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Subjective survival beliefs and the life-cycle model

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A B S T R A C T

Evidence from panel surveys of households, collected over several years and in different countries, shows that people's perception about their remaining lifetime deviates from actuarial data. This has consequences for consumption, savings and investment over an individual's financial life cycle, and in particular for retirement planning and the purchase of annuities. We use data from the U.S. Survey of Consumer Finances to estimate subjective survival probabilities at different ages. This relies on two different methods of adjusting survival probabilities from a suitable life table. We observe survival pessimism at younger ages and optimism at older ages, consistent with the literature. We optimize numerically for consumption, investment and annuitization in a life-cycle model where individuals receive stochastic labour income and invest in a risk-free asset and in stock whose returns are imperfectly correlated with wages, and where they can annuitize their wealth at retirement. We demonstrate that there is some under-saving before retirement, over-saving post-retirement, and under-annuitization when subjective survival beliefs are used, relative to objective survival expectations. These effects are fairly small, irrespective of the method employed to estimate subjective mortality. Subjective survival beliefs do not therefore fully explain household finance puzzles such as the "annuity puzzle", i.e. observed lower-than-optimal demand for annuities. This conclusion is robust to variations in risk preferences, in the labour income profile, and in the loading factored by insurers in annuity prices.

1. Introduction

Life-cycle models of consumption and investment for individuals typically involve inter-temporal expected utility maximization with survival probabilities drawn from mortality statistics. However, laypeople do not accurately estimate the probabilities that underlie financial and insurance markets. A classic illustration of this is that individuals have *subjective* survival beliefs which differ from the *objective* survival probabilities derived from actuarial and demographic data. This is repeatedly demonstrated in a number of surveys: the Survey of Consumer Finance (SCF) (Heimer et al., 2019; Puri and Robinson, 2007), the Health and Retirement Study (HRS) (Elder, 2013; Salm, 2010; Hurd and McGarry, 1995), the English Longitudinal Study of Ageing (ELSA) (O'Dea and Sturrock, 2021), and the Survey of Health, Ageing and Retirement in Europe (SHARE) (Post and Hanewald, 2013; Peracchi and Perotti, 2014). The deviation of subjective survival beliefs from objective survival expectations is an important issue because it may lead individuals to make sub-optimal investment and consumption decisions.

This phenomenon is related to bounded rationality because of limited information and cognitive capacity (Simon, 1955). People have limited information when they estimate probabilities of events and thus form subjective probabilities which deviate from the objective probabilities based on data. Indeed, Hamermesh (1985) shows that different people have different abilities to estimate their lifespan. On the one hand, O'Dea and Sturrock (2018), Gan et al. (2005), Smith et al. (2001b), and Hurd and McGarry (1995) report that people can make a reasonably accurate estimation of survival based on their health behaviours, related to smoking, substance use, diet, sleep etc. Subjective survival beliefs can serve as predictors of actual mortality (Hurd and McGarry, 2002; Smith et al., 2001a) as well as of longevity risk, i.e. uncertain future survival rates (Post and Hanewald, 2013; Perozek, 2008). On the other hand, O'Dea and Sturrock (2021), Heimer et al. (2019), Wu et al. (2015), Peracchi and Perotti (2014), Elder (2013) and others document discrepancies in subjective survival beliefs compared to objective survival probabilities: in particular, the young tend to underestimate their survival chances whereas the old overestimate them.

Subjective survival expectations influence saving and investing decisions to varying degrees. Puri and Robinson (2007) show that subjective survival belief can predict key economic behaviour, including stock market participation. They find that overly optimistic individuals tend to

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make imprudent financial decisions. Moreover, retirement decisions can be affected by subjective survival beliefs. For example, Van Solinge and Henkens (2009) show that subjective life expectations affect workers' retirement age. Hurd et al. (2004) find that people with low subjective survival beliefs are more likely to retire earlier and claim social security benefits earlier. Gan et al. (2015) illustrate that subjective survival beliefs explain bequests better than actuarial life tables although these authors disregard lifetime portfolio optimization. Heimer et al. (2019) and Wu et al. (2015) find that subjective mortality can explain undersaving for retirement and the slow decumulation of wealth towards the end of life.

Chen et al. (2021) show that subjective mortality beliefs significantly influence the decision between choosing a tontine or an annuity. An individual's subjective life expectancy, compared to the life expectancy used by the insurer for premium calculations, determines the perceived cost of the insurance product. This finding aligns with the conclusions of Wu et al. (2015). Han and Hung (2021) address an optimal consumption and annuitization problem for a decision-maker with ambiguous beliefs about mortality rates. They use a robust control framework to model ambiguity aversion and they conclude that annuity demand decreases as the individual's ambiguity aversion increases.

A specific area of lifetime investment where subjective survival beliefs may be relevant is the purchase of annuities. Pessimism about survival can potentially furnish an explanation for the "annuity puzzle" (Benartzi et al., 2011; Boyer et al., 2020). This is the observed empirical behaviour of individuals to under-annuitize in retirement, relative to the optimal level under certain idealized circumstances (Yaari, 1965). In 2023 in the U.S., \$24.0 trillion of assets were held in Individual Retirement Accounts (IRAs) and employer-sponsored Defined Contribution (DC) pension plans, but only \$385 billion worth of individual annuities were purchased, i.e. only 1.6% of assets were annuitized (LIMRA, 2024; Willis Towers Watson, 2024). The corresponding figures in the U.K. in 2023 are £660bn of assets in personal pension plans and DC pension schemes, and £5.2bn of individual annuities purchased, i.e. 0.8% of assets were annuitized (ABI, 2024; Willis Towers Watson, 2024). O'Dea and Sturrock (2021) and Wu et al. (2015) claim that under-annuitization can be explained by subjective survival beliefs. Bateman et al. (2018) also find evidence that subjective views about lifespan affect the decision to buy an annuity. Horneff et al. (2020) conduct a welfare analysis which shows that defaulting a modest portion of retirees' 401(k) assets into annuities can enhance retirement security, potentially increasing welfare by up to 20% of retirees' plan accruals. Hubener et al. (2016) find that some Americans claim Social Security retirement benefits at the earliest age of 62, thereby forgoing the option to "purchase" higher benefits by delaying their claims. Peijnenburg et al. (2017) show that the optimal annuity demand drops significantly when individuals face health cost risk. However, the annuity puzzle is unresolved and several other explanations have been offered: see Alexandrova and Gatzert (2019) for a recent review.

This paper is concerned with estimating subjective survival expectations from survey data and with the application of these expectations to optimal investment and consumption over the life cycle. The closest studies to ours are O'Dea and Sturrock (2021), Heimer et al. (2019), Wu et al. (2015), and Gan et al. (2015, 2005). Our research differentiates itself from these studies in two key ways:

1. we implement a full life-cycle model with stochastic stock returns and wages, including annuities at retirement,
2. we estimate subjective survival beliefs (a) at a full spectrum of adult ages (20 and above), (b) benchmarked to objective life table probabilities, (c) and scaled using survey respondents' reported subjective life expectancy.

With regard to point 1. above, O'Dea and Sturrock (2021), Wu et al. (2015), and Gan et al. (2015, 2005) do not have a full life-cycle model with a portfolio decision. Gan et al. (2015, 2005) do not consider annuitization at all.

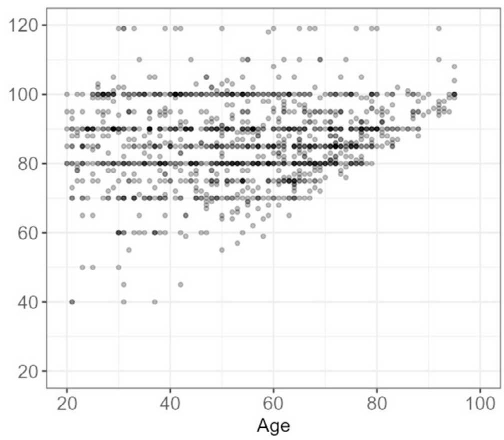
With regard to point 2.(a), survey respondents are aged 50+ in Wu et al. (2015), 60+ in O'Dea and Sturrock (2021), and 70+ in Gan et al. (2015, 2005). Their samples are insufficiently representative of both the working and retired populations to explain full life-cycle investment and consumption behaviour.

With regard to point 2.(b), several studies find that subjective survival beliefs are informative about objective survival probabilities and even about longevity risk (Post and Hanewald, 2013; Perozek, 2008; Hurd and McGarry, 2002; Smith et al., 2001a; Hurd and McGarry, 1995). Consequently, Gan et al. (2015, 2005) and Wu et al. (2015) use life table data when estimating subjective survival probabilities, as we also do. However, O'Dea and Sturrock (2021) use a Weibull distribution and Heimer et al. (2019) use a quadratic regression on age, neither with reference to objective statistical survival data.

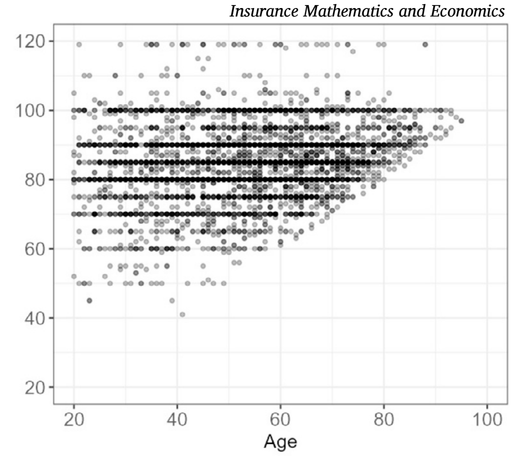
With regard to point 2.(c), we use respondents' stated subjective life expectancy, rather than their stated survival probabilities. Reported subjective survival probabilities pose a number of problems. First, respondents are asked to estimate the probability of survival to only a limited set of future ages, for practical reasons, e.g. O'Dea and Sturrock (2021) and Heimer et al. (2019) use only a few (3 or 4) reported subjective survival probabilities per respondent. Second, responses are prone to "focal bias" where respondents answer with focal point probabilities such as 0, 0.5 and 1 (Post and Hanewald, 2013; Hurd and McGarry, 2002). This is a pervasive problem and Gan et al. (2005) develop a Bayesian updating method to counter it, while other authors use imputation or discard focal responses (see e.g. Post and Hanewald, 2013). Third, Payne et al. (2013) demonstrate that responses about survival probabilities are strongly influenced by whether survey questions are framed in terms of survival to or dying by a certain age. Fourth, responses about probabilities can be incoherent. For example, fully 7% of respondents in the survey of Wu et al. (2015) provide answers which imply conditional survival probabilities that are greater than one, so that their responses have to be discarded. Wu et al. (2015) also find that reported survival probabilities lack consistency with reported life expectancy. They propose a model where survival pessimism varies with both cohort age and target survival age, but this results in non-stationary transition probabilities which means that they cannot implement a stochastic life-cycle model with asset allocation and annuitization. For these reasons, we use reported life expectancy rather than reported survival probabilities, but we combine it with life table survival probabilities.

As discussed above, our paper complements the extant literature and leads to two original contributions. The first is a method to estimate subjective survival probabilities based on subjective life expectancy and the objective life table. Our premise is that individuals can intuitively perceive and adjust survival *probabilities* whereas Gan et al. (2015, 2005) and others scale *hazard rates*. This is mathematically convenient but there is little to suggest that it describes realistically how individuals perceive their survival or mortality. First, people are more likely to estimate a *probability* than a *hazard rate*. Second, as pointed out by Wu et al. (2015), financial life-cycle decisions such as consumption, savings and investment are concerned with *survival*, and only bequests and life insurance are directly about *death*. Third, panel surveys such as the SCF couch questions in terms of survival *to* a given age, and not death *by* a given age. Our first contribution is therefore to estimate subjective survival beliefs by scaling survival probabilities and not hazard rates.

Our second contribution is to utilize these subjective survival probabilities and evaluate optimal consumption, investment and annuitization decisions in a life-cycle model with stock, risk-free asset, annuity and stochastic labour income. In contrast to O'Dea and Sturrock (2021), Heimer et al. (2019) and Wu et al. (2015), we find that subjective survival beliefs contribute little to explaining household finance puzzles such as under-saving before retirement, under-annuitization, and over-saving in retirement.



(a) Female respondents



(b) Male respondents

Fig. 1. Scatterplots for female and male respondents of their subjective expected age at death (on vertical axis) vs. age (on horizontal axis). Source: SCF (2019).

Table 1
Characteristics of the survey respondents. Source: SCF (2019). Subjective expected age at death is the age at which respondents believe that they will die. If this age is greater than 119 (the maximum age in a corresponding life table), it is truncated at 119.

Statistic	N	Mean	St. Dev.	Min	Max
Age	5,777	53.22	16.24	18	95
Subjective expected age at death					
without truncation	5,777	85.33	11.27	40	150
with truncation	5,721	85.21	10.75	40	119

2. Data

There are several longitudinal panel studies which ask respondents about their survival beliefs. In this paper, we use data from the Survey of Consumer Finance (SCF, 2019) because this survey has a large sample and a long continuous history. The SCF is a triennial cross-sectional survey of U.S. families sponsored by the United States Federal Reserve Board in cooperation with the U.S. Treasury Department. The first SCF survey took place in 1983 with 3,824 families. The SCF survey includes information such as age and other demographic characteristics, financial status (e.g. life insurance and annuity holdings), and it also explicitly asks respondents about their belief concerning the age to which they expect to survive.

Fig. 1 shows a scatterplot of individuals' subjective expected age at death versus their age, from the 2019 SCF data (SCF, 2019). The visible horizontal lines on Fig. 1 suggest that many respondents may be rounding their answers at ages that are multiples of 5 years. Although this is an example of focal bias, it is much less pronounced than the focal responses at 0, 0.5 and 1 that occur when people are asked about survival probabilities (Post and Hanewald, 2013; Gan et al., 2005; Hurd and McGarry, 2002).

For actuarial survival probabilities, we follow Heimer et al. (2019) and use the period life table from the U.S. Social Security Administration for the year 2019 (SSA, 2019). This corresponds in time to the subjective survival beliefs that are expressed by the respondents in SCF (2019). The maximum age in this life table is 119 years.

