$ i.e. $[U_2] = [u_2] = 2\pi$ as expected. However $[U_2] \neq 2\pi$ for any other allowed choice of γ_2. This means that we cannot construct a uniformly moving dislocation field which has the same strength as the static dislocation, and we therefore conclude that such a motion is impossible.

7.3 Edge dislocation (slip case)

An alternative possibility to the preceding case is that the dislocation moves uniformly along the y-axis, i.e. the direction of its Burgers' vector. If so, then a jump of amount $[u_2]$ appears behind it, displacing the quadrant $x < 0$ relative to $x > 0$ as indicated in fig. 2. Such a motion is physically possible since the two quadrants remain in contact. Usually the yOz plane is termed the slip plane, since the

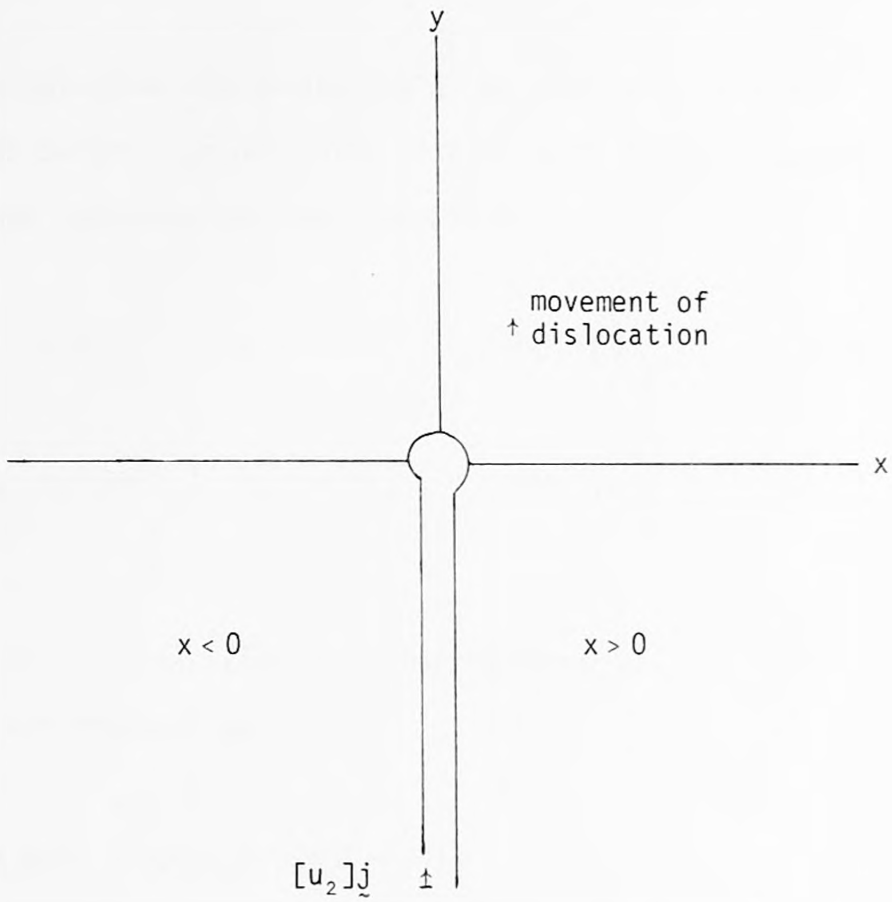

Fig. 2

Edge dislocation (coinciding with z-axis-perpendicular to the plane of the paper at the origin) of Burgers' vector $[u_2]\underset{\sim}{j}$ separating the quadrant $x > 0$ from $x < 0$. As the dislocation moves forward along the positive y-direction the quadrants remain in contact so that this would be a physically possible motion.

displacement jumps across this plane as the dislocation propagates along it.

To determine the moving field, we start with (7.2.3), (7.2.4) as before, and of course (7.2.8) still holds. However, we must now introduce the transformations :

$$y_1 = \frac{y-ct}{\gamma_1} \quad ; \quad R_1^2 = x^2+y_1^2 \ , \ \Theta_1 = \tan^{-1}\left(\frac{y_1}{x}\right) , \qquad (7.3.1)$$

$$y_2 = \frac{y-ct}{\gamma_2} \quad ; \quad R_2^2 = x^2+y_2^2 \ , \ \Theta_2 = \tan^{-1}\left(\frac{y_2}{x}\right) , \qquad (7.3.2)$$

in place of (7.2.14), (7.2.15). Correspondingly (7.2.10), (7.2.11) are replaced by

$$-x \log r = -y\theta - (x \log r - y\theta)$$

$$= y \frac{\partial}{\partial y} (x \log r - y\theta) - (x \log r - y\theta), \qquad (7.3.3)$$

$$-x\theta = y \log r - (x\theta + y \log r)$$

$$= y \frac{\partial}{\partial y} (x\theta + y \log r - y) - (x\theta + y \log r) . \qquad (7.3.4)$$

If so, then

$$\mathfrak{J} = - \frac{2(1-2\kappa)}{1-\gamma_2^2} \{a(x \log R_1 - y_1\Theta_1) - (x \log r' - y'\theta')\}, \qquad (7.3.5)$$

$$\underset{\sim}{\mathcal{A}} = \frac{2}{1-\gamma_2^2}\{b(x\Theta_2+y_2\log R_2 - y_2)-(x\theta'+y'\log r' - y')\}\underset{\sim}{k} \;, \quad (7.3.6)$$

where

$$y' = y-ct, \; (r')^2 = x^2+(y')^2 \;, \; \theta' = \tan^{-1}\left(\frac{y'}{x}\right) \;.$$

The introduction of $a(\gamma_1)$, $b(\gamma_2)$ is necessary - by contrast with (7.2.12), (7.2.13) - because of the residual harmonic functions $(x\log r - y\theta)$, $(x\theta + y\log r)$ which appear in (7.3.3), (7.3.4) respectively. By reference to Chap. 5 these may be accounted for by writing $a(x\log R_1 - y_1\Theta_1)$ in place of $(x\log R_1 - y_1\Theta_1)$ etc. Accordingly for the moving field :

$$\underset{\sim}{U} = \nabla\mathcal{S} + \nabla\wedge\underset{\sim}{\mathcal{A}} \;\; ; \quad \nabla.\underset{\sim}{\mathcal{A}} = 0. \qquad (7.3.7)$$

Inserting (7.3.5), (7.3.6) into (7.3.7) , we get

$$\underset{\sim}{U} = -\frac{2(1-2\kappa)}{1-\gamma_1^2}\nabla\{a(x\log R_1 - y_1\Theta_1)-(x\log r' - y'\theta')\}$$

$$+\frac{2}{1-\gamma_2^2}\nabla\wedge\{b(x\Theta_2+y_2\log R_2 - y_2)-(x\theta'+y'\log r' - y')\}\underset{\sim}{k}. \quad (7.3.8)$$

Bearing in mind that

$$\frac{2(1-2\kappa)}{1-\gamma_1^2} = \frac{2}{1-\gamma_2^2} \;,$$

and that

$$\nabla(x \log r' - y'\theta') = \nabla\wedge(x\theta' + y'\log r')\,\underset{\sim}{k}\ , \qquad (7.3.9)$$

the expression (7.3.8) becomes

$$\underset{\sim}{U} = -\frac{2}{1-\gamma_2^2}\,\nabla\{a(x \log R_1 - y_1\Theta_1)\}$$

$$+\frac{2}{1-\gamma_2^2}\nabla\wedge\{b(x\Theta_2 + y_2 \log R_2) - (y_2 - y')\}\underset{\sim}{k}\ . \qquad (7.3.10)$$

In components (note $\frac{\partial}{\partial y} = \frac{1}{\gamma_1}\frac{\partial}{\partial y_1}$ or $\frac{1}{\gamma_2}\frac{\partial}{\partial y_2}$ as appropriate) :

$$U_1 = -\frac{2}{1-\gamma_2^2}\left\{a(1+\log R_1) - \frac{b}{\gamma_2}(1+\log R_2) + (\frac{1}{\gamma_2} - 1)\right\}$$

$$= -\frac{2}{1-\gamma_2^2}\left\{a \log R_1 - \frac{b}{\gamma_2}\log R_2\right\} + \frac{2}{1-\gamma_2^2}\left\{1-a-\frac{1-b}{\gamma_2}\right\}, \qquad (7.3.11)$$

$$U_2 = \frac{2}{1-\gamma_2^2}\left\{\frac{a}{\gamma_1}\Theta_1 - b\Theta_2\right\}. \qquad (7.3.12)$$

