Thermal convection in channels and long boxes

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Abstract

Rayleigh's (1916) attempt to describe the experimental observations of Benard (1900) is the foundation of a large number of theoretical studies on thermal convection. Many of these investigations are based on the assumptions that the horizontal fluid layer is confined between stress-free upper and lower boundaries and is unbounded in the horizontal directions. Here, three-dimensional thermal convection in an idealistic infinite channel of rectangular cross-section with no-slip sidewalls and stress-free upper and lower boundaries, in which an adverse temperature gradient is maintained by heating the underside is investigated using both linear and nonlinear techniques. The amplitude equation for nonlinear disturbances is derived and the variation of its coefficients with both aspect ratio (the width of the channel) and Prandtl number ascertained. The results show that the amplitude of the motion undergoes a supercritical bifurcation as the Rayleigh number passes through the critical value for instability predicted by linear theory.

The effect of introducing distant endwalls is investigated using the technique developed by Daniels (1977) for the related two-dimensional problem. If the endwalls of the long three-dimensional box are rigid and the thermal conditions there are consistent with a basic state of no motion a supercritical bifurcation occurs at a new critical Rayleigh number. By determining the higher order amplitude equation the question of wavenumber selection, as developed by Cross et al (1980) for two-dimensional rolls, is extended to the three-dimensional case for the long box with finite length and aspect ratio, for Rayleigh numbers slightly above critical. The results indicate a physical behaviour similar to that observed in experiments.

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Contents

* Indicates the chapter has been written jointly with Professor P.G. Daniels.

Chapter 1 Introduction

Thermal instability often arises when a fluid is heated from below. A classic example of this is a horizontal layer of fluid in which an adverse temperature gradient is maintained by heating the underside. The temperature gradient thus maintained is qualified as adverse since, on account of thermal expansion, the fluid at the bottom is lighter than the fluid at the top. This is a top-heavy arrangement which is basically unstable and there is a tendency on the part of the fluid to re-distribute itself and remedy the weakness in its arrangement. However, this natural tendency on the part of the fluid is inhibited by its own viscosity. Thus, the adverse temperature gradient which is maintained must exceed a certain critical value before the instability can manifest itself. The earliest experiments to demonstrate the onset of thermal convection in fluids are those of Bénard in 1900, though the phenomenon of thermal convection itself had been recognized earlier by Rumford (1797) and Thomson (1882). Bénard carried out his experiments on very thin layers of fluids, about a millimetre in depth, or less, on a levelled metallic plate maintained at a constant temperature. The upper surface was free and being in contact with the air was at a lower temperature. Block (1956) showed physically and Pearson (1958) analytically that most of the motions observed by Bénard, being in very thin layers with a free surface, were driven by the variation of surface tension with temperature and not by thermal instability of light fluid below heavy fluid.

Stimulated by the experiments of Bénard, Rayleigh (1916) formulated the theory of convective instability of a layer of fluid between horizontal planes. He showed that the instability is decided by the numerical value of the non-dimensional parameter $g^* \alpha^* \beta^* d^{*4} / \kappa^* \nu^*$

where g^* denotes the acceleration due to gravity, d^* the depth of the layer, β^* the uniform adverse temperature gradient which is maintained, and α^*, K^*, and ν^* are the coefficients of volume expansion, thermometric conductivity and kinematic viscosity, respectively. This parameter is called the Rayleigh number. Rayleigh further showed that instability occurs when the Rayleigh number exceeds a certain critical value and when the Rayleigh number just exceeds this value, a stationary pattern of motions exists.

Rayleigh's (1916) attempt to describe the experimental observations of Bénard (1900) is the foundation of a large number of theoretical studies on thermal convection. Many of these investigations are based on the assumptions that the horizontal fluid layer is confined between stress-free upper and lower boundaries and is unbounded (ie. extends to infinity) in the horizontal directions. Linear theory predicts that for an infinite fluid layer with stress-free horizontal boundaries heated from below the critical Rayleigh number is $27\pi^4/4$ and the associated critical wavenumber is $\pi/\sqrt{2}$. The desire to make realistic comparisons with experimental work has led to studies of the effects of lateral bounding walls. Their importance, however distant they may be, has been realised for some time. For instance, the experimental work of Koschmieder (1966) indicates that 'the fluid has little or no ability to form a definite cell pattern on its own but the shape is defined by the form of the lateral boundaries'. Davis (1967) considered thermal convection in rigid perfectly conducting three-dimensional boxes, with width to depth and length to depth ratios in the range [¼,6]. He defined 'finite rolls' as cells with two non-zero velocity components dependent on all three spatial variables. Using a linear stability analysis based on the Galerkin method with 'finite roll' trial functions he obtained upper bounds on the critical Rayleigh number. Within his approximation general three-dimensional flows could be constructed by a linear superposition of finite rolls so, he was able to predict the preferred mode at the onset of convection. His analysis showed that the preferred mode is always some number of finite rolls with axes parallel to the short side. In addition, the critical Rayleigh number rapidly decreases to the value 1708 as the horizontal dimensions increase, consistent with the results of Low (1929) and Pellew & Southwell (1940) for the infinite horizontal layer with rigid boundaries. The results of Davis (1967) are confirmed by the experiments of Stork & Müller (1972). Drazin (1975) was able to illustrate some of Davis's ideas analytically by a linear analysis of a simplified two-dimensional model with rigid endwalls and stress-free upper and lower surfaces.

Davies-Jones (1970) considered thermal convection in an infinite rectangular channel of aspect ratio a (width/height) with no-slip sidewalls. By assuming the upper and lower surfaces to be stress-free he obtained exact solutions of the linearized perturbation equations for the convective motions in a Boussinesq fluid heated from below. He showed that the preferred modes of convection closely resemble transverse finite rolls, as predicted by Davis (1967), unless $\frac{1}{2} < a < 5$. Inside this range they show noticeable departures from roll form. The effect of bringing the sidewalls closer together was to inhibit convection and generally, to reduce the wavelength at the onset of convection.

Theoretical analysis of the effect of the lateral boundaries on the convective pattern in a Rayleigh-Bénard cell has been mostly limited to two cases. Firstly, the linearized equations have been studied to determine the onset patterns and their critical Rayleigh numbers (Davis (1967), Davies-Jones (1970)). A second class of problems that has been studied is the inclusion of the weak non-linearities close to onset by the introduction of a slowly varying complex amplitude function which describes the slow modulation of the roll pattern. An equation of motion, the 'amplitude equation', for the amplitude function may be derived from the hydrodynamic equations using the method of multiple scales and incorporating the nonlinear terms (Segel (1969), Newell & Whitehead (1969), Daniels (1977)), based on the method developed by Stuart (1960) for the related Taylor problem. The amplitude equation gives a complete description of the fluid away from the lateral walls. In addition boundary conditions on the amplitude function must be specified so that the temperature and velocity components satisfy the physical boundary conditions at the lateral walls. The amplitude equation, together with the boundary conditions, enable the influence of the lateral walls on convection close to onset to be studied.

Segel (1969) used the method of multiple scales to describe the motion at the onset of convection in a shallow three-dimensional layer bounded by rigid lateral walls. He derived the boundary conditions on the amplitude function for two cases in which the rolls are parallel and perpendicular to the walls. Later Brown & Stewartson (1977) corrected one of his results. Segel's results show that if the lateral walls of the container are rigid and the thermal conditions there are consistent with a basic state of no motion a supercritical bifurcation occurs at a new critical Rayleigh number. Daniels (1977) considered a two-dimensional model with stress-free upper and lower surfaces and rigid endwalls. Using the assumption that the horizontal distance between the endwalls is large compared with the distance between the upper and lower surfaces (ie. the aspect ratio 2L is large compared with 1), he showed that if there is a small heat transfer through the endwalls, so that the boundary conditions there are inconsistent with a state of no motion, the bifurcation is replaced by a smooth transition to finite amplitude convection. Hall & Walton (1977) investigated the fluid motion in a two-dimensional box with stress-free upper and lower boundaries and rigid perfectly insulating endwalls. They also considered the effect of allowing the endwalls to be slightly imperfect insulators. The main difference between the study of Hall & Walton (1977) and that of Daniels (1977) is that each concerns a different limiting situation. Hall & Walton (1977) were mainly concerned with the situation in which the semi-aspect ratio L, is of order one and the imperfection λ, is small. However, Daniels assumed that L >> 1 and that $\lambda = O(L^{-1})$. Brown & Stewartson (1977) extended the ideas of Daniels (1977) to accommodate rectangular containers with large, horizontal dimensions using a similar approach. In particular they showed, in agreement with earlier theories and experiments, that there is a preference for rolls parallel to the shorter side. In the studies by Daniels (1977), Hall & Walton (1977) and Brown & Stewartson (1977) it is assumed that the fluid is contained by stress-free horizontal boundaries, a situation which is difficult to achieve experimentally. Stewartson & Weinstein (1979) have considered thermal convection in a large box, with rigid horizontal boundaries and rigid endwalls but only for the two-dimensional case with L >> 1.

In the present study the ideas of Segel (1969) and Daniels (1977) are applied to a three-dimensional long box with stress-free upper and lower surfaces, rigid sidewalls at $y=\pm a$ and rigid endwalls at $x=\pm L$. Here x and y are horizontal co-ordinates non-dimensionalised with respect to the height of the box. The assumption of an idealistic theoretical model with stress-free upper and lower surfaces enables a semi-analytical solution of the linearized perturbation equations to be constructed by separation of variables, thus avoiding the difficult numerical problems associated with the more realistic model, with rigid upper and lower surfaces.

In chapter 2, thermal convection in an infinite channel of rectangular cross-section with semi-aspect ratio a, no-slip sidewalls and stress-free upper and lower boundaries, in which an adverse temperature gradient is maintained by heating the underside is investigated using a linear stability analysis equivalent to that used by Davies-Jones (1970) (section 2.2). The results of Davies-Jones are reproduced and extended to include asymptotic forms for the critical Rayleigh number and the associated critical wavenumber for thin channels (semi-aspect ratio $a \to 0$) and wide channels ($a \to \infty$) for both perfectly conducting and perfectly insulating rigid sidewalls ($y=\pm a$). In addition to the rigid model considered in section 2.2, two simplified models are investigated. The stress-free model (section 2.5) assumes that the sidewalls are stress-free and perfect insulators, while the 'finite-roll' approximation (section 2.6) assumes the y-momentum equation is neglected and the horizontal velocity component in the y direction is zero. Both these simplified models enable fully analytical solutions to be constructed, in contrast to the rigid model which involves extensive numerical calculations.

In chapter 3, a nonlinear analysis involving the derivation of the amplitude equation using the method of multiple scales as outlined by Segel (1969) and Newell & Whitehead (1969), is presented for the simple models. In the case of the stress-free model the results for the limiting case in which the sidewalls tend to infinity ($a \to \infty$), are compared with the results of Newell & Whitehead (1969). In chapter 4, the ideas of chapter 3 are extended to the rigid model. The amplitude equation for nonlinear disturbances is derived and the variation of its coefficients with both aspect ratio and Prandtl number ascertained by extensive numerical computations. The results clearly show that for the aspect ratios considered the amplitude of the motion undergoes a supercritical bifurcation as the Rayleigh number passes through the critical value for instability predicted by linear theory.

In chapter 5, the effect of introducing distant endwalls (at $x=\pm L \gg 1$) is investigated for both the simple models and the rigid model using the techniques developed by Daniels (1977) and Stewartson & Weinstein (1979). The results obtained in chapters 3 and 4, in each case, form the basis of the solution in the 'core' region which must match with the solution in the neighbourhood of each endwall. In the end-regions the amplitude of the motion is small and the linearized form of the Boussinesq equations may be used. In the case of the simple models both linear and nonlinear approaches are considered. However, the simplicity of the linear approach is lost in the case of the rigid model where only a modified nonlinear approach is used. The results show that if the endwalls of the long box are rigid and the thermal conditions there are consistent with a basic state of no motion a supercritical bifurcation occurs at a new critical Rayleigh number. For the limiting case in which the sidewalls tend to infinity ($a \to \infty$), the new critical Rayleigh number at which bifurcation occurs, approaches the two-dimensional value obtained by Daniels (1977). Imperfect end conditions at $x=\pm L$ are also considered.

Chapters 6 and 7 are devoted to finding the higher order amplitude equation for the rigid model, thus enabling the effect of rigid endwalls with zero forcing ($\lambda = 0$) to be investigated in chapter 8.

Cross et al (1980,1983) and Daniels (1981,1984) have shown that for the related two-dimensional problem, in the absence of any forcing effects at the endwalls the wavelength of the roll pattern can be altered significantly from its critical value when the Rayleigh number exceeds its critical value by an amount order L^{-1} where 2L is the distance between the endwalls. Recent experimental work by Oertel (1980) using a laser-anemointerferometer (Oertel & Bühler (1978)) has shown that for a fluid of large Prandtl number P (for instance silicone oil P=1780) the number of convection rolls within a rectangular box of non-dimensional length ten (L=5) and width four (a=2) does not change as the Rayleigh number is increased above the critical value of 1708. However, at low Prandtl numbers (for instance nitrogen P=0.71) the number of convection rolls decreases as the Rayleigh number is increased above critical. In addition to his experimental work Oertel (1980) has investigated thermal convection in a three-dimensional box (of non-dimensional length ten and width four) with rigid perfectly conducting lateral walls using a Galerkin and an explicit finite-difference technique. His numerical and experimental results were in good agreement with regard to wavenumber selection.

In chapter 8, the question of wavenumber selection, as developed by Cross et al (1980) and Daniels (1981) for two-dimensional rolls, is extended to the three-dimensional case for the long box with a finite semi-aspect ratio a and length L >> 1, for Rayleigh numbers slightly above critical. The effect of varying the closeness of the sidewalls at $y=\pm a$ with either perfectly conducting or perfectly insulating rigid endwalls at $x=\pm L$ is investigated as the Prandtl number is varied from zero to infinity. The results indicate a physical behaviour similar to that observed by Oertel (1980). However, an explicit comparison of the results obtained with those of Oertel cannot be made since in the present study the upper and lower surfaces are assumed to be stress-free. It is expected that the boundary conditions at the upper and lower surfaces play a more important role in determining the roll pattern at large aspect ratios ($a \to \infty$) than at small aspect ratios ($a \to 0$), when the conditions at the sidewalls would be expected to dominate the form and behaviour of the roll pattern as the Rayleigh number is increased above critical. It should be noted that the work presented in chapter 8 has been done jointly with Professor P.G. Daniels.

All numerical calculations were performed on The City University, Honeywell 60/66 mainframe computer. All computer programs were constructed using the Fortran computer language. For the rigid model approximately twenty five minutes of processor time was required to determine the critical Rayleigh number and the amplitude coefficients of the higher order amplitude equation for a given aspect ratio. Due to the large amount of computer programming involved (approximately 9,000 Fortran statements) it has not been convenient to include listings of the programs in the thesis.

The three-dimensional model used in the present study takes us one step closer to the ultimate goal of studying the effect of lateral walls on the transition to finite amplitude Rayleigh-Bénard convection in a fully rigid three-dimensional box. From Davis (1967) it is expected that the techniques used in this thesis may be extended to the fully rigid three-dimensional long box, with a Galerkin method used to obtain the basic solution in the 'core' of the box perpendicular to the x direction. Multiple-scale techniques could still be used to model the behaviour of the solution in the x direction.

Chapter 2 The infinite channel: linear theory for rigid and simplified models

2.0 Introduction

Rayleigh (1916) showed that for an infinite layer of fluid with stress-free horizontal boundaries heated from below linear theory predicts that at the onset of convection the critical Rayleigh number is $27\pi^4/4$ and the critical wavenumber is $\pi/\sqrt{2}$. The effect of lateral walls was considered by Davis (1967) who investigated the linear stability of a rigid, perfectly conducting rectangular box of fluid heated from below using a Galerkin technique. Davies-Jones (1970) investigated the linear stability of an infinite channel of fluid with a rectangular cross-section of semi-aspect ratio a. Davies-Jones assumed the upper and lower boundaries to be stress-free and investigated the effect of introducing rigid sidewalls at varying aspect ratios using an eigenvalue approach involving extensive numerical calculations. In this chapter the results of Davies-Jones (1970) are reproduced and extended to include asymptotic solutions for both thin channels ($a \to 0$) and wide channels ($a \to \infty$). In addition, two simplified models for which there exists completely analytical solutions are investigated.

2.1 Formulation of the problem

The following notation is used: x^* and y^* are the horizontal co-ordinates along and across the channel and z^* is the vertical co-ordinate. The superscript $*$ refers to dimensional quantities. The channel is defined by the planes $z^* = 0, d^*$ and $y^* = \pm a^*$. The components of velocity in the x^*, y^* and z^* directions are u^*, v^* and w^*. The temperature, pressure, density and time are denoted by T^*, p^*, ρ^* and t^* respectively.

The upper and lower planes are held at the constant temperatures T_1^* and T_0^* with $T_1^* < T_0^*$. The basic hydrodynamical equations of a heat-conducting viscous fluid must be supplemented by an equation of state which is assumed to be

$$\rho^* = \rho_0^* (1 - \alpha^* (T^* - T_0^*)) \tag{2.1.1}$$

where α^* is the coefficient of volume expansion and ρ_0^* is the density of the fluid at the temperature T_0^* of the lower boundary. It is assumed that the coefficient of volume expansion is small, so that the Boussinesq approximation can be used ie. the change in the density due to variations in the temperature may be neglected, except where it is multiplied by the external force. Thus the equations which govern the motion of a Boussinesq fluid under the action of gravity are as follows. The equation of continuity is

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} + \frac{\partial w^*}{\partial z^*} = 0. \tag{2.1.2}$$

The momentum equations are

$$\frac{\partial u^*}{\partial t^*} + u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} + w^* \frac{\partial u^*}{\partial z^*} = -\frac{1}{\rho_0^*} \frac{\partial p^*}{\partial x^*} + \nu^* \nabla^{*2} u^* \tag{2.1.3}$$

$$\frac{\partial v^*}{\partial t^*} + u^* \frac{\partial v^*}{\partial x^*} + v^* \frac{\partial v^*}{\partial y^*} + w^* \frac{\partial v^*}{\partial z} = -\frac{1}{\rho_0^*} \frac{\partial p^*}{\partial y^*} + \nu^* \nabla^{*2} v^* \tag{2.1.4}$$

$$\frac{\partial w^*}{\partial t^*} + u^* \frac{\partial w^*}{\partial x^*} + v^* \frac{\partial w^*}{\partial y^*} + w^* \frac{\partial w^*}{\partial z^*} = -\frac{1}{\rho_0^*} \frac{\partial p^*}{\partial z^*} - \frac{\rho^*}{\rho_0^*} g^* + \nu^* \nabla^{*2} w^* \tag{2.1.5}$$

where the Laplacian operator $\nabla^{*2} = \frac{\partial^2}{\partial x^{*2}} + \frac{\partial^2}{\partial y^{*2}} + \frac{\partial^2}{\partial z^{*2}}$, ν^* is the kinematic viscosity and g^* is the accleration due to gravity. The equation of heat conduction is

$$\frac{\partial T^*}{\partial t^*} + u^* \frac{\partial T^*}{\partial x^*} + v^* \frac{\partial T^*}{\partial y^*} + w^* \frac{\partial T^*}{\partial z^*} = K^* \nabla^{*2} T^* \tag{2.1.6}$$

where K^* is the thermal diffusivity.

Assume that the boundary conditions are consistent with a basic state of no motion so that

$$u^* = v^* = w^* = 0 \quad \text{and} \quad T^* = T^*(z^*) . \tag{2.1.7}$$

When no motions are present the Boussinesq equations require that the pressure distribution is governed by the equations

$$\frac{\partial p^*}{\partial x^*} = 0 , \quad \frac{\partial p^*}{\partial y^*} = 0 , \quad \frac{\partial p^*}{\partial z^*} = -\rho^* g^* . \tag{2.1.8}$$

The temperature distribution is governed by

$$\nabla^{*2} T^* = 0 . \tag{2.1.9}$$

The solution of (2.1.9) subject to the boundary conditions

$$T^* = T_0^* \quad \text{at} \quad z^* = 0 \quad \text{and} \quad T^* = T_1^* \quad \text{at} \quad z^* = d^* \tag{2.1.10}$$

is

$$T^* = T_0^* + (T_1^* - T_0^*) z^*/d^* = T_B^* . \tag{2.1.11}$$

The corresponding density distribution is

$$\rho^* = \rho_0^* (1 - \alpha^* (T_1^* - T_0^*) z^*/ d^* = \rho_B^* . \tag{2.1.12}$$

With this expression for ρ^*, equation (2.1.8) can be integrated to give

$$p^* = p_0^* - g^* \rho_0^* z^* (1 - \alpha^* (T_1^* - T_0^*) z^*/2d^*) = p_B^* . \tag{2.1.13}$$

The non-dimensional variables u,v,w,θ and p are defined relative to this basic state of no motion by

$$[u^*, v^*, w^*] = \frac{K^*}{d^*} [u,v,w](x,y,z,t) , \qquad T^* = T_B^* + (T_1^* - T_0^*)\theta(x,y,z,t)$$

and

$$p^* = p_B^* + \frac{\rho_0^* K^{*2}}{d^{*2}} p(x,y,z,t) \tag{2.1.14}$$

where

$$[x^*, y^*, z^*] = d^* [x,y,z] \quad \text{and} \quad t^* = d^{*2} t / K^* . \tag{2.1.15}$$

The dimensionless form of the Boussinesq equations which govern the motion is then

$$\nabla \cdot \underline{u} = 0 \tag{2.1.16}$$

$$\frac{\partial u}{\partial t} + \underline{u} \cdot \nabla u = -\frac{\partial p}{\partial x} + P \nabla^2 u \tag{2.1.17}$$

$$\frac{\partial v}{\partial t} + \underline{u} \cdot \nabla v = -\frac{\partial p}{\partial y} + P \nabla^2 v \tag{2.1.18}$$

$$\frac{\partial w}{\partial t} + \underline{u} \cdot \nabla w = -\frac{\partial p}{\partial z} + P \nabla^2 w + P R \theta \tag{2.1.19}$$

and

$$\frac{\partial \theta}{\partial t} + \underline{u} \cdot \nabla \theta = w + \nabla^2 \theta \tag{2.1.20}$$

where

$\nabla = \frac{\partial}{\partial x} \underline{i} + \frac{\partial}{\partial y} \underline{j} + \frac{\partial}{\partial z} \underline{k}$ and $\underline{u} = u\underline{i} + v\underline{j} + w\underline{k}$.

The Rayleigh number R, is defined by

$$R = g^* \alpha^* (T_0^* - T_1^*) d^{*4} / \kappa^* \nu^* \tag{2.1.21}$$

and the Prandtl number is

$$P = \nu^* / \kappa^*. \tag{2.1.22}$$

It should be noted that the Prandtl number is an intrinsic property of the fluid, not of the flow; and that the Rayleigh number is a ratio of the buoyancy to the viscous forces. The third parameter governing the system is the semi-aspect ratio of the channel

$$a = a^* / d^*. \tag{2.1.23}$$

The stability of the basic state (2.1.7) can be examined by considering perturbations equivalent to non-zero values of u,v,w,θ and p. If these are small they will satisfy the linearized forms of the equations (2.1.16)-(2.1.20), which are

$$\nabla \cdot \underline{u} = 0 \tag{2.1.24}$$

$$L_1 u = -\frac{\partial p}{\partial x} \tag{2.1.25}$$

$$L_1 v = -\frac{\partial p}{\partial y} \tag{2.1.26}$$

$$L_1 w = -\frac{\partial p}{\partial z} + PR\theta \tag{2.1.27}$$

and

$$L_2 \theta = w \tag{2.1.28}$$

where L_1 and L_2 denote the differential operators $\left(\frac{\partial}{\partial t} - P\nabla^2\right)$ and $\left(\frac{\partial}{\partial t} - \nabla^2\right)$ respectively.

It will be assumed that the principle of exchange of stabilities holds ie. all non-decaying disturbances are non-oscillatory in time. Shermann & Ostrach (1966) have proved that this does hold for convection in a fully enclosed geometry and Davies-Jones (1970) has modified their proof to show that it also holds for periodic solutions in the infinite channel considered here.

2.2 The infinite channel with rigid sidewalls

In this section thermal convection in an infinite rectangular channel with no-slip sidewalls is considered. The upper and lower surfaces are assumed to be stress-free, allowing a solution to be constructed by separation of variables; the conditions on the top and bottom boundaries are therefore

$$\theta = w = \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0 \qquad z = 0, 1. \tag{2.2.1}$$

The sidewalls are assumed to be rigid and either perfect conductors or perfect insulators. In the conducting case the conditions on the sidewalls are

$$\theta = u = v = w = 0 \qquad y = \pm a \tag{2.2.2}$$

while in the insulating case the condition on the temperature is replaced by

$$\frac{\partial \theta}{\partial y} = 0 \qquad y = \pm a. \tag{2.2.3}$$

From equations (2.1.24) and (2.1.28) the conditions (2.2.2) are equivalent to

$$\theta = v = \frac{\partial v}{\partial y} = L_2\theta = 0 \qquad y = \pm a. \tag{2.2.4}$$

Unfortunately however, the conditions on the sidewalls cannot all be expressed in terms of a single variable.

By elimination of variables in equations (2.1.24)-(2.1.28) a single sixth order differential equation in θ is obtained:

$$[\nabla^2 L_1 L_2 - PR\,\nabla_H^2]\theta = 0, \tag{2.2.5}$$

where $\nabla_H^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ and ∇^2 is the Laplacian operator. Since not all the conditions on the sidewalls are in terms of θ, the following equation relating $\frac{\partial v}{\partial z}$ and θ is also required

$$\frac{\partial}{\partial z} L_1 v = (L_1 L_2 - PR)\frac{\partial \theta}{\partial y}. \tag{2.2.6}$$

The boundary conditions (2.2.1) allow normal mode solutions to be obtained by separation of variables as

$$\left.\begin{matrix} u(x,y,z,t) \\ v(x,y,z,t) \\ p(x,y,z,t) \end{matrix}\right\} = \begin{bmatrix} U(y) \\ V(y) \\ P(y) \end{bmatrix} e^{\sigma t} e^{ikx} \cos n\pi z$$

$$\left.\begin{matrix} w(x,y,z,t) \\ \theta(x,y,z,t) \end{matrix}\right\} = \begin{bmatrix} W(y) \\ \Theta(y) \end{bmatrix} e^{\sigma t} e^{ikx} \sin n\pi z \tag{2.2.7}$$

where σ is the growth rate of the disturbance with wavenumber k in the x direction and vertical wavenumber $n\pi$ (n=1,2...). The value of σ is set to zero in order to determine the neutral stability curve.

Substituting (2.2.7) into equations (2.2.5,6) gives

$$L_3\Theta = 0, \tag{2.2.8}$$

$$L_4 V = L_5 \Theta \tag{2.2.9}$$

and

$$\alpha^2 = k^2 + n^2\pi^2$$

where the differential operators are defined by

$$L_3 = \left[\frac{d^6}{dy^6} - 3\alpha^2\frac{d^4}{dy^4} + (3\alpha^4 - R)\frac{d^2}{dy^2} - (\alpha^6 - k^2R)\right],$$

$$L_4 = n\pi\left[\frac{d^2}{dy^2} - \alpha^2\right]$$

and

$$L_5 = \left[\frac{d^5}{dy^5} - 2\alpha^2\frac{d^3}{dy^3} + (\alpha^4 - R)\frac{d}{dy}\right]. \tag{2.2.10}$$

The homogenous differential equation (2.2.8) has the general solution

$$\Theta(y) = \sum_{j=1}^{6} c_j e^{r_j y} \tag{2.2.11}$$

where r_j are the roots of the characteristic equation

$$(r^2)^3 - 3\alpha^2(r^2)^2 + (3\alpha^4 - R)(r^2) - (\alpha^6 - k^2R) = 0 \tag{2.2.12}$$

and c_j are arbitrary constants. The roots r_j are assumed to be distinct (see below). As the equation (2.2.12) has real coefficients and is bi-cubic, the roots can easily be determined using Cardin's formula (see appendix A).

Substitution of the general solution (2.2.11) into equation (2.2.9) gives a second order inhomogenous equation for V which has a particular solution

$$V_2(y) = \sum_{j=1}^{6} c_j \beta_j e^{r_j y} \tag{2.2.14}$$

where

$$\beta_j = \left[r_j^5 - 2\alpha^2 r_j^3 + (\alpha^4 - R)r_j\right] / n\pi(r_j^2 - \alpha^2). \tag{2.2.15}$$

The complementary solution is

$$V_1(y) = \sum_{j=7}^{8} c_j e^{r_j y}, \qquad r_j = \pm\alpha, \qquad (j = 7, 8). \tag{2.2.16}$$

Hence the complete solution for V(y) is given by

$$V(y)=\sum_{j=1}^{6} c_j \beta_j e^{r_j y} + \sum_{j=7}^{8} c_j e^{r_j y} . \tag{2.2.17}$$

The solutions (2.2.11) and (2.2.17) have already been derived by Davies-Jones (1970). He then proceeded by applying the eight boundary conditions (2.2.4) to equations (2.2.11,17), giving eight homogenous equations in eight unknowns. Here, symmetry properties are used to simplify the calculations.

From appendix A, by letting $r^2=\eta+\alpha^2$ the characteristic equation (2.2.12) is reduced to the form

$$\eta^3+\gamma\eta+\beta=0 \qquad \text{where } \gamma=-R \text{ and } \beta=-n^2\pi^2 R . \tag{2.2.18}$$

The sign of the term $(\beta^2/4+\gamma^3/27)$ determines the format of the roots. Three possibilities exist. Firstly if $R=27(n\pi)^4/4$ then three real roots exist, at least two of which are equal. Secondly if $R<27(n\pi)^4/4$ then there exists one real and two complex conjugate roots, and thirdly if $R>27(n\pi)^4/4$ then three distinct real roots exist. For an infinite layer of fluid with stress-free boundaries linear theory predicts that the Rayleigh number at the onset of convection is $27\pi^4/4$ and corresponds to the vertical mode n=1. The presence of the rigid sidewalls delays the onset of instability (Davies-Jones (1970)), so that only the case when n=1 and R is greater than $27\pi^4/4$ need be considered.

Since the characteristic equation is bi-cubic the general solution of equation (2.2.8) can be written as

$$\Theta(y)=\sum_{j=1}^{3} c_j e^{r_j y}+\bar{c}_j e^{-r_j y} = \sum_{j=1}^{3} d_j \cosh r_j y + \bar{d}_j \sinh r_j y$$

$$= \Theta_E(y)+\Theta_O(y) , \tag{2.2.19}$$

and accordingly

$$V(y) = V_O(y) + V_E(y) , \tag{2.2.20}$$

where

$$V_c(y) = \sum_{j=1}^{3} d_j \beta_j \sinh r_j y + d_4 \sinh \alpha y$$

and

$$V_E(y) = \sum_{j=1}^{3} \bar{d}_j \beta_j \cosh r_j y + \bar{d}_4 \cosh \alpha y$$

From the evenness of the operators L_3 and L_4, and the form of the boundary conditions (2.2.4) which have to be satisfied at $y=\pm a$, it follows that the general solution of equation (2.2.8) falls into two non-combining groups of even and odd solutions. This enables the problem to be divided into 'even' and 'odd' cases. Applying the boundary conditions (2.2.4) to the equations (2.2.19,20) for the even case gives four equations in four unknowns which can be expressed in matrix form as

$$[A][D] = [0] \tag{2.2.21}$$

where

$$A = \begin{bmatrix} \cosh r_1 a, & \cosh r_2 a, & \cosh r_3 a, & 0 \\ (r_1^2-\alpha^2)\cosh r_1 a, & (r_2^2-\alpha^2)\cosh r_2 a, & (r_3^2-\alpha^2)\cosh r_3 a, & 0 \\ \beta_1 r_1 \cosh r_1 a, & \beta_2 r_2 \cosh r_2 a, & \beta_3 r_3 \cosh r_3 a, & \alpha \cosh \alpha a \\ \beta_1 \sinh r_1 a, & \beta_2 \sinh r_2 a, & \beta_3 \sinh r_3 a, & \sinh \alpha a \end{bmatrix} \tag{2.2.22}$$

and D is the column vector $\{d_i\}$ i=1..4. For the odd case the matrix A becomes

$$A = \begin{bmatrix} \sinh r_1 a, & \sinh r_2 a, & \sinh r_3 a, & 0 \\ (r_1^2-\alpha^2)\sinh r_1 a, & (r_2^2-\alpha^2)\sinh r_2 a, & (r_3^2-\alpha^2)\sinh r_3 a, & 0 \\ \beta_1 r_1 \sinh r_1 a, & \beta_2 r_2 \sinh r_2 a, & \beta_3 r_3 \sinh r_3 a, & \alpha \sinh \alpha a \\ \beta_1 \cosh r_1 a, & \beta_2 \cosh r_2 a, & \beta_3 \cosh r_3 a, & \beta_4 \cosh \alpha a \end{bmatrix} \tag{2.2.23}$$

and the column vector D becomes $\{\bar{d}_i\}$ i=1..4. Similar forms can be constructed for the case where the sidewalls are perfect insulators.

For a non-trivial solution of equation (2.2.21) the determinant of the matrix A must be zero. These zeroes were located numerically for various values of the aspect ratio a and wavenumber k, using the following technique. For given values of a and k, two values of the Rayleigh number are found which enclose a root of the equation

$$\det[A] = 0 . \qquad (2.2.24)$$

A simple interval bi-section routine is then used to accurately locate the value of k, that corresponds to the root. As only values of $R>27\pi^4/4$ are considered, the task is made easier by the fact that the determinant of the matrix A is either purely real or purely imaginary.

It should be noted that the equation (2.2.24) has an infinite set of solutions for a given aspect ratio a and wavenumber k. The lowest value of the Rayleigh number (R_1 say), is given by the first zero in this infinite set. For all Rayleigh numbers less than R_1, disturbances with the wavenumber k are stable; these disturbances become marginally stable when the Rayleigh number equals R_1 and unstable when the Rayleigh number exceeds R_1. Thus the neutral stability curve for a given aspect ratio a is determined by plotting R as a function of k.

Having computed the eigenvalues R, the constants d_1,d_2 and d_3 can be found in terms of d_4 by means of Cramer's rule. This gives the solutions for Θ and V . The solutions for the other variables are obtained from the relations

$$w = L_2 \theta , \qquad (2.2.25)$$

$$u = \frac{i}{k}\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) \qquad (2.2.26)$$

and

$$p = \frac{i}{k} L_1 u , \qquad (2.2.27)$$

which are obtained from the equations (2.1.24)-(2.1.28).

Figures 1 and 2 show the Rayleigh numbers at which the two lowest modes become unstable as a function of the wavenumber for different aspect ratios and for conducting and insulating sidewalls respectively. In each case the curve which has the lower minimum corresponds to the lowest mode of the even solution, while the other curve corresponds to the lowest mode of the odd solution. Except for $k \lesssim 1$ the even mode is the more unstable of the two.

The critical Rayleigh number at the onset of instability, which is associated with the even mode is determined by the additional condition

$$\frac{\partial R}{\partial k} = 0 \quad . \tag{2.2.28}$$

By using a central difference equation to approximate the derivative in equation (2.2.28) the critical Rayleigh number R_o and wavenumber k_o for different aspect ratios are obtained numerically, using a simple interval bi-section method.

Figures 3 and 4 show the critical Rayleigh number and the critical wavenumber plotted against the reciprocal of the aspect ratio, for both conducting and insulating sidewalls. The values are given in table 1. The higher critical Rayleigh number in the case of conducting sidewalls is a manifestation of greater heat loss through the sidewalls, which reduces the destabilising effect of buoyancy. It should be noted that figures 1-4 are in good agreement with the results obtained by Davies-Jones (1970).

It should be noted that the y-dependent function $\Theta(y)$ is normalised such that $\Theta = 1$ at $y = 0$.

	Conducting Sidewalls		Insulating Sidewalls	
a	k_o	R_o	k_o	R_o
0.25	3.6094	6734.0354	2.7687	2715.9884
0.50	2.7021	1654.7422	2.4962	1277.5669
0.75	2.3879	1023.0677	2.3629	921.1781
1.00	2.2315	827.5656	2.2680	799.4416
1.25	2.1455	744.7721	2.1894	737.5152
1.50	2.1044	704.8865	2.1333	703.3260
1.75	2.0929	684.4800	2.1056	684.2265
2.00	2.0977	673.5975	2.1013	673.5741
2.25	2.1093	667.5498	2.1093	667.5498
2.50	2.1224	664.0411	2.1214	664.0373
2.75	2.1349	661.9182	2.1337	661.9112
3.00	2.1460	660.5828	2.1449	660.5750
3.25	2.1555	659.7127	2.1546	659.7055
3.50	2.1635	659.1279	2.1628	659.1217
3.75	2.1703	658.7240	2.1697	658.7189
4.00	2.1760	658.4382	2.1756	658.4340
4.25	2.1810	658.2315	2.1806	658.2281
4.50	2.1851	658.0791	2.1849	658.0764
4.75	2.1887	657.9649	2.1885	657.9627
5.00	2.1918	657.8780	2.1917	657.8761

Table 1. The values of the critical wavenumber and critical Rayleigh number for different aspect ratios for both conducting and insulating sidewalls.

Figures 5 and 6 show plots of the y-dependent functions Θ,U,V and W at the onset of convection for conducting and insulating sidewalls respectively, for the aspect ratios a = $\frac{1}{4}$,$\frac{1}{2}$,1 and 2. Figures 7 and 8 show plan forms of the cells at the onset of convection for the case of conducting sidewalls, showing the isopleths of vertical velocity for the aspect ratios a = $\frac{1}{4}$,$\frac{1}{2}$ and a = 2 respectively. In each case the dashed vertical lines indicate the boundaries of the convection cells.

It should be noted that in figure 5 the vertical velocity component W(y) changes direction in the neighbourhood of the sidewalls for the aspect ratios a = 1 and a = 2. A similar effect is observed in the case of insulating sidewalls (see fig. 6) for the aspect ratio a = 2. The flow in this region of 'reverse' flow is extremely weak, and therefore does not appear in any detail on the horizontal plan form of the cells given by fig. 8 for the aspect ratio a = 2 in the case of conducting sidewalls. An analytical approach considered in section 2.7 (below) confirms that this region of reverse flow does exist in the case of both conducting and insulating sidewalls at large aspect ratios. A similar effect has been observed by Oertel (1980) who considered three-dimensional thermal convection in long rigid boxes.

A more detailed picture of the flow at the onset of convection can be obtained by tracing the path followed by a particle, which is given by the equations

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} = dt, \tag{2.2.29}$$

where u,v and w are the horizontal and vertical velocity components. Here, without loss of generality,

$$u = -\frac{U(y)}{k_o} \sin k_o x \cos \pi z ,$$

$$v = V(y) \cos k_o x \cos \pi z$$

and

$$w = W(y) \cos k_o x \sin \pi z , \tag{2.2.30}$$

where U,V and W are the y-dependent functions determined above and k_o is the critical wavenumber. Solving (2.2.29) using (2.2.30) gives

$$x = \frac{1}{k_o} \sin^{-1} [\sin (k_o x_o) \exp(-\int_{y_o}^{y} \frac{U}{V} dy)] \tag{2.2.31}$$

and

$$z = \frac{1}{\pi} \sin^{-1} [\sin(\pi z_o) \exp(\pi \int_{y_o}^{y} \frac{W}{V} dy)] \tag{2.2.32}$$

where x_o, y_o and z_o represent the starting point of the particle. Equations (2.2.31) and (2.2.32) enable the path followed by a particle starting at a given point to be traced. Figure 9 shows the projected paths traced by three particles A,B and C in all three co-ordinate planes, in each case starting at the point (x_o, y_o, z_o) given by

(0.3,0.4,0.3),(0.18,0.3,0.18) and (0.3,0.18,0.18) respectively, for the case of conducting sidewalls and the aspect ratio a = ½. As expected, all three particles trace closed paths roughly parallel to the xz plane. Figure 10 shows the projected paths traced by the particles A,B and C, in each case starting at the point (x_0,y_0,z_0) given by (0.3,1.9,0.3),(0.3,1.75,0.3) and (0.18,0.3,0.18), for conducting sidewalls and the aspect ratio a = 2. As expected, point C which has a starting point distant from the region of reverse flow, traces a closed path roughly parallel to the xz plane representing a cell with x-wavelength π/k_0. However, points A and B which have starting points in and near the region of reverse flow respectively, trace three-dimensional paths representing extremely weak 'corner' cells with x-wavelength $\pi/(2k_0)$.

2.3 Asymptotic solution for large aspect ratio a>>1

As $a\to\infty$, the critical Rayleigh number aprroaches the value $27\pi^4/4$ associated with an infinite horizontal layer. Numerical results from section 2.2 suggested that, locally the neutral stability curve is given by the asymptotic forms

$$R \sim \frac{27\pi^4}{4} + \frac{D^2}{a^4} \tag{2.3.1}$$

and

$$k^2 \sim \frac{\pi^2}{2} - \frac{C}{a^2} \tag{2.3.2}$$

where C and D are finite as $a\to\infty$.

From appendix A, by letting $r^2=\eta+\alpha^2$ and using the expressions (2.3.1) and (2.3.2) for the values of R and k respectively, the characteristic equation (2.2.12) is reduced to the form of equation (2.2.18) where

$$\gamma \sim -\frac{27\pi^4}{4} - \frac{D^2}{a^4} \quad \text{and} \quad \beta \sim -\frac{27\pi^6}{4} - \frac{\pi^2 D^2}{a^4} \ .$$

The roots of the reduced cubic are then given by

$$\eta_1 = E_+ + E_- \quad \text{and} \quad \eta_2, \eta_3 = -\tfrac{1}{2}(E_+ + E_-) \pm i\frac{\sqrt{3}}{2}(E_+ - E_-) \qquad (2.3.3)$$

where

$$E_\pm^3 \sim \left(\frac{27\pi^6}{8} \pm \frac{i\sqrt{27}\,\pi^4 D}{4a^2} + \frac{\pi^2 D^2}{2a^4} \right). \qquad (2.3.4)$$

By expressing $E_\pm^3$ in polar form and using De Movire's Theorm it is found that

$$E_\pm \sim \frac{3\pi^2}{2}\left(1 \pm \frac{2iD}{9\sqrt{3}\,\pi^2 a^2} + \frac{16D^2}{3^5\pi^4 a^4} \right).$$

Hence the roots of the characteristic equation (2.2.12) are found to have the forms

$$r_1 = \frac{3\pi}{\sqrt{2}}\left(1 - c/9\pi^2 a^2\right) + O(1/a^4),$$

$$r_2 = \frac{i(D+3C)^{1/2}}{\sqrt{3}\,a} + O(1/a^2),$$

$$r_3 = \frac{(D-3C)^{\frac{1}{2}}}{\sqrt{3}\,a} + O(1/a^2)$$

and

$$\alpha^2 = \frac{3\pi^2}{2} - \frac{C}{a^2} + O(1/a^4). \qquad (2.3.5)$$

Substitution into equation (2.2.15) then gives

$$\beta_1 = \frac{1}{\sqrt{2}}\left(\frac{9\pi^2}{4} + \frac{6C}{a^2}\right) + O\left(\frac{1}{a^4}\right), \qquad \beta_1 r_1 = \frac{3\pi}{2}\left(\frac{9\pi^4}{4} + \frac{6C}{a^2}\right) + O\left(\frac{1}{a^4}\right),$$

$$\beta_2 = \frac{i\sqrt{3}\,\pi}{a}(D+3C)^{\frac{1}{2}} + O\left(\frac{1}{a^2}\right), \qquad \beta_2 r_2 = -\frac{\pi}{a^2}(D+3C) + O\left(\frac{1}{a^4}\right)$$

and

$$\beta_3 = \frac{\sqrt{3}\,\pi}{a}(D-3C)^{\frac{1}{2}} + O\left(\frac{1}{a^2}\right), \qquad \beta_3 r_3 = \frac{\pi}{a^2}(D-3C) + O\left(\frac{1}{a^4}\right). \qquad (2.3.6)$$

The results (2.3.1)-(2.3.6) apply to both conducting and insulating sidewalls.

Since the numerical results show that the most unstable mode is the even mode (see figures 1 and 2), equations (2.3.5,6) are substituted into equation (2.2.21), in order to obtain the relationship between C and D that defines the neutral curve for large aspect ratios. For the case of conducting sidewalls this gives

$$[\bar{A}] \begin{bmatrix} d_1 \cosh \Gamma_1 a \\ d_2 \cosh \Gamma_2 a \\ d_3 \cosh \Gamma_3 a \\ d_4 \cosh \alpha a \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{2.3.7}$$

where the matrix $\bar{A}$ is given by

$$\bar{A} \sim \begin{bmatrix} 1, & 1, & 1, & 0 \\ 3\pi^2, & -\left[\frac{3\pi^2}{2} + \frac{D}{3a^2}\right], & -\left[\frac{3\pi^2}{2} - \frac{D}{3a^2}\right], & 0 \\ \frac{3\pi^2}{2}\left[\frac{9\pi^2}{4} + \frac{6C}{a^2}\right], & -\pi[D+3C]/a^2, & \pi[D-3C]/a^2, & \alpha \\ \frac{1}{\sqrt{2}}\left[\frac{9\pi^2}{4} + \frac{6C}{a^2}\right], & -\frac{\sqrt{3}}{a}(D+3C)^{\frac{1}{2}}\pi \tan\frac{(D+3C)^{\frac{1}{2}}}{\sqrt{3}}, & \frac{\sqrt{3}}{a}(D-3C)^{\frac{1}{2}}\pi \tanh\frac{(D-3C)^{\frac{1}{2}}}{\sqrt{3}}, & 1 \end{bmatrix}, \tag{2.3.8}$$

as $a \to \infty$.

For the case of insulating sidewalls the first row of the matrix $\bar{A}$ becomes

$$\left[3\pi/\sqrt{2}, \ \frac{-(D+3C)^{\frac{1}{2}}}{\sqrt{3}\,a} \tan\frac{(D+3C)^{\frac{1}{2}}}{\sqrt{3}}, \ \frac{(D-3C)^{\frac{1}{2}}}{\sqrt{3}\,a} \tanh\frac{(D-3C)^{\frac{1}{2}}}{\sqrt{3}}, \ 0 \right], \tag{2.3.9}$$

and all other terms are unchanged.

A non-trivial solution of equation (2.3.7) exists if the determinant of the matrix $\bar{A}$ is zero. On expanding the determinant of $\bar{A}$, in the case of conducting sidewalls the leading terms $O(a^{\circ})$, are found to be zero and the first non-zero terms occur at $O(\frac{1}{a})$. In the case of insulating sidewalls the leading order terms are of $O(\frac{1}{a})$ and non-zero. In both cases equating the determinant of $\bar{A}$ with zero at $O(\frac{1}{a})$, shows that C and D must satisfy

$$(D+3C)^{\frac{1}{2}} \tan\frac{(D+3C)^{\frac{1}{2}}}{\sqrt{3}} + (D-3C)^{\frac{1}{2}} \tanh\frac{(D-3C)^{\frac{1}{2}}}{\sqrt{3}} = 0. \tag{2.3.10}$$

The neutral stability curve in the neighbourhood of the critical Rayleigh number is defined by the lowest branch.

The critical Rayleigh number for the onset of instability is determined by the additional condition (2.2.28) which here is equivalent to

$$\frac{dD^2}{dC} = 0. \tag{2.3.11}$$

Applying this to equation (2.3.10) gives

$$(D+3C)^{-\frac{1}{2}}\left[\tan\frac{(D+3C)^{\frac{1}{2}}}{\sqrt{3}} + \frac{(D+3C)^{\frac{1}{2}}}{\sqrt{3}}\sec^2\frac{(D+3C)^{\frac{1}{2}}}{\sqrt{3}}\right]$$

$$- (D-3C)^{-\frac{1}{2}}\left[\tanh\frac{(D-3C)^{\frac{1}{2}}}{\sqrt{3}} + \frac{(D-3C)^{\frac{1}{2}}}{\sqrt{3}}\operatorname{sech}^2\frac{(D-3C)^{\frac{1}{2}}}{\sqrt{3}}\right] = 0. \tag{2.3.12}$$

For a given value of C the equation (2.3.10) has an infinite set of solutions $D=D_m$ ($m=1..\infty$). The minimum points (C_c,D_c), for the first two branches of this even mode were found numerically and checked by substitution into (2.3.12). They are given as follows.

m	C_c	D_c	D_c^2
1	3.1959	13.905	193.36
2	18.0152	58.363	3406.24

The results for m=1 show that the critical Rayleigh number and wavenumber, for both conducting and insulating sidewalls, are given by

$$R_0 \sim \frac{27\pi^4}{4} + \frac{193.36}{a^4} \tag{2.3.14}$$

and

$$k_c \sim \frac{\pi}{\sqrt{2}}(1 - 3.1959/\pi^2 a^2) \tag{2.3.15}$$

as $a \to \infty$.

In table 2 below a comparison of the numerical solutions with the asymptotic solutions for both conducting and insulating sidewalls is given for aspect ratios 4.25 to 5.0.

	Numerical Solution		Asymptotic
Aspect Ratio	Conducting Case	Insulating Case	Solution
a	R_o	R_o	R_o
4.25	658.231	658.228	658.104
4.50	658.079	658.076	657.983
4.75	657.965	657.963	657.891
5.00	657.878	657.876	657.821
a	k_o	k_o	k_o
4.25	2.18096	2.18061	2.18125
4.50	2.18514	2.18486	2.18563
4.75	2.18873	2.18851	2.18933
5.00	2.19184	2.19166	2.19248

Table 2. A comparison of the numerical results for R_o and k_o with the asymptotic results for large aspect ratios.

Equation (2.3.10) enables a non-trivial solution of the matrix system (2.3.7) to be determined. However, this solution is not valid in the neighbourhood of the sidewalls at $y=\pm a$ $(a \gg 1)$ where $y_1 = y \pm a \sim 1$, since equation (2.3.10) corresponds to assuming the unknown constants d_1 and d_4 in (2.3.7) are zero. Using the overall minimum values for C_c and D_c obtained at m=1, and solving the matrix system (2.3.7), the solution for the temperature θ in the case of both conducting and insulating sidewalls is given by

$$\theta = \frac{1}{\theta_0}\left[\frac{\left(\cosh (D_c - 3C_c)^{1/2}/\sqrt{3}\right)}{\cos (D_c + 3C_c)^{1/2}/\sqrt{3}} \cos \frac{(D_c+3C_c)^{1/2}y}{\sqrt{3}a} - \cosh \frac{(D_c-3C_c)^{\frac{1}{2}}y}{\sqrt{3}a}\right] e^{ik_0 x} \sin \pi z \tag{2.3.16}$$

where

$$\theta_0 = \left[\cosh (D_c - 3C_c)^{1/2}/\sqrt{3} - \cos(D_c+3C_c)^{\frac{1}{2}}/\sqrt{3}\right] \Big/ \cos(D_c+3C_c)^{\frac{1}{2}}/\sqrt{3}\,, \tag{2.3.17}$$

$a \gg 1$ and $y \sim a$. It is noted that the solution (2.3.16) has the property that it and its first derivative vanish at the sidewalls. For the aspect ratio a = 5, the analytical solution for the temperature (2.3.16) and the related numerical solutions for conducting and insulating sidewalls (determined as outlined in section 2.2) were in good agreement 'away' from the sidewalls. The order of magnitude of the velocity components is $w = O(1)$, $u = O(1)$ and $v = O(a^{-1})$, $a \gg 1$.

In order to determine the analytical form of the solution in the neighbourhood of the sidewalls using the asymptotic approach outlined above, the higher order terms that have been neglected in determining equation (2.3.10) would need to be taken into consideration, which would be very complicated. However, in section 2.7 (below) the solution in the neighbourhood of the sidewalls (at $y = \pm a$, $a \gg 1$) is determined using a matched asymptotic expansion approach, which requires the flow field to be divided into a core region $-1 < y/a < 1$, and end regions near the sidewalls at $y = \pm a$ ($a \gg 1$). In addition, equation (2.3.10) and the related equation for the odd mode of instability (not considered here) are obtained.

2.4 Asymptotic solution for small aspect ratio $a \ll 1$

Section 2.3 shows that when the aspect ratio is large ($a \to \infty$), the critical Rayleigh number is the same for both conducting and insulating sidewalls. As indicated by figure 3, this is no longer true when the aspect ratio is small ($a \to 0$), since now the conditions at the sidewalls have a strong influence on the flow. Two cases are considered.

2.4(1) Perfectly conducting sidewalls

Numerical results from section 2.2 and a 'finite-roll' approximation given in section 2.6 below, suggest that for small aspect ratios the neutral curve is given by

$$R \sim \frac{D_1^4}{a^4} + \frac{D_2}{a^3} \tag{2.4.1}$$

with

$$k^2 \sim \frac{C_1}{a} \tag{2.4.2}$$

where D_1, D_2 and C_1 are finite as $a \to 0$.

Substituting (2.4.1) and (2.4.2) into the characteristic equation (2.2.12) and using Cardin's formula from appendix A, the roots of the characteristic equation are found to have the forms

$$r_1 \sim \frac{D_1}{a}\left(1 + \frac{1}{2}\left(\frac{D_2}{2D_1^2} + C_1\right)\frac{a}{D_1^2}\right),$$

$$r_2 \sim \frac{iD_1}{a}\left(1 + \frac{1}{2}\left(\frac{D_2}{2D_1^2} - C_1\right)\frac{a}{D_1^2}\right),$$

$$r_3 \sim \sqrt{\frac{C_1}{a}}$$

and

$$\alpha \sim \sqrt{\frac{C_1}{a}}\,(1 + \pi^2 a/2C_1). \tag{2.4.3}$$

Substitution into equation (2.2.15) gives

$$\beta_1 \sim \frac{D_1\pi}{a}, \quad \beta_2 \sim \frac{iD_1\pi}{a}, \quad \beta_3 \sim \frac{D_1^4}{\pi^3 a^4}\sqrt{\frac{C_1}{a}},$$

$$\beta_1 r_1 \sim \frac{D_1^2\pi}{a^2}, \quad \beta_2 r_2 \sim -\frac{D_1^2\pi}{a^2}, \quad \beta_3 r_3 \sim \frac{D_1^4 C_1}{\pi^3 a^5}. \tag{2.4.4}$$

Figure 1 clearly shows that when the aspect ratio is small, the critical Rayleigh number is obtained from the even mode solution. It is convenient to re-write equation (2.2.21) in the form

$$[\bar{A}]\begin{bmatrix} d_1 \cosh r_1 a \\ d_2 \cosh r_2 a \\ d_3 \cosh r_3 a \\ d_4 \cosh \alpha a \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{2.4.5}$$

where

$$\bar{A} = \begin{bmatrix} 1, & 1, & 1, & 0 \\ (r_1^2 - \alpha^2), & (r_2^2 - \alpha^2), & (r_3^2 - \alpha^2), & 0 \\ \beta_1 r_1, & \beta_2 r_2, & \beta_3 r_3, & \alpha \\ \beta_1 \tanh r_1 a, & \beta_2 \tanh r_2 a, & \beta_3 \tanh r_3 a, & \tanh \alpha a \end{bmatrix}. \tag{2.4.6}$$

Expressing the elements to leading order gives

$$\bar{A} \sim \begin{bmatrix} 1, & 1, & 1, & 0 \\ D_1^2/a^2, & -D_1^2/a^2, & -\pi^2, & 0 \\ D_1^2\pi/a^2, & -D_1^2\pi/a^2, & C_1 D_1^4/\pi^2 a^5, & (C_1/a)^{\frac{1}{2}} \\ \frac{\pi D_1}{a}\tanh D_1, & -\frac{\pi D_1}{a}\tan D_1, & \frac{C_1^{1/2} D_1^4}{\pi^3 a^{9/2}}\tanh (C_1 a)^{1/2}, & \tanh (C_1 a)^{\frac{1}{2}} \end{bmatrix} \tag{2.4.7}$$

as $a \to 0$. This shows that the dominant terms are $\beta_3 r_3$ and $\beta_3 \tanh r_3 a$ in column 3. Hence expanding the determinant of $\bar{A}$ along the third column, from (2.4.6) and (2.4.7) gives

$$\det \bar{A} = (r_2^2 - r_1^2)\, D_T - \frac{i\pi\sqrt{C_1}\, D_1^3}{a^{7/2}} \tanh r_2 a + O(a^{-7/2}),$$

where

$$D_T = \beta_3 r_3 \tanh \alpha a - \beta_3 \alpha \tanh r_3 a . \tag{2.4.8}$$

Unfortunately D_T is zero at leading order. However, since D_T just depends on $\alpha \left(= (k^2+\pi^2)^{\frac{1}{2}}\right)$ and the root r_3, it is only necessary to evaluate these at higher orders. This can be done as follows. The expansions (2.4.1) and (2.4.2) are extended to the forms

$$R \sim \frac{D_1^4}{a^4} + \frac{D_2}{a^3} + \frac{D_3}{a^2} + \frac{D_4}{a} \quad \text{and} \quad k^2 \sim \frac{C_1}{a} + C_2 + C_3 a \tag{2.4.9}$$

respectively. By letting $r^2 = q + \alpha^2$, the characteristic equation is reduced to the form of equation (2.2.18) above, (see appendix A) where $\gamma = -R$ and $\beta = -\pi^2 R$. The leading order term in r_3 indicates that the corresponding root of the reduced cubic (2.2.18) has the form

$$q_3 \sim -\pi^2 + \delta_1 a + \delta_2 a^2 + \delta_3 a^3 + \delta_4 a^4 . \tag{2.4.10}$$

Substituting (2.4.9,10) into (2.2.18) and equating coefficients at ascending powers of a gives

$$\delta_1 = \delta_2 = \delta_3 = 0 \quad , \quad \delta_4 = \pi^6 / D_1^4 \tag{2.4.11}$$

which implies

$$r_3 = \sqrt{\frac{C_1}{a}} \left(1 + \frac{C_2 a}{2C_1} + \left(\frac{C_3}{2C_1} - \frac{C_2^2}{8C_1^2} \right) a^2 + O(a^3) \right). \tag{2.4.12}$$

Using (2.4.9) and (2.4.12) it can be shown that D_T is given by

$$D_T \sim 0 \times a^{-9/2} + 0 \times a^{-7/2} - \frac{1}{3\pi} D_1^4 C_1 \sqrt{C_1}\, a^{-5/2}, \tag{2.4.14}$$

which gives

$$\det \bar{A} = \frac{2 C_1 \sqrt{C_1}\, D_1^6}{3\pi a^{9/2}} - \frac{i\sqrt{C_1}\, D_1^3 \pi}{a^{7/2}} \tanh r_2 a + O(a^{-7/2}). \tag{2.4.15}$$

Thus the condition that the determinant of $\bar{A}$ is zero can only be satisfied at $O(a^{-9/2})$ if

$$\tanh r_2 a \sim i \tan D_1 = O(a^{-1}), \tag{2.4.16}$$

ie. the tangent is infinite. More specifically, from (2.4.3,15)

$$\frac{C_1}{a} = \frac{-3\pi^2}{2D_1^3} \tan\left(D_1 + (D_2 - 2D_1^2 C_1)\, a/4D_1^3\right) + O(1), \tag{2.4.17}$$

which implies that

$$D_1 = m\pi/2 \qquad m = 1, 3, 5 \ldots \tag{2.4.18}$$

and

$$D_2 = \frac{\pi^2 C_1}{2}\left[m + 12/C_1^2\right]. \tag{2.4.19}$$

From (2.4.18) the lowest mode of instability occurs when m=1 and $D_1 = \pi/2$. In order to obtain the critical Rayleigh number at the onset of instability the additional condition

$$\frac{dD_2}{dC_1} = 0, \tag{2.4.20}$$

which is equivalent to (2.2.28), must be satisfied. The critical values of D_2 and C_1 are therefore given by

$$C_{1c} = \sqrt{12} \quad \text{and} \quad D_{2c} = \pi^2\sqrt{12}. \tag{2.4.21}$$

Thus when the sidewalls are perfect conductors the critical Rayleigh number and wavenumber are given by

$$R_0 \sim \frac{\pi^4}{16a^4} + \frac{\pi^2\sqrt{12}}{a^3}, \quad k_0 \sim (12)^{1/4}/a^{\frac{1}{2}} \tag{2.4.22}$$

as $a \to 0$. The closeness of the sidewalls severly restricts the instability leading to the large value of R_0. Although the wavelength of the instability is small compared to the height of the channel, it is large compared to the width.

By assuming that the unknown constant d_2 in equation (2.4.5) is order one, it can easily be shown that the temperature profile is given by

$$\Theta(y) \sim d_2 \cos \pi y/2a \tag{2.4.23}$$

and substituting this into equation (2.1.28) gives to leading order

$$W(y) = O(a^{-2}).$$

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$$V(y) = O(a^{-1}) \text{ and } U(y) = O(a^{-3/2}). \tag{2.4.24}$$

2.4(2) Perfectly insulating sidewalls

Numerical results from section 2.2 suggest that when the sidewalls are perfect insulators, and the aspect ratio is small the neutral stability curve is given by

$$R \sim \frac{\bar{D}_1^2}{a^2} + \frac{\bar{D}_2}{a}, \qquad k^2 \sim \bar{C}_1 \tag{2.4.25}$$

where $\bar{D}_1, \bar{D}_2$ and $\bar{C}_1$ are finite as $a \to 0$. Substituting these forms for R and k into the characteristic equation (2.2.12) and using Cardin's formula (see appendix A) gives

$$r_1 \sim \sqrt{\frac{\bar{D}_1}{a}}\left(1 + \frac{1}{2}\left(\frac{\bar{D}_2}{2\bar{D}_1} + \bar{C}_1 + \frac{3\pi^2}{2}\right)\frac{a}{\bar{D}_1}\right),$$

$$r_2 \sim i\sqrt{\frac{\bar{D}_1}{a}}\left(1 + \frac{1}{2}\left(\frac{\bar{D}_2}{2\bar{D}_1} - \bar{C}_1 - \frac{3\pi^2}{2}\right)\frac{a}{\bar{D}_1}\right),$$

$$r_3 \sim \sqrt{\bar{C}_1}$$

and

$$\alpha^2 \sim \bar{C}_1 + \pi^2. \tag{2.4.26}$$

Substituting these into equation (2.2.15) gives

$$\beta_1 \sim \pi\sqrt{\frac{\bar{D}_1}{a}}\left(1 + \frac{1}{2}\left(\frac{\bar{D}_2}{2\bar{D}_1} + \bar{C}_1 - \frac{\pi^2}{2}\right)\frac{a}{\bar{D}_1}\right), \quad \beta_1 r_1 \sim \frac{\pi \bar{D}_1}{a}\left(1 + \left(\frac{\bar{D}_2}{2\bar{D}_1} + \bar{C}_1 + \frac{\pi^2}{2}\right)\frac{a}{\bar{D}_1}\right),$$

$$\beta_2 \sim i\pi\sqrt{\frac{\bar{D}_1}{a}}\left(1 + \frac{1}{2}\left(\frac{\bar{D}_2}{2\bar{D}_1} - \bar{C}_1 + \frac{\pi^2}{2}\right)\frac{a}{\bar{D}_1}\right), \quad \beta_2 r_2 \sim -\frac{\pi \bar{D}_1}{a}\left(1 + \left(\frac{\bar{D}_2}{2\bar{D}_1} - \bar{C}_1 - \frac{\pi^2}{2}\right)\frac{a}{\bar{D}_1}\right),$$

and

$$\beta_3 \sim \frac{\sqrt{\bar{C}_1}\,\bar{D}_1^2}{\pi^3 a^2}, \qquad \beta_3 r_3 \sim \frac{\bar{C}_1 \bar{D}_1^2}{\pi^3 a^2}. \tag{2.4.27}$$

Figure 2 clearly shows that for small aspect ratios and insulating sidewalls, the critical Rayleigh number is obtained from the even mode solution. Applying the boundary conditions (2.2.3) to equations (2.2.19,20) and expressing in matrix form gives

$$[\bar{A}] \begin{bmatrix} \bar{d}_1 \cosh r_1 a \\ \bar{d}_2 \cosh r_2 a \\ \bar{d}_3 \cosh r_3 a \\ \bar{d}_4 \cosh \alpha a \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{2.4.28}$$

where the matrix $\bar{A}$ is given by (2.4.6), except that the first row now becomes

$$[\, r_1 \tanh r_1 a,\ r_2 \tanh r_2 a,\ r_3 \tanh r_3 a,\ 0\,]. \tag{2.4.29}$$

Expanding the determinant of the matrix $\bar{A}$ along the third column and using (2.4.25)-(2.4.27) gives

$$\det \bar{A} \sim A_1 + \tfrac{2}{3}\pi\,(\bar{C}_1+\pi^2)^{3/2}\,\bar{D}_1^3\, a, \tag{2.4.30}$$

where

$$A_1 = [\, r_1(r_2^2-\alpha^2)\tanh r_1 a - r_2(r_1^2-\alpha^2)\tanh r_2 a\,]\, D_T$$

and D_T is given by (2.4.8) above. As in the conducting case D_T is found to be zero at leading order. By extending the expansion (2.4.25) to the form

$$k^2 \sim \bar{C}_1 + \bar{C}_2 a + \bar{C}_3 a^2 \tag{2.4.31}$$

and using a similar approach as in the conducting case (2.4(1)), it can be shown

$$r_3 \sim \sqrt{\bar{C}_1}\left(1 + \frac{\bar{C}_2 a}{2\bar{C}_1} + \left(\frac{\bar{C}_3 + \pi^6/\bar{D}_1^2}{2\bar{C}_1} - \frac{\bar{C}_2^2}{8\bar{C}_1^2}\right)a^2\right), \tag{2.4.32}$$

which implies

$$A_1 \sim -\frac{2\bar{C}_1\bar{D}_1^3}{9\pi}(\bar{C}_1+\pi^2)^{\frac{1}{2}}\left(\bar{D}_1^2 - 3(\bar{C}_1+\pi^2)\right)a. \tag{2.4.33}$$

Substituting (2.4.33) into (2.4.30) and equating the determinant of the matrix $\bar{A}$ to zero at $O(a)$ gives

$$\bar{D}_1^2 - \frac{3}{\bar{C}_1}(\bar{C}_1+\pi^2)^2 = 0. \tag{2.4.34}$$

The critical Rayleigh number at the onset of instability is determined by the additional condition (2.2.28) which here is equivalent to

$$\frac{d\bar{D}_1^2}{d\bar{C}_1} = 0. \tag{2.4.35}$$

Applying this to equation (2.4.34) gives the critical values of $\bar{D}_1$ and $\bar{C}_1$ as

$$\bar{D}_{1c} = \sqrt{12}\,\pi \quad \text{and} \quad \bar{C}_{1c} = \pi^2. \tag{2.4.36}$$

It should be noted that equation (2.4.34) defines only one mode of instability since for a given value of k there exists only one value of R satisfying (2.4.34). In order to show that this is the lowest mode of instability it is necessary to consider R and k to have the same scaling as in the case of the conducting sidewalls ie. R and k are defined by (2.4.1) and (2.4.2) respectively. Assuming this to be the case, the results (2.4.1-4) will now also apply to insulating sidewalls. Equation (2.4.7), which expresses the elements of $\bar{A}$ to leading order, will also apply to the insulating case except that the first row now becomes

$$\left[\frac{D_1}{a}\tanh D_1, \; -\frac{D_1}{a}\tan D_1, \; \sqrt{\frac{C_1}{a}}\tanh\sqrt{C_1 a}, \; 0 \right]. \tag{2.4.37}$$

Expanding the determinant of the matrix $\bar{A}$ now gives

$$\det \bar{A} \sim \frac{D_1^3}{a^3}\left(-\tanh D_1 + \tan D_1\right) D_T + O(a^{-7/2}), \tag{2.4.38}$$

where D_T is defined by (2.4.8) above. Using the result (2.4.14) and equating the determinant of $\bar{A}$ to be zero at $O(a^{-11/2})$ gives

$$\tanh D_1 - \tan D_1 = 0, \tag{2.4.39}$$

which has solutions $D_1 = D^{(m)}$ (m=1,2..). The lowest mode of instability (m=1) is given by $D^{(1)}=0$, which implies that for this mode the scaling of R given by (2.4.1) is inappropriate. The scaling (2.4.1) is the correct scaling for determining all the higher modes of instability ($m \geq 2$) but, it is necessary to use the scaling of R as defined by (2.4.25) in order to obtain the neutral stability curve given by (2.4.34).

Thus when the sidewalls are perfect insulators and the aspect ratio is small, the critical Rayleigh number and wavenumber are given by

$$R_0 \sim \frac{12\pi^2}{a^2} \quad \text{and} \quad k_0 \sim \pi \quad \text{as} \quad a \to 0. \tag{2.4.40}$$

The wavelength of the instability, which is independent of the aspect ratio when the sidewalls are insulators, is larger than in the conducting case. Since the wavenumber k_0 in the x-direction is the same as the wavenumber in the z-direction, the resulting rolls have a 'square' cross-section in the xz plane.

In table 3 a comparison of the numerical results with the asymptotic results for the critical Rayleigh number and wavenumber are given for various small aspect ratios for both conducting and insulating sidewalls.

	Conducting Sidewalls		Insulating Sidewalls	
Aspect Ratio	Numerical	Asymptotic	Numerical	Asymptotic
a	R_0	R_0	R_0	R_0
0.2	0.12251E5	0.80787E4	0.37994E4	0.29609E4
0.1	0.10843E6	0.95070E5	0.12718E5	0.11844E5
0.05	0.12955E7	0.12476E7	0.48260E5	0.47374E5
a	k_0	k_0	k_0	k_0
0.1	4.0223	4.1618	2.8542	3.1416
0.2	5.7281	5.8857	3.0407	3.1416
0.05	8.1904	8.3236	3.1133	3.1416

Table 3. Comparison of the critical Rayleigh number and wavenumber from the numerical and asymptotic results for various small aspect ratios for conducting and insulating sidewalls.

2.5 The infinite channel with stress-free sidewalls

In order to avoid the large amount of numerical work associated with the rigid channel of section 2.2 a simpler model may be considered. By assuming that the sidewalls are stress-free and perfect insulators a fully analytical solution for Θ can be obtained. As in the case of the rigid channel the upper and lower surfaces are assumed to be stress-free, allowing a solution to be constructed by separation of variables; the boundary conditions on the top and bottom surfaces are then given by (2.2.1), which allows the normal mode solutions given by (2.2.7) to be obtained. The conditions on the perfectly insulating stress-free sidewalls are

$$\frac{\partial \theta}{\partial y} = \frac{\partial w}{\partial y} = v = \frac{\partial u}{\partial y} = 0 \qquad y = \pm a. \tag{2.5.1}$$

From equations (2.1.24)-(2.1.28) these are equivalent to

$$\frac{\partial \theta}{\partial y} = \frac{\partial}{\partial y} L_2 \theta = v = \frac{\partial^2 v}{\partial y^2} = 0 \qquad y = \pm a, \tag{2.5.2}$$

where the operator L_2 is defined in section 2.1 above.

Clearly, the solutions satisfying the equations (2.2.8) and (2.2.9) and the boundary conditions (2.5.2) are

$$\Theta(y) = \begin{cases} \cos \dfrac{q\pi y}{2a} & q = 0, 2, 4 \ldots \quad (\text{Even Modes}) \\ \sin \dfrac{q\pi y}{2a} & q = 1, 3, 5 \ldots \quad (\text{Odd Modes}) \end{cases} \tag{2.5.3}$$

By defining

$$Y = y + a \tag{2.5.4}$$

the solutions (2.5.3) can be re-written in a more convenient form as

$$\Theta(y) = \cos \frac{m\pi Y}{\bar{a}} \qquad (0 \leq Y \leq \bar{a}), \quad m = 0, 1, 2 \ldots \tag{2.5.5}$$

where

$$\bar{a} = 2a. \tag{2.5.6}$$

Substituting the solution (2.5.5) into equation (2.2.8) leads to the characteristic equation

$$R_{nm} = (k_{nm}^2 + N^2\pi^2)^3 / (k_{nm}^2 + m^2\pi^2/\bar{a}^2), \quad (n = 1, 2, 3 \ldots, \; m = 0, 1, 2 \ldots) \tag{2.5.7}$$

where

$$N^2 = n^2 + m^2/\bar{a}^2, \tag{2.5.8}$$

$m\pi/\bar{a}$ and $n\pi$ are the horizontal and vertical wavenumbers in the y and z directions respectively, k_{nm} is the wavenumber in the x direction and R_{nm} is the Rayleigh number for the mode of instability (n,m).

The critical Rayleigh number and wavenumber for the mode (n,m) are

$$R_{nmc} = \frac{27n^4\pi^4}{4} \quad \text{and} \quad k_{nmc} = \frac{\pi}{\sqrt{2}}\left(n^2 - 2m^2/\bar{a}^2\right)^{1/2}, \tag{2.5.9}$$

which are obtained by applying the condition (2.2.28) to equation (2.5.7). The critical Rayleigh number R_{nmc}, is independent of m and the lowest mode of instability occurs when n=1. This implies that the overall critical Rayleigh number and wavenumber for a stress-free channel are

$$R_c = \frac{27\pi^4}{4} \quad \text{and} \quad k_c = \frac{\pi}{\sqrt{2}}\left(1 - m^2/2a^2\right)^{\frac{1}{2}}, \tag{2.5.10}$$

so that all y-modes such that

$$m \leqslant a\sqrt{2} \tag{2.5.11}$$

appear simultaneously at this point. This contrasts with the rigid channel where for a given aspect ratio, the higher y-modes of instability occur at increasing Rayleigh numbers. In the stress-free channel the mode m=0 is present for any a and corresponds to a two-dimensional motion parallel to the sidewalls of the channel. The higher y-dependent modes (m=1,2,..) are only present if the aspect ratio is greater than $m/\sqrt{2}$. When the aspect ratio, a, equals $m/\sqrt{2}$, the mth y-mode corresponds to a set of m rolls with axes parallel to the channel walls.

At the onset of thermal instability the solutions for the other variables obtained from (2.1.24)-(2.1.28) are

$$w = \frac{3\pi^2}{2} e^{ik_0x} \sin\pi z \cos\frac{m\pi y}{2a},$$

$$u = \frac{3\pi^2 i}{\sqrt{2}} (1-m^2/2a^2)^{1/2} e^{ik_0x} \cos\pi z \cos\frac{m\pi y}{2a},$$

$$v = -\frac{3\pi^2}{2a} m e^{ik_0x} \cos\pi z \sin\frac{m\pi y}{2a}$$

and

$$p = \frac{-9P\pi^3}{2} e^{ik_0x} \cos\pi z \cos\frac{m\pi y}{2a}. \qquad (2.5.12)$$

2.6 The 'finite-roll' approximation

As in the rigid channel of section 2.2 thermal convection in an infinite channel with stress-free upper and lower surfaces and no-slip sidewalls is considered. Davis (1967) defined 'finite-rolls' as cells with two non-zero velocity components dependent on all three spatial variables. Davies-Jones (1970) showed that 'finite-rolls' are never exact solutions of the linearized equations (2.1.24)-(2.1.28), except when the rigid sidewalls are relaxed to infinity. However, the simplified structure allows fully analytical solutions to be obtained as in section 2.5. Thus in this section it is assumed that the velocity component in the y direction is zero ie. v=0. In addition the momentum equation in the y direction (2.1.26) is neglected. In order to obtain a completely analytical solution, only the case where the sidewalls are perfect conductors is considered.

The assumption that v=0 reduces the boundary conditions (2.2.4) to the form

$$\theta = L_2\theta = 0 \qquad y = \pm a \qquad (2.6.1)$$

and by eliminating the variables in equations (2.1.24,25,27,28) a single partial differential equation in θ is obtained:

$$[L_1L_2\nabla_1^2 - PR\frac{\partial^2}{\partial x^2}]\theta = 0 \qquad (2.6.2)$$

where $\nabla_1^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}$, and the operators L_1 and L_2 are defined in section 2.2 above. This equation is sixth order in x and z and fourth order in y.

By using the same method of solution as for the rigid channel and defining Y = y + a, it can be shown that solutions of (2.6.2) satisfying the boundary conditions (2.6.1) are

$$\theta = e^{ikx} \sin n\pi z \sin \frac{m\pi Y}{\bar{a}}, \ (0 \leq Y \leq \bar{a}), \quad n=1,2,3\ldots, \ m=1,2,3\ldots \tag{2.6.3}$$

where $\bar{a}$ = 2a. Substituting the solution (2.6.3) into (2.6.2) leads to the characteristic equation

$$R_{nm} = (k_{nm}^2+n^2\pi^2)(k_{nm}^2+n^2\pi^2+m^2\pi^2/\bar{a}^2)^2/k_{nm}^2, \tag{2.6.4}$$

where $n\pi$ and $m\pi/\bar{a}$ are the vertical and horizontal wavenumbers in the z and y directions respectively and $k=k_{nm}$ is the wavenumber in the x direction.

The critical Rayleigh number and wavenumber at the onset of instability for the mode (n,m) are determined by the condition (2.2.28) as

$$R_{nmc} = \frac{1}{16}(\beta_{nm}+3n^2\pi^2)(\beta_{nm}+3n^2\pi^2+4m^2\pi^2/\bar{a}^2)^2/(\beta_{nm}-n^2\pi^2) \tag{2.6.6}$$

and

$$k_{nmc} = \frac{1}{2}(\beta_{nm}-n^2\pi^2)^{\frac{1}{2}} \tag{2.6.7}$$

where

$$\beta_{nm} = (9n^4\pi^4+8m^2n^2\pi^4/\bar{a}^2)^{\frac{1}{2}}. \tag{2.6.8}$$

This implies that the lowest mode of instability occurs when n=m=1 and that the overall critical Rayleigh number and wavenumber for the 'finite-roll' approximation are given by

$$R_o = \frac{1}{16}(\beta_{11}+3\pi^2)(\beta_{11}+3\pi^2+\pi^2/a^2)^2/(\beta_{11}-\pi^2), \quad k_o = \frac{1}{2}(\beta_{11}-\pi^2)^{\frac{1}{2}} \tag{2.6.9}$$

where

$$\beta_{11} = (9\pi^4+2\pi^4/a^2)^{1/2}.$$

The results (2.6.9) have already been derived by Davies-Jones (1970).

At the onset of thermal instability the solutions for the other variables are

$$w = \sigma_1 e^{ik_0 x} \sin\pi z \sin\frac{\pi y}{2a},$$

$$u = \frac{i\pi\sigma_1}{k_0} e^{ik_0 x} \cos\pi z \sin\frac{\pi y}{2a}$$

and

$$p = \frac{-\pi\sigma_1^2 P}{k_0^2} e^{ik_0 x} \cos\pi z \sin\frac{\pi y}{2a} \qquad (2.6.10)$$

where

$$\sigma_1 = \frac{1}{4}[\beta_{11} + 3\pi^2 + \pi^2/a^2]. \qquad (2.6.11)$$

Figures 3 and 4 indicate that the 'finite-roll' solution is an extremely good approximation to the rigid solution when the aspect ratio is small. Indeed, the leading order terms in the expansions for R_o and k_o as $a \to 0$ from (2.6.9) are

$$R_o \sim \frac{\pi^4}{16a^4} \quad \text{and} \quad k_o \sim \frac{\pi}{2^{3/4} a^{1/2}} \simeq \frac{1.87}{a^{1/2}}. \qquad (2.6.12)$$

The result for R_o is identical with the leading term in the asymptotic solution for the rigid channel with conducting sidewalls given in section 2.4(1). The formula for the wavelength is not identical but the coefficient of a agrees to within 0.007 with the asymptotic result (2.4.22).

Figure 3 indicates that the 'finite-roll' solution for the critical Rayleigh number is a good approximation to the rigid channel over the whole range of the aspect ratio a. Figure 4 indicates that as the sidewalls are moved in from infinity, in the rigid channel, the cells widen at first in the x direction ($a \gtrsim 2$), before narrowing ($0 \lesssim a \lesssim 2$). This widening is not a feature of the 'finite-roll' calculations, so it must in some way be related to the advection across the channel.

2.7 The 'edge-region' for rigid sidewalls

From the numerical results in section 2.2 (see figures 5 and 6) it is known that the profile of the vertical velocity component W(y) changes direction near the sidewalls (ie. a region of reverse flow exists) for sufficiently large aspect ratios. In the case of conducting sidewalls this occurs for aspect ratios a greater than approximately one, and in the case of insulating sidewalls for aspect ratios greater than approximately two. This change in direction near the sidewalls at large aspect ratios for both conducting and insulating sidewalls was not predicted by the asymptotic analysis considered in section 2.3, the reason being that the higher order terms that were neglected in that expansion (and are not needed in determining the critical Rayleigh number, the critical wavenumber and the solutions of the dependent variables θ ,u,v,w and p away from the sidewalls) are important in obtaining the solutions of the dependent variables near the sidewalls.

In this section it is assumed that the aspect ratio a >> 1 and the results of Segel (1969), Newell & Whitehead (1969) and Brown & Stewartson (1977) are used to re-derive the results of section 2.3 and determine the solutions of the dependent variables in the neighbourhood of the sidewalls.

Assuming the aspect ratio a is large compared with one, the solution for the temperature θ , except in the neighbourhood of the sidewalls may be expanded in the form (see Brown & Stewartson (1977))

$$\theta = e^{\frac{i\pi}{\sqrt{2}}x} A(X,Y)\sin\pi z + \varepsilon^{\frac{1}{2}}\theta_2 + \varepsilon\theta_3 + \cdots \qquad (2.7.1)$$

where

$$\varepsilon = R - \frac{27\pi^4}{4} = \frac{D^2}{a^4} \quad , \qquad (2.7.2)$$

A is a slowly varying function of the co-ordinates X and Y given by

$$X = x/a^2 \qquad \text{and} \qquad Y = y/a \quad (a \gg 1). \tag{2.7.3}$$

It should be noted that D is the constant defined in (2.3.1). Similar expressions for the other dependent variables may be obtained from the linearized Boussinesq equations (2.1.24)-(2.1.28). The equation for the function A(X,Y) is determined by consideration of the second and third terms in the interior expansion for the temperature given by (2.7.1) and the other dependent variables. Details of the calculation are given by Segel (1969), Newell & Whitehead (1969) and Brown & Stewartson (1977) and need not be repeated here. An inconsistency in the expansion at $O(\varepsilon A)$ can only be avoided if A satisfies the equation

$$-\left(2\frac{\partial}{\partial X} - \frac{i\sqrt{2}}{\pi}\frac{\partial^2}{\partial Y^2}\right)^2 A = \frac{2D^2}{4\pi^2}A \tag{2.7.4}$$

where, in the context of the present linear theory, it is assumed that $A \ll \varepsilon$ so that the nonlinear term $A|A|^2$ can be neglected. This must be solved subject to the boundary conditions derived by Brown & Stewartson (1977),

$$A = \frac{\partial A}{\partial Y} = 0 \qquad Y = \pm 1. \tag{2.7.5}$$

Solutions are expressed in the form

$$A(X,Y) = e^{-\frac{iC}{\sqrt{2}\pi}X}\,\tilde{A}(Y), \tag{2.7.6}$$

where C is equivalent to the constant defined in (2.3.2). Then

$$\tilde{A} = A^{O} \qquad \text{or} \qquad \tilde{A} = A^{E} \tag{2.7.7}$$

where

$$A^{O} = \alpha_1 \sin\frac{(D+3C)^{1/2}Y}{\sqrt{3}} + \alpha_2 \sinh\frac{(D-3C)^{1/2}Y}{\sqrt{3}}$$

and

$$A^{E} = \alpha_3 \cos\frac{(D+3C)^{1/2}Y}{\sqrt{3}} + \alpha_4 \cosh\frac{(D-3C)^{1/2}Y}{\sqrt{3}}, \tag{2.7.8}$$

and A^O and A^E denote 'odd' and 'even' solutions and α_i i=1..4 are the unknowns constants fixed by the boundary conditions (2.7.5). Applying the boundary conditions (2.7.5) to the 'odd' solution for A^O shows that a non-trivial solution for A exists only if the equation

$$-(D+3C)^{1/2}\cot\frac{(D+3C)^{1/2}}{\sqrt{3}} + (D-3C)^{1/2}\coth\frac{(D-3C)^{1/2}}{\sqrt{3}} = 0 \qquad (2.7.9)$$

is satisfied. Similarly, a non-trivial solution for A^E only exists if the equation

$$(D+3C)^{1/2}\tan\frac{(D+3C)^{1/2}}{\sqrt{3}} + (D-3C)^{1/2}\tanh\frac{(D-3C)^{1/2}}{\sqrt{3}} = 0 \qquad (2.7.10)$$

is satisfied. It is noted that the condition (2.7.10) has been previously obtained in section 2.3. As the expansion for θ given by (2.7.1) is only valid in the core region (ie. away from the sidewalls) where $Y\sim 1$, the solutions for A^O and A^E will only be valid in the core, which in turn implies that the basic eigensolution for the temperature at large aspect ratios given by (2.3.16) is only valid in that region.

The solutions for the dependent variables in the core must match as $Y\to\pm 1$ with the solutions for the dependent variables in the neighbourhood of each sidewall where $y \pm a \sim 1$. In these edge-regions the linearized form of the Boussinesq equations can be considered and corrections to the critical Rayleigh number and wavenumber

$$R = \frac{27\pi^4}{4} \qquad \text{and} \qquad k = \pi/\sqrt{2} \qquad (n=1) \qquad (2.7.11)$$

can be ignored to leading order. The dependent variables take the forms

$$\begin{bmatrix}\theta\\ w\end{bmatrix} = \begin{bmatrix}\hat{\Theta}\\ \hat{W}\end{bmatrix}(y_1)\, e^{i\left(\frac{\pi}{\sqrt{2}}x - \frac{CX}{\sqrt{2}\pi}\right)}\sin\pi z, \qquad \begin{bmatrix}u\\ v\\ p\end{bmatrix} = \begin{bmatrix}\hat{U}\\ \hat{V}\\ \hat{P}\end{bmatrix}(y_1)\, e^{i\left(\frac{\pi}{\sqrt{2}}x - \frac{CX}{\sqrt{2}\pi}\right)}\cos\pi z$$

(2.7.12)

in the edge-region where $y_1 = y \pm a$. Substituting (2.7.11) and (2.7.12) into equations (2.2.8) and (2.2.9) gives

$$\frac{d^4}{dy_1^4}\left(\frac{d^2}{dy_1^2} - 3\hat{\alpha}^2\right)\hat{\Theta} = 0$$

and

$$\pi\left(\frac{d^2}{dy_1^2} - \hat{\alpha}^2\right)\hat{V} = \left(\frac{d^5}{dy_1^5} - 2\hat{\alpha}^2\frac{d^3}{dy_1^3} + (\hat{\alpha}^4 - R)\frac{d}{dy_1}\right)\hat{\Theta} \tag{2.7.14}$$

where $\hat{\alpha}^2 = 3\pi^2/2$. Hence the solutions for $\hat{\Theta}$ and $\hat{V}$ near the sidewall $y=-a$ are

$$\hat{\Theta}(y_1) = c_0 + c_1 y_1 + c_2 y_1^2 + c_3 y_1^3 + c_4 e^{-3\pi y_1/\sqrt{2}} \tag{2.7.15}$$

and

$$\hat{V}(y_1) = 3\pi c_1 + 6\pi c_2 y_1 + \frac{6c_3}{\pi}\left(\frac{3\pi^2}{2}y_1^2 + 4\right) - \frac{9\pi^2}{4\sqrt{2}}c_4 e^{-3\pi y_1/\sqrt{2}} + c_5 e^{-\pi y_1\sqrt{3/2}} \tag{2.7.16}$$

where $y_1 = y + a$. The unknown constants c_i $i=0..5$ are independent of y_1 and two solutions which grow exponentially with y_1 have been omitted. A similar set of solutions exists near the sidewall $y=+a$. At the boundary $y_1 = 0$ the four boundary conditions in the case of conducting sidewalls are

$$\hat{\Theta} = \hat{V} = \frac{d\hat{V}}{dy_1} = \hat{W}\left[= -\left(\frac{d^2}{dy^2} - \hat{\alpha}^2\right)\hat{\Theta}\right] = 0 \quad y = \pm a. \tag{2.7.17}$$

In the case of insulating sidewalls the condition on the temperature $\hat{\Theta}$ becomes $\frac{d}{dy_1}\hat{\Theta}$ at $y_1 = 0$. Applying the conditions (2.7.17) to the solutions (2.7.15,16) gives

$$c_0 = \frac{4}{9\pi^2}c_2, \quad c_5 = \frac{9}{\sqrt{6}}c_2, \quad c_1 = -\frac{1}{\pi\sqrt{2}}\left[\frac{1}{3} + \sqrt{3}\right]c_2 - \frac{8}{\pi^2}c_3, \quad c_4 = \frac{-4}{9\pi^2}c_2. \tag{2.7.18}$$

(A similar set of four equations may be derived for the case of insulating sidewalls). Thus, there are four equations and six unknowns c_i $i=0..5$. The remaining two conditions are obtained by matching the solution for the temperature in the edge-region (2.7.15) as $y_1 \to \infty$ with the solution for the temperature in the core (2.7.1) as $Y \to -1$. This gives

$$c_3 = 0 \qquad \text{and} \qquad c_2 = \frac{1}{2a^2}\frac{\partial^2}{\partial y^2}A^E(-1). \tag{2.7.19}$$

The even mode for the solution in the core is chosen since the numerical results in section 2.2 show that the most unstable mode is the even mode (see figures 1 and 2). Hence, the solution for the vertical velocity component w in the edge-region near y=-a is given by

$$w = \hat{W}(y_1)\, e^{i\frac{\pi}{\sqrt{2}}x}\, e^{-\frac{icX}{\sqrt{2}\pi}} \sin\pi z$$

where

$$\hat{W}(y_1) = \frac{3\pi^2}{2}(c_0 + c_1 y_1) + \left(\frac{3\pi^2}{2}y_1^2 - 2\right)c_2 + \left(\frac{3\pi^2}{2}y_1^2 - 6\right)y_1 c_3 - 3\pi^2 c_4 e^{-3\pi y_1/\sqrt{2}} \tag{2.7.20}$$

and the constants c_i i=0..4 can be determined from equations (2.7.18) and (2.7.19) in the case of conducting sidewalls. The horizontal velocity component $\hat{U}(y_1)$ can now easily be obtained from the continuity equation (2.1.24).

In the core region the vertical velocity component w is given by

$$w \sim \frac{3\pi^2}{2}\left(A(X,Y)e^{i\frac{\pi}{\sqrt{2}}x} + A^*(X,Y)e^{-i\frac{\pi}{\sqrt{2}}x}\right)\sin\pi z \tag{2.7.21}$$

where A is given by (2.7.6).

The solutions (2.7.15) and (2.7.20) for the temperature and vertical velocity component w in the edge-region near y=-a, show that for large aspect ratios and for both conducting and insulating sidewalls, not only does the profile of $\hat{W}(y)$ change direction near the sidewall, but the profile of the temperature $\hat{\Theta}(y)$ also changes direction. This effect does not appear in the horizontal velocity component profiles. Numerical results obtained for the aspect ratio a = 5 (as outlined in section 2.2) verify this behaviour. Table 4 shows a comparison between the temperature profile given by $\Theta(y)$ and $\hat{\Theta}(y_1)$ for the aspect ratio a = 5 near the sidewall at y=-a for both conducting and insulating sidewalls. Since $\Theta(y)$ is normalised to be one at y = 0, α_3 and α_4 in (2.7.8) are chosen so that $A^E(0) = 1$. The corresponding comparison between the numerical and asymptotic results for the vertical velocity component is given in table 5. It can be seen that all four sets of results indicate a good agreement and the asymptotic theory provides an excellent prediction of the region of vertical flow reversal.

Aspect Ratio a = 5

	Conducting Sidewalls		Insulating Sidewalls	
y	Θ	$\hat{\Theta}$	Θ	$\hat{\Theta}$
-5.000	0.	0.	-0.537E-2	-0.720E-2
-4.875	-0.130E-2	-0.196E-2	-0.540E-2	-0.743E-2
-4.750	-0.482E-3	-0.198E-2	-0.388E-2	-0.650E-2
-4.625	0.341E-2	0.876E-3	0.454E-3	-0.302E-2
-4.500	0.170E-1	0.700E-2	0.808E-2	0.358E-2
-4.375	0.215E-1	0.166E-1	0.191E-1	0.136E-1

Table 4. A comparison of the numerical and asymptotic results for the y-dependent temperature profile in the edge-region. (E n denotes 10^n)

Aspect Ratio a = 5

	Conducting Sidewalls		Insulating sidewalls	
y	W	$\hat{W}$	W	$\hat{W}$
-5.000	0.	0.	0.	0.
-4.875	-0.161E 0	-0.159E 0	-0.187E 0	-0.194E 0
-4.750	-0.207E 0	-0.216E 0	-0.241E 0	-0.262E 0
-4.625	-0.170E 0	-0.199E 0	-0.206E 0	-0.247E 0
-4.500	-0.662E-1	-0.118E 0	-0.101E 0	-0.165E 0
-4.375	0.968E-1	0.194E-1	0.638E-1	-0.235E-1

Table 5. A comparison of the numerical and asymptotic results for the y-dependent vertical velocity profile in the edge-region.

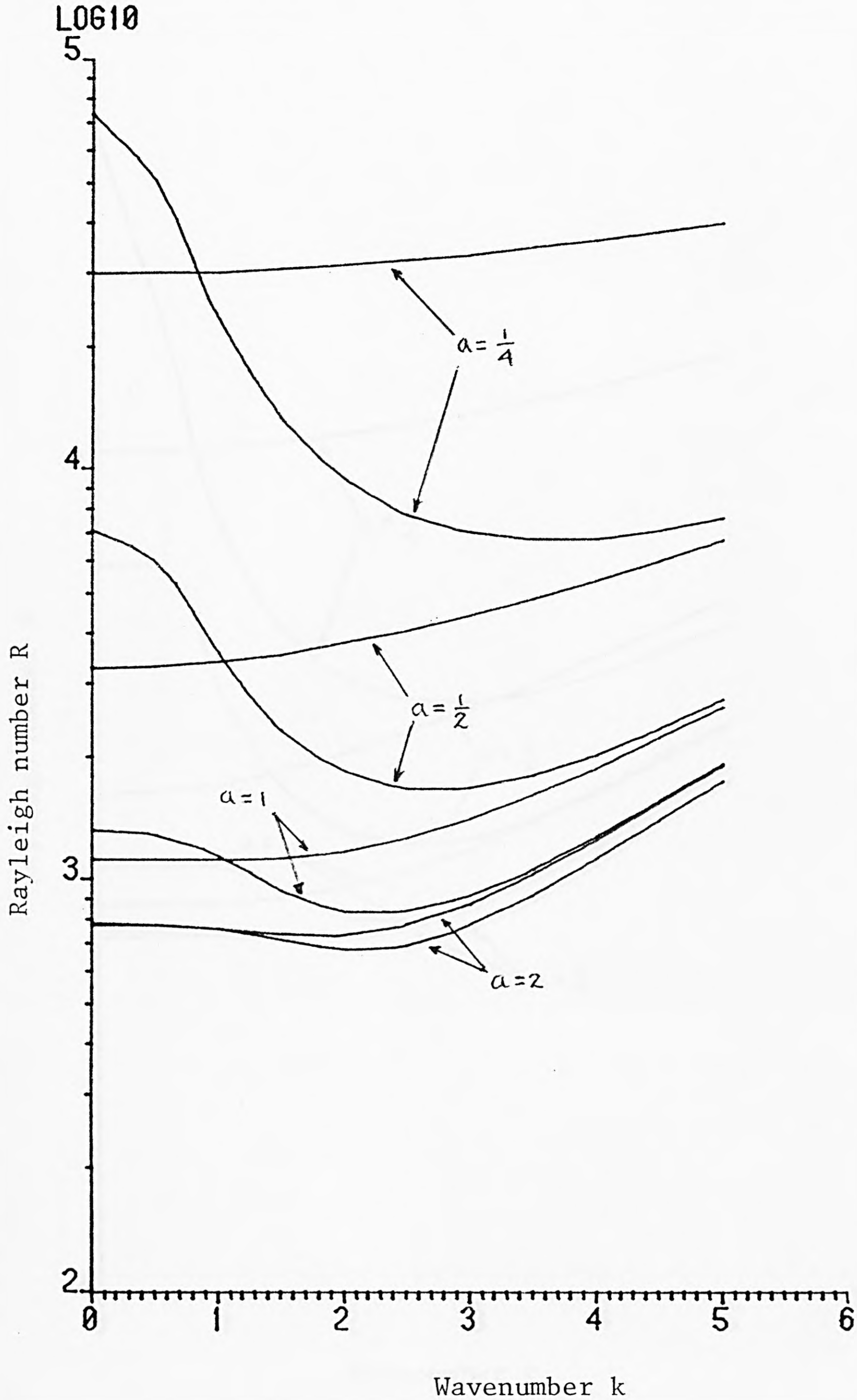


Figure 1. Rayleigh numbers at which the two lowest modes are marginally stable as functions of wavenumber at different aspect ratios for perfectly conducting sidewalls.

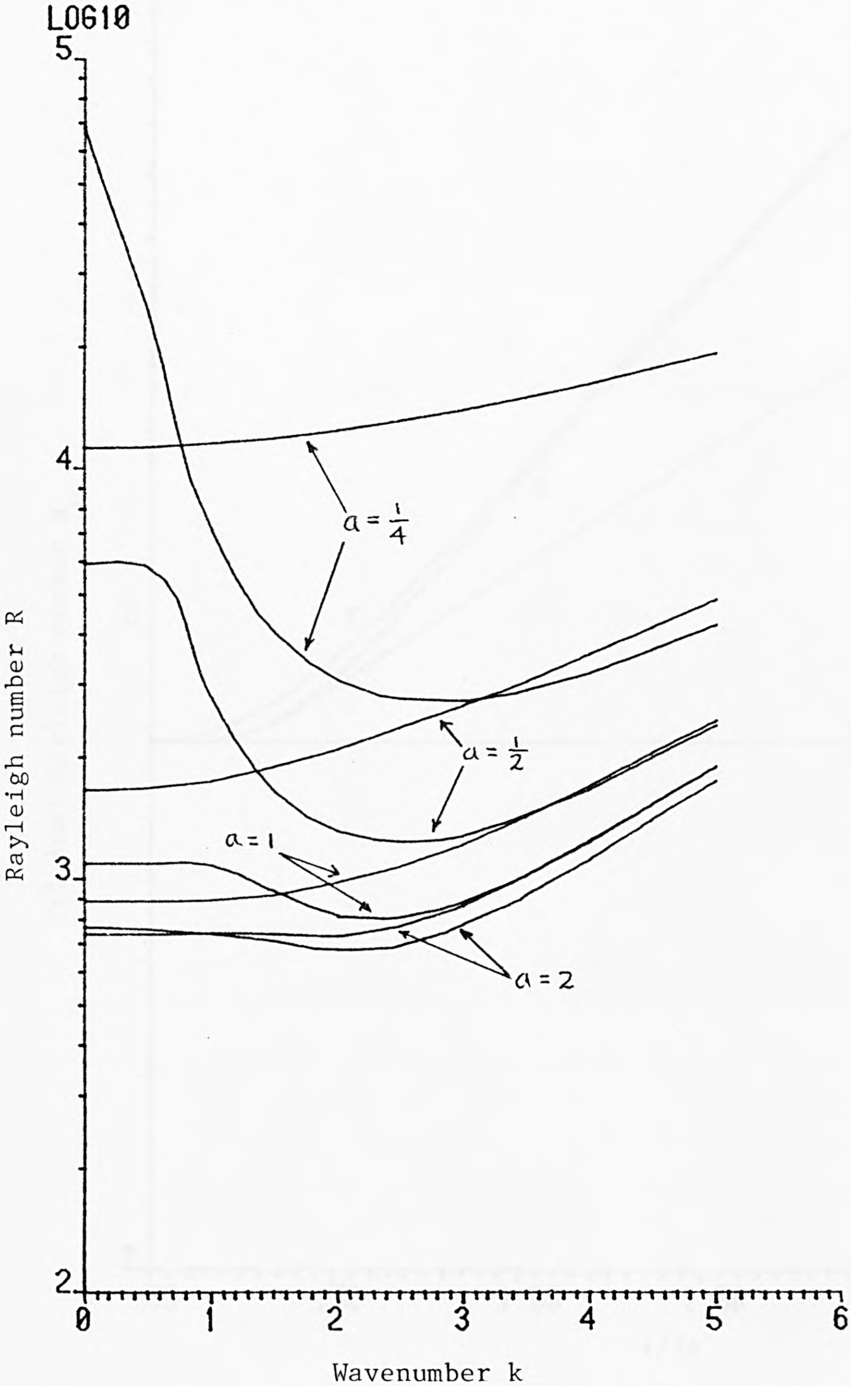


Figure 2. Rayleigh numbers at which the two lowest modes are marginally stable as functions of wavenumber at different aspect ratios for perfectly insulating sidewalls.

