Editors' Suggestion

Where Is String Theory in the Space of Scattering Amplitudes?

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We use the S-matrix bootstrap to carve out the space of unitary, crossing symmetric and supersymmetric graviton scattering amplitudes in ten dimensions. We focus on the leading Wilson coefficient α controlling the leading correction to maximal supergravity. The negative region $\alpha < 0$ is excluded by a simple dual argument based on linearized unitarity (the desert). A whole semi-infinite region $\alpha \gtrsim 0.14$ is allowed by the primal bootstrap (the garden). A finite intermediate region is excluded by nonperturbative unitarity (the swamp). Remarkably, string theory seems to cover all (or at least almost all) the garden from very large positive α—at weak coupling—to the swamp boundary—at strong coupling.

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At large distances gravity is universal. At short distances it is UV completed. The first hints of such completions come from the Wilson coefficients (WCs) governing the low energy effective action. String theory leads to some values of the WCs: other theories to other values. As we will illustrate, the S-matrix bootstrap is a powerful quantitative tool to carve out the allowed space of such WCs and thus learn about the potential UV completions of gravity (the application of the S-matrix bootstrap to the study of EFT has been initiated in [1,2] for the simpler study of 2D flux tubes and 4D massless pions, respectively). To kick off this program we focus on a simpler setup and set out to study the space of 10-dimensional gravitational theories with maximal supersymmetry. In $d \geq 5$ dimensions gravity is IR finite. The main simplification here is however supersymmetry as it allows us to relate scattering of gravitons to the much simpler scattering of its scalar superpartners. The two-to-two scattering amplitude of the graviton multiplet for any 10D theory with maximal SUSY takes the following form [see, e.g., Eq. (7.4.57) in [3] and [4] for a simple general argument. For $\mathcal{N} = (2,0)$—as in type IIB superstring theory—there is a manifestly supersymmetric representation of this prefactor as $\mathbf{R}^4 = \delta(Q)$ [5] while for $\mathcal{N} = (1,1)$—as in type IIA—such SUSY rewriting is not known as reviewed in [6]. Nonetheless, in both cases (1) holds. See [7] for a covariant representation of this prefactor in the pure spinor formalism]:

$$\mathbb{A}_{2\to 2} = \mathbf{R}^4 A(s,t,u). \tag{1}$$

By extracting different components of the $\mathbf{R}^4$ prefactor sitting in front we get access to the various scattering processes. At low energy $A(s,t,u) \sim 1/stu$ is the universal gravity behavior. In $\mathcal{N} = (2,0)$ we can scatter the charged axidilaton, for instance, by picking an s^4 factor from the $\mathbf{R}^4$ prefactor thus getting an amplitude

$$T(s,t,u) \equiv s^4 A(s,t,u) = -8\pi G_N \left(\frac{s^2}{t} + \frac{s^2}{u}\right) + \ldots, \tag{2}$$

where in the last equality we used $s + t + u = 0$. There are t and u channel poles corresponding to massless graviton exchanges between the charged scalars; there is no s channel pole since these scalars are charged and thus can not annihilate. The combination T is very important and will be the central object in this Letter since unitarity for the super amplitude turns out be equivalent to usual unitarity for this component as explained in the Supplemental Material [8], Appendix A, which includes Refs. [9–11].

The Wcs are in the dots in (2). More precisely [12],

$$\frac{T(s,t,u)}{8\pi G_N = 64\pi^7 \ell_P^8} = s^4\left(\frac{1}{stu} + \alpha \ell_P^6 + O(s)\right), \qquad (3)$$

where the $O(s)$ term is a universal one loop contribution, which we work out in the Supplemental Material [8], Appendix B, which contains Ref. [13]. At higher orders in s there are subleading WCs and higher loop contributions. Nicely, the first Wilson coefficient α appears at order s^0 and can thus be cleanly separated from the 1-loop contribution [14]. The coefficient α controls the (SUSY completion of the) Riemann^4 term in the effective action [15]. The purpose of this Letter is to study the allowed space of α compatible with the S-matrix bootstrap principles of analyticity, crossing and unitarity of the 2 to 2 scattering amplitude.