Table 1 shows relevant statistics of the respondents in SCF (2019). The data in the SCF consists of 4,484 men and 1,293 women. The mean age of respondents is 53.22 years. The survey allows respondents to state the age to which they believe they will survive as an integer between their current age and 150. We truncate this at 119, as this is the maximum age in the life table (SSA, 2019), i.e. any individual who states a subjective expected age at death which is greater than 119 is assigned a value of 119 years. The survey also asks whether the respondent has purchased an annuity. 333 respondents state that they hold an annuity product, the average value of which is \$536,227.

3. Model description

3.1. Objective survival probabilities

Consider an individual aged $x \in \mathbb{Z}_+$. The individual belongs to a population whose survival probabilities at various ages are known, either through a life table or a mortality law (e.g. the Gompertz law). We assume that the age x of individual i is rounded down to the nearest integer, i.e. it is the individual's age at her last birthday, noting that life tables usually list integer ages. Denote by p_x the probability, according to the life table or mortality law, that the individual survives for at least 1 year till at least age $x+1$, given that she is alive at age x. We refer to p_x as the objective one-year survival probability of individuals aged x since it is estimated from the lifetime data of a large sample of individuals of age x in the population. The following assumption about p_x is satisfied by all life tables, in practice. In particular, it states that human lifetime is finite and there is a pragmatically-chosen terminal age ω, e.g. 119 years.

Assumption 1. The objective one-year survival probability p_x at age $x \in \mathbb{Z}_+$ satisfies $0 < p_x < 1$ for $x \in [0, \omega - 1]$ and $p_x = 0$ for $x \in [\omega, \infty)$, where $\omega \in \mathbb{Z}_{++}$ is a terminal age beyond which no individual in the population is alive.

Assumption 1 states that no individual is alive past the age of ω, so all individuals die in the year after they turn ω years old. (The terminal age is often defined somewhat differently as the age ϖ at and beyond which no individual is alive, in which case $\omega = \varpi - 1$.) Mortality laws do not usually have a terminal age. If we implement a mortality law rather than a life table, we let $\omega \to \infty$ with suitable convergence conditions on $\{p_x\}$ as $x \to \infty$.

It is convenient to borrow compact actuarial notation and extend the above notation to a k-year survival probability. Let ${}_k p_x$ be the objective probability that an individual survives till at least age $x+k$ given that she is alive at age $x \in [0, \omega - 1]$, for $k \in [1, \omega - x]$. Clearly, ${}_1 p_x = p_x$. It is easily verified that ${}_k p_x = \prod_{j=0}^{k-1} p_{x+j}$ for $x \in [0, \omega - 1]$ and $k \in [1, \omega - x]$ (Promislow, 2010, p. 39).

The (objective) life expectancy, or expected future lifetime, of an individual aged $x \in [0, \omega - 1]$ is

$$e_x = \sum_{k=1}^{\omega - x} {}_k p_x. \tag{1}$$

See for example Promislow (2010, p. 40) and Dickson et al. (2013, p. 33). e_x is known as the curtate life expectancy as it is the expected value of the future lifetime, where the future lifetime is rounded down to the nearest integer, in keeping with the definition of age as an integer (age last birthday). The curtate life expectancy $e_x \in \mathbb{R}_+$ is not necessarily an integer, of course. The curtate life expectancy of an individual aged ω is 0, since $p_\omega = 0$ by Assumption 1.

Remark 1. By Assumption 1, $0 < {}_kp_x < 1$ for $x \in [0, \omega - 1]$ and $k \in [1, \omega - x]$, and $0 < e_x < \omega - x$ for $x \in [0, \omega - 1]$.

3.2. Subjective survival probabilities

An individual's personal belief about her survival probability may differ from the objective survival probability according to an actuarial life table. Define p_x^i as individual i's subjective probability that she survives for at least 1 year, given that she is alive at age x. Assumption 2 below rules out individuals who believe in their immortality and sets a terminal age ω for subjective survival beliefs. Without loss of generality, this terminal age is equal to the terminal age in the life table (see Assumption 1).

Assumption 2. The subjective one-year survival probability p_x^i of individual i, aged $x \in \mathbb{Z}_+$, satisfies $0 \le p_x^i \le 1$ for $x \in [0, \omega - 1]$ and $p_x^i = 0$ for $x \in [\omega, \infty)$, where ω is defined in Assumption 1.

Compare the strict inequalities satisfied by p_x in Assumption 1 with the non-strict inequalities satisfied by p_x^i in Assumption 2. An individual may believe that she is certain to die, or certain to survive, over the next year. No such certainty exists objectively except, by assumption, at the terminal age ω.

As with the objective survival probability, we can define the k-year subjective survival probability of individual i aged $x \in [0, \omega - 1]$ as ${}_kp_x^i = \prod_{j=0}^{k-1} p_{x+j}^i$ for $k \in [1, \omega - x]$. Her subjective life expectancy is (see e.g. Wu et al., 2015)

$$e_x^i = \sum_{k=1}^{\omega - x} {}_kp_x^i. \tag{2}$$

Remark 2. By Assumption 2, $0 \le {}_kp_x^i \le 1$ for $x \in [0, \omega - 1]$ and $k \in [1, \omega - x]$, and $0 \le e_x^i \le \omega - x$ for $x \in [0, \omega - 1]$.

3.3. Hazard scaling (μ_x-scaling)

The objective hazard rate or instantaneous mortality rate μ_x at age x is related to the one-year objective survival probability p_x as follows (see e.g. Dickson et al., 2013, p. 23):

$$p_x = \exp\left(-\int_0^1 \mu_{x+\tau}\, d\tau\right). \tag{3}$$

The subjective hazard rate μ_x^i of individual i at age x is similarly related to the subjective one-year survival probability p_x^i.

Gan et al. (2005, 2015) suggest that subjective survival probabilities may be derived from their objective counterparts by scaling the hazard rate. Under their hazard-scaling method (or μ_x-scaling),

$$\mu_x^i = \gamma_i\, \mu_x, \quad \text{with } \gamma_i \ge 0. \tag{4}$$

This method is used in several studies, e.g. Boyer et al. (2020). Combining eqs. (3) and (4), the relationship between objective and subjective one-year survival probabilities is:

$$p_x^i = \exp\left(-\int_0^1 \mu_{x+\tau}^i\, d\tau\right) = \exp\left(-\gamma_i \int_0^1 \mu_{x+\tau}\, d\tau\right) = (p_x)^{\gamma_i}. \tag{5}$$

γ_i may be regarded as an index of survival pessimism for individual i. We highlight five cases concerning p_x^i here: (a) If $\gamma_i = 1$, then individual i is neutral in her survival beliefs, relative to the objective survival probability, since $p_x^i = p_x$. (b) If $\gamma_i > 1$, then individual i is pessimistic in her survival beliefs since $p_x^i < p_x$. (c) If $\gamma_i < 1$, she is optimistic. (d) If $\gamma_i = 0$, then individual i is perfectly optimistic with $p_x^i = 1$. (e) If $\gamma_i \to \infty$, then individual i is perfectly pessimistic since $p_x^i \to 0$.

The subjective life expectancy of individual i aged $x \in [0, \omega - 1]$, using eqs. (2) and (5), is

$$e_x^i = \sum_{k=1}^{\omega-x} \prod_{j=0}^{k-1} p_{x+j}^i = \sum_{k=1}^{\omega-x} \prod_{j=0}^{k-1} (p_{x+j})^{\gamma_i} = \sum_{k=1}^{\omega-x} \left(\prod_{j=0}^{k-1} p_{x+j}\right)^{\gamma_i}$$
$$= \sum_{k=1}^{\omega-x} \left({}_kp_x\right)^{\gamma_i}. \tag{6}$$

We propose to use the subjective life expectancy that individual i discloses in a survey and solve eq. (6) to determine her pessimism index γ_i. Proposition 1 below is helpful for this purpose.

Proposition 1. *Suppose that Assumptions 1 and 2 hold. (a) Survival beliefs:* $\gamma_i = (>)(<)1 \Leftrightarrow e_x^i = (<)(>)e_x$. *(b) Perfect pessimism: as* $\gamma_i \to \infty$, $e_x^i \to 0$. *(c) Perfect optimism:* $\gamma_i = 0 \Leftrightarrow e_x^i = \omega - x > 0$. *(d) One-to-one correspondence:* e_x^i *is strictly decreasing wrt* γ_i.

Proof. For clarity, we suppress the subscript i where no ambiguity arises. Let $f(\gamma)$ represent the rhs of eq. (6), i.e. $f(\gamma) = \sum_{k=1}^{\omega-x} \left({}_kp_x\right)^{\gamma}$. Note that $\gamma \ge 0$ from eq. (4) and that $0 < {}_kp_x < 1$ from Remark 1. Hence, $f(\gamma) > 0$, $f'(\gamma) = \sum_{k=1}^{\omega-x} \left(\ln {}_kp_x\right)\left({}_kp_x\right)^{\gamma} < 0$ (proving part (d)), $f(0) = \omega - x > 0$ since age $x \in [0, \omega - 1]$ (proving sufficiency in part (c)), and $\lim_{\gamma\to\infty} f(\gamma) = 0$ (proving part (b)). $f(\gamma)$ is continuous wrt γ at $\gamma > 0$ and is right-continuous at $\gamma = 0$. The slope and continuity of $f(\gamma)$ wrt γ prove necessity in part (c). Part (a) is easily verified by comparing the corresponding summation terms in eqs. (1) and (6), and by exploiting part (d) for the direction of the inequalities and for necessity. □

Proposition 1 describes the subjective life expectancy e_x^i of individual i, aged x, in terms of her survival pessimism index γ_i under the hazard scaling method. In particular, e_x^i is positive, strictly decreasing, has a maximum at $\gamma_i = 0$ of $\omega - x > 0$, and converges asymptotically to an infimum of 0 as $\gamma_i \to \infty$. Part (a) of Proposition 1 states than an individual who is neutral (resp. pessimistic, optimistic) in her survival beliefs, relative to objective survival probabilities, has a life expectancy equal to (resp. less than, greater than) the objective life expectancy for her age. Part (b) of Proposition 1 states than a perfectly pessimistic individual expects to survive only for an instant. Part (c) states that a perfectly optimistic individual expects to survive up to just before the terminal age.

There is a one-to-one relation between γ_i and e_x^i, according to part (d) of Proposition 1. This is important because it confirms that we can estimate a unique value of the survival pessimism index γ_i for individual i if she states her subjective life expectancy e_x^i, given a suitable life table or mortality law of objective survival probabilities for the population to which individual i belongs.

In order to apply Proposition 1, the survey respondent's objective and subjective life expectancies are subject to Assumptions 1 and 2, and specifically to $e_x^i \le \omega - x$ (Remark 1). One may wish to censor fancifully high survey responses, or delete anomalous responses altogether. Alternatively, one can raise the terminal age ω to an arbitrarily high level. If a mortality law, rather than an actuarial life table, is used to model objective survival probabilities, the terminal age is infinite (see the discussion in the vicinity of Assumption 1). This is discussed further in Appendix A.

3.4. Survival probability scaling (p_x-scaling)

The hazard scaling method of Gan et al. (2005, 2015) described in sec. 3.3 above relates subjective survival beliefs to objective survival probabilities in a mathematically convenient way. As argued in section 1, it is more likely that individuals perceive a survival probability than a hazard rate. We suggest therefore an additional method by which subjective survival probabilities may be derived from reported life expectancies and objective survival probabilities. We refer to this method using the shorthand of p_x-scaling. The objective one-year survival probability is directly scaled and capped so that the subjective survival probability cannot exceed one:

$$p_x^i = \min\left(v_i\, p_x,\, 1\right), \quad \text{with } v_i \geq 0. \tag{7}$$

The presence of the minimum function in eq. (7) introduces a nonlinearity and so p_x-scaling is less convenient mathematically than μ_x-scaling. However, we show below that this may be overcome.

In eq. (7), v_i is the survival optimism index of individual i. (Compare with γ_i in eq. (4).) If $v_i > 1$, individual i is optimistic over the next year, relative to objective mortality. If $v_i = 1$, she is neutral in her survival beliefs whereas, if $v_i < 1$, she is pessimistic. If $v_i = 0$, individual i is perfectly pessimistic about survival over the next year since $p_x^i = 0$. If $v_i \geq 1/p_x$, individual i is perfectly optimistic with $p_x^i = 1$.

The subjective life expectancy of individual i aged $x \in [0, \omega - 1]$, using eqs. (2) and (7), is

$$e_x^i = \sum_{k=1}^{\omega-x} \prod_{j=0}^{k-1} p_{x+j}^i = \sum_{k=1}^{\omega-x} \prod_{j=0}^{k-1} \min\left(v_i\, p_{x+j},\, 1\right). \tag{8}$$

Proposition 2 below relates the subjective life expectancy e_x^i of individual i, aged $x \in [0, \omega - 1]$, to her survival optimism index v_i under the p_x-scaling method.

Proposition 2. *Suppose that Assumptions 1 and 2 hold. (a) Survival beliefs:* $v_i = (>)(<) 1 \Leftrightarrow e_x^i = (>)(<) e_x$. *(b) Perfect pessimism:* $v_i = 0 \Leftrightarrow e_x^i = 0$. *(c) Perfect optimism:* $v_i \geq 1/\underline{p}_x \Leftrightarrow e_x^i = \omega - x > 0$ *where* $\underline{p}_x = \min\{p_{x+j} : j \in [0, \omega - x - 1]\}$. *(d) Bounded one-to-one correspondence:* e_x^i *is strictly increasing wrt* v_i *if* $0 \leq v_i \leq 1/\underline{p}_x$ *and* e_x^i *is constant at* $\omega - x$ *if* $v_i \geq 1/\underline{p}_x$.

The proof of Proposition 2 is relegated to Appendix B.

Parts (a)–(c) may be interpreted in a similar way to the corresponding parts of Proposition 1. Part (d) is worthy of note. We observe that e_x^i is non-negative, has a minimum of 0 at $v_i = 0$, increases strictly up to a maximum of $\omega - x$ as v_i reaches a critical value of $1/\underline{p}_x$, and thereafter flattens out. By Remark 2, subjective survival expectancy cannot exceed the remaining years to the terminal age (i.e. e_x^i is restricted to no more than $\omega - x$). This means that e_x^i is strictly increasing wrt v_i for the meaningful values of v_i. Therefore we can estimate a unique value of the survival optimism index v_i given individual i's stated subjective life expectancy e_x^i.