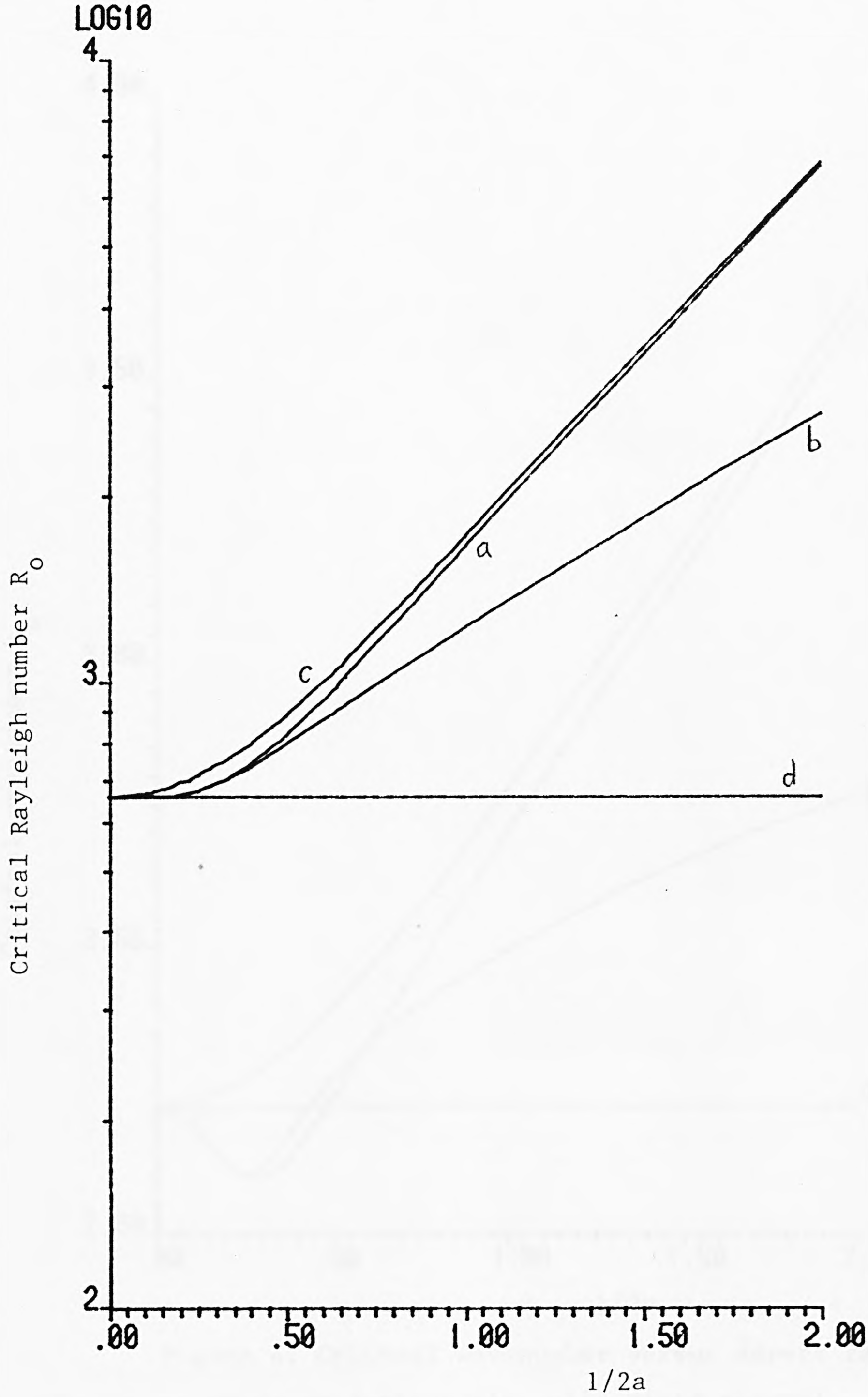


Figure 3. Critical Rayleigh number versus aspect ratio a.
curve a: conducting sidewalls rigid channel
curve b: insulating sidewalls rigid channel
curve c: 'finite-roll' approximation
curve d: stress-free channel

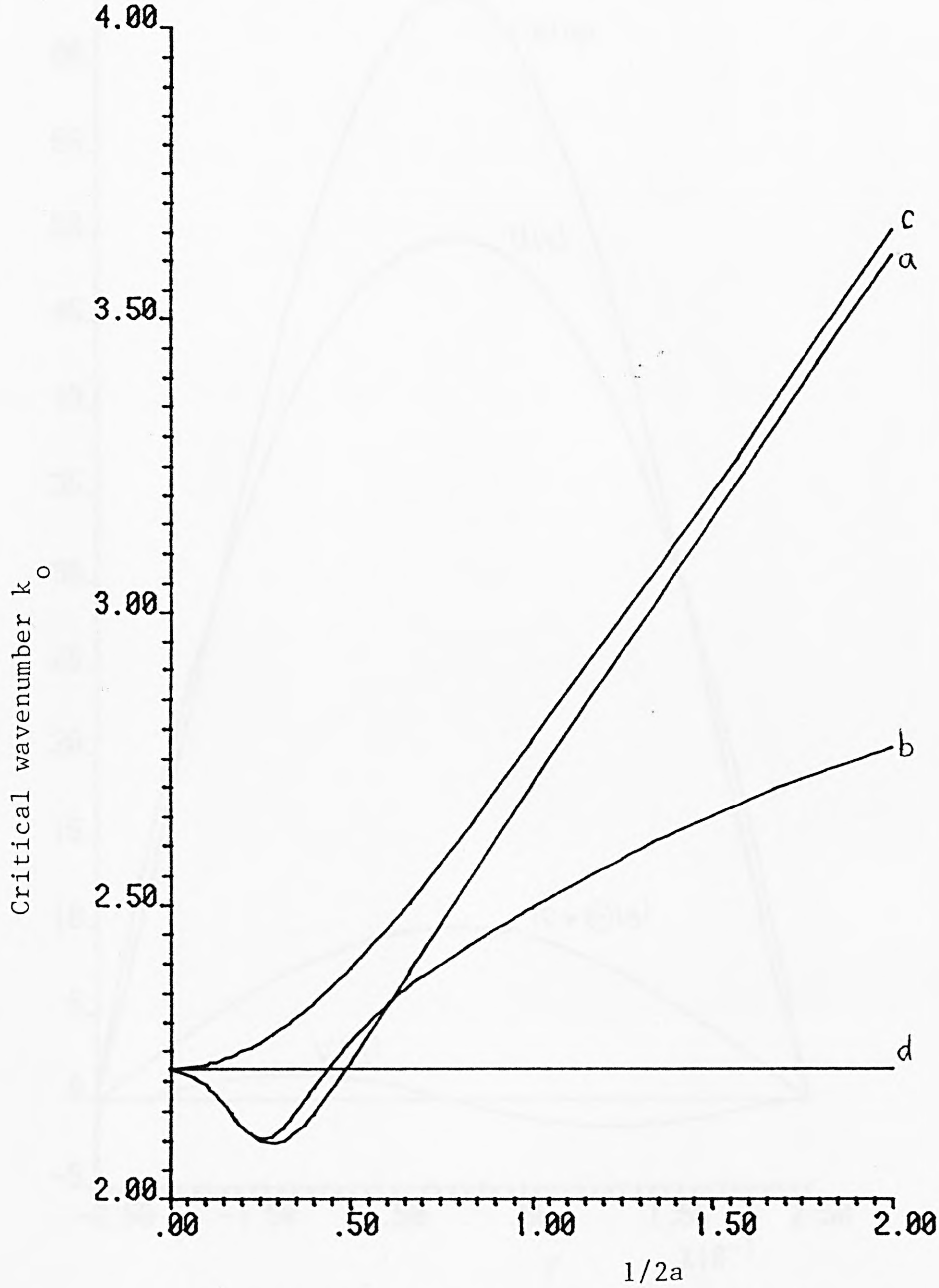


Figure 4. Critical wavenumber versus aspect ratio a.

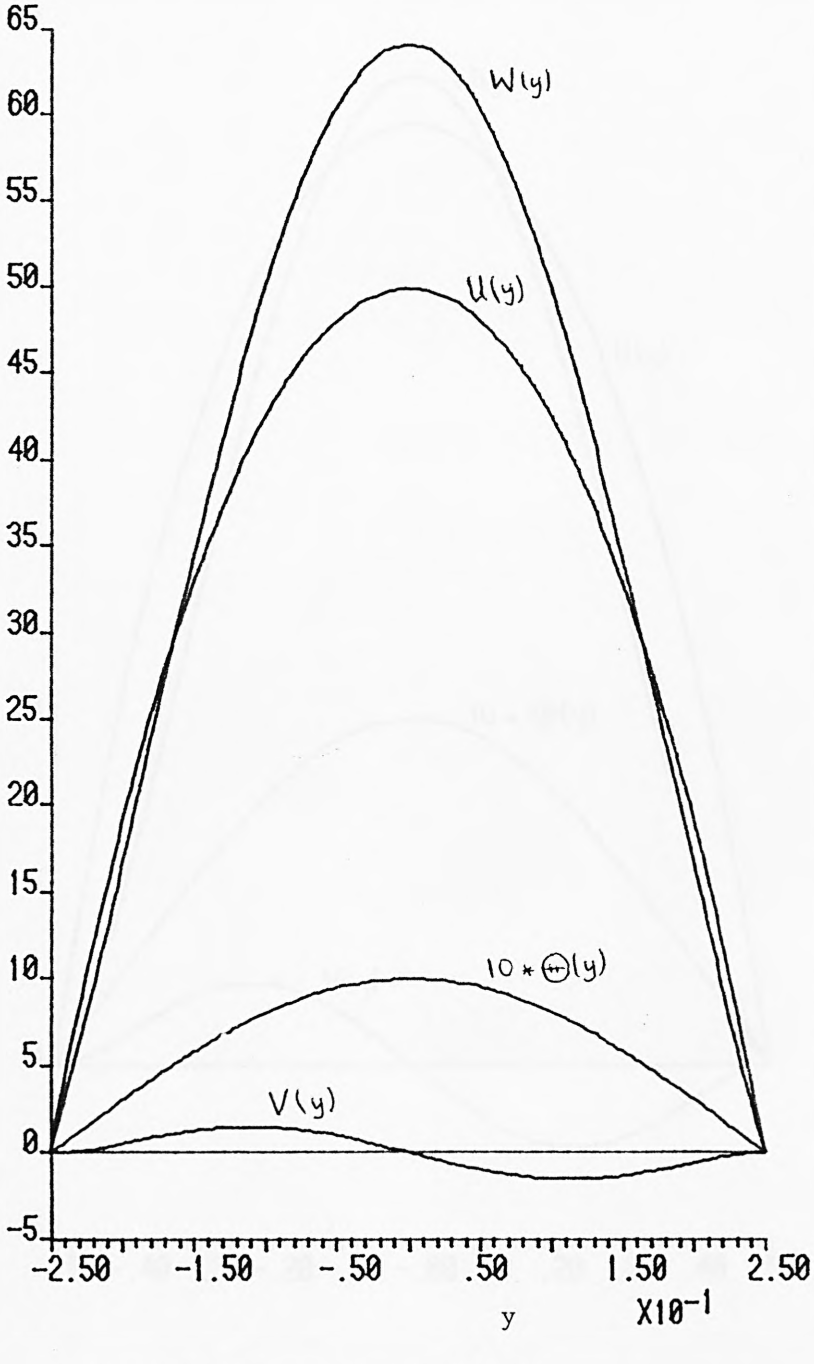
curve a: conducting sidewalls rigid channel

curve b: insulating sidewalls rigid channel

curve c: 'finite-roll' approximation

curve d: indicates the value $\pi/\sqrt{2}$

Aspect ratio a = 0.25



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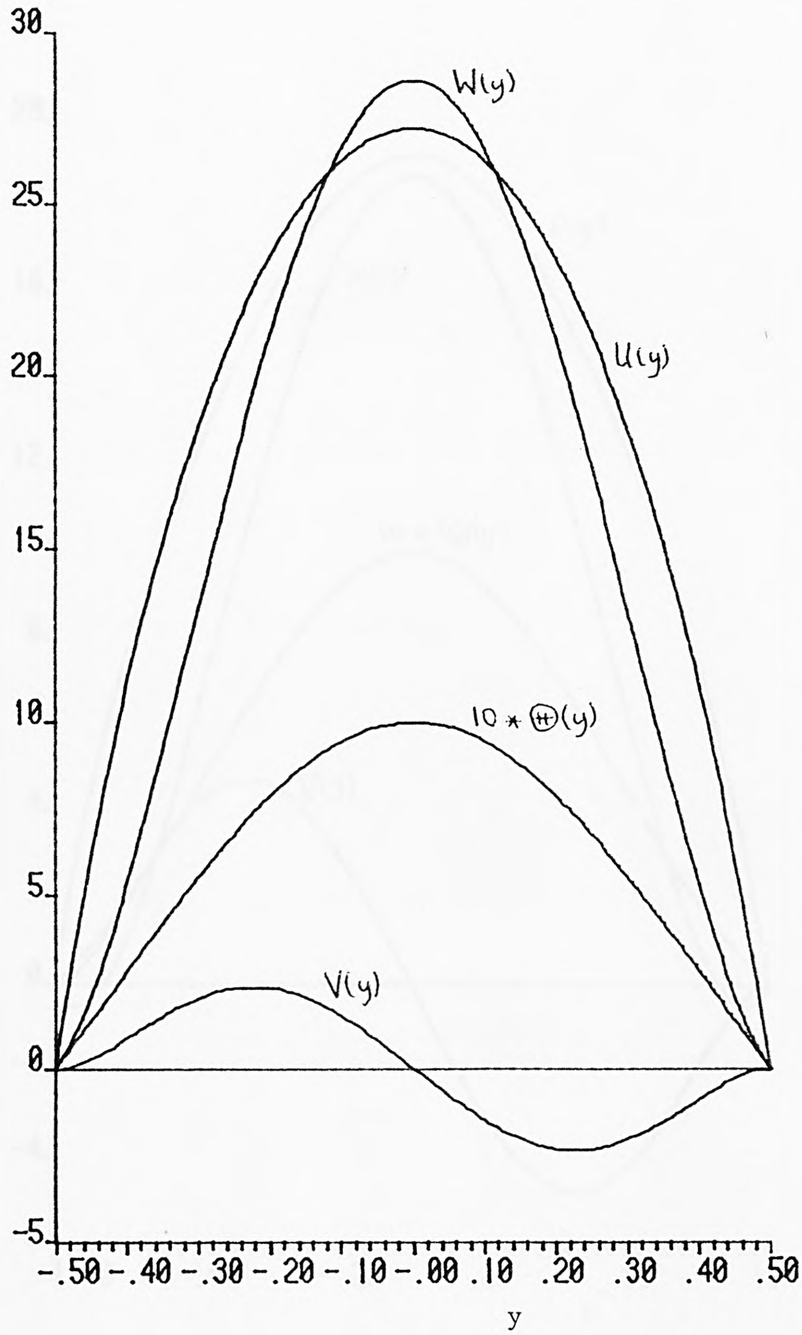
Aspect ratio a = 0.5



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Aspect ratio a = 1.0

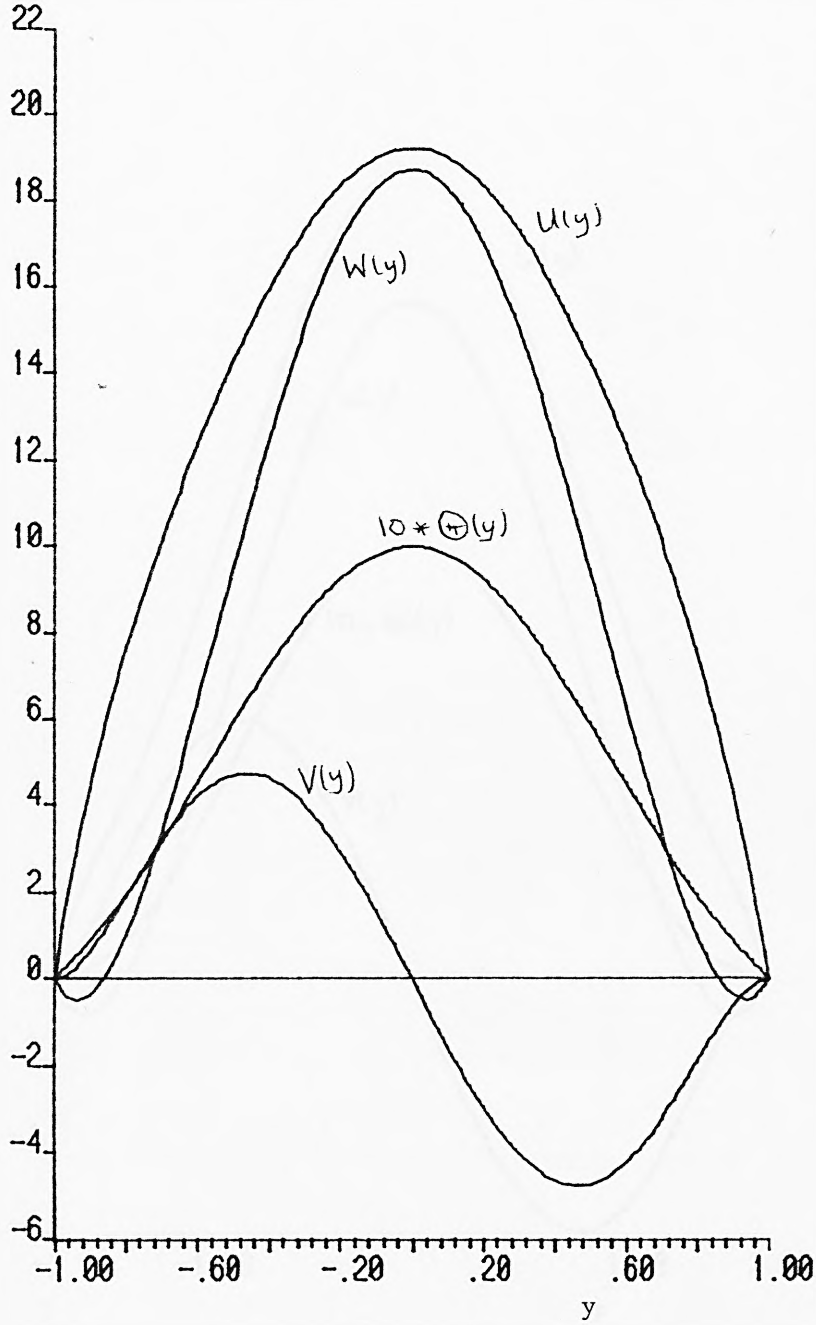


Figure 5 continued on next page

Aspect ratio a = 2.0

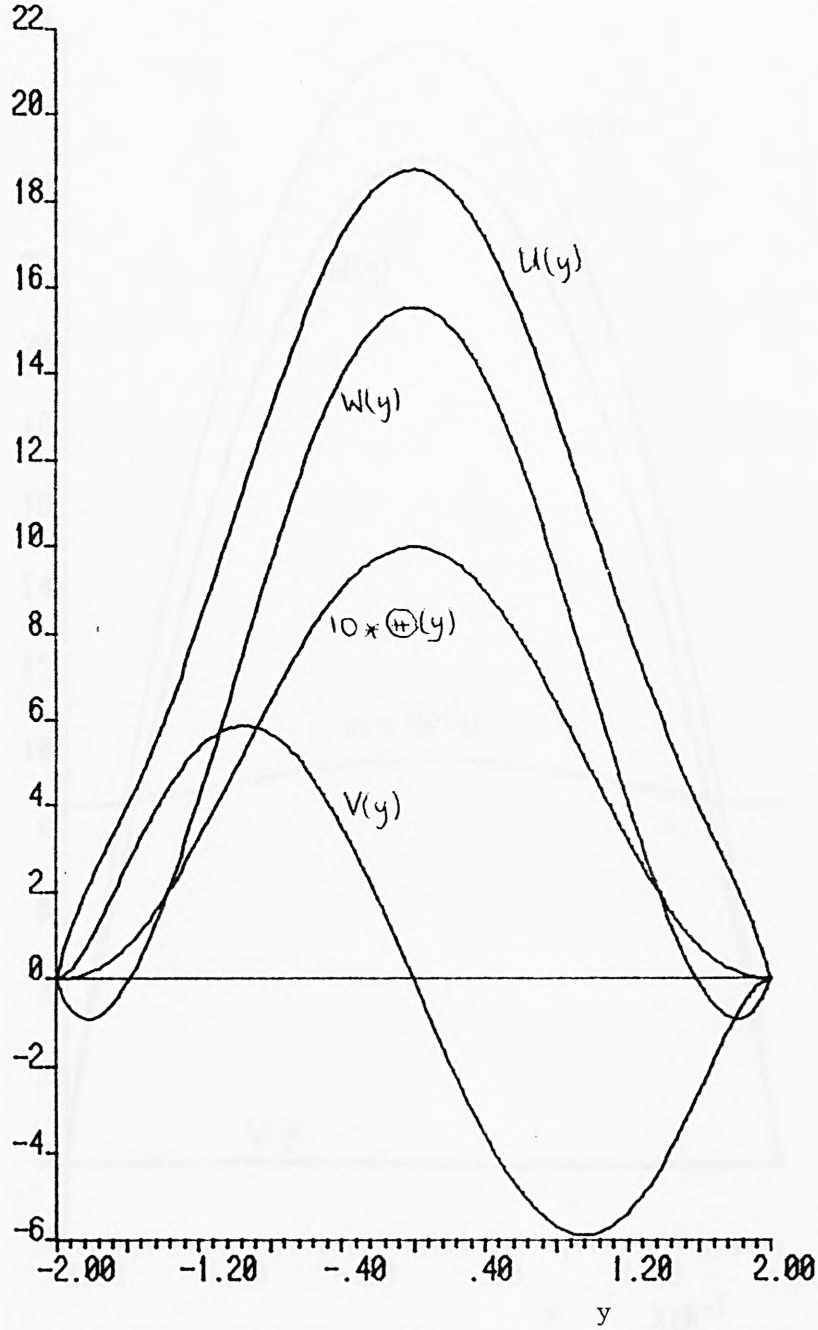
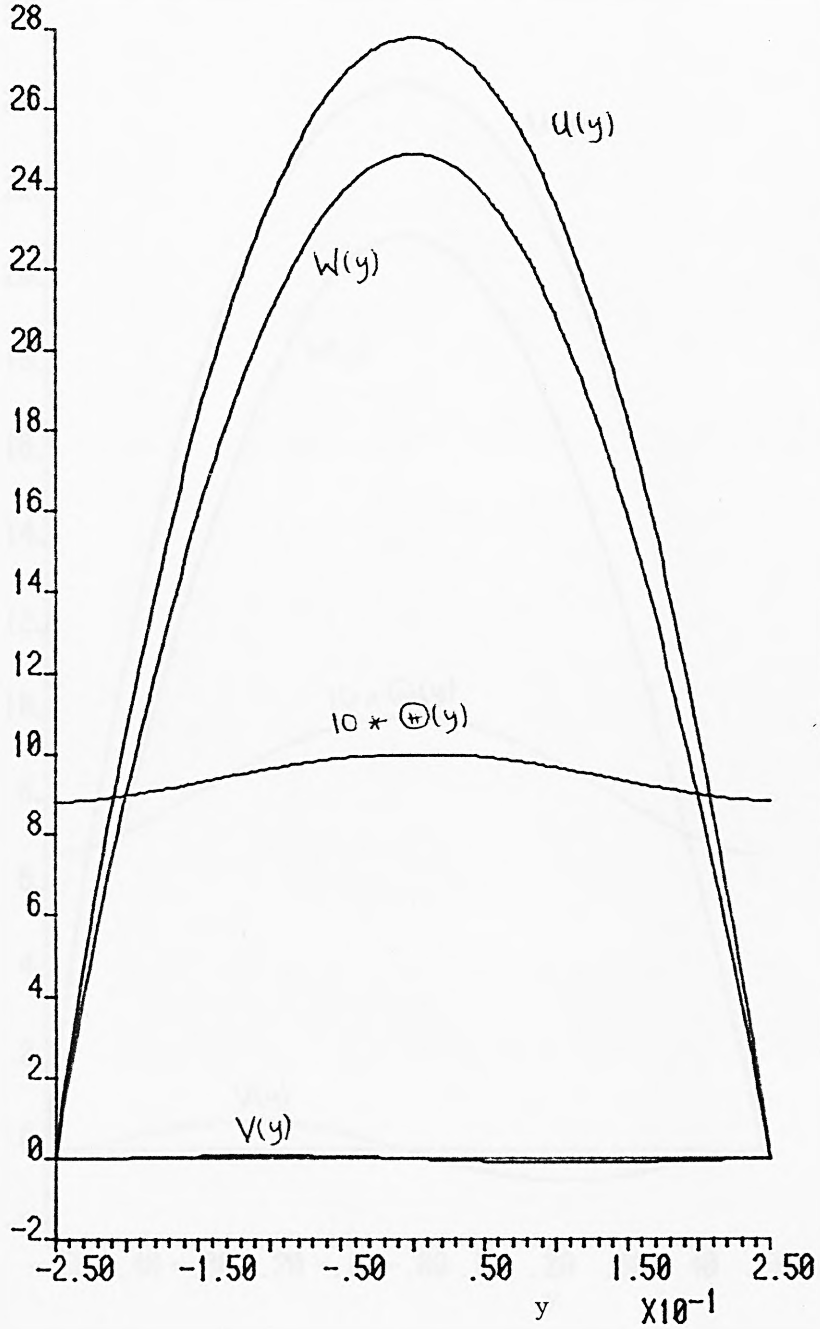


Figure 5. Profiles of the y-dependent functions Θ(y), U(y), V(y) and W(y) at the onset of convection for conducting sidewalls and aspect ratios a = 0.25, 0.5, 1.0 and 2.0.

Aspect ratio a = 0.25



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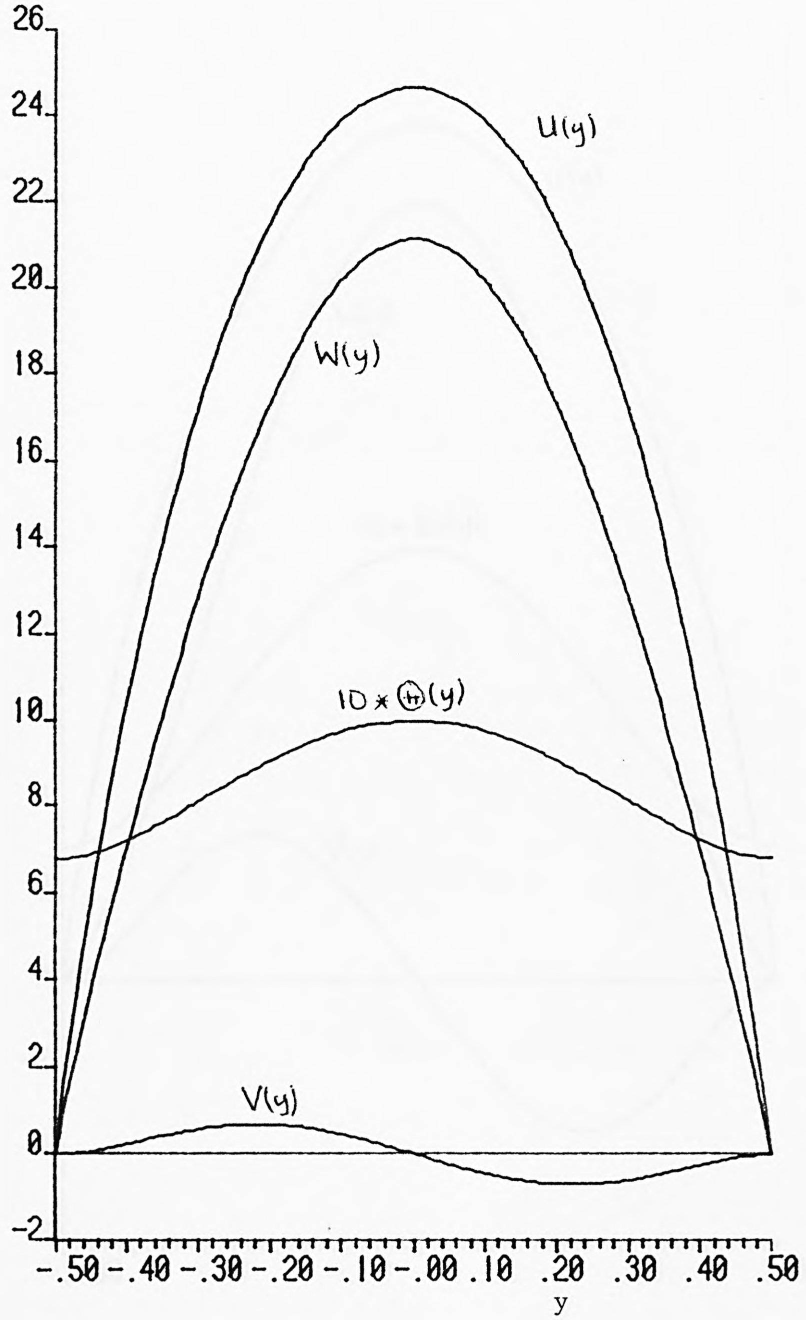
Aspect ratio a = 0.5



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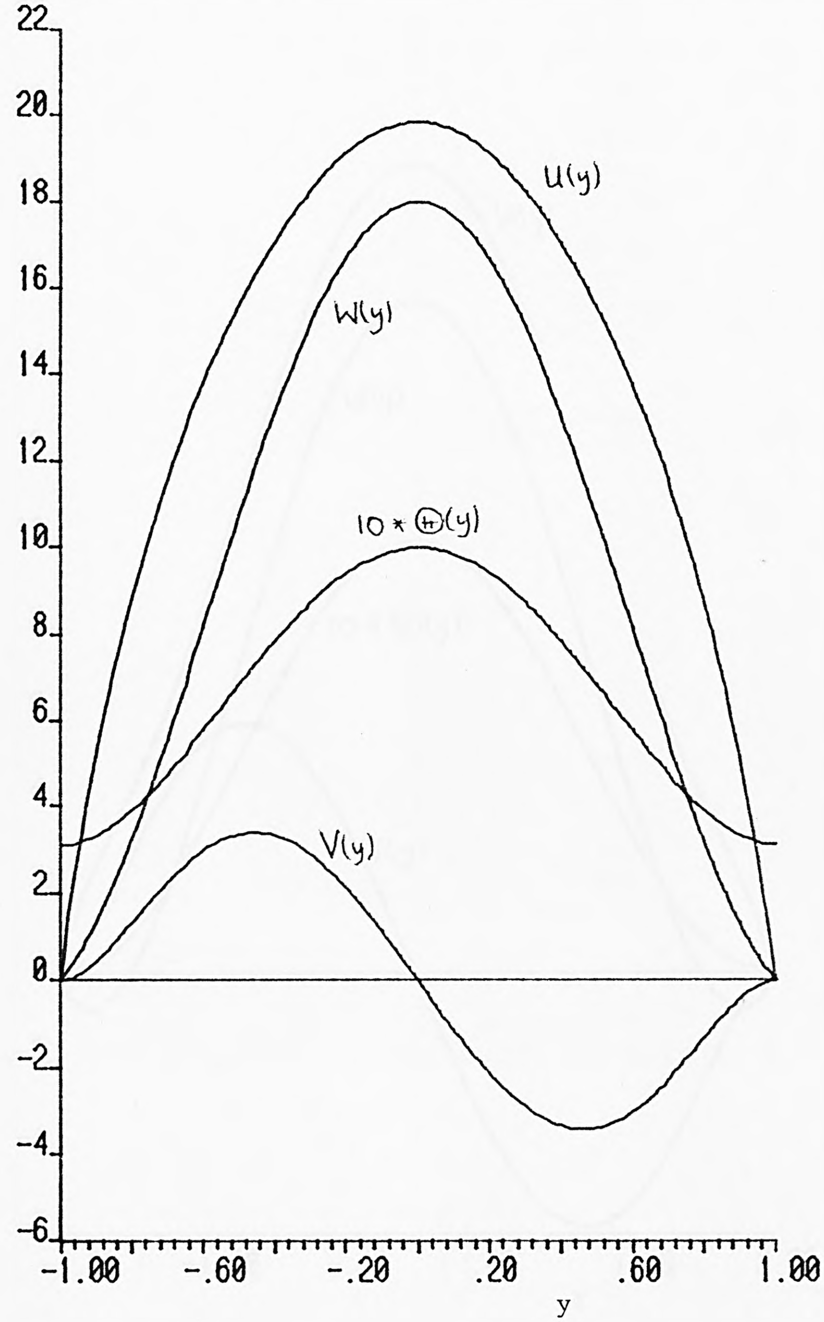
Aspect ratio a = 1.0



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Aspect ratio a = 2.0

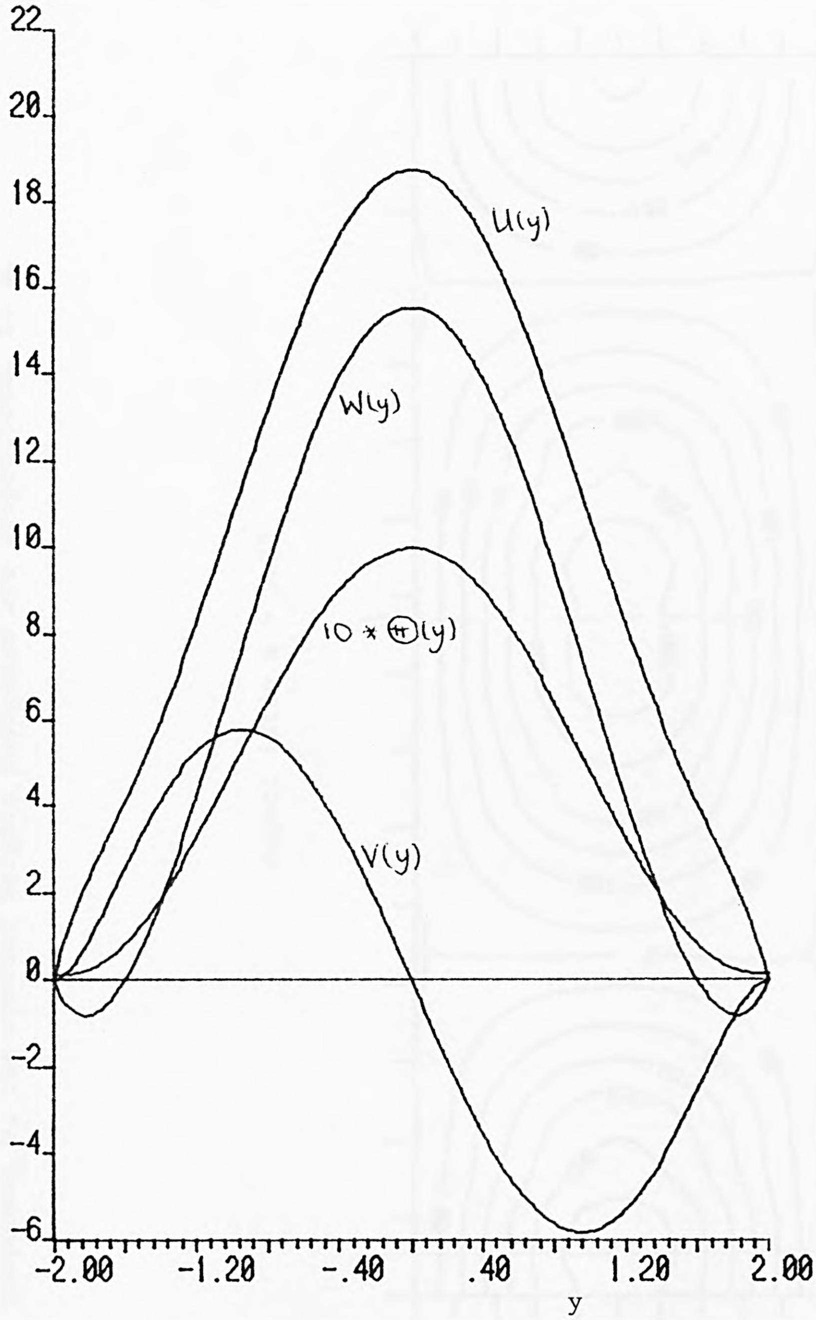

Figure 6. Profiles of the y-dependent functions Θ(y),U(y), V(y) and W(y) at the onset of convection for insulating sidewalls and aspect ratios a = 0.25,0.5,1.0 and 2.0.

It should be noted that on all diagrams showing contours of the vertical velocity component w, the contour heights indicated are proportional to w.

Aspect ratio a = 0.25

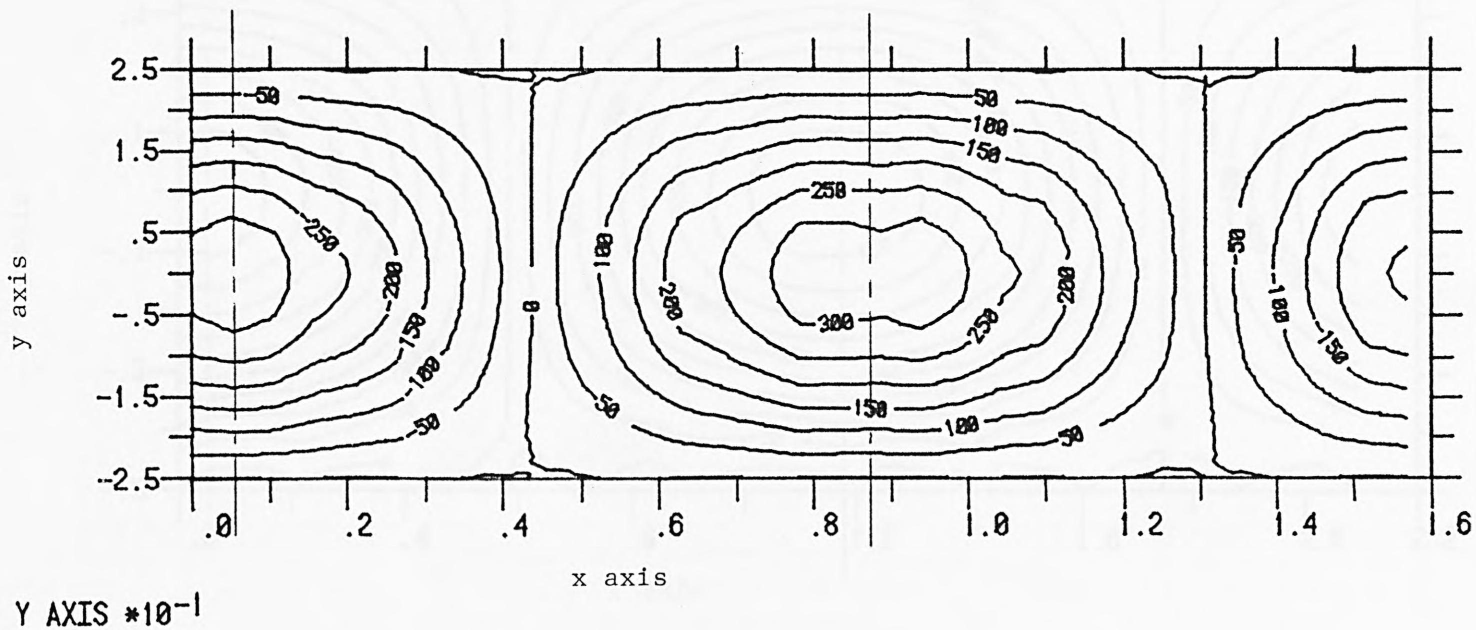

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Figure 7. Horizontal plan form of the cells at the onset of convection for the case of conducting sidewalls showing the isopleths of vertical velocity for the aspect ratios a = 0.25 and 0.5.

Aspect ratio a = 0.5

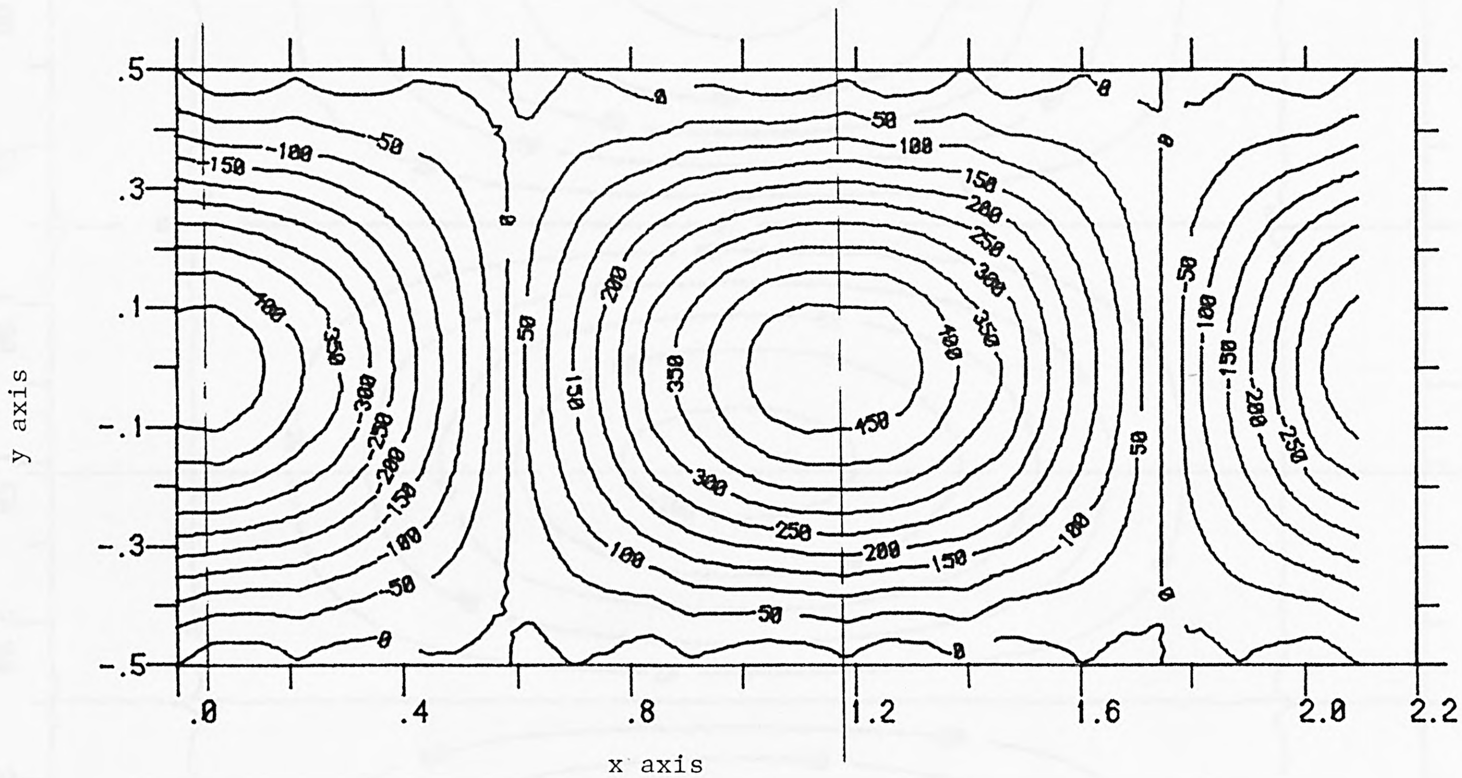
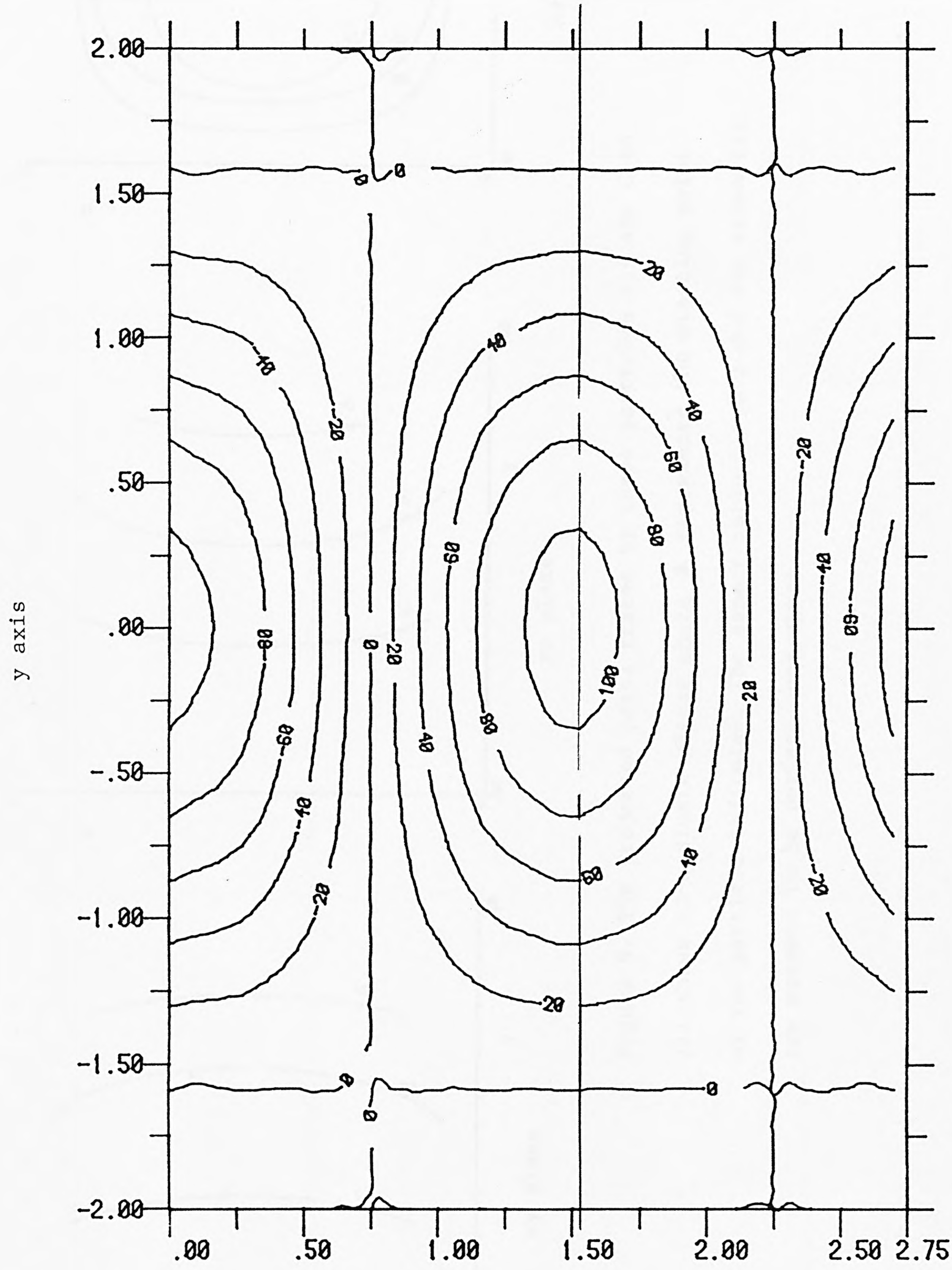


Figure 8. Horizontal plan form of the cells at the onset of convection for the case of conducting sidewalls showing the isopleths of vertical velocity for the aspect ratio a = 2.



Aspect ratio a = 2.0

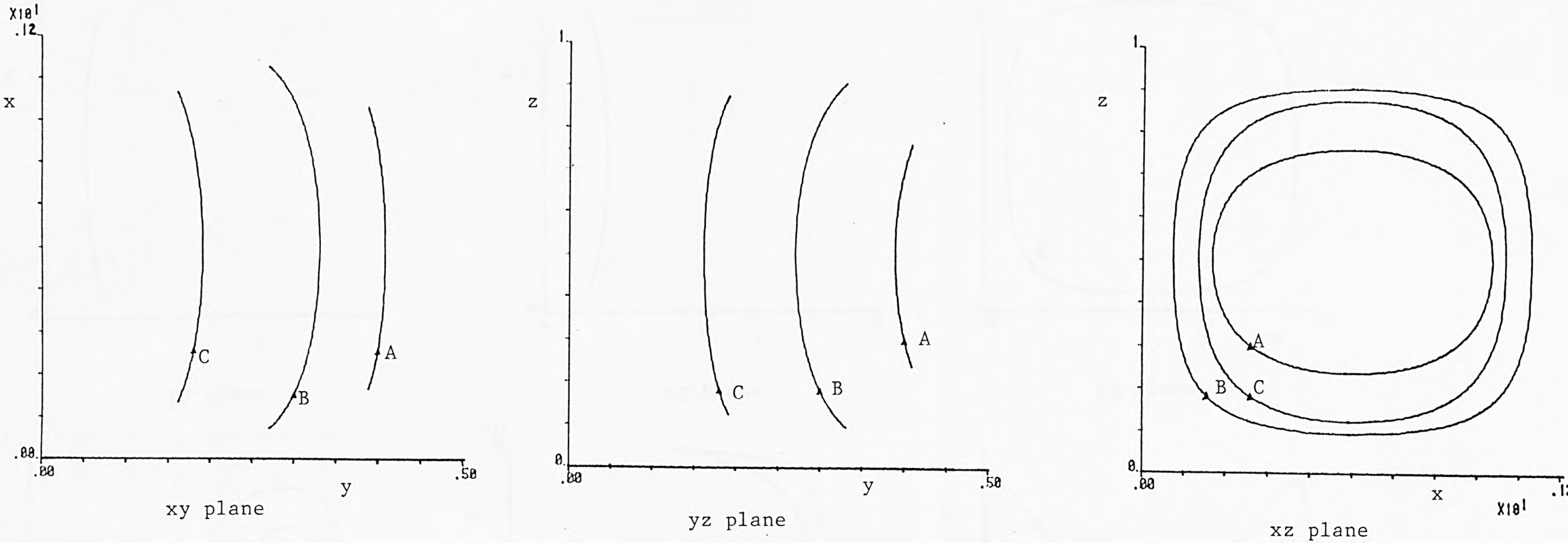


Figure 9. The projected paths traced by three particles A,B and C in all three co-ordinate planes where ▲ represents the starting point of the particle indicated. The aspect ratio a = 0.5 and the sidewalls are assumed to be perfect conductors.

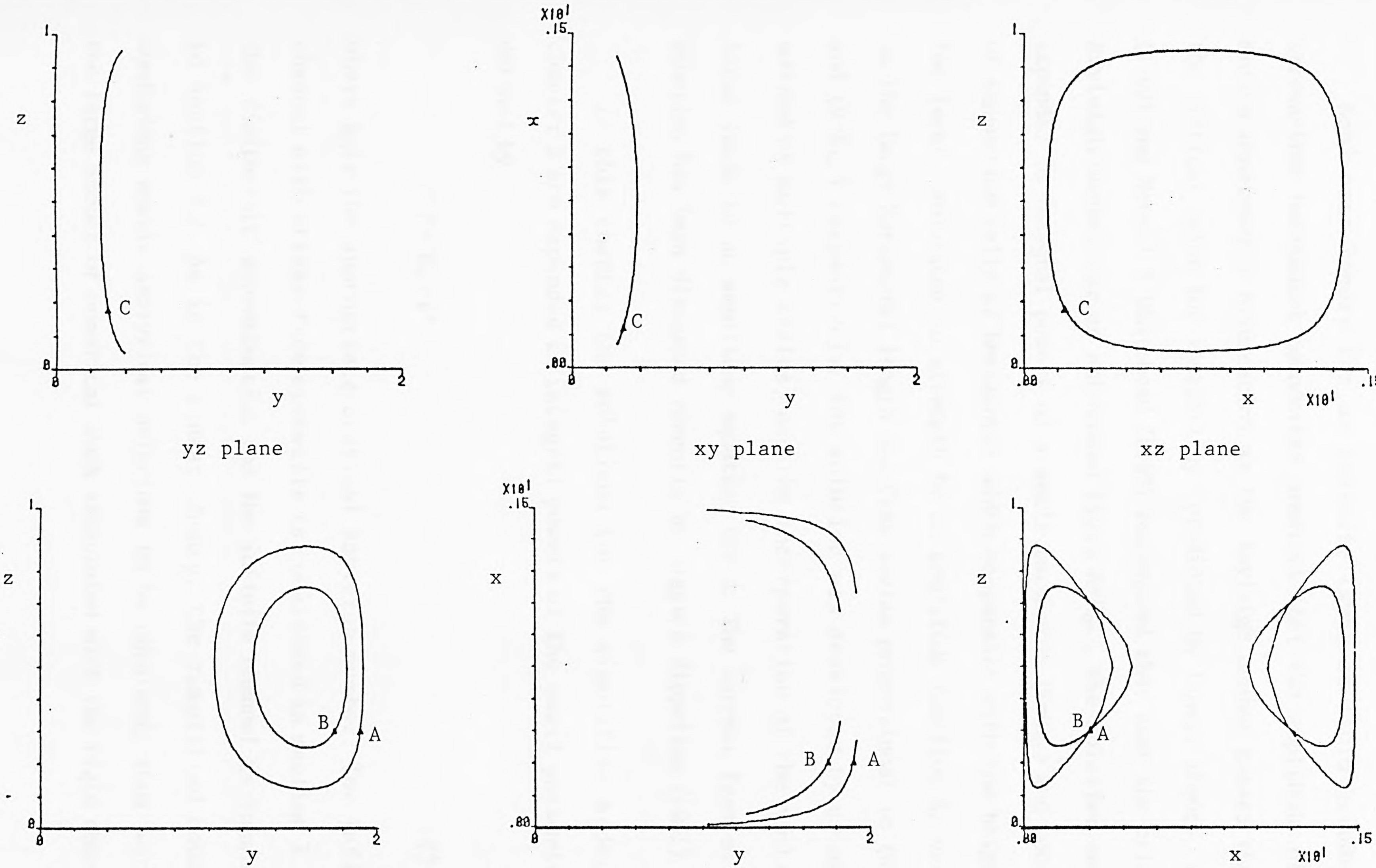


Figure 10. The projected paths traced by three particles A,B and C in all three co-ordinate planes where ▲ represents the starting point of the particle indicated. The aspect ratio a = 2 and the sidewalls are assumed to be perfect conductors.

Chapter 3. The infinite channel: nonlinear theory for simplified models

3.0 Introduction

Nonlinear theory for an infinite layer of fluid bounded by stress-free horizontal boundaries predicts that the amplitude of the motion undergoes a bifurcation as the Rayleigh number passes through the critical value for instability predicted by linear theory. Segel (1969) and Newell & Whitehead (1969) recognized that near the critical Rayleigh number for an unbounded fluid $R=R_\infty$, the solution may be expanded in integral powers of a small parameter $(R-R_\infty)^{1/2}$ and consists of convection cells of horizontal width comparable with the height of the layer, modulated in strength by an amplitude function A, varying on the large horizontal length and time scales proportional to $(R-R_\infty)^{1/2}$ and $(R-R_\infty)$ respectively. The solution was developed by using the method of multiple scales, and the incorporation of the nonlinear terms leads to an amplitude equation for A. The correct form of this equation has been discussed recently by Siggia & Zippelius (1981).

In this chapter the solutions for the simplified models of chapter 2 are expanded in integral powers of the small parameter ε , defined by

$$R - R_0 = \varepsilon^2 \tag{3.0.1}$$

where R_0 is the appropriate critical Rayleigh number. The infinite channel with stress-free sidewalls is considered in section 3.1 and the 'finite-roll' approximation for the infinite channel is considered in section 3.2. As in the linear theory, the simplified boundary conditions enable analytical solutions to be obtained, thus avoiding the large amount of numerical work associated with the rigid channel.

3.1 The infinite channel with stress-free sidewalls

Linear theory for the infinite stress-free channel (section 2.5) shows that at the critical Rayleigh number all y-modes such that $m \leqslant \bar{a}/\sqrt{2}$ appear simultaneously. If $\bar{a} < \sqrt{2}$ only the two dimensional (m=0) mode discussed by Daniels (1977) is present. For higher values of $\bar{a}$, further modes must be taken into account and here it is assumed that $\bar{a}$ is bounded such that

$$\sqrt{2} < \bar{a} < 2\sqrt{2} \qquad (\bar{a} = 2a), \tag{3.1.1}$$

when the first two modes corresponding to m=0 and m=1 will be present. Thus from (2.5.5) and (2.5.12) modified neutral solutions are

$$\theta_1 = \left(\bar{A}(\bar{X},\bar{\tau})e^{ik_0x} + \bar{A}^*(\bar{X},\bar{\tau})e^{-ik_0x}\right)\sin\pi z + \left(\bar{B}(\bar{X},\bar{\tau})e^{ik_1x} + \bar{B}^*(\bar{X},\bar{\tau})e^{-ik_1x}\right)\sin\pi z\cos\frac{\pi y}{\bar{a}}$$

$$w_1 = \frac{3\pi^2}{2}\left[\left(\bar{A}e^{ik_0x} + \bar{A}^*e^{-ik_0x}\right)\sin\pi z + \left(\bar{B}e^{ik_1x} + \bar{B}^*e^{-ik_1x}\right)\sin\pi z\cos\frac{\pi y}{\bar{a}}\right]$$

$$u_1 = 3\pi i\left[k_0\left(\bar{A}e^{ik_0x} - \bar{A}^*e^{-ik_0x}\right)\cos\pi z + k_1\left(\bar{B}e^{ik_1x} - \bar{B}^*e^{-ik_1x}\right)\cos\pi z\cos\frac{\pi y}{\bar{a}}\right]$$

$$v_1 = 0 + (-3\pi^2/\bar{a})\left(\bar{B}e^{ik_1x} + \bar{B}^*e^{-ik_1x}\right)\cos\pi z\sin\frac{\pi y}{\bar{a}}$$

and

$$p_1 = -\frac{9\pi^3}{2}P\left[\left(\bar{A}e^{ik_0x} + \bar{A}^*e^{-ik_0x}\right)\cos\pi z + \left(\bar{B}e^{ik_1x} + \bar{B}^*e^{-ik_1x}\right)\cos\pi z\cos\frac{\pi y}{\bar{a}}\right] \tag{3.1.2}$$

where

$$k_0 = \frac{\pi}{\sqrt{2}},$$

$$k_1 = \frac{\pi}{\sqrt{2}}(1 - 2/\bar{a}^2)^{1/2}, \tag{3.1.3}$$

and * denotes complex conjugate. $\bar{A}(\bar{X},\bar{\tau})$ and $\bar{B}(\bar{X},\bar{\tau})$ are the amplitude functions associated with the m=0 and m=1 modes respectively. The slow spatial and time variables $\bar{X}$ and $\bar{\tau}$ are related to x and t by

$$\bar{X} = \varepsilon x, \qquad \bar{\tau} = \varepsilon^2 t. \tag{3.1.4}$$

With these transformations the operators in x and t become

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + \varepsilon^2\frac{\partial}{\partial\bar{\tau}},$$

$$\frac{\partial}{\partial x} \to \frac{\partial}{\partial x} + \varepsilon\frac{\partial}{\partial\bar{X}} \quad\text{and}\quad \frac{\partial^2}{\partial x^2} \to \frac{\partial^2}{\partial x^2} + 2\varepsilon\frac{\partial^2}{\partial x\partial\bar{X}} + \varepsilon^2\frac{\partial^2}{\partial\bar{X}^2}. \tag{3.1.5}$$

By defining a unit vector $\hat{\lambda}$, in the z direction and using the identity

$$(\underline{u}\cdot\underline{\nabla})\underline{u} = \nabla(\tfrac{1}{2}\underline{u}\cdot\underline{u}) + \underline{\Omega}\times\underline{u} \tag{3.1.6}$$

where the vorticity $\underline{\Omega} = \underline{\nabla}\times\underline{u}$, the dimensionless momentum equations (2.1.17)-(2.1.19) can be more conveniently written as

$$\frac{\partial \underline{u}}{\partial t} + (\underline{\Omega}\times\underline{u}) = -\underline{\nabla}(p + \tfrac{1}{2}\underline{u}\cdot\underline{u}) + P\nabla^2\underline{u} + PR\theta\hat{\underline{\lambda}} \tag{3.1.7}$$

where

$$\hat{\underline{\lambda}} = (0, 0, 1), \qquad \underline{u} = (u, v, w). \tag{3.1.8}$$

A single equation in w and θ can be obtained by twice taking the curl of equation (3.1.7) and then its scalar product with $\hat{\lambda}$. The temperature dependence θ can be eliminated by applying the operator L_2 and using the heat equation (2.1.20), thus leaving a single equation for w, namely

$$(L_1L_2\nabla^2 - PR\nabla_H^2)w = -PR\nabla_H^2(\underline{u}\cdot\underline{\nabla})\theta + L_2(\hat{\underline{\lambda}}\cdot(\underline{\nabla}\times\underline{\nabla}\times(\underline{\Omega}\times\underline{u}))) \tag{3.1.8}$$

where the operators L_1, L_2 and ∇_H^2 are those defined in chapter 2. The boundary conditions on the upper and lower surfaces and on the sidewalls are

$$\theta = w = \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0 \quad z = 0, 1 \quad , \quad \frac{\partial \theta}{\partial y} = \frac{\partial w}{\partial y} = \frac{\partial u}{\partial y} = v = 0 \quad y = 0, \bar{a} \tag{3.1.9}$$

respectively. The equations (2.1.16)-(2.1.20) imply

$$w = \frac{\partial^2 w}{\partial z^2} = \frac{\partial^4 w}{\partial z^4} = 0 \quad z = 0, 1 \quad , \quad \frac{\partial w}{\partial y} = \frac{\partial^3 w}{\partial y^3} = \frac{\partial^5 w}{\partial y^5} = 0 \quad y = 0, \bar{a}. \tag{3.1.10}$$

Unlike the case of the rigid channel, here it is possible to express all the conditions on the sidewalls in terms of a single variable.

The dependent variables are expanded as power series in ε:

$$[\theta\ w\ u\ v\ p] = \varepsilon[\theta_1\ w_1\ u_1\ v_1\ p_1] + \varepsilon^2[\theta_2\ w_2\ u_2\ v_2\ p_2] + \varepsilon^3[\theta_3\ w_3\ u_3\ v_3\ p_3] + \cdots \tag{3.1.11}$$

where $[\theta_i, w_i, u_i, v_i, p_i]$ i=1,2,.. are functions of $x, \bar{X}, y, z$ and τ. The leading terms corresponding to i=1 are the modified neutral solutions of the most unstable modes.

Using the transformations (3.0.1) and (3.1.4,5,11) equation (3.1.8) becomes

$$\left(\bar{L}_1 + \varepsilon \bar{L}_2 + \varepsilon^2 \bar{L}_3 + \cdots\right)\left(w_1 + \varepsilon w_2 + \varepsilon^2 w_3 + \cdots\right) \tag{3.1.12}$$

$$= -\varepsilon P(R_0 + \varepsilon^2)\left(\nabla_H^2 + 2\varepsilon \frac{\partial^2}{\partial x \partial \bar{X}} + \varepsilon^2 \frac{\partial^2}{\partial \bar{X}^2}\right)\left[\underline{u}_1 \cdot \bar{\nabla} \theta_1 + \varepsilon\left(\underline{u}_1 \cdot \bar{\nabla} \theta_2 + \underline{u}_2 \cdot \bar{\nabla} \theta_1\right) + O(\varepsilon^2)\right]$$

$$+ \varepsilon\left[\frac{\partial}{\partial t} + \varepsilon^2 \frac{\partial}{\partial \tau} - \left(\nabla^2 + 2\varepsilon \frac{\partial^2}{\partial x \partial \bar{X}} + \varepsilon^2 \frac{\partial^2}{\partial \bar{X}^2}\right)\right]\left[\underline{\lambda} \cdot \left(\underline{\nabla} \times \underline{\nabla} \times \left((\underline{\Omega}_1 \times \underline{u}_1) + \varepsilon\left((\underline{\Omega}_2 \times \underline{u}_1) + (\underline{\Omega}_1 \times \underline{u}_2)\right) + O(\varepsilon^2)\right)\right)\right]$$

where

$$\underline{\Omega}_i = \bar{\underline{\nabla}} \times \underline{u}_i \quad i = 1,2 \quad , \quad \bar{\underline{\nabla}} = \left(\frac{\partial}{\partial x} + \varepsilon \frac{\partial}{\partial \bar{X}}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right),$$

$$\bar{L}_1 = \left[L_1 L_2 \nabla^2 - P R_0 \nabla_H^2\right],$$

$$\bar{L}_2 = 2\frac{\partial^2}{\partial x \partial \bar{X}}\left[L_1 L_2 - P L_2 \nabla^2 - L_1 \nabla^2 - P R_0\right]$$

and

$$\bar{L}_3 = \left[L_1 L_2 - L_1 \nabla^2 - P L_2 \nabla^2\right]\left[\frac{\partial^2}{\partial \bar{X}^2}\right] + \left[2\frac{\partial}{\partial t} - (1+P)\nabla^2\right]\left[\nabla^2 \frac{\partial}{\partial \tau}\right]$$

$$+ \left[P\nabla^2 + L_1 + P L_2\right]\left[2\frac{\partial^2}{\partial x \partial \bar{X}}\right]^2 - P\left[R_0 \frac{\partial^2}{\partial \bar{X}^2} + \nabla_H^2\right]. \tag{3.1.14}$$

The conditions (3.1.10) become

$$\frac{\partial^n}{\partial z^n}\left[w_1 + \varepsilon w_2 + \varepsilon^2 w_3 + \cdots\right] = 0 \qquad z = 0,1 \qquad n = 0,2,4$$

and

$$\frac{\partial^n}{\partial y^n}\left[w_1 + \varepsilon w_2 + \varepsilon^2 w_3 + \cdots\right] = 0 \qquad y = 0, \bar{a} \qquad n = 1,3,5. \tag{3.1.15}$$

The Boussinesq equations (2.1.16)-(2.1.20) become

$$\bar{\underline{\nabla}} \cdot \left[\underline{u}_1 + \varepsilon \underline{u}_2 + \varepsilon^2 \underline{u}_3 + \cdots\right] = 0 \tag{3.1.16}$$

$$\tilde{L}_1\left[u_1 + \varepsilon u_2 + \varepsilon^2 u_3 + \cdots\right] = -\left[\frac{\partial}{\partial x} + \varepsilon \frac{\partial}{\partial \bar{X}}\right]\left[p_1 + \varepsilon p_2 + \varepsilon^2 p_3 + \cdots\right] \tag{3.1.17}$$

$$\tilde{L}_1\left[v_1 + \varepsilon v_2 + \varepsilon^2 v_3 + \cdots\right] = -\frac{\partial}{\partial y}\left[p_1 + \varepsilon p_2 + \varepsilon^2 p_3 + \cdots\right] \tag{3.1.18}$$

$$\tilde{L}_1\left[w_1 + \varepsilon w_2 + \varepsilon^2 w_3 + \cdots\right] = -\frac{\partial}{\partial z}\left[p_1 + \varepsilon p_2 + \varepsilon^2 p_3 + \cdots\right] \tag{3.1.19}$$

$$+ P\left[R_0 + \varepsilon^2\right]\left[\theta_1 + \varepsilon \theta_2 + \varepsilon^2 \theta_3 + \cdots\right]$$

and

$$\tilde{L}_2[\theta_1+\varepsilon\theta_2+\varepsilon^2\theta_3+\cdots]=[w_1+\varepsilon w_2+\varepsilon^2 w_3+\cdots] \tag{3.1.20}$$

where

$$\tilde{L}_1=\left[\frac{\partial}{\partial t}+\varepsilon^2\frac{\partial}{\partial T}-P\left(\nabla^2+2\varepsilon\frac{\partial^2}{\partial x\partial X}+\varepsilon^2\frac{\partial^2}{\partial X^2}\right)+\varepsilon^2[u_1+\varepsilon u_2+\cdots]\frac{\partial}{\partial X}+\varepsilon[\underline{u}_1+\varepsilon\underline{u}_2+\cdots].\underline{\nabla}\right]$$

and (3.1.21)

$$\tilde{L}_2=\left[\frac{\partial}{\partial t}+\varepsilon^2\frac{\partial}{\partial T}-\left(\nabla^2+2\varepsilon\frac{\partial^2}{\partial x\partial X}+\varepsilon^2\frac{\partial^2}{\partial X^2}\right)+\varepsilon^2[u_1+\varepsilon u_2+\cdots]\frac{\partial}{\partial X}+\varepsilon[\underline{u}_1+\varepsilon\underline{u}_2+\cdots].\underline{\nabla}\right].$$

The procedure by which the dependent variables are found is as follows: having obtained the vertical velocity component w from equation (3.1.12), subject to the boundary conditions (3.1.15), the temperature θ can be found from the heat equation (3.1.20). This enables the pressure p to be determined from equation (3.1.19), which in turn enables the horizontal velocity components u and v to be obtained from (3.1.17) and (3.1.18) respectively. Once all three velocity components have been obtained a consistency check is provided by the continuity equation (3.1.16). Possible contributions to the pressure and horizontal velocity components independent of z require consideration of the coupled system (3.1.16-18) (see below).

Equating ascending powers of ε in equations (3.1.12-20) gives:

(1) Expansion at $O(\varepsilon^0)$

The linear balance

$$\tilde{L}_1 w_1 = 0 \tag{3.1.22}$$

with the conditions

$$w_1=\frac{\partial^2 w_1}{\partial z^2}=\frac{\partial^4 w_1}{\partial z^4}=0 \quad z=0,1 \quad \text{and} \quad \frac{\partial w_1}{\partial y}=\frac{\partial^3 w_1}{\partial y^3}=\frac{\partial^5 w_1}{\partial y^5}=0 \quad y=0,\bar{a}. \tag{3.1.23}$$

yields the neutral solution set (3.1.2) above.

(2) Expansion at $O(\varepsilon)$

At order ε, equation (3.1.12) becomes

$$\tilde{L}_1 w_2 = -\tilde{L}_2 w_1 - PR_0\nabla_H^2(\underline{u}_1\cdot\underline{\nabla})\theta_1 + L_2(\hat{\underline{z}}\cdot\underline{\nabla}\times\underline{\nabla}\times(\underline{\Omega}_1\times\underline{u}_1)) \tag{3.1.24}$$

where $\Omega_1 = \nabla \times u_1$ and the boundary conditions from (3.1.15) are

$$w_2 = \frac{\partial^2 w_2}{\partial z^2} = \frac{\partial^4 w_2}{\partial z^4} = 0 \quad z=0,1 \quad \text{and} \quad \frac{\partial w_2}{\partial y} = \frac{\partial^3 w_2}{\partial y^3} = \frac{\partial^5 w_2}{\partial y^5} = 0 \quad y = 0, \bar{a}. \tag{3.1.25}$$

Expanding the right hand side of (3.1.24) gives

$$\bar{L}_1 w_2 = -\bar{L}_2 w_1 + \Big\{ [\bar{\alpha}_1 P + \bar{\alpha}_2][\bar{B} e^{2ik_1 x} + c.c] + [\bar{\alpha}_3 P + \bar{\alpha}_4][\bar{B}\bar{B}^*] \cos\frac{2\pi y}{\bar{a}} + [\bar{\alpha}_5 P + \bar{\alpha}_6][\bar{A}\bar{B} e^{i(k_0+k_1)x} + c.c] \cos\frac{\pi y}{\bar{a}}$$

$$+ [\bar{\alpha}_7 P + \bar{\alpha}_8][\bar{A}\bar{B}^* e^{i(k_0-k_1)x} + c.c] \cos\frac{\pi y}{\bar{a}} \Big\} \sin 2\pi z \tag{3.1.26}$$

where the coefficients $\bar{\alpha}_i$ i=1..8 are given by (B1) in appendix B and

$$\bar{L}_2 w_1 = 2P \frac{\partial^2}{\partial x \partial \bar{x}} [3\nabla^4 - R_0] w_1 \tag{3.1.27}$$

is zero, since the expansion is centred about the critical Rayleigh number R_0. Therefore, no forcing functions appear on the right hand side of (3.1.26) which are eigensolutions of the basic linear operator $\bar{L}_1$ with the conditions (3.1.25). Thus

$$w_2 = \Big\{ [\alpha_1 + \frac{\alpha_2}{P}][\bar{B}^2 e^{2ik_1 x} + c.c] + [\alpha_3 + \frac{\alpha_4}{P}][\bar{B}\bar{B}^*] \cos\frac{2\pi y}{\bar{a}} + [\alpha_5 + \frac{\alpha_6}{P}][\bar{A}\bar{B} e^{i(k_0+k_1)x} + c.c] \cos\frac{\pi y}{\bar{a}}$$

$$+ [\alpha_7 + \frac{\alpha_8}{P}][\bar{A}\bar{B}^* e^{i(k_0-k_1)x} + c.c] \cos\frac{\pi y}{\bar{a}} \Big\} \sin 2\pi z + \frac{3\pi^2}{2} \Big\{ (\bar{C}(\bar{x},\bar{t}) e^{ik_0 x} + c.c) + (\bar{D}(\bar{x},\bar{t}) e^{ik_1 x} + c.c) \cos\frac{\pi y}{\bar{a}} \Big\} \sin \pi z \tag{3.1.28}$$

where α_i i=1..8 are given by (B2) in appendix B, and $\bar{C}$ and $\bar{D}$ are arbitrary amplitude functions associated with the modes m=0 and m=1 respectively, which could be omitted for the purpose of finding the amplitude equations for $\bar{A}$ and $\bar{B}$.

Having obtained w_2, the temperature θ_2 may be obtained from the heat equation (3.1.20), which at order ε is

$$L_2 \theta_2 = w_2 - \left(-2 \frac{\partial^2}{\partial x \partial \bar{x}} + u_1 \cdot \nabla \right) \theta_1 . \tag{3.1.29}$$

The conditions on θ_2 are

$$\theta_2 = 0 \quad z = 0, 1 \quad \text{and} \quad \frac{\partial \theta_2}{\partial y} = 0 \quad y = 0, \bar{a}.$$

Thus

$$\theta_2 = \Big\{ \big[\tilde{\alpha}_1 + \frac{\tilde{\alpha}_2}{P}\big]\big[\bar{B}^2 e^{2ik_1x} + c.c\big] + \big[\tilde{\alpha}_3 + \frac{\tilde{\alpha}_4}{P}\big]\big[\bar{B}\bar{B}^*\big]\cos\frac{2\pi Y}{\bar{a}} + \big[\tilde{\alpha}_5 + \frac{\tilde{\alpha}_6}{P}\big]\big[\bar{A}\bar{B}e^{i(k_0+k_1)x} + c.c\big]\cos\frac{\pi Y}{\bar{a}}$$

$$+ \big[\tilde{\alpha}_7 + \frac{\tilde{\alpha}_8}{P}\big]\big[\bar{A}\bar{B}^* e^{i(k_0-k_1)x} + c.c\big]\cos\frac{\pi Y}{\bar{a}} + \tilde{\alpha}_9\big[\bar{A}\bar{A}^*\big] + \tilde{\alpha}_{10}\big[\bar{B}\bar{B}^*\big]\Big\}\sin 2\pi z + \Big\{\tilde{\alpha}_{11}\big[\bar{A}_{\bar{X}} e^{ik_0x} - c.c\big]$$

$$+ \tilde{\alpha}_{12}\big[\bar{B}_{\bar{X}} e^{ik_1x} - c.c\big]\cos\frac{\pi Y}{\bar{a}} + \big[\bar{C}e^{ik_0x} + c.c\big] + \big[\bar{D}e^{ik_1x} + c.c\big]\cos\frac{\pi Y}{\bar{a}}\Big\}\sin\pi z + T_c \qquad (3.1.30)$$

where T_c is a possible complementary solution which must satisfy

$$\nabla^2 T_c = 0 \qquad T_c = 0 \quad z = 0,1 \quad ; \quad \frac{\partial T_c}{\partial y} = 0 \quad Y = 0, \bar{a} \qquad (3.1.31)$$

and the coefficients $\tilde{\alpha}_i$ i=1..12 are given by (B3) in appendix B. The only relevant solution for T_c satisfying the conditions (3.1.31) is the trivial solution

$$T_c = 0. \qquad (3.1.32)$$

The solution for the pressure p_2 is obtained from the w_2 momentum equation (3.1.19) which at order ε, is

$$L_1 w_2 = -\frac{\partial}{\partial z}p_2 + P R_0 \theta_2 - \big(-2P\frac{\partial^2}{\partial x \partial \bar{X}} + \underline{u}_1 \cdot \underline{\nabla}\big) w_1 \; . \qquad (3.1.33)$$

The velocity components v_2 and u_2 can then be obtained using the horizontal momentum equations (3.1.18) and (3.1.17) which at order ε are

$$L_1 v_2 = -\frac{\partial}{\partial y}p_2 - \big(-2P\frac{\partial^2}{\partial x \partial \bar{X}} + \underline{u}_1 \cdot \underline{\nabla}\big) v_1 \qquad (3.1.34)$$

and

$$L_1 u_2 = -\frac{\partial}{\partial x}p_2 - \frac{\partial}{\partial \bar{X}}p_1 - \big(-2P\frac{\partial^2}{\partial x \partial \bar{X}} + \underline{u}_1 \cdot \underline{\nabla}\big) u_1 \; . \qquad (3.1.35)$$

The solutions for v_2 and u_2 must satisfy the conditions

$$\frac{\partial v_2}{\partial z} = \frac{\partial u_2}{\partial z} = 0 \quad z = 0,1 \quad \text{and} \quad v_2 = \frac{\partial u_2}{\partial y} = 0 \quad Y = 0, \bar{a}. \qquad (3.1.36)$$

Having obtained all three velocity components a consistency check is provided by the continuity equation (3.1.18) which at order ε is

$$\frac{\partial u_2}{\partial x} + \frac{\partial u_1}{\partial \bar{X}} + \frac{\partial v_2}{\partial y} + \frac{\partial w_2}{\partial z} = 0 \; . \qquad (3.1.37)$$

Equation (3.1.33) gives

$$P_2=\{[\alpha_{14}+\alpha_{15}P][\bar{B}^2e^{2ik_1x}+c.c]+[\alpha_{16}+\alpha_{17}P][\bar{B}\bar{B}^*]\cos\frac{2\pi Y}{\bar{a}}+[\alpha_{18}+\alpha_{19}P][\bar{A}\bar{B}e^{i(k_0+k_1)x}+c.c]\cos\frac{\pi Y}{\bar{a}}$$

$$+[\alpha_{20}+\alpha_{21}P][\bar{A}\bar{B}^*e^{i(k_0-k_1)x}+c.c]\cos\frac{\pi Y}{\bar{a}}+[\alpha_{22}+\alpha_{23}P][\bar{A}\bar{A}^*]+[\alpha_{24}+\alpha_{25}P][\bar{B}\bar{B}^*]\}\cos 2\pi z$$

$$+\{\alpha_{26}P[\bar{A}_{\bar{X}}e^{ik_0x}-c.c]+\alpha_{27}P[\bar{B}_{\bar{X}}e^{ik_1x}-c.c]\cos\frac{\pi Y}{\bar{a}}+\alpha_{28}P[(\bar{C}e^{ik_0x}+c.c)+(\bar{D}e^{ik_1x}+c.c)\cos\frac{\pi Y}{\bar{a}}\}\cos\pi z$$

$$+F(x,y,\bar{X},\bar{\tau}) \tag{3.1.38}$$

where the coefficients α_i i=14..28 are given by (B4) in appendix B and F is an arbitrary function of x,y,$\bar{X}$ and $\bar{\tau}$.

Expansion of the right hand sides of (3.1.34) and (3.1.35) gives

$$V_2=\{[\alpha_{29}+\frac{\alpha_{30}}{P}][\bar{B}\bar{B}^*]\sin\frac{2\pi Y}{\bar{a}}+[\alpha_{31}+\frac{\alpha_{32}}{P}][\bar{A}\bar{B}e^{i(k_0+k_1)x}+c.c]\sin\frac{\pi Y}{\bar{a}}+[\alpha_{33}+\frac{\alpha_{34}}{P}][\bar{A}\bar{B}^*e^{i(k_0-k_1)x}+c.c]\sin\frac{\pi Y}{\bar{a}}\}\cos 2\pi z$$

$$+\{\alpha_{35}[\bar{B}_{\bar{X}}e^{ik_1x}-c.c]\sin\frac{\pi Y}{\bar{a}}+\alpha_{36}[\bar{D}e^{ik_1x}+c.c]\sin\frac{\pi Y}{\bar{a}}\}\cos\pi z+\tilde{V}_2(x,y,\bar{X},\bar{\tau}) \tag{3.1.39}$$

and

$$u_2=\{[\alpha_{37}+\frac{\alpha_{38}}{P}][\bar{B}^2e^{2ik_1x}-c.c]+[\alpha_{39}+\frac{\alpha_{40}}{P}][\bar{A}\bar{B}e^{i(k_0+k_1)x}-c.c]\cos\frac{\pi Y}{\bar{a}}+[\alpha_{41}+\frac{\alpha_{42}}{P}][\bar{A}\bar{B}^*e^{i(k_0-k_1)x}-c.c]\cos\frac{\pi Y}{\bar{a}}\}\cos 2\pi z$$

$$+\{\alpha_{43}[\bar{A}_{\bar{X}}e^{ik_0x}+c.c]+\alpha_{44}[\bar{B}_{\bar{X}}e^{ik_1x}+c.c]\cos\frac{\pi Y}{\bar{a}}+\alpha_{45}[\bar{C}e^{ik_0x}-c.c]+\alpha_{46}[\bar{D}e^{ik_1x}-c.c]\cos\frac{\pi Y}{\bar{a}}\}\cos\pi z$$

$$+\tilde{U}_2(x,y,\bar{X},\bar{\tau}) \tag{3.1.40}$$

where $\tilde{V}_2$ and $\tilde{U}_2$ satisfy

$$P\nabla_H^2\tilde{V}_2=-N_v+\frac{\partial F}{\partial y}$$

and

$$P\nabla_H^2\tilde{U}_2=-N_u+\frac{\partial F}{\partial x} \tag{3.1.41}$$

where N_v and N_u are the z-independent parts of the right hand sides of (3.1.34) and (3.1.35), given by (B5) and (B6) in appendix B. In addition, $\tilde{V}_2$ and $\tilde{U}_2$ satisfy the two-dimensional continuity equation

$$\frac{\partial\tilde{U}_2}{\partial x}+\frac{\partial\tilde{V}_2}{\partial y}=0. \tag{3.1.42}$$

The coefficients α_i i=29..46 are given by (B8) and (B9) in appendix B. As in the case of the temperature θ_2, no additional complementary solutions are generated in v_2 and u_2.

In order to complete the solutions for the horizontal velocity components it is necessary to obtain the arbitrary function F, so that $\tilde{V}_2$ and $\tilde{U}_2$ may be determined. Applying the operator to equation (3.1.42) gives

$$\frac{\partial}{\partial x}\nabla_H^2 \tilde{U}_2 + \frac{\partial}{\partial y}\nabla_H^2 \tilde{V}_2 = 0 . \tag{3.1.43}$$

Thus, from (3.1.41) it is clear that the condition needed for this equation to be satisfied is that the arbitrary function F satisfies

$$\nabla_H^2 F = \frac{\partial}{\partial y} N_v + \frac{\partial}{\partial x} N_u . \tag{3.1.44}$$

A particular solution F_p for F can be found such that

$$\frac{\partial}{\partial y} F_p = N_v \qquad \text{and} \qquad \frac{\partial}{\partial x} F_p = N_u \tag{3.1.45}$$

(see (B7) in appendix B) which implies

$$P\nabla_H^2 \tilde{V}_2 = \frac{\partial}{\partial y} F_c \quad , \qquad P\nabla_H^2 \tilde{U}_2 = \frac{\partial}{\partial x} F_c \tag{3.1.46}$$

where F_c is the complementary solution for F. The conditions on $\tilde{V}_2$ and $\tilde{U}_2$ are

$$\tilde{V}_2 = \frac{\partial \tilde{U}_2}{\partial y} = 0 \qquad Y = 0, \bar{a} \tag{3.1.47}$$

which from (3.1.42) are equivalent to

$$\tilde{V}_2 = \frac{\partial^2 \tilde{V}_2}{\partial y^2} = 0 \qquad Y = 0, \bar{a} . \tag{3.1.48}$$

Eliminating $\tilde{U}_2$ in (3.1.43,46) gives the single fourth order partial differential equation

$$\nabla_H^4 \tilde{V}_2 = 0 \tag{3.1.49}$$

and the only relevant solution satisfying the conditions (3.1.48) is the trivial solution

$$\tilde{V}_2 = 0 . \tag{3.1.50}$$

This implies

$$\tilde{U}_2 = \hat{U}_2(y, \bar{X}, \bar{\tau}) \qquad \text{and} \qquad F_c = \beta x \qquad (\beta \text{ real}) \tag{3.1.51}$$

such that

$$P\frac{\partial^2}{\partial y^2}\hat{U}_2 = \beta \qquad \text{and} \qquad \frac{\partial \hat{U}_2}{\partial y} = 0 \qquad Y = 0, \bar{a} . \tag{3.1.52}$$

It should be noted that a term of the form βx in the pressure at the present order, is equivalent to a term of the form $\beta \bar{X}$ in the pressure at the previous order. The overall mass flux down the channel at this level of approximation is given by

$$\int_{Y=0}^{\bar{a}} \int_{z=0}^{1} u_2 \, dY dz \, . \tag{3.1.53}$$

From (3.1.40) this is zero if

$$\int_{Y=0}^{\bar{a}} \hat{U}_2 \, dY = 0 \, . \tag{3.1.54}$$

This condition together with (3.1.52) implies

$$\hat{U}_2 = 0 \quad \text{and} \quad \beta = 0 \tag{3.1.55}$$

which gives

$$\tilde{U}_2 = 0 \, . \tag{3.1.56}$$

Having obtained the solutions for all three velocity components u_2, v_2 and w_2, they can be shown to be consistent by substitution into the continuity equation (3.1.37), which is found to be satisfied.

In later chapters the stress-free channel will be assumed to have rigid endwalls in which case the mass flux (3.1.53) must be zero. However, it should be noted that if the flux condition is relaxed a solution is possible in which

$$\hat{U}_2 = U_0 \quad \text{and} \quad \beta = 0 \tag{3.1.57}$$

where U_0 is a constant. A solution of this type which corresponds to the stress-free analogue of a Poiseuille-type flow down the channel is not of interest in the present study.

The solutions for all the dependent variables at this order have thus been obtained and are re-stated in full in appendix B.

(3) Expansion at $O(\varepsilon^2)$

At the order ε^2, equation (3.1.12) becomes

$$L_1 w_3 = -L_2 w_2 - L_3 w_1 - PR_0\left[2\frac{\partial^2}{\partial x\partial\bar{x}}(\underline{u}_1\cdot\underline{\nabla})\theta_1 + \nabla_H^2\left(\underline{u}_1\cdot\underline{\nabla}\theta_2 + \underline{u}_2\cdot\underline{\nabla}\theta_1 + u_1\frac{\partial}{\partial\bar{x}}\theta_1\right)\right]$$

$$+ L_2\left[\hat{\lambda}\cdot\left((\underline{\nabla}\times\underline{\nabla}\times(\underline{\Omega}_2\times\underline{u}_1 + \underline{\Omega}_1\times\underline{u}_2)) + (\underline{\nabla}_L\times\underline{\nabla}\times(\underline{\Omega}_1\times\underline{u}_1)) + (\underline{\nabla}\times\underline{\nabla}_L\times(\underline{\Omega}_1\times\underline{u}_1))\right.\right.$$

$$\left.\left. + (\underline{\nabla}\times\underline{\nabla}\times(\underline{\Omega}_{1L}\times\underline{u}_1))\right)\right] - 2\frac{\partial^2}{\partial x\partial\bar{x}}\left[\hat{\lambda}\cdot(\underline{\nabla}\times\underline{\nabla}\times(\underline{\Omega}_1\times\underline{u}_1))\right] \qquad (3.1.58)$$

where

$$\underline{\Omega}_{1L} = \underline{\nabla}_L\times\underline{u}_1\,, \qquad \underline{\nabla}_L = \left(\frac{\partial}{\partial\bar{x}}, 0, 0\right).$$

The boundary conditions from (3.1.15) are

$$w_3 = \frac{\partial^2}{\partial z^2}w_3 = \frac{\partial^4 w_3}{\partial z^4} = 0 \quad z = 0, 1 \quad , \quad \frac{\partial}{\partial y}w_3 = \frac{\partial^3}{\partial y^3}w_3 = \frac{\partial^5 w_3}{\partial y^5} = 0 \quad y = 0, \bar{a}. \qquad (3.1.59)$$

It is known from Daniels (1977) that in the related two-dimensional problem a nonlinear equation for the amplitude function arises at this stage of the expansion due to the appearance of a forcing function which is an eigensolution of the basic linear operator. Thus, expanding (3.1.58) assuming

$$w_3 = e^{ik_0x}\tilde{w}_3(y,z,\bar{x},\bar{t}) + e^{ik_1x}\hat{w}_3(y,z,\bar{x},\bar{t}) + c.c + \ldots \qquad (3.1.60)$$

gives

$$P\left[\left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-k_0^2\right)^3 - R_0\left(\frac{\partial^2}{\partial y^2}-k_0^2\right)\right]\tilde{w}_3 = \tilde{N}_{10}(\bar{x},\bar{t})\sin\pi z + \tilde{\chi} \qquad (3.1.61)$$

and

$$P\left[\left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-k_1^2\right)^3 - R_0\left(\frac{\partial^2}{\partial y^2}-k_1^2\right)\right]\hat{w}_3 = \hat{N}_{11}(\bar{x},\bar{t})\sin\pi z\cos\frac{\pi y}{\bar{a}} + \hat{\chi} \qquad (3.1.62)$$

where

$$\tilde{N}_{10}(\bar{x},\bar{t}) = [\alpha_{47}+\alpha_{48}P]\bar{A}_{\bar{t}} + \alpha_{49}P\bar{A}_{\bar{x}\bar{x}} + \alpha_{50}P\bar{A} + \alpha_{51}P\bar{A}|\bar{A}|^2 + [\alpha_{52}+\alpha_{53}P+\alpha_{54}/P]\bar{A}|\bar{B}|^2\,,$$

$$\hat{N}_{11}(\bar{x},\bar{t}) = [\alpha_{55}+\alpha_{56}P]\bar{B}_{\bar{t}} + \alpha_{57}P\bar{B}_{\bar{x}\bar{x}} + \alpha_{58}P\bar{B} + \alpha_{59}P\bar{B}|\bar{B}|^2 + [\alpha_{60}+\alpha_{61}P+\alpha_{62}/P]\bar{B}|\bar{A}|^2$$

$$+ [\alpha_{63}+\alpha_{64}P+\alpha_{65}/P]\bar{B}|\bar{B}|^2\,,$$

$$\tilde{\chi} = \sum_{\substack{m=0\\(n,m)\neq(1,0)}}^{2}\sum_{n=1}^{3}\tilde{N}_{nm}(\bar{x},\bar{t})\sin n\pi z\cos\frac{m\pi y}{\bar{a}}\,, \qquad \hat{\chi} = \sum_{\substack{m=1\\(n,m)\neq(1,1)}}^{3}\sum_{n=1}^{3}\hat{N}_{nm}(\bar{x},\bar{t})\sin n\pi z\cos\frac{m\pi y}{\bar{a}}\,,$$

(3.1.63)

and α_i i=47..65 are complicated real coefficients that are known. The unknown functions $\tilde{N}_{n,m}$ and $\hat{N}_{n,m}$ do not need to be determined explicitly (see below). It should be noted that other terms with x dependencies of the form $1, e^{\pm 2ik_0x}, e^{\pm 2ik_1x}, e^{\pm 3ik_1x}, e^{\pm i(k_0\pm k_1)x}, e^{\pm i(2k_0\pm k_1)x}$ and $e^{\pm i(2k_1\pm k_0)x}$ also exist in w_3. However, these play no part in obtaining the amplitude equations. Applying the conditions (3.1.59) to (3.1.60) gives

$$\frac{\partial^n\tilde{w}_3}{\partial z^n} = \frac{\partial^n\hat{w}_3}{\partial z^n} = 0 \quad n=0,2,4 \quad z=0,1, \quad \frac{\partial^n\tilde{w}_3}{\partial y^n} = \frac{\partial^n\hat{w}_3}{\partial y^n} = 0 \quad n=1,3,5 \quad Y=0,\bar{a}. \tag{3.1.64}$$

The solvability conditions for equations (3.1.61,62) are

$$P=\int_{Y=0}^{\bar{a}}\int_{z=0}^{1}\left[\left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-k_0^2\right)^3-R_0\left(\frac{\partial^2}{\partial y^2}-k_0^2\right)\right]\tilde{w}_3\sin\pi z\,dY\,dz$$
$$=\int_{Y=0}^{\bar{a}}\int_{z=0}^{1}[\tilde{N}_{10}(\bar{X},\bar{\tau})\sin n\pi z+\tilde{\chi}]\sin\pi z\,dY\,dz=\tfrac{1}{2}\bar{a}\tilde{N}_{10}(\bar{X},\bar{\tau}) \tag{3.1.65}$$

and

$$P\int_{Y=0}^{\bar{a}}\int_{z=0}^{1}\left[\left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-k_1^2\right)^3-R_0\left(\frac{\partial^2}{\partial y^2}-k_1^2\right)\right]\hat{w}_3\sin\pi z\cos\frac{\pi Y}{\bar{a}}\,dY\,dz$$
$$=\int_{Y=0}^{\bar{a}}\int_{z=0}^{1}\left[\hat{N}_{11}(\bar{X},\bar{\tau})\sin\pi z\cos\frac{\pi Y}{\bar{a}}+\hat{\chi}\right]\sin\pi z\cos\frac{\pi Y}{\bar{a}}\,dY\,dz=\tfrac{1}{4}\bar{a}\hat{N}_{11}(\bar{X},\bar{\tau}). \tag{3.1.66}$$

Expanding (3.1.65,66) using (3.1.64) and repeated integration by parts gives

$$P\int_{Y=0}^{\bar{a}}\int_{z=0}^{1}[(-\pi^2-k_0^2)^3+R_0k_0^2]\tilde{w}_3\sin\pi z\,dY\,dz=\tfrac{1}{2}\bar{a}\tilde{N}_{10}(\bar{X},\bar{\tau})$$

and

$$P\int_{Y=0}^{\bar{a}}\int_{z=0}^{1}\left[\left(-\pi^2-\frac{\pi^2}{\bar{a}^2}-k_1^2\right)^3+R_0\left(\frac{\pi^2}{\bar{a}^2}+k_1^2\right)\right]\hat{w}_3\sin\pi z\cos\frac{\pi Y}{\bar{a}}\,dY\,dz=\tfrac{1}{4}\bar{a}\hat{N}_{11}(\bar{X},\bar{\tau}). \tag{3.1.67}$$

Thus, the solvability conditions (3.1.65) and (3.1.66) are

$$\tilde{N}_{10}(\bar{X},\bar{\tau})=\hat{N}_{11}(\bar{X},\bar{\tau})=0. \tag{3.1.68}$$

Using the transformations

$$\bar{A}=\frac{4\sqrt{2}\,\hat{\delta}^{1/2}A}{3\pi^2},\quad \bar{B}=\frac{4\sqrt{2}\,\hat{\delta}^{1/2}B}{3\pi^2},\quad \bar{X}=X/\hat{\delta}^{1/2}\ \text{and}\ \bar{\tau}=(1+P)\tau/4P\hat{\delta} \tag{3.1.69}$$

where

$$\hat{\delta}=1/18\pi^2,$$

the conditions (3.1.68) imply that the scaled amplitude functions A and B, satisfy the equations

$$-A_\tau + A_{xx} + A = A|A|^2 + \tfrac{1}{2}[\beta_1 + \beta_2/P + \beta_3/P^2]\,A|B|^2 \tag{3.1.70}$$

and

$$-B_\tau + (1-2/\bar{a}^2)B_{xx} + B = [\beta_1 + \beta_2/P + \beta_3/P^2]\,B|A|^2 + \tfrac{1}{2}[1+\beta_4+\beta_5/P+\beta_6/P^2]\,B|B|^2. \tag{3.1.71}$$

The amplitude coefficients β_i i=1..6, which are functions of the aspect ratio a (=$\bar{a}$/2) are

$$\beta_1 = [1 - 2\pi^2[(1-D-1/4a^2)\frac{\phi_+^2}{\phi_1} + (1+D-1/4a^2)\frac{\phi_-^2}{\phi_2}]],$$

$$\beta_2 = \frac{-9\pi^6}{2a^2}\left[\frac{(1-D)}{\phi_1} + \frac{(1+D)}{\phi_2}\right], \qquad \beta_3 = \frac{-3\pi^4}{2a^2}\left[\frac{\phi_+(1-D)}{\phi_1} + \frac{\phi_-(1+D)}{\phi_2}\right],$$

$$\beta_4 = -8\pi^2\left[\frac{\pi^2}{\phi_4}(1+1/4a^2)^2D^4 + \frac{1}{4a^4\phi_3}(k_1^2+\pi^2)^2\right],$$

$$\beta_5 = \frac{-4\pi^4}{2a^2}\left[\frac{D^4}{\phi_4} + \frac{k_1^2}{a^2\phi_3}\right] \quad \text{and} \quad \beta_6 = -\frac{6\pi^2}{a^2}\left[\frac{\pi^2}{\phi_4}(1+1/4a^2)D^4 + \frac{k_1^2}{a^2\phi_3}(k_1^2+\pi^2)\right] \tag{3.1.72}$$

where D, $\phi_\pm$ and ϕ_i i=1..4 are given by (B1a) and (B2a) in appendix B. Figures 11 and 12 show profiles of the amplitude coefficients β_i i=1..6 as the aspect ratio is varied from $\bar{a}=\sqrt{2}$ to $\bar{a}=5\sqrt{2}$.

As expected, if the amplitude function B is assumed to be zero, the equations (3.1.70,71) reduce to the single equation

$$-A_\tau + A_{xx} + A = A|A|^2 \tag{3.1.73}$$

equivalent to that used by Daniels (1977).