As regards the strength of dislocation, we note that

$$[U_1] = -\frac{2}{1-\gamma_2^2}\{a[\log R_1] - \frac{b}{\gamma_2}[\log R_2]\} = 0, \qquad (7.3.13)$$

and that

$$[U_2] = \frac{2}{1-\gamma_2^2}\{\frac{a}{\gamma_1}[\Theta_1] - b[\Theta_2]\} = \frac{2}{1-\gamma_2^2}\left(\frac{a}{\gamma_1} - b\right)2\pi, \qquad (7.3.14)$$

by virtue of (3.2.2). There is no difficulty in choosing the parameters $a(\gamma_1)$, $b(\gamma_2)$ to satisfy the condition $[U_2] = 2\pi$, which ensures that the strength of the uniformly moving dislocation equals that of the static dislocation for every choice of $c < c_2$. This condition gives

$$\frac{2}{1-\gamma_2^2}\left(\frac{a}{\gamma_1} - b\right)2\pi = 2\pi,$$

$$\text{i.e.} \quad \frac{a(\gamma_1)}{\gamma_1} - b(\gamma_2) = \frac{1}{2}(1-\gamma_2^2),$$

$$\text{i.e.} \quad \frac{a(\gamma_1)}{\gamma_1} = b(\gamma_2) + \frac{1}{2}(1-\gamma_2^2). \tag{7.3.15}$$

Since equation (7.3.15) must hold for every allowed choice of γ_1, γ_2 , each side must have the same constant value d, i.e.

$$\frac{a(\gamma_1)}{\gamma_1} = d \quad , \quad b(\gamma_2) + \frac{1}{2}(1-\gamma_2^2) = d \tag{7.3.16}$$

from which

$$a = \gamma_1 d \ , \ b = d - \frac{1}{2} + \frac{1}{2}\gamma_2^2 \ . \tag{7.3.17}$$

Bearing in mind that $a(\gamma_1) = 1$ when $\gamma_1 = 1$ and $b(\gamma_2) = 1$ when $\gamma_2 = 1$, it follows that $d = 1$, showing that

$$a = \gamma_1 \ , \ b = \frac{1}{2}(1+\gamma_2^2). \tag{7.3.18}$$

It may be noted that $\frac{1}{2}(1+\gamma_2^2) = \gamma_2 + \frac{1}{2}(1-\gamma_2)^2$ in accordance with the general form $b = \gamma_2 + O(1-\gamma_2)^2$. Inserting the result (7.3.18) into (7.3.11), (7.3.12) and noting that

$$\frac{2}{1-\gamma_2^2}\left(1-a-\frac{1-b}{\gamma_2}\right) = \frac{2}{1-\gamma_2^2}\left\{1-\gamma_1-\frac{1-\frac{1}{2}(1+\gamma_2^2)}{\gamma_2}\right\}$$

$$= \frac{2}{1-\gamma_2^2}\left(\frac{1-\gamma_1^2}{1+\gamma_1}-\frac{1-\gamma_2^2}{2\gamma_2}\right) = \frac{2(1-2\kappa)}{1+\gamma_1}-\frac{1}{\gamma_2} ,$$

by virtue of (6.1.12), we get

$$U_1 = -\frac{2}{1-\gamma_2^2}\left\{\gamma_1 \log R_1 - \frac{\frac{1}{2}(1+\gamma_2^2)}{\gamma_2}\log R_2\right\}+\left\{\frac{2(1-2\kappa)}{1+\gamma_1}-\frac{1}{\gamma_2}\right\} \quad (7.3.19)$$

$$U_2 = \frac{2}{1-\gamma_2^2}\left\{\Theta_1 - \frac{1}{2}(1+\gamma_2^2)\,\Theta_2\right\} . \quad (7.3.20)$$

For a given value of $c < c_2$ the final term in (7.3.19) is a constant which can be omitted without loss of generality. It may be verified that (7.3.19), (7.3.20) reduce to the corresponding static expressions (7.2.3), (7.2.4) respectively as $c \to 0$. Our solution agrees with that of Eshelby (1949) on interchanging the role of x and y as indicated in fig. 3.

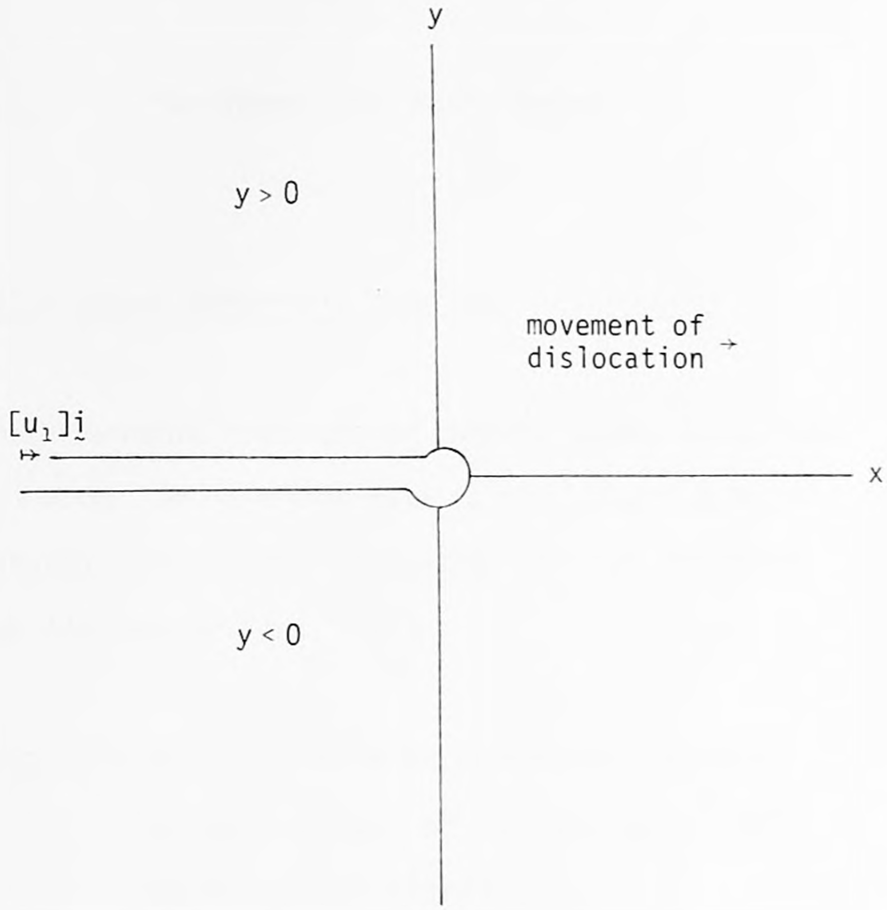

Fig. 3

This is equivalent to Fig. 2. rotated clockwise through 90^0, so that u_2 transforms to u_1 and u_1 transforms to $-u_2$, which gives Eshelby's results directly.

Chapter 8

Two-Dimensional Point Forces

8.1 Multi-valued biharmonic functions

The biharmonic functions of chapter 7 were all single-valued. However, an important role is also played by multi-valued biharmonic functions, since they have the following properties (Jaswon and Symm, 1977) :

$-\left[\frac{\partial\chi}{\partial x}\right]$ = resultant force in y-direction produced by the tractions acting upon any circuit enclosing the origin 0 ; (8.1.1)

$\left[\frac{\partial\chi}{\partial y}\right]$ = resultant force in x-direction produced by the tractions acting upon any circuit enclosing the origin 0 ; (8.1.2)

$\left[\chi - x\frac{\partial\chi}{\partial x} - y\frac{\partial\chi}{\partial y}\right]$ = resultant moment produced by the tractions acting upon any circuit around the origin 0. (8.1.3)

Choosing $\chi = -y\theta$, we note that

$$\frac{\partial\chi}{\partial y} = -\left[\theta + y\frac{\partial\theta}{\partial y}\right] = -[\theta]-\left[y\frac{\partial\theta}{\partial y}\right] = -2\pi \ , \qquad (8.1.4)$$

so providing a resultant force of magnitude -2π(*) pointing in the x-direction. For the limiting case of a vanishingly small circuit, we may think of a point force of magnitude -2π pointing in the y-direction at 0.