Remark 3. No assumption is made about the slope of $\{p_x\}$ in Propositions 1, 3 and 2.

The relevance of Remark 3 is that the sequence of survival probabilities $\{p_x\}$ typically decreases with age x but there are empirical regularities, which go counter to this. Perinatal mortality is particularly high and mortality then declines into early childhood, and young adult mortality (at ages around 18–25) also peaks because of accidents (Dickson et al., 2013, p. 51; Pitacco et al., 2009, p. 97). Mortality rates at very old ages also decline very slowly and possibly reach a 'plateau' (Pitacco et al., 2009, pp. 75, 103).

Finally, the case of a mortality law, when the terminal age of the life table is removed to infinity, is discussed further in Appendix C.

3.5. Life-cycle model

Armed with the subjective survival probabilities derived in the earlier sections, we solve numerically for the optimal consumption and investment over an individual's lifetime. The life-cycle model that we build is based on the classic model, with stochastic labour income correlated to stock returns, of Cocco et al. (2005), Campbell et al. (2001) and Gomes et al. (2008). Our model also features annuities (Horneff et al., 2008, 2010) and social security (Inkmann et al., 2011).

Consider an individual investor i who turns x years old at time 0. She retires at age x_r at time $\tau = x_r - x$ and lives to a maximum age of ω at time $\omega - x$, as per Assumption 1. The risk preferences of this investor are given by additive time-separable power utility:

$$\mathbb{E}\left[\sum_{k=1}^{\omega-x} \beta^{k-1}\left({}_k p_x^i\right)\frac{C_k^{1-\delta}}{1-\delta} + \sum_{k=1}^{\omega-x} v\left({}_{k-1}p_x^i\right)\left(1 - p_{x+k-1}^i\right)\beta^{k-1}\frac{W_k^{1-\delta}}{1-\delta}\right], \tag{9}$$

where $0 < \beta < 1$ is a time preference coefficient or discount factor, C_k is the individual's consumption at the end of year $(k-1, k)$, δ is the coefficient of relative risk aversion of the individual, and v is a bequest preference parameter. The individual has boundedly rational survival expectations and uses her subjective survival probabilities in eq. (9).

During her working life, the individual receives labour income at the end of year $(k-1, k)$, for $k \in [1, \tau]$. During her retirement, she receives a fixed pension income (which may comprise both social security and a defined-benefit pension) at the end of year $(k-1, k)$ for $k \in [\tau + 1, \omega]$. Both labour and pension income at time k are denoted by Y_k and

$$Y_k = \begin{cases} \exp(w\, f(x+k)\, P_k\, U_k) & \text{for } k \in [1, \tau] \\ \kappa \exp(w\, f(x_r)\, P_\tau) & \text{for } k \in [\tau + 1, \omega], \end{cases} \tag{10}$$

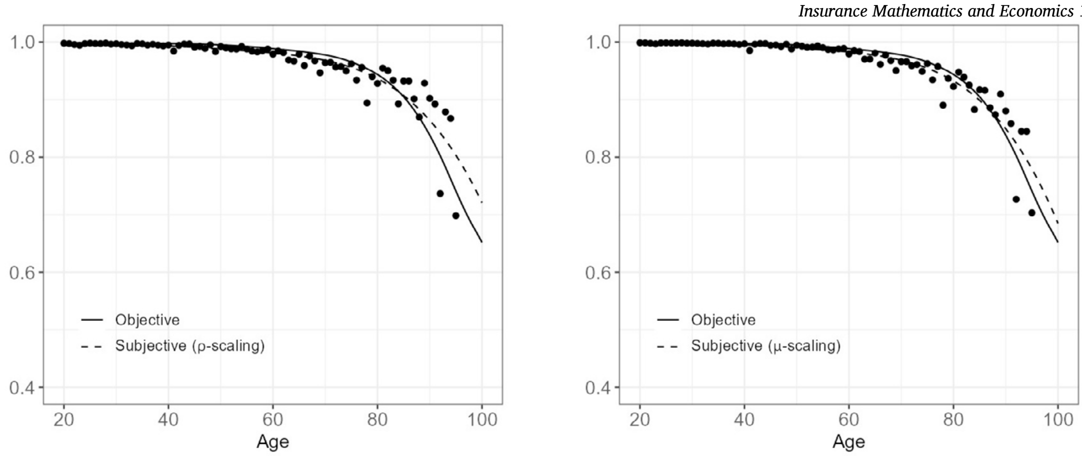
where w is a wage rate, $f(x+k)$ is a deterministic function of age $x+k$ which captures the hump shape of labour income over working lifetime (and possibly other characteristics such as individual i's education level of), P_k is a persistent productivity shock given by $P_k = P_{k-1} + \epsilon_k$ with innovation $\epsilon_k \sim \text{iid N}(0, \sigma_\epsilon^2)$, $U_k \sim \text{iid N}(0, \sigma_u^2)$ is a transitory productivity shock uncorrelated with ϵ_k, and $0 < \kappa < 1$ so that pension is a fixed proportion of the permanent component of labour income in the last year of work (Cocco et al., 2005; Horneff et al., 2008, 2010).

At retirement, the individual can purchase an annuity which pays \$1 at the end of every year while she remains alive. The price of this annuity is

$$s = (1 + \xi) \sum_{k=1}^{\omega - x_r} {}_k p_{x_r} (1 + r_f)^{-k}, \tag{11}$$

where r_f is a risk-free rate and ξ represents a loading factor charged by an insurer to cover its expenses and profit margin on top of the fair actuarial premium for the annuity. Three remarks may be made here: (1) The insurer uses objective survival probabilities to price the annuity. The more pessimistic about survival that individual i is, the more she will perceive the annuity to be expensive. (2) The annuity can only be purchased at retirement. (3) The annuity pays out immediately, and has no deferral period.

During her lifetime, the individual invests her financial wealth W_k, at time $k \in [0, \omega - x]$ in an investment portfolio consisting of a dollar amount B_k of a risk-free asset ("bond") and a dollar amount S_k of a risky asset ("stock"), so that $W_k = B_k + S_k$. The individual may have an initial endowment W_0 known w.p. 1. The risk-free asset earns the risk-free rate r_f and the risky asset earns $r_k = r_f + \mu_r + \vartheta_k$ with $\vartheta_k \sim \text{iid N}(0, \sigma_\vartheta^2)$ and correlation between wages and stock return: $\text{Cov}\left[\epsilon_j, \vartheta\right] = \rho\sigma_\epsilon\sigma_\vartheta$ (Cocco et al., 2005; Fagereng et al., 2017; Campanale et al., 2015). The proportion of financial wealth held in stock is $\pi_k = S_k / W_k$. At retirement (time $k = \tau$), a proportion $\tilde{\pi}_\tau$ of financial wealth is used to buy annuities which will then pay $Z_k = \tilde{\pi}_\tau W_\tau / s$ annually in retirement $(k > \tau)$ while the investor is alive.



(a) p_x-scaling

(b) μ_x-scaling

Fig. 2. Subjective one-year survival probabilities averaged over all males at each age (dots), spline curve fitted to subjective survival probabilities (dashed line), and corresponding actuarial survival probabilities (solid line).

Short-sales and borrowing constraints apply. Thus,

$$W_k \geq 0, \quad 0 \leq \pi_k \leq 1, \quad 0 \leq \tilde{\pi}_\tau \leq 1, \quad \tilde{\pi}_k = 0 \text{ for } k \neq \tau. \tag{12}$$

The budget constraint for the investor is

$$W_{k+1} = (1 + r_f + \pi_k(r_k - r_f))(1 - \tilde{\pi}_k)W_k + Y_{k+1} + Z_{k+1} - C_{k+1}. \tag{13}$$

The investor's optimization problem is therefore to maximize the discounted expected utility in eq. (9) wrt the portfolio, annuity purchase and consumption decision variables (π_k, $\tilde{\pi}_k$, C_k respectively) subject to the constraints in eqs. (12) and (13). As is well known, this problem does not admit an analytical solution and numerical dynamic programming is therefore required (Cocco et al., 2005; Horneff et al., 2008; Fagereng et al., 2017). The numerical method is described in Appendix D.

4. Model calibration

4.1. Estimation of subjective survival probabilities

For every single individual i in the SCF survey described in sec. 2, we use his stated subjective life expectancy e_x^i, and calculate his pessimism index γ_i by numerically solving eq. (6). Likewise, we calculate his optimism index ν_i by solving eq. (8). The numerical method is detailed in Appendix D. We can then compute the subjective one-year survival probability p_x^i for every individual i aged x, according to both the μ_x-scaling and p_x-scaling methods, described in sections 3.3 and 3.4 respectively.

We assume that there is a representative agent at every age. Individuals draw their subjective survival probabilities from the same distribution. We find the mean of p_x^i for each age and fit a spline curve to build the set of subjective survival probabilities. More specifically, we employ I-splines (Ramsay, 1988) with a cubic polynomial to capture the monotonic behaviour of survival probability wrt age, following the method of Wang and Yan (2021).

Fig. 2 depicts, as dots, the mean of p_x^i over all males at each age. Actuarial survival probabilities and subjective survival probabilities fitted using splines are shown as solid and dashed lines respectively. Subjective survival probabilities with p_x- and μ_x-scaling are plotted in Figs. 2a and 2b respectively. It is immediately apparent, under both p_x- and μ_x-scaling, that older individuals are optimistic about their survival. At older ages, the p_x curve is higher than the μ_x curve, but the opposite is true at younger ages.

This is examined further in Fig. 3 which shows the survival curves for the male representative agent at four different ages. Underestimation of survival probabilities at younger ages gives way to overestimation at older ages. This pattern is well-documented in other studies (Gan et al., 2005; Wu et al., 2015; Heimer et al., 2019; O'Dea and Sturrock, 2021). Furthermore, we observe that p_x-scaling generates more pronounced underestimation at younger ages and overestimation at older ages than μ_x-scaling. Similar effects are seen in the female data in Appendix E.

Life expectancies at different ages under objective and subjective longevity are recorded in Table 2 and graphically depicted in Fig. 4. This confirms the pattern of survival pessimism when young and survival optimism when old. Survival pessimism appears to reduce as individuals age, then switches to optimism by about age 85. We observe again that p_x-scaling suggests greater pessimism at younger ages than μ_x-scaling but also greater optimism at older ages. This appears to be because the subjective survival probability is more sensitive to a change in ν_i, which is effectively capped at $1/p_x$ (eq. (7)), than it is to a change in γ_i, which is unbounded above (eq. (4)).[1] It is therefore worth studying the effects of subjective mortality under both μ_x- and p_x-scaling.

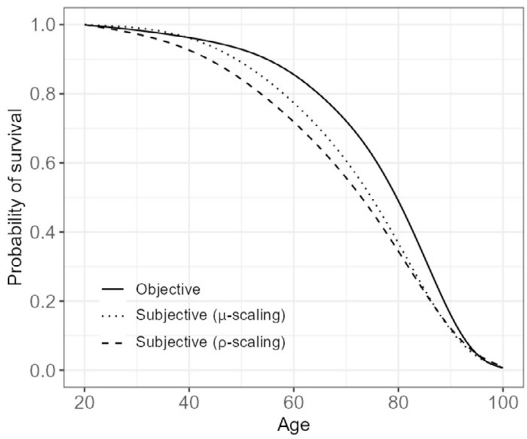
4.2. Data estimation over time

The results that we describe above are based solely on the 2019 wave of the SCF survey. It is useful to consider whether the results continue to hold based on other waves. Tables 3 and 4 show objective and estimated subjective life expectancies using the 1995 and 2007 waves. Comparing with Table 2, we find minor variations but the observation that there is survival pessimism among the young, which reduces as they age, holds for all three waves of the SCF survey. The SCF provides cross-sectional data only and not panel data, so the updating of subjective beliefs by individuals as they age cannot be analyzed fully. However, the stability of the trend that we observe across all three waves gives greater credence to our results.

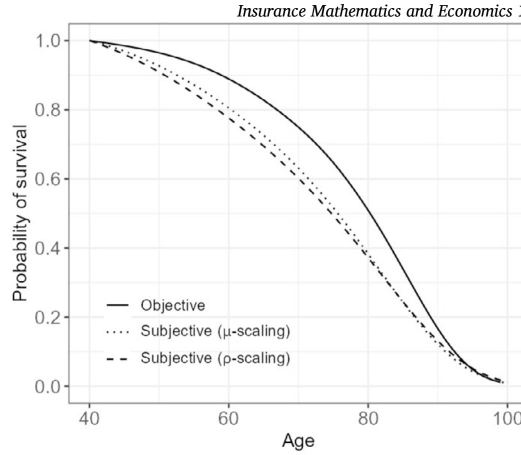
4.3. Calibration of life-cycle model

The life-cycle model described in sec. 3.5 is calibrated to U.S. markets using the parameter values of Heimer et al. (2019) and Love (2013),

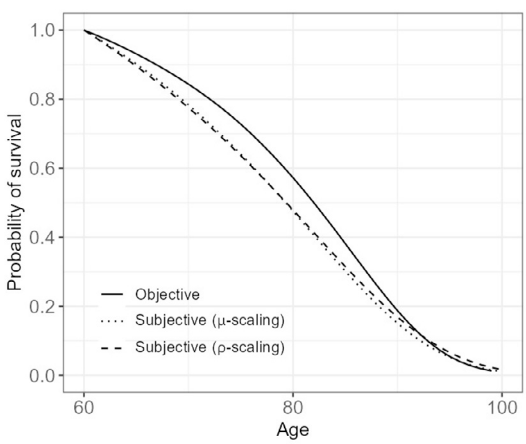
[1] We observe that the magnitude of the sensitivity of p_x^i to γ_i in eq. (4) is $\left|(d/d\gamma_i)(p_x)^{\gamma_i}\right| = \left|(\ln p_x)(p_x)^{\gamma_i}\right| \leq -\ln p_x$ because $\gamma_i \geq 0$ and $0 < p_x < 1$. It is readily shown that $-\ln p_x < p_x = (d/d\nu_i)(\nu_i p_x)$ for $p_x > W_0(1) \approx 0.567$ using Lambert's W-function. Therefore, $\left|(d/d\gamma_i)(p_x)^{\gamma_i}\right| < \left|(d/d\nu_i)(\nu_i p_x)\right|$ holds for most values of p_x. For a quantum of over- or under-estimation of survival probability, represented by an equal change in γ_i and ν_i, the subjective survival probability estimate with p_x-scaling will tend to overshoot, in the direction of optimism or pessimism, the corresponding estimate with μ_x-scaling.