In the limit

$$a \to \infty \text{ and } P \to \infty \tag{3.1.74}$$

the results agree with those obtained by Newell & Whitehead (1969). The amplitude coefficients (3.1.72) are

$$\beta_1 = 2, \quad \beta_2 = \beta_3 = 0, \quad \beta_4 = 1/2, \quad \beta_5 = \beta_6 = 0 \tag{3.1.75}$$

and the modified neutral solution for the velocity component w_1 is

$$w_1 = 2\sqrt{2}\,\hat{\delta}^{\frac{1}{2}}[A(X,\tau)e^{ik_0x} + B(X,\tau)e^{ik_0x}\cos\frac{\pi Y}{\bar{a}} + c.c]\sin\pi z \quad (k_0 = \frac{\pi}{\sqrt{2}}) \tag{3.1.76}$$

where, from (3.1.70,71), the amplitude functions A and B satisfy the equations

$$-A_\tau + A_{xx} + A = A|A|^2 + A|B|^2 \tag{3.1.77}$$

and

$$-B_\tau + B_{xx} + B = \frac{3}{4} B|B|^2 + 2B|A|^2 . \tag{3.1.78}$$

In the Newell & Whitehead (1969) analysis the equivalent vertical velocity can be expressed as

$$\hat{w}_1 = 2\sqrt{2}\,\hat{\delta}^{\frac{1}{2}} [A(x,\tau)e^{ik_0x} + \hat{C}(x,\tau)e^{i(k_0x+\pi y/\bar{a})} + \hat{E}(x,\tau)e^{i(k_0x-\pi y/\bar{a})} + c.c]\sin\pi z \tag{3.1.79}$$

where in the limit (3.1.74), the arbitrary amplitude functions A,$\hat{C}$ and $\hat{E}$ satisfy the equations

$$-A_\tau + A_{xx} + A = A|A|^2 + 2A|\hat{C}|^2 + 2A|\hat{E}|^2 , \tag{3.1.80}$$

$$-\hat{C}_\tau + \hat{C}_{xx} + \hat{C} = \hat{C}|\hat{C}|^2 + 2\hat{C}|A|^2 + 2\hat{C}|\hat{E}|^2 , \tag{3.1.81}$$

and

$$-\hat{E}_\tau + \hat{E}_{xx} + \hat{E} = \hat{E}|\hat{E}|^2 + 2\hat{E}|A|^2 + 2\hat{E}|\hat{C}|^2 . \tag{3.1.82}$$

Considering the case

$$\hat{E}(x,\tau) \rightarrow \hat{C}(x,\tau) \quad \text{and} \quad 2\hat{C}(x,\tau) \rightarrow B(x,\tau) \tag{3.1.83}$$

gives

$$\hat{w}_1 = 2\sqrt{2}\,\hat{\delta}^{\frac{1}{2}} [A(x,\tau)e^{ik_0x} + B(x,\tau)e^{ik_0x}\cos\frac{\pi y}{\bar{a}} + c.c]\sin\pi z \equiv w_1 , \tag{3.1.84}$$

while the three equations (3.1.80,81,82) reduce identically to the equations (3.1.77,78).

3.2 The 'finite-roll' approximation

At the onset of thermal instability in the stress-free channel the number of y-dependent modes that appear is determined by the aspect ratio and is always equal to or greater than one. However, in the 'finite-roll' approximation only one y-mode appears, whatever the aspect ratio, thereby providing a better model of the situation likely to be relevant in a real channel with rigid sidewalls. From (2.6.3) and (2.6.10) the modified neutral solutions are

$$\theta_1 = (\bar{A}(\bar{X},\bar{\tau})e^{ik_1x} + \bar{A}^*(\bar{X},\bar{\tau})e^{-ik_1x}) \sin\pi z \sin\frac{\pi y}{\bar{a}}$$

$$w_1 = \sigma_1 (\bar{A}e^{ik_1x} + \bar{A}^*e^{-ik_1x}) \sin\pi z \sin\frac{\pi y}{\bar{a}}$$

$$u_1 = \frac{\sigma_1\pi i}{k_1} (\bar{A}e^{ik_1x} - \bar{A}^*e^{-ik_1x}) \cos\pi z \sin\frac{\pi y}{\bar{a}}$$

$$v_1 = 0$$

and

$$P_1 = \frac{-\sigma_1^2\pi P}{k_1^2} (\bar{A}e^{ik_1x} + \bar{A}^*e^{-ik_1x}) \cos\pi z \sin\frac{\pi y}{\bar{a}} \qquad (3.2.1)$$

where

$$k_1 = \tfrac{1}{2}[\beta_{11} - \pi^2]^{\frac{1}{2}}, \quad \sigma_1 = [k_1^2 + \pi^2 + \pi^2/\bar{a}^2], \quad \beta_{11} = [9\pi^4 + 8\pi^4/\bar{a}^2]^{\frac{1}{2}} \qquad (3.2.2)$$

and * denotes complex conjugate. $\bar{A}$ and k_1 are the amplitude function and wavenumber associated with the most unstable mode. The slow spatial and time variables $\bar{X}$ and $\bar{\tau}$ are related to x and t as given by (3.1.4).

Since the 'finite-roll' approximation assumes that the velocity component in the y direction, v, is zero and that the y momentum equation is neglected, the following simplified form of the single differential equation for w derived in the previous section is obtained:

$$[L_1L_2\nabla_1^2 - PR\frac{\partial^2}{\partial x^2}]w = -PR\frac{\partial^2}{\partial x^2}(\underline{u}\cdot\underline{\nabla}_1)\theta + L_2\frac{\partial}{\partial x}[\frac{\partial}{\partial z}(\underline{u}\cdot\underline{\nabla}_1)u - \frac{\partial}{\partial x}(\underline{u}\cdot\underline{\nabla}_1)w] \qquad (3.2.3)$$

where the operators L_1, L_2 and ∇_1^2 are those defined in chapter 2. This equation is sixth order order in x and z and fourth order in y. The boundary conditions on the upper and lower surfaces and on the rigid sidewalls are

$$\theta = w = \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0 \quad z = 0,1 \quad \text{and} \quad \theta = w = u = v = 0 \quad y = 0, \bar{a} \tag{3.2.4}$$

respectively, which from equations (2.1.16,17,19,20) infer that

$$w = \frac{\partial^2 w}{\partial z^2} = \frac{\partial^4 w}{\partial z^4} = 0 \quad z = 0,1 \quad \text{and} \quad w = \frac{\partial^2 w}{\partial y^2} = 0 \quad y = 0, \bar{a}. \tag{3.2.5}$$

As in the previous section, by expanding the dependent variables as a power series in ε given by (3.1.11) and using the transformations (3.0.1) and (3.1.4,5), equation (3.2.3) becomes

$$\begin{aligned}
&(\tilde{L}_1 + \varepsilon \tilde{L}_2 + \varepsilon^2 \tilde{L}_3 + \cdots)(w_1 + \varepsilon w_2 + \varepsilon^2 w_3 + \cdots) \\
&= -\varepsilon P(R_0 + \varepsilon^2)\left(\frac{\partial^2}{\partial x^2} + 2\varepsilon \frac{\partial^2}{\partial x \partial \bar{X}} + \varepsilon^2 \frac{\partial^2}{\partial \bar{X}^2}\right) \tilde{L}(\theta) \\
&+ \varepsilon\left[\frac{\partial}{\partial t} + \varepsilon^2 \frac{\partial}{\partial \bar{\tau}} - \left(\nabla^2 + 2\varepsilon \frac{\partial^2}{\partial x \partial \bar{X}} + \varepsilon^2 \frac{\partial^2}{\partial \bar{X}^2}\right)\right]\left[\frac{\partial}{\partial x} + \varepsilon \frac{\partial}{\partial \bar{X}}\right]\left[\frac{\partial}{\partial z} \tilde{L}(u) - \left(\frac{\partial}{\partial x} + \varepsilon \frac{\partial}{\partial \bar{X}}\right) \tilde{L}(w)\right]
\end{aligned} \tag{3.2.6}$$

where

$$\tilde{L}(f) = \left[(\underline{u}_1 \cdot \underline{\tilde{\nabla}}_1) f_1 + \varepsilon \left((\underline{u}_1 \cdot \underline{\tilde{\nabla}}_1) f_2 + (\underline{u}_2 \cdot \underline{\tilde{\nabla}}_1) f_1 \right) + O(\varepsilon^2) \right],$$

$$\underline{\tilde{\nabla}}_1 = \left[\frac{\partial}{\partial x} + \varepsilon \frac{\partial}{\partial \bar{X}}, 0, \frac{\partial}{\partial z} \right]$$

and

$$\tilde{L}_i = \bar{L}_i\left(L_1, L_2, \nabla_1^2, \frac{\partial^2}{\partial x^2}\right) \quad \text{where} \quad \bar{L}_i = \bar{L}_i(L_1, L_2, \nabla^2, \nabla_H^2) \quad i = 1,2,3. \tag{3.2.7}$$

The conditions (3.2.5) become

$$\frac{\partial^n}{\partial z^n}[w_1 + \varepsilon w_2 + \varepsilon^2 w_3 + \cdots] = 0 \quad n = 0,2,4 \quad z = 0,1$$

and (3.2.8)

$$\frac{\partial^n}{\partial y^n}[w_1 + \varepsilon w_2 + \varepsilon^2 w_3 + \cdots] = 0 \quad n = 0,2 \quad y = 0, \bar{a}.$$

The Boussinesq equations (2.1.16,17,19,20) are now given by the equations (3.1.16,17,19,20) in the previous section.

As the horizontal velocity component v is assumed to be zero, the procedure by which the dependent variables are determined is simpler than in the case of the stress-free channel. Once the vertical velocity component w has been determined from (3.2.6) subject to the conditions (3.2.8), the temperature θ is found from the heat equation (3.1.20) and the horizontal velocity component u is obtained from the continuity equation (3.1.16).

Equating ascending powers of ϵ gives:

(1) Expansion at $O(\epsilon^0)$

The linear balance

$$\tilde{L}_1 w_1 = 0 , \tag{3.2.9}$$

with the conditions

$$w_1 = \frac{\partial^2 w_1}{\partial z^2} = \frac{\partial^4 w_1}{\partial z^4} = 0 \quad z = 0, 1 \quad \text{and} \quad w_1 = \frac{\partial^2 w_1}{\partial y^2} = 0 \quad y = 0, \bar{a} \tag{3.2.10}$$

yields the neutral solution set (3.2.1) above.

(2) Expansion at $O(\epsilon)$

At order ϵ, equation (3.2.6) becomes

$$\tilde{L}_1 w_2 = -\tilde{L}_2 w_1 - PR_0 \frac{\partial^2}{\partial x^2}(\underline{u}_1 \cdot \underline{\nabla}_1)\theta_1 + L_2 \frac{\partial}{\partial x}\left[\frac{\partial}{\partial z}(\underline{u}_1 \cdot \underline{\nabla}_1)u_1 - \frac{\partial}{\partial x}(\underline{u}_1 \cdot \underline{\nabla}_1)w_1\right] \tag{3.2.11}$$

and the boundary conditions from (3.2.8) are

$$w_2 = \frac{\partial^2 w_2}{\partial z^2} = \frac{\partial^4 w_2}{\partial z^4} = 0 \quad z = 0, 1 \quad , \quad w_2 = \frac{\partial^2 w_2}{\partial y^2} = 0 \quad y = 0, \bar{a} . \tag{3.2.12}$$

Expanding the right hand side of (3.2.11) gives

$$\tilde{L}_1 w_2 = -\tilde{L}_2 w_1$$

$$= -2P \frac{\partial^2}{\partial x \partial X}\left[\nabla^4 + 2\nabla^2 \nabla_1^2 - R_0\right] w_1 . \tag{3.2.14}$$

The right hand side of (3.2.14) is zero since the expansion is centred about the critical Rayleigh number R_0. Thus

$$w_2 = \sigma_1 \left(\bar{C}(\bar{X},\bar{t})e^{ik_1x} + \bar{C}^* e^{-ik_1x} \right) \sin\pi z \sin\frac{\pi Y}{\bar{a}} \tag{3.2.15}$$

where $\bar{C}$ is an arbitrary amplitude function which could be taken as zero for the purpose of finding the amplitude equation for $\bar{A}$. Expanding the heat equation (3.1.24) gives

$$\left(\frac{\partial}{\partial t} - \nabla^2\right)\theta_2 = -\sigma_1\pi \bar{A}\bar{A}^*\left[1 - \cos\frac{2\pi Y}{\bar{a}}\right]\sin 2\pi z + \left\{2ik_1\left(\bar{A}_{\bar{X}}e^{ik_1x} - c.c\right) + \sigma_1\left(\bar{C}e^{ik_1x} + c.c\right)\right\}\sin\pi z \sin\frac{2\pi Y}{\bar{a}} \tag{3.2.16}$$

which implies

$$\theta_2 = \frac{\sigma_1}{4\pi}\bar{A}\bar{A}^*\left[-1 + \frac{\bar{a}^2}{(1+\bar{a}^2)}\cos\frac{2\pi Y}{\bar{a}}\right]\sin 2\pi z + \left[\frac{2ik_1}{\sigma_1}\left(\bar{A}_{\bar{X}}e^{ik_1x} - c.c\right) + \left(\bar{C}e^{ik_1x} + c.c\right)\right]\sin\pi z \sin\frac{\pi Y}{\bar{a}} + G \tag{3.2.17}$$

where

$$G = \left[\sigma_1 \bar{A}\bar{A}^* \sin 2\pi z \cosh 2\pi(Y-a)\right] \Big/ \left[4\pi(1+\bar{a}^2)\cosh 2\pi a\right] \tag{3.2.18}$$

is the complementary solution of (3.2.16) chosen such that θ_2 vanishes on the rigid boundaries at $Y = 0,\bar{a}$. It should be noted that the corresponding complementary solution for θ_2 in the case of the stress-free channel was shown to be zero. Since G is independent of x, it does not affect the solution for u_2, which from the continuity equation (3.1.37) is

$$u_2 = \frac{\sigma_1\pi}{k_1}\left[-\frac{1}{k_1}\left(\bar{A}_{\bar{X}}e^{ik_1x} + c.c\right) + i\left(\bar{C}e^{ik_1x} - \bar{C}^*e^{-ik_1x}\right)\right]\cos\pi z \sin\frac{\pi Y}{\bar{a}} \, . \tag{3.2.19}$$

As v_2 is assumed to be zero, all the dependent variables that are needed at the next stage of the expansion have thus been determined.

<u>(3) Expansion at $O(\varepsilon^2)$</u>

At order ε^2, the equation (3.2.6) becomes

$$\begin{aligned}\tilde{L}_1 w_3 = &-\tilde{L}_2 w_2 - \tilde{L}_3 w_1 + \left[L_2\frac{\partial}{\partial \bar{X}} - 2\frac{\partial^3}{\partial x^2 \partial \bar{X}}\right]\left[\frac{\partial}{\partial z}(\underline{u}_1\cdot\underline{\nabla}_1)u_1 - \frac{\partial}{\partial x}(\underline{u}_1\cdot\underline{\nabla}_1)w_1\right] \\ &+ L_2\frac{\partial}{\partial x}\left[\frac{\partial}{\partial z}\left[(\underline{u}_1\cdot\underline{\nabla}_1)u_2 + (\underline{u}_2\cdot\underline{\nabla}_1)u_1\right] - \frac{\partial}{\partial x}\left[(\underline{u}_1\cdot\underline{\nabla}_1)w_2 + (\underline{u}_2\cdot\underline{\nabla}_1)w_1\right] + \frac{\partial}{\partial z}\left[(\underline{u}_1\cdot\underline{\nabla}_L)u_1\right] - \frac{\partial}{\partial x}\left[(\underline{u}_1\cdot\underline{\nabla}_L)w_1 - \frac{\partial}{\partial \bar{X}}(\underline{u}_1\cdot\underline{\nabla})w_1\right]\right] \\ &- PR_0\frac{\partial^2}{\partial x^2}\left[(\underline{u}_1\cdot\underline{\nabla}_1)\theta_2 + (\underline{u}_2\cdot\underline{\nabla}_1)\theta_1 + (\underline{u}_1\cdot\underline{\nabla}_L)\theta_1\right] - 2PR_0\frac{\partial^2}{\partial x \partial \bar{X}}(\underline{u}_1\cdot\underline{\nabla}_1)\theta_1\end{aligned} \tag{3.2.20}$$

where $\underline{\nabla}_L$ is defined by (3.1.58). The boundary conditions from (3.2.8) are

$$w_3 = \frac{\partial^2 w_3}{\partial z^2} = \frac{\partial^4 w_3}{\partial z^4} = 0 \quad z = 0, 1 \quad , \qquad w_3 = \frac{\partial^2}{\partial y^2} w_3 = 0 \quad y = 0, \bar{a}. \tag{3.2.21}$$

Following the procedure outlined in the previous section for the stress-free channel, expanding equation (3.2.20) assuming

$$w_3 = e^{ik_1 x} \hat{w}_3(y, z, \bar{X}, \bar{\tau}) + c.c + \ldots \tag{3.2.22}$$

gives

$$P[(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - k_1^2)^2(\frac{\partial^2}{\partial z^2} - k_1^2) + R_0 k_1^2]\hat{w}_3 = [\hat{N}_1 + \bar{N}_1 \cosh 2\pi(y-a)] \sin\pi z \sin\frac{\pi y}{a} + \chi \tag{3.2.23}$$

where

$$\hat{N}_1(\bar{X}, \bar{\tau}) = \bar{A}\{\sigma_1[-(1+P)\sigma_1(k_1^2+\pi^2)\frac{\partial}{\partial\bar{\tau}} + 4P(k_1^2+\pi^2+2\sigma_1)k_1^2\frac{\partial^2}{\partial\bar{X}^2} + Pk_1^2]$$

$$+ [\frac{PR_0}{4}\sigma_1^2 k_1^2(1+\bar{a}^2/2(1+\bar{a}^2))]\bar{A}\bar{A}^*\},$$

$$\bar{N}_1(\bar{X}, \bar{\tau}) = -PR_0\sigma_1^2 k_1^2 \bar{A}^2\bar{A}^*/[4(1+\bar{a}^2)\cosh 2\pi a]$$

and

$$\chi = \sum_{\substack{\hat{n}=1 \\ (\hat{n},\bar{n})\neq(1,1)}}^{3}\sum_{\bar{n}=1}^{3} \hat{N}_{\bar{n}}(\bar{X},\bar{\tau})\sin\bar{n}\pi z \sin\frac{\hat{n}\pi y}{\bar{a}} + \bar{N}_3(\bar{X},\bar{\tau})\sin 3\pi z \sin\frac{\pi y}{\bar{a}}\cosh 2\pi(y-a), \tag{3.2.24}$$

and the unknown functions $\hat{N}_{\bar{n}}$ and $\bar{N}_3$ do not need to be determined explicitly. It should be noted that other terms with x dependencies of the form 1 and $e^{\pm 2ik_1 x}$ also exist in w_3. However, these play no part in obtaining the amplitude equation.

The solvability condition for equation (3.2.23) demands

$$\hat{N}_1(\bar{X}, \bar{\tau}) + \frac{2\beta_1}{\bar{a}}\bar{N}_1(\bar{X}, \bar{\tau}) = 0 \tag{3.2.25}$$

where

$$\beta_1 = \bar{\beta}_1 \cosh 2\pi a, \qquad \bar{\beta}_1 = \tanh\pi\bar{a}/2\pi(1+\bar{a}^2). \tag{3.2.26}$$

Using the transformations

$$\bar{A} = \frac{4}{\sigma_1}\beta_2^{1/2}\beta_4^{1/2} A, \qquad \bar{X} = \beta_3^{1/2} X/\beta_4^{1/2} \quad \text{and} \quad \bar{\tau} = (1+P)\tau/4P\beta_4^{1/2} \tag{3.2.27}$$

where

$$\beta_2 = a(1+4a^2)^2/[a(1+4a^2)(1+6a^2) - \bar{\beta}_1], \quad \beta_3 = (\beta_{11}-\pi^2)(3\beta_{11}+9\pi^2+2\pi^2/a^2)/[(\beta_{11}+3\pi^2)(\beta_{11}+3\pi^2+\pi^2/a^2)],$$

$$\beta_4 = (\beta_{11} - \pi^2) \big/ [(\beta_{11} + 3\pi^2)(\beta_{11} + 3\pi^2 + \pi^2/a^2)] \qquad (3.2.28)$$

and β_{11} and σ_1 are given by (3.2.2), the condition (3.2.25) implies that the scaled amplitude function satisfies the equation

$$-A_\tau + A_{xx} + A = A|A|^2. \qquad (3.2.29)$$

Figure 14 shows plots of β_2, β_5 and β_4 as the aspect ratio is varied.

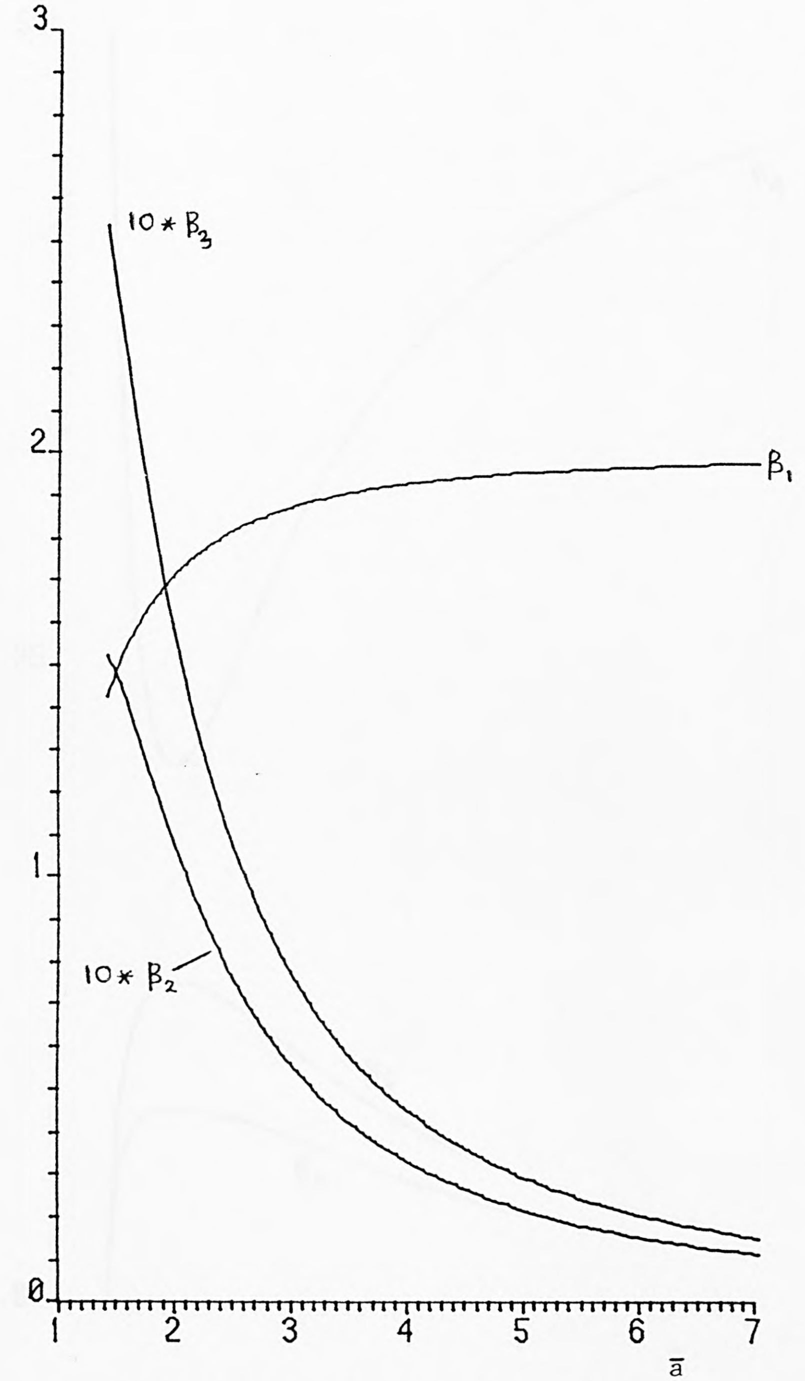


Figure 11. The amplitude coefficients $\beta_i(i)$ i=1,2,3 for the stress-free infinite channel.

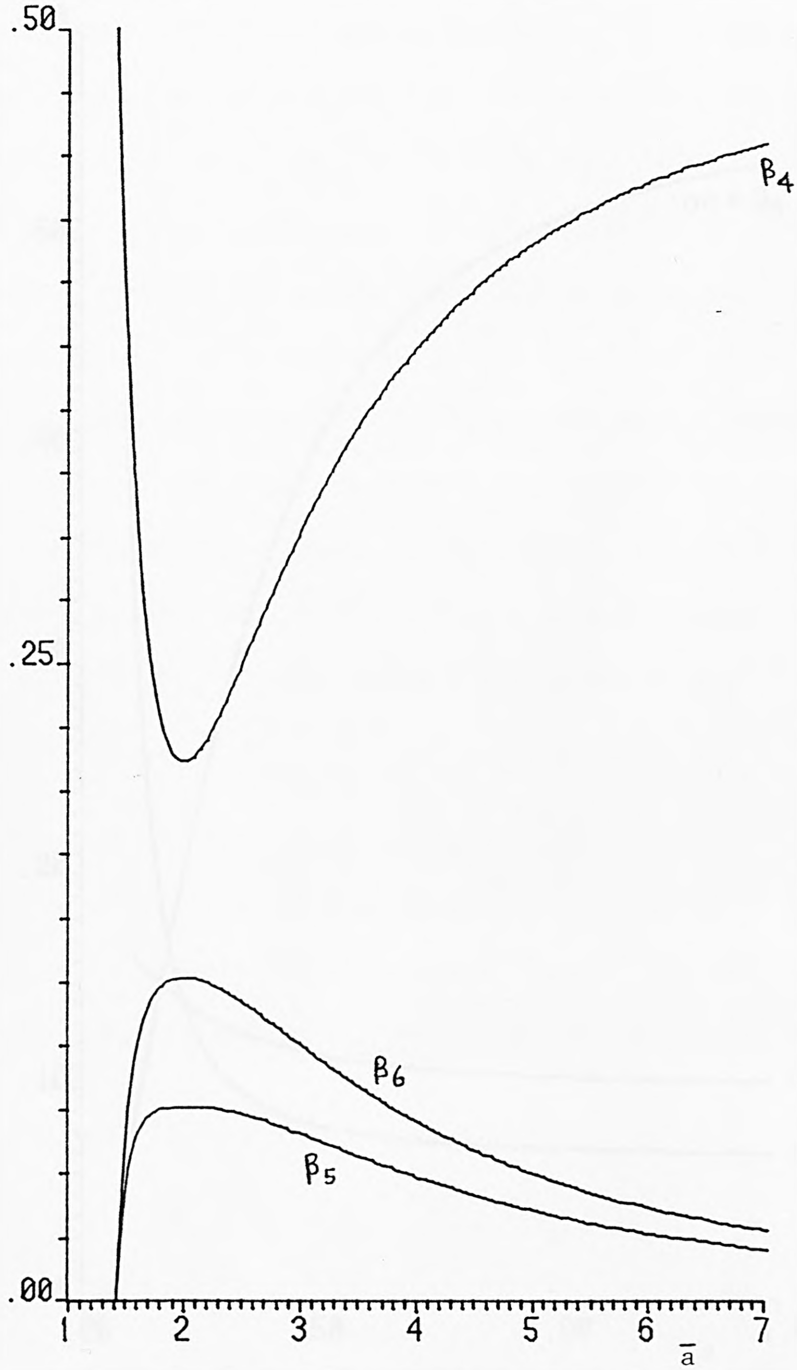


Figure 12. The amplitude coefficients β_i(i) i=4,5,6 for the stress-free infinite channel.

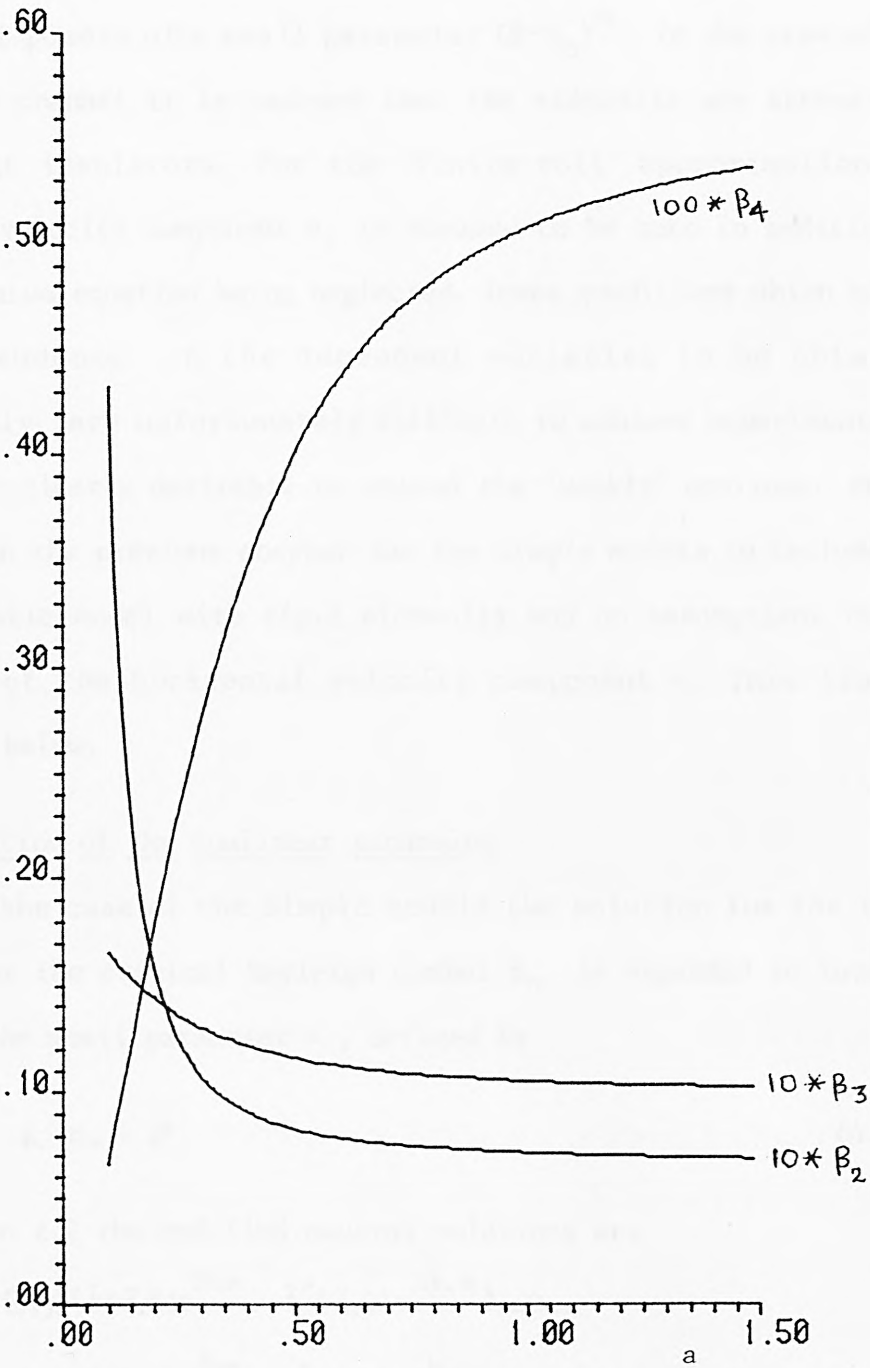


Figure 14. The coefficients β_i(i) i=2,3,4 for the 'finite-roll' approximation.

Chapter 4 The infinite channel: nonlinear theory for rigid sidewalls

4.0 Introduction

In the previous chapter the solutions for the simplified models in the neighbourhood of the critical Rayleigh number R_o, are expanded in integral powers of a small parameter $(R-R_o)^{1/2}$. In the case of the stress-free channel it is assumed that the sidewalls are stress-free and perfect insulators. For the 'finite-roll' approximation the horizontal velocity component v, is assumed to be zero in addition to the y momentum equation being neglected. These conditions which enable the y-dependence in the dependent variables to be obtained analytically are unfortunately difficult to achieve experimentally. Thus, it is clearly desirable to extend the 'weakly' nonlinear theory discussed in the previous chapter for the simple models to include the more realistic model with rigid sidewalls and no assumptions on the behaviour of the horizontal velocity component v. This task is undertaken below.

4.1 Formulation of the nonlinear expansion

As in the case of the simple models the solution for the rigid channel near the critical Rayleigh number R_o, is expanded in integral powers of the small parameter ε, defined by

$$R - R_c = \varepsilon^2 . \qquad (4.1.1)$$

From section 2.2 the modified neutral solutions are

$$\theta_1 = \Theta(y)\left(\bar{A}(\bar{x},\bar{\tau})e^{ik_c x} + \bar{A}^*(\bar{x},\bar{\tau})e^{-ik_c x}\right)\sin\pi z$$

$$w_1 = W(y)\left(\bar{A}(\bar{x},\bar{\tau})e^{ik_o x} + \bar{A}^*(\bar{x},\bar{\tau})e^{-ik_c x}\right)\sin\pi z$$

$$v_1 = V(y)\left(\bar{A}(\bar{x},\bar{\tau})e^{ik_o x} + \bar{A}^*(\bar{x},\bar{\tau})e^{-ik_c x}\right)\cos\pi z$$

and

$$u_1 = \frac{i}{k_o} U(y)\left(\bar{A}(\bar{x},\bar{\tau})e^{ik_o x} - \bar{A}^*(\bar{x},\bar{\tau})e^{-ik_o x}\right)\cos\pi z$$

$$p_1 = P(y)(\bar{A}(\bar{X},\bar{\tau})e^{ik_0x} + \bar{A}^*(\bar{X},\bar{\tau})e^{-ik_0x})\cos\pi z \tag{4.1.2}$$

where * denotes complex conjugate. $\bar{A}$ is the amplitude function and k_0 is the critical wavenumber associated with the critical Rayleigh number R_0, for a given value of the aspect ratio a. The slow spatial and time variables $\bar{X}$ and $\bar{\tau}$ are related to x and t as given by (3.1.4). The y-dependent functions in the neutral solution set are

$$\Theta(y) = \sum_{j=1}^{3} d_j \cosh r_j y \,, \qquad V(y) = \sum_{j=1}^{3} d_j \beta_j \sinh r_j y + d_4 \sinh \alpha y \,,$$

$$W(y) = (\alpha^2 - \frac{d^2}{dy^2})\Theta(y), \qquad U(y) = \frac{d}{dy}V(y) + \pi W(y) \,,$$

$$P(y) = \frac{P}{k_c^2}(\frac{d^2}{dy^2} - \alpha^2) U(y) \tag{4.1.3}$$

where

$$\alpha^2 = k_c^2 + \pi^2 \,, \qquad \beta_j = [r_j^5 - 2\alpha^2 r_j^3 + (\alpha^4 - R_0) r_j] / \pi (r_j^2 - \alpha^2) \qquad j = 1, 2, 3 \,,$$

and r_j j=1..3 are the roots of the bi-cubic characteristic equation (2.2.12). As shown in section 2.2 the arbitrary constants d_j j=1..4 are fixed by the boundary conditions on the rigid sidewalls which are

$$\theta = w = u = v = 0 \qquad y = \pm a \,. \tag{4.1.4}$$

From the heat equation (2.1.20) and the continuity equation (2.1.16) these are equivalent to

$$\theta = \frac{\partial^2 \theta}{\partial y^2} = \frac{\partial v}{\partial y} = v = 0 \qquad y = \pm a \,. \tag{4.1.5}$$

It is noted that in contrast to the simple models the boundary conditions at the sidewalls cannot all be expressed in term of a single variable. Unlike the linear theory where the rigid sidewalls were either perfect conductors or perfect insulators, here only the case where the sidewalls are perfect conductors is considered. The conditions on the stress-free upper and lower boundaries are given by (2.2.1).

As in section 3.1 by expanding the dependent variables as a power series in ε given by (3.1.11) and using the transformations (4.1.1) and (3.1.4,5) the Boussinesq equations (2.1.16)-(2.1.20) become the equations (3.1.16)-(3.1.20). The conditions (4.1.5) and (2.2.1) become

$$[\theta_1+\varepsilon\theta_2+\varepsilon^2\theta_3+\cdots] = \frac{\partial^2}{\partial y^2}[\theta_1+\varepsilon\theta_2+\varepsilon^2\theta_3+\cdots] = \frac{\partial}{\partial y}[v_1+\varepsilon v_2+\varepsilon^2 v_3+\cdots] = [v_1+\varepsilon v_2+\varepsilon^2 v_3+\cdots] = 0 \quad y=\pm a$$

and (4.1.6)

$$[\theta_1+\varepsilon\theta_2+\varepsilon^2\theta_3+\cdots] = [w_1+\varepsilon w_2+\varepsilon^2 w_3+\cdots] = \frac{\partial}{\partial z}[u_1+\varepsilon u_2+\varepsilon^2 u_3+\cdots] = \frac{\partial}{\partial z}[v_1+\varepsilon v_2+\varepsilon^2 v_3+\cdots] = 0 \quad z=0,1.$$

The procedure by which the dependent variables are obtained is as follows: by eliminating the variables in equations (3.1.16)-(3.1.20) a sixth order equation for the temperature θ may be obtained. However, since not all the conditions on the sidewalls are in terms of θ, a second order equation relating $\frac{\partial v}{\partial z}$ and θ is also required. This enables θ and v to be determined subject to the conditions (4.1.6). Having obtained θ and v the vertical velocity component w may be found from the heat equation (3.1.20). This enables the pressure p, to be determined from equation (3.1.19), which in turn allows the horizontal velocity component u to be obtained from (3.1.17). Once all three velocity components have been obtained a consistency check is provided by the continuity equation (3.1.16). Any z-independent parts of the solutions for u,v and p have to be determined seperately.

Ascending powers of ε in the equations (3.1.16-20) and (4.1.6) are now considered.

4.2 Expansion at $O(\varepsilon^\circ)$

At order ε°, the Boussinesq equations (3.1.16-20) reduce to the linearized equations (2.1.24-28), which with the conditions

$$\theta_1 = \frac{\partial^2}{\partial y^2}\theta = v_1 = \frac{\partial}{\partial y}v_1 = 0 \quad y=\pm a \quad \text{and} \quad \theta_1 = w_1 = \frac{\partial v_1}{\partial z} = \frac{\partial u_1}{\partial z} = 0 \quad z=0,1 \qquad (4.2.1)$$

yield the neutral solution set (4.1.2). The y-dependent functions Θ

and V in the temperature θ_1 and horizontal velocity component v_1 satisfy equations which it is more convenient to express in the generalised form

$$\left[\left(\frac{d^2}{dy^2} - \alpha^2\right)^3 - R_0\left(\frac{d^2}{dy^2} - k_0^2\right)\right]\Theta = M$$

and (4.2.2a)

$$\left(\frac{d^2}{dy^2} - \alpha^2\right)V - \frac{1}{\pi}\left[\frac{d^5}{dy^5} - 2\alpha^2\frac{d^3}{dy^3} + (\alpha^4 - R_0)\frac{d}{dy}\right]\Theta = N$$

where, here

$$M = N = 0. \tag{4.4.2b}$$

The boundary conditions are

$$\Theta = \frac{d^2}{dy^2}\Theta = V = \frac{dV}{dy} = 0 \qquad y = \pm a. \tag{4.4.2c}$$

By defining

$$\underline{f}(\Theta, V) = [\Theta^{(1)}, \Theta^{(3)}, \Theta^{(4)}, \Theta^{(5)}, \Theta, \Theta^{(2)}, V, V^{(1)}]^{tr} \tag{4.2.3}$$

where tr denotes transpose and the superscript (i) denotes $\frac{d^i}{dy^i}$, the equations (4.2.2a) and the conditions (4.2.2c) can be more conveniently expressed in matrix notation as

$$\left[\frac{d}{dy} - \underline{E}(k_0, \pi)\right]\underline{f}(\Theta, V) = \underline{Q}(M, N)$$

and (4.2.4)

$$\underline{m}\,\underline{f}(\Theta, V) = 0 \qquad y = \pm a$$

where

$$\underline{E}(b,c) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3\hat{\alpha}^2 & 0 & (\hat{\alpha}^6 - k_0^2 R_0), & (R_0 - 3\hat{\alpha}^4) & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{(\hat{\alpha}^4 - R_0)}{c} & \frac{-2\hat{\alpha}^2}{c} & 0 & \frac{1}{c} & 0 & 0 & \hat{\alpha}^2 & 0 \end{bmatrix},$$

$$\hat{\alpha}^2 = b^2 + c^2 ,$$

$$\underline{\varphi}(M,N) = [\, 0, 0, 0, M, 0, 0, 0, N\,]^{tr}, \tag{4.2.5}$$

and

$$\underline{\underline{m}} = \begin{bmatrix} \underline{0} & \underline{0} \\ \underline{0} & \underline{I}_4 \end{bmatrix} .$$

Here Q=0 and solutions of the system (4.2.5) are given by (4.1.3) above

4.3 Expansion at O(ε)

At order ε , the Boussinesq equations (3.1.16-20) become

$$\frac{\partial u_2}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial w_2}{\partial z} = - \frac{\partial u_1}{\partial \bar{x}} \tag{4.3.1}$$

$$L_1 u_2 + \frac{\partial p_2}{\partial x} = - \frac{\partial p_1}{\partial \bar{x}} - \left(-2P \frac{\partial^2}{\partial x \partial \bar{x}} + \underline{u}_1 \cdot \underline{\nabla}\right) u_1 \tag{4.3.2}$$

$$L_1 v_2 + \frac{\partial p_2}{\partial y} = - \left(-2P \frac{\partial^2}{\partial x \partial \bar{x}} + \underline{u}_1 \cdot \underline{\nabla}\right) v_1 \tag{4.3.3}$$

$$L_1 w_2 + \frac{\partial p_2}{\partial z} - PR_c \theta_2 = - \left(-2P \frac{\partial^2}{\partial x \partial \bar{x}} + \underline{u}_1 \cdot \underline{\nabla}\right) w_1 \tag{4.3.4}$$

and

$$L_2 \theta_2 - w_2 = - \left(-2 \frac{\partial^2}{\partial x \partial \bar{x}} + \underline{u}_1 \cdot \underline{\nabla}\right) \theta_1 \tag{4.3.5}$$

where the operators L_1 and L_2 are those defined in section 2.1. The boundary conditions from (4.1.6) are

$$\theta_2 = \frac{\partial^2}{\partial y^2}\theta_2 = v_2 = \frac{\partial v_2}{\partial y} = 0 \quad y = \pm a \quad \text{and} \quad \theta_2 = w_2 = \frac{\partial v_2}{\partial z} = \frac{\partial u_2}{\partial z} = 0 \quad z = 0, 1 . \tag{4.3.6}$$

In sub-section 4.3(1) the basic forms of the dependent variables are constructed; the numerical determination of the various unknown functions of y that occur in the solutions is described in sub-section 4.3(2) below.

4.3(1) The basic forms of the dependent variables

As in the linear theory, by eliminating the variables in equations (4.3.1)-(4.3.5) a single sixth order differential equation for the temperature θ_2 is obtained, namely

$$[\nabla^2\nabla^2\nabla^2 - R_0\nabla_H^2]\theta_2 = -2\frac{\partial^2}{\partial x\partial \bar{x}}(3\nabla^4 - R_0)\theta_1 + \nabla^4(\underline{u}_1\cdot\underline{\nabla})\theta_1 - \frac{1}{P}\nabla_H^2(\underline{u}_1\cdot\underline{\nabla})w_1$$

$$+\frac{1}{P}\frac{\partial}{\partial z}\left[\frac{\partial}{\partial x}(\underline{u}_1\cdot\underline{\nabla})u_1 + \frac{\partial}{\partial y}(\underline{u}_1\cdot\underline{\nabla})v_1\right]. \quad (4.3.7)$$

It should be noted that the $\frac{\partial}{\partial t}$ terms are not relevant and have been omitted. Expanding the right hand side of (4.3.7) gives

$$[\nabla^6 - R_0\nabla_H^2]\theta_2 = iM_1(\bar{A}_{\bar{x}}e^{ik_0x} - c.c)\sin\pi z + \frac{1}{P}M_2(\bar{A}^2e^{2ik_0x} + c.c)\sin 2\pi z + \frac{1}{P}M_3(2\bar{A}\bar{A}^*)\sin 2\pi z \quad (4.3.8)$$

which implies

$$\theta_2 = i\Theta_1(y)(\bar{A}_{\bar{x}}e^{ik_0x} - c.c)\sin\pi z + \frac{1}{P}\Theta_2(y)(\bar{A}^2e^{2ik_0x} + c.c)\sin 2\pi z + \frac{1}{P}\Theta_3(y)(2\bar{A}\bar{A}^*)\sin 2\pi z + \theta_c \quad (4.3.9)$$

where

$$\left[\left(\frac{d^2}{dy^2}-\alpha^2\right)^3 - R_0\left(\frac{d^2}{dy^2}-k_0^2\right)\right]\Theta_1 = M_1 = -2k_0\left[3\left(\frac{d^2}{dy^2}-\alpha^2\right)^2 - R_0\right]\Theta, \quad (4.3.10)$$

$$\left[\left(\frac{d^2}{dy^2}-4\alpha^2\right)^3 - R_0\left(\frac{d^2}{dy^2}-4k_0^2\right)\right]\Theta_2 = M_2 = P\left(\frac{d^2}{dy^2}-4\alpha^2\right)^2\alpha_1(y) - \left(\frac{d^2}{dy^2}-4k_0^2\right)\alpha_3(y) + 4\pi k_0\alpha_5(y) - 2\pi\frac{d}{dy}\alpha_7(y), \quad (4.3.11)$$

$$\left[\left(\frac{d^2}{dy^2}-4\pi^2\right)^3 - R_0\frac{d^2}{dy^2}\right]\Theta_3 = M_3 = P\left(\frac{d^2}{dy^2}-4\pi^2\right)^2\alpha_2(y) - \frac{d^2}{dy^2}\alpha_4(y) - 2\pi\frac{d}{dy}\alpha_9(y), \quad (4.3.12)$$

and $\alpha_i(y)$ i=1..5,7,9 are functions of Θ, W, V and U, given by (C5) in appendix C. θ_c is a possible complementary solution of (4.3.8) which for the purpose of finding the amplitude equation for $\bar{A}$ can be taken to be zero.

From the heat equation (4.3.5) the vertical velocity component w_2 has the form

$$w_2 = iW_1(y)(\bar{A}_{\bar{x}}e^{ik_0x} - c.c)\sin\pi z + W_2(y)(\bar{A}^2e^{2ik_0x} + c.c)\sin 2\pi z + W_3(y)(2\bar{A}\bar{A}^*)\sin 2\pi z \quad (4.3.14)$$

where

$$W_1 = -\left(2k_0\Theta + \left(\frac{d^2}{dy^2}-\alpha^2\right)\Theta_1\right), \quad (4.3.15)$$

$$W_2 = \left(\alpha_1(y) - \frac{1}{P}\left(\frac{d^2}{dy^2}-4\alpha^2\right)\Theta_2\right), \quad (4.3.16)$$

and

$$W_3 = \left(\alpha_2(y) - \frac{1}{P}\left(\frac{d^2}{dy^2}-4\pi^2\right)\Theta_3\right). \quad (4.3.17)$$

The form of the solution for the pressure p_2 is obtained from the w_2 momentum equation (4.3.4). The form of the horizontal velocity components v_2 and u_2 can then be obtained using the horizontal momentum equations (4.3.3) and (4.3.2). Having constructed all three velocity components a consistency check is provided by the continuity equation (4.3.1).

Equation (4.3.4) gives

$$p_2 = iPP_1(y)(\bar{A}_{\bar{X}} e^{ik_0x} - c.c)\cos\pi z + P_2(y)(\bar{A}^2 e^{2ik_0x} + c.c)\cos 2\pi z + P_3(y)(2\bar{A}\bar{A}^*)\cos 2\pi z + F(x,y,\bar{X},\bar{\tau}) \qquad (4.3.18)$$

where

$$P_1 = -\frac{1}{\pi}[2k_0W + (\frac{d^2}{dy^2}-\alpha^2)W_1 + R_0\Theta_1 , \qquad (4.3.19)$$

$$P_2 = -\frac{1}{2\pi}[-\alpha_3(y) + P(\frac{d^2}{dy^2}-4\alpha^2)W_2 + R_0\Theta_2] , \qquad (4.3.20)$$

$$P_3 = -\frac{1}{2\pi}[-\alpha_4(y) + P(\frac{d^2}{dy^2}-4\pi^2)W_3 + R_0\Theta_3] \qquad (4.3.21)$$

and F is an arbitrary function of $x, y, \bar{X}$ and $\bar{\tau}$.

Expansion of the right hand sides of (4.3.3) and (4.3.2) gives

$$v_2 = iV_1(y)(\bar{A}_{\bar{X}} e^{ik_0x} - c.c)\cos\pi z + \frac{V_2(y)}{P}(\bar{A}^2 e^{2ik_0x} + c.c)\cos 2\pi z + \frac{V_3(y)}{P}(2\bar{A}\bar{A}^*)\cos 2\pi z + \tilde{v}_2(x,y,\bar{X},\bar{\tau}) \qquad (4.3.22)$$

and

$$u_2 = U_1(y)(\bar{A}_{\bar{X}} e^{ik_0x} + c.c)\cos\pi z + \frac{i}{P}U_2(y)(\bar{A}^2 e^{2ik_0x} - c.c)\cos 2\pi z + \tilde{u}_2(x,y,\bar{X},\bar{\tau}) \qquad (4.3.23)$$

where

$$[\frac{d^2}{dy^2}-\alpha^2]V_1 = -2k_0V + \frac{d}{dy}P_1 , \qquad (4.3.24)$$

$$[\frac{d^2}{dy^2}-4\alpha^2]V_2 = \alpha_7(y) + \frac{d}{dy}P_2 , \qquad (4.3.25)$$

$$[\frac{d^2}{dy^2}-4\pi^2]V_3 = \alpha_4(y) + \frac{d}{dy}P_3 , \qquad (4.3.26)$$

$$[\frac{d^2}{dy^2}-\alpha^2]U_1 = 2U + \frac{1}{k_0^2}(\frac{d^2}{dy^2}-\alpha^2)U - k_0P_1 , \qquad (4.3.27)$$

$$[\frac{d^2}{dy^2}-4\alpha^2]U_2 = \alpha_5(y) + 2k_0P_2 \qquad (4.3.28)$$

and $\tilde{v}_2$ and $\tilde{u}_2$ satisfy

$$P\nabla_H^2\tilde{v}_2 = \frac{\partial F}{\partial y} + \alpha_8(y)(\bar{A}^2 e^{2ik_0x} + c.c) + \alpha_{10}(y)(2\bar{A}\bar{A}^*) \qquad (4.3.29)$$

and

$$P\nabla_H^2 \tilde{u}_2 = \frac{\partial F}{\partial x} + i\alpha_6(y)(\bar{A}^2 e^{2ik_0 x} - c.c) \tag{4.3.30}$$

where $\alpha_i(y)$ i=6,8,10 are given by (C5) in appendix C. In addition $\tilde{v}_2$ and $\tilde{u}_2$ satisfy the two-dimensional continuity equation

$$\frac{\partial \tilde{u}_2}{\partial x} + \frac{\partial \tilde{v}_2}{\partial y} = 0 . \tag{4.3.31}$$

In order to complete the basic forms of the horizontal velocity components it is necessary to determine $\tilde{v}_2$ and $\tilde{u}_2$. The right hand sides of equations (4.3.29,30) suggest

$$\tilde{v}_2 = \frac{1}{P} V_4(y)(\bar{A}^2 e^{2ik_0 x} + c.c) \tag{4.3.32}$$

$$\tilde{u}_2 = U_3(y,\bar{X},\bar{\tau}) + \frac{i}{P} U_4(y)(\bar{A}^2 e^{2ik_0 x} - c.c) \tag{4.3.33}$$

and

$$F = F_1(y)(\bar{A}^2 e^{2ik_0 x} + c.c) + F_2(y)(2\bar{A}\bar{A}^*) + F_0(\bar{X},\bar{\tau}) . \tag{4.3.34}$$

The boundary conditions on $\tilde{v}_2$ and $\tilde{u}_2$ are

$$\tilde{v}_2 = \tilde{u}_2 = 0 \qquad y = \pm a \tag{4.3.35}$$

which from (4.3.31) are equivalent to

$$\tilde{v}_2 = \frac{\partial}{\partial y}\tilde{v}_2 = 0 \qquad y = \pm a . \tag{4.3.36}$$

Substituting the forms (4.3.32,33,34) into the equations (4.3.29) and (4.3.30) gives

$$\left[\frac{d^4}{dy^4} - 8k_0^2 \frac{d^2}{dy^2} + 16k_0^4\right] V_4 = -2k_0\left[2k_0\alpha_8(y) - \frac{d}{dy}\alpha_6(y)\right] \tag{4.3.37}$$

where V_4 satisfies the conditions

$$V_4 = \frac{d}{dy}V_4 = 0 \qquad y = \pm a . \tag{4.3.38}$$

Also

$$U_4 = \frac{1}{2k_0}\frac{d}{dy}V_4 , \tag{4.3.39}$$

$$F_1 = \frac{1}{2k_c}\left[\left(\frac{d^2}{dy^2} - 4k_c^2\right)U_4 - \alpha_6(y)\right] \tag{4.3.40}$$

$$F_2 = -\int^y \alpha_{10}(y)\,dy \tag{4.3.41}$$

and

$$\frac{d^3}{dy^3} U_3 = 0 \tag{4.3.42}$$

where U_3 satisfies the conditions

$$U_3 = 0 \quad y = \pm a. \tag{4.3.43}$$

The overall mass flux down the channel at this level of approximation is given by

$$\int_{y=-a}^{+a}\int_{z=0}^{1} u_2 \, dy\, dz \, . \tag{4.3.44}$$

From (4.3.23) this is zero only if

$$\int_{y=-a}^{a} U_3 \, dy = 0 \, . \tag{4.3.45}$$

This condition together with (4.3.42) and (4.3.43) imply

$$U_3(y, \bar{x}, \bar{t}) = 0 \, . \tag{4.3.46}$$

The numerical solution for V_4 is considered in sub-section 4.3(2) below.

The basic forms of the solutions for all the dependent variables have thus been obtained. By applying the Laplacian operator ∇^2 to the continuity equation (4.3.1) a consistency check for the z-dependent terms in the three velocity components u_2, v_2 and w_2 is obtained, and is found to be satisfied.

In later chapters the channel will be considered to have rigid endwalls in which case the mass flux (4.3.44) must be zero. However, it should be noted that if the flux condition is relaxed a solution in which

$$U_3 = \frac{1}{2p}\beta(y-a)(y+a) \qquad (\beta \neq 0) \tag{4.3.47}$$

associated with a term of the form $\beta\bar{x}$ in the pressure at the previous order is possible. Solutions of this type which correspond to a

Poiseuille-type flow down the channel are not of interest in the present study.

4.3(2) Determination of y-dependent functions in the basic forms

(a) Θ_1, V_1

From equations (4.3.10,24,19,15) the y-dependent functions Θ_1 and V_1 satisfy the equations

$$\left[\left(\frac{d^2}{dy^2}-\alpha^2\right)^3 - R_0\left(\frac{d^2}{dy^2}-k_0^2\right)\right]\Theta_1 = -2k_0\left[3\left(\frac{d^4}{dy^4}-2\alpha^2\frac{d^2}{dy^2}+\alpha^4\right)-R_0\right]\Theta \tag{4.3.48a}$$

and

$$\left[\frac{d^2}{dy^2}-\alpha^2\right]V_1 - \frac{1}{\pi}\left[\frac{d^5}{dy^5}-2\alpha^2\frac{d^3}{dy^3}+(\alpha^4-R_0)\frac{d}{dy}\right]\Theta_1 = -2k_0\left[V+\frac{2}{\pi}\frac{d}{dy}W\right], \tag{4.3.48b}$$

which must be solved subject to the conditions

$$\Theta_1 = \frac{d^2}{dy^2}\Theta_1 = V_1 = \frac{d}{dy}V_1 = 0 \quad y = \pm a. \tag{4.3.48c}$$

It should be noted that the system (4.3.48) is a non-homogenous form of the basic linear system (4.2.2). A necessary condition for (4.3.48) to have a solution is described later. However, differentiation of the basic linear equations (2.2.8,9) with respect to k gives

$$\left[\left(\frac{\partial^2}{\partial y^2}-\alpha^2\right)^3 - R_0\left(\frac{d^2}{dy^2}-k_0^2\right)\right]\frac{\partial}{\partial k}\Theta = 2k_0\left[3\left(\frac{d^4}{dy^4}-2\alpha^2\frac{d^2}{dy^2}+\alpha^4\right)-R_0\right]\Theta \tag{4.3.49a}$$

and

$$\left[\frac{d^2}{dy^2}-\alpha^2\right]\frac{\partial}{\partial k}V - \frac{1}{\pi}\left[\frac{d^5}{dy^5}-2\alpha^2\frac{d^3}{dy^3}+(\alpha^4-R_0)\frac{d}{dy}\right]\frac{\partial}{\partial k}\Theta = 2k_0\left[V+\frac{2}{\pi}\frac{d}{dy}W\right] \tag{4.3.49b}$$

having set n=1, $k=k_0$ and used the fact that $\frac{\partial R}{\partial k}=0$ at $k=k_0$. The boundary conditions (2.2.4) imply

$$\frac{\partial\Theta}{\partial k} = \frac{\partial^2}{\partial y^2}\left(\frac{\partial}{\partial k}\Theta\right) = \frac{\partial}{\partial k}V = \frac{\partial}{\partial y}\left(\frac{\partial V}{\partial k}\right) = 0 \quad y = \pm a. \tag{4.3.49c}$$

Comparing the systems (4.3.48) and (4.3.49) implies that one solution for Θ_1 and V_1 is

$$\Theta_1 = -\frac{\partial\Theta}{\partial k}\bigg|_{k=k_0} \quad \text{and} \quad V_1 = -\frac{\partial V}{\partial k}\bigg|_{k=k_0}. \tag{4.3.50}$$

Therefore a solution for Θ_1 and V_1 exists although it is not unique since an arbitrary constant multiple of the basic linear solution for Θ and V, may be added to Θ_1 and V_1 respectively. It is known from section 2.2 that Θ is normalised such that

$$\Theta = 1 \quad \text{at} \quad y = 0. \tag{4.3.51}$$

Thus, the solution (4.3.50) for Θ_1 has the property

$$\Theta_1 = 0 \quad \text{at} \quad y = 0. \tag{4.3.52}$$

Unfortunately $\frac{\partial\Theta}{\partial k}$ and $\frac{\partial V}{\partial k}$ are difficult to obtain analytically. Alternative analytical and numerical methods for determining Θ_1 and V_1 are discussed below. However, it should be noted that the condition (4.3.52) is used as the criterion for obtaining a unique solution for Θ_1 and V_1.

An analytical method for determining Θ_1 and V_1 is as follows: solving (4.3.48a) gives

$$\Theta_1^A = -k_0 \sum_{j=1}^{3} \frac{d_j}{r_j}\, y \sinh r_j y + \sum_{j=1}^{3} \tilde{d}_j \cosh r_j y. \tag{4.3.53}$$

Substituting into (4.3.48b) gives

$$\left[\frac{d^2}{dy^2} - \alpha^2\right] V_1^A = -2k_0 V - \frac{4k_0}{\pi}\frac{d}{dy} W + \frac{1}{\pi}\sum_{j=1}^{3} \tilde{d}_j \left(r_j^5 - 2\alpha^2 r_j^3 + (\alpha^4 - R_0) r_j\right) \sinh r_j y - \frac{k_0}{\pi}\sum_{j=1}^{3} \frac{d_j}{r_j}\left\{ (5r_j^4 - 6\alpha^2 r_j^3 + (\alpha^4 - R_0)) \right.$$

$$\left. \sinh r_j y + (r_j^5 - 2\alpha^2 r_j^3 + (\alpha^4 - R_0) r_j)\, y \cosh r_j y \right\}, \tag{4.3.54}$$

which implies

$$V_1^A = -2k_0 \sum_{j=1}^{3} d_j \left[\frac{\beta_j}{(r_j^2 - \alpha^2)} - \frac{2}{\pi} r_j\right] \sinh r_j y - \frac{k_0}{\alpha} d_4\, y \cosh \alpha y + \sum_{j=1}^{3} \tilde{d}_j \beta_j \sinh r_j y + \tilde{d}_4 \sinh \alpha y$$

$$- k_0 \sum_{j=1}^{3} \frac{d_j}{r_j} \left\{ \eta_j \sinh r_j y + \beta_j y \cosh r_j y - \frac{2 r_j \beta_j}{(r_j^2 - \alpha^2)} \sinh r_j y \right\} \tag{4.3.55}$$

where

$$\eta_j = (5r_j^4 - 6\alpha^2 r_j^2 + \alpha^4 - R_0) / \pi (r_j^2 - \alpha^2) \qquad j = 1, 2, 3, \tag{4.3.56}$$

d_j, β_j j=1..3, d_4 are given by (4.1.3) and the arbitrary constants $\tilde{d}_j$ j=1..4 are determined from the boundary conditions (4.3.48c).

Applying the conditions (4.3.48c) to (4.3.53,54) gives four non-homogenous equations in four unknowns which can be expressed in matrix notation as

$$[A][\tilde{d}_1, \tilde{d}_2, \tilde{d}_3, \tilde{d}_4]^{tr} = [S_1(a), S_2(a), S_3(a), S_4(a)]^{tr} \tag{4.3.57a}$$

where A is the 4 x 4 matrix given by (2.2.22) and $S_i(a)$ i=1..4 are given by (C6) in appendix C. As the expansion is centred about the critical Rayleigh number R_o and wavenumber k_o, for a given aspect ratio a, from section 2.2 the determinant of the matrix A is zero. Assuming $\tilde{d}_4$ (say) to be a known arbitrary constant, a non-homogenous system of three equations in the three unknowns $\tilde{d}_j$ j=1..3 is obtained which can be solved using Cramer's rule. The existence of the solutions (4.3.50) is equivalent to the fact that the fourth equation is automatically satisfied. Since the value of the constant $\tilde{d}_4$ is arbitrary, the solutions for $\hat{\Theta}_1$ and $\hat{V}_1$ are not unique. However, applying the additional condition (4.3.52) gives

$$\Theta_1 = \hat{\Theta}_1 - \hat{\Theta}_1(0)\,\Theta \quad \text{and} \quad V_1 = \hat{V}_1 - \hat{\Theta}_1(0)\,V. \tag{4.3.57}$$

A solution of the system (4.3.48) may also be obtained numerically in which case, it is more convenient to express the equations (4.3.48a,48b) and the conditions (4.3.48c) in the matrix notation

$$\left[\frac{d}{dy} - \underline{E}(k_o,\pi)\right]\underline{f}(\Theta_1, V_1) = \underline{Q}(M_1, N_1), \tag{4.3.58}$$

$$\underline{m}\,\underline{f}(\Theta_1, V_1) = 0 \qquad y = \pm a$$

where

$$M_1 = -2k_o\left[3\left(\frac{d^4}{dy^4} - 2\alpha^2\frac{d^2}{dy^2} + \alpha^4\right) - R_o\right]\Theta \quad \text{and} \quad N_1 = -2k_o\left[V + \frac{2}{\pi}\frac{d}{dy}W\right].$$

Following the procedure outlined by Eagles (1980) and given in appendix D, it can be shown that the condition needed for the existence of a solution of (4.3.58) is

$$\int_{y=-a}^{+a} G(y)\,dy = 0$$

where (4.3.59)

$$G(y) = \hat{f}_4 M_1 + \hat{f}_8 N_1 ,$$

and $\hat{f}_4$ & $\hat{f}_8$ are solutions obtained from the matrix system

$$\left[\frac{d}{dy} + \underline{E}^{tr}(k_o, \pi) \right] \hat{\underline{f}} = \underline{0} \qquad (4.3.60)$$

with boundary conditions

$$\hat{\underline{m}}\, \hat{\underline{f}} = 0$$

where

$$\hat{\underline{f}} = [\hat{f}_1, \hat{f}_2, \hat{f}_3, \hat{f}_4, \hat{f}_5, \hat{f}_6, \hat{f}_7, \hat{f}_8]^{tr}$$

and

$$\hat{\underline{m}} = \begin{bmatrix} \underline{I}_4 & \underline{0} \\ \underline{0} & \underline{0} \end{bmatrix} .$$

The matrix system (4.3.60) is the adjoint system corresponding to (4.3.58), the solutions of which are also needed to obtain the amplitude equation for $\bar{A}$ at the next stage of the expansion.

Expanding (4.3.60) gives eight equations in eight unknowns, namely

$$\hat{f}_1' = -\hat{f}_5 - \frac{(\alpha^4 - R_o)}{\pi}\hat{f}_8 , \quad \hat{f}_2' = -\hat{f}_6 + \frac{2\alpha^2}{\pi}\hat{f}_8 , \quad \hat{f}_3' = -\hat{f}_2 - 3\alpha^2\hat{f}_4 , \quad \hat{f}_4' = -\hat{f}_3 - \frac{1}{\pi}\hat{f}_8 , \qquad (4.3.61a,b,c,d)$$

$$\hat{f}_5' = -\eta\hat{f}_4 , \quad \hat{f}_6' = -\hat{f}_1 + (3\alpha^4 - R_o)\hat{f}_4 , \quad \hat{f}_7' = -\alpha^2\hat{f}_8 , \quad \hat{f}_8' = -\hat{f}_7 \qquad (4.3.61e,f,g,h)$$

where

$$\eta = \alpha^6 - k_o^2 R_o$$

and prime denotes $\frac{d}{dy}$. The boundary conditions are

$$\hat{f}_1 = \hat{f}_2 = \hat{f}_3 = \hat{f}_4 = 0 \qquad y = \pm a . \qquad (4.3.62)$$

Eliminating the variables in (4.3.61a-f) gives the sixth order differential equation

$$L_3 \hat{f}_5 = -R_o \eta\, \hat{f}_8 / \pi \qquad (4.3.63)$$

where the operator L_3 is given by (2.2.10) with n=1. Equations

(4.3.61g,h) give

$$\left(\frac{d^2}{dy^2}-\alpha^2\right)\hat{f}_8=0 \tag{4.3.64}$$

which implies

$$\hat{f}_8=\bar{e}_7\sinh\alpha y+\bar{e}_8\cosh\alpha y . \tag{4.3.65}$$

Substituting for $\hat{f}_8$ in (4.3.63) gives

$$\hat{f}_5=\sum_{j=1}^{6}\bar{e}_j e^{r_j y}+\frac{\eta}{\pi^3}(\bar{e}_7\sinh\alpha y+\bar{e}_8\cosh\alpha y) \tag{4.3.66}$$

where $\bar{e}_j$ j=1..8 are arbitrary constants and r_j j=1..6 are the roots of the bi-cubic characteristic equation (2.2.12) which have been shown to be distinct in section 2.2, since the critical Rayleigh number for the infinite channel with rigid sidewalls is always greater than the critical value $27\pi^4/4$ for a layer of infinite width.

As in the linear theory, since the characteristic equation is bi-cubic the general solution (4.3.66) can be written as

$$\hat{f}_5=\hat{f}_5^E+\hat{f}_5^O \tag{4.3.67}$$

where

$$\hat{f}_5^E=\sum_{j=1}^{3}\tilde{e}_j\cosh r_j y+\frac{\eta}{\pi^3}\tilde{e}_4\cosh\alpha y \quad , \quad \hat{f}_5^O=\sum_{j=1}^{3}\hat{e}_j\sinh r_j y+\frac{\eta}{\pi^3}\hat{e}_4\sinh\alpha y . \tag{4.3.68}$$

Accordingly

$$\hat{f}_8=\hat{f}_8^E+\hat{f}_8^O=\tilde{e}_4\cosh\alpha y+\hat{e}_4\sinh\alpha y . \tag{4.3.69}$$

From the evenness of the operator L_3, the form of the solution (4.3.65), the linearity of equations (4.3.61a-d,f) and the form of the conditions (4.3.62) which have to be satisfied at $y=\pm a$, it follows that the general solution of equation (4.3.69) falls into two non-combining groups of even and odd solutions. This enables the problem to be divided into 'even' and 'odd' cases. Unfortunately, the boundary conditions (4.3.62) are in terms of the variables $\hat{f}_i$ i=1..4, which can be obtained from (4.3.61a-d,f) as

$$\hat{f}_i = \hat{f}_i^{O} + \hat{f}_i^{E} \quad i=4,2,1 \quad \text{and} \quad \hat{f}_3 = \hat{f}_3^{E} + \hat{f}_3^{O} \tag{4.3.70}$$

where

$$\hat{f}_4^{E} = -\frac{1}{\eta}\Big[\sum_{j=1}^{3} \hat{e}_j r_j \cosh r_j y + \frac{\alpha\eta}{\pi^3}\hat{e}_4 \cosh\alpha y \Big],$$

$$\hat{f}_3^{O} = \frac{1}{\eta}\Big[\sum_{j=1}^{3} \hat{e}_j r_j^2 \sinh r_j y + \frac{\eta}{\pi}\Big(\frac{\alpha^2}{\pi^2}-1\Big)\hat{e}_4 \sinh\alpha y \Big],$$

$$\hat{f}_2^{E} = \frac{1}{\eta}\Big[\sum_{j=1}^{3} \hat{e}_j (3\alpha^2 - r_j^2) r_j \cosh r_j y + \frac{\alpha\eta}{\pi}\Big(\frac{2\alpha^2}{\pi^2}+1\Big)\hat{e}_4 \cosh\alpha y \Big]$$

and

$$\hat{f}_1^{E} = \frac{1}{\eta}\Big[\sum_{j=1}^{3} \hat{e}_j \{ (3\alpha^2 - r_j^2) r_j^2 - (3\alpha^4 - R_0) \} r_j \cosh r_j y + \frac{\alpha\eta}{\pi}\Big(-\frac{\alpha^4}{\pi^2} - \alpha^2 + \frac{R_0}{\pi^2}\Big)\hat{e}_4 \cosh\alpha y \Big]. \tag{4.3.71}$$

Similar expressions exist for $\hat{f}_4^{O}$, $\hat{f}_3^{E}$, $\hat{f}_2^{O}$ and $\hat{f}_1^{O}$. However, at the critical Rayleigh number R_0 and wavenumber k_0 for the given aspect ratio a, it is found that for the case where $\hat{f}_5$ is even only the trivial solution $\tilde{e}_j = 0$ j=1..4 exists. For the case where $\hat{f}_5$ is odd, applying the boundary conditions (4.3.62) to the equations (4.3.70) gives four homogenous equations in four unknowns which can be expressed in matrix notation as

$$[\hat{S}][\hat{e}] = [0] \tag{4.3.72}$$

where $\hat{S}$ is the 4 x 4 matrix obtained from (4.3.71) and $\hat{e}$ is the column vector $\{\hat{e}_j\}$ j=1..4. The determinant of the matrix $\hat{S}$ is found to be zero, which implies that a non-trivial solution does exist. Thus the constants $\hat{e}_j$ j=1..3 can be determined in terms of $\hat{e}_4$ (say) using Cramer's rule. Hence

$$\hat{f}_5 = \sum_{j=1}^{3} \hat{e}_j \sinh r_j y + \frac{\eta}{\pi^3}\hat{e}_4 \sinh\alpha y \quad \text{and} \quad \hat{f}_7 = -\hat{e}_4 \alpha \cosh\alpha y \tag{4.3.73}$$

which implies

$$\hat{f}_4 = -\frac{1}{\eta}\Big[\sum_{j=1}^{3} \hat{e}_j r_j \cosh r_j y + \frac{\eta\alpha\hat{e}_4}{\pi^3}\cosh\alpha y \Big] \quad \text{and} \quad \hat{f}_8 = \hat{e}_4 \sinh\alpha y. \tag{4.3.74}$$

These solutions for $\hat{f}_4$ and $\hat{f}_8$ together with M_1 and N_1 imply that the integrand G of the condition (4.3.59) is an even function in y. Graphs of the integrand for the aspect ratios ½,¼,1 and 2 are given by figures 15 and 16.

Numerical integration (Simpson's rule) confirmed that the condition (4.3.59) was satisfied for the aspect ratios under consideration ie. a = $\frac{1}{4}$,$\frac{1}{2}$,1 and 2, thus providing a check that the system (4.3.58) has a solution. It should be noted that in the case of the simple models the expansions were clearly consistent at $O(\varepsilon)$, since for the stress-free channel equation (3.1.27) and for the 'finite-roll' approximation the right hand side of equation (3.2.14), were algebraically zero. The method of finding the numerical solution of the system (4.3.58) is discussed below.

(b) $\Theta_{2,3}$, $V_{2,3}$

In order to determine the y-dependent functions Θ_2 and Θ_3 in the temperature θ_2, and V_2 and V_3 in the horizontal velocity component v_2, numerical methods need to be used. From (4.3.11,25) and (4.3.12,26) the coupled functions Θ_2,V_2 and Θ_3,V_3 satisfy the matrix systems

$$\left[\frac{d}{dy} - \underline{E}(2k_0,2\pi)\right]\underline{f}(\hat{\Theta}_2,\hat{V}_2) = \underline{\Phi}(\hat{M}_2,\hat{N}_2), \quad \underline{m}\underline{f}(\hat{\Theta}_2,\hat{V}_2)=0 \quad y=\pm a, \tag{4.3.75a}$$

$$\left[\frac{d}{dy} - \underline{E}(2k_0,2\pi)\right]\underline{f}(\bar{\Theta}_2,\bar{V}_2) = \underline{\Phi}(\bar{M}_2,\bar{N}_2), \quad \underline{m}\underline{f}(\bar{\Theta}_2,\bar{V}_2)=0 \quad y=\pm a \tag{4.3.75b}$$

where

$$\Theta_2 = \hat{\Theta}_2 + P\bar{\Theta}_2 \qquad \text{and} \qquad V_2 = \hat{V}_2 + P\bar{V}_2\ , \tag{4.3.75c}$$

$$\hat{M}_2 = \left[-\left(\frac{d^2}{dy^2} - 4k_0^2\right)\alpha_3 + 4\pi k_0\alpha_5 - 2\pi\frac{d}{dy}\alpha_7\right], \qquad \hat{N}_2 = \left[\frac{1}{2\pi}\frac{d}{dy}\alpha_3 + \alpha_7\right],$$

$$\bar{M}_2 = \left(\frac{d^2}{dy^2} - 4\alpha^2\right)^2\alpha_1 \quad , \qquad \bar{N}_2 = \left[\frac{1}{2\pi}\left(4\alpha^2 - \frac{d^2}{dy^2}\right)\frac{d}{dy}\alpha_1\right], \tag{4.3.75d}$$

and

$$\left[\frac{d}{dy} - \underline{E}(0,2\pi)\right]\underline{f}(\hat{\Theta}_3,\hat{V}_3) = \underline{\Phi}(\hat{M}_3,\hat{N}_3), \quad \underline{m}\underline{f}(\hat{\Theta}_3,\hat{V}_3)=0 \quad y=\pm a, \tag{4.3.76a}$$

$$\left[\frac{d}{dy} - \underline{E}(0,2\pi)\right]\underline{f}(\bar{\Theta}_3,\bar{V}_3) = \underline{\Phi}(\bar{M}_3,\bar{N}_3), \quad \underline{m}\underline{f}(\bar{\Theta}_3,\bar{V}_3)=0 \quad y=\pm a, \tag{4.3.76b}$$

where

$$\Theta_3 = \hat{\Theta}_3 + P\bar{\Theta}_3 \quad \text{and} \quad V_3 = \hat{V}_3 + P\bar{V}_3 \ , \tag{4.3.76c}$$

$$\hat{M}_3 = \left[-\frac{d^2}{dy^2}\alpha_4 - 2\pi \frac{d}{dy}\alpha_q \right] , \qquad \hat{N}_3 = \left[\frac{1}{2\pi}\frac{d}{dy}\alpha_4 + \alpha_q \right],$$

$$\bar{M}_3 = \left[\left(\frac{d^2}{dy^2} - 4\pi^2 \right)^2 \alpha_2 \right] , \qquad \bar{N}_3 = \left[-\frac{1}{2\pi}\left(\frac{d^2}{dy^2} - 4\pi^2 \right)\frac{d}{dy}\alpha_2 \right] . \tag{4.3.76d}$$

As noted above, the y-dependent functions Θ_1 and V_1 satisfying the matrix system (4.3.58) may also be determined numerically. Thus the solutions of five eighth order systems (4.3.58,75a,75b,76a,76b) need to be determined.

In each case a fourth order Runge-Kutta process is used to compute the solution from y=-a to y=+a. A problem that arises is that each of the systems specifies four conditions at y=-a, which for instance in the case of (4.3.58) can be written as

$$\underline{f}(\Theta_1, V_1) = [q_1, q_2, q_3, q_4, 0, 0, 0, 0]^{tr} \quad \text{at } y = -a \tag{4.3.77}$$

where the constants q_i i=1..4 correspond to the unknown values of $\Theta_1^{(1)}$, $\Theta_1^{(3)}$, $\Theta_1^{(4)}$, $\Theta_1^{(5)}$ at y=-a. The other four conditions are specified at y=+a. However, in order to start the Runge-Kutta process eight conditions are needed at y=-a. This difficulty is overcome by letting q_i i=1..4, be known arbitrary constants enabling a particular solution $\underline{f}_p$ of the system to be computed. In addition by choosing four linearly independent combinations for q_i i=1..4, for instance

j	q_1	q_2	q_3	q_4
1	1	0	0	0
2	0	1	0	0
3	0	0	1	0
4	0	0	0	1

(4.3.76)

four complementary solutions $\underline{f}_j$ j=1..4 of the system are computed. The general solution $\underline{f}$, say in the case of (4.3.58) is then given by

$$\underline{f}(\Theta_1, V_1) = \hat{q}_1 \underline{f}_1(\Theta_1, V_1) + \hat{q}_2 \underline{f}_2(\Theta_1, V_1) + \hat{q}_3 \underline{f}_3(\Theta_1, V_1) + \hat{q}_4 \underline{f}_4(\Theta_1, V_1) + \underline{f}_p(\Theta_1, V_1) \tag{4.3.77}$$

where the arbitrary constants $\hat{q}_j$ j=1..4 are determined by applying the four known conditions at y=+a. This gives a non-homogenous system of four equations in four unknowns which can be expressed in matrix notation as

$$[f_c][\hat{q}] = -[f_p] \tag{4.3.78}$$

where f_c is a 4 x 4 matrix, f_p is a column vector with four elements and $\hat{q}$ is the column vector $\{q_j\}$ j=1..4. In the case of (4.3.58) the jth column (j=1..4) of the matrix f_c corresponds to the last four elements of the column vector $\underline{f}_j(\Theta_1, V_1)$ and the vector f_p corresponds to the last four elements of the vector $\underline{f}_p(\Theta_1, V_1)$.

In general the determinant of the matrix f_c is non-zero and the constants $\hat{q}_j$ j=1..4 can be uniquely determined using Cramer's rule. It is noted that the systems (4.3.75a,75b,76a,76b) fall in this category. The condition needed for

$$\det[\ f_c\ ] = 0 \tag{4.3.81}$$

is that $E = E(k_o,\pi)$; the system (4.3.58) falls in this category. In this case the value of $\hat{q}_4$(say) is assumed to be a known arbitrary constant, which leaves a non-homogenous system of three equations in three unknowns $\hat{q}_j$ j=1..3, which can be determined using Cramer's rule. The existence of the solution (4.3.50) and the solvability condition (4.3.59) which for the aspect ratios under consideration has been shown to be satisfied, ensure that the fourth equation is automatically satisfied. It should be noted that the solutions for Θ_1 and V_1 are not unique since the value of the constant $\hat{q}_4$ is arbitrary. However, application of the condition (4.3.52) gives

$$\Theta_1 = \Theta_1^N - \Theta_1^N(0)\Theta \qquad \text{and} \qquad V_1 = V_1^N - \Theta_1^N(0)V \tag{4.3.82}$$

where Θ_1^N and V_1^N are the numerical solutions. A comparison of the analytical solutions (4.3.57) and the numerical solutions (4.3.82) for

Θ_1 and V_1 showed good agreement. This provides a check that the Fortran computer program constructed to obtain the Runge-Kutta solution, as outlined above, is correct.

(c) V_4

From (4.3.37,38) the y-dependent function V_4 in the horizontal velocity component v_2 satisfies the matrix system

$$\left[\frac{d}{dy} - \tilde{\underline{E}}(2k_0)\right]\tilde{\underline{f}}(V_4) = \tilde{\underline{\Phi}}(N_4), \qquad \tilde{\underline{m}}\tilde{\underline{f}}(V_4) = 0 \quad y = \pm a \tag{4.3.83}$$

where

$$N_4 = -2k_0\left[2k_0\alpha_8 - \frac{d}{dy}\alpha_6\right], \qquad \tilde{\underline{f}}(V_4) = \left[V_4^{(2)}, V_4^{(3)}, V_4, V_4^{(1)}\right]^{tr}, \qquad \tilde{\underline{\Phi}}(N_4) = [0, N_4, 0, 0]^{tr},$$

$$\tilde{\underline{E}}(2k_0) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 8k_0 & 0 & -16k_0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \tag{4.3.84}$$

and

$$\tilde{\underline{m}} = \begin{bmatrix} \underline{0} & \underline{0} \\ \underline{0} & \underline{I}_2 \end{bmatrix}.$$

The fourth order system (4.3.83) is solved using a fourth order Runge-Kutta process as outlined above. The only difference is that only two complementary solutions need to be computed. Thus, the general solution for (4.3.83) is given by

$$\tilde{\underline{f}}(V_4) = \hat{q}_1\tilde{\underline{f}}_1(V_4) + \hat{q}_2\tilde{\underline{f}}_2(V_4) + \tilde{\underline{f}}_p(V_4) \tag{4.3.85}$$

where the arbitrary constants $\hat{q}_j$ j=1,2 may be determined uniquely from the boundary conditions at y=+a, as outlined above.

(d) The other dependent variables

Having numerically computed Θ_i i=1..3 in the temperature θ_2 and

V_i i=1..4 in the horizontal velocity component v_2, the y-dependent functions W_i i=1..3 that appear in the vertical velocity component w_2 may be obtained from the relationships

$$W_1 = -\left[2k_0 \Theta + \left(\frac{d^2}{dy^2} - \alpha^2\right) \Theta_1\right], \tag{4.3.86}$$

$$W_2 = \hat{W}_2 + \frac{1}{P}\tilde{W}_2 = \left[\alpha_1 - \left(\frac{d^2}{dy^2} - 4\alpha^2\right)\bar{\Theta}_2\right] + \frac{1}{P}\left[\left(4\alpha^2 - \frac{d^2}{dy^2}\right)\hat{\Theta}_2\right] \tag{4.3.87}$$

and

$$W_3 = \hat{W}_3 + \frac{1}{P}\tilde{W}_3 = \left[\alpha_2 - \left(\frac{d^2}{dy^2} - 4\pi^2\right)\bar{\Theta}_3\right] + \frac{1}{P}\left[\left(4\pi^2 - \frac{d^2}{dy^2}\right)\hat{\Theta}_3\right], \tag{4.3.88}$$

which are obtained from equations (4.3.15,16,17). The y-dependent functions U_1 and U_2 in the horizontal velocity component u_2, may either be obtained from the second order differential equations

$$\left[\frac{d^2}{dy^2} - \alpha^2\right]U_1 = \left[2U + \frac{1}{k_0^2}\left(\frac{d^2}{dy^2} - \alpha^2\right)U + \frac{k_0}{\pi}\left(2k_0 W + \left(\frac{d^2}{dy^2} - \alpha^2\right)W_1 + R_0\Theta_1\right)\right], \quad U_1 = 0 \quad y = \pm a \tag{4.3.89}$$

and

$$\left[\frac{d^2}{dy^2} - 4\alpha^2\right]\hat{U}_2 = \left[\alpha_5 + \frac{k_0}{\pi}\left(\alpha_3 + \left(\frac{d^2}{dy^2} - 4\alpha^2\right)^2 \hat{\Theta}_2 - R_0\hat{\Theta}_2\right)\right], \quad \hat{U}_2 = 0 \quad y = \pm a, \tag{4.3.90}$$

$$\left[\frac{d^2}{dy^2} - 4\alpha^2\right]\bar{U}_2 = \frac{k_0}{\pi}\left[\left(\frac{d^2}{dy^2} - 4\alpha^2\right)^2 \bar{\Theta}_2 - R_0\bar{\Theta}_2 - \left(\frac{d^2}{dy^2} - 4\alpha^2\right)\alpha_1\right], \quad \bar{U}_2 = 0 \quad y = \pm a \tag{4.3.91}$$

where

$$U_2 = \hat{U}_2 + P\bar{U}_2 \tag{4.3.92}$$

which are obtained from equations (4.3.27,28), or from the relationships

$$U_1 = \frac{1}{k_0}\left[-\frac{1}{k_0}U - \frac{d}{dy}V_1 - \pi W_1\right], \tag{4.3.93}$$

$$\hat{U}_2 = \frac{1}{2k_0}\left[\frac{d}{dy}\hat{V}_2 + 2\pi\tilde{W}_2\right] \quad \text{and} \quad \bar{U}_2 = \frac{1}{2k_0}\left[\frac{d}{dy}\bar{V}_2 + 2\pi\hat{W}_2\right] \tag{4.3.94}$$

which are obtained from the continuity equation (4.3.1). In practice both methods were used and good agreement of the results was obtained. The third y-dependent function U_4, that appears in u_2 may be obtained from the relationship (4.3.39). Having obtained all three velocity components u_2, v_2 and w_2 a consistency check for the z-dependent terms is provided by the continuity equation (4.3.1) which is found to be satisfied. The y-dependent functions P_i i=1..3 and F_i i=1,2 in the pressure p_2, may be obtained from the equations (4.3.19)-(4.3.21) and (4.3.40,41).

It should be noted that the y-dependent functions W_1 and U_1 in the vertical and horizontal velocity components w_2 and u_2 may also be determined analytically as follows: substituting (4.3.53) and (4.1.3) into (4.3.86) gives

$$W_1^A = k_0 \sum_{j=1}^{3} \frac{d_j}{r_j}(r_j^2-\alpha^2)\, y \sinh r_j y - \sum_{j=1}^{3} \tilde{d}_j (r_j^2-\alpha^2) \cosh r_j y. \tag{4.3.95}$$

Substituting (4.3.53,96) and (4.1.3) into (4.3.89) gives

$$\begin{aligned} U_1^A = {} & 2\sum_{j=1}^{3} d_j \left[\frac{\beta_j r_j}{(r_j^2-\alpha^2)} - \pi\right]\cosh r_j y + d_4\, y \sinh \alpha y + \frac{1}{k_0^2}\sum_{j=1}^{3} d_j\left(\beta_j r_j - \pi(r_j^2-\alpha^2)\right)\cosh r_j y \\ & + \frac{k_0}{\pi}\left[\sum_{j=1}^{3} d_j\left(\alpha^2 - r_j^2 + \frac{R_0}{(r_j^2-\alpha^2)}\right)\cosh r_j y + k_0 \sum_{j=1}^{3}\frac{d_j}{r_j}\left[r_j^2-\alpha^2-\frac{R_0}{(r_j^2-\alpha^2)}\right]\left[y\sinh r_j y - \frac{2r_j}{(r_j^2-\alpha^2)}\cosh r_j y\right]\right] \\ & + \tilde{d}_5 \cosh \alpha y. \end{aligned} \tag{4.3.96}$$

From the continuity equation (4.3.1) the unknown constant $\tilde{d}_5$ associated with the even complementary solution of (4.3.89) is given by

$$\tilde{d}_5 = -\alpha\, \tilde{d}_4 + (k_0^2-\alpha^2) d_4 / \alpha k_0 \, . \tag{4.3.97}$$

The solutions for all the dependent variables at order ε have thus been obtained. Figures 17a-17e show profiles of the y-dependent functions Θ_i i=1,2,3.

4.4 Expansion at $O(\varepsilon^2)$

At order ε^2, the Boussinesq equations (3.1.16-20) become

$$\frac{\partial u_3}{\partial x} + \frac{\partial v_3}{\partial y} + \frac{\partial w_3}{\partial z} = -\frac{\partial u_2}{\partial \bar{x}} \tag{4.4.1}$$

$$L_1 u_3 + \frac{\partial}{\partial x} p_3 = -\frac{\partial}{\partial \bar{x}} p_2 - \Gamma_1 u_2 - \Gamma_3 u_1, \tag{4.4.2}$$

$$L_1 v_3 + \frac{\partial}{\partial y} p_3 = -\Gamma_1 v_2 - \Gamma_3 v_1, \tag{4.4.3}$$

$$L_1 w_3 + \frac{\partial}{\partial z} p_3 - P R_o \theta_3 = -\Gamma_1 w_2 - \Gamma_3 w_1 + P\theta_1 \tag{4.4.4}$$

and

$$L_2 \theta_3 - w_3 = -\Gamma_2 \theta_2 - \Gamma_4 \theta_1 \tag{4.4.5}$$

where

$$\Gamma_1 = \left[-2P\frac{\partial^2}{\partial x \partial \bar{x}} + \underline{u}_1 \cdot \underline{\nabla}\right] \quad , \quad \Gamma_3 = \left[\frac{\partial}{\partial \bar{t}} - P\frac{\partial^2}{\partial \bar{x}^2} + \underline{u}_2 \cdot \underline{\nabla} + u_1 \frac{\partial}{\partial \bar{x}}\right],$$

$$\Gamma_2 = \left[-2\frac{\partial^2}{\partial x \partial \bar{x}} + \underline{u}_1 \cdot \underline{\nabla}\right] \quad , \quad \Gamma_4 = \left[\frac{\partial}{\partial \bar{t}} - \frac{\partial^2}{\partial \bar{x}^2} + \underline{u}_2 \cdot \underline{\nabla} + u_1 \frac{\partial}{\partial \bar{x}}\right]. \tag{4.4.6}$$

The boundary conditions from (4.1.6) are

$$\theta_3 = \frac{\partial^2}{\partial y^2}\theta_3 = v_3 = \frac{\partial}{\partial y} v_3 = 0 \quad y = \pm a \qquad \text{and} \quad \theta_3 = w_3 = \frac{\partial v_3}{\partial z} = \frac{\partial u_3}{\partial z} = 0 \quad z = 0, 1. \tag{4.4.7a,7b}$$

It should be noted that from (4.4.1–5), the conditions on the top and bottom boundaries (4.4.7b) are equivalent to

$$\theta_3 = \frac{\partial^2}{\partial z^2}\theta_3 = \frac{\partial^4 \theta_3}{\partial z^4} = \frac{\partial v_3}{\partial z} = 0 \qquad z = 0, 1. \tag{4.4.7c}$$

By elimination of variables in equations (4.4.1–5) a single sixth order differential equation in θ_3 is obtained:

$$P(\nabla^6 - R_o \nabla_H^2)\theta_3 = P\nabla^4 \left[\left(-2\frac{\partial^2}{\partial x \partial \bar{x}} + \underline{u}_1 \cdot \underline{\nabla}\right)\theta_2 + \left(\underline{u}_2 \cdot \underline{\nabla} + u_1 \frac{\partial}{\partial \bar{x}} - 2\frac{\partial^2}{\partial \bar{x}^2} + \frac{(1+P)}{P}\frac{\partial}{\partial \bar{t}}\right)\theta_1\right]$$

$$-\nabla_H^2\left[(\underline{u}_1 \cdot \underline{\nabla})w_2 + \left(\underline{u}_2 \cdot \underline{\nabla} + u_1 \frac{\partial}{\partial \bar{x}}\right)w_1 - P\theta_1\right] + 2P\frac{\partial^2}{\partial x \partial \bar{x}}\nabla^2 w_2 + \frac{\partial^2}{\partial x \partial z}\left[2P\frac{\partial^2}{\partial x \partial \bar{x}}u_1 + P\nabla^2 u_2 + \frac{\partial}{\partial x}p_2\right]$$

$$+\frac{\partial}{\partial z}\left[\frac{\partial}{\partial x}\left[(\underline{u}_1 \cdot \underline{\nabla})u_2 + \left(\underline{u}_2 \cdot \underline{\nabla} + u_1 \frac{\partial}{\partial \bar{x}}\right)u_1\right] + \frac{\partial}{\partial y}\left[(\underline{u}_1 \cdot \underline{\nabla})v_2 + \left(\underline{u}_2 \cdot \underline{\nabla} + u_1 \frac{\partial}{\partial \bar{x}}\right)v_1\right]\right] \tag{4.4.8}$$

Since not all the conditions on the sidewalls are in terms of θ_3, the following equation relating $\frac{\partial v_3}{\partial z}$ and θ_3 is also required.

$$-P\left[\nabla^2\frac{\partial}{\partial z}v_3 + (\nabla^4 - R_o)\frac{\partial}{\partial y}\theta_3\right] = -\frac{\partial}{\partial z}\left[\Gamma_1 v_2 + \Gamma_3 v_1\right] + \frac{\partial}{\partial y}\left[\Gamma_1 w_2 + \Gamma_3 w_1 - P\theta_1\right] - P\nabla^2\frac{\partial}{\partial y}\left[\Gamma_2\theta_2 + \Gamma_4\theta_1\right]. \tag{4.4.9}$$

Expanding the right hand sides of equations (4.4.8,9) gives

$$P(\nabla^6 - R_o\nabla_H^2)\theta_3 = \left[\left(M_5\bar{A} + M_6\bar{A}_{\bar{x}\bar{x}} + M_7\bar{A}^2\bar{A}^* + M_8\bar{A}_{\bar{t}}\right)e^{ik_o x} + c.c\right]\sin\pi z + M_9\left(\bar{A}^3 e^{3ik_o x} + c.c\right)\sin\pi z$$

$$+\left[M_{10}\left(\bar{A}\bar{A}_{\bar{x}}e^{2ik_o x} - c.c\right) + M_{11}\left(\bar{A}^*\bar{A}_{\bar{x}} - \bar{A}\bar{A}^*_{\bar{x}}\right)\right]\sin 2\pi z + \left[M_{12}\left(\bar{A}^3 e^{3ik_o x} + c.c\right) + M_{13}\left(\bar{A}^2\bar{A}^* e^{ik_o x} + c.c\right)\right]\sin 3\pi z \tag{4.4.10}$$

and

$$-P\left[\nabla^2\frac{\partial}{\partial z}v_3+(\nabla^4-R_0)\frac{\partial}{\partial y}\theta_3\right]=\left[(N_5\bar{A}+N_6\bar{A}_{\bar{X}\bar{X}}+N_7\bar{A}^2\bar{A}^*+N_8\bar{A}_{\bar{t}})e^{ik_0x}+c.c\right]\sin\pi z+N_9(\bar{A}^3e^{3ik_0x}+c.c)\sin\pi z$$

$$+\left[N_{10}(\bar{A}\bar{A}_{\bar{X}}e^{2ik_0x}-c.c)+N_{11}(\bar{A}^*\bar{A}_{\bar{X}}-\bar{A}\bar{A}^*_{\bar{X}})\right]\sin 2\pi z+\left[N_{12}(\bar{A}^3e^{3ik_0x}+c.c)+N_{13}(\bar{A}^2\bar{A}^*e^{ik_0x}+c.c)\right]\sin 3\pi z \quad (4.4.11)$$

where

$$M_5=P\bar{M}_5=P\left(\frac{d^2}{dy^2}-k_0^2\right)\Theta, \quad (4.4.12)$$

$$M_6=P\bar{M}_6=2k_0P\left[3\left(\frac{d^2}{dy^2}-\alpha^2\right)^2\Theta_1-R_0\Theta_1-6k_0W-\frac{\pi}{2k_0^3}\left(\frac{d^2}{dy^2}-\alpha^2\right)U-\frac{1}{k_0}\left[\frac{d^2}{dy^2}-\alpha^2\right]^2\Theta\right], \quad (4.4.14)$$

$$M_7=\hat{M}_7+P\bar{M}_7+\frac{1}{P}\tilde{M}_7$$

$$=\left[\left(\frac{d^2}{dy^2}-\alpha^2\right)\hat{\varepsilon}_1-\left(\frac{d^2}{dy^2}-k_0^2\right)\hat{\varepsilon}_2+\pi k_0\bar{\varepsilon}_3-\pi\bar{\varepsilon}_4'+\left(\frac{d^2}{dy^2}-\alpha^2\right)^2(\hat{\varepsilon}_5+\tilde{\varepsilon}_6)-\left(\frac{d^2}{dy^2}-k_0^2\right)(\bar{\varepsilon}_7+\hat{\varepsilon}_8)+\pi k_0(\bar{\varepsilon}_9+\hat{\varepsilon}_{10})-\pi(\bar{\varepsilon}_{11}'+\hat{\varepsilon}_{12}'\right.$$

$$+P\left[\left(\frac{d^2}{dy^2}-\alpha^2\right)^2\bar{\varepsilon}_1+\left(\frac{d^2}{dy^2}-\alpha^2\right)^2(\bar{\varepsilon}_5+\hat{\varepsilon}_6)\right]+\frac{1}{P}\left[-\left(\frac{d^2}{dy^2}-k_0^2\right)\tilde{\varepsilon}_2+\pi k_0\hat{\varepsilon}_3-\pi\hat{\varepsilon}_4'\right.$$

$$\left.-\left(\frac{d^2}{dy^2}-k_0^2\right)(\hat{\varepsilon}_7+\tilde{\varepsilon}_8)+\pi k_0(\hat{\varepsilon}_9+\tilde{\varepsilon}_{10})-\pi(\hat{\varepsilon}_{11}'+\tilde{\varepsilon}_{12}')\right], \quad (4.4.15)$$

$$M_8=\hat{M}_8+P\bar{M}_8=(1+P)\left(\frac{d^2}{dy^2}-\alpha^2\right)\Theta, \quad (4.4.16)$$

$$N_5=P\bar{N}_5=-P\frac{d}{dy}\Theta, \quad (4.4.17)$$

$$N_6=P\bar{N}_6=P\left[2k_0\left[2\left(\alpha^2\frac{d}{dy}-\frac{d^3}{dy^3}\right)\Theta_1+\pi V_1\right]-2\left[\frac{d}{dy}W+\frac{\pi}{2}V+2k_0^2\frac{d}{dy}\Theta\right]\right], \quad (4.4.18)$$

$$N_7=\hat{N}_7+P\bar{N}_7+\frac{1}{P}\tilde{N}_7$$

$$=\left[-\left(\frac{d^3}{dy^3}-\alpha^2\frac{d}{dy}\right)\hat{\varepsilon}_1+\hat{\varepsilon}_2'+\pi\bar{\varepsilon}_4-\left(\frac{d^3}{dy^3}-\alpha^2\frac{d}{dy}\right)(\hat{\varepsilon}_5+\tilde{\varepsilon}_6)+(\bar{\varepsilon}_7'+\hat{\varepsilon}_8')+\pi(\bar{\varepsilon}_{11}+\hat{\varepsilon}_{12})\right]$$

$$+P\left[-\left(\frac{d^3}{dy^3}-\alpha^2\frac{d}{dy}\right)\bar{\varepsilon}_1-\left(\frac{d^3}{dy^3}-\alpha^2\frac{d}{dy}\right)(\bar{\varepsilon}_5+\hat{\varepsilon}_6)\right]+\frac{1}{P}\left[\tilde{\varepsilon}_3'+\pi\hat{\varepsilon}_4+\hat{\varepsilon}_7'+\tilde{\varepsilon}_8'+\pi(\hat{\varepsilon}_{11}+\varepsilon_{12})\right], \quad (4.4.19)$$

$$N_8=\hat{N}_8+P\bar{N}_8=\left[-\left(\frac{d^3}{dy^3}-\alpha^2\frac{d}{dy}\right)\Theta-\pi V\right]+P\left[-\left(\frac{d^3}{dy^3}-\alpha^2\frac{d}{dy}\right)\Theta\right], \quad (4.4.20)$$

and $\varepsilon_i(y)$ i=1..12 are given by (C7) in appendix C. It should be noted that for the purpose of finding the amplitude equation for $\bar{A}$, the other y-dependent functions M_i and N_i i=9..13 that appear in equations (4.4.10,11) do not need to be determined explicitly.