In type IIB superstring theory, we have [16–19]

$$\alpha^{\mathrm{IIB}} = \frac{1}{2^6} E_{\frac{3}{2}}(\tau, \bar{\tau}) \geq \frac{1}{2^6} E_{\frac{3}{2}}(e^{i\pi/3}, e^{-i\pi/3}) \approx 0.1389, \qquad (4)$$

where the nonholomorphic Eisenstein series depends on the complexified string coupling $\tau = \chi_s + (i/g_s)$. In fact, it is always larger than a finite positive value (see the Supplemental Material [8], Appendix D, which contains Ref. [20] for more details). In type IIA superstring theory, we have (see, e.g., [17,18,21,22])

$$\alpha^{\mathrm{IIA}} = \frac{\zeta(3)}{32 g_s^{3/2}} + g_s^{1/2}\frac{\pi^2}{96} \geq \frac{\pi^{3/2}(\zeta(3))^{1/4}}{24\sqrt{3}} \approx 0.1403, \qquad (5)$$

where the string coupling $g_s \geq 0$. We conclude that the values realized in string theory are

$$\alpha \geq \alpha_{\min}^{\mathrm{ST}} \equiv \frac{1}{2^6} E_{\frac{3}{2}}(e^{i\pi/3}, e^{-i\pi/3}) \approx 0.1389. \qquad (6)$$

Our goal is to use the bootstrap to find out the allowed possible values of α. How big is the space of allowed quantum gravity UV completions and does string theory fit in this space?

First of all, we can show that α can not be negative. Indeed, in the Supplemental Material [8], Appendix C, which contains Refs. [23,24], we employ the usual contour manipulation arguments [25]—see also [26–29]—to show that

$$\alpha = \frac{1}{32\pi^8 \ell_P^{14}} \int_0^\infty \frac{ds}{s^5} \mathrm{Im} T(s + i\epsilon, t = 0). \qquad (7)$$

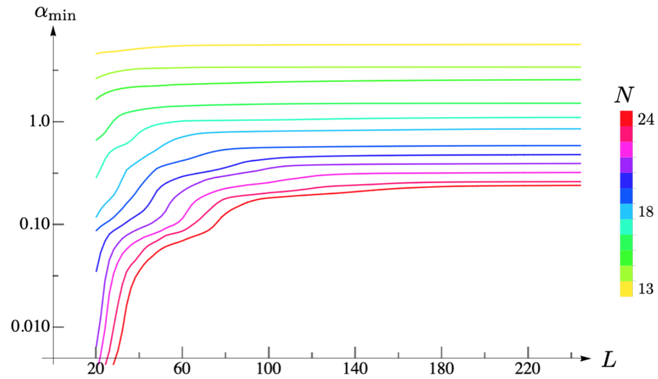

FIG. 1. Minimum $\alpha_{\min}(N, L)$. We see that the curves nicely converge towards a plateau whenever L is large enough. The larger N is, the further we need to go in L to reach this plateau. For each value of N we extrapolate these plateaus to estimate $\alpha_{\min}(N, \infty)$, which we plot in the next figure.

The optical theorem then implies

$$\alpha \geq 0. \qquad (8)$$

This is a prototypical example of a rigorous dual exclusion bound. No matter how hard we scan over putative *Ansätze* for T—as we do in the primal formulation—we will never encounter an amplitude with a negative Wilson coefficient α. Beautifully, both type IIA and IIB coefficients reviewed above are indeed always positive.

