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Multiparticle Flux-Tube S-matrix Bootstrap

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We introduce the notion of jets, states of collinear flux-tube excitations. We argue for the analyticity, crossing, and unitarity of the multiparticle scattering of these jets and, through the S-matrix bootstrap, place bounds on a set of finite-energy multiparticle sum rules. Such bounds define a matrioska with a smaller and smaller allowed regions as we impose more constraints. The Yang-Mills flux tube, as well as other interesting flux-tube theories recently studied through lattice simulations, lie inside a tiny island hundreds of times smaller than the most general space of allowed theories.

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Introduction—Scatter at high energies and particles you shall produce. Such is life; $2\to n$ amplitudes exist. And yet, we are often afraid of them, and rightfully so. Even the number of variables required to describe their kinematics is intimidating, not to speak of their intricate analytic structure, and it is perhaps not surprising that within the recent resurgence of the S-matrix bootstrap only $2\to 2$ amplitudes have been constrained thus far. In this Letter, we give a first step toward improving this state of affairs and delve into the $m\to n$ world. We will do so within the physics of long one-dimensional flux tubes.

The transverse fluctuations of long flux tubes are described by massless particles, Goldstone bosons of the nonlinearly realized Poincaré symmetry [1–4]. We call them branons. The flux-tube S-matrix bootstrap [5–7] aims at studying these flux tubes by constraining the possible S matrices of these branons. We will focus on flux tubes in three space-time dimensions so that we have a single Goldstone particle.

Its two-to-two scattering at low energies is given by

$$S_{11\to 11}(s)=e^{is\ell^2/4}\times e^{i\gamma\ell^6 s^3/768}\times[1+O(s^4)], \quad (1)$$

where ℓ is the string length (set to 1 henceforth), s is the center of mass energy squared, and we separated the amplitude into three parts: (a) a "gravitational" dressing [8–10] which would arise from the Nambu-Goto (NG) string action, (b) the first deviation from NG action parametrized by the Wilson coefficient γ, and (c) all higher corrections that UV complete this amplitude. The existence of such UV completion with $|S|\le 1$ for all positive energies together with the assumption of polynomial boundness in the upper half plane leads to a nontrivial bound on the first Wilson coefficient as [5]

$$\gamma\ge -1. \quad (2)$$

Here we suggest the use of multibranon processes to further constrain the dynamics of quantum flux tubes [11]. We will produce the first S-matrix bootstrap bounds involving multiparticle processes.

The key player in our construction is what we denote as a "branon jet," a multiparticle state with n particles moving collinearly, each with a fraction α_i of the total energy P,

$$|\alpha_1,\ldots,\alpha_n,P\rangle_n\equiv|\alpha_1\vec{P},\alpha_2\vec{P},\ldots,\alpha_n\vec{P}\rangle, \quad (3)$$

where $\alpha_1+\cdots+\alpha_n=1$.

The jet can be a left mover or right mover and within a (left) right-mover jet all constituents are (left) right movers. It is important to stress that the possibility of making sense of such a state of collinear massless particles is quite nontrivial. It is well defined due to the absence of collinear divergences for these Goldstone particles [33]. In fact, the scattering of any number of left movers (or right movers) among themselves is trivial,

$$S_{RR'}=S_{LL'}=1, \quad (4)$$

since we can boost such processes to arbitrarily low energy when they become effectively free. This is true for R and R' being jets of any number of right movers or simply fundamental right-moving individual branons. These crucial properties allow us to cross jets of multiple particles from past to future, argue for the all-loops analyticity of their scattering amplitudes, and treat them effectively as new massless fundamental particles; their internal composition can be simply thought as a new flavor index; see Secs. S1–S3 of Supplemental Material [12].

For jets of two particles (and similarly for jets of many particles) we define a discrete complete basis by averaging over the energy fraction α at fixed total energy P according to

$$|n,P\rangle \equiv \sqrt{2n+1}\int_0^1 d\alpha \frac{P_n(2\alpha-1)}{\sqrt{8\pi\alpha(1-\alpha)}}|\alpha,(1-\alpha),P\rangle_2 \qquad (5)$$

with n even since branons are identical bosons; see Sec. S3 of Supplemental Material [12] for the general treatment of N particle jets. The precise normalization and choice of Legendre polynomials for jet wave functions in (5) is tuned so that these jets are nicely normalized as one-particle states,

$$\langle m,Q|n,P\rangle = 4\pi P_0\delta_{n,m}\delta(\vec{Q}-\vec{P}).$$

We can now decompose a multiparticle amplitude into infinitely many scattering amplitudes involving these jets. We can scatter one jet against a fundamental branon, for instance, to get two possible S matrices, $S_{1n\to 1n}(s)$ and $S_{1n\to n1}(s)$ corresponding to the forward and backward scattering of the jet.