Following the procedure outlined for the stress-free channel (section 3.1), expanding the right hand sides of equations (4.4.8,9) assuming

$$\theta_3 = \hat{\theta}_3(y,z,\bar{X},\bar{\tau})e^{ik_0x} + c.c + \ldots \qquad \text{and} \qquad v_3 = \hat{v}_3(y,z,\bar{X},\bar{\tau})e^{ik_0x} + c.c + \cdots , \tag{4.4.21}$$

it is clear from (4.4.10) and (4.4.11) that

$$P[(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-k_0^2)^3 - R_0(\frac{\partial^2}{\partial y^2}-k_0^2)]\hat{\theta}_3 = \Delta_1(y,\bar{X},\bar{\tau})\sin\pi z + \Delta_3 \sin 3\pi z \tag{4.4.22a}$$

and

$$-P\{\frac{\partial}{\partial z}(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-k_0^2)\hat{v}_3 + [(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-k_0^2)^2 - R_0]\frac{\partial}{\partial y}\hat{\theta}_3\} = \Delta_2(y,\bar{X},\bar{\tau})\sin\pi z + \Delta_4 \sin 3\pi z \tag{4.4.22b}$$

where

$$\Delta_1 = [M_5\bar{A} + M_6\bar{A}_{\bar{X}\bar{X}} + M_7\bar{A}^2\bar{A}^* + M_8\bar{A}_{\bar{\tau}}] , \qquad \Delta_3 = M_{13}\bar{A}^2\bar{A}^* ,$$

$$\Delta_2 = [N_5\bar{A} + N_6\bar{A}_{\bar{X}\bar{X}} + N_7\bar{A}^2\bar{A}^* + N_8\bar{A}_{\bar{\tau}}] , \qquad \Delta_4 = N_{13}\bar{A}^2\bar{A}^* . \tag{4.4.23}$$

It should be noted that other terms with x dependencies of the form 1, $e^{\pm 2ik_0x}$ and $e^{\pm 3ik_0x}$ also exist in θ_3 and v_3. However, these play no part in obtaining the amplitude equation. Applying the conditions (4.4.7a,7c) to (4.4.21) gives

$$\hat{\theta}_3 = \frac{\partial^2}{\partial y^2}\hat{\theta}_3 = \hat{v}_3 = \frac{\partial}{\partial y}\hat{v}_3 = 0 \quad y=\pm a , \qquad \hat{\theta}_3 = \frac{\partial^2}{\partial z^2}\hat{\theta}_3 = \frac{\partial^4}{\partial z^4}\hat{\theta}_3 = \frac{\partial\hat{v}_3}{\partial z} = 0 \quad z=0,1 . \tag{4.4.24a,b}$$

The z solvability condition for the system (4.4.22,24) is

$$P\int_{z=0}^{1}[(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-k_0^2)^3 - R_0(\frac{\partial^2}{\partial y^2}-k_0^2)]\hat{\theta}_3\sin\pi z\,dz = \int_{z=0}^{1}[\Delta_1\sin\pi z + \Delta_3\sin 3\pi z]\sin\pi z\,dz = \frac{1}{2}\Delta_1 \tag{4.4.25a}$$

and

$$P\int_{z=0}^{1}\{\frac{\partial}{\partial z}(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-k_0^2)\hat{v}_3 + [(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-k_0^2)^2 - R_0]\frac{\partial}{\partial y}\hat{\theta}_3\}\sin\pi z\,dz = \int_{z=0}^{1}[\Delta_2\sin\pi z + \Delta_4\sin 3\pi z]\sin\pi z\,dz = \frac{1}{2}\Delta_2 . \tag{4.4.25b}$$

Expanding (4.4.25a,25b) using (4.4.24b) and repeated integration by parts gives

$$P\int_{z=0}^{1}[(\frac{\partial^2}{\partial y^2}-\alpha^2)^3 - R_0(\frac{\partial^2}{\partial y^2}-k_0^2)]\hat{\theta}_3\sin\pi z\,dz = \frac{1}{2}\Delta_1 \tag{4.4.26a}$$

and

$$-P\int_{z=0}^{1}\{-\pi(\frac{\partial^2}{\partial y^2}-\alpha^2)\hat{v}_3\cos\pi z + [(\frac{\partial^2}{\partial y^2}-\alpha^2)^2 - R_0]\frac{\partial}{\partial y}\hat{\theta}_3\sin\pi z\}dz = \frac{1}{2}\Delta_2 \tag{4.4.26b}$$

By defining

$$\bar{\theta}_3(y,\bar{x},\bar{\tau}) = P\int_{z=0}^{1} \hat{\theta}_3 \sin\pi z\, dz \qquad \text{and} \qquad \bar{v}_3(y,\bar{x},\bar{\tau}) = P\int_{z=0}^{1} \hat{v}_3 \cos\pi z\, dz\,, \tag{4.4.27}$$

equations (4.4.26a,26b) become

$$\left[\left(\frac{d^2}{dy^2}-\alpha^2\right)^3 - R_0\left(\frac{d^2}{dy^2}-k_0^2\right)\right]\bar{\theta}_3 = \tfrac{1}{2}\Delta_1$$

and

$$\left[\frac{d^2}{dy^2}-\alpha^2\right]\bar{v}_3 - \frac{1}{\pi}\left[\frac{d^5}{dy^5} - 2\alpha^2\frac{d^3}{dy^3} + (\alpha^4 - R_0)\right]\bar{\theta}_3 = \Delta_2/2\pi\,, \tag{4.4.28}$$

which together with the boundary conditions

$$\bar{\theta}_3 = \frac{d^2}{dy^2}\bar{\theta}_3 = \bar{v}_3 = \frac{d}{dy}\bar{v}_3 = 0 \qquad y = \pm a \tag{4.4.29}$$

can be expressed in matrix notation as

$$\left[\frac{d}{dy} - \underline{E}(k_0,\pi)\right]\underline{f}(\bar{\theta}_3,\bar{v}_3) = \underline{\Phi}\left(\tfrac{1}{2}\Delta_1, \tfrac{1}{2\pi}\Delta_2\right), \quad \underline{m}\,\underline{f}(\bar{\theta}_3,\bar{v}_3) = 0 \quad y = \pm a\,. \tag{4.4.30}$$

From the previous section, the condition needed for the existence of a solution of (4.4.30) is

$$\int_{y=-a}^{+a} H(y,\bar{x},\bar{\tau})\, dy = 0 \tag{4.4.31}$$

where

$$H(y,\bar{x},\bar{\tau}) = \hat{f}_4\Delta_1 + \hat{f}_8\Delta_2/\pi \tag{4.4.32}$$

and $\hat{f}_4$ & $\hat{f}_8$ are solutions obtained from the matrix system (4.3.60) given by (4.3.74).

Expanding (4.4.32) shows that the condition (4.4.31) is only satisfied if the amplitude function $\bar{A}$ satisfies the equation

$$C_1\bar{A} + C_2\bar{A}_{\bar{x}\bar{x}} + (C_3 + C_4/P + C_5/P^2)\bar{A}|\bar{A}|^2 + (C_6 + C_7/P)\bar{A}_{\bar{\tau}} = 0 \tag{4.4.33}$$

where

$$C_1 = \hat{H}(\bar{M}_5,\bar{N}_5)\,, \quad C_2 = \hat{H}(\bar{M}_6,\bar{N}_6)\,,$$

$$C_3 = \hat{H}(\bar{M}_7,\bar{N}_7)\,, \quad C_4 = \hat{H}(\hat{M}_7,\hat{N}_7)\,, \quad C_5 = \hat{H}(\tilde{M}_7,\tilde{N}_7)\,,$$

$$C_6 = \hat{H}(\bar{M}_8, \bar{N}_8) \ , \quad C_7 = \hat{H}(\hat{M}_8, \hat{N}_8) \tag{4.4.34}$$

and

$$\hat{H}(s,t) = \int_{y=-a}^{+a} [\hat{f}_4 s + \tfrac{1}{\pi}\hat{f}_8 t] \, dy \ . \tag{4.4.35}$$

Numerical integration (Simpson's rule) is used to determine the amplitude coefficients C_i $i=1..7$, which for the aspect ratios under consideration are given in table 7. However, it should be noted that the coefficients C_1, C_6 and C_7 may be expressed analytically as

$$C_1 = -\frac{2}{\eta}\sum_{j=1}^{4}\sum_{i=1}^{3} \tilde{d}_j d_i r_j (r_i^2 - k_0^2)\bar{\xi}_{ij} - \frac{2\tilde{d}_4}{\pi}\sum_{j=1}^{3} d_j r_j \hat{\xi}_j \ , \tag{4.4.36}$$

$$C_6 + C_7/P = (1+\tfrac{1}{P})[-\frac{2}{\eta}\sum_{j=1}^{4}\sum_{i=1}^{3} \tilde{d}_j d_i r_j (r_i^2 - r_4^2)\bar{\xi}_{ij} + 2\frac{\tilde{d}_4}{\pi}\sum_{j=1}^{3} d_j r_j (r_4^2 - r_j^2)\hat{\xi}_j \} + \frac{2\tilde{d}_4}{P}\sum_{j=1}^{4} d_j \beta_j \hat{\xi}_j \tag{4.4.37}$$

where

$$r_4 = \alpha \ , \quad \beta_4 = 1 \ ,$$

$$\bar{\xi}_{ij} = \begin{cases} [r_j \sinh r_j a \cosh r_i a - r_i \sinh r_i a \cosh r_j a]/(r_j^2 - r_i^2) & i \neq j \\ [\sinh r_i a \cosh r_i a + r_i a]/r_i & i = j \end{cases}$$

and

$$\hat{\xi}_j = \begin{cases} [r_4 \sinh r_j a \cosh r_4 a - r_j \sinh r_4 a \cosh r_j a]/(r_4^2 - r_j^2) & j \neq 4 \\ [\sinh r_4 a \cosh r_4 a - r_4 a]/r_4 & j = 4 . \end{cases} \tag{4.4.38}$$

A comparison of the analytical and numerical results for the coefficients C_i $i=1,6,7$ showed good agreement.

C_i	$a = \frac{1}{4}$	$a = \frac{1}{2}$	$a = 1$	$a = 2$
1	1.17356E-1	2.98550E 0	6.66208E 2	2.35883E 7
2	6.82607E 1	8.17820E 2	1.16230E 5	3.26389E 9
3	-9.05110E 3	-3.88266E 4	-3.12612E 6	-8.00064E10 *
4	-5.90913E 2	-3.10812E 3	-2.59926E 5	-4.27370E 9 *
5	-4.61856E 2	-3.16741E 3	-3.11862E 5	-5.77686E 9 *
6	-1.26669E 1	-1.82351E 2	-3.14470E 4	-1.04976E 9
7	-1.25073E 1	-1.77438E 2	-3.04607E 4	-1.04380E 9

Table 7. The amplitude coefficients. (E n denotes 10^n)

* Indicates that the coefficient for the aspect ratio a = 2 may be inaccurate (see section 7.4 below).

Aspect ratio a = 0.25

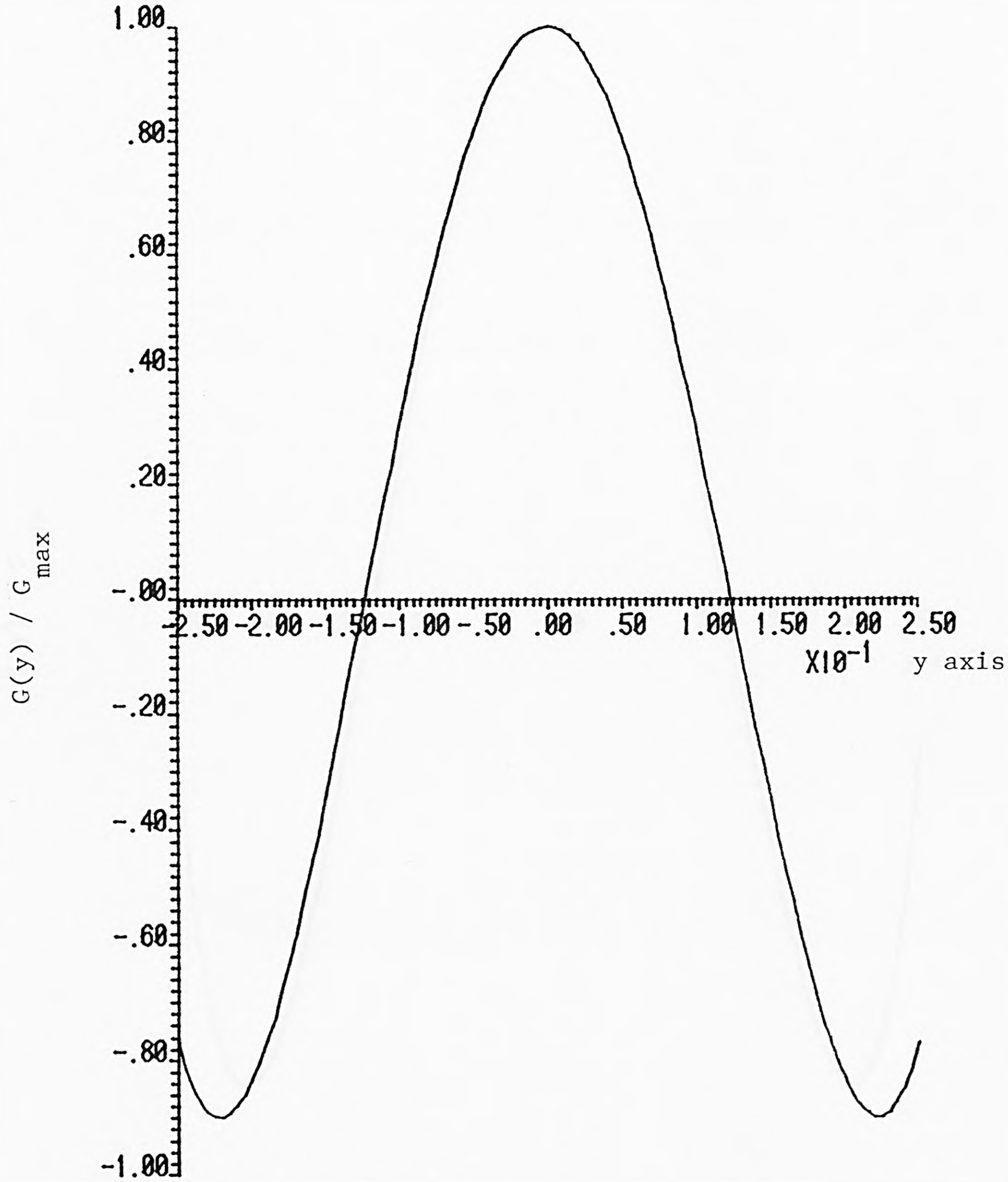


Figure 15 continued on next page

Aspect ratio a = 0.5

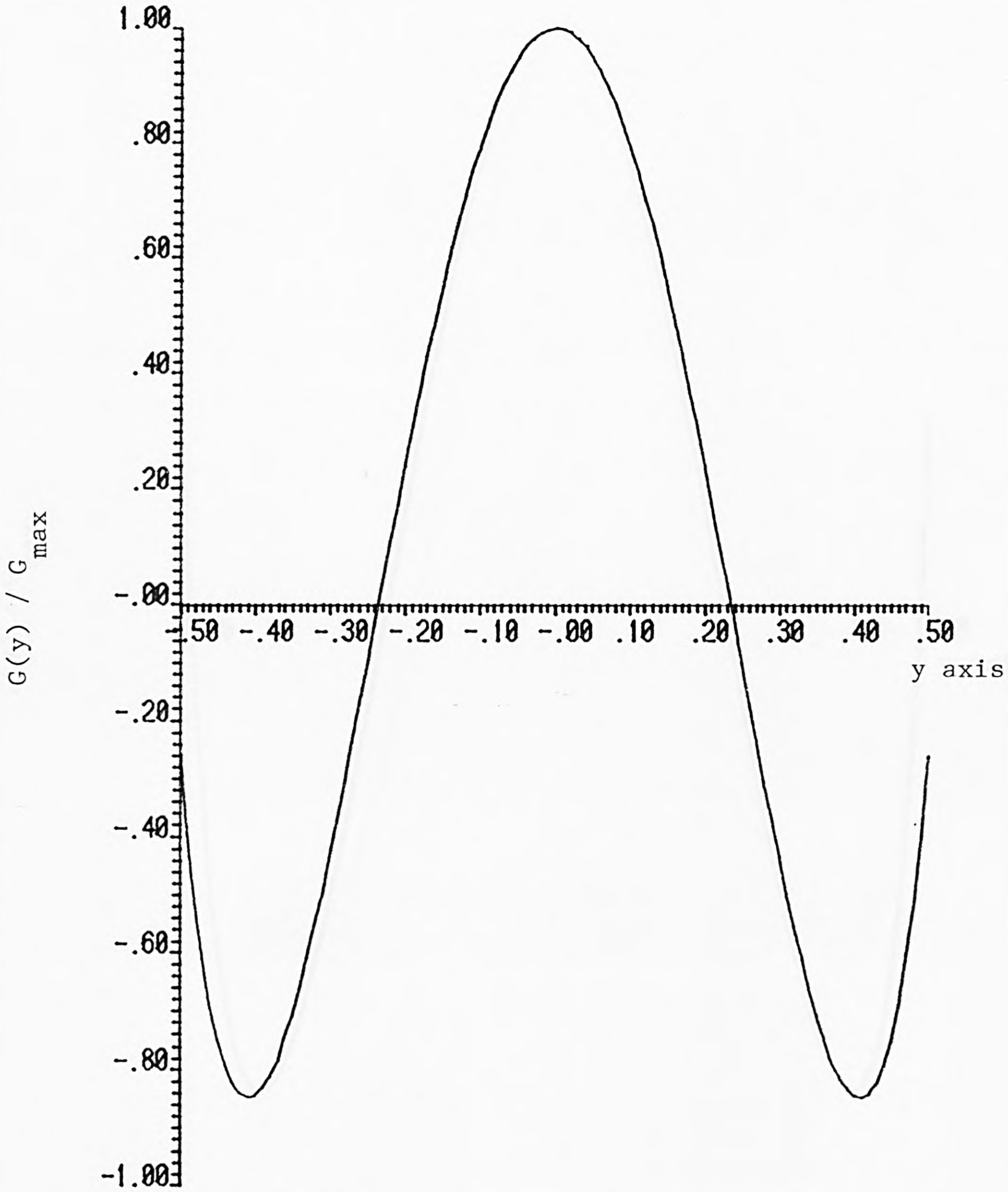


Figure 15. Graphs of the integrand G of the integral condition (4.3.59) for the aspect ratios a = 0.25 and 0.5.

Aspect ratio a = 1.0

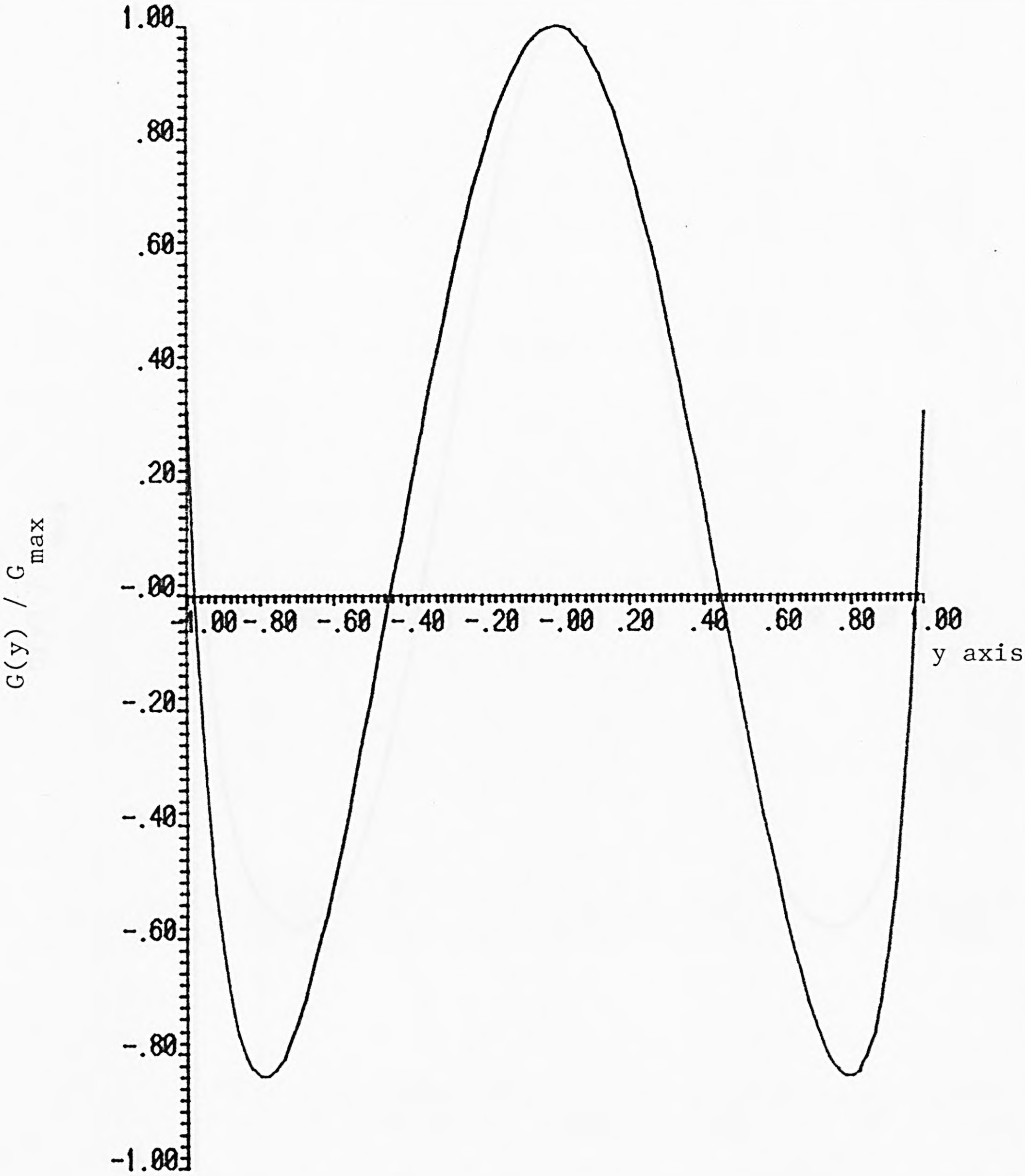


Figure 16 continued on next page

Aspect ratio a = 2.0

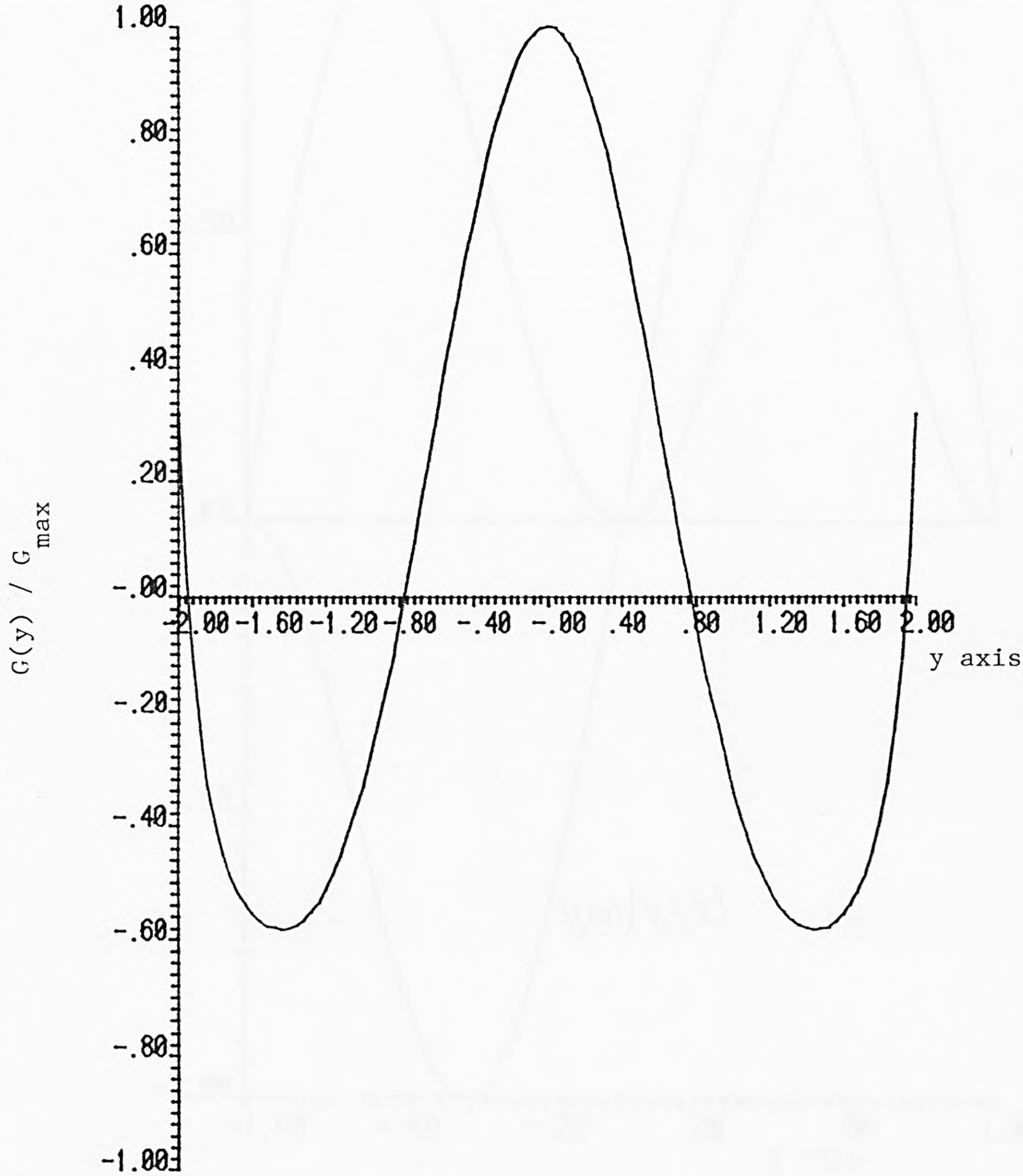


Figure 16. Graphs of the integrand G of the integral condition (4.3.59) for the aspect ratios a = 1 and 2.

$\Theta_1(0.5) = -0.09573$

$V_1(0.5) = -1.82152$

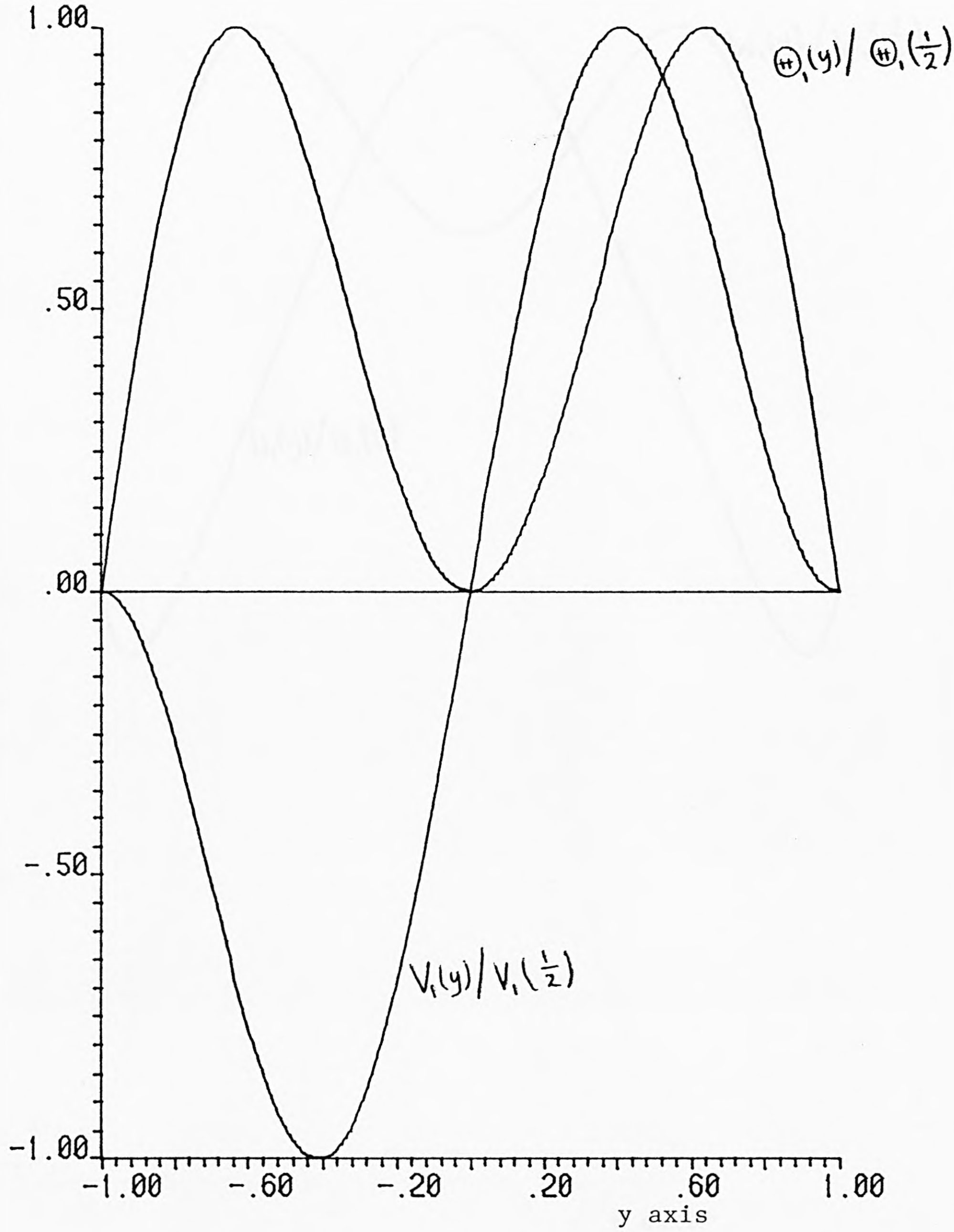


Figure 17a continued on next page

$W_1(0.5) = -5.05939$

$U_1(0.0) = -0.68526$

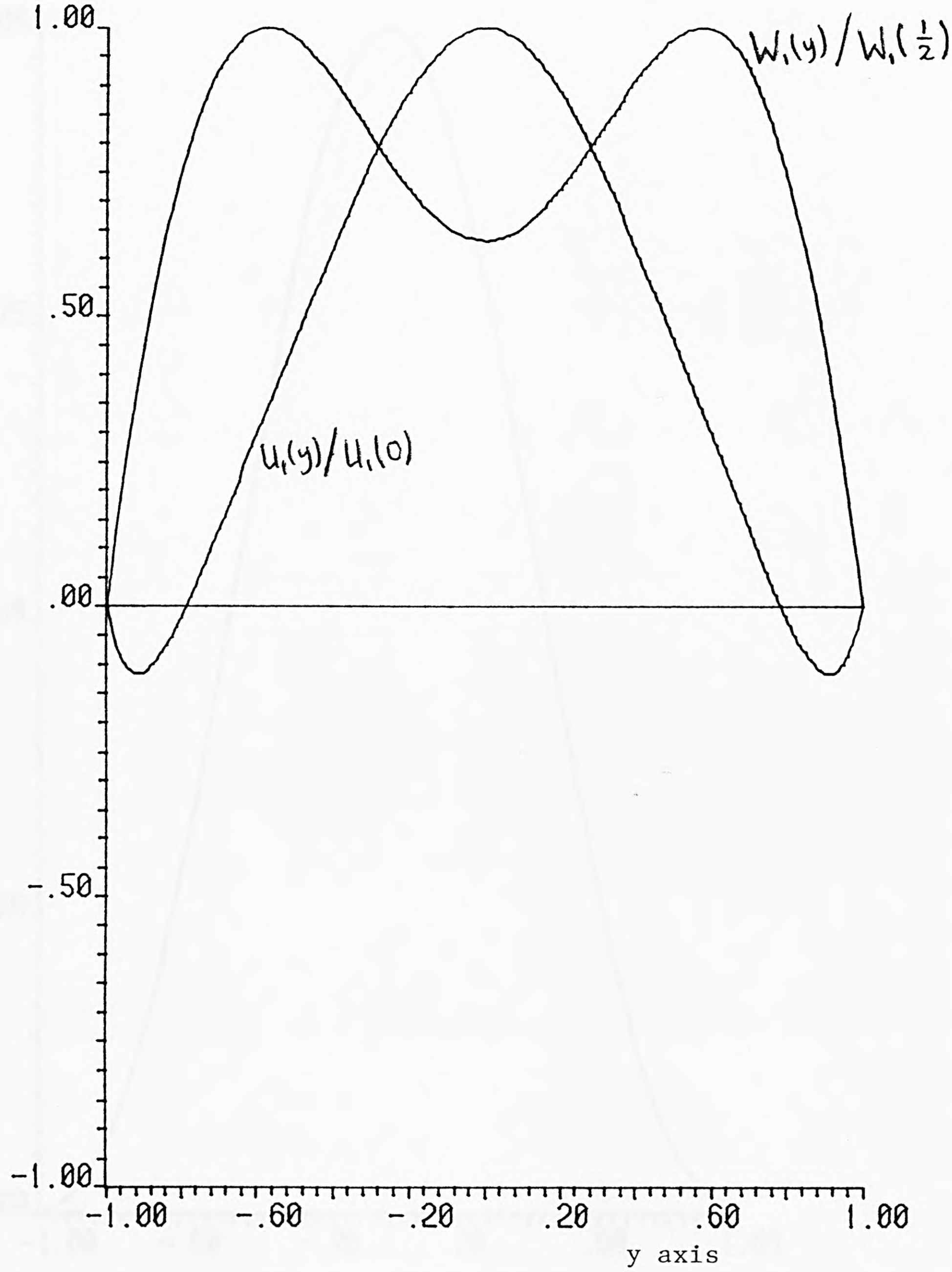


Figure 17a. Profiles of the y-dependent functions $\Theta_1(y), V_1(y), W_1(y)$ and $U_1(y)$ for the aspect ratio a = 1 and conducting sidewalls.

Aspect ratio a = 1

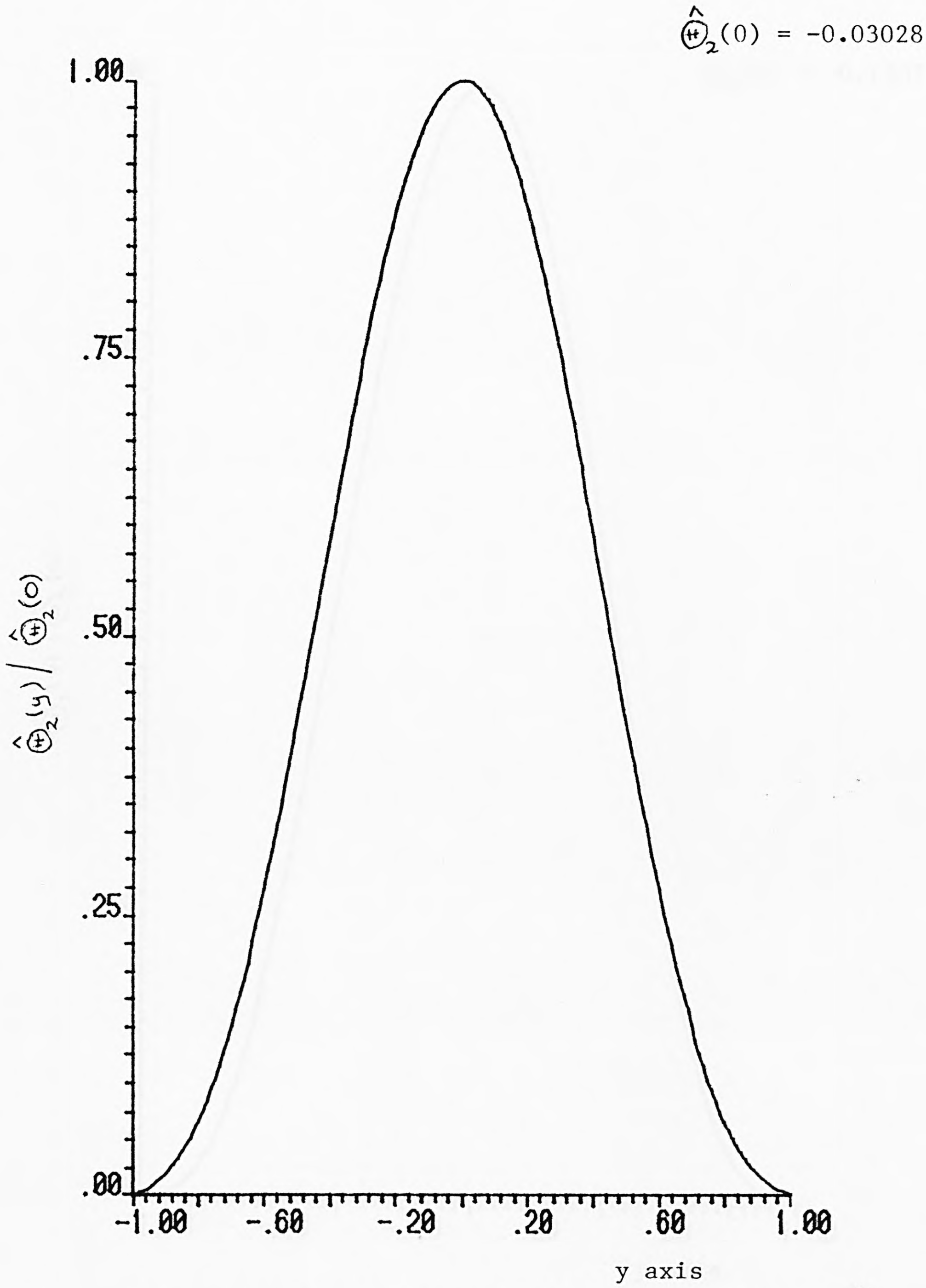


Figure 17b. Profile of the y-dependent function $\hat{\Theta}_2$ for the aspect ratio a = 1 and conducting sidewalls.

Aspect ratio a = 1

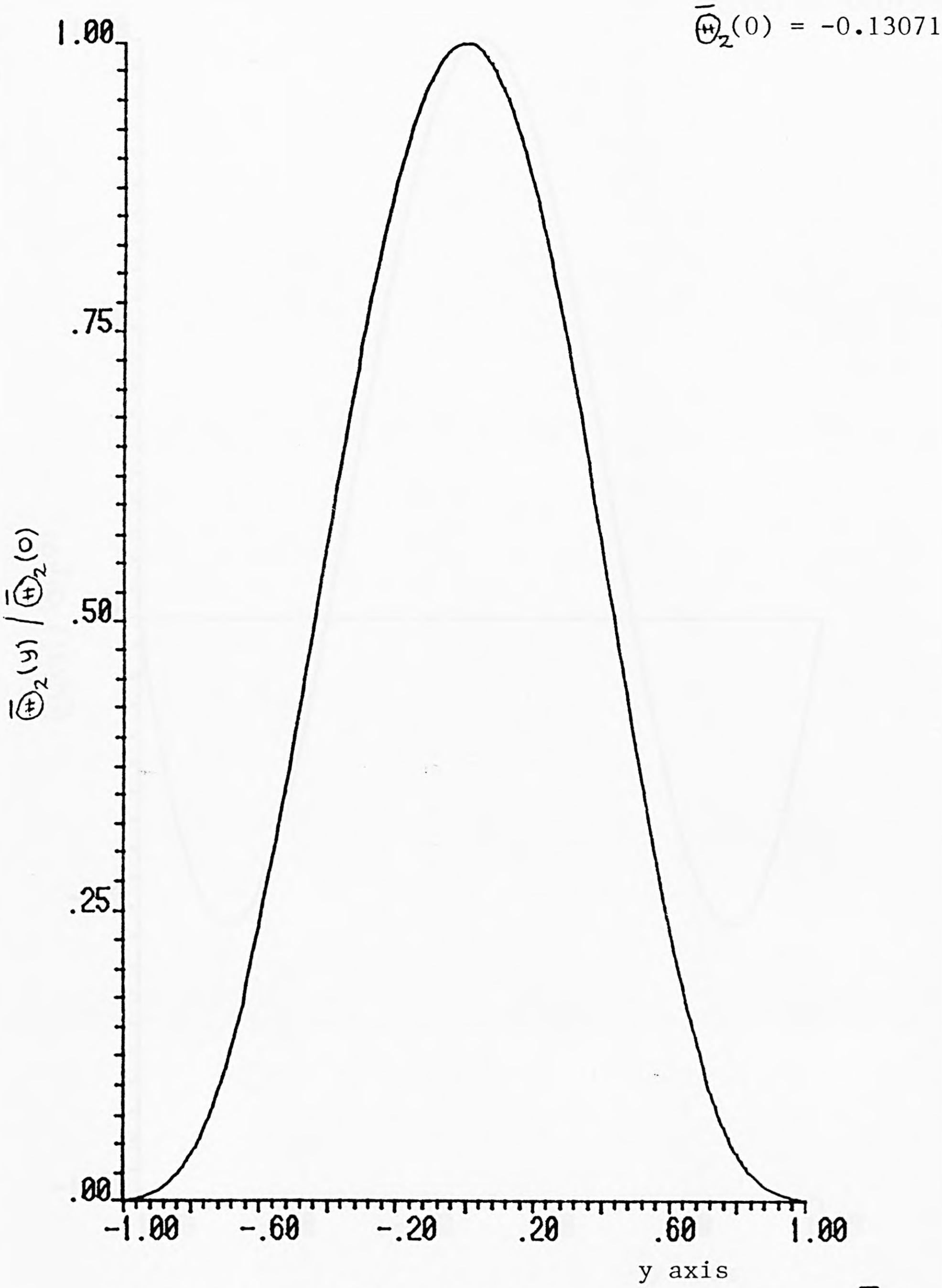


Figure 17c. Profile of the y-dependent function $\bar{\Theta}_2$ for the aspect ratio a = 1 and conducting sidewalls.

Aspect ratio a = 1

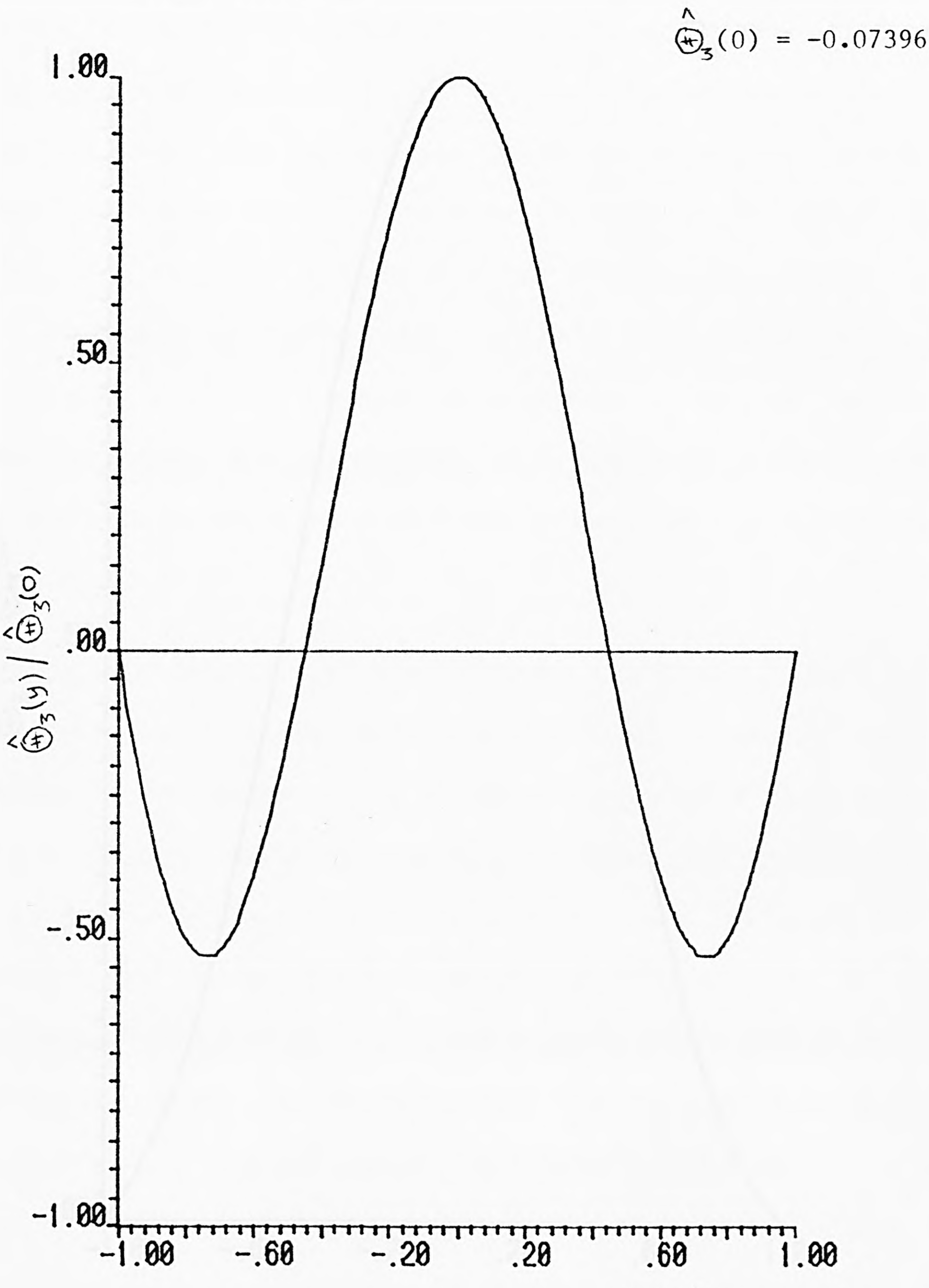


Figure 17d. Profile of the y-dependent function $\hat{\Theta}_3$ for the aspect ratio a = 1 and conducting sidewalls.

Aspect ratio a = 1.0

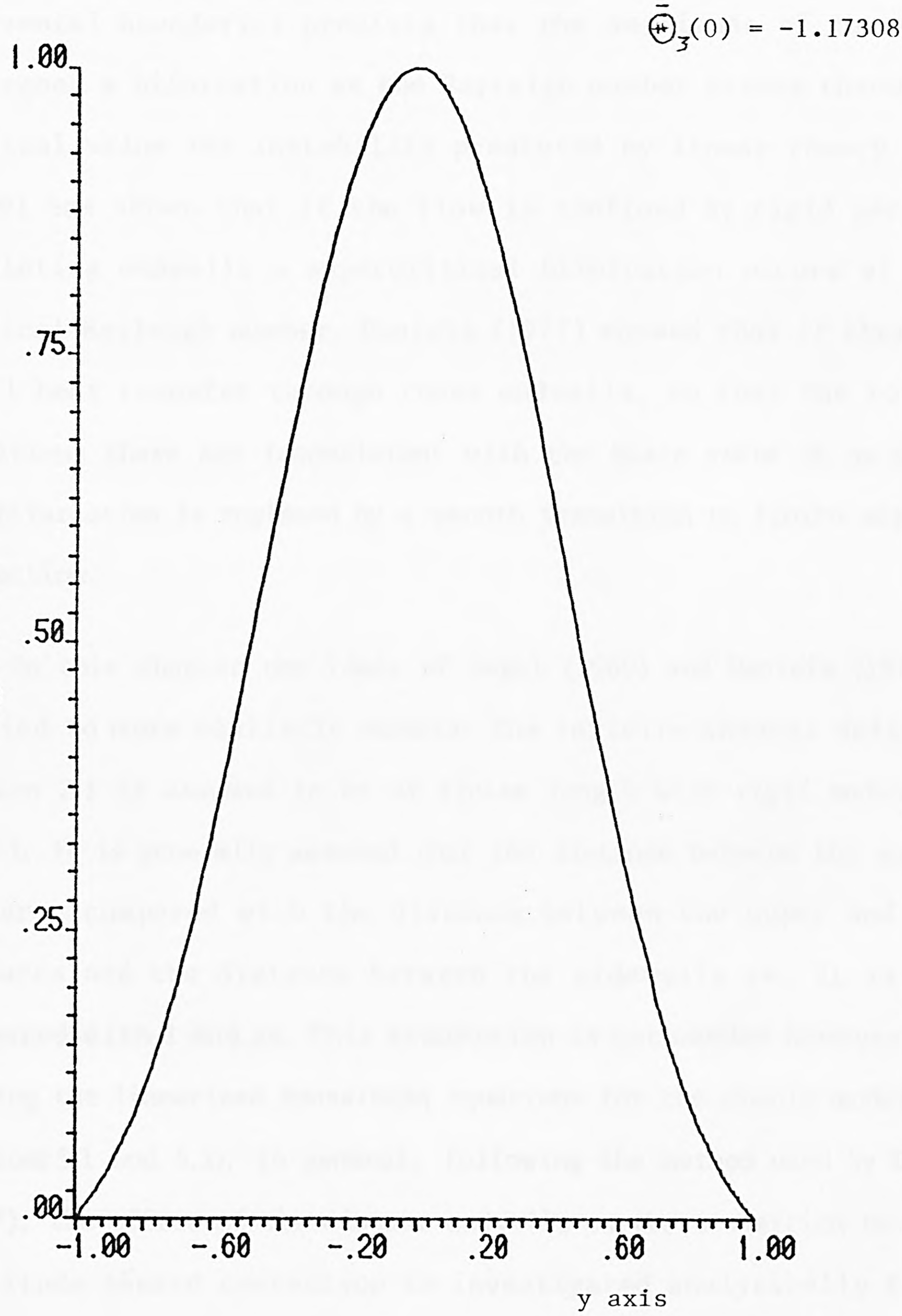


Figure 17e. Profile of the y-dependent function $\bar{\Theta}_3$ for the aspect ratio a = 1 and conducting sidewalls.

Chapter 5 The long box: simplified and rigid sidewall models

5.0 Introduction

Nonlinear theory for an infinite layer of fluid confined between horizontal boundaries predicts that the amplitude of the motion undergoes a bifurcation as the Rayleigh number passes through the critical value for instability predicted by linear theory. Segel (1969) has shown that if the flow is confined by rigid perfectly insulating endwalls a supercritical bifurcation occurs at a new critical Rayleigh number. Daniels (1977) showed that if there is a small heat transfer through these endwalls, so that the boundary conditions there are inconsistent with the basic state of no motion, the bifurcation is replaced by a smooth transition to finite amplitude convection.

In this chapter the ideas of Segel (1969) and Daniels (1977) are applied to more realistic models. The infinite channel defined in section 2.1 is assumed to be of finite length with rigid endwalls at $x = \pm L$. It is generally assumed that the distance between the endwalls is large compared with the distance between the upper and lower surfaces and the distance between the sidewalls ie. 2L is large compared with 1 and 2a. This assumption is not needed however, when solving the linearized Boussinesq equations for the simple models (see sections 5.1 and 5.3). In general, following the method used by Daniels (1977), the effect of the distant endwalls on the transition to finite amplitude Bénard convection is investigated analytically for the simple models using both linear and nonlinear techniques. However, for the long box with rigid sidewalls the simplicity of the analytical methods used for the simple models is lost and a method used by Stewartson and Weinstein (1979) involving extensive numerical calculations must be used.

5.1 Long box with stress-free sidewalls: linear theory

As in the case of the infinite stress-free channel (section 2.5) the upper and lower surfaces are assumed to be stress-free and perfect conductors, while the sidewalls are assumed to be stress-free and perfect insulators, thus allowing a solution to be constructed by separation of variables. The conditions on the top and bottom boundaries and the sidewalls are then given by (2.2.1) and (2.5.2) respectively. The endwalls are assumed to be rigid but not perfect insulators. Thus the conditions on the endwalls are

$$w = u = v = 0 \quad x = \pm L \tag{5.1.1}$$

and

$$\frac{\partial \theta}{\partial x} = \lambda L^{-1} g(Y, z) \quad x = -L \qquad \text{and} \qquad \frac{\partial \theta}{\partial x} = \lambda L^{-1} h(Y, z) \quad x = +L, \tag{5.1.2}$$

where the parameter L^{-1} is introduced for convenience, g and h are given functions of Y (=y+a) and z which are assumed to be consistent with the conditions (2.2.1) and (2.5.2), and may be decomposed into the Fourier series

$$g(Y, z) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} g_{nm} \sin n\pi z \cos\frac{m\pi Y}{a}, \quad h(Y, z) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} h_{nm} \sin n\pi z \cos\frac{m\pi Y}{a}. \tag{5.1.3}$$

In general the functions g and h and the parameter λ (≥ 0), which determines the magnitude of the heat transfer through the endwalls, are functions of both the physical properties of the containing walls and of the basic temperature field in the container. The flow is assumed to be steady. Thus, from the heat equation (2.1.20) and the continuity equation (2.1.16) the conditions (5.1.1) are equivalent to

$$\nabla^2 \theta = u = \frac{\partial u}{\partial x} = 0 \quad x = \pm L. \tag{5.1.4}$$

If the parameter λL^{-1} is assumed to be small, the dependent variables u,v,w,θ and p satisfy the linearized form of the Boussinesq equations that govern the motion, given by equations (2.1.24-28). By eliminating the variables in equations (2.1.24-28) a single sixth order differential equation in θ given by (2.2.5) is obtained. Since not all the conditions on the endwalls are in terms of θ the following equation relating u and θ is also required

$$-\frac{\partial}{\partial z}\nabla^2 u = (\nabla^4 - R)\frac{\partial \theta}{\partial x} \ . \tag{5.1.5}$$

From section 2.5 the solution satisfying the homogenous differential equation (2.2.5) and the boundary conditions (2.2.1) and (2.5.2) is

$$\theta = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \bar{\theta}_{nm}(x) \sin n\pi z \cos \frac{m\pi y}{\bar{a}} \tag{5.1.6}$$

where

$$\bar{\theta}_{nm} = \sum_{j=1}^{6} \bar{C}_{nmj}\, e^{i k_{nmj} x} , \tag{5.1.7}$$

where k_{nmj} are the roots of the characteristic equation

$$(k_{nm}^2)^3 + 3N^2\pi^2 (k_{nm}^2)^2 + (3N^4\pi^4 - R)(k_{nm}^2) + \left(N^6\pi^6 - \left(\frac{m\pi}{\bar{a}}\right)^2 R\right) = 0 \ , \quad N^2 = n^2 + m^2/\bar{a}^2 \tag{5.1.8}$$

and $\bar{C}_{nmj}$ are arbitrary constants. As equation (5.1.8) is bi-cubic and has real coefficients, the roots can be easily determined using Cardin's formula. From appendix A by letting $k_{nm}^2 = \eta - N^2\pi^2$, the characteristic equation (5.1.8) reduces to the form

$$\eta^3 + \gamma\eta + \beta = 0 \qquad \text{where} \quad \gamma = -R \quad \text{and} \quad \beta = n^2\pi^2 R. \tag{5.1.9}$$

It should be noted that the properties of equation (5.1.9) are the same as those of equation (2.2.18), which have been discussed in section 2.2. From the existence of the neutral stability curve and the properties of the characteristic equation (5.1.8) the root $k_{nm3}^2 = -\bar{k}_{nm3}^2$ remains real and negative for all values of R. The roots k_{nm1}^2 and k_{nm2}^2 are real and positive for $R > R_{nmc} = 27(n\pi)^4/4$, are equal when

$R=R_{nmc}$ ($k^2_{nm1} = k^2_{nm2} = \frac{\pi^2}{2}(n^2 - 2m^2/\bar{a}^2)$) and are complex conjugate pairs with non-zero imaginary parts for $R<R_{nmc}$. Here R_{nmc} is the critical Rayleigh number determined in section 2.5 for the mode of instability (n,m) in the infinite stress-free channel.

Since the characteristic equation is bi-cubic the solution (5.1.6) can be written as

$$\theta = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} [\,\bar{\theta}^E_{nm}(x) + \bar{\theta}^o_{nm}(x)\,] \sin n\pi z \cos\frac{m\pi y}{\bar{a}} \tag{5.1.10}$$

where

$$\bar{\theta}^E_{nm} = \hat{d}_{nm1}\cos k_{nm1}x + \hat{d}_{nm2}\cos k_{nm2}x + \hat{d}_{nm3}\cosh \bar{k}_{nm3}x \tag{5.1.11}$$

and

$$\bar{\theta}^o_{nm} = \bar{d}_{nm1}\sin k_{nm1}x + \bar{d}_{nm2}\sin k_{nm2}x + \bar{d}_{nm3}\sinh \bar{k}_{nm3}x\,. \tag{5.1.12}$$

Substituting (5.1.10) into (5.1.5) gives

$$u = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} [\,\bar{u}^o_{nm}(x) + \bar{u}^E_{nm}(x)\,] \cos n\pi z \cos\frac{m\pi y}{\bar{a}} \tag{5.1.14}$$

where

$$\bar{u}^o_{nm} = \hat{d}_{nm1}\beta_{nm1}\sin k_{nm1}x + \hat{d}_{nm2}\beta_{nm2}\sin k_{nm2}x + \hat{d}_{nm3}\beta_{nm3}\sinh \bar{k}_{nm3}x + \hat{d}_{nm4}\sinh N\pi x, \tag{5.1.15}$$

$$\bar{u}^E_{nm} = \bar{d}_{nm1}\bar{\beta}_{nm1}\cos k_{nm1}x + \bar{d}_{nm2}\bar{\beta}_{nm2}\sin k_{nm2}x + \bar{d}_{nm3}\beta_{nm3}\cosh \bar{k}_{nm3}x + \bar{d}_{nm4}\cosh N\pi x, \tag{5.1.16}$$

$$\beta_{nmi} = [k^5_{nmi} + 2N^2\pi^2 k^3_{nmi} + (N^4\pi^4 - R)k_{nmi}] \Big/ [n\pi(k^2_{nmi} + N^2\pi^2)], \quad \bar{\beta}_{nmi} = -\beta_{nmi} \quad i=1,2$$

and (5.1.17)

$$\beta_{nm3} = [\bar{k}^5_{nm3} - 2N^2\pi^2\bar{k}^3_{nm3} + (N^4\pi^4 - R)\bar{k}_{nm3}] \Big/ [n\pi(\bar{k}^2_{nm3} - N^2\pi^2)], \quad \bar{\beta}_{nm3} = \beta_{nm3}.$$

For a given mode (n,m) if $R>R_{nmc}$ the arbitrary constants $\hat{d}_{nmj}$ and $\bar{d}_{nmj}$ j=1..4 are purely real. However, if $R<R_{nmc}$ the constants $\hat{d}_{nmj}$, $\bar{d}_{nmj}$ j=3,4 remain purely real while the constants $\hat{d}_{nm1,2}$ and $\bar{d}_{nm1,2}$ become complex conjugate pairs.

From the evenness of the equations satisfied by $\bar{\theta}_{nm}$ and $\bar{u}_{nm}$ and the form of the conditions (5.1.2,4) which have to be satisfied at $x=\pm L$, it follows that the general solutions for $\bar{\theta}_{nm}$ and $\bar{u}_{nm}$ fall into two non-combining groups of even and odd solutions. This enables the problem to be divided into 'even' and 'odd' cases. The conditions (5.1.2) may be more conveniently expressed as

$$\frac{\partial\theta}{\partial x} = \frac{\lambda L^{-1}}{2}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left[(g_{nm}+h_{nm}) + (g_{nm}-h_{nm})\right]\sin n\pi z \cos\frac{m\pi y}{\bar{a}} \qquad x=-L$$

and (5.1.18)

$$\frac{\partial\theta}{\partial x} = \frac{\lambda L^{-1}}{2}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left[(g_{nm}+h_{nm}) - (g_{nm}-h_{nm})\right]\sin n\pi z \cos\frac{m\pi y}{\bar{a}} \qquad x=+L.$$

Thus, applying the boundary conditions (5.1.4,18) to the solutions (5.1.10,14), for the even case gives a non-homogenous system of four equations in four unknowns which can be expressed in matrix notation as

$$[B][D]=[S] \tag{5.1.19}$$

where

$$B = \begin{bmatrix} k_{nm1}^2+N^2\pi^2, & k_{nm2}^2+N^2\pi^2, & -\bar{k}_{nm3}^2+N^2\pi^2, & 0 \\ \beta_{nm1}\tan k_{nm1}L, & \beta_{nm2}\tan k_{nm2}L, & \beta_{nm3}\tanh \bar{k}_{nm3}L, & \tanh N\pi L \\ \beta_{nm1}k_{nm1}, & \beta_{nm2}k_{nm2}, & \beta_{nm3}\bar{k}_{nm3}, & N\pi \\ -k_{nm1}\tan k_{nm1}L, & -k_{nm2}\tan k_{nm2}L, & \bar{k}_{nm3}\tanh\bar{k}_{nm3}L, & 0 \end{bmatrix}, \tag{5.1.20}$$

$$D = \begin{bmatrix} \hat{d}_{nm1}\cos k_{nm1}L, & \hat{d}_{nm2}\cos k_{nm2}L, & \hat{d}_{nm3}\cosh\bar{k}_{nm3}L, & \hat{d}_{nm4}\cosh N\pi L \end{bmatrix}^{tr},$$

(5.1.21)

$$S = \begin{bmatrix} 0, & 0, & 0, & -\frac{\lambda L^{-1}}{2}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}(g_{nm}-h_{nm}) \end{bmatrix}^{tr}$$

and tr denotes transpose. For the odd case the matrices B,D and S become

$$B = \begin{bmatrix} k_{nm1}^2+N^2\pi^2, & k_{nm2}^2+N^2\pi^2, & -\bar{k}_{nm3}^2+N^2\pi^2, & 0 \\ \beta_{nm1}\cot k_{nm1}L, & \beta_{nm2}\cot k_{nm2}L, & -\beta_{nm3}\coth \bar{k}_{nm3}L, & -\coth N\pi L \\ \beta_{nm1}k_{nm1}, & \beta_{nm2}k_{nm2}, & \beta_{nm3}\bar{k}_{nm3}, & N\pi \\ k_{nm1}\cot k_{nm1}L, & k_{nm2}\cot k_{nm2}L, & \bar{k}_{nm3}\coth\bar{k}_{nm3}L, & 0 \end{bmatrix}, \tag{5.1.22}$$

$$D = \begin{bmatrix} \bar{d}_{nm1}\sin k_{nm1}L, & \bar{d}_{nm2}\sin k_{nm2}L, & \bar{d}_{nm3}\sinh\bar{k}_{nm3}L, & \bar{d}_{nm4}\sinh N\pi L \end{bmatrix}^{tr},$$

(5.1.23)

$$S = \begin{bmatrix} 0, & 0, & 0, & \frac{\lambda L^{-1}}{2}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}(g_{nm}+h_{nm}) \end{bmatrix}^{tr}.$$

For a given mode (n,m) when the eight constants $\hat{d}_{nmj}$ and $\bar{d}_{nmj}$ $j=1..4$ are determined from the conditions at $x=\pm L$, all the terms in the solutions (5.1.10,14) decay near the endwalls if $R<R_{nmc}$ and $L\gg 1$.

It is expected from Segel (1969) and Daniels (1977) that the change in the overall critical Rayleigh number R_0 for the infinite channel (given by (2.5.10)) due to the introduction of the distant endwalls may be investigated by considering a perturbation of the form

$$R = R_0 + \bar{\delta} L^{-2} \qquad (5.1.24)$$

where $\bar{\delta}$ is assumed to be order one and $L\gg 1$. It should be noted that R_0 corresponds to the vertical mode $n=1$ and is independent of the horizontal mode m. It can be shown from (5.1.8) that

$$k_{1m2,1}^2 \sim k_m^2\left[1 \pm 2i(R_0-R)^{1/2}/(3\pi^2(1-2m^2/\bar{a}^2))\right] \quad \text{as } R\to R_0 \qquad (5.1.25)$$

where

$$k_m^2 = \frac{\pi^2}{2}\left(1-2m^2/\bar{a}^2\right), \qquad (5.1.26)$$

so that

$$k_{1m2,1}^2 \sim k_m^2 \pm \frac{1}{3}\bar{\delta}^{\frac{1}{2}} L^{-1} \qquad (L\gg 1). \qquad (5.1.27)$$

Substituting (5.1.27) into (5.1.18) gives

$$\bar{k}_{1m3}^2 = 4\pi^2 + m^2\pi^2/\bar{a}^2 + O(L^{-2}). \qquad (5.1.28)$$

Thus, when $\bar{\delta}$ is positive the solution for the temperature $\theta(x,y,z)$ may be expanded in the form

$$\theta = -\frac{\lambda L^{-1}}{2}\Bigg\{\sum_{m=0}^{M}\frac{1}{\phi_1}\Bigg[(h_{1m}-g_{1m})\left(\frac{\phi_2\sin k_m x\,\sin\phi_4\bar{\delta}^{\frac{1}{2}}L^{-1}x}{\sin\Delta}+\frac{\phi_3\cos k_m x\,\cos\phi_4\bar{\delta}^{\frac{1}{2}}L^{-1}x}{\cos\Delta}\right)$$

$$+(h_{1m}+g_{1m})\left(\frac{\phi_3\sin\phi_4\bar{\delta}^{\frac{1}{2}}L^{-1}x\,\cos k_m x}{\sin\Delta}+\frac{\phi_2\sin k_m x\,\cos\phi_4\bar{\delta}^{\frac{1}{2}}L^{-1}x}{\cos\Delta}\right)$$

$$-\sqrt{2}N\phi_5(k_m^2+N^2\pi^2)(1-2m^2/\bar{a}^2)^{1/2}\left(g_{1m}e^{-\phi_6\pi x_1}-h_{1m}e^{\phi_6\pi x_2}\right)\Bigg]\sin\pi z\cos\frac{m\pi y}{\bar{a}}$$

$$+\sum_{m=M+1}^{\infty}\left[(a_{1m1}+a_{1m2}x_1)e^{-\bar{k}_m x_1}+a_{1m3}e^{-\phi_6\pi x_1}+(b_{1m1}+b_{1m2}x_2)e^{\bar{k}_m x_2}+b_{1m3}e^{\phi_6\pi x_2}\right]\sin\pi z\cos\frac{m\pi y}{\bar{a}}\Bigg\}$$

+ (SEE NEXT PAGE)

$$+ \lambda L^{-1} \sum_{m=0}^{\infty} \sum_{n=2}^{\infty} [G_{nm}(x_1) + H_{nm}(x_2)] \sin n\pi z \cos \frac{m\pi Y}{\bar{a}} + O(\lambda L^{-2}) \qquad (5.1.29)$$

where

$$x_1 = x + L, \quad x_2 = x - L \qquad (5.1.30)$$

and

$$\phi_1 = k_m \{ (k_m^2 + N^2\pi^2)[\beta_{1m2} (\phi_6 - N)(k_m^2 + N^2\pi^2) - N\phi_5\phi_6] + (\phi_6^2 - N^2) k_m^2 \phi_5 \},$$

$$\phi_2 = (k_m^2 + N^2\pi^2)^2 (N - \phi_6) \beta_{1m3} \tilde{\Delta} + (\phi_6^2 - N^2) k_m \phi_5 (N\pi \sin \bar{\Delta} - k_m \cos \bar{\Delta}),$$

$$\phi_3 = -(k_m^2 + N^2\pi^2)^2 (N - \phi_6) \beta_{1m3} \sin \bar{\Delta} - (\phi_6^2 - N^2) k_m \phi_5 (N\pi \tilde{\Delta} + k_m \sin \bar{\Delta}),$$

$$\phi_4 = 1/6k_m, \quad \phi_5 = (k_m^4 + 2N^2\pi^2 k_m^2 + N^4\pi^4 - R_0), \quad \bar{k}_m = \frac{\pi}{\sqrt{2}} \left(\frac{2m^2}{\bar{a}^2} - 1 \right)^{1/2}, \quad \phi_6 = (4 + m^2/\bar{a}^2)^{1/2},$$

$$\Delta = \delta^{-1/2} / (3\sqrt{2}\pi(1 - 2m^2/\bar{a}^2)^{1/2}), \quad \bar{\Delta} = \frac{\pi}{\sqrt{2}} (1 - 2m^2/\bar{a}^2)^{1/2} L, \quad \tilde{\Delta} = \cos \bar{\Delta}, \qquad (5.1.31)$$

a_{1mi}, b_{1mi} i=1..3 are real constants which may be written in terms of g_{1m} and h_{1m} respectively, and M is the largest integer such that

$$M < \bar{a} / \sqrt{2}. \qquad (5.1.32)$$

It should be noted that for the vertical mode n=1, the horizontal modes $m \leqslant M$ associated with $k_{1m2,1}$ being purely real penetrate into the interior region where $x/L \sim 1$. However, the modes $m > M$ which correspond to $k_{1m2,1}$ being equal and purely imaginary, decay in the neighbourhood of the endwalls where $x \pm L \sim 1$. The functions G_{nm} and H_{nm} in (5.1.29) have the forms

$$G_{nm}(x_1) = e^{-\omega_{nm1} x_1} (\bar{a}_{nm1} \cos \omega_{nm2} x_1 + \bar{a}_{nm2} \sin \omega_{nm2} x_1) + \bar{a}_{nm3} e^{-\phi_6 \pi x_1},$$

$$H_{nm}(x_2) = e^{\omega_{nm1} x_2} (\bar{b}_{nm1} \cos \omega_{nm2} x_2 + \bar{b}_{nm2} \sin \omega_{nm2} x_2) + \bar{b}_{nm3} e^{\phi_6 \pi x_2} \qquad (5.1.33)$$

where $\bar{a}_{nmi}$ and $\bar{b}_{nmi}$ i=1..3 are real constants which may be expressed in terms of g_{nm} and h_{nm} respectively and

$$\omega_{nm1} \pm i\omega_{nm2} = ik_{nm2,1} \qquad (5.1.34)$$

where $\omega_{nm1,2}$ are real and positive. If the forcing functions g and h given by (5.1.3) are assumed to be independent of Y, only the m=0 mode is generated and the solution (5.1.29) becomes

$$\theta = \frac{-\lambda L^{-1}}{3\sqrt{2}\pi}\Big\{(h_{1c}-g_{1o})\Big[\frac{\hat{\phi}_2 \sin\hat{k}_o x \sin\hat{\phi}_4 \bar{\delta}^{\frac{1}{2}} L^{-1}x}{\sin\hat{\Delta}} - \frac{\hat{\phi}_3 \cos\hat{k}_o x \cos\hat{\phi}_4 \bar{\delta}^{\frac{1}{2}} L^{-1}x}{\cos\hat{\Delta}}\Big]$$

$$+(g_{1o}+h_{1o})\Big[\frac{\hat{\phi}_2 \sin\hat{k}_o x \cos\hat{\phi}_4 \bar{\delta}^{\frac{1}{2}} L^{-1}x}{\cos\hat{\Delta}} - \frac{\hat{\phi}_3 \cos\hat{k}_o x \sin\hat{\phi}_4 \bar{\delta}^{\frac{1}{2}} L^{-1}x}{\sin\hat{\Delta}}\Big]$$

$$+2\sqrt{2}\big(g_{1o}e^{-\hat{\phi}_6\pi x_1} - h_{1o}e^{\hat{\phi}_6\pi x_2}\big)\Big\}\sin\pi z + \lambda L^{-1}\sum_{n=2}^{\infty}\big[G_{no}(x_1)+H_{no}(x_2)\big]\sin n\pi z + O(\lambda L^{-2}) \quad (5.1.35)$$

where

$$\hat{k}_o = \pi/\sqrt{2}, \quad \hat{\phi}_2 = \cos\tfrac{\pi}{\sqrt{2}}L - 2\sqrt{2}\sin\tfrac{\pi}{\sqrt{2}}L, \quad \hat{\phi}_3 = \sin\tfrac{\pi}{\sqrt{2}}L + 2\sqrt{2}\cos\tfrac{\pi}{\sqrt{2}}L,$$

$$\hat{\phi}_4 = 1/6\hat{k}_o, \quad \hat{\phi}_6 = 2 \quad \text{and} \quad \hat{\Delta} = \bar{\delta}^{\frac{1}{2}}/3\sqrt{2}\pi, \qquad (5.1.36)$$

equivalent to that obtained by Daniels (1977).

In contrast to the finite solution obtained for $\bar{\delta}<0$, from (5.1.29) and the form of Δ, it can be seen that the linear solution becomes infinite (ie. resonance occurs) at an infinite sequence of values of $\bar{\delta}$ given by

$$\bar{\delta} = \frac{9\pi^4}{2}j(1-m^2/2a^2) \qquad j=1,2\ldots, \quad m=0,1\ldots M. \qquad (5.1.37)$$

These states correspond to the existence of eigensolutions in the equivalent long box with null endwall boundary conditions ($\lambda=0$ see section 5.2). The first such state corresponds to the critical Rayleigh number R_c for the long box, the distant endwalls resulting in a slight increase from the critical value R_o for the infinite channel:

$$R_c = R_o + \frac{9\pi^4}{2}(1-m^2/2a^2)L^{-2} + O(L^{-3}) \qquad m=M. \qquad (5.1.38)$$

In contrast to R_o, the critical Rayleigh number R_c for the long box is dependent on the horizontal mode m and the aspect ratio a. For a given value of the aspect ratio a, the lowest mode of instability corresponds to M, the largest value of m satisfying the condition

$$m < \sqrt{2}a, \qquad (5.1.39)$$

which must be related to the fact that the extra freedom associated with choosing m (ie. the number of rolls in the y direction), in some way allows an easier 'fit' of the rolls in the x direction. As expected if the flow is assumed two-dimensional, only the m=0 mode is allowed and (5.1.38) becomes $R_c = R_0 + \frac{9\pi^4}{2} L^{-2} + O(L^{-3})$, equivalent to the result obtained by Daniels (1977).

From Daniels (1977) it is expected that the $O(L^{-3})$ correction in (5.1.38) will distinguish the resonance points of the even and odd parts of the solution (5.1.29), which are identical at order L^{-2}. Figure 23 below, shows the correction to R_0 at $O(L^{-2})$ plotted against the aspect ratio a for the horizontal modes m=0,1,2 and 3.

5.2 The long box with stress-free sidewalls: nonlinear theory

It is known from section 3.1 that for the infinite stress-free channel the solution near the critical Rayleigh number R_0 may be expanded in integral powers of a small parameter proportional to $(R-R_0)^{\frac{1}{2}}$. Here, the semi-aspect ratio L is assumed to be large compared with 1 and 2a and the solution is expanded in decaying powers of L. In addition a scaled Rayleigh number $\bar{\delta}$ defined by

$$R = R_0 + \bar{\delta} L^{-2} \qquad (5.2.1)$$

is introduced, where $\bar{\delta}$ is assumed to be order one. As in the case of the stress-free channel $\bar{a}$ (=2a) is assumed to be bounded such that $\sqrt{2} < \bar{a} < 2\sqrt{2}$. Thus only the first two modes corresponding to m=0 and m=1 will be present away from the endwalls. From section 3.1 the solutions for the dependent variables in the interior of the long box may be expanded in the forms

$$\theta = \frac{4\sqrt{2}}{3\pi^2} L^{-1} \left[(A(X,T)e^{ik_0x} + c.c) \sin\pi z + \sqrt{2} (B(X,T)e^{ik_1x} + c.c) \sin\pi z \cos\frac{\pi y}{\bar{a}} \right] + L^{-2}\theta_2 + L^{-3}\theta_3 + \cdots \qquad (5.2.2)$$

$$w = 2\sqrt{2} L^{-1} \left[(A(X,T)e^{ik_0x} + c.c) \sin\pi z + \sqrt{2} (B(X,T)e^{ik_1x} + c.c) \sin\pi z \cos\frac{\pi y}{\bar{a}} \right] + L^{-2}w_2 + L^{-3}w_3 + \cdots \qquad (5.2.3)$$

$$u = \frac{4\sqrt{2}}{\pi} i L^{-1}\left[k_0\left(A(X,\tau)e^{ik_0x} - c.c\right)\cos\pi z + \sqrt{2}k_1\left(B(X,\tau)e^{ik_1x} - c.c\right)\cos\pi z \cos\frac{\pi y}{\bar{a}}\right] + L^{-2}u_2 + L^{-3}u_3 + \cdots \quad (5.2.4)$$

$$v = \frac{-4\sqrt{2}L^{-1}}{\bar{a}}\left[\quad 0 \quad + \sqrt{2}\left(B(X,\tau)e^{ik_1x} + B^*(X,\tau)e^{-ik_1x}\right)\cos\pi z \sin\frac{\pi y}{\bar{a}}\right] + L^{-2}v_2 + L^{-3}v_3 + \cdots \quad (5.2.5)$$

and

$$p = -6\pi\sqrt{2}PL^{-1}\left[\left(A(X,\tau)e^{ik_0x} + c.c\right)\cos\pi z + \sqrt{2}\left(B(X,\tau)e^{ik_1x} + c.c\right)\cos\pi z \cos\frac{\pi y}{\bar{a}}\right] + L^{-2}p_2 + L^{-3}p_3 + \cdots \quad (5.2.6)$$

where * denotes complex conjugate, k_0 and k_1 are given by (3.1.3) and X and τ are slow spatial and time variables related to x and t by

$$X = L^{-1}x \qquad \text{and} \qquad \tau = L^{-2}4Pt/(1+P). \quad (5.2.7)$$

The equations for the amplitude functions $A(X,\tau)$ and $B(X,\tau)$ are determined from consideration of the second and third terms in the interior expansions (5.2.2-6) and the details of the calculations are essentially the same as those given for the infinite stress-free channel in section 3.1. An inconsistency in the expansion at $O(L^{-3})$ due to the appearance of forcing functions which are eigensolutions of the basic linear operator with the conditions (3.1.59) can only be avoided if A and B satisfy the equations

$$-A_\tau + A_{XX} + \bar{\delta}A = A|A|^2 + [\beta_1 + \beta_2/P + \beta_3/P^2]A|B|^2 \quad (5.2.8)$$

and

$$-B_\tau + (1 - 1/2\bar{a}^2)B_{XX} + \bar{\delta}B = [\beta_1 + \beta_2/P + \beta_3/P^2]B|A|^2 + [1 + \beta_4 + \beta_5/P + \beta_6/P^2]B|B|^2 \quad (5.2.9)$$

where

$$\bar{\delta} = 18\pi^2\delta \quad (5.2.10)$$

and the amplitude coefficients β_i i=1..6 are given by (3.1.72). It should be noted that the equations (5.2.8,9) can be obtained directly from the equations (3.1.70,71) using the transformations

$$\varepsilon \to \bar{\delta}^{-\frac{1}{2}}L^{-1}, \quad X \to \bar{\delta}^{\frac{1}{2}}X, \quad \tau \to \bar{\delta}\tau, \quad A \to A/\bar{\delta}^{1/2} \text{ and } B \to \sqrt{2}B/\bar{\delta}^{1/2}. \quad (5.2.11)$$

The solutions (5.2.2-6) must match as $X \to \pm 1$ with the solutions in the neighbourhood of each endwall where $x \pm L \sim 1$. In these regions the amplitude of the motion is small and the linearized form of the Boussinesq equations may be used. At x=+L the solution for the temperature θ has the form

$$\theta = \frac{4\sqrt{2}}{3\pi^2} L^{-1}\Big\{\big[(e_1+e_2x_2)e^{ik_0x} + (e_1^*+e_2^*x_2)e^{-ik_0x} + e_3e^{2\pi x_2}\big]\sin\pi z + \sqrt{2}\big[(e_4+e_5x_2)e^{ik_1x} + (e_4^*+e_5^*x_2)e^{-ik_1x}$$

$$+ e_6e^{c_1\pi x_2}\big]\sin\pi z\cos\frac{\pi Y}{\bar{a}}\Big\} + L^{-1}\sum_{m=2}^{\infty}\hat{H}_{1m}(x_2)\sin\pi z\cos\frac{m\pi Y}{\bar{a}} + L^{-1}\sum_{m=0}^{\infty}\sum_{n=2}^{\infty}H_{nm}(x_2)\sin n\pi z\cos\frac{m\pi Y}{\bar{a}} \qquad (5.2.12)$$

$$+ L^{-2}\bar{\theta}_2 + L^{-3}\bar{\theta}_3 + \cdots$$

where $x_2 = x - L$ and $\hat{H}_{1m}$ has the form

$$\hat{H}_{1m}(x_2) = (b_{1m1} + b_{1m2}x_2)e^{\bar{k}_mx_2} + b_{1m3}e^{c_m\pi x_2} \qquad (5.2.14)$$

where

$$\bar{k}_m = \frac{\pi}{\sqrt{2}}\left(\frac{2m^2}{\bar{a}^2}-1\right)^{1/2}, \quad c_m = (4+m^2/4a^2)^{1/2} \qquad (5.2.15)$$

and b_{1mi} i=1..3 are real constants which can be expressed in terms of h_{1m}. The complex constants e_j j=1,2,4,5 and the real constants e_3 and e_6 must be determined from the boundary conditions at x=+L and matching with the interior solutions (5.2.2-6). From the latter

$$e_2 = e_5 = 0, \quad e_1 = A(1,\tau) \quad \text{and} \quad e_4 = B(1,\tau). \qquad (5.2.16)$$

Similarly at x=-L the solution for the temperature θ is

$$\theta = \frac{4\sqrt{2}}{3\pi^2}L^{-1}\Big\{\big[A(-1,\tau)e^{ik_0x} + A^*(-1,\tau)e^{-ik_0x} + e_7e^{-2\pi x_1}\big]\sin\pi z + \sqrt{2}\big[B(-1,\tau)e^{+ik_1x} + B^*(-1,\tau)e^{-ik_1x} + e_8e^{-c_1\pi x_1}\big]\sin\pi z\cos\frac{\pi Y}{a}\Big\}$$

$$+ L^{-1}\sum_{m=2}^{\infty}\hat{G}_{1m}(x_1)\sin\pi z\cos\frac{m\pi Y}{a} + L^{-1}\sum_{m=0}^{\infty}\sum_{n=2}^{\infty}G_{nm}(x_1)\sin n\pi z\cos\frac{m\pi Y}{\bar{a}} + L^{-2}\tilde{\theta}_2 + L^{-3}\tilde{\theta}_3 + \cdots \qquad (5.2.17)$$

where $x_1 = x + L$ and e_j j=7,8 are unknown real constants. The function $\hat{G}_{1m}$ has the form

$$\hat{G}_{1m}(x_1) = (a_{1m1} + a_{1m2}x_1)e^{-\bar{k}_mx_1} + a_{1m3}e^{-c_m\pi x_1} \qquad (5.2.18)$$

where a_{1mi} i=1..3 are real constants which can be expressed in terms of g_{1m}. From (5.2.12,17), and with

$$A(x,\tau) = A_1 + iA_2 \quad \text{and} \quad B(x,\tau) = B_1 + iB_2 \qquad (5.2.19)$$

the solution for the temperature

$$\theta = \frac{4\sqrt{2}L^{-1}}{3\pi^2}\Big\{[2(A_1\cos k_0x - A_2\sin k_0x) + e_3e^{2\pi x_2} + e_7e^{-2\pi x_1}]\sin\pi z + [2\sqrt{2}(B_1\cos k_1x - B_2\sin k_1x) + \sqrt{2}e_6e^{c_1\pi x_2} + \sqrt{2}e_8e^{-c_1\pi x_1}]$$

$$\sin\pi z\cos\frac{\pi y}{\bar{a}}\Big\} + L^{-1}\sum_{m=2}^{\infty}(\hat{G}_{1m}(x_1) + \hat{H}_{1m}(x_2))\sin\pi z\cos\frac{m\pi y}{\bar{a}} + L^{-1}\sum_{m=0}^{\infty}\sum_{n=2}^{\infty}(G_{nm} + H_{nm})\sin n\pi z\cos\frac{m\pi y}{\bar{a}} + O(L^{-2}) \qquad (5.2.20)$$

may be regarded as valid to leading order throughout the fluid. From the linearized Boussinesq equations (2.1.24-28) the corresponding velocity components are

$$w = \sqrt{2}\,4L^{-1}\Big\{[A_1\cos k_0x - A_2\sin k_0x - e_3e^{2\pi x_2} - e_7e^{-2\pi x_1}]\sin\pi z + \sqrt{2}[B_1\cos k_1x - B_2\sin k_1x - e_6e^{c_1\pi x_2} - e_8e^{-c_1\pi x_1}]$$

$$\sin\pi z\cos\frac{\pi y}{\bar{a}}\Big\} + L^{-1}\sum_{m=2}^{\infty}\hat{W}_{1m}(x)\sin\pi z\cos\frac{m\pi y}{\bar{a}} + L^{-1}\sum_{m=0}^{\infty}\sum_{n=2}^{\infty}W_{nm}(x)\sin n\pi z\cos\frac{m\pi y}{\bar{a}} + O(L^{-2}), \qquad (5.2.21)$$

$$u = -L^{-1}\Big\{2\sqrt{2}\,[2\sqrt{2}(A_1\sin k_0x + A_2\cos k_0x) - (e_3e^{2\pi x_2} - e_7e^{-2\pi x_1})]\cos\pi z$$

$$+ 2[4\sqrt{2}(1-1/2a^2)^{\frac{1}{2}}(B_1\sin k_1x + B_2\cos k_1x) - c_1(e_6e^{c_1\pi x_2} - e_8e^{-c_1\pi x_1}) - (e_9e^{\bar{N}\pi x_2} + e_{10}e^{-\bar{N}\pi x_1})]\cos\pi z\cos\frac{\pi y}{\bar{a}}\Big\}$$

$$+ L^{-1}\sum_{m=2}^{\infty}\hat{U}_{1m}(x)\cos\pi z\cos\frac{m\pi y}{\bar{a}} + L^{-1}\sum_{m=0}^{\infty}\sum_{n=2}^{\infty}U_{nm}(x)\cos n\pi z\cos\frac{m\pi y}{\bar{a}} + O(L^{-2}) \qquad (5.2.22)$$

and

$$v = -L^{-1}\Big\{\frac{1}{a}[8(B_1\cos k_1x - B_2\sin k_1x) + e_6e^{c_1\pi x_2} + e_8e^{-c_1\pi x_1} + 2a^2\bar{N}(e_9e^{\bar{N}\pi x_2} - e_{10}e^{-\bar{N}\pi x_1})]\cos\pi z\sin\frac{\pi y}{\bar{a}}\Big\}$$

$$+ L^{-1}\sum_{m=2}^{\infty}\hat{V}_{1m}(x)\cos\pi z\sin\frac{m\pi y}{\bar{a}} + L^{-1}\sum_{m=0}^{\infty}\sum_{n=2}^{\infty}V_{nm}(x)\cos n\pi z\sin\frac{m\pi y}{\bar{a}} + O(L^{-2}) \qquad (5.2.23)$$

where

$$\bar{N} = (1 + 1/4a^2), \qquad (5.2.24)$$

e_{10} and e_9 are real unknown constants, and the functions $\hat{W}_{1m}$, $\hat{U}_{1m}$ and $\hat{V}_{1m}$ can be expressed in terms of $\hat{G}_{1m}(x_1)$ and $\hat{H}_{1m}(x_2)$. Similarly the unknown functions W_{nm}, U_{nm} and V_{nm} can be expressed in terms of $G_{nm}(x_1)$ and $H_{nm}(x_2)$.

The boundary conditions at x=+L give the relationships

$$A_1(1,\tau)\cos k_0L - A_2(1,\tau)\sin k_0L = e_3, \qquad B_1(1,\tau)\cos k_1L - B_2(1,\tau)\sin k_1L = e_6;$$

$$2\sqrt{2}(A_1(1,\tau)\sin k_0L + A_2(1,\tau)\cos k_0L) = e_3, \quad 4\sqrt{2}(1-1/2a^2)^{1/2}(B_1(1,\tau)\sin k_1L + B_2(1,\tau)\cos k_1L) = c_1e_6 + e_9,$$

$$\frac{4\sqrt{2}}{3\pi^2}\left[-\sqrt{2}\pi\left(A_1(1,\tau)\sin k_0 L + A_2(1,\tau)\cos k_0 L\right) + 2\pi e_3\right] = \lambda h_{10}, \qquad 18 e_6 + 4a^2 \bar{N} e_4 = 0,$$

$$\frac{8}{3\pi^2}\left[-\sqrt{2}\pi\left(1 - 1/2a^2\right)^{\frac{1}{2}}\left(B_1(1,\tau)\sin k_1 L + B_2(1,\tau)\cos k_1 L\right) + c_1 \pi e_6\right] = \lambda h_{11}. \tag{5.2.25}$$

Hence

$$A(1,\tau) = a_1 \lambda e^{ib_1}, \qquad B(1,\tau) = a_3 \lambda e^{ib_2} \tag{5.2.26}$$

where

$$a_1 = 3h_{10}\pi/8, \qquad b_1 = -k_0 L + \tan^{-1} 1/2\sqrt{2},$$

$$a_3 = \pi h_{11}\left(1 + a_5^2\right)^{1/2} / \left(2c_1 + 3/a^2\bar{N}\right), \quad b_2 = -k_1 L + \tan^{-1} a_5, \quad a_5 = \frac{2c_1 a^2 \bar{N} - 9}{8\sqrt{2}\, a^2 \bar{N}\left(1 - 1/2a^2\right)^{1/2}}. \tag{5.2.27}$$

Similarly the conditions at x=-L give

$$A(-1,\tau) = -a_2 \lambda e^{-ib_1}, \qquad B(-1,\tau) = -a_4 \lambda e^{-ib_2} \tag{5.2.28}$$

where

$$a_2 = 3g_{10}\pi/8, \qquad a_4 = \pi g_{11}\left(1 + a_5^2\right)^{1/2} / \left(2c_1 + 3/a^2\bar{N}\right). \tag{5.2.29}$$

It should be noted that if the endwalls are perfect insulators, the condition on the endwalls are

$$\frac{\partial\theta}{\partial x} = 0 \qquad x = \pm L, \tag{5.2.30}$$

which is equivalent to λ=0. Thus the conditions (5.2.26,28) become

$$A = B = 0 \qquad X = \pm 1. \tag{5.2.31}$$

The critical Rayleigh number R_c for the long box determined in the previous section using linear theory may also be determined for the horizontal modes m=0 and m=1 as follows: the nonlinear terms in equations (5.2.8,9) may be neglected since at the critical Rayleigh number the amplitude of the motion is small. In addition, by neglecting the slow time dependence τ the equations (5.2.8,9) become

$$A_{xx} + \delta A = 0 \qquad \text{and} \qquad \left(1 - 1/2a^2\right)B_{xx} + \delta B = 0. \tag{5.2.32}$$

Solving the equation (5.2.32) subject to the conditions (5.2.31) gives

$$\delta = \frac{\pi^2}{4} j^2 (1 - m^2/2a^2) \qquad j = 1, 2 \ldots \;, \quad m = 0, 1, \tag{5.2.33}$$

which agrees with the result (5.1.37). Here it is assumed that $\frac{1}{\sqrt{2}} < a < \sqrt{2}$ and therefore the critical Rayleigh number R_c for the long box corresponds to j=1 and m=1, with

$$R_c = R_o + \frac{9\pi^4}{2} (1 - m^2/2a^2) L^{-2} + \ldots \qquad m = 1, \quad (L >> 1), \tag{5.2.34}$$

as obtained in the previous section (M=1). The correction to R_o plotted against the aspect ratio a is shown in figure 23 below.

5.3 'Finite-roll' approximation: linear and nonlinear theory

5.3(1) Linear theory

As in the case of the 'finite-roll' approximation for the infinite channel (section 2.6) the upper and lower surfaces are assumed stress-free and perfect conductors, while the sidewalls are assumed to be rigid and perfect conductors, thus enabling a solution to be constructed by separation of variables. The conditions on the top and bottom boundaries and the sidewalls are given by (2.2.1) and (2.6.1) respectively. As in sections 5.1 and 5.2 the endwalls are assumed to be rigid but not perfect insulators. The thermal conditions on the endwalls are given by (5.1.2) where g and h are given functions of Y (=y+a) and z, which are assumed to be consistent with the conditions (2.2.1) and (2.5.2), and may be decomposed into the Fourier series

$$g(Y,z) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} g_{nm} \sin n\pi z \sin \frac{m\pi Y}{a}, \quad h(Y,z) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} h_{nm} \sin n\pi z \sin \frac{m\pi Y}{a}. \tag{5.3.1}$$

The conditions on the velocity components at $x = \pm L$ are given by (5.1.1). It is assumed that the flow is steady.

If the parameter λL^{-1} in the thermal condition (5.1.2) is assumed to be small the dependent variables satisfy the linearized form of the Boussinesq equations given by (2.1.24-28). Since the 'finite-roll' approximation assumes that the horizontal velocity component in the y-direction, v, is zero and that the y momentum equation is neglected, eliminating the variables in equations (2.1.24,25,27,28) gives a single partial differential equation in θ given by (2.6.2).

From section 2.6 the solution satisfying the homogenous differential equation (2.6.2) and the boundary conditions (2.2.1) and (2.6.1) is

$$\theta = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} [\tilde{\theta}_{nm}^{E}(x) + \tilde{\theta}_{nm}^{O}(x)] \sin n\pi z \sin \frac{m\pi y}{\bar{a}} \tag{5.3.2}$$

where

$$\tilde{\theta}_{nm}^{E} = \hat{q}_{nm1} \cos k_{nm1} x + \hat{q}_{nm2} \cos k_{nm2} x + \hat{q}_{nm3} \cosh \bar{k}_{nm3} x,$$

$$\tilde{\theta}_{nm}^{O} = \bar{q}_{nm1} \sin k_{nm1} x + \bar{q}_{nm2} \sin k_{nm2} x + \bar{q}_{nm3} \sinh \bar{k}_{nm3} x, \tag{5.3.3}$$

and $k_{nm1}^2, k_{nm2}^2, k_{nm3}^2 = -\bar{k}_{nm3}^2$ are the roots of the bi-cubic characteristic equation

$$(k_{nm}^2)^3 + (3n^2\pi^2 + 2m^2\pi^2/\bar{a}^2)(k_{nm}^2)^2 + [(n^2\pi^2 + m^2\pi^2/\bar{a}^2)(3n^2\pi^2 + m^2\pi^2/\bar{a}^2) - R](k_{nm}^2) + n^2\pi^2(n^2\pi^2 + m^2\pi^2/\bar{a}^2)^2 = 0. \tag{5.3.4}$$

As in section 5.1, from the existence of the neutral stability curve and the properties of the characteristic equation (5.3.4) the root $k_{nm3}^2 = -\bar{k}_{nm3}^2$ remains real and negative for all values of R. The roots k_{nm1}^2 and k_{nm2}^2 are real and positive if $R > R_{nmc}$, are equal when $R = R_{nmc}$ ($k_{nm1}^2 = k_{nm2}^2 = \frac{1}{4}(9n^4\pi^4 + 8m^2n^2\pi^4/\bar{a}^2)^{\frac{1}{2}} - \frac{n^2\pi^2}{4}$) and are complex conjugate pairs with non-zero imaginary parts for $R < R_{nmc}$. Here

$$R_{nmc} = \frac{1}{16}(\beta_{nm} + 3n^2\pi^2)(\beta_{nm} + 3n^2\pi^2 + 4m^2\pi^2/\bar{a}^2)^2/(\beta_{nm} - n^2\pi^2), \quad (\beta_{nm} = (9\pi^4 n^4 + 8m^2n^2\pi^4/\bar{a}^2)^{1/2}) \tag{5.3.5}$$

is the critical Rayleigh number determined in section 2.6 for the mode of instability (n,m) in the infinite rigid channel using the 'finite-roll' approximation. Thus for a given mode (n,m) if $R > R_{nmc}$ the constants $\hat{q}_{nmj}$ and $\bar{q}_{nmj}$ j=1..3 are purely real. However, if $R<R_{nmc}$ the constants $\hat{q}_{nm3}$, $\bar{q}_{nm3}$ remain real while the constants $\hat{q}_{nm1,2}$ and $\bar{q}_{nm1,2}$ become complex conjugate pairs.

From the linearized heat equation (2.1.28) the vertical velocity component w is

$$w = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} [\tilde{w}^E_{nm}(x) + \tilde{w}^o_{nm}(x)] \sin n\pi z \sin \frac{m\pi y}{\bar{a}} \tag{5.3.6}$$

where

$$\tilde{w}^E_{nm} = \hat{q}_{nm1}(k^2_{nm1}+c_4)\cos k_{nm1}x + \hat{q}_{nm2}(k^2_{nm2}+c_4)\cos k_{nm2}x - \hat{q}_{nm3}(\bar{k}^2_{nm3}-c_4)\cosh \bar{k}_{nm3}x,$$

$$\tilde{w}^o_{nm} = \bar{q}_{nm1}(k^2_{nm1}+c_4)\sin k_{nm1}x + \bar{q}_{nm2}(k^2_{nm2}+c_4)\sin k_{nm2}x - \bar{q}_{nm3}(\bar{k}^2_{nm3}-c_4)\sinh \bar{k}_{nm3}x \tag{5.3.7}$$

and

$$c_4 = n^2\pi^2 + m^2\pi^2/\bar{a}^2. \tag{5.3.8}$$

Since the horizontal velocity component v is assumed to be zero, the horizontal velocity component u from the continuity equation (2.1.24) is

$$u = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} [\tilde{u}^o_{nm}(x) + \tilde{u}^E_{nm}(x)] \cos n\pi z \sin \frac{m\pi y}{\bar{a}} \tag{5.3.9}$$

where

$$\tilde{u}^o_{nm} = -\hat{q}_{nm1}c_1 \sin k_{nm1}x - \hat{q}_{nm2}c_2 \sin k_{nm2}x + \hat{q}_{nm3}c_3 \sinh \bar{k}_{nm3}x,$$

$$\tilde{u}^E_{nm} = \bar{q}_{nm1}c_1 \cos k_{nm1}x + \bar{q}_{nm2}c_2 \cos k_{nm2}x + \bar{q}_{nm3}c_3 \cosh \bar{k}_{nm3}x \tag{5.3.10}$$

and

$$c_i = n\pi(k^2_{nmi}+c_4)/k_{nmi} \quad i=1,2, \quad c_3 = n\pi(\bar{k}^2_{nm3}-c_4)/\bar{k}_{nm3}. \tag{5.3.11}$$

From the evenness of the equations satisfied by θ_{nm}, the linearity of the equations (2.1.24,28) and the form of the conditions (5.1.2,4) which have to be satisfied at $x=\pm L$, it follows that the general solution for θ_{nm} falls into two non-combining groups of even and odd solutions. This enables the problem to be divided into 'even' and 'odd' cases. The conditions (5.1.2) may therefore be more conveniently expressed as

$$\frac{\partial\theta}{\partial x} = \frac{\lambda L^{-1}}{2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}[(g_{nm}+h_{nm})+(g_{nm}-h_{nm})]\sin n\pi z \sin\frac{m\pi y}{\bar{a}} \quad x=-L$$

and (5.3.12)

$$\frac{\partial\theta}{\partial x} = \frac{\lambda L^{-1}}{2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}[(g_{nm}+h_{nm})-(g_{nm}-h_{nm})]\sin n\pi z \sin\frac{m\pi y}{\bar{a}} \quad x=+L.$$

Thus applying the conditions (5.3.12) and (5.1.1) to the equations (5.3.2,6,9), for the even case gives a non-homogenous system of three equations in three unknowns which can be expressed in matrix notation as

$$[\hat{B}][q] = [\hat{S}] \tag{5.3.14}$$

where

$$\hat{B} = \begin{bmatrix} c_1 & , & c_2 & , & c_3 \\ (k_{nm1}^2+c_4)\cot k_{nm1}L & , & (k_{nm2}^2+c_4)\cot k_{nm2}L & , & -(\bar{k}_{nm3}^2-c_4)\coth\bar{k}_{nm3}L \\ k_{nm1} & , & k_{nm2} & , & -\bar{k}_{nm3} \end{bmatrix},$$

$$q=[\hat{q}_{nm1}\sin k_{nm1}L, \hat{q}_{nm2}\sin k_{nm2}L, \hat{q}_{nm3}\sinh\bar{k}_{nm3}L]^{tr}, \quad \hat{S}=[0, 0, \tfrac{\lambda L^{-1}}{2}(g_{nm}-h_{nm})]^{tr}. \tag{5.3.15}$$

For the odd case the matrices $\hat{B}$, q and $\hat{S}$ become

$$\hat{B} = \begin{bmatrix} c_1 & , & c_2 & , & c_3 \\ (k_{nm1}^2+c_4)\tan k_{nm1}L & , & (k_{nm2}^2+c_4)\tan k_{nm2}L & , & -(\bar{k}_{nm3}^2-c_4)\tanh\bar{k}_{nm3}L \\ k_{nm1} & , & k_{nm2} & , & \bar{k}_{nm3} \end{bmatrix},$$

$$q=[\bar{q}_{nm1}\cos k_{nm1}L, \bar{q}_{nm2}\cos k_{nm2}L, \bar{q}_{nm3}\cosh\bar{k}_{nm3}L]^{tr}, \quad \hat{S}=[0, 0, \tfrac{\lambda L^{-1}}{2}(g_{nm}+h_{nm})]^{tr}. \tag{5.3.16}$$

As in the case of the stress-free long box, for a given mode (n,m) when the six constants $\hat{q}_{nmj}$ and $\bar{q}_{nmj}$ j=1..3 are determined from the conditions (5.1.1,2) all the terms in the solutions (5.3.2,6,9) decay near the endwalls if $R<R_{nmc}$ and $L>>1$.