The optimal bound must therefore be somewhere between the dual bound (8) and the string theory realization (6). To look for it we turn to the primal S-matrix bootstrap formulation and construct the most general amplitude compatible with maximal SUSY, Lorentz invariance, crossing, analyticity and unitarity following [2,30–34]. The key representation is given by

$$\frac{T}{8\pi G_N} = s^4\Bigg(\underbrace{\frac{1}{stu}}_{\texttt{SUGRA}} + \underbrace{\prod_{A=s,t,u}(\rho_A+1)^2 \sum_{a+b+c\leq N}{}' \alpha_{(abc)}\rho_s^a \rho_t^b \rho_u^c}_{\texttt{UVcompletion}}\Bigg), \qquad (9)$$

which follows the notation introduced in those references and is discussed in detail in the Supplemental Material [8], Appendix E (also Appendices F and G).

We minimize α for any fixed N (related to the number of parameters in a primal ansatz) and L (maximum spin up to which we impose unitarity of the partial waves) using a semidefinite program solver for the conformal bootstrap (SDPB) solver [35,36]. We run SDPB on 80 cores: the running time scales linearly with L and quadratically with N. We assume the optimal solution is found when the duality gap is $\Delta < 10^{-16}$ [the *Ansatz* (9) is a polynomial of high degree, in general we observe large cancellations

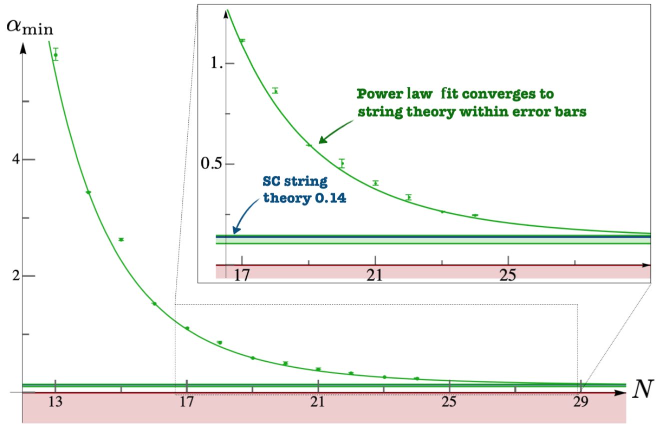

FIG. 2. Minimum $\alpha_{\min}(N,\infty)$ obtained by extrapolating the various plateaus in Fig. 1. We estimate the error bars here by scanning over a large number of such fits as explained in the Supplemental Material [8], Appendix H. We then extrapolate these points to estimate $\alpha_{\min}^{\mathrm{Boot}} = \alpha_{\min}(\infty,\infty) \simeq 0.13$ with an uncertainty represented by the green strip. It nicely embraces the strong coupling string prediction depicted by the solid blue line.

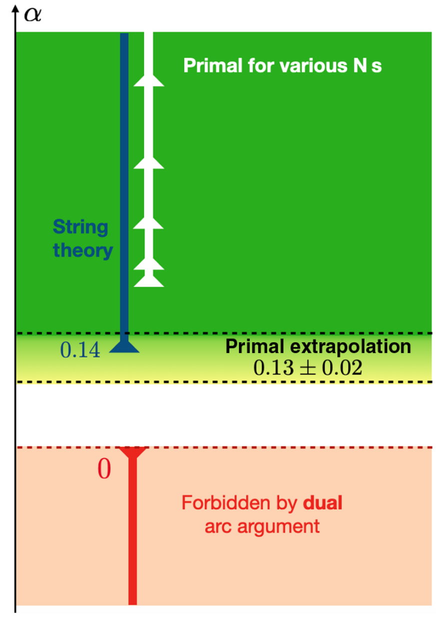

FIG. 3. String theory covers all or almost all the allowed quantum gravity theory space.

among the various terms, therefore we work at fixed high precision ($\sim 10^3$ binary digits)].