At low energies, all of these S matrices are again given by a gravitational dressing part $e^{i\ell^2 P_L P_R/4}$, where P_L/P_R is the total left (right) momenta of the incoming (or outgoing) state together with the first deviation from this Nambu-Goto behavior, governed by γ and captured concisely by the generating function [10]

$$L_\gamma = \frac{\gamma}{3\times 2^{13}}(\partial_+^2 x)^2(\partial_-^2 x)^2(16\ell^6 + 24\ell^8(\partial_+ x)(\partial_- x) + 21\ell^{10}(\partial_+ x)^2(\partial_- x)^2 + \cdots), \qquad (6)$$

where x in this equation indicates the transverse coordinate to the flux tube. Combining both factors we can read off the low-energy behavior of all scattering processes involving two-particle jets and fundamental particles as detailed in Sec. S4 of Supplemental Material [12].

The behavior of all these S matrices at energies $s\sim\ell^{-2}$ is nontrivial. For most flux-tube theories of interest, such as pure Yang-Mills theory, their dynamics is expected to be strongly coupled at those energies. In this short Letter, we will place bounds on the simplest possible multiparticle observables capturing the dynamics at the string scale. For that we will consider only the fundamental branon—for which we use the label 1—together with the first branon jet given by $n=0$ in (5)—which we will henceforth indicate by the label 2.

This nomenclature, where we now have two effective particles with labels 1 and 2, is quite convenient. Indeed, note that we now have a simple S-matrix bootstrap setup with two particles and $\mathbb{Z}_2$ symmetry (with 1 being the odd particle and 2 the even one) and four independent amplitudes for which a great deal of technology has been developed in [34–36].

A triplet of finite-energy observables—Any S-matrix bootstrap study starts with the choice of wise observables to bootstrap. We suggest

$$(X,Y,Z) \equiv (S_{11\to 11}(i), S_{22\to 22}(i), \mathrm{Re}[S_{11\to 22}(i)]) \qquad (7)$$

as an interesting triplet of physical observables, a finite-energy section of the infinite-dimensional space of two-dimensional S matrices.

The amplitudes $S_{AB\to CD}(s)$ are analytic in the upper half plane, so a natural observable measuring their strength can be defined as the value of these amplitudes somewhere in this upper plane; here we pick an energy $s=i$.

Let us note that (X,Y,Z) can be measured from simple sum rules probing scattering at all energies. For example, we can write

$$X = \int_{\mathbb{R}} \frac{ds}{\pi(s^2+1)} S_{11\to 11}(s) \qquad (8)$$

and similarly for Y and Z [37].

A flux-tube experimentalist might like this sum rule: from several scattering outcomes at various energies s, one simply adds them up to produce a nice approximation to (8). Of course, depending on the quality and quantity of such scattering data for different energy ranges, such experimentalist could also prefer other sum rules. Indeed, in (8) we considered a single subtraction s_0, which is always enough for one dimension; we could write infinitely many equivalent sum rules for X, Y, or Z with more subtractions such as

$$Z = SR_n \equiv \mathrm{Re}\int_{\mathbb{R}} \frac{ds}{\pi(s^2+1)}\left(\frac{3i}{s+2i}\right)^n S_{11\to 22}(s). \qquad (9)$$

All these integrals must evaluate to the same thing, although they weigh the real s regions very differently. For large n we suppress high energy but the low-energy contribution becomes highly enhanced, while for small n one needs good high-energy data but the low-energy data need not be resolved so much. An intermediate n could be ideal for an experimentalist with reasonable but not ideal low- and high-energy data.