From formula (7.1.10), it follows that

$$2\mu \underset{\sim}{u} = -2(1-\nu)\langle \log r, \theta\rangle - \nabla(-y\theta) \qquad (8.1.5)$$

i.e.

$$2\mu[u_2] = -2(1-\nu)[\theta] + \left[\frac{\partial}{\partial y}(y\theta)\right]$$

$$= \{-2(1-\nu)+1\}2\pi = -2(1-2\nu)\pi \quad , \qquad (8.1.6)$$

showing the presence of an edge dislocation of Burgers' vector $-\frac{1-2\nu}{\mu}\pi \underset{\sim}{j}$. To remove this dislocation, we replace

$\chi = -y\theta$ by

$$\chi = -y\theta + Nx\log r$$

$$= -(1-N)y\theta + N(x\log r - y\theta), \qquad (8.1.7)$$

where $\left[\frac{\partial\chi}{\partial y}\right] = \left[-\frac{\partial}{\partial y}(y\theta)\right] + N\left[\frac{\partial}{\partial y}(x\log r)\right] = -2\pi$ as before

and N is a constant to be determined subject to $[u_2] = 0$. Formula (7.1.10) now gives

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(*) For a force of magnitude F we replace 2π by $\frac{F}{2\pi}$,

i.e. choosing $\chi = \frac{F}{2\pi}y\theta$ etc.

$$2\mu \underset{\sim}{u} = -2(1-\nu)(1-N)\langle \log r\, ,\theta\rangle - \nabla(-y\theta+Nx\log r). \tag{8.1.8}$$

In components :

$$2\mu u_1 = -2(1-\nu)(1-N)\log r - \frac{\partial}{\partial x}(-y\theta+Nx\log r), \tag{8.1.9}$$

$$2\mu u_2 = -2(1-\nu)(1-N)\theta - \frac{\partial}{\partial y}(-y\theta+Nx\log r). \tag{8.1.10}$$

Accordingly

$$2\mu[u_2] = 2\{-2(1-\nu)(1-N)+1\}\pi$$

$$= 0 \quad \text{if } N = \frac{1-2\nu}{2(1-\nu)} . \tag{8.1.11}$$

Therefore, (8.1.9), (8.1.10) become

$$u_1 = -\frac{1}{2\mu}\log r - \frac{1}{2\mu}\frac{\partial}{\partial x}(-y\theta+Nx\log r)$$

$$= -\frac{1+N}{2\mu}\log r + \frac{1-N}{2\mu}\frac{x^2}{r^2} - \frac{1}{2\mu} , \tag{8.1.12}$$

$$u_2 = -\frac{1}{2\mu}\theta - \frac{1}{2\mu}\frac{\partial}{\partial y}(-y\theta+Nx\log r)$$

$$= \frac{1-N}{2\mu}\frac{xy}{r^2} , \tag{8.1.13}$$

on bearing in mind that

$$2(1-\nu)(1-N) = 1 \quad \text{for } N = \frac{1-2\nu}{2(1-\nu)} . \tag{8.1.14}$$

Similarly, we construct the dislocation-free biharmonic function

$$\chi = x\theta + My \log r \ ; \ M = \frac{1-2\nu}{2(1-\nu)} \equiv N, \tag{8.1.15}$$

for which

$$-\left[\frac{\partial\chi}{\partial x}\right] = -2\pi \quad , \left[\frac{\partial\chi}{\partial y}\right] = 0 \quad , \left[\chi - \frac{\partial\chi}{\partial x} - \frac{\partial\chi}{\partial y}\right] = 0. \tag{8.1.16}$$

8.2 Uniformly moving point force (longitudinal case)

Corresponding with (8.1.7), i.e.

$$\phi = -(1-N)\theta \ , \ \overline{\phi} = (1-N)\log r \ , \psi = N(x \log r - y\theta), \tag{8.2.1}$$

formula (7.1.18) gives

$$S = -\frac{1-2\nu}{2\mu}(1-N)y\theta - \frac{N}{2\mu}(x \log r - y\theta)$$

$$= -\frac{N}{2\mu}y\theta - \frac{N}{2\mu}(x \log r - y\theta) = -\frac{N}{2\mu}x \log r, \tag{8.2.2}$$

since

$$(1-2\nu)(1-N) = N \quad \text{for} \quad N = \frac{1-2\nu}{2(1-\nu)} \quad . \tag{8.2.3}$$

Also,

$$\underset{\sim}{A} = -\frac{1-\nu}{\mu}(1-N)(y \log r)\underset{\sim}{k} = -\frac{1}{2\mu}(y \log r)\underset{\sim}{k} \ , \tag{8.2.4}$$

by virtue of (8.1.14). It will be noted that $\nabla . \underset{\sim}{A} = 0$.
Accordingly, we may write

$$2\mu \underset{\sim}{u} = -N\nabla(x \log r) - \nabla_{\wedge}(y \log r)\underset{\sim}{k} . \qquad (8.2.5)$$

It may be verified that (8.2.5) is equivalent to (8.1.8).

Now,

$$+ x \log r = x \frac{\partial}{\partial x} (x \log r - y\theta - x), \qquad (8.2.6)$$

$$- y \log r = x\theta - (x\theta + y \log r)$$

$$= x \frac{\partial}{\partial x} (x\theta + y \log r) - (x\theta + y \log r). \qquad (8.2.7)$$

Therefore, it follows that

$$\mathcal{S} = -\frac{N}{2\mu} \frac{2}{1-\gamma_1^2} \{(x_1 \log R_1 - y\Theta_1 - x_1) - (x' \log r' - y\theta' - x')\}, \quad (8.2.8)$$

$$\underset{\sim}{\mathcal{A}} = \frac{1}{2\mu} \frac{2}{1-\gamma_2^2} \{b(x_2\Theta_2 + y \log R_2) - (x'\theta' + y \log r')\}\underset{\sim}{k}, \qquad (8.2.9)$$

where

$$x_1 = \frac{x-ct}{\gamma_1} \quad ; \quad R_1^2 = x_1^2 + y^2 \quad , \Theta_1 = \tan^{-1}\left(\frac{y}{x_1}\right) ,$$

$$x_2 = \frac{x-ct}{\gamma_2} \quad ; \quad R_2^2 = x_2^2 + y^2 \quad , \Theta_2 = \tan^{-1}\left(\frac{y}{x_2}\right) ,$$

$$x' = x-ct \quad ; \quad (r')^2 = (x')^2 + y^2 \quad , \theta' = \tan^{-1}\left(\frac{y}{x'}\right) .$$

For the moving field, we write

$$\underset{\sim}{U} = \nabla \mathcal{S} + \nabla_\wedge \underset{\sim}{\mathcal{A}} ; \qquad \nabla . \underset{\sim}{\mathcal{A}} = 0. \tag{8.2.10}$$

Substituting (8.2.8), (8.2.9) into (8.2.10) gives

$$\underset{\sim}{U} = -\frac{1}{\mu}\frac{N}{1-\gamma_1^2} \nabla \{ (x_1 \log R_1 - y\Theta_1 - x_1) - (x' \log r' - yR' - x') \}$$

$$+ \frac{1}{\mu}\frac{1}{1-\gamma_2^2} \nabla \{ b(x_2\Theta_2 + y \log R_2) - (x'\theta' + y \log r') \} \underset{\sim}{k}. \tag{8.2.11}$$

Noting that $N = \dfrac{1-2\nu}{2(1-\nu)} = 1-2\kappa$; $\kappa = \dfrac{1}{4(1-\nu)}$,

so that

$$\frac{N}{1-\gamma_1^2} = \frac{1-2\kappa}{1-\gamma_1^2} = \frac{1}{1-\gamma_2^2} \quad - \text{ see } (6.1.12). \tag{8.2.12}$$