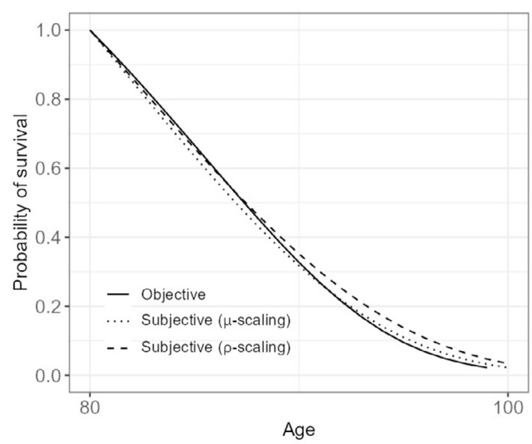
(a) Survival curves at age 20



(b) Survival curves at age 40



(c) Survival curves at age 60



(d) Survival curves at age 80

Fig. 3. Survival curves for males at different ages with actuarial and subjective survival probabilities.

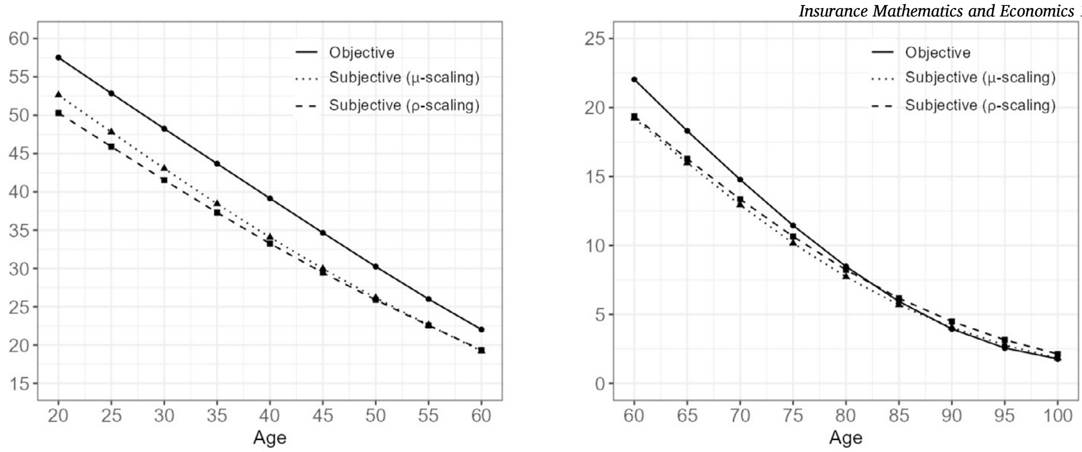
Table 2
Life expectancy for males and females at different ages with objective mortality and subjective mortality (under both p_x- and μ_x-scaling).

	e_{20}	e_{40}	e_{60}	e_{80}	e_{85}	e_{90}	e_{95}	e_{100}
Males								
Objective	57.51	39.14	22.03	8.47	5.94	3.95	2.56	1.78
Subjective (p_x-scaling)	50.29	33.24	19.36	8.23	6.18	4.45	3.16	2.13
Subjective (μ_x-scaling)	52.62	34.06	19.22	7.72	5.69	4.05	2.78	1.82
Females								
Objective	62.41	43.20	25.13	9.95	7.06	4.76	3.13	2.15
Subjective (p_x-scaling)	48.12	31.47	18.07	8.74	7.08	5.66	4.45	3.39
Subjective (μ_x-scaling)	53.77	35.45	19.92	8.55	6.61	5.02	3.74	2.71

Table 3
Life expectancy for males at different ages with objective mortality and subjective mortality (under both p_x- and μ_x-scaling) using 1995 SCF data.

	e_{20}	e_{40}	e_{60}	e_{80}	e_{85}	e_{90}	e_{95}	e_{100}
Objective	54.25	36.04	19.29	7.07	5.03	3.51	2.43	1.70
Subjective (p_x-scaling)	44.53	29.19	16.99	7.23	5.40	3.89	2.70	1.79
Subjective (μ_x-scaling)	47.49	30.40	17.09	6.78	4.95	3.48	2.34	1.49

to whom we refer readers for underlying calibration work and justification. The baseline parameter values are summarized in Table 5. Note in particular that, for the labour income process, we follow Love (2013) who uses data from the 1970–2007 waves of the U.S. Panel Study of Income Dynamics (PSID) to estimate the income profiles for individuals with different levels of education. We use the parameter values for college graduates in our baseline model. We also use objective and subjective survival probabilities for males, since the survey data in SCF (2019) comprises more males than females, as described in sec. 2.



(a) Life expectancy at ages 20–60

(b) Life expectancy at ages 60–100

Fig. 4. Life expectancy for males at different ages with actuarial and subjective survival probabilities.

Table 4
Life expectancy for males at different ages with objective mortality and subjective mortality (under both p_x- and μ_x-scaling) using 2007 SCF data.

	e_{20}	e_{40}	e_{60}	e_{80}	e_{85}	e_{90}	e_{95}	e_{100}
Objective	56.83	38.26	21.19	7.91	5.56	3.72	2.43	1.69
Subjective (p_x-scaling)	49.13	33.10	18.35	8.25	6.58	5.19	4.05	3.08
Subjective (μ_x-scaling)	52.25	34.78	18.74	7.50	5.69	4.24	3.09	2.19

Table 5
Baseline parameter values for the life-cycle model.

Parameter	Value
Individual	
Risk aversion coefficient δ	6
Discount factor β	0.98
Retirement age	65
Minimum age	20
Maximum age	100
Sex	male
Financial	
Risk-free rate r_f	2%
Equity risk premium μ_r	4%
Equity return volatility σ_ϑ	18%
Labor income	
Wage-stock return correlation ρ	0.1
Persistent wage shock volatility σ_ϵ	0.1095
Transitory wage shock volatility σ_u	0.2917
Education level	college grad.
Wage rate w	26,695
Other	
Life table	SSA 2019 male
Pension replacement ratio κ	0.7567
Annuity loading factor ξ	0
Bequest preference parameter ν	0

5. Results

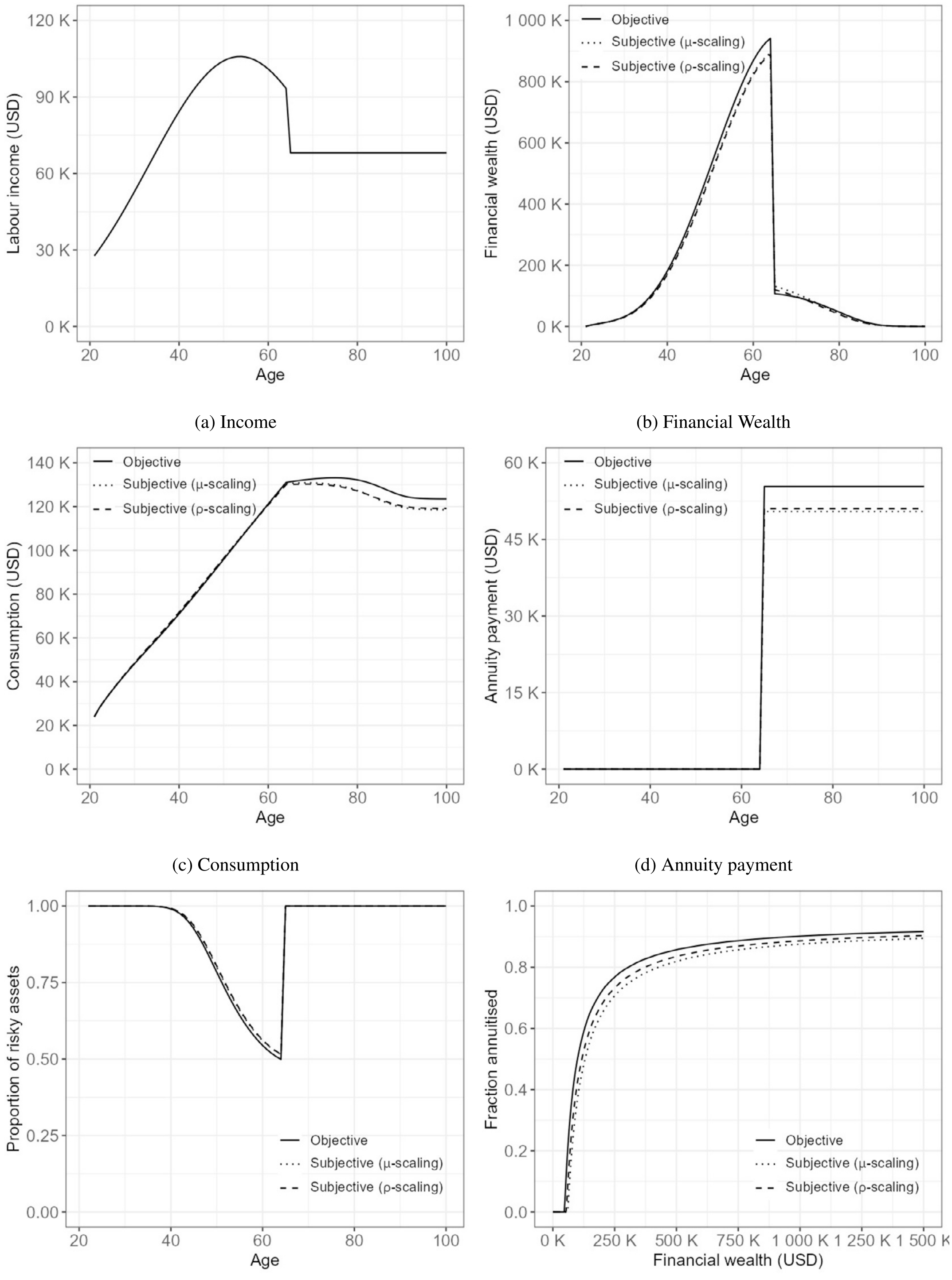
In this section, we solve the life-cycle model with risk-free asset, risky asset and an annuity at retirement using numerical stochastic dynamic programming. The method follows Fehr and Kindermann (2018), as set out in Appendix D.

5.1. Baseline case

We first consider results from the model with baseline parameter values. Average values, at different ages, of income, financial wealth, consumption, annuity payment, equity weight in portfolio, and fraction of wealth annuitized are shown in Fig. 5. Some of these average values are also tabulated in Table 6. Recall that income consists of labour income prior to retirement and pension income (social security and/or defined benefit pension) after retirement. Financial wealth consists of wealth invested in the portfolio of stock and risk-free asset. Typical life-cycle results are visible: (1) average financial wealth and consumption grow during working lifetime (Figs. 5b and 5c); (2) wealth is partially annuitized at retirement (Figs. 5b and 5d); (3) the composition of the investment portfolio shifts away from risky asset as the bond-like holding in human capital declines with age before retirement, but the availability of a risk-free pension income and an annuity means that it is optimal to allocate post-retirement wealth fully to stock (Fig. 5e); (4) the larger financial wealth is just before retirement, the larger the fraction of wealth that is annuitized (Fig. 5f).

We observe, in both Fig. 5 and Table 6, that there is a difference between the average life-cycle paths in the objective and subjective mortality cases. On average, there is some under-saving of about 1% prior to retirement, some over-saving of about 4% post-retirement, and under-annuitization of about 8% when individuals exhibit boundedly rational survival expectations than when they are fully rational. This discrepancy is not very large, so subjective mortality beliefs cannot fully explain stylized facts such as the annuity puzzle. It is noteworthy that the average annuity purchases of between \$763,702 and \$837,959 in Table 6 are significantly larger than the average annuity value of \$536,227 held by annuity-holders in the SCF data as reported in section 2. Further, subjective mortality under p_x- and μ_x-scaling produce roughly similar average consumption, investment and annuitization decisions over time.

The optimal paths are stochastic, of course, so Fig. 6 shows quantiles related to the life-cycle paths. There is no marked difference in these quantile profiles with age between the objective and subjective mortality cases. (There is also very little difference between p_x- and μ_x-scaling, so we only show the former.) We do observe that the quantiles of annuitization in Fig. 6d are lower in the subjective mortality case compared to the objective case, but again this is not substantial enough to fully explain the annuity puzzle.