To conclude: our triplet choice (X,Y,Z) can be thought as a measure of the various contact multipoint couplings of

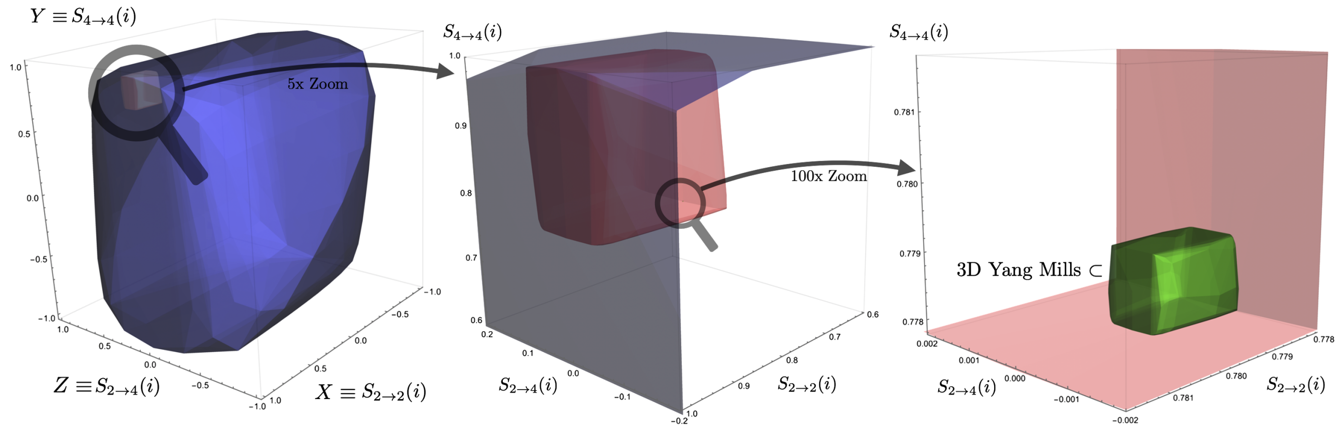

FIG. 1. The branon matrioska: allowed space of (X, Y, Z) for various two-dimensional theories where jets make sense. The blue shape assumes only unitarity and analyticity. Inside it is a much smaller red surface where we also impose nonlinearly realized Lorentz by fixing the low-energy behavior of the S matrices. Inside it, in a yet much smaller green region, is the space of such S matrices with first Wilson coefficient $\gamma \lesssim 0.8$, a conservative upper bound that should contain most of the interesting flux-tube theories according to the recent lattice estimates. In this figure, we use $S_{n\to m}$ to denote the processes involving in total n and m particles in the initial and final states; $S_{4\to4}$, for instance, refers to the scattering of two jets (each with two particles) in the past yielding two jets in the future, that is, $S_{4\to4} = S_{22\to22} = Y$ and so on. In each matrioska doll, some directions can be bounded from analytic single-component Schwarz-type arguments, see Sec. S9 of Supplemental Material [12].

the theory at some nonperturbative finite (complex) energy as well as a set of sum rules probing scattering across all (real) energy scales. We now turn to the allowed (X, Y, Z) space.

The branon matrioska—What values can (X, Y, Z) take? If we impose very little, we will get that (X, Y, Z) must leave inside a big (albeit compact) three-dimensional "rock." If we impose more conditions, we will obtain a smaller rock inside that one. And so on. We call this sequence of allowed regions—one inside the other—the "branon matrioska." We will now describe a matrioska with three such regions.

The first physical constraint we will impose is unitarity. Unitarity can be stated as positive semidefiniteness of two simple matrices. One is obtained by constructing all possible scalar products of even in and out states so that

$$\begin{pmatrix} 1 & S_{11\to11} & 0 & S_{11\to22} \\ S^*_{11\to11} & 1 & S^*_{22\to11} & 0 \\ 0 & S_{22\to11} & 1 & S_{22\to22} \\ S^*_{11\to22} & 0 & S^*_{22\to22} & 1 \end{pmatrix} \succeq 0 \qquad (10)$$

for any $s > 0$. The other condition, for the odd sector, takes a similar form with 11 and 22 replaced by 12 and 21, respectively. These two unitarity conditions talk to each other through crossing [34] since $S_{11\to22} = S^*_{12\to21}$. It is this interplay between crossing and unitarity that leads to nontrivial bounds.

The positivity conditions (10) are imposed numerically using SDPB [38]. In the primal formulation we explore a (truncated version of an) ansatz [5],

$$S_{AB\to CD} = \sum_{m=0}^{\infty} c_m^{(AB\to CD)} \left(\frac{s-i}{s+i}\right)^m \qquad (11)$$

subject to unitarity and optimize over the constants $c_m^{(AB\to CD)}$ to explore the allowed (X, Y, Z) space. The truncation amounts to replacing ∞ by a large $n_{\max}$ cutoff. For a dual formulation, see Secs. S7 and S8 of Supplemental Material [12].