Since

$$\nabla(x' \log r' - y\theta') = \nabla_\wedge (x'\theta' + y \log r') \underset{\sim}{k} \quad , \tag{8.2.13}$$

the expression (8.2.11) becomes

$$\underset{\sim}{U} = -\frac{1}{\mu}\frac{1}{1-\gamma_2^2} \nabla \{ (x_1 \log R_1 - y\Theta_1) - (x_1 - x') \}$$

$$+ \frac{1}{\mu}\frac{1}{1-\gamma_2^2} \nabla_\wedge \{ b(x_2\Theta_2 + y \log R_2) \} \underset{\sim}{k} \; . \tag{8.2.14}$$

In components (note $\frac{\partial}{\partial x} = \frac{1}{\gamma_1}\frac{\partial}{\partial x_1}$ or $\frac{1}{\gamma_2}\frac{\partial}{\partial x_2}$ as appropriate) :

$$U_1 = -\frac{1}{\mu(1-\gamma_2^2)}\left\{\frac{1}{\gamma_1}(1+\log R_1) - \left(\frac{1}{\gamma_1} - 1\right) - b(1+\log R_2)\right\}$$

$$= -\frac{1}{\mu(1-\gamma_2^2)}\left\{\frac{1}{\gamma_1}\log R_1 - b\log R_2\right\} - \frac{1-b}{\mu(1-\gamma_2^2)} , \quad (8.2.15)$$

$$U_2 = \frac{1}{\mu(1-\gamma_2^2)}\left\{\Theta_1 - \frac{b}{\gamma_2}\Theta_2\right\} . \quad (8.2.16)$$

These expressions reduce to the corresponding static expressions (8.1.12), (8.1.13) respectively as $c \to 0$, provided that $b = \gamma_2 + O(1-\gamma_2)^2$. To determine b more precisely, we note that

$$[U_2] = \frac{1}{\mu(1-\gamma_2^2)}\left\{[\Theta_1] - \frac{b}{\gamma_2}[\Theta_2]\right\} = \frac{2\pi}{\mu(1-\gamma_2^2)}\left(1-\frac{b}{\gamma_2}\right),$$

$$= 0 \text{ only if } b = \gamma_2 . \quad (8.2.17)$$

Inserting $b = \gamma_2$ into (8.2.15), (8.2.16) and noting that

$$\frac{1-b}{\mu(1-\gamma_2^2)} = \frac{1-\gamma_2}{\mu(1-\gamma_2^2)} = \frac{1}{\mu(1+\gamma_2)} \to \frac{1}{2\mu} \text{ when } \gamma_2 = 1$$

as in the static solution, we get

$$U_1 = -\frac{1}{\mu(1-\gamma_2^2)}\left\{\frac{1}{\gamma_1}\log R_1 - \gamma_2 \log R_2\right\} - \frac{1}{\mu(1+\gamma_2)}, \qquad (8.2.18)$$

$$U_2 = \frac{1}{\mu(1-\gamma_2^2)}\{\Theta_1 - \Theta_2\} \quad . \qquad (8.2.19)$$

For a given value of c the final term in (8.2.18) is a constant which can be omitted without loss of generality.

This problem has been previously attacked by Eason *et al* (1956) using Fourier transforms, but Eason gave only the stress component p_{12} and the stress combinations $p_{11}+p_{22}$, $p_{11}-p_{22}$. From the expressions (8.2.18), (8.2.19), we obtain

$$e_{11} = \frac{\partial U_1}{\partial x} = -\frac{1}{\mu(1-\gamma_2^2)}\left(\frac{1}{\gamma_1^2}\frac{x_1}{R_1^2} - \frac{x_2}{R_2^2}\right)$$

$$e_{22} = \frac{\partial U_2}{\partial y} = \frac{1}{\mu(1-\gamma_2^2)}\left(\frac{x_1}{R_1^2} - \frac{x_2}{R_2^2}\right)$$

$$\Delta = e_{11} + e_{22} = \frac{1}{\mu(1-\gamma_2^2)}\left\{\left(1-\frac{1}{\gamma_1^2}\right)\frac{x_1}{R_1}\right\}$$

$$= -\frac{1}{\mu}\frac{1-\gamma_1^2}{1-\gamma_2^2}\frac{x_1}{\gamma_1^2 R_1^2} = -\frac{N}{\mu\gamma_1}\frac{x-ct}{(x-ct)^2+\gamma_1^2 y^2} \text{ — see 8.2.12}$$

$$e_{12} = \frac{1}{2}\left(\frac{\partial U_1}{\partial y} + \frac{\partial U_2}{\partial x}\right)$$

$$= -\frac{1}{2\mu(1-\gamma_2^2)}\left\{\frac{2y}{\gamma_1 R_1^2} - \left(\gamma_2 + \frac{1}{\gamma_2}\right)\frac{y}{R_2^2}\right\}$$

$$= -\frac{1}{\mu(1-\gamma_2^2)}\left\{\frac{\gamma_1 y}{(x-ct)^2+\gamma_1^2 y^2} - \frac{\frac{1}{2}(1+\gamma_2^2)\gamma_2 y}{(x-ct)^2+\gamma_2^2 y^2}\right\} .$$

Accordingly,

$$p_{12} = 2\mu e_{12} = -\frac{2}{1-\gamma_2^2}\left\{\frac{\gamma_1 y}{(x-ct)^2+\gamma_1^2 y^2} - \frac{\frac{1}{2}(1+\gamma_2^2)\gamma_2 y}{(x-ct)^2+\gamma_2^2 y^2}\right\} ,$$

$$p_{11}+p_{22} = 2\mu(e_{11}+e_{22}) + 2\lambda\Delta = 2(\lambda+\mu)\Delta .$$

Since $N = \frac{1-2\nu}{2(1-\nu)} = \frac{\mu}{\lambda+2\mu}$ – see (1.4.6) ,

i.e. $\frac{1}{N} = \frac{\lambda+2\mu}{\mu} = 1 + \frac{\lambda+\mu}{\mu}$,

we obtain

$$p_{11}+p_{22} = 2\mu\,\frac{1-N}{N}\,\Delta \tag{8.2.20}$$

$$= 2\mu\,\frac{1-N}{N}\left\{\frac{-N}{\mu\gamma_1}\;\frac{(x-ct)}{(x-ct)^2+\gamma_1^2 y^2}\right\}$$

$$= -\frac{2(1-N)}{\gamma_1}\;\frac{(x-ct)}{(x-ct)^2+\gamma_1^2 y^2} ,$$

$$p_{11}-p_{22} = 2\mu(e_{11}-e_{22})$$

$$= -\frac{2}{1-\gamma_2^2}\left\{\left(\frac{1}{\gamma_1^2}+1\right)\frac{x_1}{R_1^2} - \frac{2x_2}{R_2^2}\right\}$$

$$= -\frac{4}{1-\gamma_2^2}\left\{\frac{\frac{1}{2}(1+\gamma_1^2)\gamma_1^{-1}(x-ct)}{(x-ct)^2+\gamma_1^2 y^2} - \frac{\gamma_2(x-ct)}{(x-ct)^2+\gamma_2^2 y^2}\right\} .$$

All these results are in agreement with Eason et al (1956).

8.3 Uniformly moving point force (transverse case)

We start with the biharmonic function $\chi = -y\theta + Nx \log r$ as before, but the point force now moves with a uniform velocity c along the y-axis (transverse to its line of action). If so, the relevant transformations are

$$y_1 = \frac{y-ct}{\gamma_1} \quad ; \quad R_1^2 = x^2+y_1^2 \quad , \ \Theta_1 = \tan^{-1}\left(\frac{y_1}{x}\right) \quad ,$$

$$y_2 = \frac{y-ct}{\gamma_2} \quad ; \quad R_2^2 = x^2+y_2^2 \quad , \ \Theta_2 = \tan^{-1}\left(\frac{y_2}{x}\right) .$$

The static formulae (8.2.2), (8.2.4) still hold, and we now note that

$$\begin{aligned} -x\log r &= -y\theta - (x\log r - y\theta) \\ &= y\frac{\partial}{\partial y}(x\log r - y\theta) - (x\log r - y\theta), \qquad (8.3.1) \end{aligned}$$

$$+ y\log r = y\frac{\partial}{\partial y}(x\theta + y\log r - y). \tag{8.3.2}$$

Accordingly for the moving field :

$$\mathcal{S} = \frac{N}{2\mu}\frac{2}{1-\gamma_1^2}\{a(x\log R_1 - y_1\Theta_1) - (x\log r' - y'\theta')\}\ , \tag{8.3.3}$$

$$\underset{\sim}{\mathcal{A}} = -\frac{1}{2\mu}\frac{2}{1-\gamma_2^2}\{(x\Theta_2 + y_2\log R_2 - y_2) - (x\theta' - y'\log r' - y')\}\underset{\sim}{k}\ ; \tag{8.3.4}$$

$$y' = y - ct\ , \quad (r')^2 = x^2 + (y')^2,\ \theta' = \tan^{-1}\left(\frac{y'}{x}\right)\ .$$