(a) Income

(b) Financial Wealth

(c) Consumption

(d) Annuity payment

(e) Proportion of risky assets

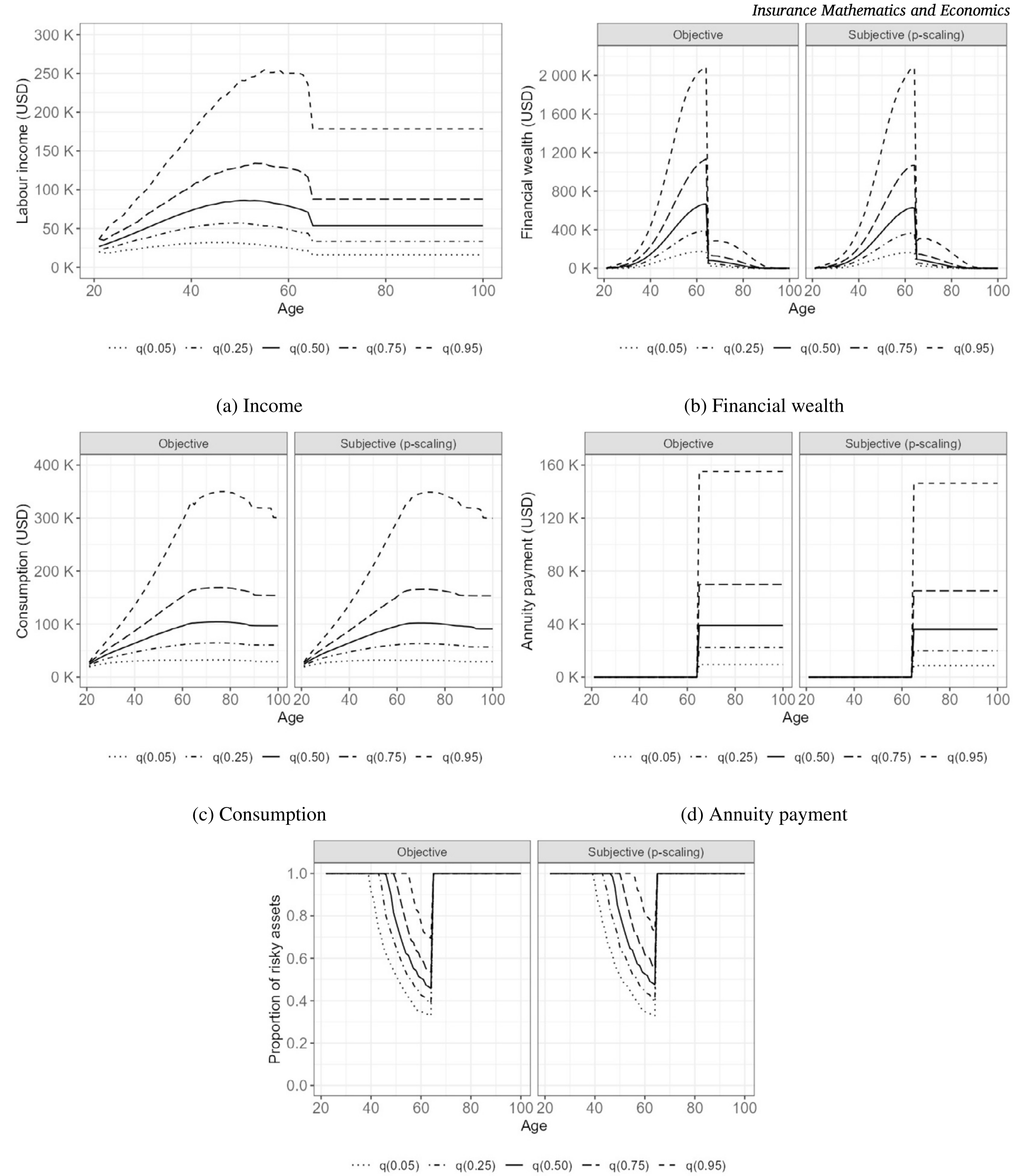
(f) Fraction of wealth annuitized

Fig. 5. Average optimal life-cycle paths with objective and subjective survival probabilities.

5.2. Variations on the baseline case

In this section, we consider how the optimal life-cycle paths change with reasonable variations in parameter values. We vary one parameter at a time while all other parameter values remain as in the baseline set out in sec. 4 (see particularly Table 5).

First, we consider the annuity loading, which represents a charge made by the annuity-provider to cover for expenses and provide a profit margin. A common explanation for the lower-than-optimal annuity demand seen in practice is that annuities are too expensive (Mitchell et al., 1999; Finkelstein and Poterba, 2002). It is therefore worth investigating if subjective mortality beliefs, in combination with higher annuity loadings, can explain the annuity puzzle. In Fig. 7, the annuity loading factor ξ, which appears in eq. (11), is varied from 0 to 0.5. As may be anticipated, the larger the annuity loading factor, the more expensive the annuity is, so the less of it is purchased at retirement (Figs. 7c and 7e), less financial wealth is annuitized (Fig. 7a), post-retirement consumption is lower (Fig. 7b), and the post-retirement investment portfolio is



(a) Income

(b) Financial wealth

(c) Consumption

(d) Annuity payment

(e) Proportion of risky asset

Fig. 6. Quantiles of the optimal life-cycle paths with objective and subjective survival probabilities.

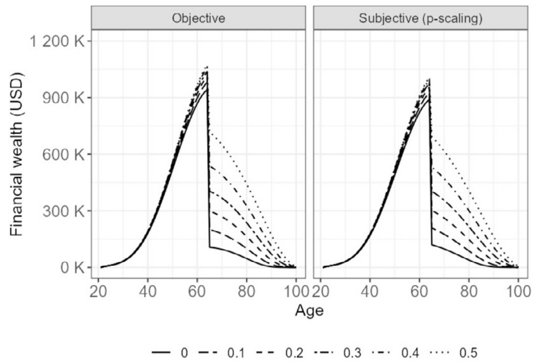
less risky to compensate for riskier overall retirement income (Fig. 7d). However, our principal observation from Fig. 7 is that the average optimal paths are not very different between the objective and subjective mortality cases. Although slightly less annuity is purchased on average under subjective mortality beliefs than objective mortality expectations (Fig. 7c), the difference is small compared to the drop in annuitization which occurs as the annuity loading factor increases.

In Fig. 8, we also vary the labour income profile based on education level, to consider not just college graduates (who earn the most) but also high school graduates and high school dropouts (who earn the least), using the parameterizations of Love (2013) as described in sec. 4.3. In Fig. 9, we vary the risk aversion coefficient δ (see eq. (9)). Again, the results are as anticipated. Our chief observation, from Figs. 8 and 9, is that there is little difference between the average optimal life-cycle paths in the objective and subjective mortality cases. In particular, annuitization is lower on average with subjective mortality beliefs than objective mortality expectations, but not substantially so (Figs. 8d and 9c).

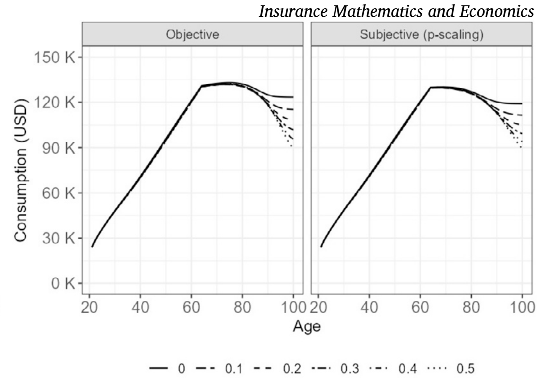
The baseline case of section 5.1 comprises no bequest motive, similar to the baseline reference case of Horneff et al. (2008). A willingness to bequeathe wealth is expected to depress annuitization, so this is an important factor to investigate. In Table 7, we show the average values, at different ages, of annuity purchases and payments at retirement as well as financial wealth at age 80, for different values of the bequest preference parameter ν (see eq. (9)). As anticipated, we observe that annuitization is negatively related to the bequest motive, whereas financial holdings in retirement are positively related. Under subjective beliefs, annuitization and financial wealth in retirement are lower than objectively, but again the discrepancy is not very wide. Under-annuitization is about 6–8% on average.

6. Conclusion

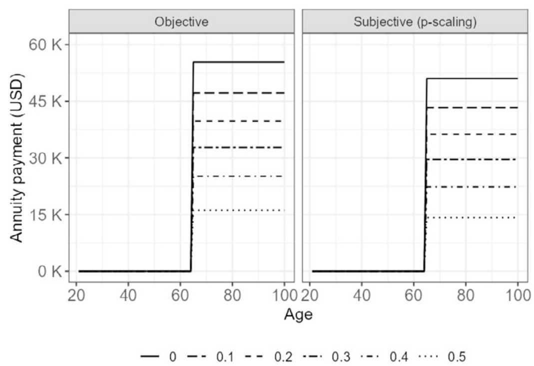
Several studies show that subjective beliefs about survival differ from actuarial survival rates based on population data. In particular,



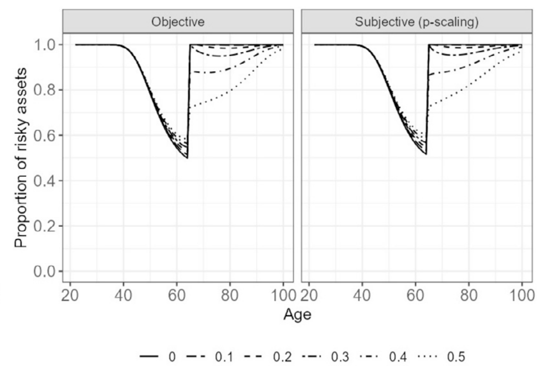
(a) Financial wealth



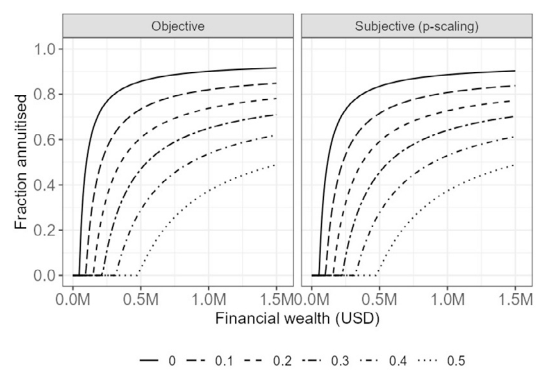
(b) Consumption



(c) Annuity payment



(d) Proportion of risky asset



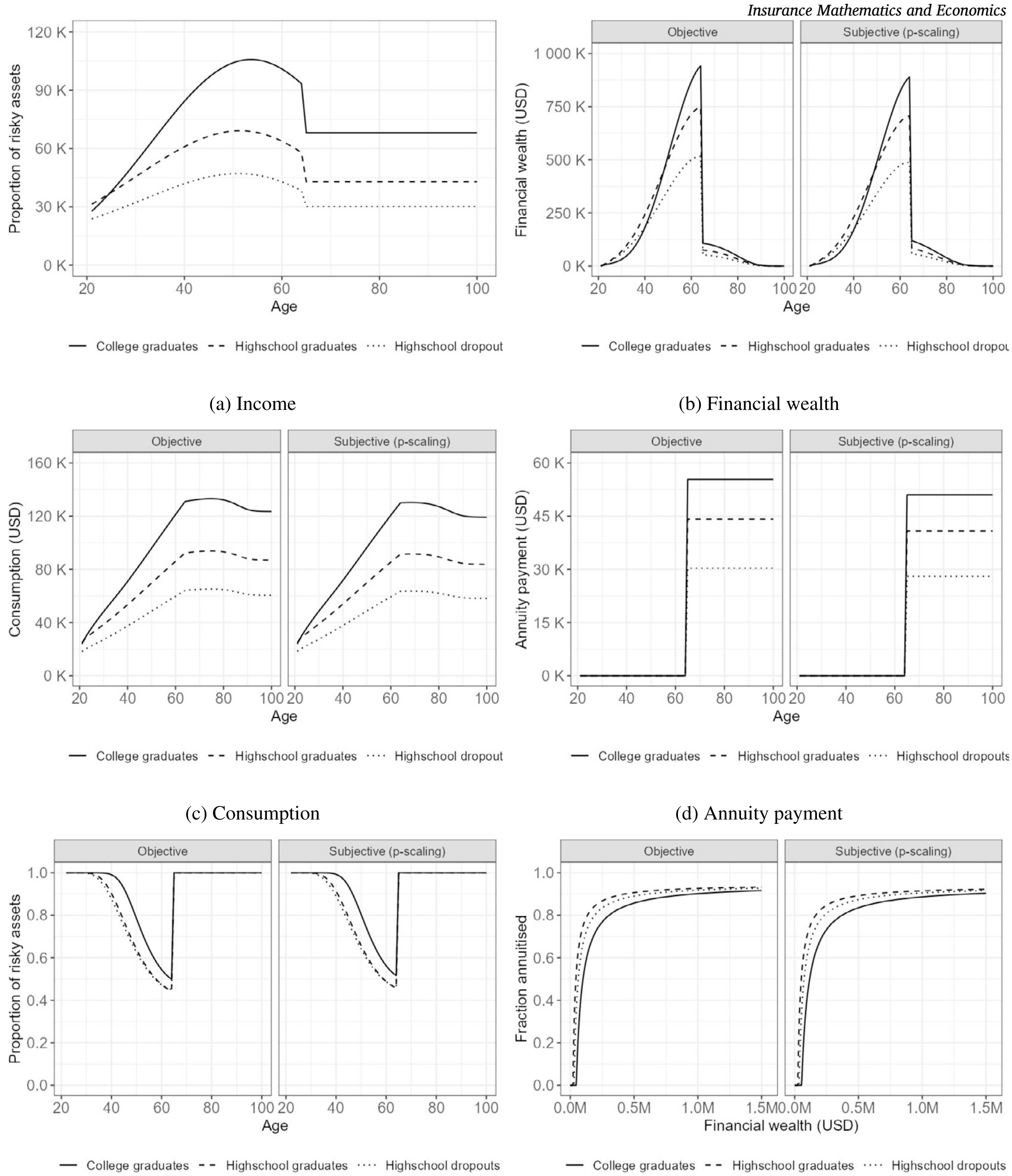
(e) Fraction of wealth annuitised

Fig. 7. Average optimal life-cycle paths with annuity loading factor $\xi = 0, 0.1, \ldots, 0.5$.

younger individuals are pessimistic about survival, this pessimism decreases with age, and older individuals turn optimistic. Heimer et al. (2019) and Wu et al. (2015) argue that subjective survival beliefs can explain empirically verified under-saving before retirement and over-saving in retirement, relative to optimal financial decisions. O'Dea and Sturrock (2021) and Wu et al. (2015) also argue that subjective survival beliefs can explain the annuity puzzle, i.e. lower-than-optimal annuity demand that is observed in practice. We investigate this further by (1) estimating subjective survival probabilities using two different methods, and (2) solving for optimal consumption, investment and annuitization decisions in a life-cycle model, under both objective and subjective survival expectations.

To this end, we use self-reported life expectancy data from the U.S. Survey of Consumer Finances. Some studies use self-reported survival probabilities, but these can be inconsistent and can suffer from focal responses. We transform subjective life expectancies into subjective survival probabilities by scaling hazard rates, an established method used in several studies (e.g. Gan et al., 2005, 2015; Wu et al., 2015; Boyer et al., 2020). We also rigorously establish a second method, where survival probabilities are scaled rather than hazard rates, on the basis that individuals are more likely to perceive a probability than a hazard rate, and that most financial life-cycle decisions are concerned with survival rather than death. A unique value of an index of survival optimism/pessimism is estimated for each survey respondent. Under both methods, we find that there is indeed survival pessimism at younger ages which reduces with age and turns to optimism at older ages well into retirement, a finding which is consistent with the literature.