For the outer doll of the matrioska, this is all we impose: unitarity and crossing symmetry (plus analyticity implicitly). Importantly, we do *not* impose any low-energy behavior. We call this problem the "blue" problem as we will present its results in a blue 3D plot. Note right away that a particular obvious consequence of unitarity is that $|X| < 1$, $|Y| < 1$, and $|Z| < 1$ since these are all probabilities, and as such the blue rock must be a compact shape inside a cube of length 2 centered at the origin of the (X, Y, Z) space; indeed, this is what we find numerically, see blue shape in Fig. 1.

The "red" problem is the setup where we impose unitarity together with the leading universal effective field theory (EFT),

$$S_{ab\to ab} = e^{is/4} + O(s^2), \qquad S_{\text{other components}} = O(s^2), \qquad (12)$$

imposing dominance of the elastic components over the reflection and creation amplitudes at low energies. This is the universal behavior predicted by the leading EFT given by Nambu and Goto and is agnostic about the first correction to it governed by γ. Imposing a low-energy behavior can dramatically improve our finite-energy bounds as reviewed in a few simple analytic examples in

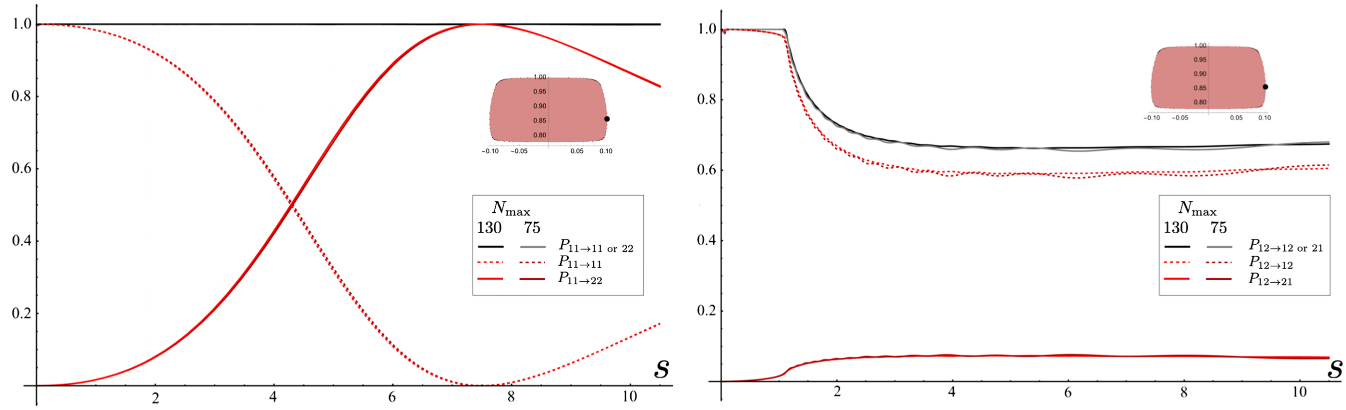

FIG. 2. At very low energy, the diagonal processes $11 \to 11$ and $12 \to 12$ dominate since the theory is free in the IR. As we crank up the energy, jet production kicks in. On the even sector depicted on the left we see that around $s = 4$ the jet production $11 \to 22$ even starts dominating for this rightmost point of the red matrioska. The solid curve on the left panel at $P = 1$ is simply the sum over the two possible even outcomes. On the right, we have the odd sector and that same sum no longer adds up to 1. That means that around $s = 1$ the optimal S matrices at this point of the matrioska choose to produce some odd state outside the 12 branon-jet system. It would be fascinating to find out what these states are in case they have a physical meaning. This is the first instance of an S-matrix bootstrap where unitarity is perfectly saturated in a finite-energy interval after which it is not. In all other examples we know of unitarity wants to converge to 1. Note that, in these plots we depicted two very different values of (very large) $n_{\max}$ to be sure that there is no convergence issue.

Sec. S5 of Supplemental Material [12]. Indeed, it does. We immediately obtain a red rock about 10 times smaller than the blue rock once we impose these extra constraints, see middle panel in Fig. 1.

We know the behavior of all flux-tube S matrices a few orders further in the low-energy expansion up to the terms governed by the leading Wilson correction γ; see Sec. S4 of Supplemental Material [12]. The reader could wonder whether imposing that subleading behavior would generate another rock inside the red rock. We have indeed tried that but all numerical evidence we found is that it does not improve the bounds further; the resulting rock seems to converge to the very same red rock as we increase our primal truncation $n_{\max}$. This is fine: not all pointwise constraints lead to bound improvements, as explained in the toy examples of Sec. S5 of Supplemental Material [12]. This seems to be one more such example—as long as we do not commit to any value for γ.