It is expected from section 5.1 that the change in the overall critical Rayleigh number R_0 for the 'finite-roll' approximation of the infinite channel (given by (2.6.9)) due to the introduction of the distant endwalls may be investigated by considering a perturbation of the form

$$R = R_o + \bar{\delta} L^{-2} \tag{5.3.17}$$

where $\bar{\delta}$ is assumed to be order one and $L \gg 1$. It should be noted that R_o corresponds to the vertical mode n=1 and the horizontal mode m=1 in the y-direction. It can be shown from (5.3.5) that

$$k_{112,1}^2 \sim k_1^2 \left[1 \pm 2i(R_o - R)^{1/2} / \left(k_1(3\beta_{11} + 9\pi^2 + 8\pi^2/\bar{a}^2)^{1/2}\right)\right] \quad \text{as } R \to R_o, \tag{5.3.18}$$

where

$$k_1^2 = \frac{1}{4}(\beta_{11} - \pi^2) \quad \text{and} \quad \beta_{11} = (9\pi^4 + 8\pi^4/\bar{a}^2)^{\frac{1}{2}}, \tag{5.3.19}$$

so that

$$k_{112,1}^2 \sim k_1^2 \pm K\bar{\delta}^{\frac{1}{2}} L^{-1} \qquad (L \gg 1) \tag{5.3.20}$$

where

$$K = 2k_1 / (3\beta_{11} + 9\pi^2 + 8\pi^2/\bar{a}^2)^{1/2}. \tag{5.3.21}$$

Substituting (5.3.20) into (5.3.5) gives

$$\bar{k}_{113}^2 = \frac{1}{2}\left[5\pi^2 + 4\pi^2/\bar{a}^2 + \beta_{11}\right] + O(L^{-2}). \tag{5.3.22}$$

Thus, when $\bar{\delta}$ is positive the solution for the temperature $\theta(x,y,z)$ may be expanded in the form

$$\theta = \frac{\lambda L^{-1}}{4\pi^2(1+1/\bar{a}^2)(k_1^2 + \bar{k}_{113}^2)} \Big\{ (\bar{k}_{113}^2 - \hat{c}_4)\Big[(h_{11} - g_{11})\Big(\frac{\bar{\phi}_1 \cos k_1 x \cos \Delta x}{\cos \Delta L} - \frac{\bar{\phi}_2 \sin k_1 x \sin \Delta x}{\sin \Delta L}\Big)$$

$$+ (h_{11} + g_{11})\Big(\frac{\bar{\phi}_1 \sin \Delta x \cos k_1 x}{\sin \Delta L} - \frac{\bar{\phi}_2 \sin k_1 x \cos \Delta x}{\cos \Delta L}\Big)\Big] - 4\bar{k}_{113}(k_1^2 + \hat{c}_4)\left[g_{11} e^{-\bar{k}_{113} x_1} - h_{11} e^{\bar{k}_{113} x_2}\right]\Big\} \sin \pi z \sin \frac{\pi y}{\bar{a}}$$

$$+ \lambda L^{-1} \sum_{\substack{m=1 \\ (mn) \neq (11)}}^{\infty} \sum_{n=1}^{\infty} \left[\tilde{G}_{nm}(x_1) + \tilde{H}_{nm}(x_2)\right] \sin n\pi z \sin \frac{m\pi y}{\bar{a}} + O(\lambda L^{-2}) \tag{5.3.23}$$

where

$$x_1 = x + L, \quad x_2 = x - L, \quad \hat{c}_4 = \pi^2 + \pi^2/\bar{a}^2,$$

$$\bar{\phi}_1 = k_1 \cos k_1 L - \bar{k}_{113} \sin k_1 L, \; \bar{\phi}_2 = k_1 \sin k_1 L + \bar{k}_{113} \cos k_1 L, \; \Delta = \frac{1}{2k_1} \bar{\delta}^{1/2} L^{-1} K, \tag{5.3.24}$$

and the functions $\tilde{G}_{nm}$ and $\tilde{H}_{nm}$ have the forms

$$\tilde{G}_{nm}(x_1) = e^{-\tilde{\omega}_{nm1} x_1}(\tilde{a}_{nm1} \cos \tilde{\omega}_{nm2} x_1 + \tilde{a}_{nm2} \sin \tilde{\omega}_{nm2} x_1) + \tilde{a}_{nm3} e^{-\bar{k}_{113} x_1},$$

and (5.3.25)

$$\tilde{H}_{nm}(x_2) = e^{\tilde{\omega}_{nm1}x_2}(\tilde{b}_{nm1}\cos\tilde{\omega}_{nm2}x_2 + \tilde{b}_{nm2}\sin\tilde{\omega}_{nm2}x_2) + \tilde{b}_{nm3}e^{k_{113}x_2}$$

where $\tilde{a}_{nmi}, \tilde{b}_{nmi}$ $i=1..3$ are real constants which may be written in terms of g_{nm} and h_{nm} respectively and

$$\tilde{\omega}_{nm1} \pm i\tilde{\omega}_{nm2} = ik_{nm2,1} \quad (5.3.26)$$

where $\tilde{\omega}_{nm1,2}$ are real and positive constants.

In contrast to the finite solution obtained for $\bar{\delta} < 0$, from (5.3.23) and the form of Δ, it can be seen that the linear solution becomes infinite at an infinite sequence of values of $\bar{\delta}$ given by

$$\bar{\delta} = \frac{\pi^2 j^2}{4}(3\beta_{11} + 9\pi^2 + 2\pi^2/a^2) \qquad j = 1, 2 \ldots \quad (5.3.27)$$

As shown for the stress-free long box, these states correspond to the existence of eigensolutions in the equivalent long box with null endwall boundary conditions ($\lambda=0$). The first such state corresponds to the critical Rayleigh number R_c for the long box:

$$R_c = R_0 + \frac{\pi^2}{4}(3\beta_{11} + 9\pi^2 + 2\pi^2/a^2)L^{-2} + O(L^{-3}) \quad (5.3.28)$$

where

$$\beta_{11} = \pi^2(9 + 2/a^2)^{1/2}. \quad (5.3.29)$$

The correction to R_0 plotted against the aspect ratio a is shown in figure 23 below.

5.3(2) Nonlinear theory

It is known from section 3.2 that in the case of the 'finite-roll' approximation for the infinite rigid channel the solution near the critical Rayleigh number R_0, may be expanded in integral powers of a small parameter proportional to $(R-R_0)^{\frac{1}{2}}$. Here, as in the case of the stress-free long box, the semi-aspect ratio L is assumed to be large compared with 1 and 2a, and the solution is expanded in descending powers of L. By introducing the scaled Rayleigh number $\bar{\delta}$ defined by

$$R = R_0 + \bar{\delta} L^{-2} \tag{5.3.30}$$

where $\bar{\delta}$ is assumed to be order one, from section 3.2 the solutions for the temperature θ and the velocity components w and u, in the interior of the long box may be expanded in the forms

$$\theta = \frac{4}{\sigma_1}(\beta_2\beta_3)^{\frac{1}{2}} L^{-1}\left(A(X,\tau)e^{ik_1x} + A^*(X,\tau)e^{-ik_1x}\right)\sin\pi z \sin\frac{\pi y}{\bar{a}} + L^{-2}\theta_2 + L^{-3}\theta_3 + \cdots \tag{5.3.31}$$

$$w = 4(\beta_2\beta_3)^{\frac{1}{2}} L^{-1}\left(A(X,\tau)e^{ik_1x} + A^*(X,\tau)e^{-ik_1x}\right)\sin\pi z \sin\frac{\pi y}{\bar{a}} + L^{-2}w_2 + L^{-3}w_3 + \cdots \tag{5.3.32}$$

and

$$u = \frac{4\pi i}{k_1}(\beta_2\beta_3)^{\frac{1}{2}} L^{-1}\left(A(X,\tau)e^{ik_1x} - A^*(X,\tau)e^{-ik_1x}\right)\cos\pi z \sin\frac{\pi y}{\bar{a}} + L^{-2}u_2 + L^{-3}u_3 + \cdots \tag{5.3.33}$$

where * denotes complex conjugate, k_1 and σ_1 are given by (3.2.2), β_2 and β_3 are given by (3.2.28) and X and τ are slow spatial and time variables related to x and t by

$$X = L^{-1}x, \qquad \tau = L^{-2}\, 4P\beta_3 t/(1+P). \tag{5.3.34}$$

The 'finite-roll' approximation assumes that the horizontal velocity component, v, is zero. The equation for the amplitude function $A(X,\tau)$ is determined from consideration of the second and third terms in the interior expansions (5.3.31-35). An inconsistency in the expansion at $O(L^{-3})$ due to the appearance of a forcing function which is an eigensolution of the basic linear operator with the conditions (3.2.21), can only be avoided if A satisfies the equation

$$-A_\tau + A_{XX} + \delta A = A|A|^2 \tag{5.3.35}$$

where

$$\bar{\delta} = \delta \beta_3/\beta_4 \tag{5.3.36}$$

where β_4 is given by (3.2.28). It should be noted that the equation

(5.3.35) can be obtained directly from equation (3.2.29) using the transformations

$$\varepsilon \to \delta^{-\frac{1}{2}} L^{-1}, \quad X \to \delta^{\frac{1}{2}} X, \quad \tau \to \delta\tau \quad \text{and} \quad A \to A/\delta^{\frac{1}{2}}. \tag{5.3.37}$$

The solutions (5.3.31-35) must match as $X \to \pm 1$ with the solutions in the neighbourhood of each endwall where $x \pm L \sim 1$. In these end regions the amplitude of the motion is small and the linearized form of the Boussinesq equations may be used. At x=+L the solution for the temperature θ has the form

$$\theta = \frac{4}{\sigma_1}(\beta_2\beta_3)^{\frac{1}{2}} L^{-1}\{[(\hat{e}_1 + \hat{e}_2 x_2)e^{ik_1 x} + (\hat{e}_1^* + \hat{e}_2^* x_2)e^{-ik_1 x} + \hat{e}_3 e^{cx_2}] \sin\pi z \sin\frac{\pi Y}{a}\}$$

$$+ L^{-1}\sum_{\substack{m=1 \\ (mn)\neq(11)}}^{\infty}\sum_{n=1}^{\infty} \tilde{H}_{nm}(x_2)\sin n\pi z \sin\frac{m\pi Y}{a} + L^{-2}\bar{\theta}_2 + L^{-3}\bar{\theta}_3 + \cdots \tag{5.3.38}$$

where

$$x_2 = x - L, \qquad c = \frac{1}{\sqrt{2}}(\beta_{11} + S\pi^2 + \pi^2/a^2)^{\frac{1}{2}} \tag{5.3.39}$$

and the complex constants $\hat{e}_j$ j=1,2 and the real constant $\hat{e}_3$ must be determined from the boundary conditions at x=+L and matching with the interior solutions (5.3.31-35). From the latter

$$\hat{e}_2 = 0 \qquad \text{and} \qquad \hat{e}_1 = A(1,\tau). \tag{5.3.40}$$

Similarly at x=-L the solution for the temperature θ is

$$\theta = \frac{4}{\sigma_1}(\beta_2\beta_3)^{\frac{1}{2}} L^{-1}\{[A(-1,\tau)e^{ik_1 x} + A^*(-1,\tau)e^{-ik_1 x} + \hat{e}_4 e^{-cx_1}] \sin\pi z \sin\frac{\pi Y}{a}\}$$

$$+ L^{-1}\sum_{\substack{m=1 \\ (mn)\neq(11)}}^{\infty}\sum_{n=1}^{\infty} \tilde{G}_{nm}(x_1)\sin n\pi z \sin\frac{m\pi Y}{a} + L^{-2}\tilde{\theta}_2 + L^{-3}\tilde{\theta}_3 + \cdots \tag{5.3.41}$$

where $x_1 = x + L$, and the constant $\hat{e}_4$ is real. From (5.3.38,41) and with $A = A_1 + iA_2$ the solution

$$\theta = \frac{4}{\sigma_1}(\beta_2\beta_3)^{\frac{1}{2}} L^{-1}[2(A_1\cos k_1 x - A_2 \sin k_1 x) + \hat{e}_3 e^{cx_2} + \hat{e}_4 e^{-cx_1}] \sin n\pi z \sin\frac{\pi Y}{a}$$

$$+ L^{-1}\sum_{\substack{m=1 \\ (mn)\neq(11)}}^{\infty}\sum_{n=1}^{\infty} [\tilde{G}_{nm}(x_1) + \tilde{H}_{nm}(x_2)] \sin n\pi z \sin\frac{m\pi Y}{a} + O(L^{-2}) \tag{5.3.42}$$

may be regarded as valid to leading order throughout the fluid. The vertical velocity component w can be obtained from the linearized heat equation (2.1.28), which inturn enables the horizontal velocity component u to be obtained from the continuity equation (2.1.24), since the horizontal velocity component v is assumed to be zero.

The boundary conditions at x=+L give the relationships

$$A_1 \cos k_1 L - A_2 \sin k_1 L = \frac{1}{2\sigma_1}\left(c^2 - \pi^2 - \pi^2/4a^2\right)\hat{e}_3 ,$$

$$A_1 \sin k_1 L - A_2 \cos k_1 L = \frac{k_1}{2\sigma_1 c}\left(c^2 - \pi^2 - \pi^2/4a^2\right)\hat{e}_3 ,$$

and

$$\frac{4}{\sigma_1}(\beta_2\beta_3)^{\frac{1}{2}}\left[-2k_1(A_1 \sin k_1 L + A_2 \cos k_1 L) + c\hat{e}_3\right] = \lambda h_{11} . \tag{5.3.43}$$

Thus

$$A(1,\tau) = a_1 \lambda e^{ib_1} \tag{5.3.44}$$

where

$$a_1 = \frac{h_{11}(c^2 - \pi^2 - \pi^2/4a^2)(c^2 + k_1^2)^{1/2}}{8(\beta_2\beta_3)^{1/2}\left[c^2 - \frac{k_1^2}{\sigma_1}(c^2 - \pi^2 - \pi^2/4a^2)\right]} \quad \text{and} \quad b_1 = -k_1 L + \tan^{-1}\frac{k_1}{c} . \tag{5.3.45}$$

Similarly the conditions at x=-L give

$$A(-1,\tau) = -a_2 \lambda e^{-ib_1} \tag{5.3.46}$$

where

$$a_2 = g_{11}(c^2 - \pi^2 - \pi^2/4a^2)(c^2 + k_1^2)^{1/2} \Big/ 8(\beta_2\beta_3)^{\frac{1}{2}}\left[c^2 - \frac{k_1^2}{\sigma_1}(c^2 - \pi^2 - \pi^2/4a^2)\right] . \tag{5.3.47}$$

It should be noted that the change in the critical Rayleigh number R_0, due to the introduction of the distant endwalls may be investigated by assuming that the rigid endwalls are perfect insulators. The thermal condition (5.1.2) becomes

$$\frac{\partial\theta}{\partial x} = 0 \quad x = \pm L , \tag{5.3.48}$$

which is equivalent to λ=0. Hence the conditions (5.3.44,46) on the amplitude function A(X,τ) become

$$A(X,\tau) = 0 \qquad X = \pm 1. \tag{5.3.49}$$

As in the case of the stress-free long box, neglecting the nonlinear term and the slow time dependence in equation (5.3.35) gives

$$A_{XX} + \delta A = 0\,. \tag{5.3.50}$$

Solving (5.3.50) subject to the conditions (5.3.49) gives

$$\delta^2 = j^2\pi^2/4 \qquad j = 1, 2\ldots \tag{5.3.51}$$

The value j=1 corresponds to the critical Rayleigh number R_c for the long box with rigid sidewalls using the 'finite-roll' approximation:

$$R_c = R_0 + \frac{\pi^2}{4}(3\beta_{11} + 9\pi^2 + 2\pi^2/a^2)L^{-2} + \ldots \qquad (L \gg 1)\,, \tag{5.3.52}$$

as obtained using the linear method above.

5.4 Long box with rigid sidewalls

When the sidewalls at y=±a are assumed to be rigid and no conditions are imposed on the behaviour of the horizontal velocity component v, the simplicity of the previous approaches is lost. This is because the eigenfunctions in y are no longer orthogonal and there is no obvious way of reducing the boundary conditions at x=±L to determine the coefficients of a finite number of eigenfunctions. Thus for the long box with rigid sidewalls only a modified version of the nonlinear approach used for the stress-free long box in section 5.2 and the 'finite-roll' approximation for the long box in section 5.3(2), is considered.

As in the case of the infinite rigid channel (section 2.2) the upper and lower surfaces are assumed to be stress-free and perfect conductors, thus enabling a solution to be constructed by separation of variables. The sidewalls at y=±a are assumed to be rigid and only the case where they are perfect conductors is considered. The conditions on the top and bottom boundaries and the rigid sidewalls are given by (2.2.1) and (2.2.4) respectively. The distant endwalls are assumed to be rigid but not perfect insulators. The thermal conditions on the endwalls are

$$\frac{\partial\theta}{\partial x} = \lambda L^{-1}\hat{g}(y,z) \quad x=-L \quad \text{and} \quad \frac{\partial\theta}{\partial x} = \lambda L^{-1}\hat{h}(y,z) \quad x=+L, \tag{5.4.1}$$

where the parameter L^{-1} is introduced for convenience, $\hat{g}$ and $\hat{h}$ are given functions of y and z which are assumed to be consistent with the conditions (2.2.1,4) and may be decomposed into the Fourier series

$$\hat{g}(y,z) = \sum_{n=1}^{\infty} \hat{g}_n(y)\sin n\pi z, \quad \hat{h}(y,z) = \sum_{n=1}^{\infty} \hat{h}_n(y)\sin n\pi z. \tag{5.4.2}$$

The parameter λ is that introduced in section 5.1. The conditions on the velocity components at $x=\pm L$ are given by (5.1.1). It is assumed that 2L is large compared with 1 and 2a, and the flow is steady.

It is known from chapter 4 that for the infinite rigid channel the solution near the critical Rayleigh number R_0 may be expanded in integral powers of a small parameter proportional to $(R-R_0)^{\frac{1}{2}}$. Here, as in sections 5.2 and 5.3 the solution is expanded in descending powers of the semi-aspect ratio L. By introducing the scaled Rayleigh number $\bar{\delta}$ defined by

$$R = R_0 + \bar{\delta}L^{-2} \tag{5.4.3}$$

where $\bar{\delta}$ is assumed to be order one, and assuming that L>>1 the solution for the temperature θ in the interior of the long box may be expanded in the form

$$\theta = L^{-1}\Theta(y)\left(\bar{A}(X,\tau)e^{ik_0x} + \bar{A}^*(X,\tau)e^{-ik_0x}\right)\sin\pi z + L^{-2}\theta_2 + L^{-3}\theta_3 + \cdots \tag{5.4.4}$$

where * denotes complex conjugate, k_0 is the wavenumber associated with the critical Rayleigh number R_0, and X and τ are slow spatial and time variables related to x and t by

$$X = L^{-1}x \;, \qquad \tau = L^{-2}t. \tag{5.4.5}$$

Similar expressions also exist for the other dependent variables w,u,v and p. The equation for the amplitude function $\bar{A}(X,\tau)$ is determined from consideration of the second and third terms in the interior expansions for the dependent variables, as given by (5.4.4) for the temperature θ. An inconsistency in the expansion at $O(L^{-3})$ due to the appearance of a forcing function which is an eigensolution of the basic linear system with the conditions (4.4.7), can only be avoided if $\bar{A}$ satisfies the equation

$$C_1\delta\bar{A} + C_2\bar{A}_{XX} + (C_3 + C_4/P + C_5/P^2)\bar{A}|\bar{A}|^2 + (C_6 + C_7/P)\bar{A}_\tau = 0. \tag{5.4.6}$$

The amplitude coefficients C_i i=1..7 are determined numerically from equations (4.4.34). For the aspect ratios under consideration (ie. a=¼,½,1 and 2) the coefficients are displayed in table 7. It should be noted that equation (5.4.6) can be obtained directly from equation (4.4.33) using the transformations

$$\varepsilon \to \delta^{-\frac{1}{2}}L^{-1}, \quad X \to \delta^{-\frac{1}{2}}X, \quad \tau \to \delta\tau \text{ and } \bar{A} \to \bar{A}/\delta^{\frac{1}{2}}. \tag{5.4.7}$$

In order to complete the specification of $\bar{A}(X,\tau)$, boundary conditions at $X=\pm1$ which are fixed by the conditions at the endwalls are needed. Initial conditions at $\tau=0$ would also be required in time-dependent situations. The solutions for the dependent variables in the interior of the long box must match as $X\to\pm1$ with the solutions in the neighbourhood of each endwall where $x \pm L \sim 1$. In these regions the amplitude of the motion is small and the linearized form of the Boussinesq equations given by (2.1.24-28) may be used. Here the solution at x=-L is considered; the solution near x=+L has a similar form.

In the case of the 'finite-roll' approximation for the long box (section 5.3) the solution at the critical Rayleigh number R_o, near $x=-L$ where $x_1 = x + L = O(1)$, is expressed as a sum of eigenfunctions in y and z each multiplied by three exponential functions of x_1 which remain bounded as $x_1 \to \infty$:

$$\theta \sim L^{-1} \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \tilde{\theta}_{nm}(x) \sin n\pi z \sin \frac{m\pi y}{a} \tag{5.4.8}$$

where, for $(n,m) \neq (1,1)$

$$\tilde{\theta}_{nm}(x) = \hat{E}_1 e^{ik_{nm1}x_1} + \hat{E}_2 e^{-ik_{nm2}x_1} + \hat{E}_3 e^{-\bar{k}_{nm3}x_1} \tag{5.4.9}$$

where $\hat{E}_i$ $i=1..3$ are arbitrary constants and k^2_{nm1}, k^2_{nm2}, $k^2_{nm3}=-\bar{k}^2_{nm3}$ are the roots of the bi-cubic characteristic equation (5.3.5) with $R=R_o$. As the eigenfunctions in y and the eigenfunctions in z are orthogonal, when applying the boundary conditions at $x_1=0$ each mode (n,m) may be considered independently of the other modes. It is known from section 5.3 that if $(n,m) \neq (1,1)$ the roots k^2_{nm1} and k^2_{nm2} are complex conjugate pairs with non-zero imaginary parts and thus, $\tilde{\theta}_{nm}$ consists of six possible exponential terms, of which three decay and three grow as $x_1 \to \infty$. The latter are therefore discarded in (5.4.9). If however, $(n,m) = (1,1)$, the roots k^2_{111} and k^2_{112} are purely real and equal. Therefore $\tilde{\theta}_{11}$ has the general form

$$\tilde{\theta}_{11} = [(\hat{e}_1+\hat{e}_2 x_1)e^{ik_{111}x} + (\hat{e}_1^* + \hat{e}_2^* x_1)e^{-ik_{111}x} + \hat{e}_3 e^{-\bar{k}_{113}x_1} + \hat{e}_4 e^{\bar{k}_{113}x_1}] \tag{5.4.10}$$

where $\hat{e}_j$ $j=1,2$ are complex constants and $\hat{e}_j$ $j=3,4$ are real constants. Of the six terms in (5.4.10), one can be excluded on the grounds that it is exponentially large at infinity and two because they are algebraically large, since as shown in section 5.3(2) the matching condition requires that their numerical coefficients be zero. The three unknown numerical coefficients are determined from the boundary conditions at $x_1=0$. In this way $\bar{A}(-1,\tau)$, and similarly $\bar{A}(1,\tau)$, are fixed.

In principle the same method may be used here but there is one essential difference. The eigenfunctions in z are orthogonal and thus only the n=1 case needs to be considered. However, the eigenfunctions in y are not orthogonal and strictly speaking there is no obvious way of reducing the conditions at $x_1=0$ to enable the coefficients of a finite number of eigenfunctions to be found. Following the method outlined by Stewartson and Weinstein (1979) for the related two-dimensional long box with rigid lower and upper boundaries (z=0,1), an infinite set of eigenfunctions in y must be considered. A typical element of the temperature θ is given by

$$\theta = \psi(y)\, e^{ikx_1} \sin \pi z \tag{5.4.11}$$

where, for an even solution in y

$$\psi = d_1 \cosh r_1 y + d_2 \cosh r_2 y + d_3 \cosh r_3 y \, . \tag{5.4.12}$$

Since not all the conditions (2.2.4) are in terms of θ, the corresponding element of the horizontal velocity component v is also needed:

$$v = (d_1\beta_1 \sinh r_1 y + d_2\beta_2 \sinh r_2 y + d_3\beta_3 \sinh r_3 y + d_4 \sinh r_4 y)\, e^{ikx_1} \cos \pi z \tag{5.4.13}$$

where β_i i=1..3 are defined by (2.2.15). The roots r_i i=1..3 satisfy

$$[\,(r^2-k^2-\pi^2)^3 - R_0(r^2-k^2)\,] = 0 \tag{5.4.14}$$

and $r_4^2 = k^2+\pi^2$, where R_0 is the critical Rayleigh number for a given aspect ratio a. It should be noted that the full end-zone solution is

$$\theta = \theta^E + \theta^O \qquad \text{and} \qquad v = v^O + v^E \, . \tag{5.4.15}$$

However, attention may be restricted to the part of θ that is even in y, denoted by θ^E, since no component of the odd part in y, denoted by

θ°, remains finite as $x_1 \to \infty$ when $R=R_0$ and, therefore, cannot affect the interior solution.

Applying the boundary conditions (2.2.4) to (5.4.12,13) gives a homogenous system of four equations in four unknowns which can be expressed in matrix notation as

$$[D][d] = [0] \tag{5.4.16}$$

where

$$D = \begin{bmatrix} \cosh r_1 a & , & \cosh r_2 a & , & \cosh r_3 a & , & 0 \\ (r_1^2 - k^2 - \pi^2)\cosh r_1 a & , & (r_2^2 - k^2 - \pi^2)\cosh r_2 a & , & (r_3^2 - k^2 - \pi^2)\cosh r_3 a & , & 0 \\ \beta_1 r_1 \cosh r_1 a & , & \beta_2 r_2 \cosh r_2 a & , & \beta_3 r_3 \cosh r_3 a & , & (k^2+\pi^2)^{1/2}\cosh r_4 a \\ \beta_1 \sinh r_1 a & , & \beta_2 \sinh r_2 a & & \beta_3 \sinh r_3 a & , & \sinh r_4 a \end{bmatrix}, \tag{5.4.17}$$

and d is the column vector $\{d_i\}$ i=1..4. For a non-trivial solution of (5.4.16) to exist the determinant of the (generally complex) matrix D must be zero.

From the (n,m) = (1,1) mode for the stress-free long box (section 5.2) it is anticipated that there are three distinct groups of eigenfunctions, one where k is complex in which case $\psi = \psi_m$ and $k=k_m$, and two groups where k is positive and purely imaginary in which case $\psi = \bar{\psi}_m, \hat{\psi}_m$ and $k=\bar{k}_m, \hat{k}_m$ respectively. The numerical computation of ψ_m is tedious, since it requires a search in the complex plane of k. It starts at $k=k_0$ and the curve through this point on which the real part of the determinant of the matrix D vanishes, is computed. Once a new point on this curve is found the imaginary part of the determinant of D is computed, thus enabling the points where the complete determinant vanishes to be computed. The numerical computation of $\bar{\psi}_m$ and $\hat{\psi}_m$ is simpler since $\bar{k}_m$ and $\hat{k}_m$ are purely imaginary and, therefore the determinant of D is purely real. A simple interval bi-section method was used to find $\bar{k}_m$ and $\hat{k}_m$. A problem that arises is that since $\bar{k}_m$ and

$\hat{k}_m$ are both purely imaginary there is no definite way of dividing them into two distinct groups. To some extent this difficulty may be overcome by counting the zeroes of $\bar{\psi}_m, \hat{\psi}_m$ and the related terms in v, and grouping them accordingly. Unfortunately, this approach does not conclusively distinguish $\bar{k}_m$ and $\hat{k}_m$ into two groups. However, it should be noted that it is not really necessary to distinguish $\bar{k}_m$ and $\hat{k}_m$ into definite distinct groups, since this just corresponds to a re-ordering of the exponentially decaying solutions that exist in the end region where $x_1 \sim 1$ (see below). The figures 18,19,20 and 21 show plots of $k_m, \bar{k}_m$ and $\hat{k}_m$, m=0,1..19 for the aspect ratios $a=\frac{1}{4},\frac{1}{2},1$ and 2. The values of $k_m, \bar{k}_m$ and $\hat{k}_m$ are displayed in the tables 8,9,10 and 11 for $a=\frac{1}{4},\frac{1}{2},1$ and 2 respectively.

The solution of the linearized Boussinesq equations (2.1.24-28) at $R=R_0$, near x=-L, can now be written as $\theta = \theta^E + \theta^0$ where

$$\theta^E = L^{-1}\sum_{m=0}^{\infty}\left[E_m\psi_m(y)e^{ik_m x_1} + E_m^*\psi_m^*(y)e^{-ik_m^* x_1} + \bar{E}_m\bar{\psi}_m(y)e^{i\bar{k}_m x_1} + \hat{E}_m\hat{\psi}_m(y)e^{i\hat{k}_m x_1}\right]\sin\pi z$$

$$+ L^{-1}\sum_{m=0}^{\infty}\sum_{n=2}^{\infty} G_{nm}(x,y)\sin n\pi z + O(L^{-2}). \qquad (5.4.18)$$

Here E_m are complex constants and $\bar{E}_m, \hat{E}_m$ are real constants which need to be determined, $i\bar{k}_m$ and $i\hat{k}_m < 0$, $\mathrm{Im}(k_m) > 0$, $(m>0)$, ψ_m is complex, $\bar{\psi}_m$ and $\hat{\psi}_m$ are real, and all the functions G_{nm} decay as $x_1 \to \infty$. The corresponding velocity components u,v and w can be easily obtained from the linearized Boussinesq equations. It should be noted that when m=0, k_0 is purely real and there exists an additional eigenfunction of the form

$$e^{ik_0 x_1}\tilde{E}_0\left(x_1\psi_0(y) + \tilde{\psi}_0(y)\right) \qquad (5.4.19)$$

where

$$\psi_0 = \Theta \quad \text{and} \quad \tilde{\psi}_0 = -i\left.\frac{\partial\Theta}{\partial k}\right|_{k=k_0} = i\Theta_1 , \qquad (5.4.20)$$

Θ is given by (4.1.3) and Θ_1 is the solution obtained from the matrix system (4.3.48) subject to the additional condition (4.3.52). However, the matching requirement as $x_1 \to \infty$ implies that $\tilde{E}_0=0$. In contrast to the (n,m) = (1,1) mode for the stress-free long box where the solution for the temperature θ given by (5.2.17), contains three exponential functions of x_1, here the solution for the temperature θ given by (5.1.18), contains four exponential functions of x_1. This is because here, in order to determine the related eigenfunctions in y, it is necessary to consider a coupled eighth order system in θ and v, for which there are eight boundary conditions, four in terms of θ and four in terms of v. For the stress-free case the related eigenfunctions in y can be determined directly from a sixth order differential equation in θ, for which there exists six boundary conditions in terms of θ. The corresponding fourth set of exponential functions in x are then confined to the solutions for u and v.

By applying the boundary conditions (5.1.1) and (5.4.1) to (5.4.18) and the corresponding forms for the velocity components u,v and w, a quadruply-infinite set of equations for $E_m, \bar{E}_m$ and $\hat{E}_m$ is obtained. Since there are no obvious orthogonality relationships connecting $\psi_m, \bar{\psi}_m$ and $\hat{\psi}_m$, the unknown constants $E_m, \bar{E}_m$ and $\hat{E}_m$ need to be determined using a collocation method. This involves satisfying the boundary conditions on θ, u and w at a set of points distributed along the line x=-L, $|y| \leq$ a and, on v at another set of points. The infinite series in (5.4.18) is truncated so that there are as many equations for $E_m, \bar{E}_m$ and $\hat{E}_m$ as there are unknowns. Thus, replacing the ∞ in (5.4.18) by m_∞, the conditions on θ^E, u^E and w^E need to be satisfied at the set of points

$$y = am / m_\infty \qquad m = 0, 1, 2 \cdots m_\infty, \tag{5.4.21}$$

while the conditions on v^0 need to be satisfied at the points

$$y = a(m+1)/(m_\infty+1) \qquad m=0,1,2\ldots m_\infty, \tag{5.4.22}$$

the different sets arising since $v^0=0$ automatically at y=0.

The matching conditions which enable the specification of $\bar{A}(X,\tau)$ to be completed are

$$\bar{A}(-1,\tau) = E_0 e^{ik_0L} \qquad \text{and} \qquad \bar{A}(1,\tau) = E_0 e^{-ik_0L}. \tag{5.4.23}$$

As a first example, the function $\hat{g}_1(y)$ in the thermal condition (5.4.1) at x = 0 was taken to be

$$\hat{g}_1 = a^2 - y^2 \quad , \quad \lambda = 1 \tag{5.4.24}$$

and the values of $E_0, \bar{E}_0$ and $\hat{E}_0$ computed. The value of m_∞ was allowed to vary from zero to fourteen, thereby providing a check on the convergence of the estimates for $E_0, \bar{E}_0$ and $\hat{E}_0$ as $m_\infty \to \infty$. The results are displayed in tables 12,13,14 and 15 for a=¼,½,1 and 2 respectively. In all cases the gross convergence is rapid and the values of $E_0, \bar{E}_0$ and $\hat{E}_0$ seem to have settled down when m_∞ reaches fourteen. If $\hat{g}_1$ is chosen such that it is incompatible with the conditions at y=±a ie. $\hat{g}_1 \neq 0$ y=±a, an effect known as Gibb's phenomenon is obtained. For example, for the aspect ratio a=1 and $\hat{g}_1$ given by

$$\hat{g}_1 = \tfrac{1}{2} \quad , \qquad \lambda = 1 \, , \tag{5.4.25}$$

the values computed for $E_0, \bar{E}_0$ and $\hat{E}_0$ as m_∞ varies from zero to fourteen are given in table 16. From figure 24, which shows the computed profile obtained at x=-L (it should be the line $\frac{\partial\theta}{\partial x} = \frac{1}{2}$), it can be seen that near y=+a the series (5.4.18) is a relatively poor approximation to (5.4.25).

In summary, the amplitude equation for $\bar{A}(X,\tau)$ given by

$$C_1 \bar{S} \bar{A} + C_2 \bar{A}_{XX} + (C_3 + C_4/P + C_5/P^2)\bar{A}|\bar{A}|^2 + (C_6 + C_7/P)\bar{A}_\tau = 0 \tag{5.4.26}$$

where the amplitude coefficients C_i i=1..7 are determined numerically from the equations (4.4.34), must be solved subject to the boundary conditions derived above,

$$\bar{A}(-1,\tau) = E_0 e^{ik_0 L} \quad \text{and} \quad \bar{A}(1,\tau) = E_0 e^{-ik_0 L} . \qquad (5.4.27)$$

The change in the critical Rayleigh number R_0 for the infinite rigid channel due to the introduction of the distant endwalls may be investigated by assuming that the rigid endwalls are perfect insulators. Thus the thermal condition (5.4.1) becomes

$$\frac{\partial \theta}{\partial x} = 0 \quad x = \pm L, \qquad (5.4.28)$$

which is equivalent to $\lambda = 0$. Hence the conditions (5.4.27) on the amplitude function $\bar{A}(X,\tau)$ become

$$\bar{A}(X,\tau) = 0 \quad X = \pm 1. \qquad (5.4.29)$$

As in the case of the stress-free long box, neglecting the nonlinear term and the slow time dependence τ in equation (5.4.26) gives

$$C_1 \bar{\delta} \bar{A} + C_2 \bar{A}_{XX} = 0 . \qquad (5.4.30)$$

Solving (5.4.30) subject to the conditions (5.4.29) gives

$$\bar{\delta} = \; j^2 \pi^2 C_2 / 4C_1 \quad j = 1,2 \cdots \qquad (5.4.31)$$

The value j=1 corresponds to the critical Rayleigh number R_c for the long box with rigid sidewalls:

$$R_c = R_0 + \frac{\pi^2 C_2}{4C_1} L^{-2} + \cdots \qquad (L \gg 1) . \qquad (5.4.32)$$

Figure 22 shows the correction to R_0 plotted against the reciprocal of the aspect ratio. The values are displayed in table 17. From figure 22 it is clear that as the aspect ratio, $a \to \infty$, the correction to R_0 does approach $\frac{9\pi^4}{2}$, as obtained for the two-dimensional case. As the sidewalls are moved in from infinity the correction to R_0 decreases at first ($a \gtrsim 2$) before increasing ($0 \lesssim a \lesssim 2$), which must in some way be related to the fact that the critical wavenumber k_0 behaves in an almost identical manner (see figure 3). Figure 23 shows a comparison of the correction to R_0 for the rigid case with both the 'finite-roll' and stress-free approximations. It can be seen that for small a, the correction to R_0 for the rigid case approaches that obtained for the corresponding 'finite-roll' approximation. This is expected, since in section 2.4(1) it was shown that as $a \to 0$ the basic eigensolution for the infinite rigid channel with perfectly conducting sidewalls is equivalent (to leading order) to the basic eigensolution for the 'finite-roll' approximation for the infinite channel.

For $a \to \infty$ the limiting value of C_2/C_1 is the same for the rigid sidewall case and the 'finite-roll' approximation (see figure 23). For any fixed value of m, it is also the same for the stress-free sidewall model. Thus all the curves in figure 23 tend to the value $\frac{4\pi^4}{2}$ as $a \to \infty$, although the asymptotic approach to this limit is different in each case. The latter behaviour corresponds to different basic profiles in y/a in each case.

a = 0.25

m	k_{mr}	k_{mi}	$-i\bar{k}_m$	$-i\hat{k}_m$
0	3.6094	0.	11.609	16.806
1	5.4533	15.281	21.176	30.241
2	6.6672	27.965	32.854	43.152
3	7.4128	40.594	45.018	55.906
4	7.9530	53.199	57.357	68.590
5	8.3772	65.793	69.778	81.238
6	8.7267	78.380	82.243	93.864
7	9.0240	90.962	94.735	106.476
8	9.2826	103.541	107.244	119.078
9	9.5116	116.117	119.765	131.673
10	9.7170	128.693	132.295	144.263
11	9.9032	141.266	144.831	156.850
12	10.0735	153.839	157.372	169.433
13	10.2305	166.411	169.918	182.014
14	10.3760	178.982	182.465	194.593
15	10.5116	191.553	195.015	207.171
16	10.6386	204.123	207.567	219.747
17	10.7581	216.693	220.120	232.323
18	10.8707	229.263	232.676	244.897
19	10.9774	241.832	245.232	257.470

Table 8. Eigenvalues of ψ_m, $\bar{\psi}_m$ and $\hat{\psi}_m$ for aspect ratio a = $\frac{1}{4}$.

a = 0.5

m	k_{mr}	k_{mi}	$-i\bar{k}_m$	$-i\hat{k}_m$
0	2.7021	0.	8.0409	7.6419
1	2.7017	8.0134	11.8969	14.6941
2	3.2950	14.2156	17.2748	21.2783
3	3.6802	20.4661	23.1261	27.7236
4	3.9585	26.7322	29.1618	34.1085
5	4.1756	33.0052	35.2854	40.4616
6	4.3535	39.2819	41.4575	46.7957
7	4.5043	45.5606	47.6589	53.1178
8	4.6352	51.8407	53.8794	59.4316
9	4.7507	58.1217	60.1130	65.7395
10	4.8543	64.4032	66.3561	72.0431
11	4.9480	70.6851	72.6061	78.3434
12	5.0337	76.9673	78.8613	84.6412
13	5.1125	83.2497	85.1207	90.9369
14	5.1856	89.5323	91.3834	97.2311
15	5.2537	95.8150	97.6487	103.5239
16	5.3174	102.0978	103.9162	109.8156
17	5.3773	108.3807	110.1855	116.1063
18	5.4339	114.6636	116.4563	122.3963
19	5.4873	120.9466	122.7283	128.6856

Table 9. Eigenvalues of ψ_m, $\bar{\psi}_m$ and $\hat{\psi}_m$ for aspect ratio a = ½.

a = 1.0

m	k_{mr}	k_{mi}	$-i\bar{k}_m$	$-i\hat{k}_m$
0	2.2315	0.	6.7124	3.4176
1	1.3625	4.2257	8.0569	6.8013
2	1.6900	7.4024	10.1006	10.2676
3	1.8427	10.4864	12.7210	13.4966
4	1.9683	13.5853	15.4981	16.7455
5	2.0745	16.6931	18.3973	19.9653
6	2.1642	19.8081	21.3681	23.1657
7	2.2410	22.9286	24.3836	26.3529
8	2.3079	26.0532	27.4285	29.5308
9	2.3670	29.1810	30.4937	32.7019
10	2.4199	32.3111	33.5734	35.8680
11	2.4677	35.4431	36.6639	39.0302
12	2.5113	38.5766	39.7626	42.1895
13	2.5514	41.7112	42.8676	45.3463
14	2.5884	44.8467	45.9777	48.5011
15	2.6229	47.9830	49.0918	51.6544
16	2.6552	51.1200	52.2093	54.8063
17	2.6855	54.2574	55.3295	57.9572
18	2.7140	57.3953	58.4521	61.1071
19	2.7410	60.5336	61.5766	64.2561

Table 10. Eigenvalues of ψ_m, $\bar{\psi}_m$ and $\hat{\psi}_m$ for aspect ratio a = 1.

a = 2.0

m	k_{mr}	k_{mi}	$-i\bar{k}_m$	$-i\hat{k}_m$
0	2.0977	0.	6.3573	3.2066
1	1.4483	1.3091	6.7342	4.1713
2	0.6986	3.3727	7.4330	5.0061
3	0.8693	5.3170	8.3678	6.4575
4	1.0083	6.9370	9.4665	8.0450
5	1.0770	8.5070	10.6972	9.6584
6	1.1220	10.0656	12.0049	11.2637
7	1.1538	11.6212	13.3657	12.8681
8	1.1785	13.1783	14.7669	14.4674
9	1.2001	14.7376	16.2064	16.0550
10	1.2204	16.2984	17.5967	17.7126
11	1.2400	17.8602	19.0879	19.2860
12	1.2588	19.4226	20.5789	20.8724
13	1.2768	20.9855	22.0790	22.4600
14	1.2940	22.5488	23.5882	24.0469
15	1.3103	24.1125	25.1050	25.6327
16	1.3259	25.6766	26.6284	27.2172
17	1.3407	27.2411	28.1573	28.8007
18	1.3547	28.8060	29.6907	30.3832
19	1.3681	30.3713	31.2281	31.9647

Table 11. Eigenvalues of $\psi_m, \bar{\psi}_m$ and $\hat{\psi}_m$ for aspect ratio a = 2.

a = 0.25

m	E_{or}	E_{oi}	$\bar{E}_o$	$\hat{E}_o$
0	0.02746030	-0.00742569	0.02420469	-0.01724982
1	-0.00439593	0.00139250	-0.00638784	0.00015642
2	-0.00440161	0.00138201	-0.00639104	0.00018774
3	-0.00440168	0.00137866	-0.00639073	0.00019303
4	-0.00440173	0.00137780	-0.00639066	0.00019432
5	-0.00440179	0.00137757	-0.00639067	0.00019475
6	-0.00440184	0.00137751	-0.00639069	0.00019493
7	-0.00440187	0.00137751	-0.00639072	0.00019501
8	-0.00440189	0.00137751	-0.00639073	0.00019505
9	-0.00440190	0.00137750	-0.00639074	0.00019508
10	-0.00440188	0.00137745	-0.00639071	0.00019510
11	-0.00440330	0.00138024	-0.00639210	0.00019471
12	-0.00440203	0.00137773	-0.00639086	0.00019508
13	-0.00440200	0.00137768	-0.00639083	0.00019509
14	-0.00440200	0.00137766	-0.00639083	0.00019509

Table 12. Variation of computed values of E_o, $\bar{E}_o$ and $\hat{E}_o$ with the number of eigenfunctions used ($\hat{g}_1 = a^2 - y^2$) where the eigenfunctions ψ_o , $\bar{\psi}_o$ and $\hat{\psi}_o$ have been normalised such that $\psi_o = \bar{\psi}_o = \hat{\psi}_o = 1$ at y = 0.

a = 0.5

m	E_{or}	E_{oi}	$\bar{E}_o$	$\hat{E}_o$
0	-0.1780620	0.0523167	-0.1477422	0.0857448
1	-0.0329854	0.0113967	-0.0435282	0.0101714
2	-0.0330306	0.0112742	-0.0439423	0.0086192
3	-0.0330153	0.0112149	-0.0439156	0.0086647
4	-0.0330097	0.0111966	-0.0439069	0.0086762
5	-0.0330078	0.0111903	-0.0439038	0.0086801
6	-0.0330073	0.0111879	-0.0439027	0.0086818
7	-0.0330072	0.0111870	-0.0439022	0.0086826
8	-0.0330072	0.0111865	-0.0439020	0.0086831
9	-0.0330070	0.0111860	-0.0439017	0.0086833
10	-0.0330060	0.0111844	-0.0439010	0.0086834
11	-0.0330328	0.0112234	-0.0439187	0.0086859
12	-0.0330104	0.0111907	-0.0439038	0.0086839
13	-0.0330096	0.0111895	-0.0439033	0.0086839
14	-0.0330093	0.0111891	-0.0439031	0.0086839

Table 13. Variation of computed values of E_o, $\bar{E}_o$ and $\hat{E}_o$ with the number of eigenfunctions used ($\hat{g}_1 = a^2 - y^2$).

a = 1.0

m	E_{or}	E_{oi}	$\bar{E}_o$	$\hat{E}_o$
0	-0.155257	0.063734	-0.216068	0.048540
1	-0.197681	0.059431	-0.218273	0.011666
2	-0.196359	0.060284	-0.210219	0.093519
3	-0.195950	0.059447	-0.209833	0.091031
4	-0.195800	0.059147	-0.209773	0.090408
5	-0.195738	0.059026	-0.209747	0.090180
6	-0.195709	0.058972	-0.209734	0.090082
7	-0.195695	0.058945	-0.209728	0.090034
8	-0.195686	0.058929	-0.209724	0.090005
9	-0.195677	0.058915	-0.209719	0.089976
10	-0.195653	0.058884	-0.209704	0.089907
11	-0.196017	0.059318	-0.209929	0.090933
12	-0.195732	0.058978	-0.209754	0.090130
13	-0.195717	0.058958	-0.209744	0.090085
14	-0.195711	0.058951	-0.209741	0.090070

Table 14. Variation of computed values of E_o, $\bar{E}_o$ and $\hat{E}_o$ with the number of eigenfunctions used ($\hat{g}_1 = a^2 - y^2$).

a = 2.0

m	E_{or}	E_{oi}	$\bar{E}_o$	$\hat{E}_o$
0	-0.743599	0.505081	-0.869939	-0.183541
1	-0.943520	0.300359	-0.825737	-0.179522
2	-1.008968	0.247698	-0.815937	-0.366162
3	-1.016348	0.252770	-0.815702	-0.389689
4	-1.013393	0.251916	-0.814255	-0.365890
5	-1.011833	0.251032	-0.813924	-0.357882
6	-1.011094	0.250568	-0.813718	-0.354684
7	-1.010689	0.250306	-0.813597	-0.353138
8	-1.010436	0.250143	-0.813518	-0.352259
9	-1.010235	0.250019	-0.813455	-0.351613
10	-1.009910	0.249841	-0.813353	-0.350630
11	-1.012082	0.250926	-0.814014	-0.357028
12	-1.010566	0.250156	-0.813550	-0.352550
13	-1.010411	0.250074	-0.813502	-0.352091
14	-1.010348	0.250039	-0.813482	-0.351903

Table 15. Variation of computed values of E_o, $\bar{E}_o$ and $\hat{E}_o$ with the number of eigenfunctions used ($\hat{g}_1 = a^2 - y^2$).

a = 1.0

m	E_{or}	E_{oi}	$\bar{E}_o$	$\hat{E}_o$
0	-0.077629	0.031867	-0.108034	0.024270
1	-0.115210	0.032937	-0.128818	0.076875
2	-0.118142	0.034669	-0.128347	0.067918
3	-0.119055	0.034309	-0.129417	0.068204
4	-0.119479	0.034156	-0.130190	0.068676
5	-0.119719	0.034096	-0.130618	0.069002
6	-0.119869	0.034071	-0.130878	0.069224
7	-0.119969	0.034061	-0.131047	0.069378
8	-0.120039	0.034055	-0.131163	0.069487
9	-0.120087	0.034050	-0.131245	0.069561
10	-0.120113	0.034034	-0.131300	0.069588
11	-0.120355	0.034289	-0.131478	0.070235
12	-0.120213	0.034092	-0.131413	0.069809
13	-0.120223	0.034082	-0.131437	0.069814
14	-0.120235	0.034079	-0.131459	0.069831

Table 16. Variation of computed values of E_o, $\bar{E}_o$ and $\hat{E}_o$ with the number of eigenfunctions used ($\hat{g}_1 = 0.5$).

a	R_o	$\bar{\delta}$ Correction to R_o
0.25	6734.035	1435.180
0.50	1654.742	675.897
0.75	1023.068	508.046
1.00	827.566	430.475
1.25	744.772	381.955
1.50	704.887	353.283
1.75	684.480	341.578
2.00	673.598	341.410
2.50	664.041	354.656
3.00	660.583	369.391
3.50	659.128	380.759

Table 17. Correction to the critical Rayleigh number R_o due to the introduction of the distant endwalls for various aspect ratios.

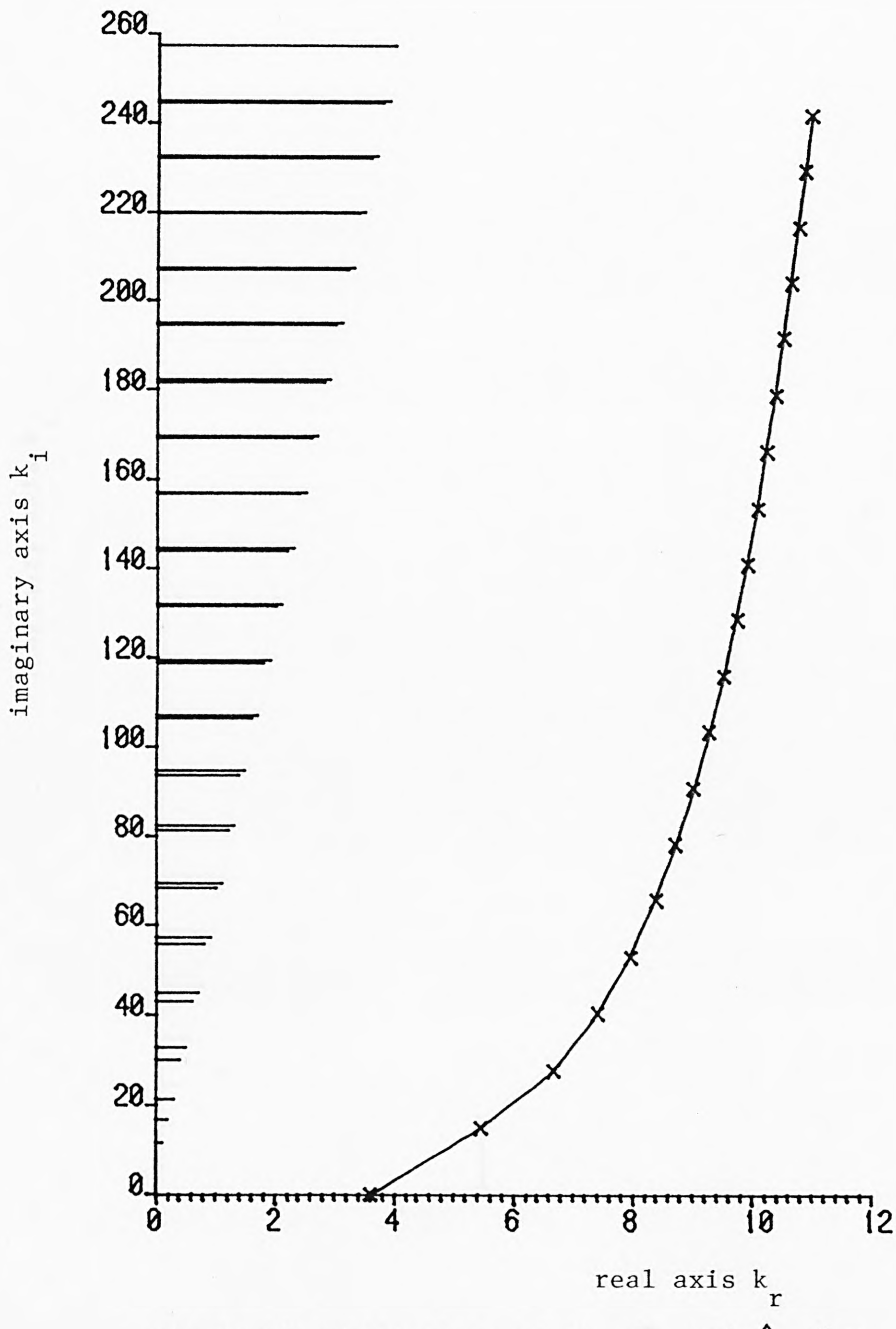


Figure 18. Eigenvalues of ψ_m, $\bar{\psi}_m$ and $\hat{\psi}_m$ for the aspect ratio a = 0.25.

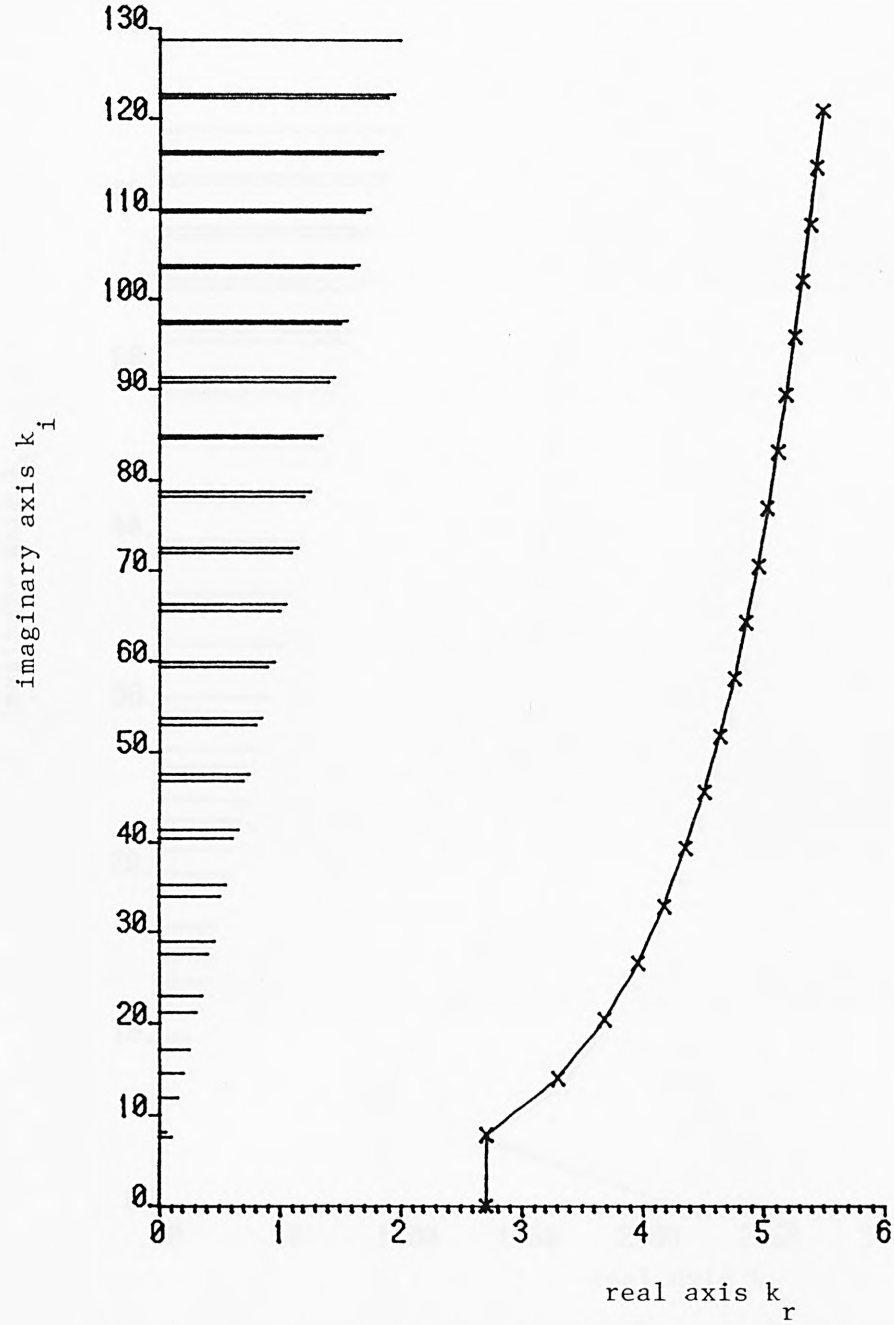


Figure 19. Eigenvalues of ψ_m, $\bar{\psi}_m$ and $\hat{\psi}_m$ for the aspect ratio a = 0.5.

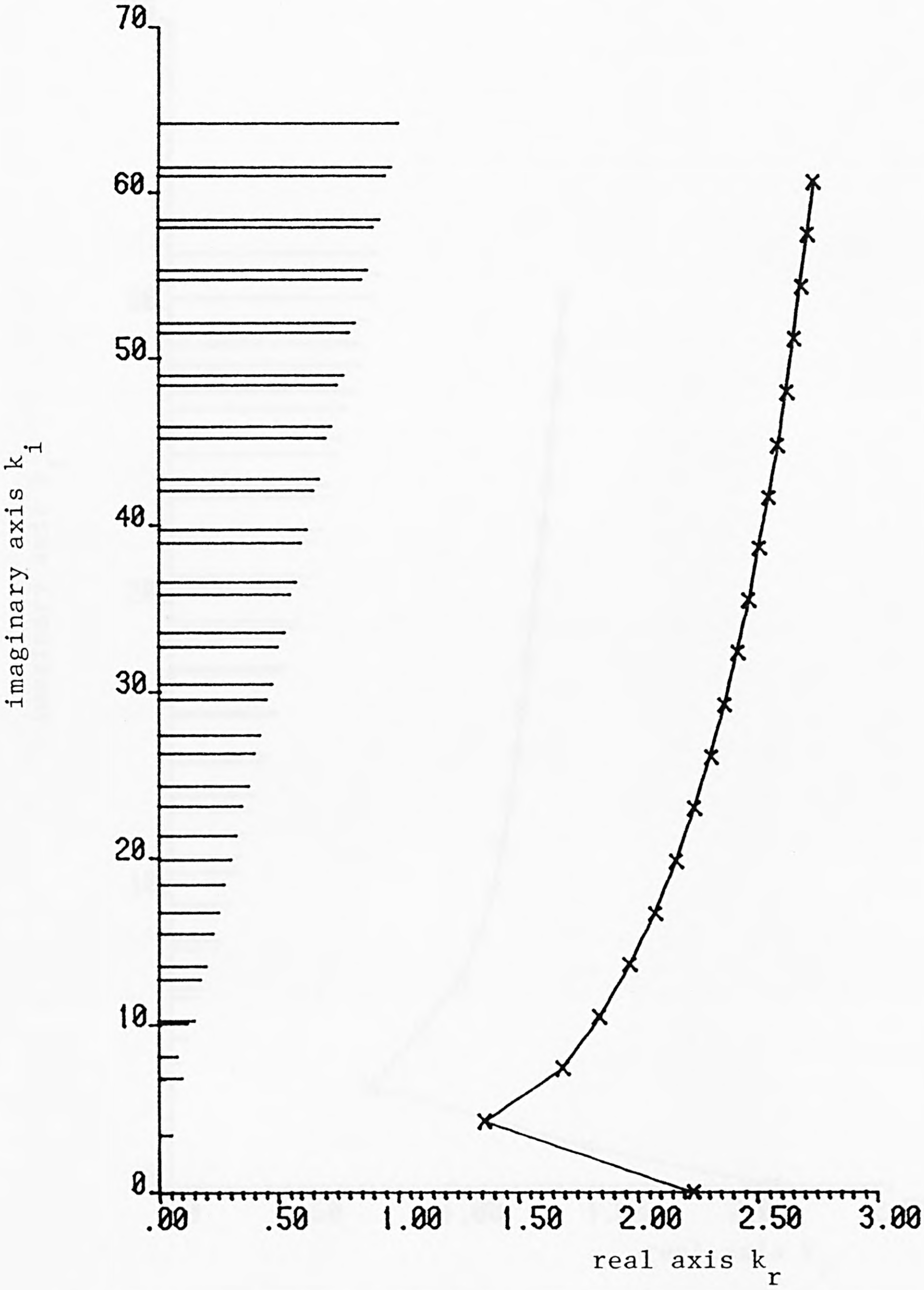


Figure 20. Eigenvalues of ψ_m, $\bar{\psi}_m$ and $\hat{\psi}_m$ for the aspect ratio a = 1.

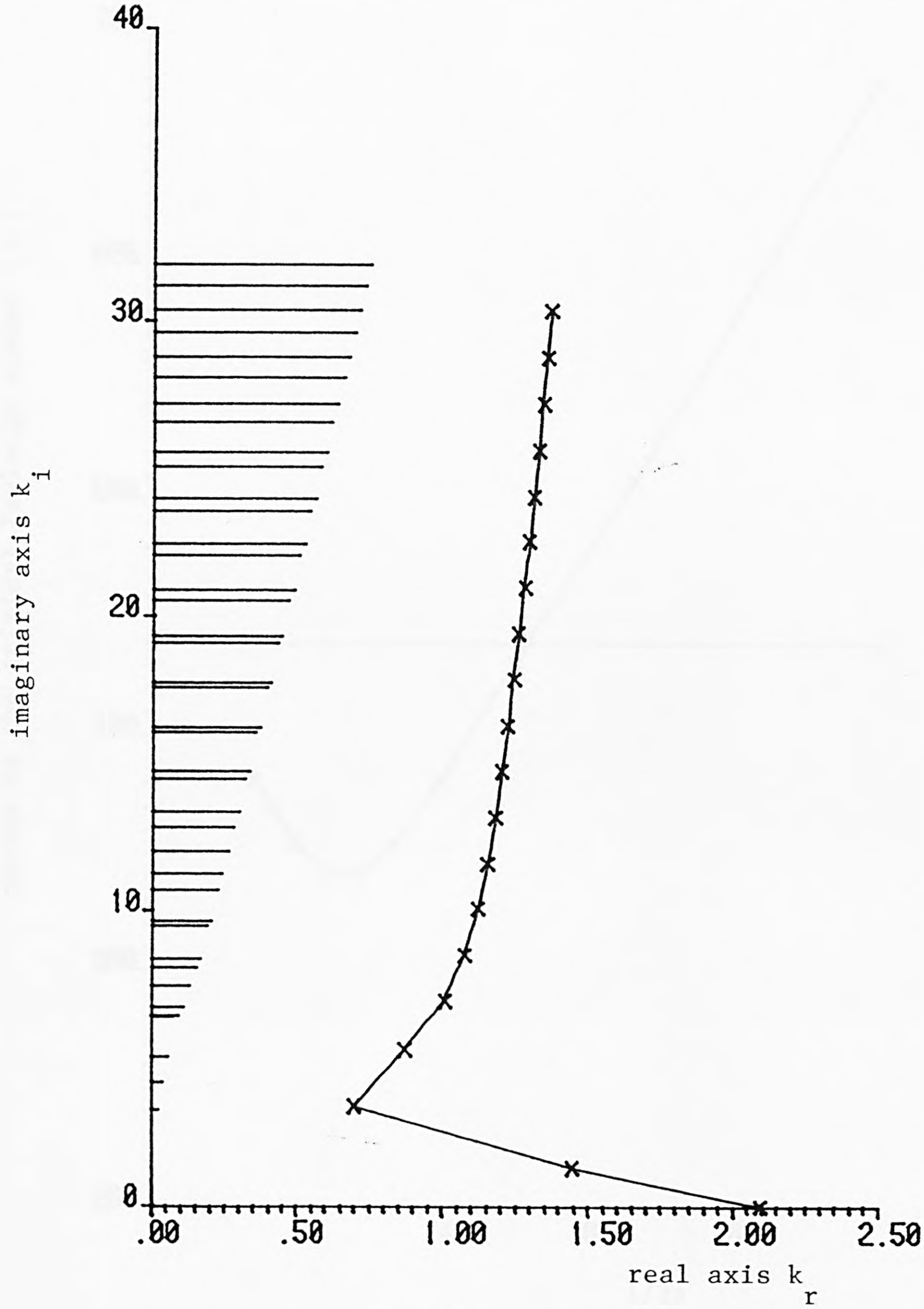


Figure 21. Eigenvalues of ψ_m, $\bar{\psi}_m$ and $\hat{\psi}_m$ for the aspect ratio a = 2.

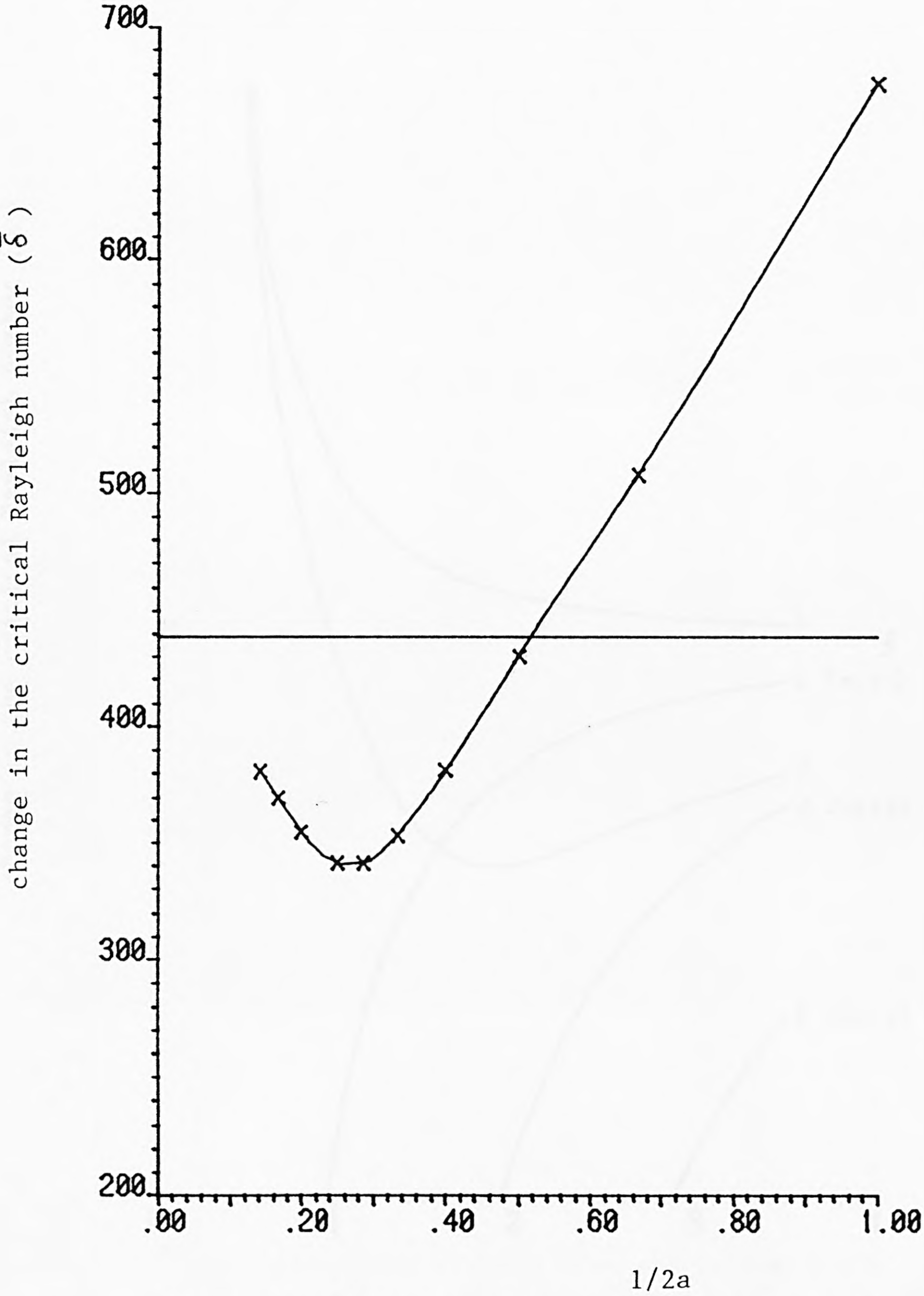


Figure 22. Correction to R_o plotted against the aspect ratio.

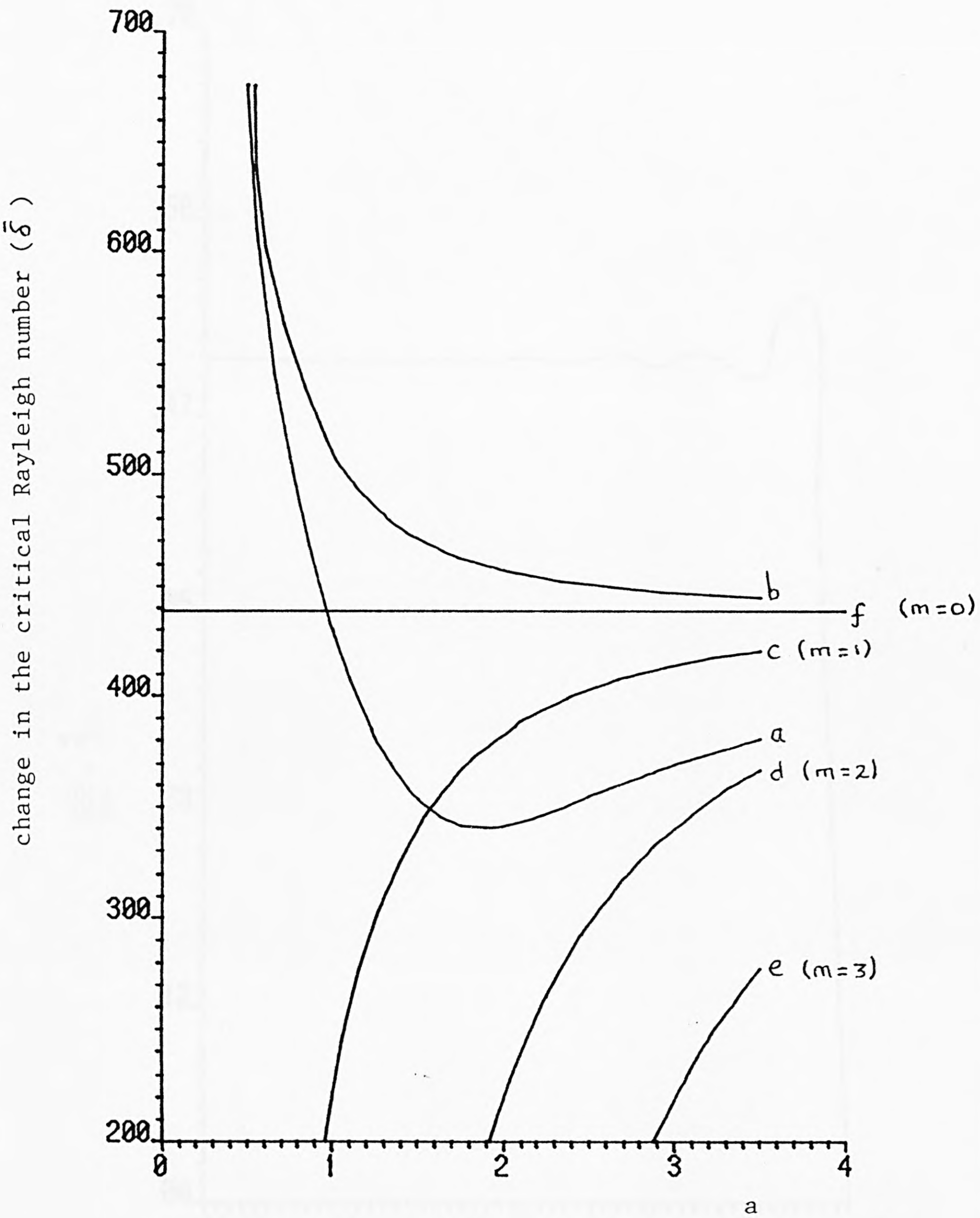


Figure 23. A comparison of the correction to R_0 for the rigid case (curve a) with both the 'finite-roll' (curve b) and stress-free (curves c,d,e) approximations. Curve f indicates the limiting value $9\pi^4/2$ as $a\to\infty$.

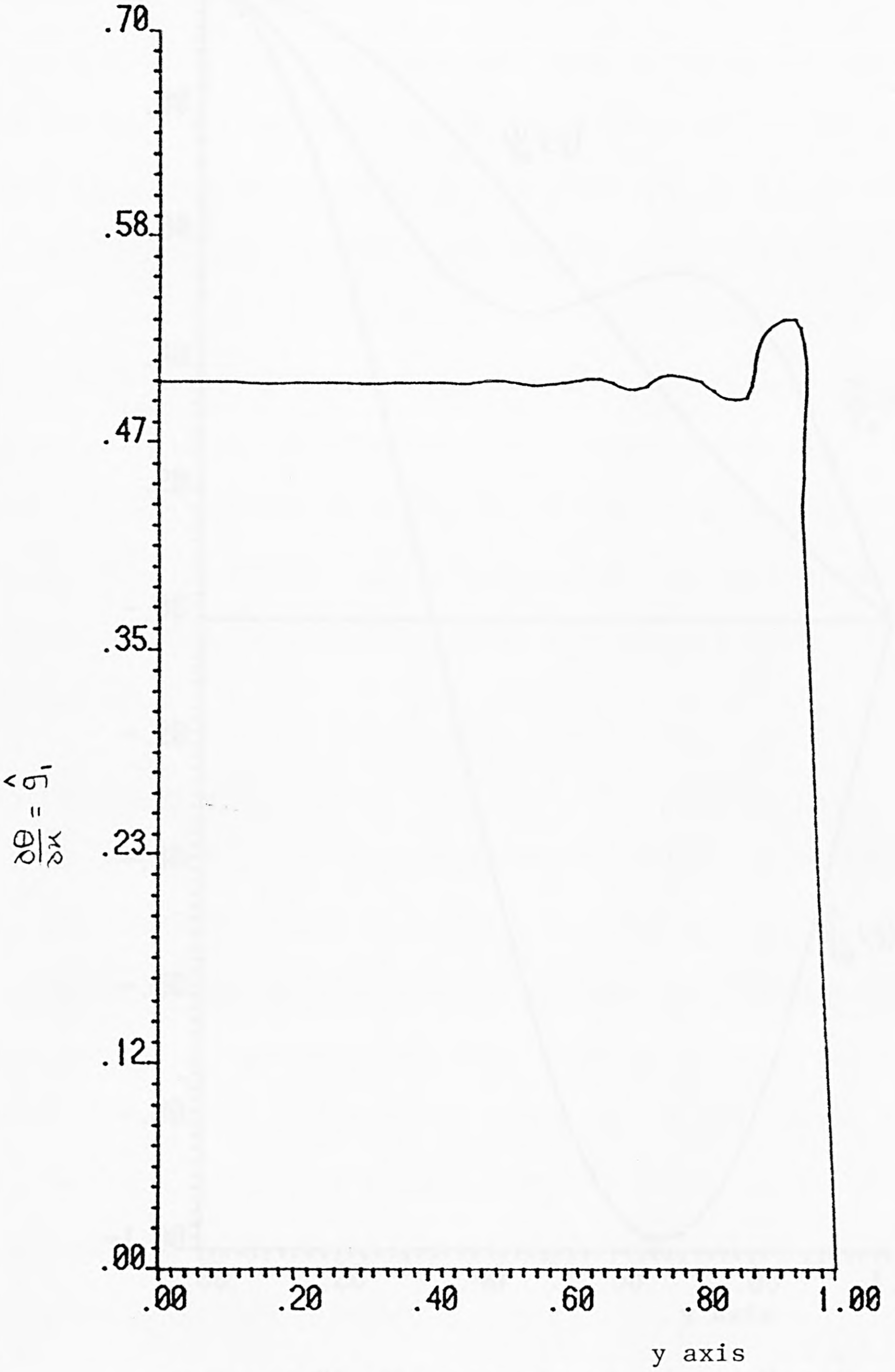


Figure 24. Computed profile at the endwall x = -L for the aspect ratio a = 1 illustrating the Gibb's phenomenon.

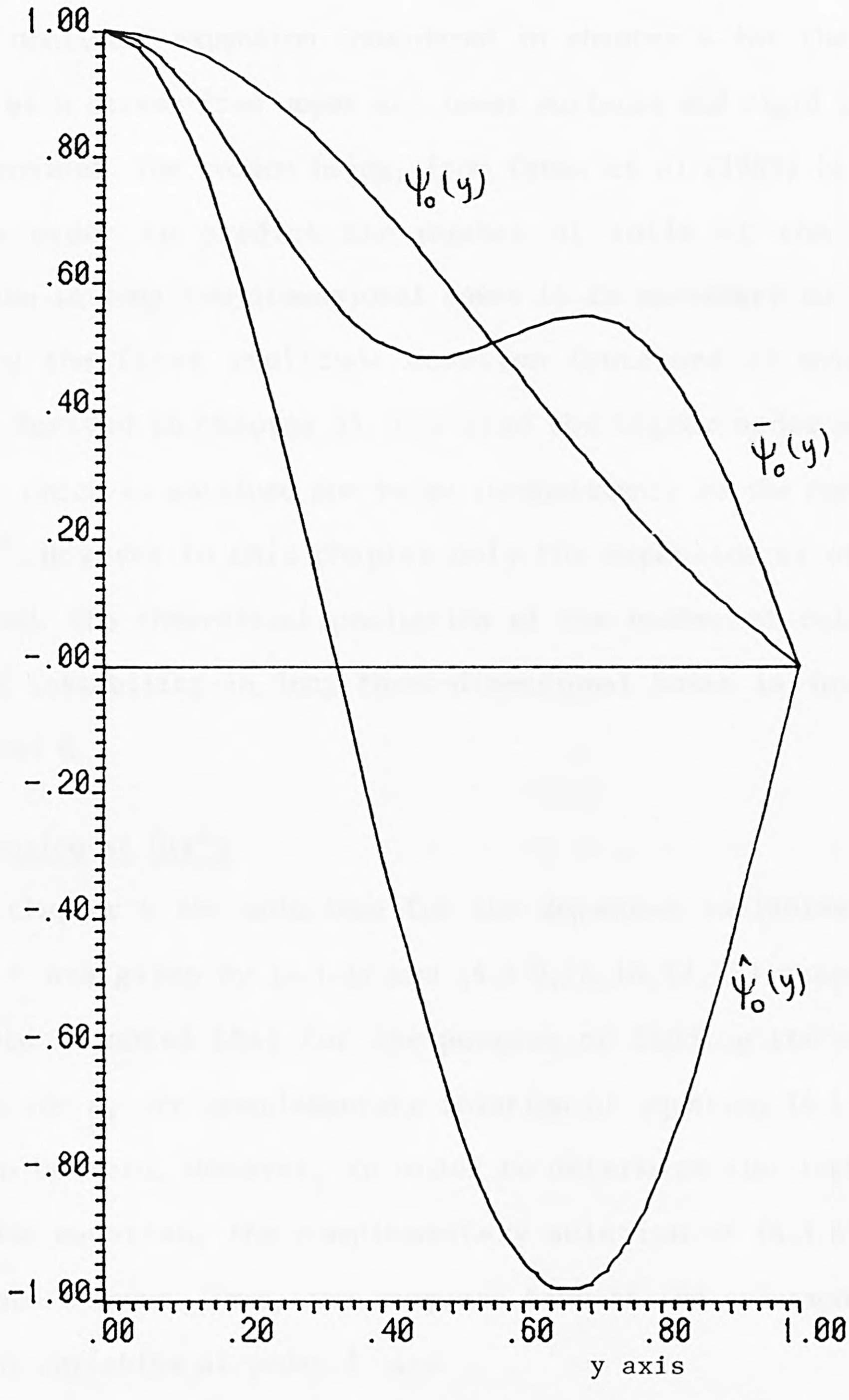


Figure 25. The even eigenfunctions $\psi_0, \bar{\psi}_0$ and $\hat{\psi}_0$ for the aspect ratio a = 1 and conducting sidewalls.

Chapter 6 The infinite channel: nonlinear theory for rigid sidewalls Part II

6.0 Introduction

In this chapter the y-dependent functions at order ε^2 in the 'weakly' nonlinear expansion considered in chapter 4 for the infinite channel with stress-free upper and lower surfaces and rigid sidewalls, are determined. The reason being, from Cross et al (1983) it is known that in order to predict the number of rolls at the onset of convection in long two-dimensional boxes it is necessary to determine not only the first amplitude equation (obtained at order ε^2 and already derived in chapter 4), but also the higher order amplitude equation which is obtained due to an inconsistency in the expansion at order ε^3. However in this chapter only the expansion at order ε^2 is considered. The theoretical prediction of the number of rolls at the onset of instability in long three-dimensional boxes is investigated in chapter 8.

6.1 Expansion at $O(\varepsilon^2)$

In chapter 4 the solutions for the dependent variables at order ε^0 and ε are given by (4.1.2) and (4.3.9,14,18,22,23) respectively. It should be noted that for the purpose of finding the amplitude equation for $\bar{A}$, the complementary solution of equation (4.3.8) can be taken to be zero. However, in order to determine the higher order amplitude equation, the complementary solution of (4.3.8) must be taken into account. Thus, the complete form of the solutions for the dependent variables at order ε are

$$\theta_2 = \Theta(y)\left(\bar{B}(\bar{X},\bar{T})e^{ik_0x} + c.c\right)\sin\pi z + i\Theta_1(y)\left(\bar{A}_{\bar{X}}e^{ik_0x} - c.c\right)\sin\pi z + \frac{1}{P}\Theta_2(y)\left(\bar{A}^2e^{2ik_0x} + c.c\right)\sin 2\pi z + \frac{1}{P}\Theta_3(y)\left(2\bar{A}\bar{A}^*\right)\sin 2\pi z;$$

$$w_2 = W(y)\left(\bar{B}(\bar{X},\bar{T})e^{ik_0x} + c.c\right)\sin\pi z + iW_1(y)\left(\bar{A}_{\bar{X}}e^{ik_0x} - c.c\right)\sin\pi z + \frac{1}{P}W_2(y)\left(\bar{A}^2e^{2ik_0x} + c.c\right)\sin 2\pi z + \frac{1}{P}W_3(y)\left(2\bar{A}\bar{A}^*\right)\sin 2\pi z$$

$$v_2 = V(y)(\bar{B}(\bar{X},\bar{\tau})e^{ik_0x} + c.c)\cos\pi z + iV_1(y)(\bar{A}_{\bar{X}}e^{ik_0x} - c.c)\cos\pi z + \frac{1}{P}V_2(y)(\bar{A}^2e^{2ik_0x} + c.c)\cos 2\pi z$$

$$+ \frac{1}{P}V_3(y)(2\bar{A}\bar{A}^*)\cos 2\pi z + \frac{1}{P}V_4(y)(\bar{A}^2e^{2ik_0x} + c.c)$$

$$u_2 = \frac{i}{k_0}U(y)(\bar{B}(\bar{X},\bar{\tau})e^{ik_0x} - c.c)\cos\pi z + U_1(y)(\bar{A}_{\bar{X}}e^{ik_0x} + c.c)\cos\pi z + \frac{i}{P}U_2(y)(\bar{A}^2e^{2ik_0x} - c.c)\cos 2\pi z + \frac{i}{P}U_4(y)(\bar{A}^2e^{2ik_0x} - c.c)$$

and

$$p_2 = P_0(y)(\bar{B}(\bar{X},\bar{\tau})e^{ik_0x} + c.c)\cos\pi z + iPP_1(y)(\bar{A}_{\bar{X}}e^{ik_0x} - c.c)\cos\pi z + P_2(y)(\bar{A}^2e^{2ik_0x} + c.c)\cos 2\pi z \qquad (6.1.1)$$

$$+ P_3(y)(2\bar{A}\bar{A}^*)\cos 2\pi z + F(x,y,\bar{X},\bar{\tau})$$

where F is given by (4.3.34) and $\bar{B}(\bar{X},\bar{\tau})$ is an arbitrary amplitude function.

At order ε^2, the Boussinesq equations (3.1.16-20) and the boundary conditions (4.1.6) yield the equations (4.4.1-5) and the conditions (4.4.7). The dependent variables are determined by following the procedure outlined in section 4.3, where the dependent variables at $O(\varepsilon)$ are determined. Thus, in section 6.2 the basic forms of the dependent variables are constructed; the numerical determination of the various unknown functions of y that occur in the solutions is described in section 6.3.

6.2 The basic forms of the dependent variables

As in the linear theory, by eliminating the variables in equations (4.4.1-5) a single sixth order differential equation for the temperature θ_3, given by equation (4.4.8) is obtained. It should be noted that the $\frac{\partial}{\partial t}$ terms are not relevant here and have been omitted. Expanding the right hand side of (4.4.8) gives

$$[\nabla^6 - R_0\nabla_H^2]\theta_3 = iM_1(\bar{B}_{\bar{X}}e^{ik_0x} - c.c)\sin\pi z + \frac{1}{P}[M_5(\bar{A}e^{ik_0x} + c.c) + M_6(\bar{A}_{\bar{X}\bar{X}}e^{ik_0x} + c.c) + M_7(\bar{A}\bar{A}^*)(\bar{A}e^{ik_0x} + c.c)$$

$$+ M_8(\bar{A}_{\bar{\tau}}e^{ik_0x} + c.c) + M_9(\bar{A}^3e^{3ik_0x} + c.c)]\sin\pi z + \frac{1}{P}[M_{10}(\bar{A}\bar{A}_{\bar{X}}e^{2ik_0x} - c.c) + M_{11}(\bar{A}^*\bar{A}_{\bar{X}} - c.c) + M_{12}(\bar{A}\bar{B}e^{2ik_0x} + c.c)$$

$$+ M_{13}(\bar{A}\bar{B}^* + c.c)]\sin 2\pi z + \frac{1}{P}[M_{14}(\bar{A}^3e^{3ik_0x} + c.c) + M_{15}(\bar{A}\bar{A}^*)(\bar{A}e^{ik_0x} + c.c)]\sin 3\pi z \qquad (6.2.1)$$

where * denotes complex conjugate and M_i i=1..15 are given in (6.2.4)-(6.2.17) below. From the amplitude equation (4.4.33) it is known that

$$\bar{A}_{\bar{X}\bar{X}} = -\frac{1}{C_2}\left[C_1\bar{A} + (C_3 + C_4/P + C_5/P^2)\bar{A}|\bar{A}|^2 + (C_6 + C_7/P)\bar{A}_{\bar{\tau}}\right] \tag{6.2.2}$$

where C_i i=1..7 are the amplitude coefficients which are determined numerically from equations (4.4.34). For convenience substituting (6.2.2) into (6.2.1) gives

$$[\nabla^6 - R_c\nabla_H^2]\theta_3 = iM_1(\bar{B}_{\bar{X}}e^{ik_0x} - c.c)\sin\pi z + \frac{1}{P}\left[(M_5 - \frac{C_1}{C_2}M_6)(\bar{A}e^{ik_0x} + c.c) + (M_7 - \frac{M_6}{C_2}(C_3 + C_4/P + C_5/P^2))(\bar{A}\bar{A}^*)(\bar{A}e^{ik_0x} + c.c)\right.$$

$$\left. + (M_8 - \frac{M_6}{C_2}(C_6 + C_7/P))(\bar{A}_{\bar{\tau}}e^{ik_0x} + c.c) + M_9(\bar{A}^3e^{3ik_0x} + c.c)\right]\sin\pi z + \frac{1}{P}\left[M_{10}(\bar{A}\bar{A}_{\bar{X}}e^{2ik_0x} - c.c) + M_{11}(\bar{A}^*\bar{A}_{\bar{X}} - c.c)\right.$$

$$\left. + M_{12}(\bar{A}\bar{B}e^{2ik_0x} + c.c) + M_{13}(\bar{A}\bar{B}^* + c.c)\right]\sin 2\pi z + \frac{1}{P}\left[M_{14}(\bar{A}^3e^{3ik_0x} + c.c) + M_{15}(\bar{A}\bar{A}^*)(\bar{A}e^{ik_0x} + c.c)\right]\sin 3\pi z \tag{6.2.3}$$

which implies

$$\theta_3 = i\Theta_1(\bar{B}_{\bar{X}}e^{ik_0x} - c.c)\sin\pi z + \frac{1}{P}\left[\Theta_5(\bar{A}e^{ik_0x} + c.c) + \Theta_7(\bar{A}\bar{A}^*)(\bar{A}e^{ik_0x} + c.c) + \Theta_8(\bar{A}_{\bar{\tau}}e^{ik_0x} + c.c) + \Theta_9(\bar{A}^3e^{3ik_0x} + c.c)\right]\sin\pi z$$

$$+ \frac{1}{P}\left[\Theta_{10}(\bar{A}\bar{A}_{\bar{X}}e^{2ik_0x} - c.c) + \Theta_{11}(\bar{A}\bar{A}^*_{\bar{X}} - c.c) + \Theta_{12}(\bar{A}\bar{B}e^{2ik_0x} + c.c) + \Theta_{13}(\bar{A}\bar{B}^* + c.c)\right]\sin 2\pi z$$

$$+ \frac{1}{P}\left[\Theta_{14}(\bar{A}^3e^{3ik_0x} + c.c) + \Theta_{15}(\bar{A}\bar{A}^*)(\bar{A}e^{ik_0x} + c.c)\right]\sin 3\pi z + \tilde{\theta}_c(x, y, z, \bar{X}, \bar{\tau}) \tag{6.2.4}$$

where

$$\left[\left(\frac{d^2}{dy^2} - \alpha^2\right)^3 - R_0\left(\frac{d^2}{dy^2} - k_0^2\right)\right]\Theta_1 = M_1 = -2k_0\left[3\left(\frac{d^4}{dy^4} - 2\alpha^2\frac{d^2}{dy^2} + \alpha^4\right) - R_0\right]\Theta, \tag{6.2.5}$$

$$\left[\left(\frac{d^2}{dy^2} - \alpha^2\right)^3 - R_0\left(\frac{d^2}{dy^2} - k_0^2\right)\right]\Theta_5 = M_5 - \frac{C_1}{C_2}M_6 = P\left(\frac{d^2}{dy^2} - k_0^2\right)\Theta - \frac{C_1}{C_2}M_6, \tag{6.2.6}$$

$$\left[\left(\frac{d^2}{dy^2} - \alpha^2\right)^3 - R_0\left(\frac{d^2}{dy^2} - k_0^2\right)\right]\Theta_7 = M_7 - \frac{M_6}{C_2}(C_3 + C_4/P + C_5/P^2)$$

$$= \left(\frac{d^2}{dy^2} - \alpha^2\right)^2\left[\varepsilon_8 + \varepsilon_{57} + 2(\varepsilon_{10} + \varepsilon_{59})\right] - \left(\frac{d^2}{dy^2} - k_0^2\right)\left[\varepsilon_{21} + 2\varepsilon_{23} + \frac{1}{P}(\varepsilon_{67} + 2\varepsilon_{69})\right]$$

$$+ \frac{\pi}{P}\left[k_0(\varepsilon_{35} + \varepsilon_{81} + 2\varepsilon_{83}) - \frac{d}{dy}(\varepsilon_{47} + \varepsilon_{45} + 2\varepsilon_{49} + 2\varepsilon_{97})\right] - \frac{M_6}{C_2}(C_3 + C_4/P + C_5/P^2), \tag{6.2.7}$$

$$\left[\left(\frac{d^2}{dy^2} - \alpha^2\right)^3 - R_0\left(\frac{d^2}{dy^2} - k_0^2\right)\right]\Theta_8 = M_8 - \frac{M_6}{C_2}(C_6 + C_7/P) = (1 + P)\left(\frac{d^2}{dy^2} - \alpha^2\right)^2\Theta - \frac{M_6}{C_2}(C_6 + C_7/P), \tag{6.2.8}$$

$$\left[\left(\frac{d^2}{dy^2} - \pi^2 - 9k_0^2\right)^3 - R_0\left(\frac{d^2}{dy^2} - 9k_0^2\right)\right]\Theta_9 = M_9 = \left(\frac{d^2}{dy^2} - \pi^2 - 9k_0^2\right)^2\left[\varepsilon_7 + \varepsilon_{56}\right] - \left(\frac{d^2}{dy^2} - 9k_0^2\right)\left(\varepsilon_{20} + \frac{1}{P}\varepsilon_{66}\right)$$

$$+ \frac{\pi}{P}\left[3k_0(\varepsilon_{34} + \varepsilon_{80}) - \frac{d}{dy}(\varepsilon_{46} + \varepsilon_{94})\right], \tag{6.2.9}$$

$$\left[\left(\frac{d^2}{dy^2} - 4\alpha^2\right)^3 - R_0\left(\frac{d^2}{dy^2} - 4k_0^2\right)\right]\Theta_{10} = M_{10} = i\left\{\left(\frac{d^2}{dy^2} - 4\alpha^2\right)^2\left[-8k_0\Theta_2 + \varepsilon_{52} + P(\varepsilon_3 + \frac{1}{2k_0}U\Theta)\right]\right.$$

$$+ 4\left(\frac{d^2}{dy^2} - 4\alpha^2\right)(2Pk_0W_2 - \pi U_2) - \left(\frac{d^2}{dy^2} - 4k_0^2\right)\left[\varepsilon_{16} + \frac{1}{P}\varepsilon_{62} + \frac{1}{2k_0}UW\right] + (\text{P.T.O})$$

$$-\frac{2\pi}{P}\left[2k_0(\varepsilon_{28}+\varepsilon_{74})+\frac{d}{dy}(\varepsilon_{40}+\varepsilon_{88})\right]+\frac{\pi}{k_0}\left[-8k_0^2P_2+2U^2-\frac{d}{dy}(UV)\right]\Big\}, \tag{6.2.10}$$

$$\left[\left(\frac{d^2}{dy^2}-4\pi^2\right)^3-R_0\frac{d^2}{dy^2}\right]\Theta_{11}=M_{11}=i\Big\{\left(\frac{d^2}{dy^2}-4\pi^2\right)^2\left[P\varepsilon_{14}+\varepsilon_{53}-\frac{P}{2k_0}U\Theta\right]$$

$$-\frac{d^2}{dy^2}\left[\varepsilon_{17}+\frac{1}{P}\varepsilon_{63}-\frac{1}{2k_0}UW\right]+\frac{\pi}{k_0}\frac{d}{dy}(UV)-\frac{2\pi}{P}\frac{d}{dy}\left[\varepsilon_{41}+\varepsilon_{89}\right]\Big\}, \tag{6.2.11}$$

$$\left[\left(\frac{d^2}{dy^2}-4\alpha^2\right)^3-R_0\left(\frac{d^2}{dy^2}-4k_c^2\right)\right]\Theta_{12}=M_{12}=P\left(\frac{d^2}{dy^2}-4\alpha^2\right)^2(\varepsilon_1+\varepsilon_{50})-\left(\frac{d^2}{dy^2}-4k_c^2\right)(\varepsilon_{14}+\varepsilon_{60})$$

$$+2\pi\left[2k_0(\varepsilon_{24}+\varepsilon_{70})-\frac{d}{dy}(\varepsilon_{36}+\varepsilon_{84})\right], \tag{6.2.12}$$

$$\left[\left(\frac{d^2}{dy^2}-4\pi^2\right)^3-R_c\frac{d^2}{dy^2}\right]\Theta_{13}=M_{13}=P\left(\frac{d^2}{dy^2}-4\pi^2\right)^2(\varepsilon_2+\varepsilon_{51})-\frac{d^2}{dy^2}(\varepsilon_{15}-\varepsilon_{61})-2\pi\frac{d}{dy}(\varepsilon_{37}+\varepsilon_{85}), \tag{6.2.14}$$

$$\left[\left(\frac{d^2}{dy^2}-9\alpha^2\right)^3-R_0\left(\frac{d^2}{dy^2}-9k_c^2\right)\right]\Theta_{14}=M_{14}=\left(\frac{d^2}{dy^2}-9\alpha^2\right)^2(\varepsilon_5+\varepsilon_{54})-\left(\frac{d^2}{dy^2}-9k_c^2\right)\left(\varepsilon_{18}+\frac{1}{P}\varepsilon_{64}\right)$$

$$+\frac{3\pi}{P}\left[3k_c(\varepsilon_{32}+\varepsilon_{78})-\frac{d}{dy}(\varepsilon_{44}+\varepsilon_{92})\right], \tag{6.2.15}$$

$$\left[\left(\frac{d^2}{dy^2}-9\pi^2-k_c^2\right)^3-R_0\left(\frac{d^2}{dy^2}-k_c^2\right)\right]\Theta_{15}=M_{15}=\left(\frac{d^2}{dy^2}-9\pi^2-k_c^2\right)^2\left[\varepsilon_6+\varepsilon_{35}+2(\varepsilon_9+\varepsilon_{58})\right]-\left(\frac{d^2}{dy^2}-k_c^2\right)\left[\varepsilon_{19}+2\varepsilon_{22}+\frac{1}{P}(\varepsilon_{65}+2\varepsilon_{68})\right]$$

$$+\frac{3\pi}{P}\left[k_c(\varepsilon_{33}+\varepsilon_{79}+2\varepsilon_{82})-\frac{d}{dy}(\varepsilon_{45}+\varepsilon_{93}+2\varepsilon_{48}+2\varepsilon_{96})\right], \tag{6.2.16}$$

$$M_6=2k_cP\Big\{\left(\frac{d^2}{dy^2}-\alpha^2\right)^2\left[3\Theta_1-\frac{1}{k_c}\Theta\right]-\frac{\pi}{2k_c^3}\left(\frac{d^2}{dy^2}-\alpha^2\right)U-R_0\Theta_1-6k_0W\Big\} \tag{6.2.17}$$

and the ε's, which are real functions of the dependent variables obtained at order ε^0 and ε, are given by (E1) in appendix E. $\tilde{\theta}_c$ is a possible complementary solution of (6.2.3) which for the purpose of finding the amplitude equation for $\bar{B}$ can be taken to be zero.