We then use our estimated subjective survival probabilities within a full life-cycle model. Individuals receive stochastic labour income and invest their savings in a portfolio consisting of risk-free asset and stock whose returns are imperfectly correlated with wages. At retirement, they can purchase an annuity, which is priced by an annuity-provider using



(a) Income

(b) Financial wealth

(c) Consumption

(d) Annuity payment

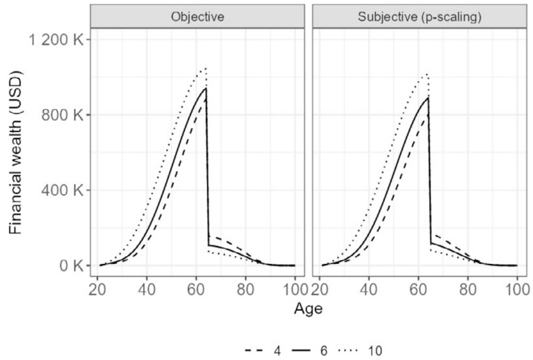
(e) Proportion of risky asset

(f) Fraction of wealth annuitised

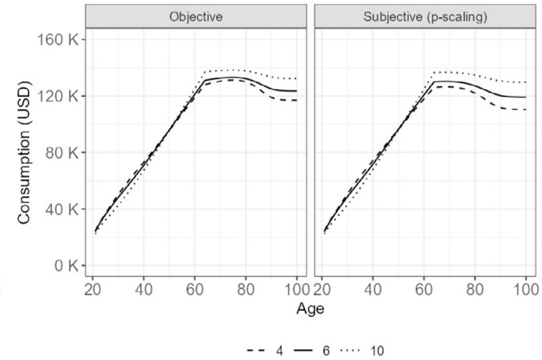
Fig. 8. Average optimal life-cycle paths with different labour incomes profile based on different levels of education.

objective (actuarial) survival probabilities. Individuals exhibit boundedly rational survival expectations and they maximize their expected discounted CRRA utility allowing for their subjective survival probabilities. When we solve the model numerically, we find that there is some under-saving (about 1%) before retirement, some over-saving (about 4%) after retirement, and under-annuitization (about 8%), on average, when individuals operate under their subjective survival beliefs, relative to objective survival expectations. This effect is fairly small, whichever method is used to estimate subjective mortality, so subjective mortality does not fully explain these household finance puzzles. This conclusion holds for a wide range of parameter values. This conclusion also runs counter to recent results in the literature (O'Dea and Sturrock, 2021; Heimer et al., 2019; Wu et al., 2015). This is likely to be because we allow investment portfolio decisions over a full spectrum of adult ages in our life-cycle model, whereas some authors do not allow this. It could also be because our subjective survival probability estimates are benchmarked to objective life table probabilities, as opposed to a Weibull distribution or a quadratic regression on age in some of the earlier literature.

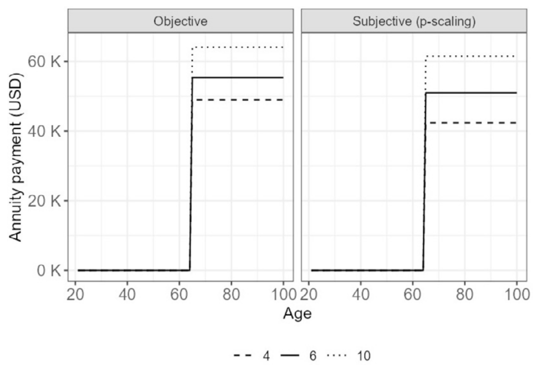
A number of limitations of our analysis should be pointed out, even though these limitations are also present in the extant literature. First, we assume that agents are boundedly rational in that their age-related subjective survival beliefs deviate from objective expectations in a consistent way over time. There may instead be a time inconsistency in survival beliefs and this should be investigated using panel data to observe how individuals update their beliefs. Second, there is considerable heterogeneity in beliefs across education levels, wealth and income, whereas we aggregate individuals across these socio-economic factors. There is also strong heterogeneity in *objective* survival probabilities across these factors (Carannante et al., 2023). Capturing this heterogeneity fully poses significant data collection, estimation and computational challenges.



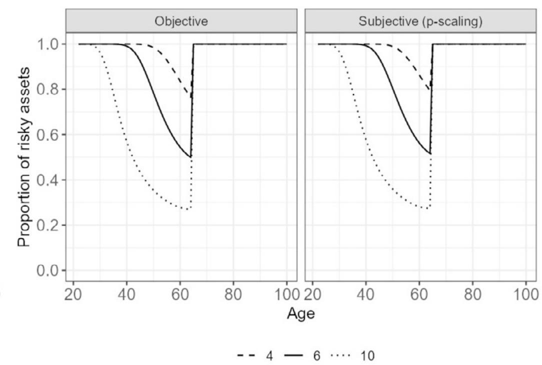
(a) Financial wealth



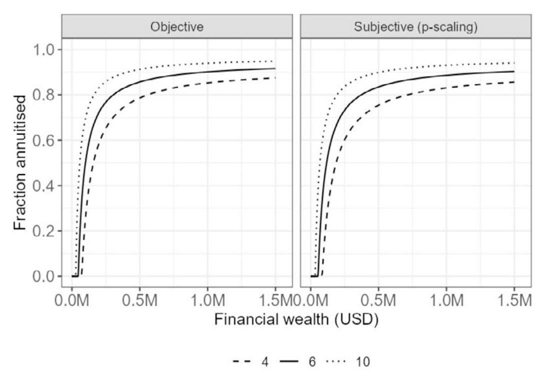
(b) Consumption



(c) Annuity payment



(d) Proportion of risky asset



(e) Fraction of wealth annuitised

Fig. 9. Average optimal life-cycle paths with risk aversion coefficient $\delta = 4$, 6, 10.

There are several promising avenues of research that are being carried out to extend the present model. First, health risk is an important factor in that it discourages annuitization not only through its impact on expectations but also due to uncertain healthcare costs and related precautionary savings (Munnell et al., 2022; Peijnenburg et al., 2017). The differential impact of health risk on subjective and objective survival expectations deserves greater scrutiny. Second, annuities should be available at various ages rather than only at retirement (Horneff et al., 2008). Again, survival optimism at older ages may increase annuitization. Third, annuities with a deferral period should be available (Horneff et al., 2010; Owadally et al., 2021a). Greater choice and flexibility should lead to welfare gain but survival pessimism in the young may confound this. Finally, financial life-cycle decisions are highly sensitive to interest rates and inflation (Owadally et al., 2021b) and the effect of subjective survival beliefs may be tempered or amplified at different stages of the economic cycle. These features will be included in subsequent studies.

CRediT authorship contribution statement

Seung Yeon Jeong: Writing – original draft, Visualization, Software, Methodology, Investigation, Formal analysis, Data curation, Conceptualization. **Iqbal Owadally:** Writing – review & editing, Writing – original draft, Visualization, Validation, Supervision, Software, Project administration, Methodology, Investigation, Formal analysis, Conceptualization. **Steven Haberman:** Writing – review & editing, Supervision, Formal analysis, Conceptualization. **Douglas Wright:** Supervision, Conceptualization.

Declaration of competing interest

There is no competing interests.

Table 6
Selected values from the average optimal life-cycle paths with objective and subjective survival probabilities.

	Objective	Subjective	
		p_x-scaling	μ_x-scaling
Consumption			
at 21	23,934	23,946	23,937
at 40	70,943	71,730	71,639
at 60	121,178	120,710	121,126
at 80	132,193	127,131	127,353
at 100	123,471	119,136	118,571
Financial wealth			
at 64	941,764	889,483	893,470
Annuity purchase			
at 65	837,959	771,858	763,702
Annuity payment			
after 65	55,373	51,005	50,466

Table 7
Selected values from the average optimal life-cycle paths with objective and subjective survival probabilities for different values of bequest preference parameter $v = 0, 5, 15, 30$.

	Objective	Subjective (p_x-scaling)
Annuity purchase at 65		
$v = 0$	837,959	771,858
$v = 5$	826,952	778,746
$v = 15$	818,037	768,937
$v = 30$	811,846	762,179
Annuity payment after 65		
$v = 0$	55,373	51,005
$v = 5$	54,646	51,460
$v = 15$	54,057	50,812
$v = 30$	53,647	50,365
Financial wealth at 80		
$v = 0$	46,999	39,201
$v = 5$	221,292	216,545
$v = 15$	239,447	234,527
$v = 30$	251,763	246,909

Appendix A. Hazard scaling under a mortality law

If a mortality law, rather than an actuarial life table, is used to model objective survival probabilities, the terminal age is infinite. Proposition 1 is then amended as shown below.

Proposition 3. *Suppose that Assumptions 1 and 2 hold with $\omega \to \infty$ so that $0 < p_x < 1$ and $0 \le p_x^i \le 1$ for $x \in \mathbb{Z}_+$. (Here, $\lim \equiv \lim_{\omega\to\infty}$ unless specified otherwise.) (a) Survival beliefs: $\gamma_i = (>)(<) 1 \Leftrightarrow \lim e_x^i = (<)(>) \lim e_x$. (b) Perfect pessimism: $\lim_{\gamma_i\to\infty} \lim e_x^i = 0$. (c) Perfect optimism: $\gamma_i = 0 \Leftrightarrow \lim e_x^i = \infty$. (d) Fineteness: $\gamma_i > 0 \Leftrightarrow \lim e_x^i < \infty$. (e) One-to-one correspondence: $\lim e_x^i$ is strictly decreasing wrt γ_i.*

Proof. For clarity, we suppress the subscript i where no ambiguity arises. Let $f_\infty(\gamma)$ represent the rhs of eq. (6) with $\omega \to \infty$, i.e. $f_\infty(\gamma) = \sum_{k=1}^{\infty} \left({}_kp_x\right)^\gamma$. Consider $\gamma = 0$. $f_\infty(0) = \sum_{k=1}^{\infty} 1$ diverges to ∞, proving sufficiency in part (c). Consider $\gamma > 0$. An application of the ratio test (d'Alembert criterion) for convergent positive series (Rudin, 1976, p. 66; Ito 1993, p. 1758) shows that $\left|({}_{k+1}p_x)^\gamma/({}_kp_x)^\gamma\right| = (p_{x+k})^\gamma$, and $\limsup_{k\to\infty}(p_{x+k})^\gamma < 1$, given that $p_{x+k} < 1$ for $k \in \mathbb{Z}_+$. Hence $f_\infty(\gamma)$ is finite for $\gamma > 0$, proving sufficiency in part (d). $\lim_{\gamma\to\infty} f_\infty(\gamma) = \sum_{k=1}^{\infty} \lim_{\gamma\to\infty} \left({}_kp_x\right)^\gamma = \sum_{k=1}^{\infty} 0 = 0$, proving part (b). The rest of the proof proceeds along similar lines to the proof of Proposition 1, noting that the objective life expectancy is also finite, $f_\infty(1) = \lim e_x < \infty$. □

Appendix B. Proof of Proposition 2

Proof of Proposition 2. For clarity, we suppress the subscript i where there is no ambiguity. Let $g(v, x, \omega)$ represent the rhs of eq. (8):

$$g(v, x, \omega) = \sum_{k=1}^{\omega-x} \prod_{j=0}^{k-1} \min\left(v p_{x+j}, 1\right). \tag{B.1}$$

Since x and ω are constant for the rest of this proof, we omit them as arguments and use $g(v)$.

The terms inside the sum-product for $g(v)$ in eq. (B.1) above are either 1 (if $vp_{x+j} \ge 1$) or less than 1 and positive (if $vp_{x+j} < 1$, noting that $0 < p_{x+j} < 1$ by Assumption 1). The relative size of v and the objective survival probabilities p_{x+j} determines what happens to each term in the sum-product in eq. (B.1). It is convenient to define

$$\underline{p}_x = \min\{p_{x+j} : j \in [0, \omega - x - 1]\} \tag{B.2a}$$

$$\overline{p}_x = \max\{p_{x+j} : j \in [0, \omega - x - 1]\} \tag{B.2b}$$

Four cases arise and we examine each separately. (If p_{x+j} is constant for $j \in [0, \omega - x - 1]$, which is an unlikely occurrence for objective survival probabilities in a life table, then $p_{x+j} = \underline{p}_x = \overline{p}_x$ and Case 4 below is redundant.)

Case 1. $v = 0$. In this case, all the terms inside the sum-product for $g(0)$ in eq. (B.1) are zero and $g(0) = 0$, verifying sufficiency of part (b) in Proposition 2.

Case 2. $0 < v < 1/\overline{p}_x$. In this case, all the terms inside the sum-product for $g(v)$ in eq. (B.1) are less than 1 and positive, since $0 < vp_{x+j} \le v\overline{p}_x < 1$ for $j \in [0, \omega - x - 1]$, and $\min(vp_{x+j}, 1) = vp_{x+j}$. In eq. (B.1),

$$g(v) = \sum_{k=1}^{\omega-x} v^k \prod_{j=0}^{k-1} p_{x+j} = \sum_{k=1}^{\omega-x} v^k {}_kp_x. \tag{B.3}$$

Now, $g'(v) = \sum_{k=1}^{\omega-x} k v^{k-1} {}_kp_x > 0$, noting that $0 < {}_kp_x < 1$ for $x \in [0, \omega - 1]$ and $k \in [1, \omega - x]$ in Remark 1. Hence, $g(v)$ is strictly increasing wrt v for $0 < v < 1/\overline{p}_x$.

Case 3. $v \ge 1/\underline{p}_x$. In this case, all the terms inside the sum-product for $g(v)$ in eq. (B.1) are equal to 1, since $vp_{x+j} \ge v\underline{p}_x \ge 1$ for $j \in [0, \omega - x - 1]$, and $\min(vp_{x+j}, 1) = 1$. In eq. (B.1),

$$g(v) = \sum_{k=1}^{\omega-x} 1 = \omega - x > 0. \tag{B.4}$$

Hence, $g(v)$ is constant wrt v for $v \ge 1/\underline{p}_x$. This verifies sufficiency of part (c) in Proposition 2.

Case 4. $1/\overline{p}_x \le v < 1/\underline{p}_x$. In this case, some of the terms inside the sum-product for $g(v)$ in eq. (B.1) will be equal to 1 and some will be less than 1 and positive. In general, the larger v is, the more there are 1's inside this sum-product since v exceeds p_{x+j} more often.