Figure 1 depicts various curves for the minimum value of α for various N as a function of L—see Supplemental Material, Appendix E for details. We see that as N grows the primal *Ansatz* is capable of minimizing α better and better as expected. But we also see that for each N it is crucial to impose unitarity up to very large spin L to observe convergence of the bound. For the $N = 24$, for instance, we see that we only converge for spin L around 220; for lower L there are important violations of unitarity in partial waves with spin greater than L. For each N we extrapolate the curves in Fig. 1 to estimate the result at $L = \infty$. Next we fit in N to estimate the $N = \infty$ final value as depicted in Fig. 2.

Note that these fits introduce error bars. More precisely, fitting these curves is a bit of an art as we can *a priori* pick different number of fitting points and different fit *Ansätze*. We took a large family of plausible fits and weight them by how well they approximate the various numerical points (see the Supplemental Material [8], Appendix H for details). The spread is an estimate of the final error. In this way we estimate that the $L \to \infty$ extrapolation leads to error bars attached to the points in Fig. 2 and those error bars, in turn lead to the uncertainty window in the large N extrapolation denoted by the green shaded region in this figure. In this way, we estimate that

$$\alpha_{\min}^{\mathrm{Boot}} \equiv \lim_{N \to \frac{\infty}{L} \to \infty} \alpha_{\min}(N, L) \approx 0.13 \pm 0.02. \qquad (10)$$

Comparing with (6) suggests that string theory realizes all values of α compatible with the S-matrix bootstrap principles. It would be useful to increase our numerical precision to check if indeed $\alpha_{\min}^{\mathrm{Boot}} = \alpha_{\min}^{\mathrm{ST}}$ or if there is some allowed space not realized in superstring theory. It would also be fascinating to develop a dual S-matrix bootstrap problem (see, e.g., [37–39]), which would extend the simple red excluded region derived above—the desert—into the swamp, which currently separates it from the green garden included by the primal problem as summarized in Fig. 3.

As usual with primal problems, it is fascinating to see what physical features the optimal solutions have. In this case, how do phase shifts for theories of quantum gravity living at the boundary of the garden look like? We are investigating this in more detail and hope to report on a more extensive study soon but two fascinating features seem to be robust: (i) there are infinitely many resonances (for large spin they seem to lie on a curved Regge trajectory with $s_* = m_*^2 \sim \ell^{3/2}$ as predicted by unitarity, see the Supplemental Material [8], Appendix F), (ii) the lightest resonance is a spin zero resonance which we show in Fig. 4. This scalar resonance is reminiscent of the graviball recently found in [40] using an approximate method to unitarize perturbative amplitudes (see the Supplemental Material, Appendix I, which contains Refs. [41,42], for a similar approach in our setup). In a strongly coupled string theory with $g_s \sim 1$ we expect excited states to show up at the scale $\sim \ell_P^{-1}$. It is therefore tempting to identify the graviball as the first excited string state.

We explored here the one dimensional space of α, the leading Wilson coefficient. Would be fascinating to explore combined space of the first two leading Wilson coefficients. What structures do we find in this richer two-dimensional

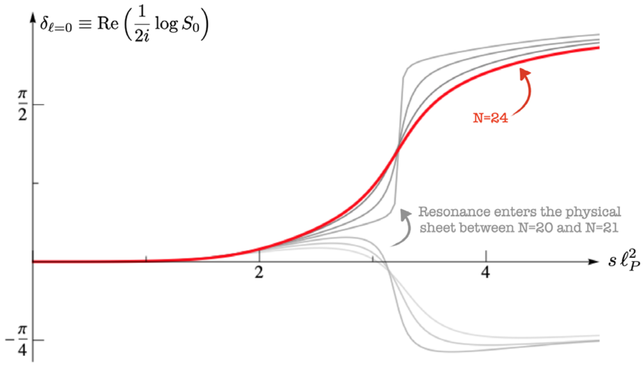

FIG. 4. Spin zero phase shift as we increase N from 18, …, 23 (in gray) until 24 (in red). Between $N = 20$ and $N = 21$ a zero enters the physical sheet—it is the lightest resonance we encounter. Our best numerics with $N = 24$ seem to be close to converging in this energy range and hint at a mass of around $m^2\ell_P^2 \simeq 3.2 + 0.3i$.

space? Is closed superstring theory again in a privileged position? Would be interesting to investigate open strings as well (and possible UV completions of Yang-Mills theory in higher dimensions).