Accordingly, we obtain

$$\underset{\sim}{U} = \frac{N}{\mu(1-\gamma_1^2)}\nabla\{a(x\log R_1 - y_1\Theta_1) - (x\log r' - y'\theta')\}$$

$$- \frac{1}{\mu(1-\gamma_2^2)}\nabla_\wedge\{x\Theta_2 + y_2\log R_2 - y_2) - (x\theta' + y'\log r' - y')\}\underset{\sim}{k}. \tag{8.3.5}$$

Bearing in mind (8.2.12) and that

$$\nabla(x\log r' - y'\theta') = \nabla_\wedge(x\theta' + y'\log r')\underset{\sim}{k}\quad , \tag{8.3.6}$$

the expression (8.3.5) becomes

$$\underset{\sim}{U} = \frac{1}{\mu(1-\gamma_2^2)}\nabla\{a(x\log R_1 - y_1\Theta_1)\}$$

$$- \frac{1}{\mu(1-\gamma_2^2)}\nabla_\wedge\{(x\theta_2 + y_2\log R_2) - (y_2 - y)\}\underset{\sim}{k}. \tag{8.3.7}$$

In components (note $\frac{\partial}{\partial y} = \frac{1}{\gamma_1}\frac{\partial}{\partial y_1}$ or $\frac{1}{\gamma_1}\frac{\partial}{\partial y_2}$ as appropriate) :

$$U_1 = \frac{1}{\mu(1-\gamma_2^2)}\left\{a(1+\log R_1) - \frac{1}{\gamma_2}(1+\log R_2) + \left(\frac{1}{\gamma_2} - 1\right)\right\}$$

$$= \frac{1}{\mu(1-\gamma_2^2)}\left\{a \log R_1 - \frac{1}{\gamma_2}\log R_2\right\} - \frac{1-a}{\mu(1-\gamma_2^2)} , \qquad (8.3.8)$$

$$U_2 = -\frac{1}{\mu(1-\gamma_2^2)}\left\{\frac{a}{\gamma_1}\,\Theta_1 - \Theta_2\right\} . \qquad (8.3.9)$$

These expressions reduce to the corresponding static expressions (8.1.12), (8.1.13) as $c \to 0$, provided that $a = \gamma_1 + O(1-\gamma_1)^2$. To determine a more precisely, we note that

$$[U_2] = -\frac{1}{\mu(1-\gamma_2^2)}\left\{\frac{a}{\gamma_1}[\Theta_1] - [\Theta_2]\right\}$$

$$= -\frac{2\pi}{\mu(1-\gamma_2^2)}\left(\frac{a}{\gamma_1} - 1\right) \quad \text{— see (3.2.2)}$$

$$= 0 \quad \text{only if } a = \gamma_1.$$

Inserting $a = \gamma_1$ into (8.3.8), (8.3.9) and noting that

$$\frac{1-a}{\mu(1-\gamma_2^2)} = \frac{1-\gamma_1}{\mu(1-\gamma_2^2)} = \frac{1-\gamma_1^2}{\mu(1-\gamma_2^2)(1+\gamma_1)} = \frac{N}{\mu(1+\gamma_1)} \to \frac{N}{2\mu}$$

when $\gamma_1 = 1$, on bearing in mind (8.2.12), we get

$$U_1 = \frac{1}{\mu(1-\gamma_2^2)}\left\{\gamma_1 \log R_1 - \frac{1}{\gamma_2}\log R_2\right\} - \frac{N}{\mu(1+\gamma_1)}, \quad (8.3.10)$$

$$U_2 = -\frac{1}{\mu(1-\gamma_2^2)}\{\Theta_1-\Theta_2\} \quad . \quad (8.3.11)$$

For a given value of c the final term in (8.3.10) is a constant which can be omitted without loss of generality. As before, we calculate the stress component p_{12} and the stress combinations $p_{11}+p_{22}$, $p_{11}-p_{22}$. From the expressions (8.3.10), (8.3.11), we obtain

$$e_{11} = \frac{\partial U_1}{\partial x} = \frac{1}{\mu(1-\gamma_2^2)}\left\{\gamma_1\frac{x}{R_1^2} - \frac{1}{\gamma_2}\frac{x}{R_2^2}\right\},$$

$$e_{22} = \frac{\partial U_2}{\partial y} = -\frac{1}{\mu(1-\gamma_2^2)}\left\{\frac{1}{\gamma_1}\frac{x}{R_1^2} - \frac{1}{\gamma_2}\frac{x}{R_2^2}\right\},$$

$$\Delta = e_{11}+e_{22} = -\frac{1}{\mu(1-\gamma_2^2)}\left\{\left(\frac{1}{\gamma_1}-\gamma_1\right)\frac{x}{R_1^2}\right\}$$

$$= -\frac{1}{\mu}\frac{1-\gamma_1^2}{1-\gamma_2^2}\left\{\frac{x\gamma_1}{\gamma_1^2x^2+(y-ct)^2}\right\}$$

$$= -\frac{N}{\mu}\left\{\frac{x\gamma_1}{\gamma_1^2x^2+(y-ct)^2}\right\},$$

$$e_{12} = \frac{1}{2}\left(\frac{\partial U_1}{\partial y}+\frac{\partial U_2}{\partial x}\right)$$

$$= - \frac{1}{2\mu(1-\gamma_2^2)} \left\{\left(1 + \frac{1}{\gamma_2^2}\right) \frac{y_2}{R_2^2} - \frac{2y_1}{R_1^2}\right\}$$

$$= - \frac{1}{\mu(1-\gamma_2^2)} \left\{\frac{\frac{1}{2}(1+\gamma_2^2)}{\gamma_2} \frac{y-ct}{\gamma_2^2 x^2+(y-ct)^2} - \frac{\gamma_1(y-ct)}{\gamma_1^2 x^2+(y-ct)^2}\right\} .$$

Accordingly

$$p_{12} = 2\mu e_{12} = - \frac{2}{1-\gamma_2^2} \left\{\frac{\frac{1}{2}(1+\gamma_2^2)\gamma_2^{-1}(y-ct)}{\gamma_2^2 x^2+(y-ct)^2} - \frac{\gamma_1(y-ct)}{\gamma_1^2 x^2+(y-ct)^2}\right\}$$

$$p_{11}+p_{22} = 2\mu \frac{1-N}{N} \Delta \quad \text{from (8.2.20)}$$

$$= - 2(1-N) \frac{x\gamma_1}{\gamma_1^2 x^2+(y-ct)^2}$$

$$p_{11}-p_{22} = 2\mu(e_{11}-e_{22})$$

$$= - \frac{2}{1-\gamma_2^2} \left\{\frac{2}{\gamma_2} \frac{x}{R_2^2} - \left(\gamma_1 + \frac{1}{\gamma_1}\right) \frac{x}{R_1^2}\right\}$$

$$= - \frac{4}{1-\gamma_2^2} \left\{\frac{\gamma_2 x}{\gamma_2^2 x^2+(y-ct)^2} - \frac{\frac{1}{2}\gamma_1(1+\gamma_1^2)x}{\gamma_1^2 x^2+(y-ct)^2}\right\} .$$

All these results are in agreement with Eason et al (1956).