From the heat equation (4.4.5) the vertical velocity component w_3 has the form

$$w_3=iW_1(\bar{B}_{\bar{X}}e^{ik_0x}-c.c)\sin\pi z+\left[W_5(\bar{A}e^{ik_0x}+c.c)+W_7(\bar{A}\bar{A}^*)(\bar{A}e^{ik_0x}+c.c)+W_8(\bar{A}_{\bar{Z}}e^{ik_0x}+c.c)+W_9(\bar{A}^3e^{3ik_0x}+c.c)\right]\sin\pi z$$

$$+\left[W_{10}(\bar{A}\bar{A}_{\bar{X}}e^{2ik_0x}-c.c)+W_{11}(\bar{A}^*\bar{A}_{\bar{X}}-c.c)+W_{12}(\bar{A}\bar{B}e^{2ik_0x}+c.c)+W_{13}(\bar{A}\bar{B}^*+c.c)\right]\sin 2\pi z$$

$$+\left[W_{14}(\bar{A}^3e^{3ik_0x}+c.c)+W_{15}(\bar{A}\bar{A}^*)(\bar{A}e^{ik_0x}+c.c)\right]\sin 3\pi z \tag{6.2.18}$$

where

$$W_1=-\left[\left(\frac{d^2}{dy^2}-\alpha^2\right)\Theta_1+2k_0\Theta\right], \qquad W_5=-\frac{1}{P}\left(\frac{d^2}{dy^2}-\alpha^2\right)\Theta_5+\frac{C_1}{C_2}(\Theta-2k_c\Theta_1), \tag{6.2.19,20}$$

$$W_7 = \frac{1}{P}\left[-\left(\frac{d^2}{dy^2}-\alpha^2\right)\Theta_7 + \varepsilon_8 + \varepsilon_{57} + 2(\varepsilon_{10}+\varepsilon_{59})\right] + \frac{1}{C_2}(C_3 + C_4/P + C_5/P^2)(\Theta - 2k_0\Theta_1), \qquad (6.2.21)$$

$$W_8 = -\frac{1}{P}\left(\frac{d^2}{dy^2}-\alpha^2\right)\Theta_8 + \Theta + \frac{1}{C_2}(C_6 + C_7/P)(\Theta - 2k_0\Theta_1), \qquad (6.2.22)$$

$$W_9 = \frac{1}{P}\left[-\left(\frac{d^2}{dy^2}-\pi^2-9k_0^2\right)\Theta_9 + \varepsilon_7 + \varepsilon_{56}\right], \quad W_{10} = -\frac{1}{P}\left(\frac{d^2}{dy^2}-4\alpha^2\right)\Theta_{10} + i\left(\frac{8k_0}{P}\Theta_2 + \varepsilon_3 + \frac{1}{P}\varepsilon_{52} + \frac{1}{2k_0}U\Theta\right), \qquad (6.2.23,24)$$

$$W_{11} = -\frac{1}{P}\left(\frac{d^2}{dy^2}-4\pi^2\right)\Theta_{11} + i\left(\varepsilon_4 + \frac{1}{P}\varepsilon_{53} - \frac{1}{2k_0}U\Theta\right), \qquad (6.2.25)$$

$$W_{12} = \left[-\frac{1}{P}\left(\frac{d^2}{dy^2}-4\alpha^2\right)\Theta_{12} + \varepsilon_1 + \varepsilon_{50}\right], \quad W_{13} = -\frac{1}{P}\left(\frac{d^2}{dy^2}-4\pi^2\right)\Theta_{13} + \varepsilon_2 + \varepsilon_{51}, \qquad (6.2.26,27)$$

$$W_{14} = \frac{1}{P}\left[-\left(\frac{d^2}{dy^2}-9\alpha^2\right)\Theta_{14} + \varepsilon_5 + \varepsilon_{54}\right], \quad W_{15} = \frac{1}{P}\left[-\left(\frac{d^2}{dy^2}-9\pi^2-k_0^2\right)\Theta_{15} + \varepsilon_6 + \varepsilon_{55} + 2(\varepsilon_9+\varepsilon_{58})\right] \qquad (6.2.28,29)$$

and the ε 's are given by (E1) in appendix E. The form of the solution for the pressure p_3 is obtained from the w_3 momentum equation (4.4.4). The form of the horizontal velocity components v_3 and u_3 can then be obtained using the horizontal momentum equations (4.4.3) and (4.4.2). Having constructed all three velocity components a consistency check is provided by the continuity equation (4.4.1).

Equation (4.4.4) gives

$$\begin{aligned} p_3 = {} & iPP_1(\bar{B}_{\bar{X}}e^{ik_0x} - c.c)\cos\pi z + [P_5(\bar{A}e^{ik_0x}+c.c) + P_7(\bar{A}\bar{A}^*)(\bar{A}e^{ik_0x}+c.c) + P_8(\bar{A}_{\bar{\tau}}e^{ik_0x}+c.c) + P_9(\bar{A}^3e^{3ik_0x}+c.c)]\cos\pi z \\ & + [P_{10}(\bar{A}\bar{A}_{\bar{X}}e^{2ik_0x} - c.c) + P_{11}(\bar{A}^*\bar{A}_{\bar{X}} - c.c) + P_{12}(\bar{A}\bar{B}e^{2ik_0x}+c.c) + P_{13}(\bar{A}\bar{B}^* + c.c)]\cos 2\pi z \\ & + [P_{14}(\bar{A}^3e^{3ik_0x}+c.c) + P_{15}(\bar{A}\bar{A}^*)(\bar{A}e^{ik_0x}+c.c)]\cos 3\pi z + G(x,y,\bar{X},\bar{\tau}) \end{aligned} \qquad (6.2.30)$$

where

$$P_1 = -\frac{1}{\pi}\left[2k_0W + \left(\frac{d^2}{dy^2}-\alpha^2\right)W_1 + R_0\Theta_1\right], \quad P_5 = -\frac{1}{\pi}\left[P\Theta + R_0\Theta_5 + P\left(\frac{d^2}{dy^2}-\alpha^2\right)W_5 - \frac{C_1}{C_2}(-2Pk_0W_1 + PW)\right], \qquad (6.2.31,32)$$

$$P_7 = -\frac{1}{\pi}\left[R_0\Theta_7 + P\left(\frac{d^2}{dy^2}-\alpha^2\right)W_7 - \varepsilon_{21} - 2\varepsilon_{23} - \frac{1}{P}(\varepsilon_{67}+2\varepsilon_{69}) - \frac{1}{C_2}(C_3 + C_4/P + C_5/P^2)(-2Pk_0W_1 + PW)\right], \qquad (6.2.33)$$

$$P_8 = -\frac{1}{\pi}\left[R_0\Theta_8 + P\left(\frac{d^2}{dy^2}-\alpha^2\right)W_8 - W - \frac{1}{C_2}(C_6 + C_7/P)(-2Pk_0W_1 + PW)\right], \qquad (6.2.34)$$

$$P_9 = -\frac{1}{\pi}\left[R_0\Theta_9 + P\left(\frac{d^2}{dy^2}-\pi^2-9k_0^2\right)W_9 - \varepsilon_{20} - \frac{1}{P}\varepsilon_{66}\right], \qquad (6.2.35)$$

$$P_{10} = -\frac{1}{2\pi}\left[R_0\Theta_{10} + P\left(\frac{d^2}{dy^2}-4\alpha^2\right)W_{10} + i\left(8k_0PW_2 - \varepsilon_{16} - \frac{1}{P}\varepsilon_{62} - \frac{1}{2k_0}UW\right)\right], \qquad (6.2.36)$$

$$P_{11} = -\frac{1}{2\pi}\left[R_0\Theta_{11} + P\left(\frac{d^2}{dy^2} - 4\pi^2\right)W_{11} + i\left(-\varepsilon_{17} - \frac{1}{P}\varepsilon_{63} + \frac{1}{2k_0}UW\right)\right], \tag{6.2.37}$$

$$P_{12} = -\frac{1}{2\pi}\left[R_0\Theta_{12} + P\left(\frac{d^2}{dy^2} - 4\alpha^2\right)W_{12} - \varepsilon_{14} - \varepsilon_{60}\right], \quad P_{13} = -\frac{1}{2\pi}\left[R_0\Theta_{13} + P\left(\frac{d^2}{dy^2} - 4\pi^2\right)W_{13} - \varepsilon_{15} - \varepsilon_{61}\right], \tag{6.2.38,39}$$

$$P_{14} = -\frac{1}{3\pi}\left[R_0\Theta_{14} + P\left(\frac{d^2}{dy^2} - 9\alpha^2\right)W_{14} - \varepsilon_{18} - \frac{1}{P}\varepsilon_{64}\right], \quad P_{15} = -\frac{1}{3\pi}\left[R_0\Theta_{15} + P\left(\frac{d^2}{dy^2} - 9\pi^2 - k_0^2\right)W_{15} - \varepsilon_{19} - 2\varepsilon_{22} - \frac{1}{P}(\varepsilon_{65} + 2\varepsilon_{68})\right] \tag{6.2.40,41}$$

and G is an arbitrary function of x,y,$\bar{X}$ and $\bar{\tau}$.

Expanding the right hand sides of (4.4.3) and (4.4.2) gives

$$\begin{aligned} v_3 = {} & iV_1(\bar{B}_{\bar{X}}e^{ik_0x} - c.c)\cos\pi z + \frac{1}{P}\left[V_5(\bar{A}e^{ik_0x} + c.c) + V_7(\bar{A}\bar{A}^*)(\bar{A}e^{ik_0x} + c.c) + V_8(\bar{A}_{\bar{\tau}}e^{ik_0x} + c.c) + V_9(\bar{A}^3e^{3ik_0x} + c.c)\right]\cos\pi z \\ & + \frac{1}{P}\left[V_{10}(\bar{A}\bar{A}_{\bar{X}}e^{2ik_0x} - c.c) + V_{11}(\bar{A}^*\bar{A}_{\bar{X}} - c.c) + V_{12}(\bar{A}\bar{B}e^{2ik_0x} + c.c) + V_{13}(\bar{A}\bar{B}^* + c.c)\right]\cos 2\pi z \\ & + \frac{1}{P}\left[V_{14}(\bar{A}^3e^{3ik_0x} + c.c) + V_{15}(\bar{A}\bar{A}^*)(\bar{A}e^{ik_0x} + c.c)\right]\cos 3\pi z + \tilde{v}_3(x,y,\bar{X},\bar{\tau}) \end{aligned} \tag{6.2.42}$$

and

$$\begin{aligned} u_3 = {} & U_1(\bar{B}_{\bar{X}}e^{ik_0x} + c.c)\cos\pi z + \frac{1}{P}\left[U_5(\bar{A}e^{ik_0x} - c.c) + U_7(\bar{A}\bar{A}^*)(\bar{A}e^{ik_0x} - c.c) + U_8(\bar{A}_{\bar{\tau}}e^{ik_0x} - c.c) + U_9(\bar{A}^3e^{3ik_0x} - c.c)\right]\cos\pi z \\ & + \frac{1}{P}\left[U_{10}(\bar{A}\bar{A}_{\bar{X}}e^{2ik_0x} + c.c) + U_{11}(\bar{A}^*\bar{A}_{\bar{X}} + c.c) + U_{12}(\bar{A}\bar{B}e^{2ik_0x} - c.c) + U_{13}(-\bar{A}\bar{B}^* + \bar{A}^*\bar{B})\right]\cos 2\pi z \\ & + \frac{1}{P}\left[U_{14}(\bar{A}^3e^{3ik_0x} - c.c) + U_{15}(\bar{A}\bar{A}^*)(\bar{A}e^{ik_0x} - c.c)\right]\cos 3\pi z + \tilde{u}_3(x,y,\bar{X},\bar{\tau}) \end{aligned} \tag{6.2.43}$$

where

$$\left[\frac{d^2}{dy^2} - \alpha^2\right]V_1 = -2k_0V + \frac{d}{dy}P_1 = N_1, \quad \left[\frac{d^2}{dy^2} - \alpha^2\right] = N_5 = \frac{d}{dy}P_5 - \frac{C_1}{C_2}(2k_0PV_1 - PV), \tag{6.2.44,45}$$

$$\left[\frac{d^2}{dy^2} - \alpha^2\right]V_7 = N_7 = \frac{d}{dy}P_7 + \frac{1}{P}(\varepsilon_{47} + \varepsilon_{95} + 2(\varepsilon_{49} + \varepsilon_{97})) - \frac{1}{C_2}(C_3 + C_4/P + C_5/P^2)(2k_0PV_1 - PV), \tag{6.2.46}$$

$$\left[\frac{d^2}{dy^2} - \alpha^2\right]V_8 = N_8 = \frac{d}{dy}P_8 + V - \frac{1}{C_2}(C_6 + C_7/P)(2k_0PV_1 - PV), \tag{6.2.47}$$

$$\left[\frac{d^2}{dy^2} - \pi^2 - 9k_0^2\right]V_9 = N_9 = \frac{d}{dy}P_9 + \frac{1}{P}(\varepsilon_{46} + \varepsilon_{94}), \tag{6.2.48}$$

$$\left[\frac{d^2}{dy^2} - 4\alpha^2\right]V_{10} = N_{10} = \frac{d}{dy}P_{10} + i\left(-8k_0V_2 + \frac{1}{P}\varepsilon_{40} + \frac{1}{P}\varepsilon_{88} + \frac{1}{2k_0}UV\right), \tag{6.2.49}$$

$$\left[\frac{d^2}{dy^2} - 4\pi^2\right]V_{11} = N_{11} = \frac{d}{dy}P_{11} + i\left(\frac{1}{P}\varepsilon_{41} + \frac{1}{P}\varepsilon_{89} - \frac{1}{2k_0}UV\right), \tag{6.2.50}$$

$$\left[\frac{d^2}{dy^2} - 4\alpha^2\right]V_{12} = N_{12} = \frac{d}{dy}P_{12} + \varepsilon_{36} + \varepsilon_{84}, \quad \left[\frac{d^2}{dy^2} - 4\pi^2\right]V_{13} = N_{13} = \frac{d}{dy}P_{13} + \varepsilon_{37} + \varepsilon_{85}, \tag{6.2.51,52}$$

$$\left[\frac{d^2}{dy^2} - 9\alpha^2\right]V_{14} = N_{14} = \frac{d}{dy}P_{14} + \frac{1}{P}(\varepsilon_{44} + \varepsilon_{92}), \quad \left[\frac{d^2}{dy^2} - 9\pi^2 - k_0^2\right]V_{15} = N_{15} = \frac{d}{dy}P_{15} + \frac{1}{P}(\varepsilon_{45} + \varepsilon_{93} + 2(\varepsilon_{48} + \varepsilon_{96}), \tag{6.2.53,54}$$

$$\left[\frac{d^2}{dy^2}-\alpha^2\right]U_1 = 2U + \frac{1}{k_0^2}\left(\frac{d^2}{dy^2}-\alpha^2\right)U - k_0 P_1\,, \quad \left[\frac{d^2}{dy^2}-\alpha^2\right]U_5 = i\left[k_0 P_5 - \frac{C_1}{C_2}P\left(P_1 - 2k_0 U_1 - \frac{1}{k_0}U\right)\right], \tag{6.2.55,56}$$

$$\left[\frac{d^2}{dy^2}-\alpha^2\right]U_7 = i\left[k_0 P_7 + \frac{1}{P}(\varepsilon_{35}+\varepsilon_{81}+2\varepsilon_{83}) - \frac{P}{C_2}(C_3 + C_4/P + C_5/P^2)\left(P_1 - 2k_0 U_1 - \frac{1}{k_0}U\right)\right], \tag{6.2.57}$$

$$\left[\frac{d^2}{dy^2}-\alpha^2\right]U_8 = i\left[k_0 P_8 + \frac{1}{k_0}U - \frac{P}{C_2}(C_6 + C_7/P)\left(P_1 - 2k_0 U_1 - \frac{1}{k_0}U\right)\right], \tag{6.2.58}$$

$$\left[\frac{d^2}{dy^2}-\pi^2-9k_0^2\right]U_9 = i\left[3k_0 P_9 + \frac{1}{P}(\varepsilon_{34}+\varepsilon_{80})\right], \tag{6.2.59}$$

$$\left[\frac{d^2}{dy^2}-4\alpha^2\right]U_{10} = 2ik_0 P_{10} + 2P_2 + 8k_0 U_2 + \frac{1}{P}\varepsilon_{28} + \frac{1}{P}\varepsilon_{74} - \frac{1}{2k_0^2}U^2, \tag{6.2.60}$$

$$\left[\frac{d^2}{dy^2}-4\pi^2\right]U_{11} = 2P_3 + \frac{1}{P}\varepsilon_{29} + \frac{1}{P}\varepsilon_{75} + \frac{1}{2k_0^2}U^2, \tag{6.2.61}$$

$$\left[\frac{d^2}{dy^2}-4\alpha^2\right]U_{12} = i(2k_0 P_{12} + \varepsilon_{24} + \varepsilon_{70}), \quad \left[\frac{d^2}{dy^2}-4\pi^2\right]U_{13} = i(\varepsilon_{25}+\varepsilon_{71}), \tag{6.2.62,63}$$

$$\left[\frac{d^2}{dy^2}-9\alpha^2\right]U_{14} = i\left[3k_0 P_{14} + \frac{1}{P}(\varepsilon_{32}+\varepsilon_{78})\right], \quad \left[\frac{d^2}{dy^2}-9\pi^2-k_0^2\right]U_{15} = i\left[k_0 P_{15} + \frac{1}{P}(\varepsilon_{33}+\varepsilon_{79} + 2\varepsilon_{82})\right] \tag{6.2.64.65}$$

and $\tilde{v}_3$ and $\tilde{u}_3$ satisfy

$$P\nabla_H^2 \tilde{v}_3 = \frac{\partial G}{\partial y} + [\varepsilon_{38}+\varepsilon_{86}](\bar{A}\bar{B}e^{2ik_0x} + c.c) + [\varepsilon_{39}+\varepsilon_{87}](\bar{A}\bar{B}^* + c.c) + i\left[-\varepsilon k_0 V_4 + \frac{1}{P}(\varepsilon_{42}+\varepsilon_{90}) + \frac{1}{2k_0}UV\right](\bar{A}\bar{A}_{\bar{x}}e^{2ik_0x} - c.c) + i\left[\frac{1}{P}(\varepsilon_{43}+\varepsilon_{91}) - \frac{1}{2k_0}UV\right](\bar{A}^*\bar{A}_{\bar{x}} - c.c) \tag{6.2.66}$$

and

$$P\nabla_H^2 \tilde{u}_3 = \frac{\partial G}{\partial x} + i[\varepsilon_{26}+\varepsilon_{72}](\bar{A}\bar{B}e^{2ik_0x} - c.c) + \left[8k_0 U_4 + \frac{1}{P}(\varepsilon_{30}+\varepsilon_{76}) - \frac{1}{2k_0^2}U^2 + 2F_1\right](\bar{A}\bar{A}_{\bar{x}}e^{2ik_0x} + c.c) + \left[\frac{1}{P}(\varepsilon_{31}+\varepsilon_{77}) + \frac{1}{2k_0^2}U^2 + 2F_2\right](\bar{A}^*\bar{A}_{\bar{x}} + c.c) + \frac{\partial F_0}{\partial \bar{x}} \tag{6.2.67}$$

where the ε's are given by (E1) in appendix E. In addition $\tilde{v}_3$ and $\tilde{u}_3$ satisfy the two-dimensional continuity equation

$$\frac{\partial \tilde{u}_3}{\partial x} + \frac{\partial \tilde{v}_3}{\partial y} = -\frac{\partial \tilde{u}_2}{\partial \bar{x}}\,. \tag{6.2.68}$$

In order to complete the basic forms of the horizontal velocity components it is necessary to determine $\tilde{v}_3$ and $\tilde{u}_3$. The right hand sides of equations (6.2.66,67) suggest

$$\tilde{v}_3 = \frac{1}{P}\left[V_{16}(\bar{A}\bar{B}e^{2ik_0x} + c.c) + iV_{17}(\bar{A}\bar{A}_{\bar{x}}e^{2ik_0x} - c.c)\right], \tag{6.2.69}$$

$$\tilde{u}_3 = \frac{1}{P}[iU_{16}(\bar{A}\bar{B}e^{2ik_0x} - c.c) + U_{17}(\bar{A}\bar{A}_{\bar{x}}e^{2ik_0x} + c.c) + U_{18}(\bar{A}^*\bar{A}_{\bar{x}} + c.c)] \quad (6.2.70)$$

and

$$g = G_1(\bar{A}\bar{B}e^{2ik_0x} + c.c) + iG_2(\bar{A}\bar{A}_{\bar{x}}e^{2ik_0x} - c.c) + G_3(\bar{A}\bar{B}^* + c.c) + iG_4(\bar{A}^*\bar{A}_{\bar{x}} - c.c) + G_0(\bar{x},\bar{t}) \quad (6.2.71)$$

where G_i i=1..4 are real functions of y. The boundary conditions on $\tilde{v}_3$ and $\tilde{u}_3$ are

$$\tilde{v}_3 = \tilde{u}_3 = 0 \quad y = \pm a \quad (6.2.72)$$

which from (6.2.68) are equivalent to

$$\tilde{v}_3 = \frac{\partial}{\partial y}\tilde{v}_3 = 0 \quad y = \pm a. \quad (6.2.73)$$

Substituting the forms (6.2.69,70,71) into equations (6.2.66,67) gives

$$\left[\frac{d^4}{dy^4} - 8k_0^2\frac{d^2}{dy^2} + 16k_0^4\right]V_{16} = -2k_0\left[2k_0(\varepsilon_{38} + \varepsilon_{86}) - \frac{d}{dy}(\varepsilon_{26} + \varepsilon_{72})\right], \quad (6.2.74)$$

$$\left[\frac{d^4}{dy^4} - 8k_0^2\frac{d^2}{dy^2} + 16k_0^4\right]V_{17} = -2\left(\frac{d^2}{dy^2} - 4k_0^2\right)\frac{d}{dy}U_4 - 2k_0\left[2k_0\left(-8k_0V_4 + \frac{1}{P}(\varepsilon_{42} + \varepsilon_{90}) + \frac{1}{2k_0}UV\right)\right.$$
$$\left. + \frac{d}{dy}\left(8k_0U_4 + \frac{1}{P}(\varepsilon_{30} + \varepsilon_{76}) - \frac{1}{2k_0^2}U^2 + 2F_1\right)\right] \quad (6.2.75)$$

where V_i i=16,17 satisfy the conditions

$$V_{16} = V_{17} = \frac{d}{dy}V_{16} = \frac{d}{dy}V_{17} = 0 \quad y = \pm a. \quad (6.2.76)$$

Also

$$U_{16} = \frac{1}{2k_0}\frac{d}{dy}V_{16}, \qquad U_{17} = -\frac{1}{k_0}\left[U_4 + \frac{1}{2}\frac{d}{dy}V_{17}\right], \quad (6.2.77,78)$$

$$G_1 = \frac{1}{2k_0}\left[\left(\frac{d^2}{dy^2} - 4k_0^2\right)U_{16} - (\varepsilon_{26} + \varepsilon_{72})\right], \quad (6.2.79)$$

$$G_2 = \frac{1}{2k_0}\left[-\left(\frac{d^2}{dy^2} - 4k_0^2\right)U_{17} + \left(8k_0U_4 + \frac{1}{P}(\varepsilon_{30} + \varepsilon_{76}) - \frac{1}{2k_0^2}U^2 + 2F_1\right)\right], \quad (6.2.80)$$

$$G_3 = -\int_0^y(\varepsilon_{39} + \varepsilon_{87})\,dy \quad \text{and} \quad G_4 = -\int_0^y\left[\frac{1}{P}(\varepsilon_{43} + \varepsilon_{91}) - \frac{1}{2k_0}UV\right]dy. \quad (6.2.81,82)$$

The overall mass flux down the channel at this level of approximation is given by

$$\int_{y=-a}^{+a}\int_{z=0}^{1} u_3 \, dy \, dz \tag{6.2.83}$$

and it is easily shown from (6.2.77,78) that U_i i=16,17 make no contribution. In chapter 8 the infinite channel is considered to have rigid endwalls in which case the overall mass flux must be zero. However, there is generally a contribution to the overall mass flux from U_{18} unless the pressure gradient $\frac{\partial F_0}{\partial X}$, arising from the pressure at the previous order, drives a Poiseuille-type component of flow of sufficient strength to counteract the forcing terms that generate U_{18}. Thus

$$F_0 = C|\bar{A}|^2 + \text{constant} \tag{6.2.84}$$

where C is a constant to be determined, so that from (6.2.67)

$$\frac{d^2}{dy^2} U_{18} = \frac{1}{P}(\varepsilon_{31} + \varepsilon_{77}) + \frac{1}{2k_c^2} U^2 + 2F_2 + C. \tag{6.2.85}$$

It is now convenient to write

$$U_{18} = \bar{U}_{18} + \hat{U}_{18} \tag{6.2.86}$$

where

$$\frac{d^2}{dy^2}\bar{U}_{18} = \frac{1}{P}(\varepsilon_{31} + \varepsilon_{77}) + \frac{1}{2k_c^2}U^2 + 2F_2, \quad \bar{U}_{18} = 0 \quad y = \pm a, \tag{6.2.87}$$

and

$$\frac{d^2}{dy^2}\hat{U}_{18} = C, \quad \hat{U}_{18} = 0 \quad y = \pm a. \tag{6.2.88}$$

The solution of (6.2.87) is obtained numerically while that of (6.2.88) is the Poiseuille flow

$$\hat{U}_{18} = \frac{1}{2}C(y+a)(y-a). \tag{6.2.89}$$

The constant C must be chosen as

$$C = \frac{3}{2a^3}\int_{y=-a}^{+a} \bar{U}_{18} \, dy, \tag{6.2.90}$$

so that the overall mass flux (6.2.83) is zero.

The basic forms of the solutions for all the dependent variables have thus been obtained. By applying the Laplacian operator ∇^2, to the continuity equation (4.4.1) a consistency check for the z-dependent terms in the three velocity components u_3, v_3 and w_3 is obtained, and is found to be satisfied.

6.3 Determination of the y-dependent functions in the basic forms

(a) Θ_i, V_i i=1,5,7,8

The y-dependent functions Θ_1 and V_1 have been determined in section 4.3. In order to determine the y-dependent functions Θ_i i=5,7,8 in the temperature θ_3, and V_i i=5,7,8 in the horizontal velocity component v_3, numerical methods need to be used. From (6.2.6,45) the coupled functions Θ_5, V_5 are given by

$$\Theta_5 = P\bar{\Theta}_5 \qquad \text{and} \qquad V_5 = P\bar{V}_5 \tag{6.3.1}$$

where

$$\left[\frac{d}{dy} - \underline{E}(k_0,\pi)\right]\underline{f}(\bar{\Theta}_5,\bar{V}_5) = \underline{\Phi}\left(\bar{M}_5 - \frac{C_1}{C_2}\bar{M}_6, \bar{N}_5 - \frac{C_1}{C_2}\bar{N}_6\right), \quad \underline{m}\underline{f}(\bar{\Theta}_5,\bar{V}_5) = 0 \quad y = \pm a \tag{6.3.2}$$

and

$$\bar{M}_5 = \left(\frac{d^2}{dy^2} - k_0^2\right)\Theta, \qquad \bar{N}_5 = -\frac{1}{\pi}\frac{d}{dy}\Theta, \tag{6.3.3}$$

$$\bar{M}_6 = 2k_0\left\{\left(\frac{d^2}{dy^2} - \alpha^2\right)^2\left[3\Theta_1 - \frac{1}{k_0}\Theta\right] - \frac{\pi}{2k_0^3}\left(\frac{d^2}{dy^2} - \alpha^2\right)U - R_0\Theta_1 - 6k_0W\right\}, \tag{6.3.4}$$

$$\bar{N}_6 = \frac{2}{\pi}\left\{2k_0\left(\alpha^2 - \frac{d^2}{dy^2}\right)\frac{d}{dy}\Theta_1 - \frac{d}{dy}(W + 2k_0^2\Theta) + k_0\pi V_1 - \frac{\pi}{2}V\right\}. \tag{6.3.5}$$

Similarly from (6.2.7,46)

$$\Theta_7 = \hat{\Theta}_7 + P\bar{\Theta}_7 + \frac{1}{P}\tilde{\Theta}_7 \qquad \text{and} \qquad V_7 = \hat{V}_7 + P\bar{V}_7 + \frac{1}{P}\tilde{V}_7 \tag{6.3.6}$$

where

$$\left[\frac{d}{dy} - \underline{E}(k_0,\pi)\right]\underline{f}(\hat{\Theta}_7,\hat{V}_7) = \underline{\Phi}\left(\hat{M}_7 - \frac{C_4}{C_2}\bar{M}_6, \hat{N}_7 - \frac{C_4}{C_2}\bar{N}_6\right), \quad \underline{m}\underline{f}(\hat{\Theta}_7,\hat{V}_7) = 0 \quad y = \pm a, \tag{6.3.7}$$

$$\left[\frac{d}{dy} - \underline{E}(k_0,\pi)\right]\underline{f}(\bar{\Theta}_7,\bar{V}_7) = \underline{\Phi}\left(\bar{M}_7 - \frac{C_3}{C_2}\bar{M}_6, \bar{N}_7 - \frac{C_3}{C_2}\bar{N}_6\right), \quad \underline{m}\underline{f}(\bar{\Theta}_7,\bar{V}_7) = 0 \quad y = \pm a, \tag{6.3.8}$$

$$\left[\frac{d}{dy} - \underline{E}(k_0,\pi)\right]\underline{f}(\tilde{\Theta}_7,\tilde{V}_7) = \underline{\Phi}\left(\tilde{M}_7 - \frac{C_5}{C_2}\bar{M}_6, \tilde{N}_7 - \frac{C_5}{C_2}\bar{N}_6\right), \quad \underline{m}\underline{f}(\tilde{\Theta}_7,\tilde{V}_7) = 0 \quad y = \pm a \tag{6.3.9}$$

and

$$\hat{M}_7=\left(\frac{d^2}{dy^2}-\alpha^2\right)^2\left[\hat{\varepsilon}_8+\hat{\varepsilon}_{57}+2(\hat{\varepsilon}_{10}+\hat{\varepsilon}_{59})\right]-\left(\frac{d^2}{dy^2}-k_c^2\right)\left[\hat{\varepsilon}_{21}+\bar{\varepsilon}_{67}+2(\hat{\varepsilon}_{23}+\bar{\varepsilon}_{69})\right]+\pi\left[k_0(\bar{\varepsilon}_{35}+\bar{\varepsilon}_{81}+2\bar{\varepsilon}_{83})-\frac{d}{dy}(\bar{\varepsilon}_{47}+\bar{\varepsilon}_{95}+2\bar{\varepsilon}_{49}+2\bar{\varepsilon}_{97})\right],$$

$$\hat{N}_7=\frac{1}{\pi}\left\{-\left(\frac{d^2}{dy^2}-\alpha^2\right)\frac{d}{dy}\left[\hat{\varepsilon}_8+\hat{\varepsilon}_{57}+2(\hat{\varepsilon}_{10}+\hat{\varepsilon}_{59})\right]+\frac{d}{dy}\left[\hat{\varepsilon}_{21}+\bar{\varepsilon}_{67}+2(\hat{\varepsilon}_{23}+\bar{\varepsilon}_{69})\right]+\pi\left[\bar{\varepsilon}_{47}+\bar{\varepsilon}_{95}+2(\bar{\varepsilon}_{49}+\bar{\varepsilon}_{97})\right]\right.$$

$$\bar{M}_7=\left(\frac{d^2}{dy^2}-\alpha^2\right)^2\left[\bar{\varepsilon}_8+\bar{\varepsilon}_{57}+2(\bar{\varepsilon}_{10}+\bar{\varepsilon}_{59})\right],\quad \bar{N}_7=-\frac{1}{\pi}\left(\frac{d^2}{dy^2}-\alpha^2\right)\frac{d}{dy}\left[\bar{\varepsilon}_8+\bar{\varepsilon}_{57}+2(\bar{\varepsilon}_{10}+\bar{\varepsilon}_{59})\right],$$

$$\tilde{M}_7=-\left(\frac{d^2}{dy^2}-k_c^2\right)\left[\tilde{\varepsilon}_{21}+\hat{\varepsilon}_{67}+2(\tilde{\varepsilon}_{23}+\hat{\varepsilon}_{69})\right]+\pi\left[k_0(\hat{\varepsilon}_{35}+\hat{\varepsilon}_{81}+2\hat{\varepsilon}_{83})-\frac{d}{dy}(\hat{\varepsilon}_{47}+\hat{\varepsilon}_{95}+2(\hat{\varepsilon}_{49}+\hat{\varepsilon}_{97}))\right],$$

$$\tilde{N}_7=\frac{1}{\pi}\left\{\frac{d}{dy}\left[\tilde{\varepsilon}_{21}+\hat{\varepsilon}_{67}+2(\tilde{\varepsilon}_{23}+\hat{\varepsilon}_{69})\right]+\pi\left[\hat{\varepsilon}_{47}+\hat{\varepsilon}_{95}+2(\hat{\varepsilon}_{49}+\hat{\varepsilon}_{97})\right]\right\}. \tag{6.3.10}$$

Finally, from (6.2.8,47)

$$\Theta_8=\hat{\Theta}_8+P\bar{\Theta}_8 \quad \text{and} \quad V_8=\hat{V}_8+P\bar{V}_8 \tag{6.3.11}$$

where

$$\left[\frac{d}{dy}-\underline{E}(k_c,\pi)\right]\underline{f}(\hat{\Theta}_8,\hat{V}_8)=\underline{\Phi}\left(\hat{M}_8-\frac{C_7}{C_2}\bar{M}_6,\hat{N}_8-\frac{C_7}{C_2}\bar{N}_6\right),\quad \underline{m}\underline{f}(\hat{\Theta}_8,\hat{V}_8)=0\quad y=\pm a, \tag{6.3.12}$$

$$\left[\frac{d}{dy}-\underline{E}(k_c,\pi)\right]\underline{f}(\bar{\Theta}_8,\bar{V}_8)=\underline{\Phi}\left(\bar{M}_8-\frac{C_6}{C_2}\bar{M}_6,\bar{N}_8-\frac{C_6}{C_2}\bar{N}_6\right),\quad \underline{m}\underline{f}(\bar{\Theta}_8,\bar{V}_8)=0\quad y=\pm a \tag{6.3.14}$$

and

$$\hat{M}_8=\left(\frac{d^2}{dy^2}-\alpha^2\right)^2\Theta,\qquad \hat{N}_8=\frac{1}{\pi}\left[\pi V+\frac{d}{dy}W\right],$$

$$\bar{M}_8=\hat{M}_8,\qquad \bar{N}_8=-\frac{1}{\pi}\left(\frac{d^2}{dy^2}-\alpha^2\right)\frac{d}{dy}\Theta. \tag{6.3.15}$$

It should be noted that each of the matrix systems (6.3.2,7,8,9,12,14) is a non-homogenous form of the basic linear system (4.2.4). From section 4.3 and appendix D, it is known that the condition needed for the existence of a solution of (6.3.2) say, is

$$\int_{y=-a}^{+a}\left[\hat{f}_4\left(\bar{M}_5-\frac{C_1}{C_2}\bar{M}_6\right)+\hat{f}_8\left(\bar{N}_5-\frac{C_1}{C_2}\bar{N}_6\right)\right]dy=0 \tag{6.3.16}$$

where $\hat{f}_4$ and $\hat{f}_8$ are solutions obtained from the matrix system (4.3.60). Similar conditions exist for the matrix systems (6.3.7,8,9,12,14). The amplitude coefficients C_1 and C_2 are given by

$$C_1=\int_{y=-a}^{+a}\left[\hat{f}_4\bar{M}_5+\hat{f}_8\bar{N}_5\right]dy \quad \text{and} \quad C_2=\int_{y=-a}^{+a}\left[\hat{f}_4\bar{M}_6+\hat{f}_8\bar{N}_6\right]dy. \tag{6.3.17}$$

Thus, it is clear that the condition (6.3.16) is satisfied and a solution of the system (6.3.2) exists. Similarly, the solvability conditions for the systems (6.3.7,8,9,12,14) can be shown to be satisfied. In all, the solutions of six eighth order systems need to be determined.

In each case a fourth order Runge-Kutta process is used to compute the solution from y=-a through to y=+a, following the procedure outlined in section 4.3. The general solution, say in the case of (6.3.2) is therefore given by

$$\underline{f}(\bar{\Theta}_5,\bar{V}_5) = \hat{q}_1\underline{f}_1(\bar{\Theta}_5,\bar{V}_5)+\hat{q}_2\underline{f}_2(\bar{\Theta}_5,\bar{V}_5)+\hat{q}_3\underline{f}_3(\bar{\Theta}_5,\bar{V}_5)+\hat{q}_4\underline{f}_4(\bar{\Theta}_5,\bar{V}_5)+\underline{f}_p(\bar{\Theta}_5,\bar{V}_5) \qquad (6.3.18)$$

where the arbitrary constants $\hat{q}_j$ j=1..4 are determined by applying the four known conditions at y=+a. This gives a non-homogenous system of four equations in four unknowns which can be expressed in matrix notation as given by (4.3.80). If $\underline{E} \neq \underline{E}(k_o,\pi)$, the determinant of the matrix f_c in equation (4.3.80) is non-zero and the constants $\hat{q}_j$ j=1..4 can be determined uniquely. However, if $\underline{E} = \underline{E}(k_o,\pi)$, as in the case for the matrix systems (6.3.2,7,8,9,12,14), the determinant of the matrix f_c is zero. In this case the value of $\hat{q}_4$(say) is assumed to be a known arbitrary constant, thus enabling $\hat{q}_j$ j=1..3 to be determined using Cramers' rule. It should be noted that the solutions for Θ_i,V_i i=5,7,8 are not unique since in each case the value of the constant q_4 is arbitrary. This non-uniqueness is equivalent to the addition of an arbitrary constant multiple of the basic linear solution for Θ and V to θ_3 and v_3, which plays no part in finding the amplitude equation for $\bar{B}$.

<u>(b) Θ_i,V_i i=9..15</u>

The y-dependent functions Θ_i i=9..15 in the temperature θ_3, and V_i i=9..15 in the horizontal velocity component v_3 are determined numerically. From (6.2.9,48) the coupled functions Θ_q, V_q are given by

$$\Theta_q = \hat{\Theta}_q + P\bar{\Theta}_q + \frac{1}{P}\tilde{\Theta}_q \quad \text{and} \quad V_q = \hat{V}_q + P\bar{V}_q + \frac{1}{P}\tilde{V}_q \tag{6.3.19}$$

where

$$\left[\frac{d}{dy} - \underline{E}(3k_0, \pi)\right]\underline{f}(\hat{\Theta}_q, \hat{V}_q) = \underline{\mathcal{L}}(\hat{M}_q, \hat{N}_q), \quad \underline{m}\,\underline{f}(\hat{\Theta}_q, \hat{V}_q) = 0 \quad y = \pm a, \tag{6.3.20}$$

$$\left[\frac{d}{dy} - \underline{E}(3k_0, \pi)\right]\underline{f}(\bar{\Theta}_q, \bar{V}_q) = \underline{\mathcal{L}}(\bar{M}_q, \bar{N}_q), \quad \underline{m}\,\underline{f}(\bar{\Theta}_q, \bar{V}_q) = 0 \quad y = \pm a, \tag{6.3.21}$$

$$\left[\frac{d}{dy} - \underline{E}(3k_0, \pi)\right]\underline{f}(\tilde{\Theta}_q, \tilde{V}_q) = \underline{\mathcal{L}}(\tilde{M}_q, \tilde{N}_q), \quad \underline{m}\,\underline{f}(\tilde{\Theta}_q, \tilde{V}_q) = 0 \quad y = \pm a \tag{6.3.22}$$

and

$$\hat{M}_q = \left(\frac{d^2}{dy^2} - \pi^2 - 9k_0^2\right)^2[\hat{\varepsilon}_7 + \hat{\varepsilon}_{56}] - \left(\frac{d^2}{dy^2} - 9k_0^2\right)[\hat{\varepsilon}_{20} + \bar{\varepsilon}_{66}] + \pi\left[3k_0(\bar{\varepsilon}_{34} + \bar{\varepsilon}_{80}) - \frac{d}{dy}(\bar{\varepsilon}_{46} + \bar{\varepsilon}_{94})\right],$$

$$\hat{N}_q = \frac{1}{\pi}\left\{-\left(\frac{d^2}{dy^2} - \pi^2 - 9k_0^2\right)\frac{d}{dy}[\hat{\varepsilon}_7 + \hat{\varepsilon}_{56}] + \frac{d}{dy}[\hat{\varepsilon}_{20} + \bar{\varepsilon}_{66}] + \pi[\bar{\varepsilon}_{46} + \bar{\varepsilon}_{94}]\right\},$$

$$\bar{M}_q = \left(\frac{d^2}{dy^2} - \pi^2 - 9k_0^2\right)^2[\bar{\varepsilon}_7 + \bar{\varepsilon}_{56}], \quad \tilde{M}_q = -\left(\frac{d^2}{dy^2} - 9k_0^2\right)[\tilde{\varepsilon}_{20} + \hat{\varepsilon}_{66}] + \pi\left[3k_0(\hat{\varepsilon}_{34} + \hat{\varepsilon}_{80}) - \frac{d}{dy}(\hat{\varepsilon}_{46} + \hat{\varepsilon}_{94})\right],$$

$$\bar{N}_q = -\frac{1}{\pi}\left(\frac{d^2}{dy^2} - \pi^2 - 9k_0^2\right)^2\frac{d}{dy}[\bar{\varepsilon}_7 + \bar{\varepsilon}_{56}], \quad \tilde{N}_q = \frac{1}{\pi}\left\{\frac{d}{dy}[\tilde{\varepsilon}_{20} + \hat{\varepsilon}_{66}] + \pi[\hat{\varepsilon}_{46} + \hat{\varepsilon}_{94}]\right\}. \tag{6.3.23}$$

From (6.2.10,49)

$$\Theta_{10} = i\hat{\Theta}_{10} + iP\bar{\Theta}_{10} \quad \text{and} \quad V_{10} = i\hat{V}_{10} + iP\bar{V}_{10} \tag{6.3.24}$$

where

$$\left[\frac{d}{dy} - \underline{E}(2k_0, 2\pi)\right]\underline{f}(\hat{\Theta}_{10}, \hat{V}_{10}) = \underline{\mathcal{L}}(\hat{M}_{10}, \hat{N}_{10}), \quad \underline{m}\,\underline{f}(\hat{\Theta}_{10}, \hat{V}_{10}) = 0 \quad y = \pm a, \tag{6.3.25}$$

$$\left[\frac{d}{dy} - \underline{E}(2k_0, 2\pi)\right]\underline{f}(\bar{\Theta}_{10}, \bar{V}_{10}) = \underline{\mathcal{L}}(\bar{M}_{10}, \bar{N}_{10}), \quad \underline{m}\,\underline{f}(\bar{\Theta}_{10}, \bar{V}_{10}) = 0 \quad y = \pm a \tag{6.3.26}$$

and

$$\hat{M}_{10} = -8k_0\left(\frac{d^2}{dy^2} - 4\alpha^2\right)^2\hat{\varepsilon}_{98} + 4\left(\frac{d^2}{dy^2} - 4\alpha^2\right)[3k_0\tilde{\varepsilon}_{99} - \pi\hat{\varepsilon}_{100}] - \left(\frac{d^2}{dy^2} - 4k_0^2\right)\left[\hat{\varepsilon}_{16} - \bar{\varepsilon}_{62} + \frac{1}{2k_0}\hat{\varepsilon}_{102}\right]$$

$$+ 4k_0[-\hat{\varepsilon}_{101} + R_0\hat{\varepsilon}_{98} - \pi(\bar{\varepsilon}_{28} + \bar{\varepsilon}_{74})] - 2\pi\left[-\frac{1}{k_0}\hat{\varepsilon}_{103} + \frac{d}{dy}\left(\frac{1}{2k_0}\hat{\varepsilon}_{105} + \bar{\varepsilon}_{40} + \bar{\varepsilon}_{88}\right)\right],$$

$$\hat{N}_{10} = \frac{1}{2\pi}\left\{8k_0\left(\frac{d^2}{dy^2} - 4\alpha^2\right)\frac{d}{dy}\hat{\varepsilon}_{98} + \frac{d}{dy}\left[-8k_0\tilde{\varepsilon}_{99} + \hat{\varepsilon}_{16} + \bar{\varepsilon}_{62} + \frac{1}{2k_0}\hat{\varepsilon}_{102}\right] + 2\pi\left[-8k_0\hat{\varepsilon}_{106} + \bar{\varepsilon}_{40} + \bar{\varepsilon}_{88} + \frac{1}{2k_0}\hat{\varepsilon}_{105}\right]\right\},$$

$$\bar{M}_{10} = \left(\frac{d^2}{dy^2} - 4\alpha^2\right)^2\left[-8k_0\bar{\varepsilon}_{98} + \hat{\varepsilon}_3 + \bar{\varepsilon}_{52} + \frac{1}{2k_0}\hat{\varepsilon}_{104}\right] + 4\left(\frac{d^2}{dy^2} - 4\alpha^2\right)[3k_0\hat{\varepsilon}_{99} - \pi\hat{\varepsilon}_{100}] + 4R_0k_0\bar{\varepsilon}_{98}\},$$

$$\bar{N}_{10} = \frac{1}{2\pi}\left\{\left(\frac{d^2}{dy^2} - 4\alpha^2\right)\frac{d}{dy}\left[8k_0\bar{\varepsilon}_{98} - \hat{\varepsilon}_3 - \bar{\varepsilon}_{52} - \frac{1}{2k_0}\hat{\varepsilon}_{104}\right] - 8k_0\frac{d}{dy}\hat{\varepsilon}_{99} - 16k_0\pi\bar{\varepsilon}_{106}\right\}. \tag{6.3.27}$$

From (6.2.11,50)

$$\Theta_{11} = i\hat{\Theta}_{11} + iP\bar{\Theta}_{11} \quad \text{and} \quad V_{11} = i\hat{V}_{11} + iP\bar{V}_{11} \tag{6.3.28}$$

where

$$\left[\frac{d}{dy} - \underline{E}(0,2\pi)\right]\underline{f}(\hat{\Theta}_{11},\hat{V}_{11}) = \underline{\mathcal{L}}(\hat{M}_{11},\hat{N}_{11}), \quad \underline{m}\,\underline{f}(\hat{\Theta}_{11},\hat{V}_{11}) = 0 \quad y=\pm a, \tag{6.3.29}$$

$$\left[\frac{d}{dy} - \underline{E}(0,2\pi)\right]\underline{f}(\bar{\Theta}_{11},\bar{V}_{11}) = \underline{\mathcal{L}}(\bar{M}_{11},\bar{N}_{11}), \quad \underline{m}\,\underline{f}(\bar{\Theta}_{11},\bar{V}_{11}) = 0 \quad y=\pm a \tag{6.3.30}$$

and

$$\hat{M}_{11} = \frac{d^2}{dy^2}\left[\frac{1}{2k_0}\hat{\varepsilon}_{102} - (\hat{\varepsilon}_{17}+\bar{\varepsilon}_{63})\right] + \pi\frac{d}{dy}\left[\frac{1}{k_0}\hat{\varepsilon}_{105} - 2(\bar{\varepsilon}_{41}+\bar{\varepsilon}_{89})\right],$$

$$\hat{N}_{11} = \frac{1}{2\pi}\left\{\frac{d}{dy}\left(\hat{\varepsilon}_{17}+\bar{\varepsilon}_{63}-\frac{1}{2k_0}\hat{\varepsilon}_{102}\right) + 2\pi\left(\bar{\varepsilon}_{41}+\bar{\varepsilon}_{89}-\frac{1}{2k_0}\hat{\varepsilon}_{105}\right)\right\},$$

$$\bar{M}_{11} = \left(\frac{d^2}{dy^2}-4\pi^2\right)^2\left(\hat{\varepsilon}_4+\bar{\varepsilon}_{53}-\frac{1}{2k_0}\hat{\varepsilon}_{104}\right), \quad \bar{N}_{11} = -\frac{1}{2\pi}\left(\frac{d^2}{dy^2}-4\pi^2\right)\frac{d}{dy}\left(\hat{\varepsilon}_4+\bar{\varepsilon}_{53}-\frac{1}{2k_0}\hat{\varepsilon}_{104}\right). \tag{6.3.31}$$

From (6.2.12,51)

$$\Theta_{12} = \hat{\Theta}_{12} + P\bar{\Theta}_{12} \qquad \text{and} \qquad V_{12} = \hat{V}_{12} + P\bar{V}_{12} \tag{6.3.32}$$

where

$$\left[\frac{d}{dy} - \underline{E}(2k_0,2\pi)\right]\underline{f}(\hat{\Theta}_{12},\hat{V}_{12}) = \underline{\mathcal{L}}(\hat{M}_{12},\hat{V}_{12}), \quad \underline{m}\,\underline{f}(\hat{\Theta}_{12},\hat{V}_{12}) = 0 \quad y=\pm a, \tag{6.3.33}$$

$$\left[\frac{d}{dy} - \underline{E}(2k_0,2\pi)\right]\underline{f}(\bar{\Theta}_{12},\bar{V}_{12}) = \underline{\mathcal{L}}(\bar{M}_{12},\bar{N}_{12}), \quad \underline{m}\,\underline{f}(\bar{\Theta}_{12},\bar{V}_{12}) = 0 \quad y=\pm a \tag{6.3.34}$$

and

$$\hat{M}_{12} = -\left(\frac{d^2}{dy^2}-4k_0^2\right)(\hat{\varepsilon}_{14}+\hat{\varepsilon}_{60}) + 2\pi\left[2k_0(\hat{\varepsilon}_{24}+\hat{\varepsilon}_{70}) - \frac{d}{dy}(\hat{\varepsilon}_{36}+\hat{\varepsilon}_{84})\right], \quad \hat{N}_{12} = \frac{1}{2\pi}\frac{d}{dy}(\hat{\varepsilon}_{14}+\hat{\varepsilon}_{60}) + (\hat{\varepsilon}_{36}+\hat{\varepsilon}_{84}),$$

$$\bar{M}_{12} = \left(\frac{d^2}{dy^2}-4\alpha^2\right)^2(\hat{\varepsilon}_1+\hat{\varepsilon}_{50}), \quad \bar{N}_{12} = -\frac{1}{2\pi}\left(\frac{d^2}{dy^2}-4\alpha^2\right)\frac{d}{dy}(\hat{\varepsilon}_1+\hat{\varepsilon}_{50}). \tag{6.3.35}$$

From (6.2.14,52)

$$\Theta_{13} = \hat{\Theta}_{13} + P\bar{\Theta}_{13} \qquad \text{and} \qquad V_{13} = \hat{V}_{13} + P\bar{V}_{13} \tag{6.3.36}$$

where

$$\left[\frac{d}{dy} - \underline{E}(0,2\pi)\right]\underline{f}(\hat{\Theta}_{13},\hat{V}_{13}) = \underline{\mathcal{L}}(\hat{M}_{13},\hat{N}_{13}), \quad \underline{m}\,\underline{f}(\hat{\Theta}_{13},\hat{V}_{13}) = 0 \quad y=\pm a, \tag{6.3.37}$$

$$\left[\frac{d}{dy} - \underline{E}(0,2\pi)\right]\underline{f}(\bar{\Theta}_{13},\bar{V}_{13}) = \underline{\mathcal{L}}(\bar{M}_{13},\bar{N}_{13}), \quad \underline{m}\,\underline{f}(\bar{\Theta}_{13},\bar{V}_{13}) = 0 \quad y=\pm a \tag{6.3.38}$$

and

$$\hat{M}_{13} = -\frac{d^2}{dy^2}(\hat{\varepsilon}_{15}+\hat{\varepsilon}_{61}) - 2\pi\frac{d}{dy}(\hat{\varepsilon}_{37}+\hat{\varepsilon}_{85}), \quad \hat{N}_{13} = \frac{1}{2\pi}\left\{\frac{d}{dy}(\hat{\varepsilon}_{15}+\hat{\varepsilon}_{61}) + 2\pi(\hat{\varepsilon}_{37}+\hat{\varepsilon}_{85})\right\},$$

$$\bar{M}_{13} = \left(\frac{d^2}{dy^2}-4\pi^2\right)^2(\hat{\varepsilon}_2+\hat{\varepsilon}_{51}), \quad \bar{N}_{13} = -\frac{1}{2\pi}\left(\frac{d^2}{dy^2}-4\pi^2\right)\frac{d}{dy}(\hat{\varepsilon}_2+\hat{\varepsilon}_{51}). \tag{6.3.39}$$

From (6.2.15,53)

$$\Theta_{14} = \hat{\Theta}_{14} + P\bar{\Theta}_{14} + \frac{1}{P}\tilde{\Theta}_{14} \qquad \text{and} \qquad V_{14} = \hat{V}_{14} + P\bar{V}_{14} + \frac{1}{P}\tilde{V}_{14} \tag{6.3.40}$$

where

$$\left[\frac{d}{dy}-\underline{E}(3k_0,3\pi)\right]\underline{f}(\hat{\Theta}_{14},\hat{V}_{14})=\underline{\mathcal{Q}}(\hat{M}_{14},\hat{N}_{14}),\qquad \underline{m}\,\underline{f}(\hat{\Theta}_{14},\hat{V}_{14})=0\quad y=\pm a, \tag{6.3.41}$$

$$\left[\frac{d}{dy}-\underline{E}(3k_0,3\pi)\right]\underline{f}(\bar{\Theta}_{14},\bar{V}_{14})=\underline{\mathcal{Q}}(\bar{M}_{14},\bar{N}_{14}),\qquad \underline{m}\,\underline{f}(\bar{\Theta}_{14},\bar{V}_{14})=0\quad y=\pm a, \tag{6.3.42}$$

$$\left[\frac{d}{dy}-\underline{E}(3k_0,3\pi)\right]\underline{f}(\tilde{\Theta}_{14},\tilde{V}_{14})=\underline{\mathcal{Q}}(\tilde{M}_{14},\tilde{N}_{14}),\qquad \underline{m}\,\underline{f}(\tilde{\Theta}_{14},\tilde{V}_{14})=0\quad y=\pm a \tag{6.3.43}$$

and

$$\hat{M}_{14}=\left(\frac{d^2}{dy^2}-9\alpha^2\right)^2(\hat{\varepsilon}_5+\hat{\varepsilon}_{54})-\left(\frac{d^2}{dy^2}-9k_0^2\right)(\hat{\varepsilon}_{18}+\bar{\varepsilon}_{64})+3\pi\left[3k_0(\bar{\varepsilon}_{32}+\bar{\varepsilon}_{78})-\frac{d}{dy}(\bar{\varepsilon}_{44}+\bar{\varepsilon}_{92})\right],$$

$$\hat{N}_{14}=\frac{1}{3\pi}\left\{-\left(\frac{d^2}{dy^2}-9\alpha^2\right)\frac{d}{dy}(\hat{\varepsilon}_5+\hat{\varepsilon}_{54})+\frac{d}{dy}(\hat{\varepsilon}_{18}+\bar{\varepsilon}_{64})+3\pi(\bar{\varepsilon}_{44}+\bar{\varepsilon}_{92})\right\},$$

$$\bar{M}_{14}=\left(\frac{d^2}{dy^2}-9\alpha^2\right)^2(\bar{\varepsilon}_5+\bar{\varepsilon}_{54}),\qquad \bar{N}_{14}=-\frac{1}{3\pi}\left(\frac{d^2}{dy^2}-9\alpha^2\right)\frac{d}{dy}(\bar{\varepsilon}_5+\bar{\varepsilon}_{54}),$$

$$\tilde{M}_{14}=-\left(\frac{d^2}{dy^2}-9k_0^2\right)(\tilde{\varepsilon}_{18}+\hat{\varepsilon}_{64})+3\pi\left[3k_0(\hat{\varepsilon}_{32}+\hat{\varepsilon}_{78})-\frac{d}{dy}(\hat{\varepsilon}_{44}+\hat{\varepsilon}_{92})\right],$$

$$\tilde{N}_{14}=\frac{1}{3\pi}\left\{\frac{d}{dy}[\tilde{\varepsilon}_{18}+\hat{\varepsilon}_{64}]+3\pi[\hat{\varepsilon}_{44}+\hat{\varepsilon}_{92}]\right\}. \tag{6.3.44}$$

Finally, from (6.2.16,54)

$$\Theta_{15}=\hat{\Theta}_{15}+P\bar{\Theta}_{15}+\frac{1}{P}\tilde{\Theta}_{15}\qquad\text{and}\qquad V_{15}=\hat{V}_{15}+P\bar{V}_{15}+\frac{1}{P}\tilde{V}_{15} \tag{6.3.45}$$

where

$$\left[\frac{d}{dy}-\underline{E}(k_0,3\pi)\right]\underline{f}(\hat{\Theta}_{15},\hat{V}_{15})=\underline{\mathcal{Q}}(\hat{M}_{15},\hat{N}_{15}),\qquad \underline{m}\,\underline{f}(\hat{\Theta}_{15},\hat{V}_{15})=0\quad y=\pm a, \tag{6.3.46}$$

$$\left[\frac{d}{dy}-\underline{E}(k_0,3\pi)\right]\underline{f}(\bar{\Theta}_{15},\bar{V}_{15})=\underline{\mathcal{Q}}(\bar{M}_{15},\bar{N}_{15}),\qquad \underline{m}\,\underline{f}(\bar{\Theta}_{15},\bar{V}_{15})=0\quad y=\pm a, \tag{6.3.47}$$

$$\left[\frac{d}{dy}-\underline{E}(k_0,3\pi)\right]\underline{f}(\tilde{\Theta}_{15},\tilde{V}_{15})=\underline{\mathcal{Q}}(\tilde{M}_{15},\tilde{N}_{15}),\qquad \underline{m}\,\underline{f}(\tilde{\Theta}_{15},\tilde{V}_{15})=0\quad y=\pm a \tag{6.3.48}$$

and

$$\hat{M}_{15}=\left(\frac{d^2}{dy^2}-9\pi^2-k_0^2\right)^2[\hat{\varepsilon}_6+\hat{\varepsilon}_{55}+2(\hat{\varepsilon}_9+\hat{\varepsilon}_{58})]-\left(\frac{d^2}{dy^2}-k_0^2\right)[\hat{\varepsilon}_{19}+\bar{\varepsilon}_{65}+2(\hat{\varepsilon}_{21}+\bar{\varepsilon}_{68})]+3\pi\Big[k_0(\bar{\varepsilon}_{33}+\bar{\varepsilon}_{79}+2\bar{\varepsilon}_{82})$$
$$-\frac{d}{dy}(\bar{\varepsilon}_{45}+\bar{\varepsilon}_{93}+2\bar{\varepsilon}_{96}+2\bar{\varepsilon}_{48})\Big],$$

$$\hat{N}_{15}=\frac{1}{3\pi}\left\{-\left(\frac{d^2}{dy^2}-9\pi^2-k_0^2\right)\frac{d}{dy}[\hat{\varepsilon}_6+\hat{\varepsilon}_{55}+2(\hat{\varepsilon}_9+\hat{\varepsilon}_{58})]+\frac{d}{dy}[\hat{\varepsilon}_{19}+\bar{\varepsilon}_{65}+2(\hat{\varepsilon}_{22}+\bar{\varepsilon}_{68})]+3\pi[\bar{\varepsilon}_{45}+\bar{\varepsilon}_{93}+2(\bar{\varepsilon}_{48}+\bar{\varepsilon}_{96})]\right\},$$

$$\bar{M}_{15}=\left(\frac{d^2}{dy^2}-9\pi^2-k_0^2\right)^2[\bar{\varepsilon}_6+\bar{\varepsilon}_{55}+2(\bar{\varepsilon}_9+\bar{\varepsilon}_{58})],\qquad \bar{N}_{15}=-\frac{1}{3\pi}\left(\frac{d^2}{dy^2}-9\pi^2-k_0^2\right)\frac{d}{dy}[\bar{\varepsilon}_6+\bar{\varepsilon}_{55}+2(\bar{\varepsilon}_9+\bar{\varepsilon}_{58})],$$

$$\tilde{M}_{15}=-\left(\frac{d^2}{dy^2}-k_0^2\right)[\tilde{\varepsilon}_{19}+\hat{\varepsilon}_{65}+2(\tilde{\varepsilon}_{22}+\hat{\varepsilon}_{68})]+3\pi\left[k_0(\hat{\varepsilon}_{33}+\hat{\varepsilon}_{79}+2\hat{\varepsilon}_{82})-\frac{d}{dy}(\hat{\varepsilon}_{45}+\hat{\varepsilon}_{93}+2(\hat{\varepsilon}_{48}+\hat{\varepsilon}_{96}))\right],$$

$$\tilde{N}_{15}=\frac{1}{3\pi}\left\{\frac{d}{dy}[\tilde{\varepsilon}_{19}+\hat{\varepsilon}_{65}+2(\tilde{\varepsilon}_{22}+\hat{\varepsilon}_{68})]+3\pi[\hat{\varepsilon}_{45}+\hat{\varepsilon}_{93}+2(\hat{\varepsilon}_{48}+\hat{\varepsilon}_{96})]\right\}. \tag{6.3.49}$$

In all, the solution of fourteen eighth order systems need to be determined. In each case a fourth order Runge-Kutta process is used to compute the solution from y=-a through to y=+a, following the procedure outlined in section 4.3. The general solution, say in the case of (6.3.41) is therefore given by

$$\underline{f}(\hat{\Theta}_{14},\hat{V}_{14}) = \hat{q}_1\underline{f}_1(\hat{\Theta}_{14},\hat{V}_{14}) + \hat{q}_2\underline{f}_2(\hat{\Theta}_{14},\hat{V}_{14}) + \hat{q}_3\underline{f}_3(\hat{\Theta}_{14},\hat{V}_{14}) + \hat{q}_4\underline{f}_4(\hat{\Theta}_{14},\hat{V}_{14}) + \underline{f}_p(\hat{\Theta}_{14},\hat{V}_{14}) \quad (6.3.50)$$

where the arbitrary constants $\hat{q}_j$ j=1..4 are determined from the boundary conditions at y=+a. In contrast to the solutions for Θ_i, V_i i=5,7,8 the solutions for Θ_i, V_i i=9..15 are unique since $\underline{E} \neq \underline{E}(k_o,\pi)$, thus the arbitrary constants $\hat{q}_j$ j=1..4 can be determined uniquely (see section 4.3).

It should be noted that a comparison of the matrix systems (6.2.33,34,37,38) with the systems (4.3.75a,75b,76a,76b) respectively, shows that

$$\Theta_{12} = 2\Theta_2 , \quad V_{12} = 2V_2 \qquad \text{and} \qquad \Theta_{13} = 2\Theta_3 , \quad V_{13} = 2V_3 . \quad (6.3.51)$$

(c) $V_{16,17}$

From (6.2.74,75) the y-dependent functions V_{16} and V_{17} in the horizontal velocity component v_3 satisfy the matrix systems

$$\left[\frac{d}{dy} - \tilde{\underline{E}}(2k_o)\right]\tilde{\underline{f}}(V_{16}) = \tilde{\underline{\Phi}}(N_{16}) , \quad \tilde{\underline{m}}\tilde{\underline{f}}(V_{16}) = 0 \quad y = \pm a \quad (6.3.52)$$

where

$$N_{16} = -2k_o\left[2k_o(\hat{\varepsilon}_{38} + \hat{\varepsilon}_{86}) - \frac{d}{dy}(\hat{\varepsilon}_{26} + \hat{\varepsilon}_{72})\right] \quad (6.3.53)$$

and

$$\left[\frac{d}{dy} - \tilde{\underline{E}}(2k_o)\right]\tilde{\underline{f}}(V_{17}) = \tilde{\underline{\Phi}}(N_{17}) , \quad \tilde{\underline{m}}\tilde{\underline{f}}(V_{17}) = 0 \quad y = \pm a \quad (6.3.54)$$

where

$$N_{17} = -2\frac{d^3}{dy^3}U_4 + 4k_o^2(6k_oV_4 - \bar{\varepsilon}_{42} - \bar{\varepsilon}_{90}) - 2k_o(\hat{\varepsilon}_{105} + \bar{\varepsilon}_{76} - 2\hat{\varepsilon}_{86}) - 2k_o\frac{d}{dy}\left(8k_oU_4 + \bar{\varepsilon}_{30} - \frac{\hat{\varepsilon}_{103}}{2k_o^2}\right) \quad (6.3.55)$$

respectively. The two fourth order systems (6.3.52,54) are solved using a fourth order Runge-Kutta process. In contrast to the eighth order systems considered above, here only two complementary solutions need to be computed. The general solution for (6.3.52) say, is therefore given by

$$\underline{\tilde{f}}(V_{16}) = \hat{q}_1 \underline{\tilde{f}}_1(V_{16}) + \hat{q}_2 \underline{\tilde{f}}_2(V_{16}) + \underline{\tilde{f}}_p(V_{16}) \tag{6.3.56}$$

where the arbitrary constants $\hat{q}_j$ j=1,2 may be determined uniquely from the boundary conditions at y=+a. A comparison of the matrix system (6.3.52) with the system (4.4.83) shows that

$$V_{16} = 2V_4 . \tag{6.3.57}$$

(d) The other dependent variables

Having numerically computed Θ_i i=1,5,7,8..15 in the temperature θ_3 and V_i i=1,5,7,8..15 in the horizontal velocity component v_3, the y-dependent functions W_i i=1,5,7,8..15 that appear in the vertical velocity component w_3, may be obtained from the relationships

$$W_1 = \hat{W}_1 = -\left[\, 2k_c \Theta + \left(\frac{d^2}{dy^2} - \alpha^2\right)\Theta_1 \right], \tag{6.3.58}$$

$$W_5 = \hat{W}_5 = -\left(\frac{d^2}{dy^2} - \alpha^2\right)\bar{\Theta}_5 - \frac{C_1}{C_2}(2k_c\Theta_1 - \Theta), \tag{6.3.59}$$

$$\begin{aligned} W_7 &= \bar{W}_7 + \frac{1}{P}\hat{W}_7 + \frac{1}{P^2}\tilde{W}_7 \\ &= \left[-\left(\frac{d^2}{dy^2}-\alpha^2\right)\bar{\Theta}_7 + \bar{\varepsilon}_8 + \bar{\varepsilon}_{57} + 2(\bar{\varepsilon}_{10}+\bar{\varepsilon}_{59}) - \frac{C_4}{C_2}(2k_0\Theta_1 - \Theta)\right] + \frac{1}{P}\left[-\left(\frac{d^2}{dy^2}-\alpha^2\right)\hat{\Theta}_7 + \hat{\varepsilon}_8 + \hat{\varepsilon}_{57} + 2(\hat{\varepsilon}_{10}+\hat{\varepsilon}_{59}) \right. \\ &\left. - \frac{C_3}{C_2}(2k_0\Theta_1 - \Theta)\right] + \frac{1}{P^2}\left[-\left(\frac{d^2}{dy^2}-\alpha^2\right)\tilde{\Theta}_7 - \frac{C_5}{C_2}(2k_0\Theta_1 - \Theta)\right], \end{aligned} \tag{6.3.60}$$

$$W_8 = \bar{W}_8 + \frac{1}{P}\hat{W}_8 = \left[-\left(\frac{d^2}{dy^2}-\alpha^2\right)\bar{\Theta}_8 + \Theta - \frac{C_7}{C_2}(2k_0\Theta_1 - \Theta)\right] + \frac{1}{P}\left[-\left(\frac{d^2}{dy^2}-\alpha^2\right)\hat{\Theta}_8 - \frac{C_6}{C_2}(2k_0\Theta_1 - \Theta)\right], \tag{6.3.61}$$

$$\begin{aligned} W_9 &= \bar{W}_9 + \frac{1}{P}\hat{W}_9 + \frac{1}{P^2}\tilde{W}_9 = \left[-\left(\frac{d^2}{dy^2} - \pi^2 - 9k_c^2\right)\bar{\Theta}_9 + \bar{\varepsilon}_7 + \bar{\varepsilon}_{56}\right] \\ &+ \frac{1}{P}\left[-\left(\frac{d^2}{dy^2} - \pi^2 - 9k_c^2\right)\hat{\Theta}_9 + \hat{\varepsilon}_7 + \hat{\varepsilon}_{56}\right] + \frac{1}{P^2}\left[-\left(\frac{d^2}{dy^2} - \pi^2 - 9k_c^2\right)\tilde{\Theta}_9\right], \end{aligned} \tag{6.3.62}$$

$$\begin{aligned} W_{10} = i\bar{W}_{10} + \frac{i}{P}\hat{W}_{10} = i\left[-\left(\frac{d^2}{dy^2} - 4\alpha^2\right)\bar{\Theta}_{10} + \hat{\varepsilon}_3 + \bar{\varepsilon}_{52} - 8k_0\bar{\varepsilon}_{48} + \frac{1}{2k_0}\hat{\varepsilon}_{104}\right] + \frac{i}{P}\left[-\left(\frac{d^2}{dy^2} - 4\alpha^2\right)\hat{\Theta}_{10} \right. \\ \left. - 8k_0\hat{\varepsilon}_{48}\right], \end{aligned} \tag{6.3.63}$$

$$W_{11} = i\bar{W}_{11} + \frac{i}{P}\hat{W}_{11} = i\left[-\left(\frac{d^2}{dy^2} - 4\pi^2\right)\bar{\Theta}_{11} + \hat{\mathcal{E}}_4 + \bar{\mathcal{E}}_{53} - \frac{1}{2k_o}\hat{\mathcal{E}}_{104}\right] + \frac{i}{P}\left[-\left(\frac{d^2}{dy^2} - 4\pi^2\right)\hat{\Theta}_{11}\right], \tag{6.3.64}$$

$$W_{12} = \bar{W}_{12} + \frac{1}{P}\hat{W}_{12} = \left[-\left(\frac{d^2}{dy^2} - 4\alpha^2\right)\bar{\Theta}_{12} + \hat{\mathcal{E}}_1 + \hat{\mathcal{E}}_{50}\right] + \frac{1}{P}\left[-\left(\frac{d^2}{dy^2} - 4\alpha^2\right)\hat{\Theta}_{12}\right], \tag{6.3.65}$$

$$W_{13} = \bar{W}_{13} + \frac{1}{P}\hat{W}_{13} = \left[-\left(\frac{d^2}{dy^2} - 4\pi^2\right)\bar{\Theta}_{13} + \hat{\mathcal{E}}_2 + \hat{\mathcal{E}}_{51}\right] + \frac{1}{P}\left[-\left(\frac{d^2}{dy^2} - 4\pi^2\right)\hat{\Theta}_{13}\right], \tag{6.3.66}$$

$$W_{14} = \bar{W}_{14} + \frac{1}{P}\hat{W}_{14} + \frac{1}{P^2}\tilde{W}_{14} = \left[-\left(\frac{d^2}{dy^2} - 9\alpha^2\right)\bar{\Theta}_{14} + \bar{\mathcal{E}}_5 + \bar{\mathcal{E}}_{54}\right]$$
$$+ \frac{1}{P}\left[-\left(\frac{d^2}{dy^2} - 9\alpha^2\right)\hat{\Theta}_{14} + \hat{\mathcal{E}}_5 + \hat{\mathcal{E}}_{54}\right] + \frac{1}{P^2}\left[-\left(\frac{d^2}{dy^2} - 9\alpha^2\right)\tilde{\Theta}_{14}\right] \tag{6.3.67}$$

and

$$W_{15} = \bar{W}_{15} + \frac{1}{P}\hat{W}_{15} + \frac{1}{P^2}\tilde{W}_{15} = \left[-\left(\frac{d^2}{dy^2} - 9\pi^2 - k_o^2\right)\bar{\Theta}_{15} + \bar{\mathcal{E}}_6 + \bar{\mathcal{E}}_{55} + 2(\bar{\mathcal{E}}_9 + \bar{\mathcal{E}}_{58})\right]$$
$$+ \frac{1}{P}\left[-\left(\frac{d^2}{dy^2} - 9\pi^2 - k_o^2\right)\hat{\Theta}_{15} + \hat{\mathcal{E}}_6 + \hat{\mathcal{E}}_{55} + 2(\hat{\mathcal{E}}_9 + \hat{\mathcal{E}}_{58})\right] + \frac{1}{P^2}\left[-\left(\frac{d^2}{dy^2} - 9\pi^2 - k_o^2\right)\tilde{\Theta}_{15}\right] \tag{6.3.68}$$

which are obtained from equations (6.2.19-29). The y-dependent functions U_i $i=1,5,7,8..15$ in the horizontal velocity component u_3 may be obtained from the second order differential equations below. From (6.2.55) U_1 satisfies the second order differential equation

$$\left[\frac{d^2}{dy^2} - \alpha^2\right]U_1 = \left[2U + \frac{1}{k_o^2}\left(\frac{d^2}{dy^2} - \alpha^2\right)U + \frac{k_o}{\pi}\left(2k_oW + \left(\frac{d^2}{dy^2} - \alpha^2\right)W_1 + R_c\Theta_1\right)\right], \quad U_1 = 0 \quad y = \pm a. \tag{6.3.69}$$

From (6.2.56)

$$U_5 = iP\bar{U}_5 \tag{6.3.70}$$

where

$$\left[\frac{d^2}{dy^2} - \alpha^2\right]\bar{U}_5 = -\frac{k_c}{\pi}\left[R_c\bar{\Theta}_5 - \left(\frac{d^2}{dy^2} - \alpha^2\right)\bar{\Theta}_5 + \Theta\right] - \frac{C_1}{C_2}U_R, \quad \bar{U}_5 = 0 \quad y = \pm a \tag{6.3.71}$$

and

$$U_R = -\frac{k_c}{\pi}\left[2k_c\left(\frac{d^2}{dy^2} - \alpha^2\right)\Theta_1 - \left(\frac{d^2}{dy^2} - \alpha^2\right)\Theta - 2k_cW_1 + W\right]$$
$$- \left[2k_cU_1 + \frac{1}{k_c}U + \frac{1}{\pi}\left(R_c\Theta_1 + \left(\frac{d^2}{dy^2} - \alpha^2\right)W_1 + 2k_cW\right)\right]. \tag{6.3.72}$$

From (6.2.57)

$$U_7 = i\left[\hat{U}_7 + P\bar{U}_7 + \frac{1}{P}\tilde{U}_7\right] \tag{6.3.73}$$

where

$$\left[\frac{d^2}{dy^2}-\alpha^2\right]\hat{U}_7 = -\frac{k_0}{\pi}\Big[R_0\hat{\Theta}_7-\left(\frac{d^2}{dy^2}-\alpha^2\right)^2\hat{\Theta}_7+\left(\frac{d^2}{dy^2}-\alpha^2\right)\left[\hat{\varepsilon}_8+\hat{\varepsilon}_{57}+2(\hat{\varepsilon}_{10}+\hat{\varepsilon}_{59})\right]-\left[(\hat{\varepsilon}_{21}+\bar{\varepsilon}_{67})+2(\hat{\varepsilon}_{23}+\bar{\varepsilon}_{69})\right]\Big]$$

$$+\bar{\varepsilon}_{35}+\bar{\varepsilon}_{81}+2\bar{\varepsilon}_{83}-\frac{C_4}{C_2}U_R, \quad \hat{U}_7=0 \quad y=\pm a, \tag{6.3.74}$$

$$\left[\frac{d^2}{dy^2}-\alpha^2\right]\bar{U}_7 = -\frac{k_0}{\pi}\left\{R_0\bar{\Theta}_7-\left(\frac{d^2}{dy^2}-\alpha^2\right)^2\bar{\Theta}_7+\left(\frac{d^2}{dy^2}-\alpha^2\right)\left[\bar{\varepsilon}_8+\bar{\varepsilon}_{57}+2(\hat{\varepsilon}_{10}+\hat{\varepsilon}_{59})\right]\right\}-\frac{C_3}{C_2}U_R, \quad \bar{U}_7=0 \quad y=\pm a \tag{6.3.75}$$

and

$$\left[\frac{d^2}{dy^2}-\alpha^2\right]\tilde{U}_7 = -\frac{k_c}{\pi}\left\{R_c\tilde{\Theta}_7-\left(\frac{d^2}{dy^2}-\alpha^2\right)^2\tilde{\Theta}_7-\left[\tilde{\varepsilon}_{21}+\hat{\varepsilon}_{67}+2(\tilde{\varepsilon}_{23}+\hat{\varepsilon}_{69})\right]\right\}-\frac{C_5}{C_2}U_R, \quad \tilde{U}_7=0 \quad y=\pm a. \tag{6.3.76}$$

From (6.2.58)

$$U_8 = i(\hat{U}_8+P\bar{U}_8) \tag{6.3.77}$$

where

$$\left[\frac{d^2}{dy^2}-\alpha^2\right]\hat{U}_8 = -\frac{k_c}{\pi}\left[R_c\hat{\Theta}_8-\left(\frac{d^2}{dy^2}-\alpha^2\right)^2\hat{\Theta}_8-W\right]+\frac{1}{k_c}U-\frac{C_7}{C_2}U_R, \quad \hat{U}_8=0 \quad y=\pm a \tag{6.3.78}$$

and

$$\left[\frac{d^2}{dy^2}-\alpha^2\right]\bar{U}_8 = -\frac{k_c}{\pi}\left[R_c\bar{\Theta}_8-\left(\frac{d^2}{dy^2}-\alpha^2\right)^2\bar{\Theta}_8-W\right]-\frac{C_6}{C_2}U_R, \quad \bar{U}_8=0 \quad y=\pm a. \tag{6.3.79}$$

From (6.2.59)

$$U_9 = i\left(\hat{U}_9+P\bar{U}_9+\frac{1}{P}\tilde{U}_9\right) \tag{6.3.80}$$

where

$$\left[\frac{d^2}{dy^2}-\pi^2-9k_c^2\right]\hat{U}_9 = \frac{3k_c}{\pi}\Big\{-R_c\hat{\Theta}_9+\left(\frac{d^2}{dy^2}-\pi^2-9k_c^2\right)^2\hat{\Theta}_9-\left(\frac{d^2}{dy^2}-\pi^2-9k_c^2\right)\left[\hat{\varepsilon}_7+\hat{\varepsilon}_{56}\right]$$

$$+\hat{\varepsilon}_{20}+\bar{\varepsilon}_{66}+\frac{\pi}{3k_c}(\bar{\varepsilon}_{34}+\bar{\varepsilon}_{80})\Big\}, \quad \hat{U}_9=0 \quad y=\pm a, \tag{6.3.81}$$

$$\left[\frac{d^2}{dy^2}-\pi^2-9k_c^2\right]\bar{U}_9 = \frac{3k_c}{\pi}\left\{-R_c\bar{\Theta}_9+\left(\frac{d^2}{dy^2}-\pi^2-9k_c^2\right)^2\bar{\Theta}_9-\left(\frac{d^2}{dy^2}-\pi^2-9k_c^2\right)(\bar{\varepsilon}_7+\bar{\varepsilon}_{56})\right\}, \quad \bar{U}_9=0 \quad y=\pm a \tag{6.3.82}$$

and

$$\left[\frac{d^2}{dy^2}-\pi^2-9k_c^2\right]\tilde{U}_9 = \frac{3k_0}{\pi}\left\{-R_c\tilde{\Theta}_9+\left(\frac{d^2}{dy^2}-\pi^2-9k_0^2\right)^2\tilde{\Theta}_9+\tilde{\varepsilon}_{20}+\hat{\varepsilon}_{66}\right\}+\hat{\varepsilon}_{34}+\hat{\varepsilon}_{80}, \quad \tilde{U}_9=0 \quad y=\pm a. \tag{6.3.83}$$

From (6.2.60)

$$U_{10} = \hat{U}_{10}+P\bar{U}_{10} \tag{6.3.84}$$

where

$$\left[\frac{d^2}{dy^2}-4\alpha^2\right]\hat{U}_{10} = \frac{k_0}{\pi}\left[R_c\hat{\Theta}_{10}-\left(\frac{d^2}{dy^2}-4\alpha^2\right)^2\hat{\Theta}_{10}-8k_0\left(\frac{d^2}{dy^2}-4\alpha^2\right)\hat{\varepsilon}_{98}+8k_c\tilde{\varepsilon}_{99}-\hat{\varepsilon}_{16}-\bar{\varepsilon}_{62}-\frac{1}{2k_0}\hat{\varepsilon}_{102}\right]$$

$$+\frac{1}{\pi}\left[\hat{\varepsilon}_{101}-\left(\frac{d^2}{dy^2}-4\alpha^2\right)\tilde{\varepsilon}_{99}-R_0\hat{\varepsilon}_{98}+\pi\left[8k_c\hat{\varepsilon}_{100}+\bar{\varepsilon}_{28}+\bar{\varepsilon}_{74}-\frac{1}{2k_c^2}\hat{\varepsilon}_{103}\right\}, \quad \hat{U}_{10}=0 \quad y=\pm a \tag{6.3.85}$$

and

$$\left[\frac{d^2}{dy^2}-4\alpha^2\right]\bar{U}_{10}=\frac{k_0}{\pi}\left\{R_0\bar{\Theta}_{10}-\left(\frac{d^2}{dy^2}-4\alpha^2\right)^2\bar{\Theta}_{10}+\left(\frac{d^2}{dy^2}-4\alpha^2\right)\left[-8k_0\bar{\mathcal{E}}_{98}+\hat{\mathcal{E}}_3+\bar{\mathcal{E}}_{52}+\frac{1}{2k_0}\hat{\mathcal{E}}_{104}\right]+8k_0\hat{\mathcal{E}}_{99}\right\}$$

$$+\left\{-\frac{1}{\pi}\left(\frac{d^2}{dy^2}-4\alpha^2\right)\hat{\mathcal{E}}_{99}-\frac{R_0}{\pi}\bar{\mathcal{E}}_{98}+8k_0\bar{\mathcal{E}}_{100}\right\},\quad \bar{U}_{10}=0\quad y=\pm a. \tag{6.3.86}$$

From (6.2.61)

$$U_{11}=\hat{U}_{11}+P\bar{U}_{11} \tag{6.3.87}$$

where

$$\left[\frac{d^2}{dy^2}-4\pi^2\right]\hat{U}_{11}=-\frac{1}{\pi}\left[-\hat{\mathcal{E}}_{61}+\left(\frac{d^2}{dy^2}-4\pi^2\right)\hat{W}_3+R_0\hat{\Theta}_3\right]+\bar{\mathcal{E}}_{29}+\bar{\mathcal{E}}_{75}+\frac{\hat{\mathcal{E}}_{103}}{2k_c^2},\quad \hat{U}_{11}=0\quad y=\pm a \tag{6.3.88}$$

and

$$\left[\frac{d^2}{dy^2}-4\pi^2\right]\bar{U}_{11}=-\frac{1}{\pi}\left[\left(\frac{d^2}{dy^2}-4\pi^2\right)\hat{W}_3+R_0\bar{\Theta}_3\right],\quad \bar{U}_{11}=0\quad y=\pm a. \tag{6.3.89}$$

From (6.2.62)

$$U_{12}=i(\hat{U}_{12}+P\bar{U}_{12}) \tag{6.3.90}$$

where

$$\left[\frac{d^2}{dy^2}-4\alpha^2\right]\hat{U}_{12}=-\frac{k_0}{\pi}\left[R_0\hat{\Theta}_{12}-\left(\frac{d^2}{dy^2}-4\alpha^2\right)^2\hat{\Theta}_{12}-\hat{\mathcal{E}}_{14}-\hat{\mathcal{E}}_{60}\right]+\hat{\mathcal{E}}_{24}+\hat{\mathcal{E}}_{70},\quad \hat{U}_{12}=0\quad y=\pm a \tag{6.3.91}$$

and

$$\left[\frac{d^2}{dy^2}-4\alpha^2\right]\bar{U}_{12}=-\frac{k_0}{\pi}\left[R_0\bar{\Theta}_{12}-\left(\frac{d^2}{dy^2}-4\alpha^2\right)^2\bar{\Theta}_{12}+\left(\frac{d^2}{dy^2}-4\alpha^2\right)(\hat{\mathcal{E}}_1+\hat{\mathcal{E}}_{50})\right],\quad \bar{U}_{12}=0\quad y=\pm a. \tag{6.3.92}$$

From (6.2.63)

$$U_{13}=0. \tag{6.3.93}$$

From (6.2.64)

$$U_{14}=i\left(\hat{U}_{14}+P\bar{U}_{14}+\frac{1}{P}\tilde{U}_{14}\right) \tag{6.3.94}$$

where

$$\left[\frac{d^2}{dy^2}-9\alpha^2\right]\hat{U}_{14}=-\frac{k_0}{\pi}\left\{R_0\hat{\Theta}_{14}-\left(\frac{d^2}{dy^2}-9\alpha^2\right)^2\hat{\Theta}_{14}+\left(\frac{d^2}{dy^2}-9\alpha^2\right)(\hat{\mathcal{E}}_5+\hat{\mathcal{E}}_{54})-\hat{\mathcal{E}}_{18}-\hat{\mathcal{E}}_{64}\right\}+\bar{\mathcal{E}}_{32}+\bar{\mathcal{E}}_{78},\quad \hat{U}_{14}=0\quad y=\pm a, \tag{6.3.95}$$

$$\left[\frac{d^2}{dy^2}-9\alpha^2\right]\bar{U}_{14}=-\frac{k_0}{\pi}\left\{R_0\Theta_{14}-\left(\frac{d^2}{dy^2}-9\alpha^2\right)^2\bar{\Theta}_{14}+\left(\frac{d^2}{dy^2}-9\alpha^2\right)(\bar{\mathcal{E}}_5+\bar{\mathcal{E}}_{54})\right\},\quad \bar{U}_{14}=0\quad y=\pm a \tag{6.3.96}$$

and

$$\left[\frac{d^2}{dy^2}-9\alpha^2\right]\tilde{U}_{14}=-\frac{k_0}{\pi}\left[R_0\tilde{\Theta}_{14}-\left(\frac{d^2}{dy^2}-9\alpha^2\right)^2\tilde{\Theta}_{14}-\tilde{\mathcal{E}}_{18}-\hat{\mathcal{E}}_{64}\right]+\hat{\mathcal{E}}_{32}+\hat{\mathcal{E}}_{78},\quad \tilde{U}_{14}=0\quad y=\pm a. \tag{6.3.97}$$

Finally, from (6.2.65)

$$U_{15}=i\left(\hat{U}_{15}+P\bar{U}_{15}+\frac{1}{P}\tilde{U}_{15}\right) \tag{6.3.98}$$

where

$$\left[\frac{d^2}{dy^2}-9\pi^2-k_o^2\right]\hat{U}_{15}=-\frac{k_o}{\pi}\Big\{R_o\hat{\Theta}_{15}-\left(\frac{d^2}{dy^2}-9\pi^2-k_o^2\right)^2\hat{\Theta}_{15}+\left(\frac{d^2}{dy^2}-9\pi^2-k_o^2\right)\left(\hat{\varepsilon}_6+\hat{\varepsilon}_{55}+2(\hat{\varepsilon}_9+\hat{\varepsilon}_{58})\right)$$

$$-\left[\hat{\varepsilon}_{19}+\bar{\varepsilon}_{65}+2(\hat{\varepsilon}_{22}+\bar{\varepsilon}_{68})\right]\Big\}+\bar{\varepsilon}_{33}+\bar{\varepsilon}_{79}+2\bar{\varepsilon}_{82},\quad \hat{U}_{15}=0\quad y=\pm a, \tag{6.3.99}$$

$$\left[\frac{d^2}{dy^2}-9\pi^2-k_o^2\right]\bar{U}_{15}=-\frac{k_o}{3\pi}\Big\{R_o\bar{\Theta}_{15}-\left(\frac{d^2}{dy^2}-9\pi^2-k_o^2\right)^2\bar{\Theta}_{15}+\left(\frac{d^2}{dy^2}-9\pi^2-k_o^2\right)\left[\bar{\varepsilon}_6+\bar{\varepsilon}_{55}+2(\bar{\varepsilon}_9+\bar{\varepsilon}_{58})\right]\Big\}, \tag{6.3.100}$$

and

$$\bar{U}_{15}=0\quad y=\pm a$$

$$\left[\frac{d^2}{dy^2}-9\pi^2-k_o^2\right]\tilde{U}_{15}=\frac{-k_o}{3\pi}\Big\{R_o\tilde{\Theta}_{15}-\left(\frac{d^2}{dy^2}-9\pi^2-k_o^2\right)^2\tilde{\Theta}_{15}-\left[\tilde{\varepsilon}_{19}+\hat{\varepsilon}_{65}+2(\tilde{\varepsilon}_{22}+\hat{\varepsilon}_{68})\right]\Big\} \tag{6.3.101}$$

$$+\hat{\varepsilon}_{33}+\hat{\varepsilon}_{79}+2\hat{\varepsilon}_{82},\quad \tilde{U}_{15}=0\quad y=\pm a.$$

Alternatively, the y-dependent functions U_i i=1,5,7,8,9,10,12,14 and 15 may be obtained from the relationships

$$U_1=\frac{1}{k_o}\left[-\frac{1}{k_o}U-\frac{d}{dy}V_1-\pi W_1\right], \tag{6.3.102}$$

$$U_5=iP\bar{U}_5=\frac{iP}{k_o}\left[\frac{d}{dy}\bar{V}_5+\pi W_5-\frac{C_1}{C_2}U_1\right], \tag{6.3.103}$$

$$U_7=i(\hat{U}_7+P\bar{U}_7+\frac{1}{P}\tilde{U}_7)=\frac{i}{k_o}\left[\frac{d}{dy}\hat{V}_7+\pi\hat{W}_7-\frac{C_4}{C_2}U_1\right]+\frac{iP}{k_o}\left[\frac{d}{dy}\bar{V}_7+\pi\bar{W}_7-\frac{C_3}{C_2}U_1\right] \tag{6.3.104}$$

$$+\frac{i}{k_oP}\left[\frac{d}{dy}\tilde{V}_7+\pi\tilde{W}_7-\frac{C_5}{C_2}U_1\right],$$

$$U_8=i(\hat{U}_8+P\bar{U}_8)=\frac{i}{k_o}\left[\frac{d}{dy}\hat{V}_8+\pi\hat{W}_8-\frac{C_7}{C_2}U_1\right]+\frac{i}{k_oP}\left[\frac{d}{dy}\bar{V}_8+\pi\bar{W}_8-\frac{C_6}{C_2}U_1\right], \tag{6.3.105}$$

$$U_9=i(\hat{U}_9+P\bar{U}_9+\frac{1}{P}\tilde{U}_9)=\frac{i}{3k_o}\left[\frac{d}{dy}\hat{V}_9+\pi\hat{W}_9\right]+\frac{iP}{3k_o}\left[\frac{d}{dy}\bar{V}_9+\pi\bar{W}_9\right]+\frac{i}{3k_oP}\left[\frac{d}{dy}\tilde{V}_9+\pi\tilde{W}_9\right], \tag{6.3.106}$$

$$U_{10}=\hat{U}_{10}+P\bar{U}_{10}=\frac{-1}{2k_o}\left[2\hat{\varepsilon}_{100}+\frac{d}{dy}\hat{V}_{10}+2\pi\hat{W}_{10}\right]-\frac{P}{2k_o}\left[2\bar{\varepsilon}_{100}+\frac{d}{dy}\bar{V}_{10}+2\pi\hat{W}_{10}\right], \tag{6.3.107}$$

$$U_{12}=i(\hat{U}_{12}+P\bar{U}_{12})=\frac{i}{2k_o}\left[\frac{d}{dy}\hat{V}_{12}+2\pi\hat{W}_{12}\right]+\frac{iP}{2k_o}\left[\frac{d}{dy}\bar{V}_{12}+2\pi\bar{W}_{12}\right], \tag{6.3.108}$$

$$U_{14}=i(\hat{U}_{14}+P\bar{U}_{14}+\frac{1}{P}\tilde{U}_{14})=\frac{i}{3k_o}\left[\frac{d}{dy}\hat{V}_{14}+3\pi\hat{W}_{14}\right]+\frac{iP}{3k_o}\left[\frac{d}{dy}\bar{V}_{14}+3\pi\bar{W}_{14}\right]+\frac{i}{3k_oP}\left[\frac{d}{dy}\tilde{V}_{14}+3\pi\tilde{W}_{14}\right], \tag{6.3.109}$$

and

$$U_{15}=i(\hat{U}_{15}+P\bar{U}_{15}+\frac{1}{P}\tilde{U}_{15})=\frac{i}{k_o}\left[\frac{d}{dy}\hat{V}_{15}+3\pi\hat{W}_{15}\right]+\frac{iP}{k_o}\left[\frac{d}{dy}\bar{V}_{15}+3\pi\bar{W}_{15}\right] \tag{6.3.110}$$

$$+\frac{i}{k_oP}\left[\frac{d}{dy}\tilde{V}_{15}+3\pi\tilde{W}_{15}\right]$$

which are obtained from the continuity equation (4.4.1). In practice both methods were used and good agreement of the results was obtained. The y-dependent functions U_{16} and U_{17} that appear in u_3 may be obtained from the relationships (6.2.77,78), while U_{18} is determined by solving the system (6.2.87-90). Having obtained all three velocity components u_3, v_3 and w_3 a consistency check for the z-dependent terms is provided by the continuity equation (4.4.1), which is found to be satisfied. The y-dependent functions P_i i=1,5,7,8,15 and G_i i=1..4 in the pressure p_3 may be obtained from the equations (6.2.31-41) and (6.2.79-82).

The solutions for all the dependent variables at order ϵ^2 have thus been obtained.

Chapter 7 The infinite channel: nonlinear theory for rigid sidewalls Part III

7.0 Introduction

The results of chapters 4 and 6 enable the higher order amplitude equation for the infinite channel with stress-free upper and lower boundaries and rigid sidewalls to be determined in this chapter. It arises at order ε^3 due to an inconsistency in the 'weakly' nonlinear expansion and enables the question of wavenumber selection to be investigated, which is considered in chapter 8.