At high energy we expect black holes in any theory of quantum gravity. It would be very nice to see how they fit in our analysis. The Wilson coefficients will likely not be affected dramatically by modifying the high energy inelasticity to the expected behavior but other quantities such as the positions of the various resonances alluded to above might change more significantly.

It would also be instructive to see how all this fits in AdS/CFT. The analog of the supergraviton scattering amplitude is the four-point function of the stress tensor multiplet in the dual super conformal field theory. This has been studied in the large N expansion with maximal supersymmetry [43–48]. The analog of our nonperturbative S-matrix bootstrap is the superconformal bootstrap [49–54]. It would be interesting to explore this connection in detail.

It would also be very interesting to repeat our analysis in other dimensions. In 11 dimensions we should make contact with M theory whose scattering amplitudes have no free parameters. It would also be fascinating to consider less or even no supersymmetry. In that case we have to deal with all the pain and glory of gravitons as spinning particles (one will need to generalize the recent work [32] from 4D to higher dimensions. The formalism developed in [55,56] should also be useful). It would be amazing if the theory of quantum gravity describing our Universe would be at a premium location, identifiable through the S-matrix bootstrap.

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[1] J. Elias Miró, A. L. Guerrieri, A. Hebbar, J. Penedones, and P. Vieira, Flux Tube S-matrix Bootstrap, Phys. Rev. Lett. **123**, 221602 (2019).

[2] A. Guerrieri, J. Penedones, and P. Vieira, S-matrix bootstrap for effective field theories: Massless pions, J. High Energy Phys. 06 (2021) 088.

[3] M. B. Green, J. H. Schwarz, and E. Witten, *Superstring Theory: Introduction*, Cambridge Monographs On Mathematical Physics (Cambridge University Press, Cambridge, England, 1987), Vol. 1, p. 469.

[4] L. F. Alday and J. Martin Maldacena, Gluon scattering amplitudes at strong coupling, J. High Energy Phys. 06 (2007) 064.

[5] R. H. Boels and D. O'Connell, Simple superamplitudes in higher dimensions, J. High Energy Phys. 06 (2012) 163.

[6] Y. Wang and X. Yin, Constraining higher derivative supergravity with scattering amplitudes, Phys. Rev. D **92**, 041701 (2015).

[7] G. Policastro and D. Tsimpis, R^4, purified, Classical Quantum Gravity **23**, 4753 (2006).

[8] See the Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.127.081601 for additional details on super unitarity, perturbation theory, dispersion relations, numerical setup and data analysis.

[9] Z. Bern, L. J. Dixon, D. C. Dunbar, M. Perelstein, and J. S. Rozowsky, On the relationship between Yang-Mills theory and gravity and its implication for ultraviolet divergences, Nucl. Phys. **B530**, 401 (1998).

[10] M. B. Green, J. G. Russo, and P. Vanhove, Low energy expansion of the four-particle genus-one amplitude in type II superstring theory, J. High Energy Phys. 02 (2008) 020.

[11] In $\mathcal{N} = (1, 1)$, we have no charged axidilaton and T is not an individual component but unitarity for the superamplitude is nonetheless the same when expressed in terms of T defined through (2).

[12] We use $\ell_P^4 = g_s \ell_s^4$ and $8\pi G_N = \frac{1}{2}(2\pi)^7 g_s^2 \ell_s^8 = 64\pi^7 \ell_P^8$ to facilitate the comparison with string theory.