APPENDICES

Appendix I

Equivalence of certain harmonic fields

We prove that the gradient of a harmonic function may be expressed as a curl of solenoidal harmonic vector, i.e.

$$\nabla h = \nabla_\wedge \underset{\sim}{f} \quad ; \quad \nabla^2 h = 0, \quad \nabla^2 f = 0, \quad \nabla \underset{\sim}{f} = 0.$$

Given a vector ∇h where h is a harmonic function, we show how to write

$$\nabla h = \nabla_\wedge \underset{\sim}{f} \; ; \quad \nabla . \underset{\sim}{f} = 0 \tag{1}$$

where $\underset{\sim}{f}$ is a vector to be determined. Putting

$$\underset{\sim}{f} = \nabla_\wedge \underset{\sim}{F} \quad ; \quad \nabla . \underset{\sim}{F} = 0, \tag{2}$$

then (1) becomes

$$\nabla h = \nabla_\wedge \nabla_\wedge \underset{\sim}{F} = \nabla(\nabla . \underset{\sim}{F}) - \nabla^2 \underset{\sim}{F} = -\nabla^2 \underset{\sim}{F},$$

i.e. $$\nabla^2 \langle F_1, F_2, F_3 \rangle = - \left\langle \frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}, \frac{\partial h}{\partial z} \right\rangle .$$

This is a vector Poisson equation for F_1, F_2, F_3 which has a solution :

$$F_1 = -\frac{1}{2} x h + \eta_1 \; ; \quad \nabla^2 \eta_1 = 0, \text{ etc.},$$

i.e. $\underset{\sim}{F} = -\frac{1}{2}\underset{\sim}{r}h + \underset{\sim}{\eta}$; $\nabla^2\underset{\sim}{\eta} = 0,$ (3)

where $\underset{\sim}{\eta}$ is a harmonic vector function to be determined so that $\nabla.\underset{\sim}{F} = 0$. We have

$$0 = \nabla.\underset{\sim}{F} = -\frac{1}{2}\nabla.(\underset{\sim}{r}h) + \nabla.\underset{\sim}{\eta} \quad ,$$

i.e.

$$h\nabla.\underset{\sim}{r} + \underset{\sim}{r}.\nabla h = 2\frac{\partial\eta_1}{\partial x} \quad , \tag{4}$$

on choosing $\underset{\sim}{\eta} = \langle\eta_1, 0, 0\rangle$. We illustrate with the following examples .

Example 1 : h = 1, in this case (4) becomes

$$3 = 2\frac{\partial\eta_1}{\partial x}$$

i.e. $\eta_1 = \frac{3}{2}x + \psi(y,z).$

Here we may choose $\psi = 0$ since $\frac{3}{2}x$ is a harmonic function. So that

$$\underset{\sim}{\eta} = \langle\frac{3}{2}x, 0, 0\rangle \; .$$

Therefore from (3), $\underset{\sim}{F} = \langle -\frac{1}{2}x + \frac{3}{2}x\, ,\ -\frac{1}{2}y\, ,\ -\frac{1}{2}z\rangle$

$$= \langle x, -\frac{1}{2}y, -\frac{1}{2}z \rangle ,$$

$$\nabla . \underset{\sim}{F} = 1 - \frac{1}{2} - \frac{1}{2} = 0$$

$$\underset{\sim}{f} = \nabla \wedge \underset{\sim}{F} = 0.$$

Example 2 : $h = x$, in which case (4) becomes

$$3x + x = 2 \frac{\partial \eta_1}{\partial x}$$

$$\text{i.e.} \quad \eta_1 = x^2 + \psi(y,z) .$$

Therefore

$$0 = \nabla^2 \eta_1 = 2 + \nabla^2 \psi(y,z),$$

from which it follows that

$$\nabla^2 \psi(y,z) = -2.$$

A possible solution is

$$\psi = -y^2$$

so that

$$\underset{\sim}{\eta} = \langle x^2-y^2, 0, 0 \rangle \; .$$

Therefore from (3), $\underset{\sim}{F} = \langle -\frac{1}{2} x^2-y^2, -\frac{1}{2}xy, -\frac{1}{2}xz \rangle$.

It is easy to verify that $\nabla.\underset{\sim}{F} = 0$, and so $\underset{\sim}{f}$ can be constructed from (2) .

Example 3 : $h = \frac{1}{r}$; $r^2 = x^2+y^2+z^2$ in which case (4) can be written as

$$3h + r\frac{\partial h}{\partial r} = 2\frac{\partial \eta_1}{\partial x}$$

i.e. $\frac{3}{r} - \frac{1}{r} = \frac{2}{r} = 2\frac{\partial \eta_1}{\partial x}$

i.e. $\eta_1 = \int \frac{dx}{r} = \log(x+r) + \psi(y,z).$

Hence

$$0 = \nabla^2\eta_1 = \nabla^2\log(x+r) + \nabla^2\psi(y,z).$$

Since $\nabla^2\log(x+r)$ is a harmonic function, it follows that

$\nabla^2\psi = 0$. A possible solution is $\psi = 0$, hence

$$\underset{\sim}{\eta} = \langle \log(x+r), 0, 0 \rangle \; .$$

Therefore

$$\underset{\sim}{F} = <-\frac{x}{2r} + \log(x+r)\ ,\ -\frac{y}{2r}\ ,\ -\frac{z}{2r}> ,$$

It may be verified that $\nabla.\underset{\sim}{F} = 0$, and so $\underset{\sim}{f}$ follows from (2).

Example 4 : Here we examine the 2-dimensional case in which (4) can be written as

$$2h + r\frac{\partial h}{\partial r} = 2\frac{\partial \eta_1}{\partial x} \quad ; \quad r^2 = x^2+y^2 \quad . \qquad (5)$$

An example is $h = \log r$, so that from (5), we get

$$\frac{\partial \eta_1}{\partial x} = \frac{1}{2} + \log r ,$$

i.e. $$\eta_1 = \int \log r \, dx + \int \frac{1}{2} dx$$

$$= (x \log r - y\theta - x) + \frac{1}{2}x + \psi(y).$$

In this case

$$0 = \nabla^2 \eta_1 = \nabla^2(x \log r - y\theta - x) + \frac{1}{2}\nabla^2 x + \nabla^2 \psi$$

$$= \nabla^2(x \log r - y\theta - x) + \nabla^2 \psi \quad .$$

Since $(x \log r - y\theta - x)$ is a harmonic function, it follows that $0 = \nabla^2 \psi$. A possible solution is $\psi = 0$, hence

$$\underset{\sim}{\eta} = \langle x \log r - y\theta - \frac{1}{2}x , 0 \rangle .$$

Therefore $$\underset{\sim}{F} = \langle \frac{1}{2}x \log r - y\theta - \frac{1}{2}x , -\frac{1}{2}y \log r \rangle .$$

It may be verified that $\nabla . \underset{\sim}{F} = 0$ and so $\underset{\sim}{f}$ can be constructed from (2).

Appendix II

Helmholtz resolution of Papkovich-Neuber representation

The Papkovich-Neuber representation can be transformed into Helmholtz resolution via (4.2.2) i.e.

$$\underset{\sim}{h}-\kappa\nabla(\underset{\sim}{r}.\underset{\sim}{h}+f) = \nabla S + \nabla_{\wedge}\underset{\sim}{A} \; ; \quad \nabla.\underset{\sim}{A} = 0. \tag{1}$$

Therefore, to determine $S, \underset{\sim}{A}$, first we operate with $\nabla.$ on (1) which gives

$$\nabla^2 S = \nabla.\underset{\sim}{h} - \kappa\nabla^2(\underset{\sim}{r}.\underset{\sim}{h}+f). \tag{2}$$

Since $\underset{\sim}{h}$ is a harmonic vector, we note that

$$\nabla^2(\underset{\sim}{r}.\underset{\sim}{h}) = 2\nabla.\underset{\sim}{h}$$

i.e. $$\nabla.\underset{\sim}{h} = \frac{1}{2}\nabla^2(\underset{\sim}{r}.\underset{\sim}{h}). \tag{3}$$

Substituting (3) into (2), we get

$$\nabla^2 S = \frac{1}{2}\nabla^2(\underset{\sim}{r}.\underset{\sim}{h}) - \kappa\nabla^2(\underset{\sim}{r}.\underset{\sim}{h}+f),$$

which is immediately seen to have the particular solution

$$S = \frac{1}{2}(\underset{\sim}{r}.\underset{\sim}{h}) - \kappa(\underset{\sim}{r}.\underset{\sim}{h}+f)$$

$$= \frac{1}{2}(1-2\kappa)(\underset{\sim}{r}.\underset{\sim}{h}) - \kappa f. \tag{4}$$