7.1 Amplitude equation for $\overline{B}$

At order ε^3, the Boussinesq equations (3.1.16)-(3.1.20) become

$$\frac{\partial u_4}{\partial x}+\frac{\partial v_4}{\partial y}+\frac{\partial w_4}{\partial z}=-\frac{\partial u_3}{\partial \overline{X}}\,, \tag{7.1.1}$$

$$L_1 u_4+\frac{\partial}{\partial x}p_4=-\frac{\partial}{\partial \overline{X}}p_3-\Gamma_1 u_3-\Gamma_3 u_2-\Gamma_5 u_1\,, \tag{7.1.2}$$

$$L_1 v_4+\frac{\partial}{\partial y}p_4=-\Gamma_1 v_3-\Gamma_3 v_2-\Gamma_5 v_1\,, \tag{7.1.3}$$

$$L_1 w_4+\frac{\partial}{\partial z}p_4-PR_0\theta_4=-\Gamma_1 w_3-\Gamma_3 w_2-\Gamma_5 w_1+P\theta_2 \tag{7.1.4}$$

and

$$L_2\theta_4-w_4=-\Gamma_2\theta_3-\Gamma_4\theta_2-\Gamma_5\theta_1 \tag{7.1.5}$$

where

$$\Gamma_1=\left[-2P\frac{\partial^2}{\partial x\partial \overline{X}}+\underline{u}_1\cdot\underline{\nabla}\right],\qquad \Gamma_2=\left[-2\frac{\partial^2}{\partial x\partial \overline{X}}+\underline{u}_1\cdot\underline{\nabla}\right],$$

$$\Gamma_3=\left[\frac{\partial}{\partial \tau}-P\frac{\partial^2}{\partial \overline{X}^2}+\underline{u}_2\cdot\underline{\nabla}+u_1\frac{\partial}{\partial \overline{X}}\right],\qquad \Gamma_4=\left[\frac{\partial}{\partial \tau}-\frac{\partial^2}{\partial \overline{X}^2}+\underline{u}_2\cdot\underline{\nabla}+u_1\frac{\partial}{\partial \overline{X}}\right],$$

$$\Gamma_5=\left[\underline{u}_3\cdot\underline{\nabla}+u_2\frac{\partial}{\partial \overline{X}}\right] \tag{7.1.6}$$

and the operators L_1 and L_2 are those defined in section 2.1. The boundary conditions from (4.1.6) are

$$\theta_4=\frac{\partial^2}{\partial y^2}\theta_4=v_4=\frac{\partial}{\partial y}v_4=0\quad y=\pm a\,,\quad \theta_4=w_4=\frac{\partial v_4}{\partial z}=\frac{\partial u_4}{\partial z}\quad z=0,1\,. \tag{7.1.7a,b}$$

From equations (7.1.1)-(7.1.5) the conditions on the top and bottom boundaries (7.1.7b) are equivalent to

$$\theta_4 = \frac{\partial^2}{\partial z^2}\theta_4 = \frac{\partial^4}{\partial z^4}\theta_4 = \frac{\partial}{\partial z}v_4 \qquad z = 0,1. \tag{7.1.7c}$$

By elimination of variables in equations (7.1.1)-(7.1.5) a single sixth order differential equation in θ_4 is obtained:

$$P[\nabla^6 - R_0\nabla_H^2]\theta_4 = P\nabla^4(\Gamma_5\theta_1 + \Gamma_4\theta_2 + \Gamma_2\theta_3) - \nabla_H^2(\Gamma_5 w_1 + \Gamma_3 w_2 + \Gamma_1 w_3) + \frac{\partial}{\partial z}\Big[\frac{\partial}{\partial x}(\Gamma_5 u_1 + \Gamma_3 u_2 + \Gamma_1 u_3)$$

$$+ \frac{\partial}{\partial y}(\Gamma_5 v_1 + \Gamma_3 v_2 + \Gamma_1 v_3)\Big] + \frac{\partial^2}{\partial \bar{X}\partial z}\Big(P\nabla^2 u_3 + \frac{\partial}{\partial x}p_3\Big) + P\nabla_H^2\theta_2 \,. \tag{7.1.8}$$

It should be noted that the $\frac{\partial}{\partial t}$ terms are not relevant here and have been omitted. Since not all the conditions on the sidewalls are in terms of θ_4, the following equation relating $\frac{\partial}{\partial z}v_4$ and θ_4 is also required:

$$P\Big[-\nabla^2\frac{\partial}{\partial z}v_4 - (\nabla^4 - R_c)\frac{\partial}{\partial y}\theta_4\Big] = -P\nabla^2\frac{\partial}{\partial y}(\Gamma_5\theta_1 + \Gamma_4\theta_2 + \Gamma_2\theta_3) + \frac{\partial}{\partial y}[\Gamma_5 w_1 + \Gamma_3 w_2 + \Gamma_1 w_3]$$

$$-\frac{\partial}{\partial z}(\Gamma_5 v_1 + \Gamma_3 v_2 + \Gamma_1 v_3) - P\frac{\partial}{\partial y}\theta_2 \,. \tag{7.1.9}$$

Following the procedure by which the amplitude equation for $\bar{A}$ was obtained in section 4.4, the amplitude equation for $\bar{B}$ is now obtained by expanding the right hand sides of equations (7.1.8,9) and assuming

$$\theta_4 = \hat{\theta}_4(y,z,\bar{X},\bar{\tau})e^{ik_0x} + c.c + \cdots \qquad \text{and} \qquad v_4 = \hat{v}_4(y,z,\bar{X},\bar{\tau})e^{ik_0x} + c.c + \cdots \tag{7.1.10}$$

This gives the equations

$$P\Big[\Big(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - k_c^2\Big)^3 - R_c\Big(\frac{\partial^2}{\partial y^2} - k_c^2\Big)\Big]\hat{\theta}_4 = \hat{\Delta}_1(y,\bar{X},\bar{\tau})\sin\pi z + \hat{\Delta}_3(y,z,\bar{X},\bar{\tau}) \tag{7.1.11}$$

and

$$-P\Big[\frac{\partial}{\partial z}\Big(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - k_c^2\Big)\hat{v}_4 + \Big[\Big(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - k_c^2\Big)^2 - R_c\Big]\frac{\partial}{\partial y}\hat{\theta}_4 \tag{7.1.12}$$

$$= \hat{\Delta}_2(y,\bar{X},\bar{\tau})\sin\pi z + \hat{\Delta}_3(y,z,\bar{X},\bar{\tau})$$

for $\hat{\theta}_4$ and $\hat{v}_4$ where

$$\hat{\Delta}_1 = M_{19}\bar{B} + M_{20}\bar{B}_{\bar{X}\bar{X}} + M_{21}|\bar{A}|^2\bar{B} + M_{22}\bar{A}^2\bar{B}^* + M_{23}\bar{B}_{\bar{\tau}} + M_{24}\bar{A}_{\bar{X}} + M_{25}\bar{A}_{\bar{X}\bar{X}\bar{X}} \tag{7.1.14}$$

$$+ M_{26}|\bar{A}|^2\bar{A}_{\bar{X}} + M_{27}\bar{A}^2\bar{A}^*_{\bar{X}} + M_{28}\bar{A}_{\bar{X}\bar{\tau}} \,,$$

$$\hat{\Delta}_2 = N_{19}\bar{B} + N_{20}\bar{B}_{\bar{X}\bar{X}} + N_{21}|\bar{A}|^2\bar{B} + N_{22}\bar{A}^2\bar{B}^* + N_{23}\bar{B}_{\bar{\tau}} + N_{24}\bar{A}_{\bar{X}} + N_{25}\bar{A}_{\bar{X}\bar{X}\bar{X}}$$
$$+ N_{26}|\bar{A}|^2\bar{A}_{\bar{X}} + N_{27}\bar{A}^2\bar{A}^*_{\bar{X}} + N_{28}\bar{A}_{\bar{X}\bar{\tau}}, \tag{7.1.15}$$

$$\hat{\Delta}_3 = \sum_{n=2}^{4} \psi_n(\bar{X},\bar{\tau}) \sin n\pi z \quad \text{and} \quad \hat{\Delta}_4 = \sum_{n=2}^{4} \hat{\psi}_n(\bar{X},\bar{\tau}) \sin n\pi z. \tag{7.1.16}$$

For the purpose of finding the amplitude equation for $\bar{B}$, the unknown functions $\psi_n(\bar{X},\bar{\tau})$ and $\hat{\psi}_n(\bar{X},\bar{\tau})$ do not need to be determined explicitly (see below). It should be noted that other terms with x dependencies of the form 1, $e^{\pm 2ik_0x}$, $e^{\pm 3ik_0x}$ and $e^{\pm 4ik_0x}$ also exist in θ_4 and v_4. However, these play no part in obtaining the amplitude equation for $\bar{B}$. The functions M_i, N_i i=19..28, which depend on y and the Prandtl number P, are

$$M_{19} = P\left(\frac{d^2}{dy^2} - k_0^2\right)\Theta, \tag{7.1.17}$$

$$M_{20} = P\left(\frac{d^2}{dy^2} - \alpha^2\right)^2 e_1 - \left(\frac{d^2}{dy^2} - k_0^2\right)e_{10} - \pi(e_{19} + e'_{28}) - \pi P\left(\frac{d^2}{dy^2} - \alpha^2\right)U_1 + \pi P k_0 P_1, \tag{7.1.18}$$

$$M_{21} = P\left(\frac{d^2}{dy^2} - \alpha^2\right)^2 e_2 - \left(\frac{d^2}{dy^2} - k_0^2\right)e_{11} - \pi(e_{20} + e'_{29}), \tag{7.1.19}$$

$$M_{22} = P\left(\frac{d^2}{dy^2} - \alpha^2\right)^2 e_3 - \left(\frac{d^2}{dy^2} - k_0^2\right)e_{12} - \pi(e_{21} + e'_{30}), \tag{7.1.20}$$

$$M_{23} = P\left(\frac{d^2}{dy^2} - \alpha^2\right)^2 e_8 - \left(\frac{d^2}{dy^2} - k_0^2\right)e_{17} - \pi(e_{26} + e'_{35}), \tag{7.1.21}$$

$$M_{24} = P\left(\frac{d^2}{dy^2} - \alpha^2\right)^2 e_4 - \left(\frac{d^2}{dy^2} - k_0^2\right)e_{13} - \pi(e_{22} + e'_{31}) - \pi\left(\frac{d^2}{dy^2} - \alpha^2\right)U_5 + iP\left(\frac{d^2}{dy^2} - k_0^2\right)\Theta_1 - \pi i k_0 P_5, \tag{7.1.22}$$

$$M_{25} = P\left(\frac{d^2}{dy^2} - \alpha^2\right)^2 e_5 - \left(\frac{d^2}{dy^2} - k_0^2\right)e_{14} - \pi(e_{23} + e'_{32}), \tag{7.1.23}$$

$$M_{26} = P\left(\frac{d^2}{dy^2} - \alpha^2\right)^2 e_6 - \left(\frac{d^2}{dy^2} - k_0^2\right)e_{15} - \pi(e_{24} + e'_{33}) - 2\pi\left(\frac{d^2}{dy^2} - \alpha^2\right)U_7 - 2\pi i k_0 P_7, \tag{7.1.24}$$

$$M_{27} = P\left(\frac{d^2}{dy^2} - \alpha^2\right)^2 e_7 - \left(\frac{d^2}{dy^2} - k_0^2\right)e_{16} - \pi(e_{25} + e'_{34}) - \pi\left(\frac{d^2}{dy^2} - \alpha^2\right)U_7 - \pi i k_0 P_7, \tag{7.1.25}$$

$$M_{28} = P\left(\frac{d^2}{dy^2} - \alpha^2\right)^2 e_9 - \left(\frac{d^2}{dy^2} - k_0^2\right)e_{18} - \pi(e_{27} + e'_{36}) - \pi\left(\frac{d^2}{dy^2} - \alpha^2\right)U_8 - \pi i k_0 P_8, \tag{7.1.26}$$

$$N_{19} = -P\frac{d}{dy}\Theta, \qquad N_{20} = -P\left(\frac{d^2}{dy^2} - \alpha^2\right)\frac{d}{dy}e_1 + e'_{10} + \pi e_{28}, \tag{7.1.27,28}$$

$$N_{21} = -P\left(\frac{d^2}{dy^2} - \alpha^2\right)\frac{d}{dy}e_2 + e'_{11} + \pi e_{24}, \quad N_{22} = -P\left(\frac{d^2}{dy^2} - \alpha^2\right)\frac{d}{dy}e_3 + e'_{12} + \pi e_{30}, \tag{7.1.29,30}$$

$$N_{23} = -P\left(\frac{d^2}{dy^2} - \alpha^2\right)\frac{d}{dy}e_8 + e'_{17} + \pi e_{35}, \quad N_{24} = -P\left(\frac{d^2}{dy^2} - \alpha^2\right)\frac{d}{dy}e_4 + e'_{13} + \pi e_{31} - iP\Theta'_1, \tag{7.1.31,32}$$

$$N_{25} = -P\left(\frac{d^2}{dy^2}-\alpha^2\right)\frac{d}{dy}e_5 + e_{14}' + \pi e_{32}, \quad N_{26} = -P\left(\frac{d^2}{dy^2}-\alpha^2\right)\frac{d}{dy}e_6 + e_{15}' + \pi e_{33}, \qquad (7.1.33,34)$$

$$N_{27} = -P\left(\frac{d^2}{dy^2}-\alpha^2\right)\frac{d}{dy}e_7 + e_{16}' + \pi e_{34}, \quad N_{28} = -P\left(\frac{d^2}{dy^2}-\alpha^2\right)\frac{d}{dy}e_9 + e_{18}' + \pi e_{36}. \qquad (7.1.35,36)$$

The functions $e_i(y)$ i=1..36, which are either purely real or purely imaginary are

$$e_1 = -\Theta + 2k_o\Theta_1, \qquad e_2 = \phi_{44} + 2\phi_4 + \phi_{27}, \qquad e_3 = \phi_{45} + \phi_3 - \phi_1 + \phi_{28},$$

$$e_4 = \frac{-2ik_o}{P}\Theta_5, \qquad e_5 = -i\Theta_1, \qquad e_6 = \phi_{43} + 2\phi_6 + \phi_{26} - \frac{i}{k_oP}\left(U\Theta_2 - U\Theta_3 + 4k_o^2\Theta_7\right),$$

$$e_7 = \phi_{46} + \phi_2 - \phi_5 + \phi_{29} + \frac{i}{P}\left(-\tfrac{1}{2}\Theta U_2 + \Theta U_4 + \frac{1}{k_o}U\Theta_3 - 2k_o\Theta_7\right), \quad e_8 = \Theta, \quad e_9 = i\left(\Theta_1 - \frac{2k_o}{P}\Theta_8\right),$$

$$e_{10} = P(W + 2k_oW_1), \qquad e_{11} = \phi_{48} + 2\phi_{10} + \phi_{31}, \qquad e_{12} = \phi_{49} + \phi_9 - \phi_7 + \phi_{32},$$

$$e_{13} = -2ik_oPW_5, \qquad e_{14} = -iPW_1, \qquad e_{15} = \phi_{47} + 2\phi_{12} + \phi_{30} - \frac{i}{k_o}\left(-UW_2 + UW_3 + 4k_o^2PW_7\right),$$

$$e_{16} = \phi_{50} + \phi_8 - \phi_{11} + \phi_{33} + \frac{i}{P}\left(-\tfrac{1}{2}WU_2 + WU_4 + \frac{P}{k_o}UW_2 - 2k_oP^2W_7\right), \quad e_{17} = W, \quad e_{18} = i(W_1 - 2k_oPW_8),$$

$$e_{19} = P(U + 2k_o^2U_1), \qquad e_{20} = ik_o(\phi_{52} + 2\phi_{18} + \phi_{36}), \qquad e_{21} = ik_o(\phi_{53} + \phi_{16} - \phi_{14} + \phi_{37}),$$

$$e_{22} = 2k_o^2U_5, \qquad e_{23} = -ik_oU_1, \qquad e_{24} = ik_o\left[\phi_{51} + 2\phi_{19} + \phi_{35} + \frac{1}{P}\left(UU_2 + 2UU_4 + \frac{4k_oP}{i}U_7\right)\right],$$

$$e_{25} = ik_o\left[\phi_{54} + \phi_{15} - \phi_{17} + \phi_{38} + \frac{1}{P}\left(\frac{1}{2k_o}UU_2 + \frac{1}{k_o}UU_4 - 2ik_oPU_7\right)\right], \quad e_{26} = -U, \quad e_{27} = k_o\left(\frac{i}{P}U_1 + 2k_oU_8\right),$$

$$e_{28} = P(-V + 2k_oV_1), \qquad e_{29} = \phi_{56} + 2\phi_{23} + \phi_{40}, \qquad e_{30} = \phi_{57} + \phi_{22} - \phi_{20} + \phi_{41},$$

$$e_{31} = -2ik_oV_5, \qquad e_{32} = -iV_1, \qquad e_{33} = \phi_{55} + 2\phi_{25} + \phi_{39} + \frac{i}{k_oP}\left(-UV_2 + UV_3 - 2UV_4 - 4k_o^2PV_7\right),$$

$$e_{34} = \phi_{58} + \phi_{21} - \phi_{24} + \phi_{42} + \frac{i}{P}\left(\tfrac{1}{2}VU_2 + VU_4 + \frac{1}{k_o}UV_3 - 2k_oPV_7\right), \quad e_{35} = V, \quad e_{36} = i\left(\frac{1}{P}V_1 - 2k_oV_8\right),$$

(7.1.37)

where the functions $\phi_i(y)$ i=1..58 (i≠13,34), (which are either purely real or purely imaginary) are given by (F1) in appendix F.

A comparison of the coupled functions M_i, N_i i=19,20,23 given by (7.1.17,27), (7.1.18,28) and (7.1.21,31), with the coupled functions M_i, N_i i=5,6,8 given by (4.4.12,17), (4.4.14,18) and (4.4.16,20), shows that

$$(M_{19},N_{19})=(M_5,N_5),\quad (M_{20},N_{20})=(M_6,N_6)\quad \text{and } (M_{23},N_{23})=(M_8,N_8). \qquad (7.1.38)$$

Since it is known from chapter 6 that

$$\Theta_{12}=2\Theta_2,\quad W_{12}=2W_2,\quad V_{12}=2V_2,\quad iU_{12}=2U_2,$$

$$\Theta_{13}=2\Theta_3,\quad W_{13}=2W_3,\quad V_{13}=2V_3,\quad U_{13}=0,\quad V_{16}=2V_4 \text{ and } U_{16}=2U_4, \qquad (7.1.39)$$

a comparison of M_{21}, N_{21} given by (7.1.19,29), with M_7,N_7 given by (4.4.15,19) gives

$$(M_{21},N_{21})=2(M_7,N_7). \qquad (7.1.40)$$

Similarly

$$(M_{22},N_{22})=(M_7,N_7)=\tfrac{1}{2}(M_{21},N_{21}) \qquad (7.1.41)$$

where the coupled functions M_{22},N_{22} are given by (7.1.20,30).

7.2 The computer program

The procedure by which the functions e_i i=1..36 in (7.1.37) are coded onto the computer involves four steps which are as follows:

Step 1 Coding the dependent variables

The names of all the dependent variables obtained at order ε^0, ε and ε^2 are stored in a vector called NAME say. The index of the vector is used as the unique integer to identify a particular dependent variable. For instance, in example (7.2.1) below V_2 would be identified by the integer '3'. In addition a two-dimensional vector called FORM say, is used to store firstly the Prandtl dependence of the dependent variables (for example from section 4.3 it is known that $V_2 = \hat{V}_2 + P\bar{V}_2$), secondly whether the dependent variable is purely real of purely imaginary and thirdly the integer associated with the permanent file in which the computed values of the dependent variable are stored.

Index	NAME		FORM 1	2	3	4	5	6	7
1	Θ	1	0	/	/	/	0	44	1
2	U	2	0	/	/	/	0	44	1
3	V_2	3	0	+1	/	/	0	47	2
4	U_2	4	0	+1	/	/	0	47	2
5	Θ_2	5	0	+1	/	/	0	47	2
6	U_4	6	0	/	/	/	0	49	1
↓									

Columns 2–4: Prandtl dependence $0 = P^0$, $+1 = P^1$, $-1 = P^{-1}$, / = not defined; column 5: 0 = real, 1 = imag.; column 6: File code; column 7: No. of terms

(7.2.1)

Step 2 Coding the functions ϕ_i given by (F1) in appendix F

It should be noted that each function ϕ_i consists of one or more terms and each term is a product of two dependent variables. Each function ϕ_i is coded using the following format:

(1) Name of ϕ_i (the integer i is used)

(2) Number of terms

(3) For term 1: the names of the two dependent variables, the Prandtl coefficient, the numerical coefficient, coefficient real or imaginary, order of derivative of each of the two dependent variables

(4) Repeat line (3) for the remaining terms.

In the case of ϕ_1 say, the following code would be generated:

1

3

$U\Theta_2$, -1, -1, 0, 0, 0

ΘU_2, -1, $\frac{k_0}{2}$, 0, 0, 0

ΘU_4, -1, $-k_0$, 0, 0, 0 (7.2.2)

Step 3 Manipulation of the code generated by step 2

The coded form for each ϕ_i is used as input to a subprogram, which together with the information stored by step 1, would in the case of ϕ_1 produce the code

FORM1

i=1..NTERM ↓	1	2	3	4	5	6	7	8
1	-1	0	2	5	0	0	44	47
2	-1	0	1	4	0	0	44	47
3	-1	0	1	6	0	0	44	49

COEFF

-1
$k_o/2$
$-k_o$

Numerical coeff.

NAME1: 1   NTERM: 3

Prandtl coeff. [i, (1)]   Coeff real or imaginary. [i, (2)]

[i, (3,4)] - Integers associated with dependent variables.

[i, (5,6)] - Order of derivative.

[i, (7,8)] - File codes.

(7.2.3)

The information stored in the two-dimensional vector FORM1 is now further processed by a complicated subprogram, which using the information stored in FORM determines the Prandtl dependence of the function ϕ_i and whether ϕ_i is purely real or purely imaginary. The results are stored in a three-dimensional vector PFORM say. In the case of ϕ_1 it is determined that

$$\phi_1 = \hat{\phi}_1 + \frac{1}{P}\tilde{\phi}_1 \qquad (7.2.4)$$

where

$$\hat{\phi}_1 = -U\bar{\Theta}_2 + \frac{k_o}{2}\bar{u}_2\Theta \qquad \text{and} \qquad \tilde{\phi}_1 = -U\hat{\Theta}_2 + \frac{k_o}{2}\hat{u}_2\Theta - \frac{k_o}{2}u_4\Theta . \qquad (7.2.5)$$

The information stored in PFORM can be made readily avaiable for future use by writing it onto a permanent random file, the integer variable NAME1 being used as the unique record number.

Step 4 Coding the functions e_i in (7.1.37)

The vectors NAME and FORM in step 1 are extended to include the functions ϕ_i i=1..58 (i≠13,34) and their corresponding Prandtl forms. Repeating steps 2 and 3 with ϕ_i replaced by e_i enables the Prandtl dependence of the function e_i, and whether e_i is purely real or purely imaginary, to be determined. Two problems arise. Firstly, the functions ϕ_i do not have a related file code and secondly not every term in the functions e_i is a product of two functions. The former problem is overcome by introducing a dummy file code which is also used as an indicator to distinguish a function ϕ_i from a dependent variable. The latter problem is solved by introducing a dummy variable D where

$$D = 1, \quad |y| \leq a \qquad \text{and} \qquad \frac{d^n}{dy^n} D = 0 \quad |y| \leq a, \; n \geq 1. \tag{7.2.6}$$

This enables the functions e_i to be expressed in the form required by step 2. For example e_1 would now be written as $e_1 = -\Theta D + 2k_0 \Theta_1 D$. It is determined that

$$\begin{aligned}
&e_1 = \hat{e}_1, && e_2 = \hat{e}_2 + \tfrac{1}{P}\tilde{e}_2, && e_3 = \hat{e}_3 + \tfrac{1}{P}\tilde{e}_3, && e_4 = i\hat{e}_4, \\
&e_5 = i\hat{e}_5, && e_6 = i(\bar{e}_6 + \tfrac{1}{P}\hat{e}_6 + \tfrac{1}{P^2}\tilde{e}_6), && e_7 = i(\bar{e}_7 + \tfrac{1}{P}\hat{e}_7 + \tfrac{1}{P^2}\tilde{e}_7), && e_8 = \hat{e}_8, \\
&e_9 = i(\hat{e}_9 + \tfrac{1}{P}\tilde{e}_9), && e_{10} = P\bar{e}_{10}, && e_{11} = \hat{e}_{11} + \tfrac{1}{P}\tilde{e}_{11}, && e_{12} = \hat{e}_{12} + \tfrac{1}{P}\tilde{e}_{12}, \\
&e_{13} = iP\bar{e}_{13}, && e_{14} = iP\bar{e}_{14}, && e_{15} = i(\hat{e}_{15} + P\bar{e}_{15} + \tfrac{1}{P}\tilde{e}_{15}), && e_{16} = i(\hat{e}_{16} + P\bar{e}_{16} + \tfrac{1}{P}\tilde{e}_{16}), \\
&e_{17} = \hat{e}_{17}, && e_{18} = i(\hat{e}_{18} + P\bar{e}_{18}), && e_{19} = P\bar{e}_{19}, && e_{20} = \hat{e}_{20} + \tfrac{1}{P}\tilde{e}_{20}, \\
&e_{21} = \hat{e}_{21} + \tfrac{1}{P}\tilde{e}_{21}, && e_{22} = iP\bar{e}_{22}, && e_{23} = iP\bar{e}_{23}, && e_{24} = i(\hat{e}_{24} + P\bar{e}_{24} + \tfrac{1}{P}\tilde{e}_{24}), \\
&e_{25} = i(\hat{e}_{25} + P\bar{e}_{25} + \tfrac{1}{P}\tilde{e}_{25}), && e_{26} = \hat{e}_{26}, && e_{27} = i(\hat{e}_{27} + P\bar{e}_{27}), && e_{28} = P\bar{e}_{28}, \\
&e_{29} = \hat{e}_{29} + \tfrac{1}{P}\tilde{e}_{29}, && e_{30} = \hat{e}_{30} + \tfrac{1}{P}\tilde{e}_{30}, && e_{31} = iP\bar{e}_{31}, && e_{32} = iP\bar{e}_{32}, \\
&e_{33} = i(\hat{e}_{33} + P\bar{e}_{33} + \tfrac{1}{P}\tilde{e}_{34}), && e_{34} = i(\hat{e}_{34} + P\bar{e}_{34} + \tfrac{1}{P}\tilde{e}_{34}), && e_{35} = \hat{e}_{35}, && e_{36} = i(\hat{e}_{36} + P\bar{e}_{36}).
\end{aligned} \tag{7.2.7}$$

The functions $\hat{e}_i$, $\bar{e}_i$ and $\tilde{e}_i$ in (7.2.7) are real and independent of P. Their explicit forms are not given here due to their complexity and the fact that, in any case they are determined directly from (7.1.37) by the computer program outlined above.

By further extending the vectors NAME and FORM in step 1 above, to include the functions e_i i=1..36 and their corresponding Prandtl forms, the Prandtl dependencies of the functions M_i, N_i i=19..28 may be determined by following a slightly modified form of steps 2 and 3. It is also determined whether the functions M_i, N_i i=19..28 are purely real or purely imaginary. As expected from (4.4.12,14-20) and (7.1.38,40,41), it was determined that

$$M_{19}=P\bar{M}_{19}, \quad N_{19}=P\bar{N}_{19}, \quad M_{20}=P\bar{M}_{20}, \quad N_{20}=P\bar{N}_{20},$$

$$M_{21}=\hat{M}_{21}+P\bar{M}_{21}+\frac{1}{P}\tilde{M}_{21}, \quad N_{21}=\hat{N}_{21}+P\bar{N}_{21}+\frac{1}{P}\tilde{N}_{21},$$

$$M_{22}=\hat{M}_{22}+P\bar{M}_{22}+\frac{1}{P}\tilde{M}_{22}, \quad N_{22}=\hat{N}_{22}+P\bar{N}_{22}+\frac{1}{P}\tilde{N}_{22}, \quad M_{23}=\hat{M}_{23}+P\bar{M}_{23}, \quad N_{23}=\hat{N}_{23}+P\bar{N}_{23}, \tag{7.2.8}$$

where $\hat{M}_{21..23}$, $\hat{N}_{21..23}$, $\bar{M}_{19..23}$, $\bar{N}_{19..23}$ and $\tilde{M}_{21,22}$, $\tilde{N}_{21,22}$ are real and independent of P. A comparison with (4.4.12,14-20) provided a check that the computer program was correct. In addition it was determined that

$$M_{24}=iP\bar{M}_{24}, \quad N_{24}=iP\bar{N}_{24}, \quad M_{25}=iP\bar{M}_{25}, \quad N_{25}=iP\bar{N}_{25},$$

$$M_{26}=i(\hat{M}_{26}+P\bar{M}_{26}+\frac{1}{P}\tilde{M}_{26}), \quad N_{26}=i(\hat{N}_{26}+P\bar{N}_{26}+\frac{1}{P}\tilde{N}_{26}), \quad M_{27}=i(\hat{M}_{27}+P\bar{M}_{27}+\frac{1}{P}\tilde{M}_{27}),$$

$$N_{27}=i(\hat{N}_{27}+P\bar{N}_{27}+\frac{1}{P}\tilde{N}_{27}), \quad M_{28}=i(\hat{M}_{28}+P\bar{M}_{28}), \quad N_{28}=i(\hat{N}_{28}+P\bar{N}_{28}), \tag{7.2.9}$$

where $\hat{M}_{26..28}$, $\hat{N}_{26..28}$, $\bar{M}_{24..28}$, $\bar{N}_{24..28}$ and $\tilde{M}_{26,27}$, $\tilde{N}_{26,27}$ are real and independent of P. Their explicit forms are not given here due to their complexity and the fact that they are determined directly from (7.1.17-36) by the computer program.

7.3 Determination of the coefficients of the amplitude equation

Applying the conditions (7.1.7a,7c) to (7.1.10) gives

$$\hat{\theta}_4=\frac{\partial^2}{\partial y^2}\hat{\theta}_4=\hat{v}_4=\frac{\partial}{\partial y}\hat{v}_4=0 \quad y=\pm a, \quad \hat{\theta}_4=\frac{\partial^2}{\partial z^2}\hat{\theta}_4=\frac{\partial^4}{\partial z^4}\hat{\theta}_4=\frac{\partial \hat{v}_4}{\partial z}=0 \quad z=0,1. \tag{7.3.1a,b}$$

The z solvability condition for the system (7.1.11,12) and (7.3.1) is

$$P\int_{z=0}^{1}\left[\left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-k_0^2\right)^3-R_0\left(\frac{\partial^2}{\partial y^2}-k_0^2\right)\right]\hat{\theta}_4\sin\pi z\,dz=\int_{z=0}^{1}[\hat{\Delta}_1\sin\pi z+\hat{\Delta}_3]\sin\pi z\,dz=\tfrac{1}{2}\hat{\Delta}_1 \tag{7.3.2a}$$

and

$$-P\int_{z=}^{1}\left\{\frac{\partial}{\partial z}\left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-k_o^2\right)\hat{v}_4+\left[\left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-k_o^2\right)^2-R_o\right]\frac{\partial}{\partial y}\hat{\theta}_4\right\}\sin\pi z\,dz=\int_{z=0}^{1}[\hat{\Delta}_2\sin\pi z+\hat{\Delta}_4]\sin\pi z\,dz \qquad (7.3.2b)$$

$$= \tfrac{1}{2}\hat{\Delta}_2 .$$

Expanding (7.3.2a,2b) using (7.3.1b) and repeated integration by parts gives

$$P\int_{z=0}^{1}\left[\left(\frac{\partial^2}{\partial y^2}-\alpha^2\right)^3-R_o\left(\frac{\partial^2}{\partial y^2}-k_o^2\right)\right]\hat{\theta}_4\sin\pi z\,dz=\tfrac{1}{2}\hat{\Delta}_1 \qquad (7.3.3a)$$

and

$$-P\int_{z=0}^{1}\left\{-\pi\left(\frac{\partial^2}{\partial y^2}-\alpha^2\right)\hat{v}_4\cos\pi z+\left[\left(\frac{\partial^2}{\partial y^2}-\alpha^2\right)^2-R_o\right]\frac{\partial}{\partial y}\hat{\theta}_4\sin\pi z\right\}dz=\tfrac{1}{2}\hat{\Delta}_2 . \qquad (7.3.3b)$$

By defining

$$\bar{\theta}_4(y,\bar{X},\bar{\tau})=P\int_{z=0}^{1}\hat{\theta}_4\sin\pi z\,dz \quad \text{and} \quad \bar{v}_4(y,\bar{X},\bar{\tau})=P\int_{z=0}^{1}\hat{v}_4\cos\pi z\,dz \qquad (7.3.4)$$

equations (7.3.3a,3b) become

$$\left[\left(\frac{d^2}{dy^2}-\alpha^2\right)^3-R_o\left(\frac{d^2}{dy^2}-k_o^2\right)\right]\bar{\theta}_4=\tfrac{1}{2}\hat{\Delta}_1 \qquad (7.3.5a)$$

and

$$\left(\frac{d^2}{dy^2}-\alpha^2\right)\bar{v}_4-\frac{1}{\pi}\left[\frac{d^5}{dy^5}-2\alpha^2\frac{d^3}{dy^3}+(\alpha^4-R_o)\frac{d}{dy}\right]\bar{\theta}_4=\frac{\hat{\Delta}_2}{2\pi}, \qquad (7.3.5b)$$

which together with the boundary conditions

$$\bar{\theta}_4=\frac{d^2}{dy^2}\bar{\theta}_4=\bar{v}_4=\frac{d}{dy}\bar{v}_4=0 \qquad y=\pm a \qquad (7.3.6)$$

can be expressed in matrix notation as

$$\left[\frac{d}{dy}-\underline{E}(k_o,\pi)\right]\underline{f}(\bar{\theta}_4,\bar{v}_4)=\underline{g}\left(\tfrac{1}{2}\hat{\Delta}_1,\tfrac{1}{2\pi}\hat{\Delta}_2\right), \qquad \underline{m}\,\underline{f}(\bar{\theta}_4,\bar{v}_4)=0 \quad y=\pm a . \qquad (7.3.7)$$

From section 4.3, the condition needed for the existence of a solution of (7.3.7) is

$$\int_{y=-a}^{+a} H(y,\bar{X},\bar{\tau})\,dy=0 \qquad (7.3.8)$$

where

$$H(y,\bar{X},\bar{\tau})=\hat{f}_4\hat{\Delta}_1+\hat{f}_8\hat{\Delta}_2/\pi \qquad (7.3.9)$$

and $\hat{f}_4$ and $\hat{f}_8$ are solutions obtained from the matrix system (4.3.60) given by (4.3.74).

Expanding (7.3.9) shows that the condition (7.3.8) is only satisfied if the amplitude function $\bar{B}$ satisfies the equation

$$b_1\bar{B} + b_2\bar{B}_{\bar{X}\bar{X}} + (b_3 + b_4/P + b_5/P^2)|\bar{A}|^2\bar{B} + (b_6 + b_7/P + b_8/P^2)\bar{A}^2\bar{B}^* + (b_9 + b_{10}/P)\bar{B}_{\bar{\tau}}$$

$$+ i[b_{11}\bar{A}_{\bar{X}} + b_{12}\bar{A}_{\bar{X}\bar{X}\bar{X}} + (b_{13} + b_{14}/P + b_{15}/P^2)|\bar{A}|^2\bar{A}_{\bar{X}} + (b_{16} + b_{17}/P + b_{18}/P^2)\bar{A}^2\bar{A}^*_{\bar{X}} \qquad (7.3.10)$$

$$+ (b_{19} + b_{20}/P)\bar{A}_{\bar{X}\bar{\tau}}] = 0,$$

where the amplitude function $\bar{A}$ satisfies equation (4.4.33). The amplitude coefficients b_i $i=1..20$ are independent of P and are given by

$$b_1 = \hat{H}(\bar{M}_{19}, \bar{N}_{19}), \quad b_2 = \hat{H}(\bar{M}_{20}, \bar{N}_{20}),$$

$$b_3 = \hat{H}(\bar{M}_{21}, \bar{N}_{21}), \quad b_4 = \hat{H}(\hat{M}_{21}, \hat{N}_{21}), \quad b_5 = \hat{H}(\tilde{M}_{21}, \tilde{N}_{21}), \quad b_6 = \hat{H}(\bar{M}_{22}, \bar{N}_{22}),$$

$$b_7 = \hat{H}(\hat{M}_{22}, \hat{N}_{22}), \quad b_8 = \hat{H}(\tilde{M}_{22}, \tilde{N}_{22}), \quad b_9 = \hat{H}(\bar{M}_{23}, \bar{N}_{23}), \quad b_{10} = \hat{H}(\hat{M}_{23}, \hat{N}_{23}),$$

$$b_{11} = \hat{H}(\bar{M}_{24}, \bar{N}_{24}), \quad b_{12} = \hat{H}(\bar{M}_{25}, \bar{N}_{25}),$$

$$b_{13} = \hat{H}(\bar{M}_{26}, \bar{N}_{26}), \quad b_{14} = \hat{H}(\hat{M}_{26}, \hat{N}_{26}), \quad b_{15} = \hat{H}(\tilde{M}_{26}, \tilde{N}_{26}), \quad b_{16} = \hat{H}(\bar{M}_{27}, \bar{N}_{27})$$

$$b_{17} = \hat{H}(\hat{M}_{27}, \hat{N}_{27}), \quad b_{18} = \hat{H}(\tilde{M}_{27}, \tilde{N}_{27}), \quad b_{19} = \hat{H}(\bar{M}_{28}, \bar{N}_{28}), \quad b_{20} = \hat{H}(\hat{M}_{28}, \hat{N}_{28}) \qquad (7.3.11)$$

where

$$\hat{H}(s,t) = \int_{y=-a}^{+a} [\hat{f}_4 s + \tfrac{1}{\pi}\hat{f}_8 t]\, dy. \qquad (7.3.12)$$

7.4 Details of the numerical calculations

For a given aspect ratio a, in order for the dependent variables at order ε to be evaluated by a fourth order Runge-Kutta process (as outlined in section 4.3) using 161 points (which corresponds to a step size of a/80), the dependent variables at order ε^0 need to be evaluated at 321 points. This presents no problems since the variables at order ε^0 and their derivatives may be evaluated from (4.1.3) at any given value of y such that $|y| \leq a$. Function values of all the dependent variables at order ε^0 and ε^1 were evaluated at 161 points and stored in permanent random files on the computer. This enables the dependent variables at order ε^2 to be evaluated by a fourth order Runge-Kutta process (as outlined in section 6.3) using 81 points (step size of a/40), without using any interpolation techniques. It should be noted that the values of the derivatives of the dependent variables at order ε^0 and ε were also evaluated at 161 points using the necessary equations from chapter 4 and stored, so that when evaluating the dependent variables at order ε^2 any derivative of the variables at order ε^0 and ε that may be required is available without further numerical approximation. Function values of all the dependent variables at order ε^2 were evaluated at 81 points and stored in permanent random files. Values of the derivatives of the variables at order ε^2 were also obtained using the necessary equations from chapter 6 and stored, so that the functions e_i i=1..36 given in (7.1.37) could be evaluated at 81 points without any further numerical approximation. Having evaluated all the functions $\hat{e}_i$, $\bar{e}_i$ and $\tilde{e}_i$ in (7.2.7) at 81 points their function values are stored.

The amplitude coefficients b_i i=1..20 can now be determined. For example, the amplitude coefficient b_2 is determined as follows. From equations (7.1.18,28),(7.2.7) and (7.3.11,12)

$$b_2 = \bar{b}_2 + \bar{\bar{b}}_2 \qquad (7.4.1)$$

where

$$\bar{b}_2 = \int_{y=-a}^{+a} \left\{ \hat{f}_4 \left(\frac{d^4}{dy^4} - 2\alpha^2 \frac{d^2}{dy^2} \right) \hat{e}_1 + \frac{1}{\pi} \hat{f}_8 \left(\alpha^2 \frac{d}{dy} - \frac{d^3}{dy^3} \right) \hat{e}_1 \right\} dy \qquad (7.4.2)$$

and

$$\bar{\bar{b}}_2 = \int_{y=-a}^{+a} \left\{ \hat{f}_4 \left[\alpha^4 \hat{e}_1 - \left(\frac{d^2}{dy^2} - k_0^2 \right) \bar{e}_{10} - \pi(\bar{e}_{19} + \bar{e}'_{28}) - \pi\left(\frac{d^2}{dy^2} - \alpha^2 \right) u_1 + \pi k_0 P_1 \right] + \frac{1}{\pi} \hat{f}_8 \left[\bar{e}'_{10} + \pi \bar{e}_{28} \right] \right\} dy. \qquad (7.4.3)$$

In order to evaluate $\bar{b}_2$ and $\bar{\bar{b}}_2$ the derivatives of the functions $\hat{e}_1$, $\bar{e}_{10}$ and $\bar{e}_{28}$, which have not been computed, are needed. Approximating them using finite difference techinques over the entire range y=-a to y=+a can be avoided by expanding $\bar{b}_2$ and $\bar{\bar{b}}_2$ using repeated integration by parts. In the case of $\bar{b}_2$ this gives

$$\bar{b}_2 = \int_{y=-a}^{+a} \left\{ \hat{e}_1 \left(\frac{d^4}{dy^4} - 2\alpha^2 \frac{d^2}{dy^2} \right) \hat{f}_4 - \frac{1}{\pi} \hat{e}_1 \left(\alpha^2 \frac{d}{dy} - \frac{d^3}{dy^3} \right) \hat{f}_8 \right\} dy$$

$$+ \left[\left(\hat{f}_4 \left(\frac{d^3}{dy^3} - 2\alpha^2 \frac{d}{dy} \right) - \frac{d}{dy} \hat{f}_4 \cdot \left(\frac{d^2}{dy^2} - 2\alpha^2 \right) + \frac{d^2}{dy^2} \hat{f}_4 \frac{d}{dy} - \frac{d^3}{dy^3} \hat{f}_4 \right) \hat{e}_1 \right]_{-a}^{+a}$$

$$+ \left[\left(\hat{f}_8 \left(\alpha^2 - \frac{d^2}{dy^2} \right) + \frac{d}{dy} \hat{f}_8 \frac{d}{dy} - \frac{d^2}{dy^2} \hat{f}_8 \right) \hat{e}_1 \right]_{-a}^{+a} \tag{7.4.4}$$

where $\frac{d^n \hat{f}_4}{dy^n}$ n=1..4 and $\frac{d^n \hat{f}_8}{dy^n}$ n=1,2,3 may be determined from $\hat{f}_4$ and $\hat{f}_8$, which are solutions obtained from the matrix system (4.3.60) given analytically in (4.3.74). The first and second derivatives of $\hat{e}_1$ at y=-a can be obtained by approximating the function $\hat{e}_1$ near y=-a by the cubic

$$\hat{e}_1 = g_0 + g_1 y + g_2 y^2 + g_3 y^3 . \tag{7.4.5}$$

Hence

$$\frac{d}{dy}\hat{e}_1 = g_1 + 2g_2 y + 3g_3 y^2 \qquad \text{and} \qquad \frac{d^2}{dy^2}\hat{e}_1 = 2g_2 + 6g_3 y . \tag{7.4.6}$$

The real coefficients g_i i=0..3 are determined from the matrix system

$$\begin{bmatrix} \hat{e}_1(y_0) \\ \hat{e}_1(y_1) \\ \hat{e}_1(y_2) \\ \hat{e}_1(y_3) \end{bmatrix} = \begin{bmatrix} 1, & y_0, & y_0^2, & y_0^3 \\ 1, & y_1, & y_1^2, & y_1^3 \\ 1, & y_2, & y_2^2, & y_2^3 \\ 1, & y_3, & y_3^2, & y_3^3 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} \tag{7.4.7}$$

where y_0=-a and $y_j = y_0$ + ja/40 j=1,2,3. The first and second derivatives of $\hat{e}_1$ at y=+a can be obtained similarly. It should be noted that the third derivative of $\hat{e}_1$ at y=±a does not need to be determined, since it is known from (4.3.62) that $\hat{f}_4$ = 0 at y= a. In this way integrals involving derivatives of the functions $\hat{e}_1$, $\bar{e}_{1c}$ and $\bar{e}_{2\ell}$ can be reduced to integrals involving only the functions $\hat{e}_1$, $\bar{e}_{1c}$ and $\bar{e}_{2\ell}$. These can be easily evaluated using Simpsons' rule (81 points) as the functions $\hat{e}_1$, $\bar{e}_{1c}$ and $\bar{e}_{2\ell}$ have been computed at 81 points and stored in permanent random files on the computer. Thus, the coefficient b_2 is determined. The coefficients b_i i=1,3..20 are determined similarly. For the aspect ratios a = ¼,½,1 and 2 the amplitude coefficients b_i i=1..20 are displayed in table 18. It should be noted that in section 4.4 the method used to determine the coefficients C_i i=1..7 in the amplitude equation for $\bar{A}$, would correspond to determining the coefficient b_2 directly from (7.4.1,2,3) by computing the derivatives of the functions $\hat{e}_1$, $\bar{e}_{10}$ and $\bar{e}_{28}$ from the necessary equations in chapter 6. It should be noted that in table 18 the amplitude coefficients b_i i=3..8,11..16 for the aspect ratio a = 2 may be inaccurate due to numerical error in the Runge-Kutta scheme used to determine various dependent variables at order ε and ε^2. These coefficients have been marked by an asterisk.

b_i	a = ¼	a = ½	a = 1	a = 2
1	1.17356E-1	2.98550E 0	6.66208E 2	2.35883E 7
2	6.82607E 1	8.17819E 2	1.16230E 5	3.26386E 9
3	-1.81022E 4	-7.76529E 4	-6.25224E 6	-1.60040E11 *
4	-1.18181E 3	-6.21610E 3	-5.19836E 5	-8.66427E 9 *
5	-9.23709E 2	-6.33480E 3	-6.23720E 5	-1.12505E10 *
6	-9.05108E 3	-3.88264E 4	-3.12612E 6	-8.00250E10 *
7	-5.90903E 2	-3.10804E 3	-2.59917E 5	-4.22758E 9 *
8	-4.61854E 2	-3.16740E 3	-3.11860E 5	-5.76471E 9 *
9	-1.26669E 1	-1.82351E 2	-3.14470E 4	-1.04976E 9
10	-1.25073E 1	-1.77438E 2	-3.04607E 4	-1.04380E 9
11	-2.60452E-2	-8.13816E-1	-1.74901E 2	-3.70855E 6
12	-9.68575E-2	-5.82072E 0	-4.60229E 3	-4.24332E 8
13	3.52319E 3	1.97157E 4	1.31542E 6	8.97646E 8 *
14	2.99373E 2	3.71255E 3	2.78021E 5	2.60400E10 *
15	7.87173E 2	4.74106E 3	3.50963E 5	2.08587E10 *
16	4.76464E 2	1.37257E 3	-3.70624E 5	-3.92738E10 *
17	-1.50959E 1	1.40592E 3	1.27473E 5	2.70132E10 *
18	5.84465E 2	3.32658E 3	2.95692E 5	2.27685E10 *
19	1.34766E 0	1.36785E 1	8.87639E 2	-7.97468E 7
20	1.41038E 0	1.68378E 1	1.92113E 3	-7.71180E 7

Table 18. The amplitude coefficients b_i i=1..20 (E n denotes 10^n)

From equations (7.1.38,40,41) it is expected that

$$b_i = C_i \;\; i=1,2, \;\; b_i = 2b_{i+3} = 2C_i \;\; i=3,4,5 \;\; b_9 = C_6 \text{ and } b_{10} = C_7. \quad (7.4.8)$$

A comparison of the coefficients b_i i=1,2,6..10 in table 18 with the coefficients C_i i=1..7 in table 7 show good agreement. The reason they are not identical is that the b coefficients and the C coefficients are determined using different techniques (see above).

The relationships (7.4.8) arise because the linear terms involving $\bar{B}$ in (7.3.10) are equivalent to an expansion of the leading amplitude function $\bar{A}$ in the form

$$\bar{A}(\bar{X},\bar{\tau}) = \tilde{A}(\bar{X},\bar{\tau}) + \varepsilon \tilde{B}(\bar{X},\bar{\tau}) + \cdots \tag{7.4.9}$$

where $\tilde{A}$ and $\tilde{B}$ are arbitrary amplitude functions. Substituting (7.4.9) into the amplitude equation for $\bar{A}$ and equating ascending powers of ε gives

$$C_1\tilde{A} + C_2\tilde{A}_{\bar{X}\bar{X}} + (C_3+C_4/P+C_5/P^2)|\tilde{A}|^2\tilde{A} + (C_6+C_7/P)\tilde{A}_{\bar{\tau}} = 0 \tag{7.4.10}$$

and

$$C_1\tilde{B} + C_2\tilde{B}_{\bar{X}\bar{X}} + (C_3+C_4/P+C_5/P^2)(2|\tilde{A}|^2\tilde{B} + \tilde{A}^2\tilde{B}^*) + (C_6+C_7/P)\tilde{B}_{\bar{\tau}} = 0. \tag{7.4.11}$$

Comparing the equations (7.4.10),(7.4.11) with (4.4.33) and (7.3.10) gives the relationships (7.4.8).

It should be noted that the purely imaginary terms in (7.3.10) are linearly dependent because

$$C_1\bar{A} + C_2\bar{A}_{\bar{X}\bar{X}} + (C_3+C_4/P+C_5/P^2)\bar{A}|\bar{A}|^2 + (C_6+C_7/P)\bar{A}_{\bar{\tau}} = 0. \tag{7.4.12}$$

Thus

$$\bar{A}_{\bar{X}\bar{X}\bar{X}} = -\frac{1}{C_2}[C_1\bar{A}_{\bar{X}} + (C_3+C_4/P+C_5/P^2)(2\bar{A}_{\bar{X}}|\bar{A}|^2 + \bar{A}_{\bar{X}}^*\bar{A}^2) + (C_6+C_7/P)\bar{A}_{\bar{X}\bar{\tau}}]. \tag{7.4.14}$$

Therefore the amplitude coefficients b_i i=11..20 are not unique in (7.3.10). A more useful form of (7.3.10) is, replacing $\bar{A}_{\bar{X}\bar{X}\bar{X}}$ by (7.4.14),

$$b_1\bar{B} + b_2\bar{B}_{\bar{X}\bar{X}} + (b_3+b_4/P+b_5/P^2)|\bar{A}|^2\bar{B} + (b_6+b_7/P+b_8/P^2)\bar{A}^2\bar{B}^* + (b_9+b_{10}/P)\bar{B}_{\bar{\tau}}$$
$$+i[\hat{b}_{11}\bar{A}_{\bar{X}} + (\hat{b}_{13}+\hat{b}_{14}/P+\hat{b}_{15}/P^2)|\bar{A}|^2\bar{A}_{\bar{X}} + (\hat{b}_{16}+\hat{b}_{17}/P+\hat{b}_{18}/P^2)\bar{A}^2\bar{A}_{\bar{X}}^* + (\hat{b}_{19}+\hat{b}_{20}/P)\bar{A}_{\bar{X}\bar{\tau}}] = 0, \tag{7.4.15}$$

where

$$\hat{b}_{11} = b_{11} - \frac{c_1}{c_2} b_{12},$$

and

$$\hat{b}_i = b_i - \frac{2c_{i-10}}{c_2} b_{12} \qquad i = 13,14,15$$

$$\hat{b}_i = b_i - \frac{c_{i-13}}{c_2} b_{12} \qquad i = 16..20. \tag{7.4.16}$$

The coefficients $\hat{b}_i$ i=11,13..20 are 'unique', whereas the coefficients b_i i=11..20 are not.

A useful consistency check on the numerical evaluation of b_i i=11..20 is provided by making use of the arbitrariness in the definition of the dependent function Θ_1 in the solution for θ_2 together with the corresponding terms in u_2, v_2, w_2 and p_2. In section 4.3(2) Θ_1 is determined uniquely by the condition

$$\Theta_1 = 0 \qquad \text{at } y = 0, \tag{7.4.17}$$

which fixes the otherwise arbitrary component of Θ in the solution. It leads to the 'unique' set of amplitude coefficients $\hat{b}_i$ i=11,13..20 above.

Now suppose that Θ_1 is replaced by Θ_1^N defined by the new condition

$$\Theta_1^N = d \qquad \text{at } y = 0, \tag{7.4.18}$$

in place of (7.4.17). This implies that

$$\Theta_1^N = \Theta_1 + d\,\Theta \tag{7.4.19}$$

and in general it can be expected that this will lead to a new set of amplitude coefficients $\tilde{b}_i$ i=11,13..20 say, in place of $\hat{b}_i$. However there is a connection between $\tilde{b}_i$ and $\hat{b}_i$ which can be found as follows. Since the overall solution for θ_2 based on the use of Θ_1^N must be the same as the original solution, the new amplitude function $\bar{B}^N$ associated with Θ_1^N and satisfying

$$b_1\bar{B}^N+b_2\bar{B}^N_{\bar{x}\bar{x}}+(b_3+b_4/P+b_5/P^2)\bar{B}^N|\bar{A}|^2+(b_6+b_7/P+b_8/P^2)\bar{A}^2\bar{B}^{N*}+(b_9+b_{10}/P)\bar{B}^N_{\bar{t}}$$

$$+i[\tilde{b}_{11}\bar{A}_{\bar{x}}+(\tilde{b}_{13}+\tilde{b}_{14}/P+\tilde{b}_{15}/P^2)|\bar{A}|^2\bar{A}_{\bar{x}}+(\tilde{b}_{16}+\tilde{b}_{17}/P+\tilde{b}_{18}/P^2)\bar{A}^2\bar{A}^*_{\bar{x}}+(\tilde{b}_{19}+\tilde{b}_{20}/P)\bar{A}_{\bar{x}\bar{t}}]=0 \quad (7.4.20)$$

must be related to the original amplitude function $\bar{B}$ by

$$\bar{B}=\bar{B}^N+id\bar{A}_{\bar{x}}\,. \quad (7.4.21)$$

Substituting (7.4.21) into (7.4.15) and comparing with (7.4.2) gives

$$\hat{b}_i=\tilde{b}_i \qquad i=11,13,14,15,19,20$$

and

$$\hat{b}_i=\tilde{b}_i+2d\,C_{i-13}/C_2 \qquad i=16,17,18\,. \quad (7.4.22)$$

The consistency check outlined above was used to validate the numerical results obtained for the amplitude coefficients b_i i=1..20 and was found to be satisfied for all the aspect ratios considered (ie. ¼, ½, 1 and 2). Table 19 shows two sets of amplitude coefficients corresponding to the different conditions (7.4.17) and (7.4.18), clearly related by the relations (7.4.22), for the aspect ratio a=1.

Aspect Ratio a = 1

i	b_i	$\hat{b}_i$	$\check{b}_i$	$\tilde{b}_i$
1	6.66208E 2	–	6.66208E 2	–
2	1.16230E 5	–	1.16230E 5	–
3	-6.25224E 6	–	-6.25224E 6	–
4	-5.19836E 5	–	-5.19836E 5	–
5	-6.23720E 5	–	-6.23720E 5	–
6	-3.12612E 6	–	-3.12612E 6	–
7	-2.59917E 5	–	-2.59917E 5	–
8	-3.11860E 5	–	-3.11860E 5	–
9	-3.14470E 4	–	-3.14470E 4	–
10	-3.04607E 4	–	-3.04607E 4	–
11	-1.74901E 2	-1.48521E 2	-2.06231E 2	-1.48521E 2
12	-4.60229E 3	–	-1.00682E 4	–
13	1.31542E 6	1.06785E 6	1.60944E 6	1.06785E 6
14	2.78021E 5	2.57437E 5	3.02466E 5	2.57437E 5
15	3.50963E 5	3.26266E 5	3.80295E 5	3.26266E 5
16	-3.70624E 5	2.33468E 9	-7.75631E 5	-1.04642E 6
17	1.27473E 5	1.94272E 8	9.37987E 4	7.12839E 4
18	2.95692E 5	2.33239E 8	2.55289E 5	2.28277E 5
19	8.87639E 2	-3.57551E 2	2.36649E 3	-3.57550E 2
20	1.92113E 3	7.14993E 2	3.35360E 3	7.14994E 2

Table 19. Two sets of non-unique (b_i and $\check{b}_i$) and unique ($\hat{b}_i$ and $\tilde{b}_i$) amplitude coefficients corresponding to the different conditions (7.4.17) and (7.4.18) where the unknown constant d = 0.08824115.

Chapter 8 Phase-winding solutions for the long box

8.0 Introduction

Recent theoretical work concerned with the effect of lateral walls parallel to the axes of two-dimensional rolls has shown that in the absence of any forcing effects the wavelength of the roll pattern can be altered significantly from its critical value only when the Rayleigh number exceeds its critical value by an amount order L^{-1} where $2L$ is the distance between the lateral walls. The theoretical description of the flow (Cross et al 1980, 1983, Daniels 1981, 1984) requires a subdivision of the flow field into a core region $-L < x < L$ and double-structured end regions near the lateral walls at $x=\pm L$. It also requires the determination of the first and second amplitude equations equivalent to those now obtained (chapters 4,6 and 7) for the full three-dimensional flow with stress-free upper and lower boundaries. In this chapter these results are used to obtain the stationary phase-winding solutions for the long three-dimensional box with rigid, perfectly conducting sidewalls at $y=\pm a$ and rigid, perfectly conducting (or insulating) endwalls at $x=\pm L$. The solutions, equivalent to the two-dimensional ones originally obtained by Cross et al (1980), provide a prediction of the change in wavelength with increasing Rayleigh number and the results, for different Prandtl numbers and aspect ratios, are compared with experimental observations of roll pattern adjustments in long boxes.

8.1 Amplitude equations

The results of chapters 4 and 7 show that if

$$R = R_0 + \bar{\varepsilon}^2 \qquad (8.1.1)$$

where R_0 is the critical Rayleigh number for an infinite channel of aspect ratio 2a and $\bar{\varepsilon} << 1$, the solution for the temperature can be expanded in the form

$$\theta = \bar{\varepsilon}[\bar{A}_0(\bar{X},\bar{\tau})e^{ik_0x}\Theta(y)\sin\pi z + c.c]$$

$$+\bar{\varepsilon}^2[(\bar{A}_1(\bar{X},\bar{\tau})\Theta(y) + i\bar{A}_{0\bar{X}}(\bar{X},\bar{\tau})\Theta_1(y))e^{ik_0x}\sin\pi z + c.c + \ldots] + \ldots \quad (8.1.2)$$

where

$$\bar{X} = \bar{\varepsilon}x \quad , \quad \bar{\tau} = \bar{\varepsilon}^2 t, \quad (8.1.3)$$

k_o is the critical wavenumber corresponding to R_o, $\Theta(y)$ is normalised such that

$$\Theta(0) = 1, \quad (8.1.4)$$

and Θ_1 is uniquely specified by

$$\Theta_1 = -\frac{\partial\Theta}{\partial k}\bigg|_{k=k_0}. \quad (8.1.5)$$

Then, for steady flow, $\bar{A}_o$ and $\bar{A}_1$ (which here replace $\bar{A}$ and $\bar{B}$ of chapters 4,6 and 7) satisfy the amplitude equations

$$\bar{A}_{0\bar{X}\bar{X}} + d_1\bar{A}_0 - (d_2 + d_3/P + d_4/P^2)\bar{A}_0|\bar{A}_0|^2 = 0 \quad (8.1.6)$$

and

$$\bar{A}_{1\bar{X}\bar{X}} + d_1\bar{A}_1 - (d_2+d_3/P+d_4/P^2)(2|\bar{A}_0|^2\bar{A}_1 + \bar{A}_0^2\bar{A}_1^*) =$$

$$i[d_5\bar{A}_{0\bar{X}} + d_6\bar{A}_{0\bar{X}\bar{X}\bar{X}} - (d_7+d_8/P+d_9/P^2)|\bar{A}_0|^2\bar{A}_{0\bar{X}} - (d_{10}+d_{11}/P+d_{12}/P^2)\bar{A}_0^2\bar{A}_{0\bar{X}}^*] \quad (8.1.7)$$

where the coefficients d_i i=1..12 depend only on the aspect ratio a and are given for various values of a by table 20. It should be noted that

$$d_i = \begin{cases} b_i/b_2 & i=1 \\ -b_{i+j}/b_2 & i=2..6 \quad j=4\ i<5,\ j=7\ i>4 \\ b_{i+6}/b_2 & i=7..12 \end{cases} \quad (8.1.8)$$

where b_i are the amplitude coefficients given by table 18.

d_i	a = ¼	a = ½	a = 1	a = 2
1	1.7192E-3	3.6506E-3	5.7318E-3	7.2271E-3
2	1.3260E 2	4.7476E 1	2.6896E 1	2.4519E 1 *
3	8.6566E 0	3.8004E 0	2.2362E 0	1.2953E 0 *
4	6.7660E 0	3.8730E 0	2.6831E 0	1.7662E 0 *
5	3.8155E-4	9.9511E-4	1.5048E-3	1.1362E-3
6	1.4189E-3	7.1174E-3	3.9596E-2	1.3001E-1
7	5.1614E 1	2.4108E 1	1.1317E 1	2.7503E-1 *
8	4.3857E 0	4.5396E 0	2.3920E 0	7.9783E 0 *
9	1.1532E 1	5.7972E 0	3.0196E 0	6.3908E 0 *
10	6.9801E 0	1.6783E 0	-3.1887E 0	-1.2033E 1 *
11	-2.2115E-1	1.7191E 0	1.0967E 0	8.2765E 0 *
12	8.5623E 0	4.0676E 0	2.5440E 0	6.9760E 0 *

Table 20. The 'normalised' amplitude coefficients.

* Indicates that the result for the aspect ratio a = 2 may be inaccurate (see section 7.4).

Here it is convenient to make the transformations

$$\bar{\varepsilon}^{2} = \varepsilon / d_1 \,, \qquad \bar{X} = X/\sqrt{d_1} \tag{8.1.9}$$

and

$$\bar{A}_0 = \left(\frac{d_1}{d_2 + d_3/\rho + d_4/\rho^2}\right)^{1/2} A_0 \,, \quad \bar{A}_1 = \left(\frac{d_1^2}{d_2 + d_3/\rho + d_4/\rho^2}\right)^{1/2} A_1 \tag{8.1.10}$$

so that

$$R = R_0 + \varepsilon / d_1 \,, \tag{8.1.11}$$

$$x = \varepsilon^{-1/2} X \tag{8.1.12}$$

and

$$A_{0xx} + A_0 - A_0 |A_0|^2 = 0 \,, \tag{8.1.13}$$

$$A_{1xx} + A_1 - 2|A_0|^2 A_1 - A_0^2 A_1^* =$$

$$i[\tilde{k}_1 A_{0x} + \tilde{k}_2 A_{0xxx} - \tilde{k}_3 |A_0|^2 A_{0x} - \tilde{k}_4 A_0^2 A_{0x}^*] \tag{8.1.14}$$

where

$$\tilde{k}_1 = d_5 / d_1 \,, \quad \tilde{k}_2 = d_6 \,, \quad \tilde{k}_3 = (d_7 + d_8/\rho + d_9/\rho^2)/(d_2 + d_3/\rho + d_4/\rho^2) \,,$$

$$\tilde{k}_4 = (d_{10} + d_{11}/\rho + d_{12}/\rho^2) / (d_2 + d_3/\rho + d_4/\rho^2) \,. \tag{8.1.15}$$

The steady form of the solution (8.1.2) is now given by

$$\theta = (d_2 + d_3/P + d_4/P^2)^{-\frac{1}{2}}\{[\varepsilon^{1/2} A_0(X) + \varepsilon A_1(X) + \cdots]\,\Theta(y)e^{ik_0x}\sin\pi z + c.c$$

$$+ \varepsilon[iA_{0X}\Theta_1(y)e^{ik_0x}\sin\pi z + c.c + \cdots] + \cdots\}. \quad (8.1.16)$$

In their complete forms, the equations (8.1.13,14) govern the flow in boundary-layer regions near each end of the box where $x \pm L = O(\varepsilon^{-\frac{1}{2}})$. Since it is subsequently assumed that

$$\varepsilon = O(L^{-1}) \quad (8.1.17)$$

the length of these regions is small compared to the total length of the box, 2L. Between the boundary-layer regions, the core zone is defined by the longer x scaling, $x = O(L) = O(\varepsilon^{-1})$, so that X derivative terms in (8.1.13,14) are appropriately higher order. Finally, there are linear end regions where $x \pm L = O(1)$ immediately adjacent to each end wall where the solution is based on that developed in section 5.4.

8.2 Core region

A core variable $\hat{X}$ is defined by

$$x = \varepsilon^{-1}\hat{X} \qquad (-\varepsilon L < \hat{X} < \varepsilon L) \quad (8.2.1)$$

and the solution expanded in the form

$$\theta = (d_2 + d_3/P + d_4/P^2)^{-1/2}\{[\varepsilon^{1/2}\hat{A}_0(\hat{X}) + \varepsilon\hat{A}_1(\hat{X}) + \varepsilon^{3/2}\hat{A}_2(\hat{X}) + \cdots]\,\Theta(y)e^{ik_0x}\sin\pi z + c.c + O(\varepsilon)\}. \quad (8.2.2)$$

At order $\varepsilon^{3/2}$ it is found that

$$\hat{A}_0 - \hat{A}_0|\hat{A}_0|^2 = 0 \quad (8.2.3)$$

so that

$$\hat{A}_0 = e^{i\phi(\hat{X})} \quad (8.2.4)$$

where the (real) phase function ϕ is as yet undetermined. At order ε^2

$$\hat{A}_1 = 2\hat{A}_1|\hat{A}_0|^2 + \hat{A}_0^2\hat{A}_1^* \tag{8.2.5}$$

and this implies

$$\hat{A}_1 = i r_1(\hat{X}) e^{i\phi(\hat{X})} \tag{8.2.6}$$

where r_1 is real. The results (8.2.3) and (8.2.5) are appropriately reduced forms of (8.1.13) and (8.1.14). At order $\varepsilon^{5/2}$ it is expected that

$$\hat{A}_2 + \hat{A}_{0\hat{X}\hat{X}} - 2\hat{A}_0|\hat{A}_1|^2 - \hat{A}_0^*\hat{A}_1^2 - 2\hat{A}_2|\hat{A}_0|^2 - \hat{A}_0^2\hat{A}_2^* - \tilde{k}_5\hat{A}_0|\hat{A}_0|^4$$

$$-\tilde{k}_6\hat{A}_0|\hat{A}_0|^2 - i[\tilde{k}_1\hat{A}_{0\hat{X}} - \tilde{k}_3|\hat{A}_0|^2\hat{A}_{0\hat{X}} - \tilde{k}_4\hat{A}_0^2\hat{A}_{0\hat{X}}^*] = 0. \tag{8.2.7}$$

Here the nonlinear terms involving the real coefficients $\tilde{k}_5$ and $\tilde{k}_6$ are of higher order than those in (8.1.13) and (8.1.14) but, along with those involving $\hat{A}_1^2$ and $|\hat{A}_1|^2$, can be expected to arise at this stage in the expansion (c.f. Cross et al 1983, Appendix E). Now let

$$\hat{A}_2 = r_2(\hat{X}) e^{i\phi(\hat{X})} \tag{8.2.8}$$

where r_2 is not assumed real. Then from (8.2.7)

$$r_2 + i\phi_{\hat{X}\hat{X}} - \phi_{\hat{X}}^2 - 2r_1^2 + r_1^2 - 2r_2 - r_2^* - \tilde{k}_5 - \tilde{k}_6 + (\tilde{k}_1 - \tilde{k}_3 + \tilde{k}_4)\phi_{\hat{X}} = 0 \tag{8.2.9}$$

and from the imaginary part of this equation it follows that

$$\phi_{\hat{X}\hat{X}} = 0. \tag{8.2.10}$$

Thus

$$\phi = Q\hat{X} + C \tag{8.2.11}$$

where Q and C are unknown constants. A non-zero value of Q is equivalent to an adjustment in the wavelength of the roll pattern from the critical value associated with R_0 (see below).

8.3 Boundary layer regions

A boundary layer variable $\tilde{X}_1$, for the region near the wall x=-L, is defined by

$$x + L = \varepsilon^{-1/2} \tilde{X}_1 \tag{8.3.1}$$

and the solution is expanded in the form

$$\theta = (d_2 + d_3/P + d_4/P^2)^{-1/2} [\varepsilon^{1/2} \tilde{A}_0 e^{ik_0 x} \Theta(y) \sin\pi z + c.c + \varepsilon(\tilde{A}_1 \Theta(y) + i\tilde{A}_{0\tilde{X}_1} \Theta_1(y)) e^{ik_0 x} \sin\pi z$$

$$+ c.c + O(\varepsilon)]. \tag{8.3.2}$$

The equations for $\tilde{A}_0$ and $\tilde{A}_1$ follow directly from (8.1.13) and (8.1.14) as

$$\tilde{A}_{0\tilde{X}_1\tilde{X}_1} + \tilde{A}_0 - \tilde{A}_0|\tilde{A}_0|^2 = 0, \tag{8.3.3}$$

$$\tilde{A}_{1\tilde{X}_1\tilde{X}_1} + \tilde{A}_1 - 2|\tilde{A}_0|^2\tilde{A}_1 - \tilde{A}_0^2 \tilde{A}_1^* =$$

$$i[\tilde{k}_1\tilde{A}_{0\tilde{X}_1} + k_2\tilde{A}_{0\tilde{X}_1\tilde{X}_1\tilde{X}_1} - \tilde{k}_3|\tilde{A}_0|^2\tilde{A}_{0\tilde{X}_1} - \tilde{k}_4\tilde{A}_0^2\tilde{A}_{0\tilde{X}_1}^*]. \tag{8.3.4}$$

The solution for $\tilde{A}_0$ that vanishes at $\tilde{X}_1 = 0$ is

$$\tilde{A}_0 = e^{i\gamma_1} \tanh\frac{\tilde{X}_1}{\sqrt{2}} \tag{8.3.5}$$

where γ_1 is a real constant. The general solution for $\tilde{A}_1$ that avoids exponentially large behaviour as $\tilde{X}_1 \to \infty$ is then

$$\tilde{A}_1 = e^{i\gamma_1}\{\tilde{a}_1 \operatorname{sech}^2\frac{\tilde{X}_1}{\sqrt{2}} + i[\tilde{b}_1(\tilde{X}_1\tanh\frac{\tilde{X}_1}{\sqrt{2}} - \sqrt{2}) + \tilde{c}_1\tanh\frac{\tilde{X}_1}{\sqrt{2}} + I(\tilde{X}_1)]\}, \tag{8.3.6}$$

where $\tilde{a}_1$, $\tilde{b}_1$, $\tilde{c}_1$ are real constants and

$$I(X) = K_1 + K_2(\tanh^2\frac{X}{\sqrt{2}} + \frac{1}{2}\operatorname{sech}^2\frac{X}{\sqrt{2}}) \tag{8.3.7}$$

where

$$K_1 = \frac{\tilde{k}_1}{\sqrt{2}} + \sqrt{2}\tilde{k}_2 - \frac{1}{\sqrt{2}}(\tilde{k}_3 + \tilde{k}_4),$$

$$K_2 = -\frac{3}{\sqrt{2}}\tilde{k}_2 + \frac{1}{\sqrt{2}}(\tilde{k}_3 + \tilde{k}_4). \tag{8.3.8}$$

Corresponding solutions for the boundary-layer region near the wall x = L are given by

$$\tilde{A}_0 = -e^{i\gamma_2} \tanh \frac{\tilde{X}_2}{\sqrt{2}} , \tag{8.3.9}$$

$$\tilde{A}_1 = e^{i\gamma_2}\{\tilde{a}_2 \operatorname{sech}^2 \frac{\tilde{X}_2}{\sqrt{2}} + i[\tilde{b}_2(\tilde{X}_2 \tanh \frac{\tilde{X}_2}{\sqrt{2}} - \sqrt{2}) + \tilde{c}_2 \tanh \frac{\tilde{X}_2}{\sqrt{2}} - I(\tilde{X}_2)]\} \tag{8.3.10}$$

where γ_2, a_2, b_2, and c_2 are real constants and

$$x - L = \varepsilon^{-1/2} \tilde{X}_2 . \tag{8.3.11}$$

Matching between the core and boundary-layer solutions requires

$$\tilde{b}_1 = -\tilde{b}_2 = \varphi , \tag{8.3.12}$$

and

$$-\varepsilon L \varphi + C = \gamma_1 , \qquad \varepsilon L \varphi + C = \gamma_2 . \tag{8.3.13}$$

8.4 End-wall regions

The endwall regions where $x \pm L = O(1)$ are governed by the linear equations to leading order and provide boundary conditions for the boundary-layer amplitude functions $\tilde{A}_0$ and $\tilde{A}_1$. For the boundary layer at x=-L it is required that

$$\tilde{A}_0 = 0 \qquad (\tilde{X}_1 = 0), \tag{8.4.1}$$

which has already been applied in order to obtain the solution (8.3.5). The end-region solution is then generated by the form of $\tilde{A}_{0\tilde{X}_1}$ as $\tilde{X}_1 \to 0$ and this leads to a condition for $\tilde{A}_1$ of the form

$$\tilde{A}_1 + \alpha^* \tilde{A}_{0\tilde{X}_1} + \beta^* \tilde{A}^*_{0\tilde{X}_1} = 0 \qquad \text{at} \quad \tilde{X}_1 = 0 . \tag{8.4.2}$$

Here α and β are complex constants whose values are obtained as follows.

In the end-region where

$$x_1 = x + L = O(1) \tag{8.4.3}$$

the solution for the temperature generated by $\tilde{A}_{0\tilde{X}_1}(0)$ has the form

$$\theta = \varepsilon(d_2+d_3|P+d_4|P^2)^{-1/2}\{E(x_1\Theta(y)+i\Theta_1(y))e^{ik_0x_1}+E^*(x_1\Theta(y)-i\Theta_1(y))e^{-ik_0x_1}$$
$$+\sum_{m=0}^{\infty}[E_m\psi_m(y)e^{ik_mx_1}+E_m^*\psi_m^*(y)e^{-ik_m^*x_1}+\hat{E}_m\hat{\psi}_m(y)e^{i\hat{k}_mx_1}+\bar{E}_m\bar{\psi}_m(y)e^{i\bar{k}_mx_1}]\}\sin\pi z \tag{8.4.4}$$
$$+O(\varepsilon^{3/2})$$

where $\psi_0(y) = \Theta(y)$. Here E and E_m ($m \geq 0$) are complex constants while $\hat{E}_m$ and $\bar{E}_m$ are real constants. The eigenfunctions $\psi = \psi_m, \hat{\psi}_m, \bar{\psi}_m$ and eigenvalues k_m, $\hat{k}_m$, $\bar{k}_m$ ($m \geq 0$) are those determined in chapter 5; only the even eigenfunctions are included since the entire solution is generated by the eigensolution $\Theta(y)$, and its k derivative $-\Theta_1$, which are both even functions of y. The odd part of the spectrum for θ is redundant. The eigenvalues $i\hat{k}_m$ and $i\bar{k}_m$ are purely real and negative while the eigenvalues k_m are complex with $\mathrm{Im}(k_m) > 0$ ($m \geq 1$). It should be noted that the solution associated with the constant E in (8.4.4) arises because at $R = R_0$, $k = k_0$ is a repeated eigenvalue. This solution was discarded in chapter 5 where the end-region solution was forced by a non-zero heat transfer at x = -L; here the end-region solution is smaller (by order $\varepsilon^{\frac{1}{2}}$) than that in the core (and boundary layers) and specification of the constant E replaces the forcing effect at $x_1 = 0$, where it is assumed that the end wall is rigid and either perfectly conducting or perfectly insulating:

$$u = v = w = 0, \quad \theta = 0 \text{ or } \frac{\partial\theta}{\partial x} = 0 \quad (x_1 = 0). \tag{8.4.5}$$

Matching with the boundary-layer solution (8.3.2) requires that

$$e^{ik_0L}E = \tilde{A}_{0\tilde{X}_1}(0) \tag{8.4.6}$$

and also gives

$$e^{ik_0L}E_0 = \tilde{A}_1(0). \tag{8.4.7}$$

It is noted that the terms in (8.3.2) and (8.4.4) that involve Θ_1

match automatically.

The solution (8.5.4) and the corresponding solutions for u,v and w must satisfy the four boundary conditions (8.4.5) and this leads to a set of equations for the unknown coefficients E_m, $\hat{E}_m$ and $\bar{E}_m$ (m=0,1,..) in terms of the (assumed known) coefficient E. The collocation method of chapter 5 is used, with the terms involving E and E^* in (8.5.4) essentially replacing the known forcing term in the earlier analysis. Thus, using a truncation $m=m_\infty$, the real and imaginary parts of E_m and the real values of $\hat{E}_m$ and $\bar{E}_m$ are expressed as linear combinations of E and E^*:

$$E_m = \alpha_m E + \beta_m E^*, \tag{8.4.8}$$

$$\hat{E}_m = \hat{\alpha}_m E + \hat{\alpha}_m^* E^*, \tag{8.4.9}$$

$$\bar{E}_m = \bar{\alpha}_m E + \bar{\alpha}_m^* E^*, \tag{8.4.10}$$

where the complex coefficients α_m, β_m, $\hat{\alpha}_m$ and $\bar{\alpha}_m$ are to be determined. For a perfectly conducting endwall, substitution into (8.4.4) and setting $x_1 = 0$ gives

$$-i\Theta_1(y) = \sum_{m=0}^{m=m_\infty} [\alpha_m \psi_m(y) + \beta_m^* \psi_m^*(y) + \hat{\alpha}_m \hat{\psi}_m(y) + \bar{\alpha}_m \bar{\psi}_m(y)] \tag{8.4.11}$$

and three similar equations are obtained from the boundary conditions on the velocity components. The real and imaginary parts of each of these then constitute eight y-dependent equations for the eight set of unknowns, which are the real and imaginary parts of α_m, β_m, $\hat{\alpha}_m$ and $\bar{\alpha}_m$. These are found as the solutions of a complex matrix equation of order $4(1+m_\infty)$ resulting from application of the collocation method of chapter 5. Tables 21-24 show the values of the coefficients α_0 and β_0 associated with the leading eigenfunction (m=0), for increasing levels of truncation up to $m_\infty = 14$ and for various aspect ratios. In table 23 $\hat{\alpha}_0$ and $\bar{\alpha}_0$ are also shown, where the eigenfunctions $\hat{\psi}_c$ and $\bar{\psi}_c$ are normalised to the value one at y = 0. Results for the perfectly insulating case are shown in tables 25-28 and it is interesting to note that the value of the imaginary part of α_0 is independent of the thermal boundary condition at the wall.

a = 0.25

m	α_{or}	α_{oi}	β_{or}	$-\beta_{oi}$
0	-0.012236	-0.057030	0.012236	0.099086
1	-0.011226	-0.050827	0.011164	0.101649
2	-0.011183	-0.050303	0.011074	0.101397
3	-0.011187	-0.050232	0.011071	0.101369
4	-0.011183	-0.050219	0.011073	0.101368
5	-0.011189	-0.050216	0.011074	0.101367
6	-0.011190	-0.050216	0.011075	0.101368
7	-0.011191	-0.050219	0.011076	0.101374
8	-0.011190	-0.050218	0.011076	0.101371
9	-0.011191	-0.050220	0.011076	0.101375
10	-0.011190	-0.050218	0.011075	0.101372
11	-0.011217	-0.050227	0.011104	0.101377
12	-0.011193	-0.050220	0.011078	0.101374
13	-0.011192	-0.050220	0.011078	0.101373
14	-0.011192	-0.050220	0.011078	0.101374

Table 21. The coefficients α_0 and β_0 for conducting endwalls.

a = 0.5

m	α_{or}	α_{oi}	β_{or}	$-\beta_{oi}$
0	-0.029254	-0.037892	0.029254	0.100502
1	-0.024807	-0.023977	0.024662	0.110099
2	-0.024867	-0.022911	0.024653	0.109801
3	-0.024937	-0.022716	0.024699	0.109724
4	-0.024960	-0.022672	0.024718	0.109709
5	-0.024968	-0.022662	0.024725	0.109706
6	-0.024972	-0.022660	0.024730	0.109708
7	-0.024975	-0.022662	0.024734	0.109714
8	-0.024975	-0.022662	0.024734	0.109712
9	-0.024977	-0.022663	0.024736	0.109717
10	-0.024973	-0.022659	0.024731	0.109712
11	-0.025024	-0.022739	0.024798	0.109779
12	-0.024982	-0.022674	0.024744	0.109724
13	-0.024981	-0.022671	0.024742	0.109722
14	-0.024980	-0.022671	0.024742	0.109723

Table 22. The coefficients α_0 and β_0 for conducting endwalls.

a = 1.0

m	α_{or}	α_{oi}	β_{or}	$-\beta_{oi}$
0	-0.090577	0.050197	0.090577	0.085885
1	-0.052390	0.069680	0.051668	0.097039
2	-0.053244	0.070932	0.052842	0.097300
3	-0.053669	0.071285	0.053248	0.097221
4	-0.053812	0.071403	0.053377	0.097165
5	-0.053860	0.071443	0.053419	0.097140
6	-0.053883	0.071458	0.053440	0.097132
7	-0.053898	0.071469	0.053455	0.097133
8	-0.053880	0.071468	0.053456	0.097126
9	-0.053906	0.071477	0.053462	0.097129
10	-0.053896	0.071487	0.053448	0.097115
11	-0.053918	0.071266	0.053540	0.097320
12	-0.053906	0.071436	0.053474	0.097162
13	-0.053908	0.071445	0.053473	0.097154
14	-0.053909	0.071449	0.053473	0.097152

m	$\bar{\alpha}_{or}$	$\bar{\alpha}_{oi}$	$\hat{\alpha}_{or}$	$\hat{\alpha}_{oi}$
0	0.000000	-0.081321	0.000000	-0.054761
1	0.000181	-0.088562	0.010431	-0.133350
2	0.000740	-0.090305	0.009615	-0.134346
3	0.000852	-0.090234	0.010143	-0.133480
4	0.000898	-0.090180	0.010319	-0.133274
5	0.000912	-0.090165	0.010376	-0.133204
6	0.000916	-0.090158	0.010396	-0.133193
7	0.000918	-0.090152	0.010407	-0.133168
8	0.000919	-0.090152	0.010408	-0.133171
9	0.000920	-0.090150	0.010418	-0.133165
10	0.000924	-0.090154	0.010444	-0.133178
11	0.000855	-0.090135	0.009940	-0.133047
12	0.000908	-0.090149	0.010328	-0.133138
13	0.000911	-0.090148	0.010348	-0.133139
14	0.000912	-0.090150	0.010354	-0.133154

Table 23. The coefficients α_o, β_o, $\bar{\alpha}_o$ and $\hat{\alpha}_o$ for conducting endwalls.

a = 2.0

m	α_{or}	α_{oi}	β_{or}	$-\beta_{oi}$
0	0.022420	-0.008699	-0.022420	-0.024584
1	-0.123722	0.280327	0.072242	0.096090
2	-0.126383	0.305405	0.065819	0.105767
3	-0.125189	0.305029	0.067530	0.105960
4	-0.125369	0.305033	0.067784	0.105738
5	-0.125479	0.305020	0.067889	0.105611
6	-0.125559	0.305024	0.067933	0.105538
7	-0.125626	0.305044	0.067949	0.105478
8	-0.125642	0.305020	0.067961	0.105451
9	-0.125685	0.305040	0.067978	0.105432
10	-0.125690	0.305004	0.068006	0.105410
11	-0.125470	0.305292	0.067712	0.105617
12	-0.125633	0.305097	0.067914	0.105461
13	-0.125655	0.305074	0.067942	0.105444
14	-0.125668	0.305070	0.067953	0.105438

Table 24. The coefficients α_o and β_o for conducting endwalls.

a = 0.25

m	α_{or}	α_{oi}	β_{or}	$-\beta_{oi}$
0	-0.745832	-0.043813	-0.628043	-0.259214
1	0.104943	-0.051684	0.106600	0.168038
2	0.104727	-0.050466	0.106247	0.167548
3	0.104660	-0.050270	0.106213	0.167446
4	0.104646	-0.050227	0.106210	0.167427
5	0.104640	-0.050216	0.106209	0.167420
6	0.104640	-0.050213	0.106210	0.167421
7	0.104643	-0.050216	0.106216	0.167428
8	0.104641	-0.050214	0.106213	0.167424
9	0.104644	-0.050216	0.106217	0.167431
10	0.104641	-0.050213	0.106213	0.167424
11	0.104666	-0.050294	0.106264	0.167493
12	0.104644	-0.050222	0.106219	0.167432
13	0.104643	-0.050220	0.106218	0.167431
14	0.104644	-0.050220	0.106218	0.167431

Table 25. The coefficients α_o and β_o for insulating endwalls.

$a = 0.5$

m	α_{or}	α_{oi}	β_{or}	$-\beta_{oi}$
0	1.022300	-0.077113	0.934897	0.636327
1	0.169933	-0.026134	0.178932	0.228910
2	0.169349	-0.023458	0.178398	0.228197
3	0.168981	-0.022866	0.178310	0.227791
4	0.168870	-0.022714	0.178297	0.227671
5	0.168831	-0.022667	0.178293	0.227629
6	0.168819	-0.022653	0.178297	0.227618
7	0.168820	-0.022650	0.178308	0.227623
8	0.168814	-0.022648	0.178304	0.227615
9	0.168817	-0.022646	0.178311	0.227621
10	0.168806	-0.022633	0.178297	0.227604
11	0.169005	-0.022935	0.178509	0.227899
12	0.168839	-0.022684	0.178334	0.227653
13	0.168832	-0.022675	0.178328	0.227644
14	0.168831	-0.022672	0.178327	0.227643

Table 26. The coefficients α_o and β_o for insulating endwalls.

$a = 1.0$

m	α_{or}	α_{oi}	β_{or}	$-\beta_{oi}$
0	0.097998	0.032430	0.237246	0.205737
1	0.185082	0.070619	0.244730	0.226801
2	0.181739	0.069691	0.244810	0.226852
3	0.180377	0.070823	0.244854	0.225613
4	0.179898	0.071221	0.244837	0.225140
5	0.179712	0.071374	0.244815	0.224950
6	0.179631	0.071439	0.244811	0.224868
7	0.179595	0.071474	0.244820	0.224836
8	0.179568	0.071485	0.244808	0.224803
9	0.179553	0.071510	0.244811	0.224792
10	0.179511	0.071559	0.244762	0.224739
11	0.180198	0.070876	0.245317	0.225536
12	0.179657	0.071406	0.244891	0.224910
13	0.179624	0.071433	0.244872	0.224874
14	0.179615	0.071443	0.244866	0.224863

Table 27. The coefficients α_o and β_o for insulating endwalls.

a = 2.0

m	α_{or}	α_{oi}	β_{or}	$-\beta_{oi}$
0	0.228290	-0.060247	0.117552	0.141379
1	0.121483	0.243907	0.280908	0.204912
2	0.124657	0.306112	0.280213	0.217900
3	0.129446	0.304991	0.283698	0.220118
4	0.128367	0.304628	0.283365	0.219192
5	0.127826	0.304833	0.283068	0.218782
6	0.127542	0.304928	0.282938	0.218574
7	0.127368	0.304997	0.282845	0.218448
8	0.127265	0.304999	0.282784	0.218357
9	0.127178	0.305030	0.282756	0.218293
10	0.127082	0.305003	0.282717	0.218192
11	0.127857	0.305443	0.282835	0.219074
12	0.127300	0.305151	0.282740	0.218449
13	0.127241	0.305115	0.282740	0.218380
14	0.127210	0.305108	0.282739	0.218354

Table 28. The coefficients α_o and β_o for insulating endwalls.

The equation (8.4.2) now follows from (8.4.6,7,8) by setting

$$\alpha_0 = -\alpha^* = -\alpha_r + i\alpha_i \quad , \tag{8.4.12}$$

$$\beta_0 = -\bar{\beta}^* = -\bar{\beta}_r + i\bar{\beta}_i \quad , \tag{8.4.13}$$

and

$$\beta = \bar{\beta} e^{-2ik_cL} = |\beta| e^{i\chi} \quad , \quad \text{say.} \tag{8.4.14}$$

The solution in the endwall region near x=+L can be formulated in a similar way and further numerical calculations are not required. Here it is found that

$$\tilde{A}_0 = 0 \qquad (\tilde{X}_2 = 0), \tag{8.4.15}$$

as assumed for the solution (8.3.9), and that

$$\tilde{A}_1 - \alpha \tilde{A}_{0\tilde{X}_2} - \beta \tilde{A}^*_{0\tilde{X}_2} = 0 \qquad (\tilde{X}_2 = 0). \tag{8.4.16}$$

The result (8.4.16) can be deduced from the endwall solution at x=-L by the transformation $x_1 \rightarrow -x_2$, $E \rightarrow -E^*$ and $E_0 \rightarrow E_0^*$ where $x_2 = x - L$.

From (8.4.2,16) and the solutions (8.3.6,10), it follows that

$$\tilde{a}_1 - i\sqrt{2}\tilde{b}_1 + iI_0 = -\frac{\alpha^*}{\sqrt{2}} - \frac{\beta^*}{\sqrt{2}} e^{-2i\gamma_1} , \tag{8.4.17}$$

$$\tilde{a}_2 - i\sqrt{2}\tilde{b}_2 - iI_0 = -\frac{\alpha}{\sqrt{2}} - \frac{\beta}{\sqrt{2}} e^{-2i\gamma_2} , \tag{8.4.18}$$

where

$$I_0 = I(0) = K_1 + \tfrac{1}{2} K_2 \tag{8.4.19}$$

and the real and imaginary parts of the equations (8.4.17,18), together with (8.3.12,13) now constitute eight equations for the eight unknowns Q, C, $\tilde{a}_1$, $\tilde{a}_2$, $\tilde{b}_1$, $\tilde{b}_2$, γ_1 and γ_2. In fact the equations are identical with those derived for the two-dimensional model by Cross et al (1983) apart, of course, from the numerical values of the coefficients α and β determined by the end-region problem and the coefficients $\tilde{k}_1$, $\tilde{k}_2$, $\tilde{k}_3$ and $\tilde{k}_4$ that arise from the second order amplitude equation.

8.5 The phase-winding solutions

Elimination of $\tilde{a}_{1,2}$, $\tilde{b}_{1,2}$ and $\gamma_{1,2}$ from the equations (8.3.12,13), (8.4.17,18) leads to the two classes of phase-winding solutions found by Cross et al (1980,1983). The type 1 solutions satisfy the equation

$$2q = -\varepsilon L [\tilde{\alpha} - (-1)^n |\beta| \sin(2q - \chi)] \tag{8.5.1}$$

for q where

$$Q = q/\varepsilon L \quad , \qquad C = n\pi/2 \qquad (n = \text{Integer}) \tag{8.5.2}$$

are the physical parameters that occur in the core solution (8.2.3,11). Here

$$\tilde{\alpha} = \alpha_i - \sqrt{2} I_0 = \alpha_i - \sqrt{2}(K_1 + \tfrac{1}{2}K_2) , \tag{8.5.3}$$

is a function of both aspect ratio a and Prandtl number P. It has the form

$$\tilde{\alpha} = (\tilde{d}_1 + \tilde{d}_2/\rho + \tilde{d}_3/\rho^2) \Big/ (d_2 + d_3/\rho + d_4/\rho^2)$$

where

$$\tilde{d}_0 = \alpha_i - \tilde{k}_1 - \tfrac{1}{2}\tilde{k}_2\,, \quad \tilde{d}_1 = \tilde{d}_0 d_2 + \tfrac{1}{2}(d_7 + d_{10}),$$

$$\tilde{d}_2 = \tilde{d}_0 d_3 + \tfrac{1}{2}(d_8 + d_{11})\,, \quad \tilde{d}_3 = \tilde{d}_0 d_4 + \tfrac{1}{2}(d_9 + d_{12}). \qquad (8.5.4)$$

The coefficients $\tilde{d}_i$ i=1,2,3 and d_{i-1} i=3,4,5 depend solely on the aspect ratio a and are given for various values of a in table 29. The other

Conducting Endwalls

	a = ¼	a = ½	a = 1	a = 2
$\lvert\beta\rvert$	1.01977E-1	1.12477E-1	1.10896E-1	1.25439E-1
$\tilde{d}_1$	-0.68837E 1	-0.12936E 1	-0.16075E 1	-0.38477E 1 *
$\tilde{d}_2$	-0.27977E 1	0.19937E 1	0.12728E 1	0.82346E 1 *
$\tilde{d}_3$	0.82009E 1	0.37751E 1	0.22160E 1	0.68296E 1 *
$\tilde{d}_0$	-2.72864E-1	-2.98820E-1	-2.10882E-1	8.28457E-2

Insulating Endwalls

	a = ¼	a = ½	a = 1	a = 2
$\lvert\beta\rvert$	1.98281E-1	2.89175E-1	3.32450E-1	3.57239E-1
$\tilde{d}_1$	-0.68837E 1	-0.12937E 1	-0.16077E 1	-0.38467E 1 *
$\tilde{d}_2$	-0.27977E 1	0.19337E 1	0.12728E 1	0.82347E 1 *
$\tilde{d}_3$	0.82009E 1	0.37751E 1	0.22160E 1	0.68297E 1 *
$\tilde{d}_0$	-2.72864E-1	-2.98821E-1	-2.10888E-1	8.28835E-2

Table 29. The parameter $\lvert\beta\rvert$ and the coefficients $\tilde{d}_i$ i=0,1,2 and 3.

parameters in (8.5.1) are $\lvert\beta\rvert$, which depends only on the aspect ratio (see table 29) and χ which from (8.4.14) is given by

$$\chi = \chi_0 - 2k_0 L \qquad (8.5.5)$$

where χ_0 also depends only on the aspect ratio. Only four values of n represent distinct solutions in (8.5.1) since an increase in C by 2π has no effect on (8.2.4). The n = 0 and n = 2 solutions are equal and opposite flows with an even number of rolls while the n = 1 and n = 3 solutions are equal and opposite flows with an odd number of rolls. It should be noted that in table 29 the values of $\tilde{d}_i$ i=1,2,3 for the aspect ratio a = 2 may be inaccurate (see section 7.4) and have been marked with an asterisk.

The type 2 solutions satisfy

$$2q = \chi - \frac{\pi}{2} + m\pi \, , \quad Q = q/\varepsilon L \qquad (m = \text{Integer}) \tag{8.5.6}$$

with

$$\chi - \frac{\pi}{2} + m\pi = -\varepsilon L(\tilde{\alpha} - (-1)^m |\beta| \cos 2C) \tag{8.5.7}$$

so that for fixed q the corresponding range of values of εL is restricted to

$$|2q/\varepsilon L + \tilde{\alpha}| < |\beta| \tag{8.5.8}$$

and is traversed as the value of C varies by $\pi/2$. These solutions connect the even and odd branches of the type 1 solutions and for a given integer m (ie. a given value of q) there are four distinct solutions corresponding to the four possible ways of connecting the two even branches of the type 1 solution with the two odd branches for the given value of q.

Stability arguments (Daniels 1981) show that all of the type 2 solutions are unstable and so of relatively minor physical significance, while half of the type 1 solutions are unstable. The two types of solution (8.5.1) and (8.5.6) are most easily displayed by a graphical construction (figure 26) with 2q as abscissa Stationary solutions correspond to points of intersection of the sinusoidal curves (type 1 solutions) or vertical lines (type 2 solutions) with the sloping line representing the value of $2q/\varepsilon L$. For fixed L, the gradient of this line decreases from infinity to zero as the Rayleigh number increment ε increases from zero through values of order L^{-1}. The stable type 1 solutions are those for which the gradient of the sinusoidal curve is negative.

a = 0.25

	Conducting Endwalls		Insulating Endwalls	
P	$\tilde{\alpha}$	$\tilde{\alpha}/\lvert\beta\rvert$	$\tilde{\alpha}$	$\tilde{\alpha}/\lvert\beta\rvert$
0.1E-3	0.1212E 1	0.1188E 2	0.1212E 1	0.6112E 1
0.1E-2	0.1210E 1	0.1187E 2	0.1211E 1	0.6105E 1
0.1E-1	0.1194E 1	0.1171E 2	0.1194E 1	0.6021E 1
0.1E 0	0.9047E 0	0.8872E 1	0.9047E 0	0.4563E 1
7.1E-1	0.5683E-1	0.5573E 0	0.5683E-1	0.2866E 0
0.1E 1	0.7009E-2	0.6873E-1	0.7009E-2	0.3535E-1
0.1E 2	-0.5115E-1	-0.5016E 0	-0.5115E-1	-0.2580E 0
0.1E 3	-0.5120E-1	-0.5089E 0	-0.5190E-1	-0.2617E 0
0.1E 4	-0.5191E-1	-0.5091E 0	-0.5191E-1	-0.2618E 0
178.0E 1	-0.5191E-1	-0.5091E 0	-0.5191E-1	-0.2618E 0
0.1E 5	-0.5191E-1	-0.5091E 0	-0.5191E-1	-0.2618E 0

Table 30. The parameters $\tilde{\alpha}$ and $\tilde{\alpha}/|\beta|$ at various Prandtl numbers.

a = 0.5

	Conducting Endwalls		Insulating Endwalls	
P	$\tilde{\alpha}$	$\tilde{\alpha}/\lvert\beta\rvert$	$\tilde{\alpha}$	$\tilde{\alpha}/\lvert\beta\rvert$
0.1E-3	0.9747E 0	0.8666E 1	0.9747E 0	0.3371E 1
0.1E-2	0.9743E 0	0.8662E 1	0.9743E 0	0.3369E 1
0.1E-1	0.9691E 0	0.8616E 1	0.9691E 0	0.3351E 1
0.1E 0	0.8379E 0	0.7450E 1	0.8379E 0	0.2898E 1
7.1E-1	0.1488E 0	0.1323E 1	0.1488E 0	0.5145E 0
0.1E 1	0.8115E-1	0.7215E 0	0.8115E-1	0.2806E 0
0.1E 2	-0.2206E-1	-0.1961E 0	-0.2206E-1	-0.7629E-1
0.1E 3	-0.2680E-1	-0.2383E 0	-0.2680E-1	-0.9268E-1
0.1E 4	-0.2720E-1	-0.2419E 0	-0.2720E-1	-0.9408E-1
178.0E 1	-0.2722E-1	-0.2420E 0	-0.2722E-1	-0.9414E-1
0.1E 5	-0.2724E-1	-0.2422E 0	-0.2724E-1	-0.9422E-1

Table 31. The parameters $\tilde{\alpha}$ and $\tilde{\alpha}/|\beta|$ at various Prandtl numbers.

a = 1.0

P	Conducting Endwalls $\tilde{\alpha}$	Conducting Endwalls $\tilde{\alpha}/\|\beta\|$	Insulating Endwalls $\tilde{\alpha}$	Insulating Endwalls $\tilde{\alpha}/\|\beta\|$
0.1E-3	0.8259E 0	0.7447E 1	0.8259E 0	0.2484E 1
0.1E-2	0.8257E 0	0.7445E 1	0.8257E 0	0.2484E 1
0.1E-1	0.8229E 0	0.7420E 1	0.8229E 0	0.2475E 1
0.1E 0	0.7328E 0	0.6608E 1	0.7328E 0	0.2204E 1
7.1E-1	0.1295E 0	0.1168E 1	0.1295E 0	0.3896E 0
0.1E 1	0.5913E-1	0.5332E 0	0.5912E-1	0.1778E 0
0.1E 2	-0.5371E-1	-0.4843E 0	-0.5372E-1	-0.1616E 0
0.1E 3	-0.5924E-1	-0.5342E 0	-0.5924E-1	-0.1782E 0
0.1E 4	-0.5972E-1	-0.5385E 0	-0.5972E-1	-0.1796E 0
178.0E 1	-0.5974E-1	-0.5387E 0	-0.5974E-1	-0.1797E 0
0.1E 5	-0.5977E-1	-0.5389E 0	-0.5977E-1	-0.1798E 0

Table 32. The parameters $\tilde{\alpha}$ and $\tilde{\alpha}/|\beta|$ at various Prandtl numbers.

a = 2.0*

P	Conducting Endwalls $\tilde{\alpha}$	Conducting Endwalls $\tilde{\alpha}/\|\beta\|$	Insulating Endwalls $\tilde{\alpha}$	Insulating Endwalls $\tilde{\alpha}/\|\beta\|$
0.1E-3	0.3867E 1	0.3083E 2	0.3867E 1	0.1082E 2
0.1E-2	0.3869E 1	0.3084E 2	0.3869E 1	0.1083E 2
0.1E-1	0.3879E 1	0.3093E 2	0.3879E 1	0.1086E 2
0.1E 0	0.3557E 1	0.2835E 2	0.3557E 1	0.9956E 1
7.1E-1	0.7136E 0	0.5689E 1	0.7136E 0	0.1998E 1
0.1E 1	0.4067E 0	0.3242E 1	0.4067E 0	0.1139E 1
0.1E 2	-0.1198E 0	-0.9554E 0	-0.1198E 0	-0.3354E 0
0.1E 3	-0.1535E 0	-0.1223E 1	-0.1534E 0	-0.4295E 0
0.1E 4	-0.1566E 0	-0.1248E 1	-0.1565E 0	-0.4382E 0
178.0E 1	-0.1567E 0	-0.1250E 1	-0.1567E 0	-0.4386E 0
0.1E 5	-0.1569E 0	-0.1251E 1	-0.1569E 0	-0.4391E 0

Table 33. The parameters $\tilde{\alpha}$ and $\tilde{\alpha}/|\beta|$ at various Prandtl numbers.

* It should be noted that the results for the aspect ratio a = 2 may be inaccurate (see section 7.4).

8.6 Results

The results displayed graphically in figures 26-30 show the two distinct situations that can arise. If the sinusoidal curves lie wholly below or above the horizontal axis (corresponding to $\tilde{\alpha}/|\beta| < -1$ or $\tilde{\alpha}/|\beta| > 1$) then the number of rolls in the box, given approximately by the formula

$$\frac{2}{\pi}(k_o L + q) \tag{8.6.1}$$

must change as the Rayleigh number increases. Alternatively, if the sinusoidal curves intersect the horizontal axis ($-1 < \tilde{\alpha}/|\beta| < 1$) the number of rolls can remain the same as the Rayleigh number increases. The results shown in figure 30 correspond to the former case and are for P = 0.166, a = 1 and conducting endwalls. The length of the box is taken as 2L = 10 and the number of rolls, which decreases as the Rayleigh number is raised, is labelled accordingly. Figures 31-35 show details of flow patterns predicted by a composite expansion for the vertical velocity w. This is constructed from the solutions in the core, boundary-layer and endwall regions and has the form

$$\begin{aligned} w = \sin\pi z\,(d_2 + d_3/P + d_4/P^2)^{-1/2}\Big\{ & \varepsilon e^{-\varepsilon^{1/2} x_1}\Big[E(x_1 W(y) + iW_1(y))e^{ik_o x_1} + E^*(x_1 W^*(y) - iW_1(y))e^{-ik_o^* x_1} \\ & + \sum_{m=c}^{m\infty}\{E_m \Psi_m(y) e^{ik_m x_1} + E_m^* \Psi_m^*(y) e^{-ik_m^* x_1} + \hat{E}_m \hat{\Psi}_m(y) e^{i\hat{k}_m x_1} + \bar{E}_m \bar{\Psi}_m(y) e^{i\bar{k}_m x_1}\}\Big] \\ & + 2\varepsilon^{1/2} W(y)\cos\Big[\frac{(x_1 - L)}{L}q + c + k_o x\Big]\tanh\frac{\varepsilon^{1/2} x_1}{\sqrt{2}}\Big\} \end{aligned} \tag{8.6.2}$$

for x < 0, where Ψ_m, $\hat{\Psi}_m$ and $\bar{\Psi}_m$ are the vertical velocity eigenfunctions corresponding to the terms ψ_m, $\hat{\psi}_m$ and $\bar{\psi}_m$ in the solution for the temperature (8.4.4). Vertical velocity contours based on only the linear end-region solution and given in figures 36-40, show the flow near the endwall x=-L in greater detail. These flow patterns, and those of figures 31-35 correspond to the various points labelled in figure 30 and depict the transition in roll pattern. The solutions at A,B,C and D are stable and would be expected to be observed in the transition from 7 to 6 and then to 5 rolls as the Rayleigh number is gradually increased. The solution at E is unstable but is included for comparison. It corresponds to the situation where 6 rolls no longer fit comfortably into the box, with the two outermost rolls about to disappear. In practice the solution at E is avoided since a gradual increase in Rayleigh number is accompanied by the (time-dependent) transition from 6 rolls to 5 rolls at the point D (see Daniels (1984)).

Generally speaking, the results for $\tilde{\alpha}/|\beta|$ given in tables 30-33 (for various aspect ratios) predict that roll transitions occur for fluids of small Prandtl number and not for fluids of large Prandtl number, irrespective of the aspect ratio, and in cases where transitions occur, the number of rolls decreases with increasing Rayleigh number. Results for a = 2 are an exception, where an increase in the number of rolls is actually (marginally) predicted at large Prandtl numbers, but this may be due to difficulties associated with the accurate computation of $\tilde{k}_i$ i=1..4 at this comparatively large value of a (see section 7.4).