Consider two individuals both aged x but with survival optimism index v_1 and v_2 such that $1/\overline{p}_x \le v_1 < v_2 < 1/\underline{p}_x$. The corresponding values of $g(v)$ are

$$g(v_1) = \sum_{k=1}^{\omega-x} \prod_{j=0}^{k-1} \min\left(v_1 p_{x+j}, 1\right), \tag{B.5a}$$

$$g(v_2) = \sum_{k=1}^{\omega-x} \prod_{j=0}^{k-1} \min\left(v_2 p_{x+j}, 1\right). \tag{B.5b}$$

Since $v_2 > v_1$, there are at least as many terms that are capped at 1 in the sum-product for $g(v_2)$ as there are in the sum-product for $g(v_1)$. We would like to show that $g(v_2) > g(v_1)$. The indicator function, $\mathbb{I}\{A\} = 1$ if A is true and 0 otherwise, is helpful in this regard.

$$g(v_1) = \sum_{k=1}^{\omega-x} \prod_{j=0}^{k-1} \left(v_1 p_{x+j} \mathbb{I}\left\{ v_1 p_{x+j} < 1 \right\} + \mathbb{I}\left\{ v_1 p_{x+j} \geq 1 \right\} \right) \tag{B.6a}$$

($g(v_1)$ from eq. (B.5a) is rewritten using the indicator function)

$$\leq \sum_{k=1}^{\omega-x} \prod_{j=0}^{k-1} \left(v_1 p_{x+j} \mathbb{I}\left\{ v_2 p_{x+j} < 1 \right\} + \mathbb{I}\left\{ v_2 p_{x+j} \geq 1 \right\} \right) \tag{B.6b}$$

(terms capped at 1 inside the sum-product for $g(v_2)$ are inserted into their corresponding position inside the sum-product for $g(v_1)$)

$$< \sum_{k=1}^{\omega-x} \prod_{j=0}^{k-1} \left(v_2 p_{x+j} \mathbb{I}\left\{ v_2 p_{x+j} < 1 \right\} + \mathbb{I}\left\{ v_2 p_{x+j} \geq 1 \right\} \right) \tag{B.6c}$$

(v_1 is replaced by v_2 inside the sum-product)

$$= g(v_2). \tag{B.6d}$$

We have therefore shown that $1/\overline{p}_x \leq v_1 < v_2 < 1/\underline{p}_x \Rightarrow g(v_1) < g(v_2)$. Hence, $g(v)$ is strictly increasing wrt v for $1/\overline{p}_x \leq v < 1/\underline{p}_x$.

Continuity of $g(v)$ *wrt* v. The function $\min(v p_{x+j}, 1)$ for $v \geq 0$ in eq. (B.1) is continuous wrt v at $v > 0$ and right-continuous at $v = 0$ (and differentiable at $v > 0$ except at $v = 1/p_{x+j}$). Hence, $g(v)$ is continuous wrt v at $v > 0$ and right-continuous at $v = 0$.

Strictly increasing $g(v)$ *wrt* v. Putting together Cases 1–4 along with continuity above proves part (d) of Proposition 2.

Necessity. Combining continuity and the strictly increasing property proves necessity in parts (b) and (c) of Proposition 2. Sufficiency was shown earlier in Cases 1 and 3.

Survival beliefs. Part (a) of Proposition 2 is easily verified by comparing the corresponding summation terms in eqs. (1) and (8), noting that $v = 1$ belongs to Case 2 above, and exploiting part (d) of Proposition 2 for the direction of the inequalities and for necessity. □

Appendix C. Survival probability scaling (p_x-scaling) under a mortality law

In Proposition 4 below, the terminal age of the life table is removed to infinity, and objective survival rates can be modelled using a mortality law instead of a finite life table. A mild assumption concerning $\{p_x\}$ is required for this purpose.

Assumption 3. Let $0 < p_x < 1$ and $0 \leq p_x^i \leq 1$ for $x \in \mathbb{Z}_+$, and $\lim_{x\to\infty} p_x = \underline{p}$ where $0 \leq \underline{p} < 1$. Further, there exists $y \in \mathbb{Z}_+$ such that $y = \min\{x\colon p_{x+j+1} \geq p_{x+j} \text{ for } j \in \mathbb{Z}_+\}$.

Assumption 3 says that, from age y onwards, objective survival probabilities must decline monotonically and converge. In more detail, Assumption 3 states that there are two possible components to the age structure of objective 1-year survival probabilities. The first component is always present: for age $x \geq y$, the sequence of survival probabilities $\{p_x\}$ is monotonic decreasing (i.e. non-increasing) and convergent. This can cater for the old-age mortality plateau, if it is exists. The second component may or may not be present: for age $x < y$, the sequence $\{p_x\}$ has an unspecified slope wrt age x. This component caters for any empirical troughs and peaks in mortality at younger ages, such as the young-adult mortality peak. Note that Assumption 3 does not rule out that $\{p_x\}$ is monotonically decreasing wrt x from age 0, i.e. it is possible that $y = 0$.

Since subjective survival beliefs are based on objective survival probabilities under both p_x- and μ_x-scaling, Remark 3 and Assumption 3 show that both p_x- and μ_x-scaling can flexibly capture people's subjective survival probabilities. This is true whether one employs an empirical life table with stationary and inflection points and a finite terminal age, or a theoretical mortality law with no terminal age. Proposition 4 below is the analogue of Proposition 2 when the terminal age is removed to infinity, under Assumption 3.

Proposition 4. *Suppose that Assumption 3 holds. (Here,* $\lim \equiv \lim_{\omega\to\infty}$ *unless specified otherwise.) (a) Survival beliefs:* $v_i = (>)(<) 1 \Leftrightarrow \lim e_x^i = (>)(<) \lim e_x$. *(b) Perfect pessimism:* $v_i = 0 \Leftrightarrow \lim e_x^i = 0$. *(c) Perfect optimism:* $v_i \geq 1/\underline{p}$ *and* $\underline{p} > 0 \Leftrightarrow \lim e_x^i = \infty$. *(d) Finiteness:* $0 \leq v_i < 1/\underline{p} \Leftrightarrow \lim e_x^i < \infty$. *(e) One-to-one correspondence:* $0 \leq v_i < 1/\underline{p} \Leftrightarrow \lim e_x^i$ *is strictly increasing wrt* v_i.

The proof of Proposition 4 requires two intermediate lemmas which are stated and proven here. Throughout, the suffix i, denoting the subjective survival beliefs of individual i, is omitted from the notation if this is unambiguous, and $\lim \equiv \lim_{\omega\to\infty}$ unless otherwise specified. We re-use the function $g(v, x, y) = \sum_{k=1}^{y-x} {}_k p_x^i$, from eq. (B.1), and we define $h(v, x, y) = {}_{y-x} p_x^i$, for $y \in \mathbb{Z}_+$, $y > x$. We also define

$$g_\infty(v, x) = \lim_{\omega\to\infty} g(v, x, \omega) = \lim_{\omega\to\infty} e_x^i = \sum_{k=1}^{\infty} {}_k p_x^i = \sum_{k=1}^{\infty} \prod_{j=0}^{k-1} \min(v p_{x+j}, 1). \tag{C.1}$$

Lemma 1 below partitions g_∞ into a component before age y, when the slope of $\{p_x\}$ is unspecified, and a component at or after age y, when $\{p_x\}$ is monotonically decreasing (see Assumption 3).

Lemma 1. *Let* $y \in \mathbb{Z}_+$ *and* $x < y$. *Then* $g_\infty(v, x) = g(v, x, y) + h(v, x, y)\, g_\infty(v, y)$.

Proof of Lemma 1. (The argument v is omitted for brevity.) Since ${}_k p_x^i = \prod_{j=0}^{k-1} p_{x+j}^i$,

$$g_\infty(x) = \sum_{k=1}^{\infty} {}_k p_x^i = p_x^i + p_x^i \sum_{k=2}^{\infty} {}_{k-1} p_{x+1}^i = p_x^i + p_x^i \sum_{k=1}^{\infty} {}_k p_{x+1}^i. \tag{C.2}$$

Eq. (C.2) shows that Lemma 1 holds for $y = x + 1$. Suppose that Lemma 1 holds for $y = x + k$, $k \geq 1$.

$$\begin{aligned} g_\infty(x) &= \sum_{k=1}^{y-x} {}_k p_x^i + {}_{y-x} p_x^i \sum_{k=1}^{\infty} {}_k p_y^i \\ &= \sum_{k=1}^{y-x} {}_k p_x^i + {}_{y-x} p_x^i \left(p_y^i + p_y^i \sum_{k=1}^{\infty} {}_k p_{y+1}^i \right), \end{aligned} \tag{C.3}$$

using eq. (C.2). Therefore,

$$\begin{aligned} g_\infty(x) &= \sum_{k=1}^{y-x} {}_k p_x^i + {}_{y-x+1} p_x^i + {}_{y-x+1} p_x^i \sum_{k=1}^{\infty} {}_k p_{y+1}^i \\ &= \sum_{k=1}^{y-x+1} {}_k p_x^i + {}_{y-x+1} p_x^i \sum_{k=1}^{\infty} {}_k p_{y+1}^i \\ &= g(x, y+1) + h(x, y+1)\, g_\infty(y+1). \end{aligned} \tag{C.4}$$

By induction, since Lemma 1 holds for $y = x + 1$, it also holds for $y = x + k$, $k \geq 1$. □

Lemma 2 below examines the properties of $h(v, x, y)$ for the component of mortality before age y, i.e. for $x < y$ (see Assumption 3).

Lemma 2. *Let* $h(v) \equiv h(v, x, y)$ *with* $x < y$. *(a)* $h(0) = 0$, *(b)* $h(v)$ *is strictly increasing wrt* v *for* $0 \leq v < 1/\underline{p}_x$, *(c)* $h(v) = 1$ *for* $v \geq 1/\underline{p}_x$, *where* $\underline{p}_x = \min\{p_{x+j}\colon j \in [0, y-x-1]\}$.

Proof of Lemma 2. $h(v) = {}_{y-x} p_x^i = \prod_{j=0}^{y-x-1} \min(v p_{x+j}, 1)$. As in the proof of Proposition 2, it is useful to consider four cases. $\underline{p}_x$ is as defined in Lemma 2 and we define $\overline{p}_x = \max\{p_{x+j}\colon j \in [0, y-x-1]\}$. In the event that $\{p_{x+j}\colon j \in [0, y-x-1]\}$ is a constant sequence, then $\underline{p}_x = \overline{p}_x$ and case 4 below is redundant.

In case 1, $v = 0$, and clearly $h(0) = 0$. In case 2, $0 < v < 1/\overline{p}_x$, and $0 < v p_{x+j} \le v\overline{p}_x < 1$, so that all terms inside the product for $h(v)$ above are positive and less than 1. Therefore, $h(v) = \prod_{j=0}^{y-x-1} v p_{x+j} = v^{y-x} \prod_{j=0}^{y-x-1} p_{x+j}$. By virtue of Assumption 1 and because $y > x$, $g_2'(v) = (y-x)v^{y-x-1} \prod_{j=0}^{y-x-1} p_{x+j} > 0$. In case 3, $v \ge 1/\underline{p}_x$, and all the terms inside the product for $h(v)$ above are equal to 1, so that $h(v) = 1$.

In case 4, $1/\overline{p}_x \le v < 1/\underline{p}_x$, and some of the terms inside the product for $h(v)$ above are positive and less than 1, while the other terms are equal to 1. Proceeding as in the proof of Proposition 2, we consider $1/\overline{p}_x \le v_1 < v_2 < 1/\underline{p}_x$.

$$h(v_1) = \prod_{j=0}^{y-x-1} \left(v_1 p_{x+j} \mathbb{1}\left\{v_1 p_{x+j} < 1\right\} + \mathbb{1}\left\{v_1 p_{x+j} \ge 1\right\}\right)$$
$$\le \prod_{j=0}^{y-x-1} \left(v_1 p_{x+j} \mathbb{1}\left\{v_2 p_{x+j} < 1\right\} + \mathbb{1}\left\{v_2 p_{x+j} \ge 1\right\}\right)$$

(terms capped at 1 inside the product for $h(v_2)$ are inserted into their corresponding position inside the product for $h(v_1)$)

$$< \prod_{j=0}^{y-x-1} \left(v_2 p_{x+j} \mathbb{1}\left\{v_2 p_{x+j} < 1\right\} + \mathbb{1}\left\{v_2 p_{x+j} \ge 1\right\}\right)$$

(v_1 is replaced by v_2 inside the product)

$$= h(v_2).$$

Thus, $1/\overline{p}_x \le v_1 < v_2 < 1/\underline{p}_x \Rightarrow h(v_1) < h(v_2)$. Combining cases 1–4, $h(0) = 0$, $h(v)$ is strictly increasing wrt v for $0 \le v < 1/\underline{p}_x$, and is then constant at 1 for $v \ge 1/\underline{p}_x$. □

Lemmas 1 and 2 above are exploited in the proof of Proposition 4 below.

Proof of Proposition 4. The proof of Proposition 4 follows that of Proposition 2, except that we have to show convergence or divergence as $\omega \to \infty$. For clarity, we suppress the subscript i where there is no ambiguity. Here, $\lim \equiv \lim_{\omega\to\infty}$ unless specified otherwise. Recall that $\lim e_x^i = g_\infty(v, x)$ in eq. (C.1).

Sufficiency in part (a) of Proposition 4 is straightforward in the manner of the proof of Proposition 2, i.e. compare the summation terms on the rhs of eqs. (1) (with $\omega \to \infty$) and (C.1). Sufficiency in part (b) of Proposition 4 is also straightforward from eq. (C.1).

For the rest of the proof, we consider separately two possible scenarios depending upon the age x of individual i compared to age y (see Assumption 3).

Scenario 1: Individual i is aged $x \ge y$. By Assumption 3, $\{p_x, x \ge y\}$ is monotonic decreasing and convergent. The definitions of $\underline{p}_x$ and $\overline{p}_x$, in eqs. (B.2a) and (B.2b) respectively from the proof of Proposition 2, stand except that $\omega \to \infty$, and $\underline{p}_x = \underline{p}$ and $\overline{p}_x = p_x$. Should $\{p_x\}$ be constant at all ages (an unusual scenario for human mortality), then $\underline{p}_x = \overline{p}_x = p_x = \underline{p}$ and Case 4, in the proof of Proposition 2 as well as below, is redundant.