[13] S. Caron-Huot and A.-K. Trinh, All tree-level correlators in $\mathrm{AdS}_5 \times S_5$ supergravity: Hidden ten-dimensional conformal symmetry, J. High Energy Phys. 01 (2019) 196.

[14] This is not always the case. In pion physics, for example, such separation is more subtle as both effects come in at the same order in the low energy expansion [2].

[15] D. J. Gross and E. Witten, Superstring modifications of Einstein's equations, Nucl. Phys. **B277**, 1 (1986).

[16] M. B. Green and M. Gutperle, Effects of D instantons, Nucl. Phys. **B498**, 195 (1997).

[17] M. B. Green and P. Vanhove, D instantons, strings and M theory, Phys. Lett. B **408**, 122 (1997).

[18] M. B. Green, J. G. Russo, and P. Vanhove, Non-renormalisation conditions in type II string theory and maximal supergravity, J. High Energy Phys. 02 (2007) 099.

[19] S. M. Chester, M. B. Green, S. S. Pufu, Y. Wang, and C. Wen, Modular invariance in superstring theory from $\mathcal{N} = 4$ super-Yang-Mills, J. High Energy Phys. 11 (2020) 016.

[20] L. F. Alday and A. Bissi, Modular interpolating functions for $N = 4$ SYM, J. High Energy Phys. 07 (2014) 007.

[21] B. Pioline, D^6R^4 amplitudes in various dimensions, J. High Energy Phys. 04 (2015) 057.

[22] D. J. Binder, S. M. Chester, and S. S. Pufu, AdS_4/CFT_3 from weak to strong string coupling, J. High Energy Phys. 01 (2020) 034.

[23] A. Martin, Unitarity and high-energy behavior of scattering amplitudes, Phys. Rev. **129**, 1432 (1963).

[24] X. O. Camanho, J. D. Edelstein, J. Maldacena, and A. Zhiboedov, Causality constraints on corrections to the graviton three-point coupling, J. High Energy Phys. 02 (2016) 020.

[25] A. Adams, N. Arkani-Hamed, S. Dubovsky, A. Nicolis, and R. Rattazzi, Causality, analyticity and an IR obstruction to UV completion, J. High Energy Phys. 10 (2006) 014.

[26] N. Arkani-Hamed, T.-C. Huang, and Y.-T. Huang, The EFT-hedron, J. High Energy Phys. 05 (2021) 259.

[27] B. Bellazzini, J. Elias Miró, R. Rattazzi, M. Riembau, and F. Riva, Positive moments for scattering smplitudes, Phys. Rev. D **104**, 036006 (2021).

[28] A. J. Tolley, Z.-Y. Wang, and S.-Y. Zhou, New positivity bounds from full crossing symmetry, J. High Energy Phys. 05 (2021) 255.

[29] S. Caron-Huot and V. Van Duong, Extremal effective field theories, J. High Energy Phys. 05 (2021) 280.

[30] M. F. Paulos, J. Penedones, J. Toledo, B. C. van Rees, and P. Vieira, The S-matrix bootstrap. Part III: Higher dimensional amplitudes, J. High Energy Phys. 12 (2019) 040.

[31] A. L. Guerrieri, J. Penedones, and P. Vieira, Bootstrapping QCD Using Pion Scattering Amplitudes, Phys. Rev. Lett. **122**, 241604 (2019).

[32] A. Hebbar, D. Karateev, and J. Penedones, Spinning S-matrix Bootstrap in 4d, arXiv:2011.11708.

[33] A. Bose, P. Haldar, A. Sinha, P. Sinha, and S. S. Tiwari, Relative entropy in scattering and the S-matrix bootstrap, SciPost Phys. **9**, 081 (2020).

[34] A. Bose, A. Sinha, and S. S. Tiwari, Selection rules for the S-Matrix bootstrap, SciPost Phys. **10**, 122 (2021).