Also, operating with $\nabla\wedge$ on (1), we obtain

$$\nabla^2\underset{\sim}{A} = -\nabla\wedge\underset{\sim}{h} \tag{5}$$

on bearing in mind $\nabla.\underset{\sim}{A} = 0$. Writing $\underset{\sim}{A} = \langle A_1,A_2,A_3\rangle$, $\underset{\sim}{h} = \langle h_1,h_2,h_3\rangle$, relation (5) implies

$$\nabla^2\langle A_1,A_2,A_3\rangle = -\left\langle \frac{\partial h_3}{\partial y} - \frac{\partial h_2}{\partial z}, \frac{\partial h_1}{\partial z} - \frac{\partial h_3}{\partial x}, \frac{\partial h_2}{\partial x} - \frac{\partial h_1}{\partial y} \right\rangle$$

i.e.

$$\nabla^2 A_1 = \frac{\partial h_2}{\partial z} - \frac{\partial h_3}{\partial y}, \quad \nabla^2 A_2 = \frac{\partial h_3}{\partial x} - \frac{\partial h_1}{\partial z}, \quad \nabla^2 A_3 = \frac{\partial h_1}{\partial y} - \frac{\partial h_2}{\partial x},$$

which gives

$$A_1 = \frac{1}{2}(zh_2-yh_3) - \frac{1}{2}\psi_1 \quad ; \quad \nabla^2\psi_1 = 0$$

$$A_2 = \frac{1}{2}(xh_3-zh_1) - \frac{1}{2}\psi_2 \quad ; \quad \nabla^2\psi_2 = 0$$

$$A_3 = \frac{1}{2}(yh_1-xh_2) - \frac{1}{2}\psi_3 \quad ; \quad \nabla^2\psi_3 = 0$$

i.e.

$$\underset{\sim}{A} = -\frac{1}{2}(\underset{\sim}{r}\wedge\underset{\sim}{h}) - \frac{1}{2}\underset{\sim}{\psi} \quad ; \quad \nabla^2\underset{\sim}{\psi} = 0 . \tag{6}$$

A constraint on $\underset{\sim}{\psi}$ is $0 = \nabla.\underset{\sim}{A} = -\frac{1}{2}\nabla.(\underset{\sim}{r}\wedge h) - \frac{1}{2}\nabla.\underset{\sim}{\psi}$

i.e.

$$\nabla.\underset{\sim}{\psi} = -\nabla.(\underset{\sim}{r}\wedge\underset{\sim}{h}) = \underset{\sim}{r}.\nabla\wedge\underset{\sim}{h} - \underset{\sim}{h}.\nabla\wedge\underset{\sim}{r} = \underset{\sim}{r}.\nabla\wedge\underset{\sim}{h}\ . \qquad (7)$$

Also, on substituting for S from (4) and $\underset{\sim}{A}$ from (6) into (1), we obtain

$$\underset{\sim}{h}-\kappa\nabla(\underset{\sim}{r}.\underset{\sim}{h}+f) = \frac{1}{2}(1-2\kappa)\nabla(\underset{\sim}{r}.\underset{\sim}{h})-\kappa\nabla f - \frac{1}{2}\nabla\wedge(\underset{\sim}{r}\wedge\underset{\sim}{h}+\underset{\sim}{\psi})\ . \qquad (8)$$

Accordingly,

$$\underset{\sim}{h} = \frac{1}{2}\nabla(\underset{\sim}{r}.\underset{\sim}{h}) - \frac{1}{2}\nabla\wedge(\underset{\sim}{r}\wedge\underset{\sim}{h}) - \frac{1}{2}\nabla\wedge\underset{\sim}{\psi}\ ,$$

i.e.

$$\nabla\wedge\underset{\sim}{\psi} = \nabla(\underset{\sim}{r}.\underset{\sim}{h}) - \nabla\wedge(\underset{\sim}{r}\wedge\underset{\sim}{h}) - 2\underset{\sim}{h}. \qquad (9)$$

Conditions (7) and (9) defines the harmonic vector $\underset{\sim}{\psi}$ subject to regular behaviour at ∞ .

The most important choice of $\underset{\sim}{h}$ is $\underset{\sim}{h}(r)$; $r^2 = x^2 + y^2 + z^2$, in which case relation (7) and (9) gives respectively

$$\nabla.\underset{\sim}{\psi} = 0$$
$$\nabla\wedge\underset{\sim}{\psi} = \underset{\sim}{h} + r\frac{d\underset{\sim}{h}}{dr} = \frac{d}{dr}(r\underset{\sim}{h})$$

This shows that $\nabla \wedge \underset{\sim}{\psi} = 0$ if $h = < \frac{a}{r}, \frac{b}{r}, \frac{c}{r} >$ with a,b,c

arbitrary constants. In this case $\nabla . \underset{\sim}{\psi} = 0$, $\nabla \wedge \underset{\sim}{\psi} = 0$ i.e. we may choose $\underset{\sim}{\psi} = 0$.

Appendix III

Taylor expansion of certain wave functions

Given a harmonic function $g_0(x,y,z)$, we introduce a harmonic function $g(x,y,z')$; $z' = z-ct$, i.e. $g_0(x,y,z)$ with z replaced by z', and also a wave function $G(x,y,z_1)$;

$z_1 = \frac{z-ct}{\gamma_1}$, $\gamma_1^2 = 1 - \frac{c^2}{c_1^2}$, i.e. $g_0(x,y,z)$ with z replaced by z_1 . The wave function G has the Taylor expansion

$$G = g + \varepsilon\left(\frac{dG}{d\varepsilon}\right)_{\varepsilon=0} + \frac{1}{2}\varepsilon^2\left(\frac{d^2G}{d\varepsilon^2}\right)_{\varepsilon=0} + \ \ldots \tag{1}$$

in the neighbourhood of $\varepsilon = \frac{1}{c_1} = 0$. Now

$$\frac{dz_1}{d\varepsilon} = \frac{\partial z_1}{\partial\gamma_1}\frac{d\gamma_1}{d\varepsilon} = \frac{z'}{\gamma_1^2}\frac{\varepsilon c^2}{\gamma_1} = 0 \quad \text{when } \varepsilon = 0.$$

$$\frac{d^2z_1}{d\varepsilon^2} = c^2z'\left(\frac{1}{\gamma_1^3} + \frac{3\varepsilon^2c^2}{\gamma_1^5}\right) = c^2z' \quad \text{when } \varepsilon = 0,$$

$$\frac{d^3z_1}{d\varepsilon^3} = 0 \text{ when } \varepsilon = 0, \text{ etc.},$$

so that

$$\frac{dG}{d\varepsilon} = \frac{\partial G}{\partial z_1}\frac{dz_1}{d\varepsilon} = 0 \quad \text{when} \quad \varepsilon = 0,$$

$$\frac{d^2G}{d\varepsilon^2} = \frac{\partial G}{\partial z_1}\frac{d^2z_1}{d\varepsilon^2} + \frac{\partial^2 G}{\partial z_1^2}\left(\frac{dz_1}{d\varepsilon}\right)^2$$

$$= c^2 z' \frac{\partial g}{\partial z'} \quad \text{when} \quad \varepsilon = 0,$$

$$\frac{d^3G}{d\varepsilon^3} = 0 \text{ when } \varepsilon = 0, \text{ etc.}$$

If so, (1) becomes

$$G = g + \frac{1}{2}\varepsilon^2 c^2 z' \frac{\partial g}{\partial z'} + O(\varepsilon^2 c^2)^2$$

$$= g + \frac{1}{2}(1-\gamma_1^2) z' \frac{\partial g}{\partial z'} + O(1-\gamma_1^2)^2 \quad ; \ \varepsilon^2 c^2 = \frac{c^2}{c_1^2} = 1-\gamma_1^2.$$

This implies that

$$2\,\frac{G-g}{1-\gamma_1^2} = z'\frac{\partial g}{\partial z'} + O(1-\gamma_1^2), \qquad (2)$$

i.e.

$$2\,\frac{G-g}{1-\gamma_1^2} \to z'\frac{\partial g}{\partial z'} \quad \text{as} \quad \varepsilon \to 0 \ (\gamma_1 \to 1). \qquad (3)$$

Also,

$$2\,\frac{G-g}{1-\gamma_1^2} \rightarrow z\,\frac{\partial g_0}{\partial z} \quad \text{as} \quad c \rightarrow 0 \ (\gamma_1 \rightarrow 1), \tag{4}$$

since $z' = z-ct = z$, $g = g_0$, when $c = 0$.