In Case 1 in the proof of Proposition 2, $v = 0$. This case clearly holds in the limit as $\omega \to \infty$.

In Case 2, $0 < v < 1/p_x$ and we only need to show that $g_\infty(v, x)$ is finite. Applying the ratio test (d'Alembert criterion) for convergent positive series (Rudin, 1976, p. 66; Ito 1993, p. 1758) we find that $\left|(v^{k+1}\,{}_{k+1}p_x)/(v^k\,{}_kp_x)\right| = v p_{x+k}$, and $\limsup_{k\to\infty}(v p_{x+k}) = v \times \lim_{k\to\infty} p_{x+k} = v\underline{p} < v p_x < 1$. Hence $g_\infty(v)$ is finite for $0 < v < 1/\overline{p}_x$. The instance of $v = 1$ is included in Case 2 and corresponds to objective life expectancy, hence $\lim e_x$ is finite.

In Case 3 in the proof of Proposition 2, $v \ge 1/\underline{p}$. Unlike in Proposition 2 where $\underline{p}_x > 0$ by Assumption 1, Assumption 3 permits a non-strict inequality to hold: $\underline{p} \ge 0$. So we insist that $\underline{p} > 0$ for the case $v \ge 1/\underline{p}$ to hold. Eq. (B.4) then shows that $g_\infty(v, x) = \sum_{k=1}^{\infty} 1$ diverges.

Case 4 in the proof of Proposition 2 concerns $1/p_x \le v < 1/\underline{p}$. In this case, some of the terms inside the sum-product for $g_\infty(v, x)$ in eq. (C.1) will be equal to 1 and the remaining terms will be less than 1 and positive. Define $\underline{k} = \min\{k : p_{x+k} < 1/v\}$. Then, $p_{x+k}^i = v p_{x+k} < 1$ if $k \ge \underline{k}$, and $p_{x+k}^i = 1$ if $k < \underline{k}$. Now, ${}_kp_x^i = \prod_{j=0}^{k-1} p_{x+j}^i$. Therefore,

$$
{}_kp_x^i = \begin{cases} 1 & \text{if } k \le \underline{k} \\ \prod_{j=\underline{k}}^{k-1}(v p_{x+j}) = v^{k-\underline{k}}\,({}_kp_x)/({}_{\underline{k}}p_x) & \text{if } k > \underline{k}. \end{cases} \tag{C.5}
$$

Substituting the above into eq. (C.1) yields

$$
\begin{aligned}
g_\infty(v, x) &= \sum_{k=1}^{\underline{k}} {}_kp_x^i + \sum_{k=\underline{k}+1}^{\infty} {}_kp_x^i = \sum_{k=1}^{\underline{k}} 1 + \sum_{k=\underline{k}+1}^{\infty} v^{k-\underline{k}}\,({}_kp_x)/({}_{\underline{k}}p_x) \\
&= \underline{k} + \sum_{k=1}^{\infty} v^k\,({}_kp_{x+\underline{k}}).
\end{aligned} \tag{C.6}
$$

Applying the ratio test again shows that $\left|v^{k+1}({}_{k+1}p_{x+\underline{k}}) \big/ v^k({}_kp_{x+\underline{k}})\right| = v p_{x+\underline{k}+k}$, and $\limsup_{k\to\infty}\,(v p_{x+\underline{k}+k}) = v \times \lim_{x\to\infty} p_x = v\underline{p} < 1$. Hence $g_\infty(v, x)$ is finite for $1/p_x \le v < 1/\underline{p}$.

We have therefore shown that $g_\infty(v, x) = \lim e_x^i$ is finite when $0 \le v < 1/\underline{p}$ and is infinite when $v \ge 1/\underline{p}$. The rest of the proof, for scenario 1 ($x \ge y$), proceeds along similar lines to the proof of Proposition 2, and continuity and the strictly increasing property of $\lim e_x^i$ wrt v provide for necessity in the various parts of Proposition 4.

Scenario 2: Individual i is aged $x < y$. For parts (c) and (d) of Proposition 4, use Lemma 1, $g_\infty(v, x) = g(v, x, y) + h(v, x, y)\, g_\infty(v, y)$, and consider each of the terms on the rhs. (Recall that $g(v, x, y)$ is identical to $g(v, x, \omega)$ in eq. (B.1) in the proof of Proposition 2, except that ω is replaced by y.) $g(v, x, y)$ is finite according to Proposition 2, Lemma 2 confirms that $h(v, x, y)$ is also finite, and we have just shown in scenario 1 above that $g_\infty(v, x)$, $x \ge y$ is finite if and only if $0 \le v \le 1/\underline{p}$. This proves sufficiency in parts (c) and (d) of Proposition 4.

For part (e), consider the case $v > 0$. From Lemma 1,

$$
\begin{aligned}
\frac{\partial}{\partial v} g_\infty(v, x) &= \frac{\partial}{\partial v} g(v, x, y) + g_\infty(v, y)\,\frac{\partial}{\partial v} h(v, x, y) \\
&\quad + h(v, x, y)\,\frac{\partial}{\partial v} g_\infty(v, y).
\end{aligned} \tag{C.7}
$$

By virtue of part (d) of Proposition 2, the first term ($\partial g(v, x, y)/\partial v$) on the rhs of eq. (C.7) is non-negative (when $v > 0$). Similarly, Lemma 2 shows that $h(v, x, y)$ is positive and $\partial h(v, x, y)/\partial v$ is non-negative (when $v > 0$). In scenario 1 above, $g_\infty(v, x)$ and $\partial g_\infty(v, x)/\partial v$ for $x \ge y$ are positive (when $v > 0$). Thus, the second term on the rhs of eq. (C.7) is non-negative, while the third term is positive. Hence, $\partial g_\infty(v, x)/\partial v > 0$ on the lhs of eq. (C.7). This proves part (e) of Proposition 4, which also provides for necessity in the various other parts of the proposition. □

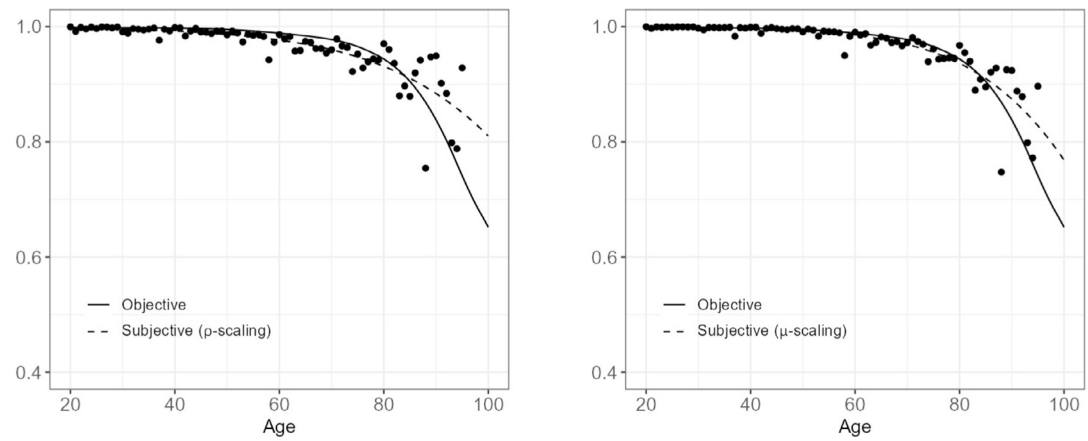
Appendix D. Numerical methods

The financial life-cycle model described in sec. 3.5 is solved numerically using the Fortran code provided by Fehr and Kindermann (2018, p. 469), which is suitably modified for our purposes. The Bellman equation is solved backwards using grid search, spline interpolation and Gaussian quadrature. The assumption in eq. (10) that the annual pension is proportional to the permanent component of final labour income means that variables can be normalized by dividing by the permanent shock $\exp(P_k)$, thus removing one state variable and simplifying the numerical dynamic programming problem (Cocco et al., 2005; Fehr and Kindermann, 2018, p. 447).

As for the subjective survival model in sections 3.3 and 3.4, Propositions 1 and 3 guarantee the existence and uniqueness of a solution to eq. (6), and likewise with Propositions 2 and 4 in relation to eq. (8). To solve for γ_i and v_i, in eqs. (6) and (8) respectively, we use Brent's algorithm, originally called the "Algol 60 procedure zero" by Brent (1973, p. 48), as implemented in **R**. This is a derivative-free method which uses combined quadratic interpolation and bisection methods to find roots of equations (Conn et al., 2009; Fehr and Kindermann, 2018, p. 63). In particular, it is fast and able to solve eq. (8) despite its local non-differentiability.

Appendix E. Subjective survival probabilities for females

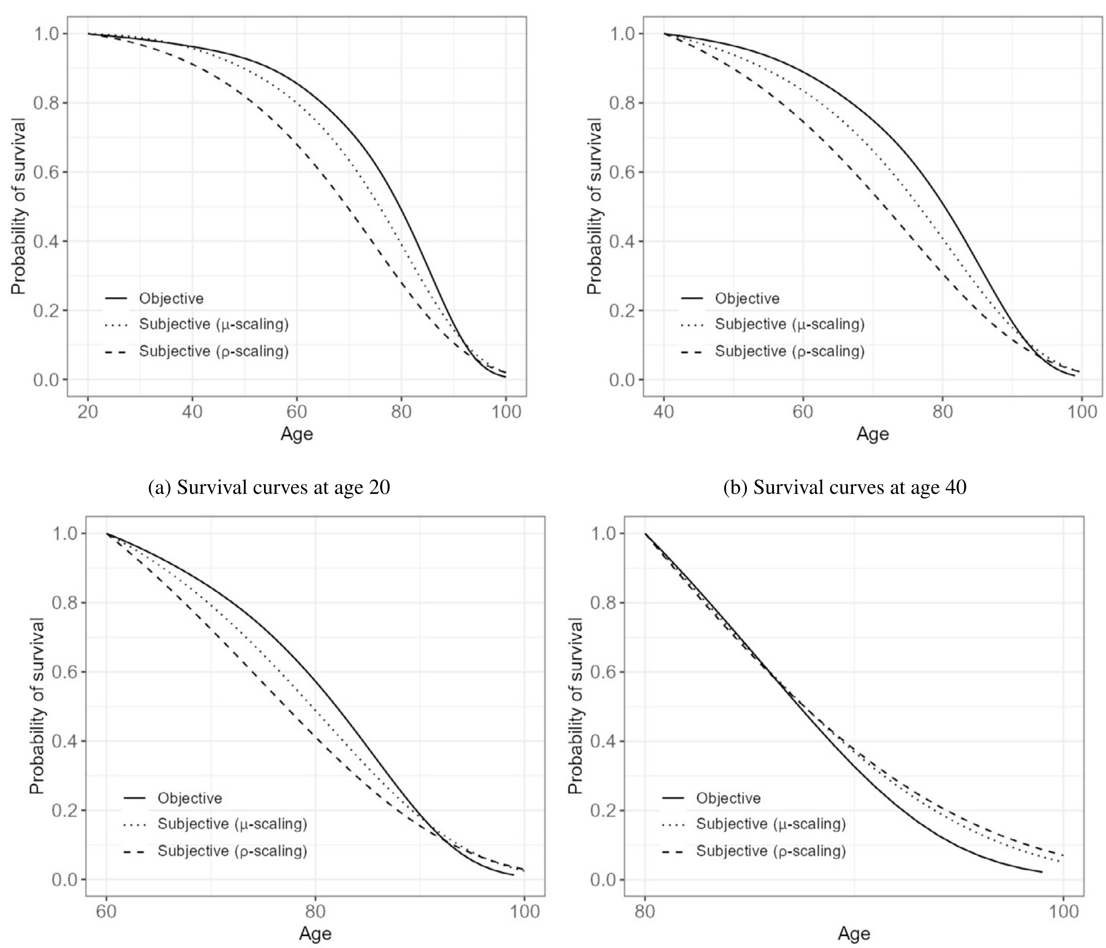
Figs. E.1–E.3 in this appendix pertain to females and correspond to Figs. 2–4 for males.



(a) p_x-scaling

(b) μ_x-scaling

Fig. E.1. Subjective one-year survival probabilities averaged over all females at each age (dots), spline curve fitted to subjective survival probabilities (dashed line), and corresponding actuarial survival probabilities (solid line).



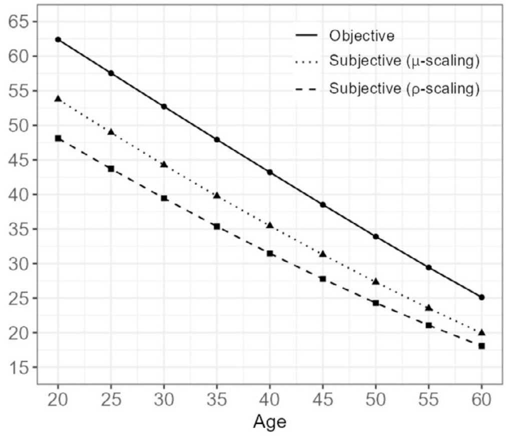
(a) Survival curves at age 20

(b) Survival curves at age 40

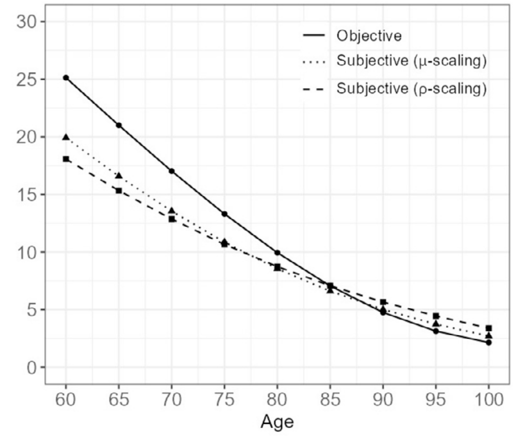
(c) Survival curves at age 60

(d) Survival curves at age 80

Fig. E.2. Survival curves for females at different ages with actuarial and subjective survival probabilities.



(a) Life expectancy at ages 20–60



(b) Life expectancy at ages 60–100

Fig. E.3. Life expectancy for females at different ages with actuarial and subjective survival probabilities.

Data availability

The data is freely available at https://www.federalreserve.gov/econres/scfindex.htm.

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