On the other hand, if we also supplement our conditions with an upper bound on the Wilson coefficient γ the situation dramatically improves. This is what we did in what we call the "green" problem. Here we imposed the constraint $\gamma < 0.768$, which we expect should include $SU(N)$ Yang-Mills theory with at least $N \leq 3$ according to various recent lattice estimates [39].

Once we impose this upper bound on γ we see that the allowed space shrinks by several orders of magnitude, leading to a tiny green rock, the smallest rock in our branon matrioska; see right panel in Fig. 1.

Discussion—What class of string theories govern the flux tube of pure three-dimensional Yang-Mills theories? What is the world sheet physics of this two-dimensional theory at finite energies? We still do not know.

Nonetheless, we find it quite remarkable that the green region is so small that we can already predict finite-energy sum rules up to a few digits for any reasonable confining gauge theory such as SU(3) pure glue. While the matrioskas in Fig. 1 were generated through primal numerics, the power of the bootstrap in constraining these sum rules can be seen analytically as well; in Sec. S9 of Supplemental Material [12], we derive several simple single-component analytic bounds including, for example, the sum rule bound

$$0.7778 \leq X \leq 0.7796,$$

for the width of the smallest branon matrioska. As far as we know, the S-matrix bootstrap is the only available machinery that allows us to quantitatively address such interesting Lorentzian quantities.

In Fig. 2 we looked for the probability for various production in the even and odd channels for the optimal S matrices on the boundary of the two-dimensional section of the red region. On the left (even) panel we observe something rather cute, albeit unsurprising, while on the right (odd) panel we observe something rather striking and unexpected (based on all previous S-matrix bootstrap studies). We find that, after some critical energy, unitarity is not saturated. The probability of finding an odd branon plus an even jet in the final state moving as in the initial state ($P_{12\to12}$) or reflected with respect to the initial state ($P_{12\to21}$) does not add up to 1. Something else is being produced. It would be fascinating—albeit probably far fetched—if they could be thought of as glue balls emitted from the flux tube when the branons reach a critical energy roughly corresponding to their mass. Even if they turn out to be more exotic objects or to have no simple physical interpretation, it is still mathematically fascinating that unitarity is not saturated along the boundary [40]. What is going on? Could something like this also happen in higher-dimensional scattering [42]? We conjecture that the answer is yes and that with a similar Z_2 scattering problem involving an odd and an even particle in higher dimensions we should observe similarly striking dips if we choose to probe the scattering in this system by a similar triplet kind of effective coupling; it would be fascinating to check that this is the case.

However, even in two dimensions, there is still much to do.

We could consider infinitely many more jets, explore other directions in the infinite-dimensional space of multiparticle S matrices, and check whether the matrioska shrinks further. We could also study flux tubes in higher-dimensional confining theories. In $D = 4$—the real world—jets could be made out of any combination of the two branons corresponding to the two transverse directions to the flux tube. In [5], when exploring the allowed space of leading Wilson coefficients, it was found that the boundary of the allowed space of flux-tube S matrices in $D = 4$ is much more interesting than in $D = 3$; the extremal S matrices contain a world sheet axion resonance whose mass and coupling match both the lattice measurements and the estimate coming from integrability [47]. In [7], it has been conjectured that the axion contributes nonperturbatively to cancel the universal particle production induced by the Polchinski-Strominger term, thus enhancing approximate low-energy integrability [48]. It would be interesting to check this mechanism by employing the full nonperturbative multiparticle bootstrap. It is thus only natural to expect that the $D = 4$ branon matrioska will probably be even richer than the $D = 3$ matrioska studied here.

What about higher dimensions? Can we attack multiparticles there? We must. Otherwise, how can we hope for the nonperturbative S-matrix bootstrap to rise to the standards of perturbative quantum field theory? A key tool in the two-dimensional explorations we initiated here was the introduction of jets, which allowed us to derive bounds on the multiparticle S matrix through exploring single variable analyticity and crossing, thus evading having to deal with the intricate analytic structure of the full multiparticle amplitude. In higher dimensions, can we develop a jet-effective field theory where we expand around highly aligned particles that will have a small effective mass and might possibly be treated as effective single particles [49]? This seems like a direction worth exploring.

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