[35] D. Simmons-Duffin, A semidefinite program solver for the conformal bootstrap, J. High Energy Phys. 06 (2015) 174.

[36] W. Landry and D. Simmons-Duffin, Scaling the semi-definite program solver SDPB, arXiv:1909.09745.

[37] L. Córdova, Y. He, M. Kruczenski, and P. Vieira, The O(N) S-matrix monolith, J. High Energy Phys. 04 (2020) 142.

[38] A. L. Guerrieri, A. Homrich, and P. Vieira, Dual S-matrix bootstrap. Part I. 2D theory, J. High Energy Phys. 11 (2020) 084.

[39] Y. He and M. Kruczenski, in Proccedings of the Bootstrap 2020 Annual Conference by M. Kruczenski, June 2020, Boston (via Zoom) (to be published).

[40] D. Blas, J. Martin Camalich, and J. Antonio Oller, Unitarization of infinite-range forces: Graviton-graviton scattering, arXiv:2010.12459.

[41] T. N. Truong, Chiral Perturbation Theory and Final State Theorem, Phys. Rev. Lett. **61**, 2526 (1988).

[42] A. Dobado and J. R. Pelaez, The inverse amplitude method in chiral perturbation theory, Phys. Rev. D **56**, 3057 (1997).

[43] F. Aprile, J. M. Drummond, P. Heslop, and H. Paul, Quantum gravity from conformal field theory, J. High Energy Phys. 01 (2018) 035.

[44] L. F. Alday and S. Caron-Huot, Gravitational S-matrix from CFT dispersion relations, J. High Energy Phys. 12 (2018) 017.

[45] O. Aharony, L. F. Alday, A. Bissi, and E. Perlmutter, Loops in AdS from conformal field theory, J. High Energy Phys. 07 (2017) 036.

[46] T. Okuda and J. Penedones, String scattering in flat space and a scaling limit of Yang-Mills correlators, Phys. Rev. D **83**, 086001 (2011).

[47] S. M. Chester and S. S. Pufu, Far beyond the planar limit in strongly-coupled $\mathcal{N} = 4$ SYM, J. High Energy Phys. 01 (2021) 103.

[48] S. M. Chester, M. B. Green, S. S. Pufu, Y. Wang, and C. Wen, New modular invariants in $\mathcal{N} = 4$ Super-Yang-Mills theory, J. High Energy Phys. 04 (2021) 212.

[49] C. Beem, L. Rastelli, and B. C. van Rees, The $\mathcal{N} = 4$ Superconformal Bootstrap, Phys. Rev. Lett. **111**, 071601 (2013).

[50] L. F. Alday and A. Bissi, The superconformal bootstrap for structure constants, J. High Energy Phys. 09 (2014) 144.

[51] S. M. Chester, J. Lee, S. S. Pufu, and R. Yacoby, The $\mathcal{N} = 8$ superconformal bootstrap in three dimensions, J. High Energy Phys. 09 (2014) 143.

[52] C. Beem, M. Lemos, L. Rastelli, and B. C. van Rees, The (2, 0) superconformal bootstrap, Phys. Rev. D **93**, 025016 (2016).

[53] C. Beem, L. Rastelli, and B. C. van Rees, More $\mathcal{N} = 4$ superconformal bootstrap, Phys. Rev. D **96**, 046014 (2017).

[54] N. B. Agmon, S. M. Chester, and S. S. Pufu, Solving M-theory with the conformal bootstrap, J. High Energy Phys. 06 (2018) 159.

[55] N. Arkani-Hamed, T.-C. Huang, and Y.-t. Huang, Scattering amplitudes for all masses and spins, arXiv:1709.04891.

[56] S. Dutta Chowdhury, A. Gadde, T. Gopalka, I. Halder, L. Janagal, and S. Minwalla, Classifying and constraining local four photon and four graviton S-matrices, J. High Energy Phys. 02 (2020) 114.