Clearly (4) becomes identical with (3) when $t = 0$.

Appendix IV

Two-dimensional Papkovich-Neuber representation

The Papkovich-Neuber representation (4.1.1) also applies to 2-dimensional field with a suitable choice for the harmonic vector $\underset{\sim}{h}$ and the scalar harmonic function f. Choosing f = 0 the representation appears as

$$\langle u_1, u_2, u_3\rangle = \langle h_1, h_2, h_3\rangle - \kappa\nabla(xh_1 + yh_2 + zh_3) \tag{1}$$

where h_1, h_2, h_3 are harmonic functions. We now choose

$$h_1 = \frac{1-\nu}{\mu}\phi , \quad h_2 = \frac{1-\nu}{\mu}\overline{\phi} , \quad h_3 = 0 \tag{2}$$

where ϕ , $\overline{\phi}$ are conjugate harmonic functions of x,y, so obtaining

$$\langle u_1, u_2, 0\rangle = \frac{1-\nu}{\mu} \langle\phi, \overline{\phi}, 0\rangle - \frac{1}{4\mu}\nabla(x\phi + y\overline{\phi}). \tag{3}$$

Now the function $y\overline{\phi} - x\phi$ is always harmonic, since

$$\nabla^2(y\overline{\phi} - x\phi) = 2\frac{\partial\overline{\phi}}{\partial y} - 2\frac{\partial\phi}{\partial x} = 0, \tag{4}$$

on bearing in mind that $\frac{\partial\phi}{\partial x} = \frac{\partial\bar{\phi}}{\partial y}$. Hence

$$y\bar{\phi} - x\phi = 2\psi \quad ; \qquad \nabla^2\psi(x,y) = 0,$$

i.e. $\quad y\bar{\phi} = x\phi+2\psi \quad .$ (5)

Accordingly

$$x\phi + y\bar{\phi} = 2x\phi + 2\psi \quad ,$$

from which (3) becomes

$$<u_1,u_2> = \frac{1-\nu}{\mu} <\phi,\bar{\phi}> - \frac{1}{2\mu}\nabla(x\phi+\psi).$$

On multiplying by 2μ and writing $\chi = x\phi+\psi$, we obtain formula (7.1.8). Similarly choosing $h_1 = -\frac{1-\nu}{\mu}\bar{\phi}$, $h_2 = \frac{1-\nu}{\mu}\phi$, $h_3 = 0$, we obtain formula (7.1.10).

Airy, G. B. (1863). "On the strains in the interior of beams". Phil. Trans. Roy. Soc., 153 , 49.

Almansi, E. (1897). "Sull' integrazione dell' equazione differenziale $\nabla^{2n} = 0$". Ann. Mat., Ser. III, 2, 1.

Bhattacharyya, P. K. (1975). "Biharmonic Potential Theory and its Applications". Ph.D. Thesis, The City University, London.

Bhattacharyya, P. K., and Symm, G. T. (1980). "A novel formulation and solution of the plane elastostatic displacement problem". Comp. & Maths. with Appls., Vol. 6, pp. 443-448.

Boggio, T. (1903). "Sull' integrazione di alcune equazioni lineari alle derivate parziali ". Ann. Mat., Ser. III, 8, 181.

Boussinesq, J. (1885). "Applications des Potentiels a l'étude de l'équilibre et du mouvement des solides élastiques". Paris.

Coker, E. G., and Filon, L. N. G. (1957). "A Treatise on Photo-Elasticity". 2nd Ed. Cambridge University Press.

Cottrell, A. H. (1953). "Dislocations in Crystals". McGraw-Hill.

Cottrell, A. H. (1964). "The Theory of Crystal Dislocations". Gordon and Breach, Science Publishers, Inc.

Eason, G., Fulton, J., and Sneddon, I. N. (1956). "The generation of waves in an infinite elastic solid by variable body forces". Phil. Trans. Roy. Soc., A, 955, 575.

Eason, G. (1965). "The stresses produced in a semi-infinite solid by a moving surface force". Int. J. Engng. Sci., Vol. 2, pp. 581-609.

Eason, G. (1966). "The displacement produced in an elastic half space by a suddenly applied surface force". J. Inst. Maths. Appls., 2, 229.

Eringen, A. C., and Suhubi, E. S. (1975). "Elastodynamics". Volume II. Academic Press.

Eshelby, J. D. (1949). "Uniformly moving dislocations". Proc. Phys. Soc., A, 62, 307.

Eubanks, R. A., and Sternberg, E. (1956). "On the completeness of the Boussinesq-Papkovich stress functions". J. Rat. Mech. Anal., 5, 735.

Foreman, A. J., Jaswon, M. A., and Wood, J. K. (1951). "Factors controlling dislocation widths". Proc. Phys. Soc., A, 64, 156.

Frank, F. C. (1949). "On the equation of motion of crystal dislocations". Proc. Phys. Soc., A, 62, 131.

Gakenheimer, D. C., and Miklowitz, J. (1969). "Transient excitation of an elastic half-space by a point load travelling on the surface". J. Appl. Mech., 505.

Iacovache, M. (1949). "O extindere a metodei lui Galerkin Pentru sistemul ecuatiilor elasticitǎtii". Bul. stiint. Acad. Rep. Pop. Romîne, Ser. A, 1, 593.

Jaswon, M. A. (1980). "What is a dislocation?". Proc. Int. Symp. Nonl. Cont. Mech. Kharagpur.

Jaswon, M. A., and Shidfar, A. (1980). "On a differential equation of Bergman and Schiffer". J. Maths. & Phys. Madras., Vol. 14, No. 3. M.A. Biot Commemorative Volume.

Jaswon, M. A., and Symm, G. T. (1977). "Integral Equation Methods in Potential Theory and Elastostatics". Academic Press.

Knops, R. J., and Payne, L. E. (1971). "Uniqueness Theorems in Linear Elasticity". Springer-Verlag, Berlin.

Lamé, G. (1852). "Leçons sur la théorie mathématique de l'élasticité des corps solids". Paris, Bachelier.

Landau, L. D., and Lifshitz, E. M. (1959). "Theory of Elasticity". 2nd Ed. Pergamon Press.

Love, A. E. H. (1927). "A Treatise on the Mathematical Theory of Elasticity". 4th Ed. Cambridge University Press. Also, (1944) Dover Ed.

Muskhelishvili, N. I. (1953). "Some Basic Problems in the Mathematical Theory of Elasticity". Noordhoff, Groningen.

Nabarro, F. R. N. (1967). "Theory of Crystal Dislocations". Clarendon, Oxford.

Neuber, H. (1934). "Ein neuer Ansatz zur Lösung räumlicher Probleme der Elastizitätstheorie". Z. Angew. Math. Mech., 14, 203.

Papadopoulos, M. (1963). "The use of integrals in wave propagation problems with applications to the point source in the semi-infinite elastic medium". Proc. Roy. Soc., A, 276, 204.

Papkovich, P. F. (1932). "Solution générale des équations differentielles fundamentales d'élasticité exprimée pour trois functions harmoniques". C. R. Acad. Sci. Paris, 195, 513.

Poisson, S. D. (1829). "Mémoire sur l'équilibre et le movement des corps élastiques". Mem. Acad. Sci. Paris, 8, 356.

Sneddon, I. N. (1951). "Fourier Transforms". McGraw-Hill.

Sneddon, I. N. (1972). "The Uses of Integral Transforms". McGraw-Hill.

Sokolnikoff, I. S. (1956). "Mathematical Theory of Elasticity". McGraw-Hill.

Sommerfeld, A. (1964). "Mechanics of Deformable Bodies". Academic Press.

Sternberg, E. (1960). "On the integration of the equation of motion in the classical theory of elasticity". Arch. Rat. Mech. Anal., 6, 34.

Sternberg, E., and Eubanks. R. A. (1957). "On stress functions for elastokinetics and the integration of the repeated wave equation". Quart. Appl. Math., 15, 194.

Stokes, G. G. (1851). "On the dynamical theory of diffraction". Trans. Camb. Phil. Soc., 9, 1.

Taylor, G. I. (1934). "The Mechanism of plastic deformation of crystals". Proc. Roy. Soc., A, 145, 362.

Timoshenko, S., and Goodier, J. N. (1951). "Theory of Elasticity". 2nd Ed. McGraw-Hill.