Results for the two-dimensional model (Cross et al (1983)) and experiments and numerical predictions by Oertel (1980) both indicate roll transitions at small Prandtl numbers and not at large Prandtl numbers, in line with the general trend of the present results. In Oertel's experiments it was observed that in a long rigid box (a=2 and L=5) containing silicone oil (P=1780) and with conducting lateral walls, the numbers of rolls remained constant as the Rayleigh number increased above it critical value of about 1800. When the silicone oil was replaced by nitrogen (P=0.71) the number of rolls decreased from 10 to 9 to 8 to 7 at values of R given by 2300, 5650 and 8900 respectively. Unfortunately, a proper comparison with Oertel's results is not possible for several reasons. Firstly, it has not been possible to obtain realiable results at the large values of a used in his experiments. Secondly, even if this had been possible, it seems likely that the present theory requires a $\ll \varepsilon^{-\frac{1}{4}}$ in which case a different approach is required at large values of a, unless ε is extremely small. Thirdly, of course, the present work is based on the assumption of stress-free horizontal boundaries, in contrast to the rigid boundaries of Oertel's experimental and numerical work. It is thought that some of these limitations can be removed in future work, but that, nevertheless the present results provide the first realistic predictions of wavelength selection in real three-dimensional flows. The general prediction of roll transitions at the smaller Prandtl numbers is encouraging and it is interesting to observe the borderline nature of the prediction in air (P=0.71) where roll transitions appear to be critically dependent on the value of the aspect ratio a.

Results for nitrogen (P=0.71) and silicone oil (P=1780) for a long box with a = 1 and L = 5 are shown in figures 26-29. For the conducting case, roll transitions in nitrogen (figure 26) are predicted to occur at values of the Rayleigh number given in table 34.

	No. of rolls	ε	$R = R_0 + \varepsilon/d_1$	R/R_0
R_0	7	0.	827.6	1.
1st transition	6	8.5	2310.5	2.79
2nd transition	5	41.0	7153.1	8.64

Table 34. Roll transition in nitrogen, P = 0.71, a = 1.0, L = 5.0.

It is believed that the present approach can be extended to the fully rigid case by a numerical treatment of the eigenvalue problem in the y,z plane and realistic comparisons with experiments would then be possible for long boxes with $a \lesssim 1$. The results described here for the stress-free case should provide a useful method of testing the numerical scheme.

sinusoidal curves - type 1 solutions

vertical dotted lines - type 2 solutions

sloping lines represent the value $2q/\varepsilon L$ for various ε as indicated

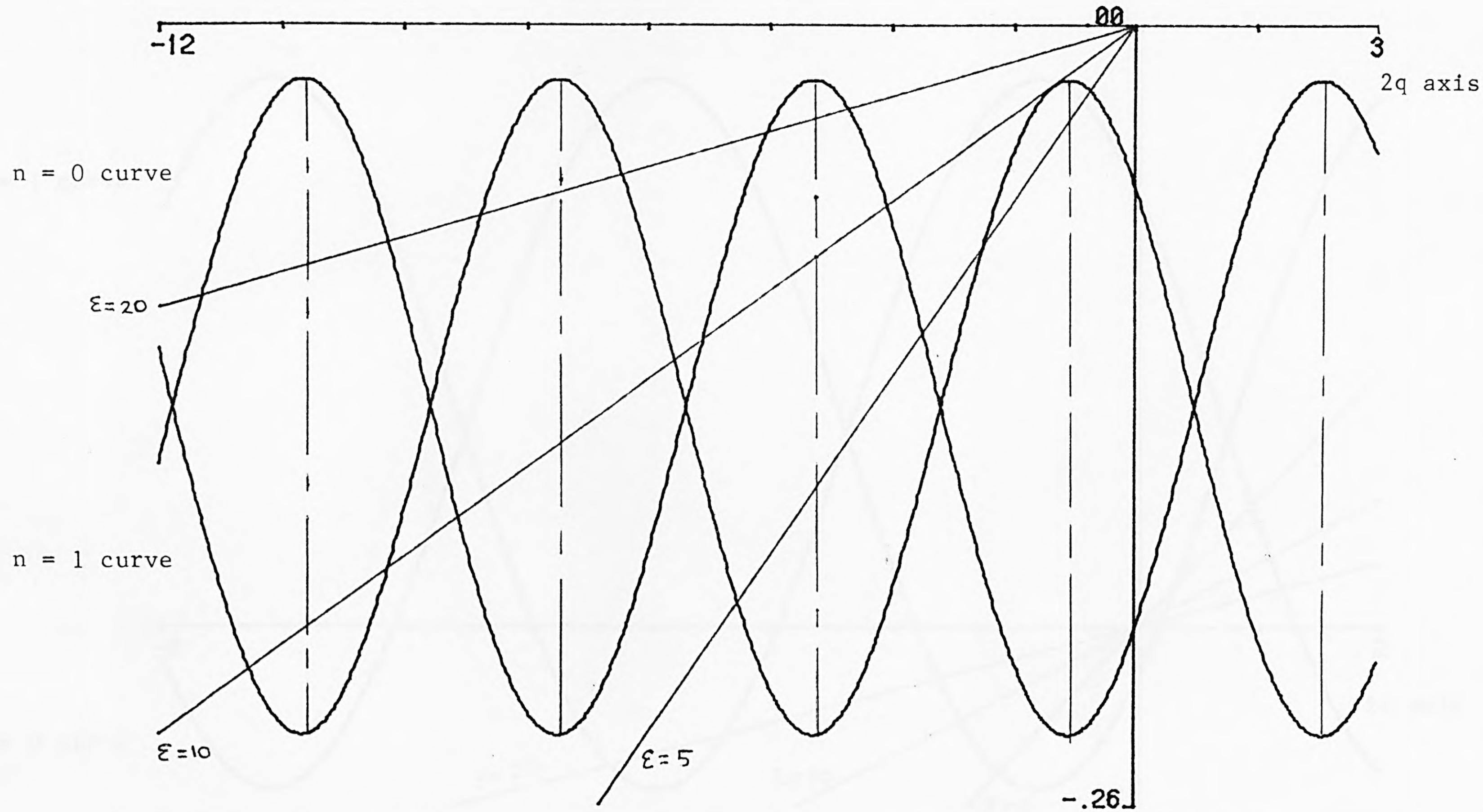


Figure 26. The two types of phase-winding solutions (8.5.1) and (8.5.6) for the case of perfectly conducting endwalls, aspect ratio a = 1, Prandtl number P = 0.71 and L = 5.

sinusoidal curves - type 1 solutions

sloping lines represent the value $2q/\varepsilon L$ for various ε as indicated

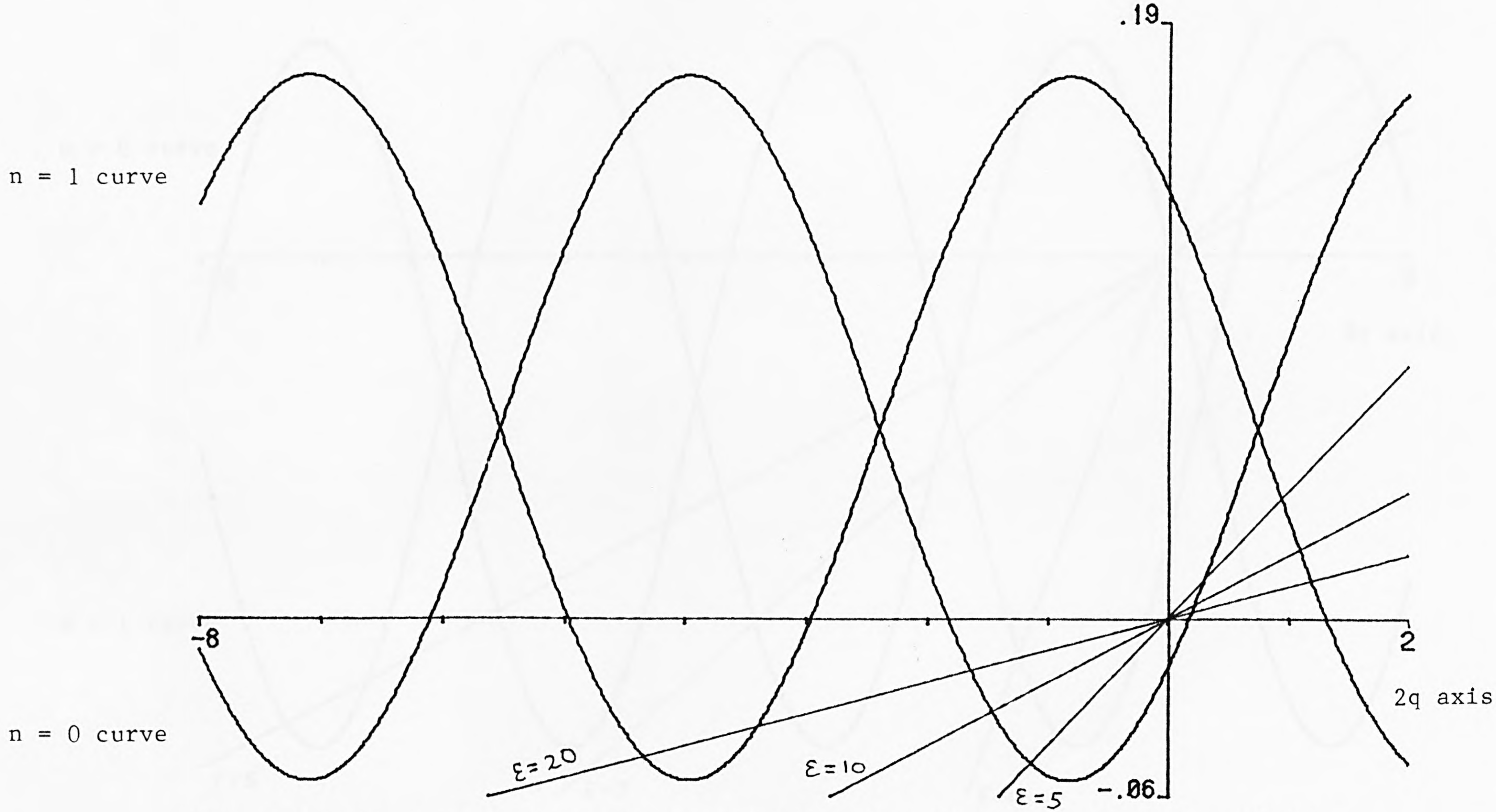


Figure 27. The type 1 phase-winding solutions (8.5.1) for the case of perfectly conducting endwalls, aspect ratio a = 1, Prandtl number P = 1780, L = 5.

sinusoidal curves - type 1 solutions

sloping lines represent the value $2q/\varepsilon L$ for various ε as indicated

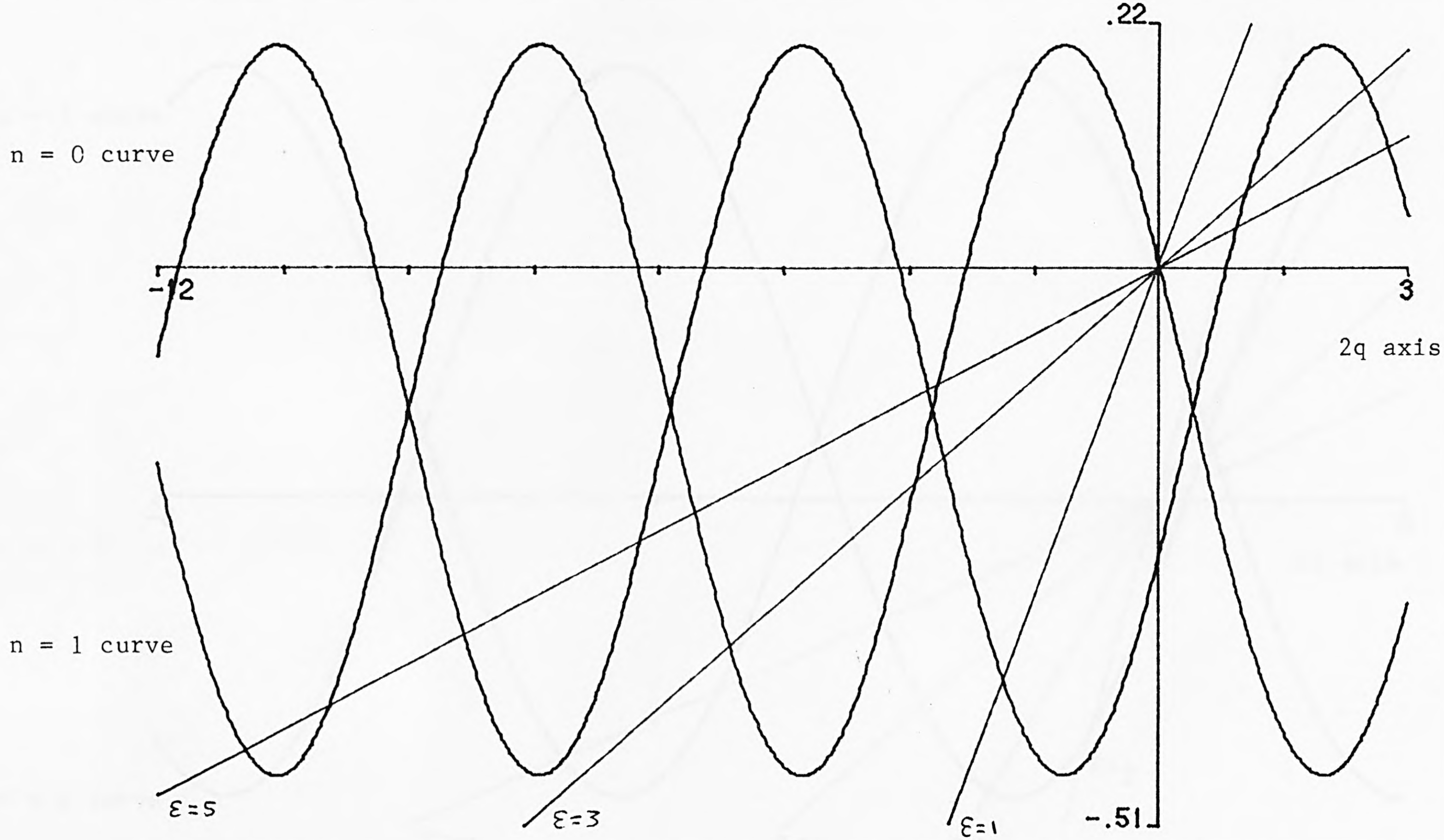


Figure 28. The type 1 solutions (8.5.1) for the case of perfectly insulating endwalls, aspect ratio a = 1, Prandtl number P = 0.71 and L = 5.

sinusoidal curves - type 1 solutions

sloping lines represent the value $2q/\varepsilon L$ for various ε as indicated

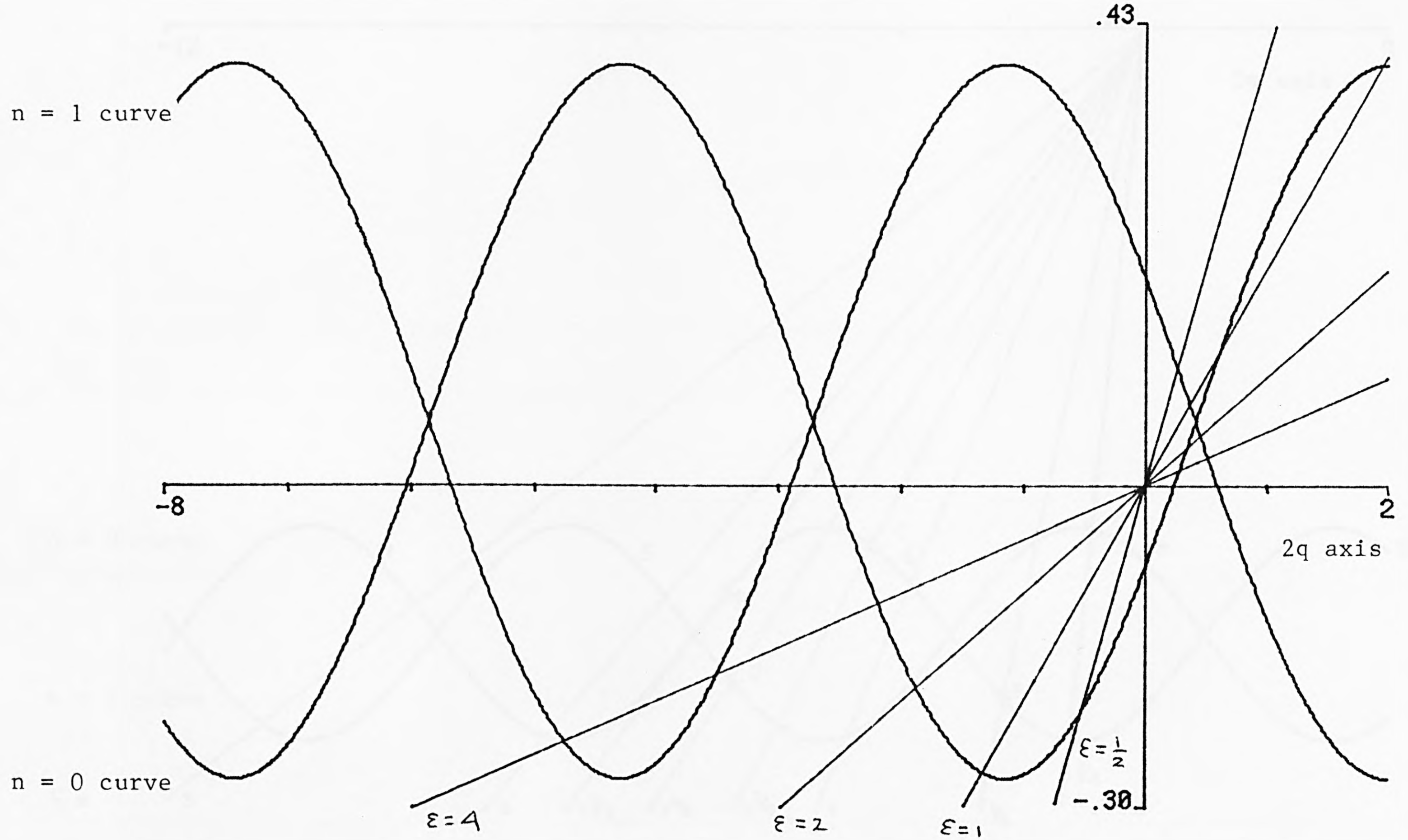


Figure 29. The type 1 phase-winding solutions (8.5.1) for the case of perfectly insulating endwalls, aspect ratio a = 1, Prandtl number P = 1780 and L = 5.

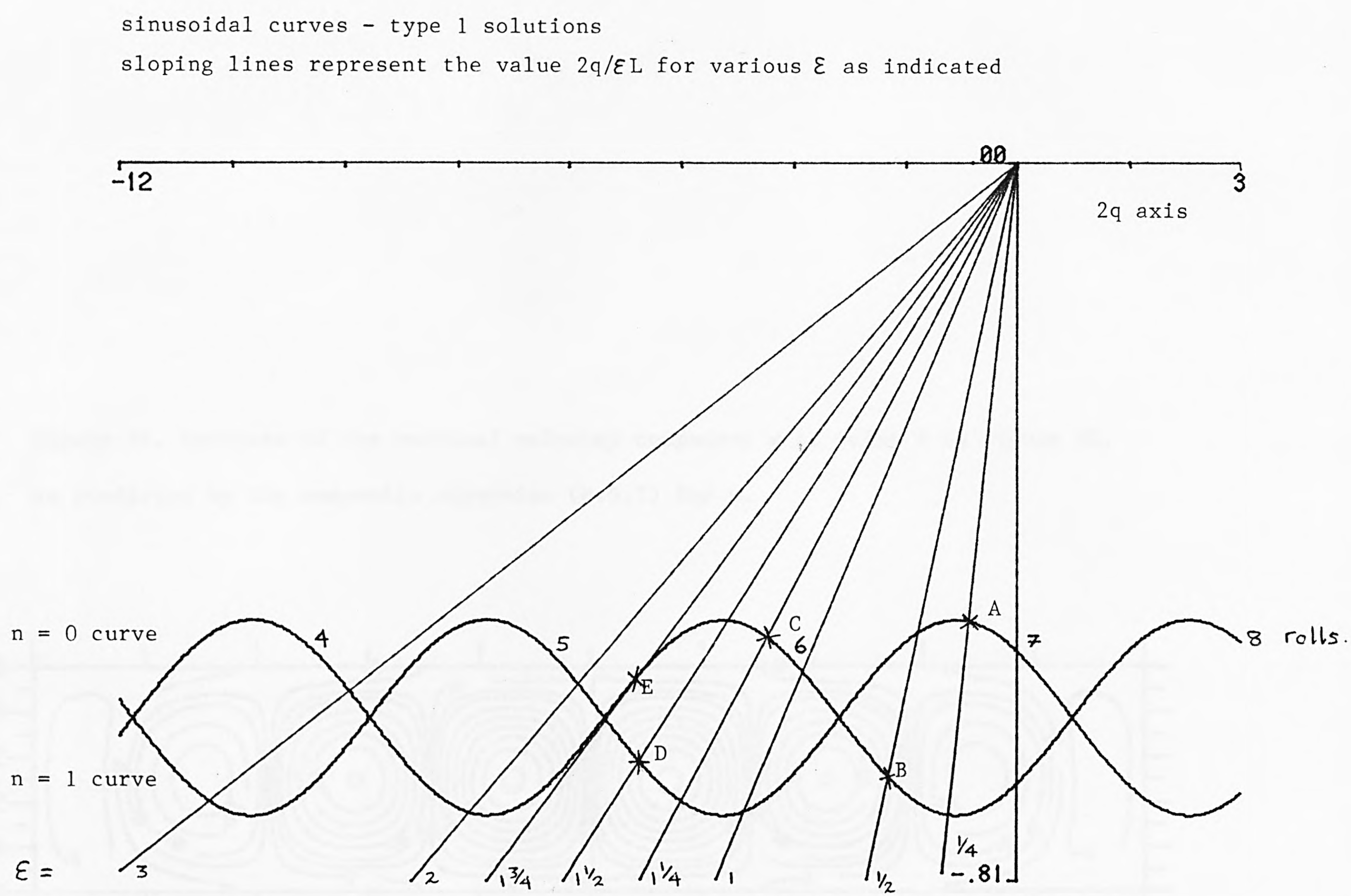


Figure 30. The type 1 phase-winding solutions (8.5.1) for the case of perfectly conducting endwalls, aspect ratio a = 1, Prandtl number P = 0.166 and L = 5.

Figure 31. Contours of the vertical velocity component w at point A in figure 30, as predicted by the composite expansion (8.6.2) for w.

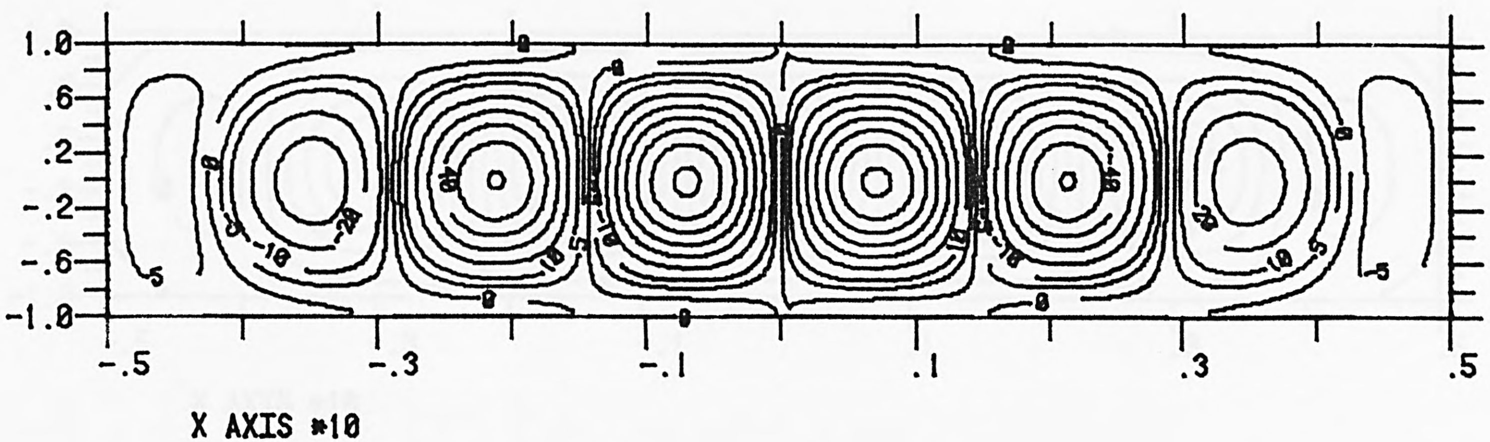


Figure 32. Contours of the vertical velocity component w at point B in figure 30, as predicted by the composite expansion (8.6.2) for w.

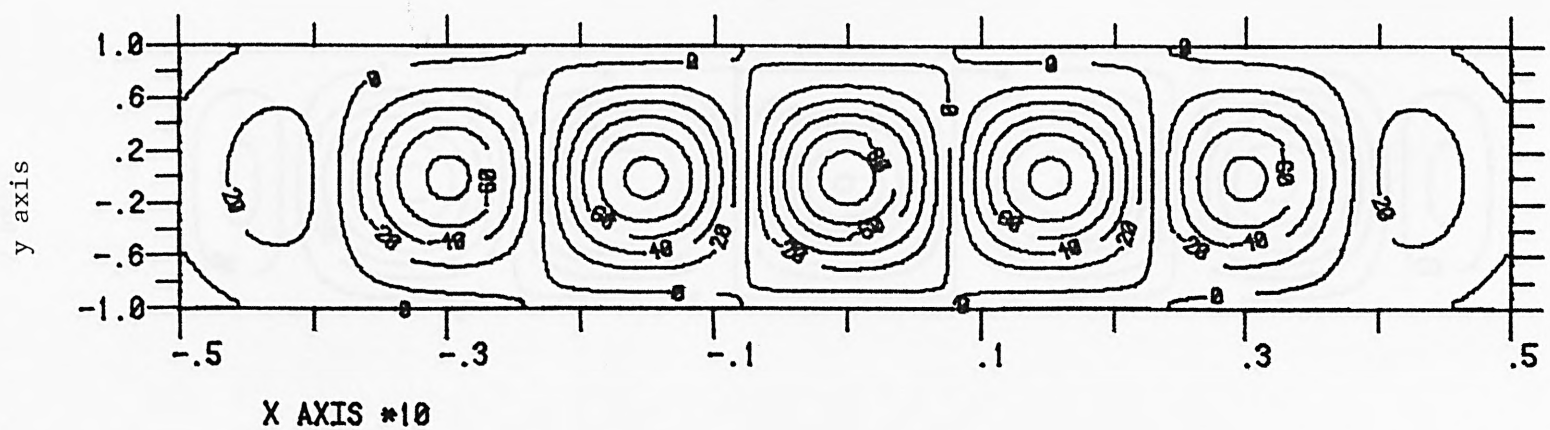


Figure 33. Contours of the vertical velocity component w at point C in figure 30, as predicted by the composite expansion (8.6.2) for w.

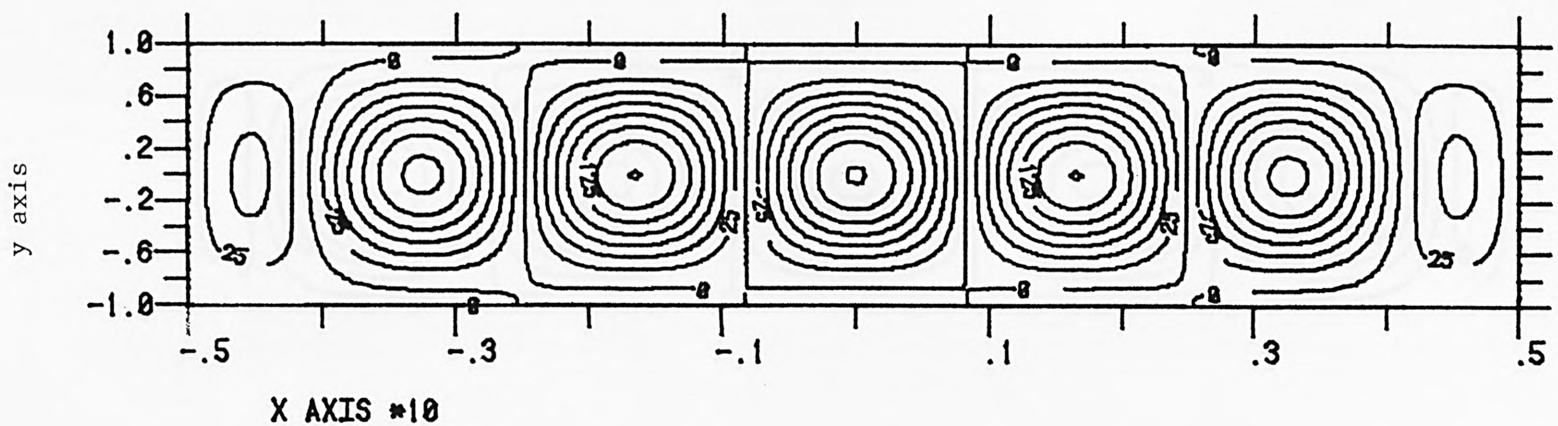


Figure 34. Contours of the vertical velocity w at point E in figure 30, as predicted by the composite expansion (8.6.2) for w.

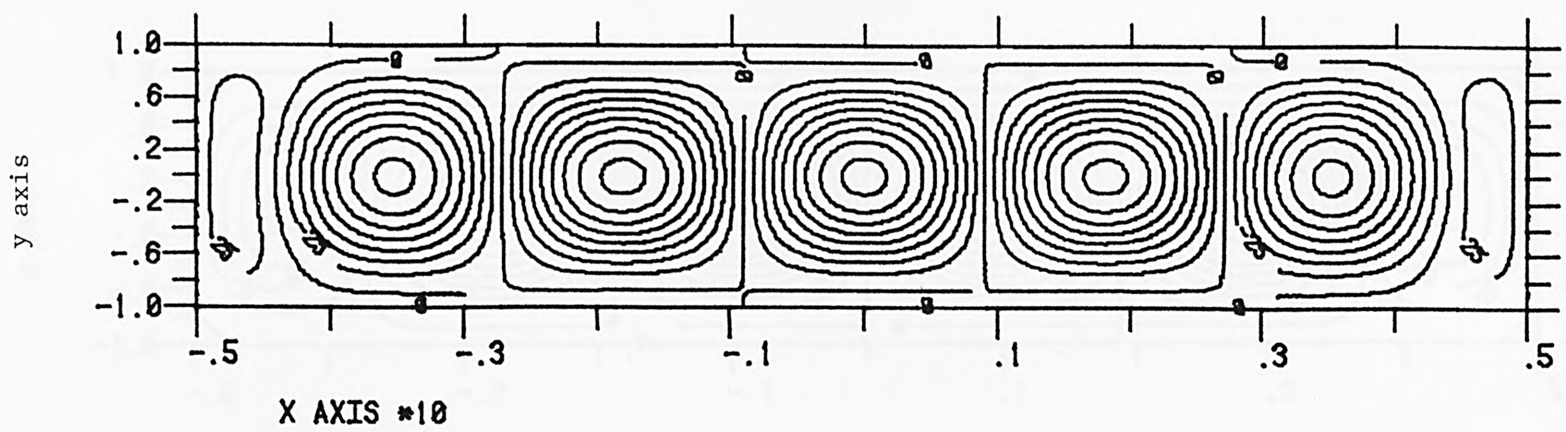


Figure 35. Contours of the vertical velocity component w at point D in figure 30, as predicted by the composite expansion (8.6.2) for w.

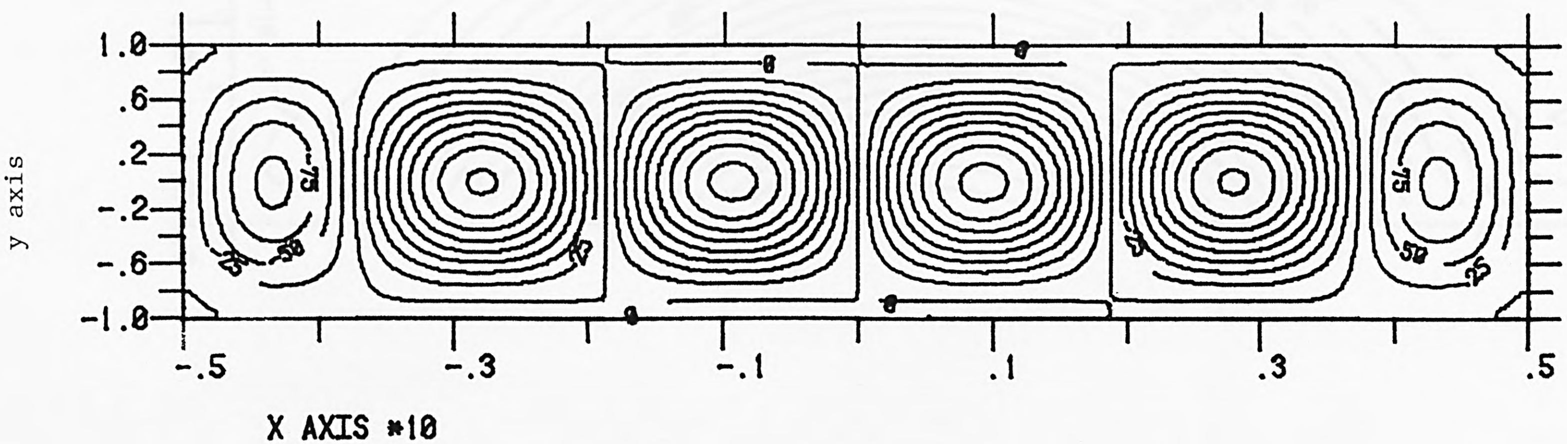


Figure 36. Contours of the vertical velocity component w at point A in figure 30, near the endwall $x_1=0$ based on the linear end-region solution for w.

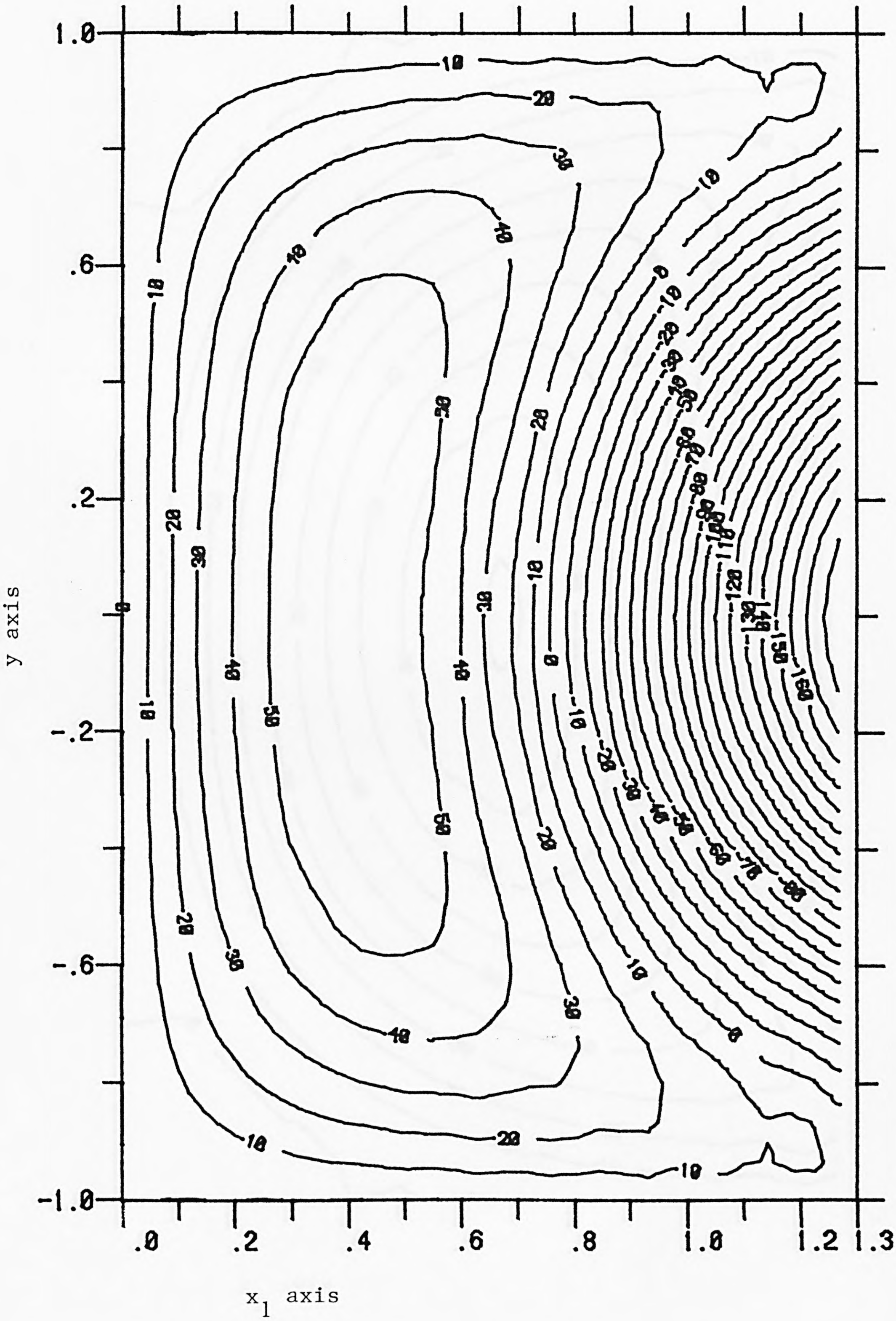


Figure 37. Contours of the vertical velocity component w at point B in figure 30, near the endwall $x_1=0$ based on the linear end-region solution for w.

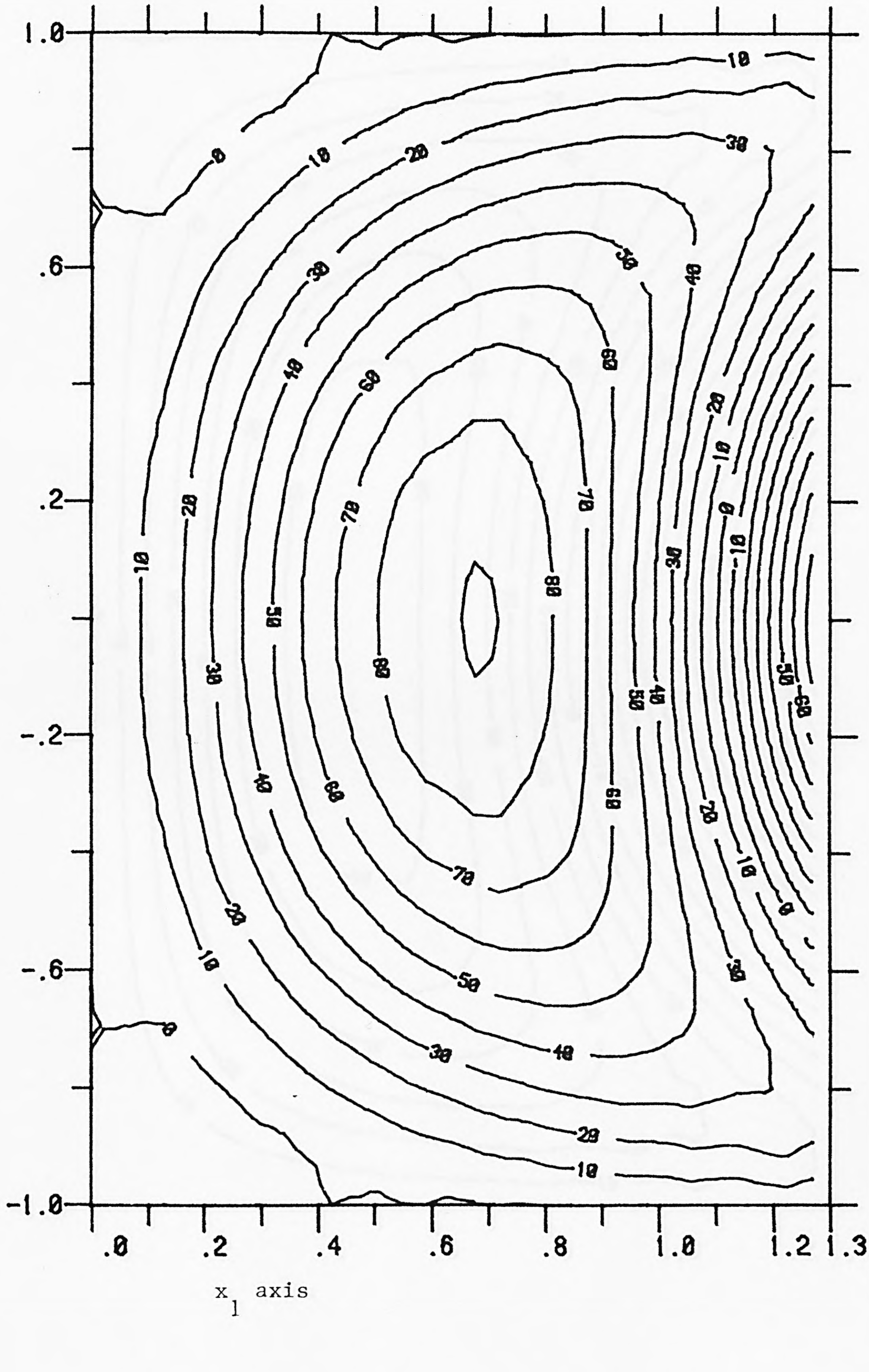


Figure 38. Contours of the vertical velocity component w at point C in figure 30, near the endwall $x_1=0$ based on the linear end-region solution for w.

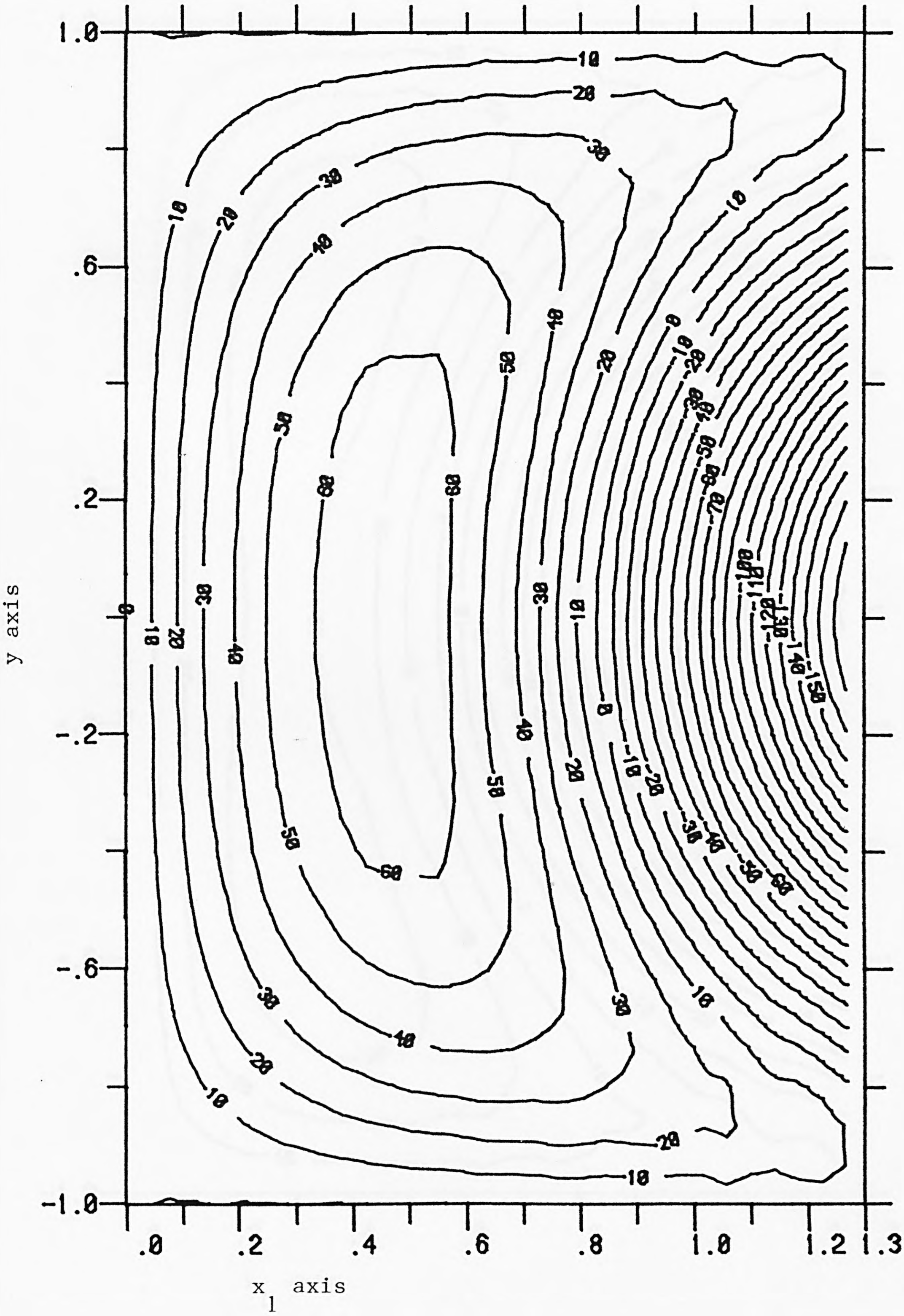


Figure 39. Contours of the vertical velocity component w at point E in figure 30, near the endwall at $x_1=0$ based on the linear end-region solution for w.

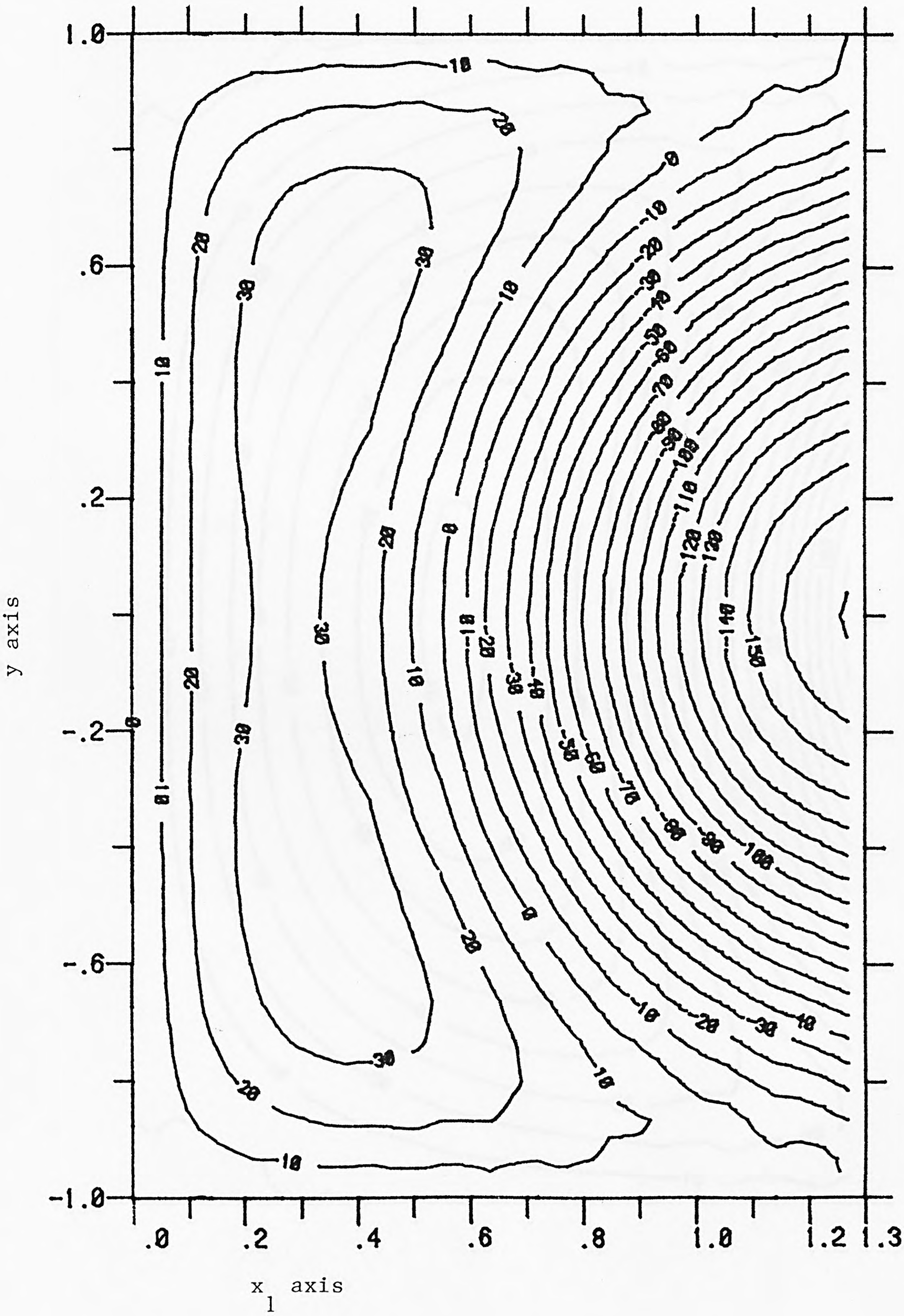
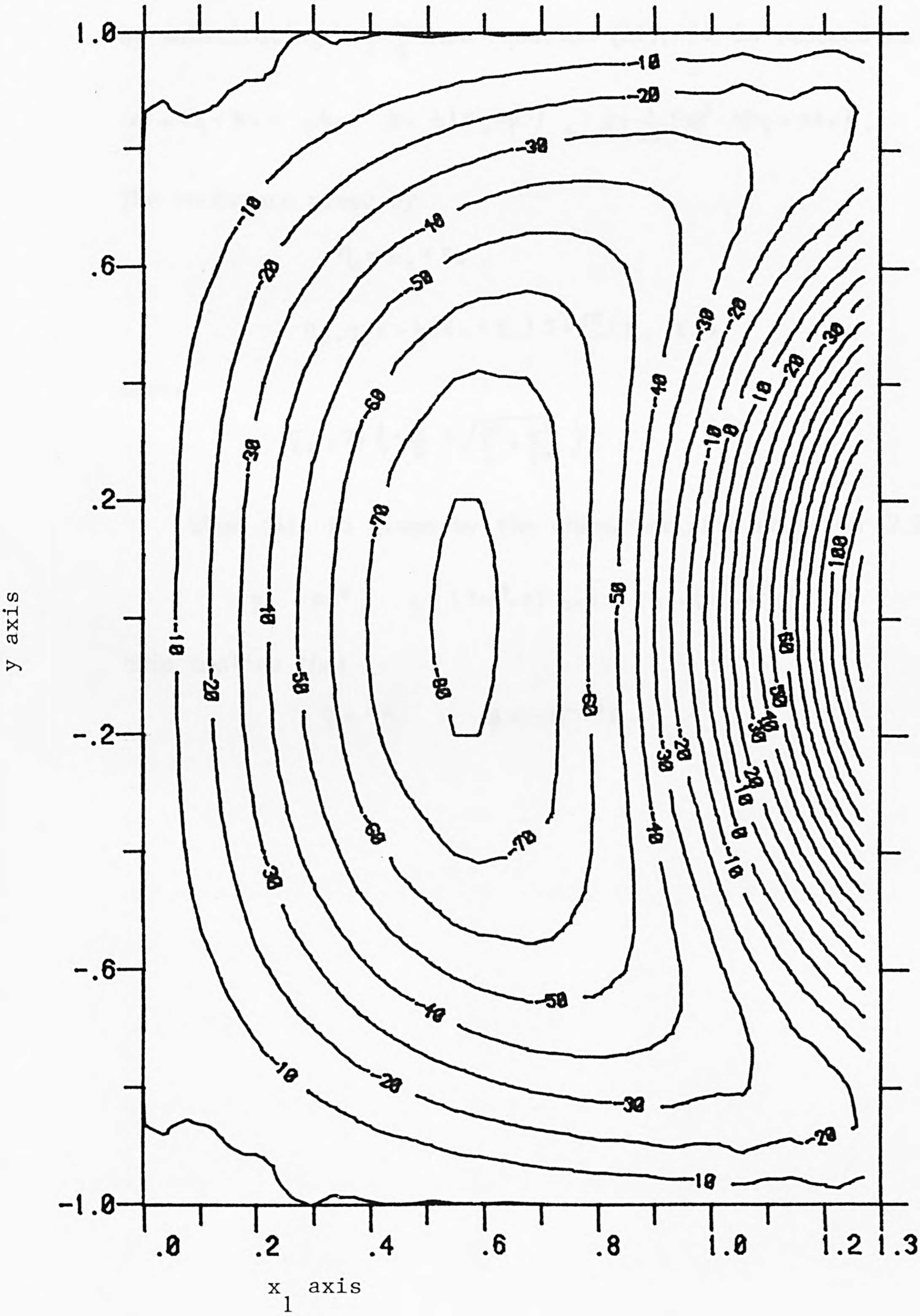


Figure 40. Contours of the vertical velocity component w at point D in figure 30, near the endwall $x_1 = 0$ based on the linear end-region solution for w.



Appendix A: Cardin's formula for the solution of a cubic equation

Consider the general form of a cubic equation, which is

$$y^3 + py^2 + qy + r = 0 . \quad \text{(A1)}$$

By substituting $y = \eta - \frac{p}{3}$ into equation (A1), it is reduced to the form

$$\eta^3 + \gamma\eta + \beta = 0 \quad \text{where} \quad \gamma = \tfrac{1}{3}(3q - p^2) \; , \quad \beta = \tfrac{1}{27}(2p^3 - 9pq + 27r) . \quad \text{(A2)}$$

The roots are given by

$$\eta_1 = E_+ + E_- , \quad \text{(A3)}$$

$$\eta_2, \eta_3 = -\tfrac{1}{2}(E_+ + E_-) \pm i\frac{\sqrt{3}}{2}(E_+ - E_-) \quad \text{(A4)}$$

where

$$E_\pm = \sqrt[3]{\left(-\frac{\beta}{2} \pm \sqrt{\frac{\beta^2}{4} + \frac{\gamma^3}{27}}\right)} .$$

When (A1) is given by the characteristic equation (2.2.12) where

$$p = -3\alpha^2 , \quad q = (3\alpha^4 - R) \text{ and } r = -(\alpha^6 - k^2 R) \quad \text{(A5)}$$

this implies that

$$\gamma = -R \quad , \quad \beta = -n^2\pi^2 R . \quad \text{(A6)}$$

Appendix B: Nonlinear solutions at order ε for the infinite stress-free channel

The complete solutions for all the dependent variables at order ε, for the infinite stress-free channel are stated below:

(1) The velocity component w_2 satisfies the sixth order differential equation

$$L_1 w_2 = -L_2 w_1 + \Big\{ [\bar{\alpha}_1 + \bar{\alpha}_2 P][\bar{B}^{*} e^{2ik_1 x} + c.c] + [\bar{\alpha}_3 + \bar{\alpha}_4 P][\bar{B}\bar{B}^{*}] \cos \frac{2\pi y}{\bar{a}} + [\bar{\alpha}_5 + \bar{\alpha}_6 P][\bar{A}\bar{B} e^{i(k_0+k_1)x} + c.c] \cos \frac{\pi}{\bar{a}}$$
$$+ [\bar{\alpha}_7 + \bar{\alpha}_8 P][\bar{A}\bar{B}^{*} e^{i(k_0-k_1)} + c.c] \cos \frac{\pi y}{\bar{a}} \Big\} \sin 2\pi z$$

and the coefficients are

$\bar{\alpha}_1 = 6 R_0 \pi^3 k_1^2 / \bar{a}^2$, $\quad \bar{\alpha}_2 = 108 \pi^5 k_1^2 (k_1^2 + \pi^2)/\bar{a}^2$, $\quad \bar{\alpha}_3 = 6 R_0 \pi^5 D^2 / \bar{a}^2$,

$\bar{\alpha}_4 = 108 \pi^9 D^2 (1 + 1/\bar{a}^2)/\bar{a}^2$, $\quad \bar{\alpha}_5 = 3 R_0 \pi^5 / \bar{a}^2$, $\quad \bar{\alpha}_6 = 27 \pi^7 \phi_+ / 2\bar{a}^2$,

$\bar{\alpha}_7 = 3 R_0 \pi^5 / \bar{a}^2$, $\quad \bar{\alpha}_8 = 27 \pi^7 \phi_- / 2\bar{a}^2$, (B1)

where

$$D = (1 - 2/\bar{a}^2)^{\frac{1}{2}}, \quad k_0 = \pi/\sqrt{2}, \quad k_1 = \frac{\pi}{\sqrt{2}}(1 - 2/\bar{a}^2)^{\frac{1}{2}},$$

$$\phi_\pm = (k_0 + k_1)^2 + 4\pi^2 + \pi^2/\bar{a}^2. \qquad \text{(B1a)}$$

Hence

$$w_2 = \Big\{ [\alpha_1 + \frac{\alpha_2}{P}][\bar{B}^2 e^{2ik_1 x} + c.c] + [\alpha_3 + \frac{\alpha_4}{P}][\bar{B}\bar{B}^{*}] \cos \frac{2\pi y}{\bar{a}} + [\alpha_5 + \frac{\alpha_6}{P}][\bar{A}\bar{B} e^{i(k_0+k_1)x} + c.c] \cos \frac{\pi y}{\bar{a}}$$
$$+ [\alpha_7 + \frac{\alpha_8}{P}][\bar{A}\bar{B}^{*} e^{i(k_0-k_1)x} + c.c] \cos \frac{\pi y}{\bar{a}} \Big\} \sin 2\pi z + \frac{3\pi^2}{2} \Big[(\bar{C}(\bar{x},\bar{t}) e^{ik_0 x} + c.c) + (\bar{D}(\bar{x},\bar{t}) e^{ik_1 x} + c.c) \cos \frac{\pi y}{\bar{a}} \Big] \sin \pi$$

and the coefficients are

$\alpha_1 = 3\pi^2 R_0 k_1^2 / 2\bar{a}^2 \phi_3$, $\quad \alpha_2 = 27\pi^5 k_1^2 (k_1^2 + \pi^2)/\bar{a}^2 \phi_3$, $\quad \alpha_3 = 3\pi^3 R_0 D^2 / 2\bar{a}^2 \phi_4$,

$\alpha_4 = 27\pi^7 D^2 (1 + 1/\bar{a}^2)/\bar{a}^2 \phi_4$, $\quad \alpha_5 = 3\pi^5 R_0 / \bar{a}^2 \phi_1$, $\quad \alpha_6 = 27\pi^7 \phi_+ / 2\bar{a}^2 \phi_1$,

$\alpha_7 = 3\pi^5 R_0 / \bar{a}^2 \phi_2$, $\quad \alpha_8 = 27\pi^7 \phi_- / 2\bar{a}^2 \phi_2$ (B2)

where $\phi_1 = R_0(\phi_+ - 4\pi^2) - \phi_+^3$, $\quad \phi_2 = R_0(\phi_- - 4\pi^2) - \phi_-^3$,

$$\phi_3 = R_0 k_1^2 - 16(k_1^2 + \pi^2)^3, \quad \phi_4 = \frac{R_0}{\bar{a}^2} - 16\pi^4 (1 + 1/\bar{a}^2)^3. \qquad \text{(B2a)}$$

(2) The temperature

$$\Theta_2 = \{[\tilde{\alpha}_1 + \frac{\tilde{\alpha}_2}{P}][\bar{B}^2 e^{2ik_1x} + c.c] + [\tilde{\alpha}_3 + \frac{\tilde{\alpha}_4}{P}][\bar{B}\bar{B}^*]\cos\frac{2\pi Y}{\bar{a}} + [\tilde{\alpha}_5 + \frac{\tilde{\alpha}_6}{P}][\bar{A}\bar{B}e^{i(k_0+k_1)x} + c.c]\cos\frac{\pi Y}{\bar{a}}$$

$$+ [\tilde{\alpha}_7 + \frac{\tilde{\alpha}_8}{P}][\bar{A}\bar{B}^* e^{i(k_0-k_1)x} + c.c]\cos\frac{\pi Y}{\bar{a}} + \tilde{\alpha}_9[\bar{A}\bar{A}^*] + \tilde{\alpha}_{10}[\bar{B}\bar{B}^*]\}\sin 2\pi z + \{\tilde{\alpha}_{11}[\bar{A}_{\bar{X}} e^{ik_0x} - c.c]$$

$$+ \tilde{\alpha}_{12}[\bar{B}_{\bar{X}} e^{ik_1x} - c.c]\cos\frac{\pi Y}{\bar{a}} + [\bar{C}e^{ik_0x} + c.c] + [\bar{D}e^{ik_1x} + c.c]\cos\frac{\pi Y}{\bar{a}}\}\sin\pi z .$$

and the coefficients are

$$\tilde{\alpha}_1 = 6\pi^3(k_1^2+\pi^2)^2/\bar{a}^2\phi_3 , \qquad \tilde{\alpha}_2 = 27\pi^5 k_1^2/4\bar{a}^2\phi_3 ,$$

$$\tilde{\alpha}_3 = 6\pi^5 D^2(1+1/\bar{a}^2)^2/\bar{a}^2\phi_4 , \qquad \tilde{\alpha}_4 = 27\pi^5 D^2/4\bar{a}^2\phi_4 ,$$

$$\tilde{\alpha}_5 = 3\pi^3(1-D)\phi_+^2/2\phi_1 , \qquad \tilde{\alpha}_6 = 27\pi^7/2\bar{a}^2\phi_1 ,$$

$$\tilde{\alpha}_7 = 3\pi^3(1+D)\phi_-^2/2\phi_2 , \qquad \tilde{\alpha}_8 = 27\pi^7/2\bar{a}^2\phi_2 , \qquad \tilde{\alpha}_9 = -3\pi/4 , \quad \tilde{\alpha}_{10} = -3\pi/8 , \tag{B3}$$

$$\tilde{\alpha}_{11} = 4ik_0/3\pi^2 , \qquad \tilde{\alpha}_{12} = 4ik_1/3\pi^2 .$$

(3) The pressure

$$p_2 = \{[\alpha_{14} + \alpha_{15}P][\bar{B}^2 e^{2ik_0x} + c.c] + [\alpha_{16} + \alpha_{17}P][\bar{B}\bar{B}^*]\cos\frac{2\pi Y}{\bar{a}} + [\alpha_{18} + \alpha_{19}P][\bar{A}\bar{B}e^{i(k_0+k_1)x} + c.c]\cos\frac{\pi Y}{\bar{a}}$$

$$+ [\alpha_{20} + \alpha_{21}P][\bar{A}\bar{B}^* e^{i(k_0-k_1)x} + c.c]\cos\frac{\pi Y}{\bar{a}} + [\alpha_{22} + \alpha_{23}P][\bar{A}\bar{A}^*] + [\alpha_{24} + \alpha_{25}P][\bar{B}\bar{B}^*]\}\cos 2\pi z$$

$$+ \{\alpha_{26}P[\bar{A}_{\bar{X}} e^{ik_0x} - c.c] + \alpha_{27}P[\bar{B}_{\bar{X}} e^{ik_1x} - c.c]\cos\frac{\pi Y}{\bar{a}} + \alpha_{28}P[(\bar{C}e^{ik_0x} + c.c) + (\bar{D}e^{ik_1x} + c.c)\cos\frac{\pi Y}{\bar{a}}]\}$$

$$\times \cos\pi z + F(x, y, \bar{X}, \bar{\tau})$$

and the coefficients are

$$\alpha_{14} = (9\pi^4/8\bar{a}^2) - (27\pi^4 k_1^2(R_0 - 16(k_1^2+\pi^2)^2)/8\bar{a}^2\phi_3) , \qquad \alpha_{15} = -3\pi^4 R_0(k_1^2+\pi^2)/\bar{a}^{-2}\phi_3 ,$$

$$\alpha_{16} = (9\pi^4 D^2/8) - (27\pi^4 D^2(R_0 - 16\pi^4(1+1/\bar{a}^2)^2)/8\bar{a}^2\phi_4) , \qquad \alpha_{17} = -3\pi^4 D^2 R_0(1+1/\bar{a}^2)/\phi_4 ,$$

$$\alpha_{18} = (9\pi^4(1-D)/8) - (27\pi^6(R_0 - \phi_+^2)/4\bar{a}^2\phi_1) , \qquad \alpha_{19} = -3\pi^4 R_0(1-D)\phi_+/\phi_1 ,$$

$$\alpha_{20} = (9\pi^4(1+D)/8) - (27\pi^6(R_0 - \phi_-^2)/4\bar{a}^2\phi_2) , \qquad \alpha_{21} = -3\pi^4 R_0(1+D)\phi_-/\phi_2 ,$$

$$\alpha_{22} = 9\pi^4/4 , \qquad \alpha_{23} = 3R_0/8 ,$$

$$\alpha_{24} = 9\pi^4/8\,, \qquad \alpha_{25} = 3R_o/16\,, \qquad \alpha_{26} = -12\pi i k_o\,,$$

$$\alpha_{27} = -12\pi i k_1\,, \qquad \alpha_{28} = -9\pi^3/2\,. \tag{B4}$$

In order to obtain the arbitrary function F, it is necessary to know the z independent terms on the right hand side of (3.1.41) which are

$$N_v = -\frac{9\pi^5}{4\bar{a}}\Big\{\Big[(\bar{B}^2 e^{2ik_1x} + c.c) + \frac{4}{\bar{a}^2}\bar{B}\bar{B}^*\Big]\sin\frac{2\pi y}{\bar{a}} + \Big[(1+D)(\bar{A}\bar{B}e^{i(k_o+k_1)x} + c.c) + (1-D)(\bar{A}\bar{B}^* e^{i(k_o-k_1)x} + c.c)\Big]\sin\frac{\pi y}{\bar{a}}\Big\} \tag{B5}$$

and

$$N_u = \frac{9\pi^4 i}{2}\Big\{k_o(\bar{A}^2 e^{2ik_ox} - c.c) + \frac{1}{2}k_1(\bar{B}^2 e^{2ik_1x} - c.c)\Big(1 - \frac{2}{\bar{a}^2} - \cos\frac{2\pi y}{\bar{a}}\Big) + \frac{k_o}{2}\Big[(1+D)^2(\bar{A}\bar{B}e^{i(k_o+k_1)x} - c.c) + (1-D)^2(\bar{A}\bar{B}^* e^{i(k_o-k_1)x} - c.c)\Big]\cos\frac{\pi y}{\bar{a}}\Big\}. \tag{B6}$$

Thus, from (3.1.45) the particular solution F_p for F is

$$F_p = \frac{9\pi^4}{2}\Big\{\frac{1}{\bar{a}^2}\bar{B}\bar{B}^*\cos\frac{2\pi y}{\bar{a}} + \frac{1}{2}(\bar{A}^2 e^{2ik_ox} + c.c) + \frac{1}{4}(\bar{B}^2 e^{2ik_1x} + c.c)\Big(1 - \frac{2}{\bar{a}^2} + \cos\frac{2\pi y}{\bar{a}}\Big) + \frac{1}{2}\Big[(1+D)(\bar{A}\bar{B}e^{i(k_o+k_1)x} + c.c) + (1-D)(\bar{A}\bar{B}^* e^{i(k_o-k_1)x} + c.c)\Big]\cos\frac{\pi y}{\bar{a}}\Big\}. \tag{B7}$$

(4) The velocity component in the y direction

$$v_2 = \Big\{\Big[\alpha_{29} + \frac{\alpha_{30}}{P}\Big][\bar{B}\bar{B}^*]\sin\frac{2\pi y}{\bar{a}} + \Big[\alpha_{31} + \frac{\alpha_{32}}{P}\Big][\bar{A}\bar{B}e^{i(k_o+k_1)x} + c.c]\sin\frac{\pi y}{\bar{a}} + \Big[\alpha_{33} + \frac{\alpha_{34}}{P}\Big][\bar{A}\bar{B}^* e^{i(k_o-k_1)x} + c.c]\sin\frac{\pi}{\bar{a}}\Big\}\cos 2\pi z + \Big\{\alpha_{35}[\bar{B}_{\bar{x}} e^{ik_1x} - c.c]\sin\frac{\pi y}{\bar{a}} + \alpha_{36}[\bar{D}e^{ik_1x} + c.c]\sin\frac{\pi y}{\bar{a}}\Big\}\cos\pi z$$

and the coefficients are

$$\alpha_{29} = -3\pi^3 D^2 R_o/2\bar{a}\phi_4\,, \qquad \alpha_{30} = -27\pi^7 D^2(1+1/\bar{a}^2)/\bar{a}\phi_4\,, \qquad \alpha_{31} = -3\pi^4 R_o(1-D)/\bar{a}\phi_1\,,$$

$$\alpha_{32} = -27\pi^7\phi_+(1-D)/2\bar{a}\phi_1\,, \qquad \alpha_{33} = -3\pi^4 R_o(1+D)/\bar{a}\phi_2\,, \qquad \alpha_{34} = -27\pi^7\phi_-(1+D)/2\bar{a}\phi_2\,,$$

$$\alpha_{35} = -12ik_1/\bar{a}\,, \qquad \alpha_{36} = -3\pi^2/\bar{a}\,. \tag{B8}$$

(5) The velocity component in the x direction

$$u_2 = \Big\{\Big[\alpha_{37} + \frac{\alpha_{38}}{P}\Big][\bar{B}^2 e^{2ik_1x} - c.c] + \Big[\alpha_{39} + \frac{\alpha_{40}}{P}\Big][\bar{A}\bar{B}e^{i(k_o+k_1)x} - c.c]\cos\frac{\pi y}{\bar{a}} + \Big[\alpha_{41} + \frac{\alpha_{42}}{P}\Big][\bar{A}\bar{B}^* e^{i(k_o-k_1)x} - c.c]\cos\frac{\pi y}{\bar{a}}\Big\}\cos 2\pi z + \Big\{\alpha_{43}[\bar{A}_{\bar{x}} e^{ik_ox} + c.c] + \alpha_{44}[\bar{B}_{\bar{x}} e^{ik_1x} + c.c]\cos\frac{\pi y}{\bar{a}} + \alpha_{45}[\bar{C}e^{ik_ox} - c.c] + \alpha_{46}[\bar{D}e^{ik_1x} - c.c]\cos\frac{\pi y}{\bar{a}}\Big\}\cos\pi z$$

and the coefficients are

$$\alpha_{37} = 3\pi^4 ik_1 R_0 / 2\bar{a}^2\phi_3 , \qquad \alpha_{38} = 27\pi^6 ik_1(k_1^2+\pi^2)/\bar{a}^2\phi_3 ,$$

$$\alpha_{39} = 6\pi^4 ik_0 R_c/\bar{a}^2\phi_1 , \qquad \alpha_{40} = 27\pi^6 ik_0\phi_+/\bar{a}^2\phi_1 ,$$

$$\alpha_{41} = 6\pi^4 ik_0 R_c/\bar{a}^2\phi_2 , \qquad \alpha_{42} = 27\pi^6 ik_0\phi_-/\bar{a}^2\phi_2 ,$$

$$\alpha_{43} = (-12k_0^2/\pi) + 3\pi , \qquad \alpha_{44} = (-12k_1^2/\pi) + 3\pi , \qquad \text{(B9)}$$

$$\alpha_{45} = 3\pi ik_0 , \qquad \alpha_{46} = 3\pi ik_1 .$$

Appendix C: Nonlinear expansion for the rigid channel

Expanding the four nonlinear terms on the right hand side of equation (4.3.7) gives

$$\underline{u}_1 \cdot \underline{\nabla}\theta_1 = [\alpha_1(y)(\bar{A}^2 e^{2ik_0x} + c.c) + \alpha_2(y)(2\bar{A}\bar{A}^*)] \sin 2\pi z, \quad \text{(C1)}$$

$$\underline{u}_1 \cdot \underline{\nabla} w_1 = [\alpha_3(y)(\bar{A}^2 e^{2ik_0x} + c.c) + \alpha_4(y)(2\bar{A}\bar{A}^*)] \sin 2\pi z, \quad \text{(C2)}$$

$$\underline{u}_1 \cdot \underline{\nabla} u_1 = i[\alpha_5(y)\cos 2\pi z + \alpha_6(y)](\bar{A}^2 e^{2ik_0x} - c.c) \quad \text{(C3)}$$

and

$$\underline{u}_1 \cdot \underline{\nabla} v_1 = [\alpha_7(y)\cos 2\pi z + \alpha_8(y)](\bar{A}^2 e^{2ik_0x} + c.c) + [\alpha_9(y)\cos 2\pi z + \alpha_{10}(y)](2\bar{A}\bar{A}^*) \quad \text{(C4)}$$

where

$$\alpha_1 = \tfrac{1}{2}[-U\Theta + V\Theta' + \pi W\Theta], \qquad \alpha_2 = \tfrac{1}{2}[U\Theta + V\Theta' + \pi W\Theta],$$

$$\alpha_3 = \tfrac{1}{2}[-UW + VW' + \pi WW], \qquad \alpha_4 = \tfrac{1}{2}[UW + VW' + \pi WW],$$

$$\alpha_5 = \tfrac{1}{2k_0}[-UU + VU' + \pi WU], \qquad \alpha_6 = \tfrac{1}{2k_0}[-UU + VU' - \pi WU],$$

$$\alpha_7 = \tfrac{1}{2}[-UV + VV' + \pi WV], \qquad \alpha_8 = \tfrac{1}{2}[-UV + VV' - \pi WV],$$

$$\alpha_9 = \tfrac{1}{2}[UV + VV' + \pi WV], \qquad \alpha_{10} = \tfrac{1}{2}[UV + VV' - \pi WV], \quad \text{(C5)}$$

where Θ, V, W and U are given by (4.1.3) and prime denotes $\frac{d}{dy}$.

The column vector on the right hand side of (4.3.57a) is

$$[S_1(a), S_2(a), S_3(a), S_4(a)]^{tr}$$

where

$$S_1(a) = k_0 \sum_{j=1}^{3} \frac{d_j}{r_j} a \sinh r_j a, \qquad S_2(a) = k_0 \sum_{j=1}^{3} \frac{d_j}{r_j}(r_j^2 - \alpha^2) a \sinh r_j a,$$

$$S_3(a) = 2k_0 \sum_{j=1}^{3} \frac{d_j \beta_j r_j}{(r_j^2 - \alpha^2)} \cosh r_j a + \frac{k_0 d_4}{\alpha}(\cosh \alpha a + \alpha a \sinh \alpha a) - \frac{4k_0}{\pi} \sum_{j=1}^{3} d_j r_j^2 \cosh r_j a$$
$$+ k_0 \sum_{j=1}^{3} \frac{d_j}{r_j}\{\eta_j r_j \cosh r_j a + \beta_j(\cosh r_j a + r_j a \sinh r_j a - \frac{2r_j^2}{(r_j^2 - \alpha^2)} \cosh r_j a\}$$

and

$$S_4(a) = k_0 \sum_{j=1}^{3} \frac{d_j \beta_j}{(r_j^2 - \alpha^2)} \sinh r_j a + \frac{k_0 d_4}{\alpha} a \cosh \alpha a - \frac{4k_0}{\pi} \sum_{j=1}^{3} d_j r_j \sinh r_j a$$
$$+ k_0 \sum_{j=1}^{3} \frac{d_j}{r_j}\{\eta_j \sinh r_j a + \beta_j(a \cosh r_j a - \frac{2r_j}{(r_j^2 - \alpha^2)} \sinh r_j a)\}. \quad \text{(C6)}$$

In equations (4.4.15,18) the y-dependent functions $\varepsilon_i(y)$ i=1..12 are

$$\varepsilon_1=\hat{\varepsilon}_1+p\bar{\varepsilon}_1=\tfrac{1}{2}[2U\hat{\Theta}_2+V(\hat{\Theta}_2'+2\hat{\Theta}_3')-2\pi W(\hat{\Theta}_2+2\hat{\Theta}_3)]+\tfrac{p}{2}[2U\bar{\Theta}_2+V(\bar{\Theta}_2'+2\bar{\Theta}_3')-2\pi W(\bar{\Theta}_2+2\bar{\Theta}_3)],$$

$$\varepsilon_2=\hat{\varepsilon}_2+\tfrac{1}{p}\tilde{\varepsilon}_2=\tfrac{1}{2}[2U\hat{W}_2+V(\hat{W}_2'+2\hat{W}_3')-2\pi W(\hat{W}_2+2\hat{W}_3)]+\tfrac{1}{2p}[2U\tilde{W}_2+V(\tilde{W}_2'+2\tilde{W}_3')-2\pi W(\tilde{W}_2+2\tilde{W}_3)],$$

$$\varepsilon_3=\hat{\varepsilon}_3+p\bar{\varepsilon}_3=\tfrac{1}{2}[2U(\hat{U}_2+2U_4)+V(\hat{U}_2'+2U_4')-2\pi W\hat{U}_2]+\tfrac{p}{2}[2U\bar{U}_2+V\bar{U}_2'-2\pi W\bar{U}_2],$$

$$\varepsilon_4=\hat{\varepsilon}_4+p\bar{\varepsilon}_4=\tfrac{1}{2}[2U(\hat{V}_2+2V_4)+V(\hat{V}_2'+2\hat{V}_3'+2V_4')-2\pi W(\hat{V}_2+2\hat{V}_3)]+\tfrac{p}{2}[2U\bar{V}_2+V(\bar{V}_2'+2\bar{V}_3')-2\pi W(\bar{V}_2+2\bar{V}_3)],$$

$$\varepsilon_5=\hat{\varepsilon}_5+p\bar{\varepsilon}_5=-\tfrac{1}{2}[k_o\Theta(\hat{U}_2-2U_4)+\Theta'[\hat{V}_2+2\hat{V}_3-2V_4]]-\tfrac{p}{2}[k_o\Theta\bar{U}_2+\Theta'(\bar{V}_2+2\bar{V}_3)],$$

$$\varepsilon_6=\hat{\varepsilon}_6+\tfrac{1}{p}\tilde{\varepsilon}_6=[\tfrac{\pi}{2}\Theta(\hat{W}_2+2\hat{W}_3)]+\tfrac{1}{p}[\tfrac{\pi}{2}\Theta(\tilde{W}_2+2\tilde{W}_3)],$$

$$\varepsilon_7=\hat{\varepsilon}_7+p\bar{\varepsilon}_7=-\tfrac{1}{2}[k_oW(\hat{U}_2-2U_4)+W'(\hat{V}_2+2\hat{V}_3-2V_4)]-\tfrac{p}{2}[k_oW\bar{U}_2+W'(\bar{V}_2+2\bar{V}_3)],$$

$$\varepsilon_8=\hat{\varepsilon}_8+\tfrac{1}{p}\tilde{\varepsilon}_8=[\tfrac{\pi}{2}W(\hat{W}_2+2\hat{W}_3)]+\tfrac{1}{p}[\tfrac{\pi}{2}W(\tilde{W}_2+2\tilde{W}_3)],$$

$$\varepsilon_9=\hat{\varepsilon}_9+p\bar{\varepsilon}_9=-\tfrac{1}{2}[U(\hat{U}_2+2U_4)+\tfrac{U'}{k_o}(\hat{V}_2-2\hat{V}_3+2V_4)]-\tfrac{p}{2}[U\bar{U}_2+\tfrac{U'}{k_o}(\bar{V}_2-2\bar{V}_3)],$$

$$\varepsilon_{10}=\hat{\varepsilon}_{10}+\tfrac{1}{p}\tilde{\varepsilon}_{10}=[\tfrac{\pi}{2k_o}U(\hat{W}_2-2\hat{W}_3)]+\tfrac{1}{p}[\tfrac{\pi}{2k_o}U(\tilde{W}_2-2\tilde{W}_3)],$$

$$\varepsilon_{11}=\hat{\varepsilon}_{11}+p\bar{\varepsilon}_{11}=\tfrac{1}{2}[k_oV(\hat{U}_2+2U_4)+V'(\hat{V}_2+2V_4+2\hat{V}_3)]+\tfrac{p}{2}[k_oV\bar{U}_2+V'(\bar{V}_2+2\bar{V}_3)],$$

and

$$\varepsilon_{12}=\hat{\varepsilon}_{12}+\tfrac{1}{p}\tilde{\varepsilon}_{12}=-[\tfrac{\pi}{2}V(\hat{W}_2+2\hat{W}_3)]-\tfrac{1}{p}[\tfrac{\pi}{2}V(\tilde{W}_2+2\tilde{W}_3)]. \tag{C7}$$

Appendix D: The generalized solvability condition

Let $\underline{f}$ be a column vector of eight elements and let $\underline{E}$ be an 8 x 8 matrix the elements of which may depend on the real variable y. Suppose that the problem

$$[\frac{d}{dy} - \underline{E}]\underline{f} = \underline{0}, \quad \underline{m}\underline{f} = \underline{0} \quad y = \pm a$$

where

$$\underline{m} = \begin{bmatrix} \underline{0}, & \underline{0} \\ \underline{0}, & \underline{I}_4 \end{bmatrix}, \tag{D1}$$

has just one non-trivial solution. Then it can be shown that the adjoint problem

$$[\frac{d}{dy} + \underline{E}^{tr}]\hat{\underline{f}} = \underline{0}, \quad \hat{\underline{m}}\hat{\underline{f}} = \underline{0} \quad y = \pm a$$

where

$$\hat{\underline{m}} = \begin{bmatrix} \underline{I}_4, & \underline{0} \\ \underline{0}, & \underline{0} \end{bmatrix}, \tag{D2}$$

also has just one non-trivial solution. A condition is required for the problem

$$[\frac{d}{dy} - \underline{E}]\underline{f} = \underline{\varphi}, \quad \underline{m}\underline{f} = 0 \quad y = \pm a, \tag{D3}$$

to have a solution. Premultplying equation (D3) by $\hat{\underline{f}}^{tr}$ and adding to the transpose of equation (D2) post-multiplied by $\underline{f}$ gives

$$\frac{d}{dy}[\hat{\underline{f}}^{tr}\underline{f}] = \hat{\underline{f}}^{tr}\underline{\varphi} \tag{D4}$$

which implies

$$[\hat{\underline{f}}^{tr}\underline{f}]_{-a}^{+a} = \int_{y=-a}^{+a} \hat{\underline{f}}^{tr}\underline{\varphi}\, dy. \tag{D5}$$

But the first four components of $\hat{\underline{f}}^{tr}$ and the last four components of $\underline{f}$ are zero at y =± a. Hence

$$0 = \int_{y=-a}^{+a} \hat{\underline{f}}^{tr}\underline{\varphi}\, dy. \tag{D6}$$

This is a necessary condition for (D3) to have a solution

Appendix E: Higher order nonlinear terms

In chapter 6 the y-dependent functions ε_i i=1..10,14..106 are as follows:

$$\varepsilon_{1,2} = \hat{\varepsilon}_{1,2} = \tfrac{1}{2}[\mp U\Theta + V\Theta' + \pi W\Theta],$$

$$\varepsilon_{3,4} = \hat{\varepsilon}_{3,4} = \tfrac{1}{2}[\mp U\Theta_1 + V\Theta_1' + \pi W\Theta_1],$$

$$\varepsilon_{5,6} = \hat{\varepsilon}_{5,6} + P\bar{\varepsilon}_{5,6} = [\mp U\hat{\Theta}_2 + \tfrac{1}{2}V\hat{\Theta}_2' + \pi W\hat{\Theta}_2] + P[\mp U\bar{\Theta}_2 + \tfrac{1}{2}V\bar{\Theta}_2' + \pi W\bar{\Theta}_2],$$

$$\varepsilon_{7,8} = \hat{\varepsilon}_{7,8} + P\bar{\varepsilon}_{7,8} = [\mp U\hat{\Theta}_2 + \tfrac{1}{2}V\hat{\Theta}_2' - \pi W\hat{\Theta}_2] + P[\mp U\bar{\Theta}_2 + \tfrac{1}{2}V\bar{\Theta}_2' - \pi W\bar{\Theta}_2],$$

$$\varepsilon_{9,10} = \hat{\varepsilon}_{9,10} + P\bar{\varepsilon}_{9,10} = [\tfrac{1}{2}V\hat{\Theta}_3' \pm \pi W\hat{\Theta}_3] + P[\tfrac{1}{2}V\bar{\Theta}_3' \pm \pi W\bar{\Theta}_3],$$

$$\varepsilon_{14,15} = \hat{\varepsilon}_{14,15} = \tfrac{1}{2}[\mp UW + VW' + \pi W^2],$$

$$\varepsilon_{16,17} = \hat{\varepsilon}_{16,17} = \tfrac{1}{2}[\mp UW_1 + VW_1' + \pi WW_1],$$

$$\varepsilon_{18,19} = \hat{\varepsilon}_{18,19} + \tfrac{1}{P}\tilde{\varepsilon}_{18,19} = [\mp U\hat{W}_2 + \tfrac{1}{2}V\hat{W}_2' + \pi W\hat{W}_2] + \tfrac{1}{P}[\mp U\tilde{W}_2 + \tfrac{1}{2}V\tilde{W}_2' + \pi W\tilde{W}_2],$$

$$\varepsilon_{20,21} = \hat{\varepsilon}_{20,21} + \tfrac{1}{P}\tilde{\varepsilon}_{20,21} = [\mp U\hat{W}_2 + \tfrac{1}{2}V\hat{W}_2' - \pi W\hat{W}_2] + \tfrac{1}{P}[\mp U\tilde{W}_2 + \tfrac{1}{2}V\tilde{W}_2' - \pi W\tilde{W}_2],$$

$$\varepsilon_{22,23} = \hat{\varepsilon}_{22,23} + \tfrac{1}{P}\tilde{\varepsilon}_{22,23} = [\tfrac{1}{2}V\hat{W}_3' \pm \pi W\hat{W}_3] + \tfrac{1}{P}[\tfrac{1}{2}V\tilde{W}_3' \pm \pi W\tilde{W}_3],$$

$$\varepsilon_{24,25} = \hat{\varepsilon}_{24,25} = \frac{1}{2k_0^2}[\mp U^2 + VU' + \pi WU],$$

$$\varepsilon_{26,27} = \hat{\varepsilon}_{26,27} = \frac{1}{2k_c^2}[\mp U^2 + VU' - \pi WU],$$

$$\varepsilon_{28,29} = P\bar{\varepsilon}_{28,29} = \frac{P}{2}[\mp UU_1 + VU_1' + \pi WU_1],$$

$$\varepsilon_{30,31} = P\bar{\varepsilon}_{30,31} = \frac{P}{2}[\mp UU_1 + VU_1' - \pi WU_1],$$

$$\varepsilon_{32,33} = \hat{\varepsilon}_{32,33} + P\bar{\varepsilon}_{32,33} = [\mp U\hat{U}_2 + \tfrac{1}{2}V\hat{U}_2' + \pi W\hat{U}_2] + P[\mp U\bar{U}_2 + \tfrac{1}{2}V\bar{U}_2' + \pi W\bar{U}_2],$$

$$\varepsilon_{34,35} = \hat{\varepsilon}_{34,35} + P\bar{\varepsilon}_{34,35} = [\mp U\hat{U}_2 \mp UU_4 + \tfrac{1}{2}V\hat{U}_2' + VU_4' - \pi W\hat{U}_2] + P[\mp U\bar{U}_2 + \tfrac{1}{2}V\bar{U}_2' - \pi W\bar{U}_2],$$

$$\varepsilon_{36,37} = \hat{\varepsilon}_{36,37} = \tfrac{1}{2}[\mp UV + VV' + \pi WV],$$

$$\varepsilon_{38,39} = \hat{\varepsilon}_{38,39} = \frac{1}{2}[\mp UV + VV' - \pi WV],$$

$$\varepsilon_{40,41} = P\bar{\varepsilon}_{40,41} = \frac{P}{2}[\mp UV_1 + VV_1' + \pi WV_1],$$

$$\varepsilon_{42,43} = P\bar{\varepsilon}_{42,43} = \frac{P}{2}[\mp UV_1 + VV_1' - \pi WV_1],$$

$$\varepsilon_{44,45} = \hat{\varepsilon}_{44,45} + P\bar{\varepsilon}_{44,45} = [\mp U\hat{V}_2 + \tfrac{1}{2}V\hat{V}_2' + \pi W\hat{V}_2] + P[\mp U\bar{V}_2 + \tfrac{1}{2}V\bar{V}_2' + \pi W\bar{V}_2],$$

$$\varepsilon_{46,47} = \hat{\varepsilon}_{46,47} + P\bar{\varepsilon}_{46,47} = [\mp U\hat{V}_2 \mp 2UV_4 + \tfrac{1}{2}V\hat{V}_2' + VV_4' - \pi W\hat{V}_2] + P[\mp U\bar{V}_2 + \tfrac{1}{2}V\bar{V}_2' - \pi W\bar{V}_2],$$

$$\varepsilon_{48,49} = \hat{\varepsilon}_{48,49} + P\bar{\varepsilon}_{48,49} = [\tfrac{1}{2}V\hat{V}_3' \pm \pi W\hat{V}_3] + P[\tfrac{1}{2}V\bar{V}_3' \pm \pi W\bar{V}_3],$$

$$\varepsilon_{50,51} = \hat{\varepsilon}_{50,51} = \frac{1}{2}[\mp \Theta U + \Theta' V + \pi \Theta W] = \hat{\varepsilon}_{1,2},$$

$$\varepsilon_{52,53} = P\bar{\varepsilon}_{52,53} = \frac{P}{2}[\pm k_0 \Theta U_1 + \Theta' V_1 + \pi \Theta W_1],$$

$$\varepsilon_{54,55} = \hat{\varepsilon}_{54,55} + P\bar{\varepsilon}_{54,55} = \frac{1}{2}[\mp k_0 \Theta \hat{U}_2 + \Theta' \hat{V}_2 + \pi \Theta \tilde{W}_2] + \frac{P}{2}[\mp k_0 \Theta \bar{U}_2 + \Theta' \bar{V}_2 + \pi \Theta \hat{W}_2],$$

$$\varepsilon_{56,57} = \hat{\varepsilon}_{56,57} + P\bar{\varepsilon}_{56,57} = \frac{1}{2}[\pm k_0 \Theta \hat{U}_2 \mp 2k_0 U_4 - \Theta' \hat{V}_2 + 2\Theta' V_4 + \pi \Theta \tilde{W}_2] + \frac{P}{2}[\pm k \bar{\Theta} U_2 - \Theta' \bar{V}_2 + \pi \Theta \hat{W}_2],$$

$$\varepsilon_{58,59} = \hat{\varepsilon}_{58,59} + P\bar{\varepsilon}_{58,59} = \frac{1}{2}[\pm \Theta' \hat{V}_3 + \pi \Theta \tilde{W}_3] + \frac{P}{2}[\pm \Theta' \bar{V}_3 + \pi \Theta \hat{W}_3],$$

$$\varepsilon_{60,61} = \hat{\varepsilon}_{60,61} = \frac{1}{2}[\mp WU + W'V + \pi W^2] = \hat{\varepsilon}_{14,15},$$

$$\varepsilon_{62,63} = P\bar{\varepsilon}_{62,63} = \frac{P}{2}[\pm k_0 WU_1 + W'V_1 + \pi WW_1],$$

$$\varepsilon_{64,65} = \hat{\varepsilon}_{64,65} + P\bar{\varepsilon}_{64,65} = \frac{1}{2}[\mp k_0 W\hat{U}_2 + W'\hat{V}_2 + \pi W\tilde{W}_2] + \frac{P}{2}[\mp k_0 W\bar{U}_2 + W'\bar{V}_2 + \pi W\hat{W}_2],$$

$$\varepsilon_{66,67} = \hat{\varepsilon}_{66,67} + P\bar{\varepsilon}_{66,67} = \frac{1}{2}[\pm k_0 W\hat{U}_2 \mp 2k_0 WU_4 - W'\hat{V}_2 + 2W'V_4 + \pi W\tilde{W}_2] + \frac{P}{2}[\pm k_0 W\bar{U}_2 - W'\bar{V}_2 + \pi W\hat{W}_2],$$

$$\varepsilon_{68,69} = \hat{\varepsilon}_{68,69} + P\bar{\varepsilon}_{68,69} = \frac{1}{2}[\pm W'\hat{V}_3 + \pi W\tilde{W}_3] + \frac{P}{2}[\pm W'\bar{V}_3 + \pi W\hat{W}_3],$$

$$\varepsilon_{70,71} = \hat{\varepsilon}_{70,71} = \frac{1}{2k_0}[-U^2 \pm U'V \pm \pi UW], \quad \hat{\varepsilon}_{70} = \hat{\varepsilon}_{24}, \quad \hat{\varepsilon}_{71} = -\hat{\varepsilon}_{25},$$

$$\varepsilon_{72,73} = \hat{\varepsilon}_{72,73} = \frac{1}{2k_0}[-U^2 \pm U'V \mp UW], \quad \hat{\varepsilon}_{72} = \hat{\varepsilon}_{26}, \quad \hat{\varepsilon}_{73} = -\hat{\varepsilon}_{27},$$

$$\varepsilon_{74,75} = P\bar{\varepsilon}_{74,75} = \frac{P}{2}[-UU_1 \mp \frac{1}{k_0}U'V_1 \mp \frac{1}{k_0}\pi UW_1],$$

$$\varepsilon_{76,77} = P\bar{\varepsilon}_{76,77} = \frac{P}{2}[-UU_1 \mp \frac{1}{k_0}U'V_1 \pm \frac{1}{k_0}\pi UW_1],$$

$$\varepsilon_{78,79}=\hat{\varepsilon}_{78,79}+p\bar{\varepsilon}_{78,79}=\tfrac{1}{2}\Big[-U\hat{u}_2\pm\frac{1}{k_0}U'\hat{v}_2\pm\frac{\pi}{k_0}U\tilde{w}_2\Big]+\frac{p}{2}\Big[-U\bar{u}_2\pm\frac{1}{k_0}U'\bar{v}_2\pm\frac{\pi}{k_0}U\hat{w}_2\Big],$$

$$\varepsilon_{80,81}=\hat{\varepsilon}_{80,81}+p\bar{\varepsilon}_{80,81}=\tfrac{1}{2}\Big[-U\hat{u}_2-2Uu_4\pm\frac{1}{k_0}(U'\hat{v}_2+2U'v_4)\mp\frac{\pi}{k_0}U\tilde{w}_2\Big]+\frac{p}{2}\Big[-U\bar{u}_2\pm\frac{1}{k_0}U'\bar{v}_2\mp\frac{\pi}{k_0}U\hat{w}_2\Big],$$

$$\varepsilon_{82,83}=\hat{\varepsilon}_{82,83}+p\bar{\varepsilon}_{82,83}=\frac{1}{2k_0}\Big[U'\hat{v}_3\pm\pi U\tilde{w}_3\Big]+\frac{p}{2k_0}\Big[U'\bar{v}_3\pm\pi U\hat{w}_3\Big],$$

$$\varepsilon_{84,85}=\hat{\varepsilon}_{84,85}=\tfrac{1}{2}\big[\mp VU+V'V+\pi VW\big]=\hat{\varepsilon}_{36,37},$$

$$\varepsilon_{86,87}=\hat{\varepsilon}_{86,87}=\tfrac{1}{2}\big[\mp VU+V'V-\pi VW\big]=\hat{\varepsilon}_{38,39},$$

$$\varepsilon_{88,89}=p\bar{\varepsilon}_{88,89}=\frac{p}{2}\big[\pm k_0Vu_1+V'v_1+\pi Vw_1\big],$$

$$\varepsilon_{90,91}=p\bar{\varepsilon}_{90,91}=\frac{p}{2}\big[\pm k_0Vu_1+V'v_1-\pi Vw_1\big],$$

$$\varepsilon_{92,93}=\hat{\varepsilon}_{92,93}+p\bar{\varepsilon}_{92,93}=\tfrac{1}{2}\big[\mp k_0V\hat{u}_2+V'\hat{v}_2+\pi V\tilde{w}_2\big]+\frac{p}{2}\big[\mp k_0V\bar{u}_2+V'\bar{v}_2+\pi V\hat{w}_2\big],$$

$$\varepsilon_{94,95}=\hat{\varepsilon}_{94,95}+p\bar{\varepsilon}_{94,95}=\tfrac{1}{2}\big[\mp k_0V\hat{u}_2\mp 2k_0Vu_4+V'\hat{v}_2+2V'v_4-\pi V\tilde{w}_2\big]+\frac{p}{2}\big[\mp k_0V\bar{u}_2+V'\bar{v}_2-\pi V\hat{w}_2\big],$$

$$\varepsilon_{96,97}=\hat{\varepsilon}_{96,97}+\bar{p}\varepsilon_{96,97}=\tfrac{1}{2}\big[V'\hat{v}_3\pm\pi V\tilde{w}_3\big]+\frac{p}{2}\big[V'\bar{v}_3\pm\pi V\hat{w}_3\big],$$

$$\varepsilon_{98}=\hat{\varepsilon}_{98}+p\bar{\varepsilon}_{98}=\hat{\Theta}_2+p\bar{\Theta}_2,\qquad \varepsilon_{99}=\hat{\varepsilon}_{99}+\frac{1}{p}\tilde{\varepsilon}_{99}=\hat{w}_2+\frac{1}{p}\tilde{\ }_2,$$

$$\varepsilon_{100}=\hat{\varepsilon}_{100}+p\bar{\varepsilon}_{100}=\hat{u}_2+p\bar{u}_2,\qquad \varepsilon_{101}=\hat{\varepsilon}_{101}=\hat{\varepsilon}_{14},$$

$$\varepsilon_{102}=\hat{\varepsilon}_{102}=UW,\qquad \varepsilon_{103}=\hat{\varepsilon}_{103}=U^2,$$

$$\varepsilon_{104}=\hat{\varepsilon}_{104}=U\Theta,\qquad \varepsilon_{105}=\hat{\varepsilon}_{105}=UV,\qquad \varepsilon_{106}=\hat{\varepsilon}_{106}+p\bar{\varepsilon}_{106}=\hat{v}_2+p\bar{v}_2 \tag{E1}$$

where ' denotes $\frac{d}{dy}$. It should be noted that $\varepsilon_{98}-\varepsilon_{106}$ have been introduced for programming purposes.

Appendix F: Infinite rigid channel: expansion at order ε^3

The y-dependent functions ϕ_i i=1..58 (i ≠ 13,34) in (7.1.37) are

$$\phi_1 = \frac{1}{P}\left[-U\Theta_2 + \frac{k_0}{2}\Theta U_2 - k_0 U_4 \Theta\right],$$

$$\phi_2 = \frac{ik_0}{P}\left[U_1\Theta_2 + \frac{1}{2}U_2\Theta_1 - U_4\Theta_1\right],$$

$$\phi_3 = \frac{1}{P}\left[\frac{1}{2}V\Theta_2' - \frac{1}{2}V_2\Theta' + V_4\Theta' - \pi W\Theta_2 + \frac{\pi P}{2}W_2\Theta\right],$$

$$\phi_4 = \frac{1}{P}\left[\frac{1}{2}V\Theta_3' - \frac{1}{2}V_3\Theta' - \pi W\Theta_3 + \frac{\pi P}{2}W_3\Theta\right],$$

$$\phi_5 = \frac{i}{P}\left[\frac{1}{2}V_1\Theta_2' - \frac{1}{2}V_2\Theta_1' + V_4\Theta_1' - \pi W_1\Theta_2 + \frac{\pi P}{2}W_2\Theta_1\right],$$

$$\phi_6 = \frac{i}{P}\left[\frac{1}{2}V_1\Theta_3' - \frac{1}{2}V_3\Theta_1' - \pi W_1\Theta_3 + \frac{\pi P}{2}W_3\Theta_1\right],$$

$$\phi_7 = \frac{k_0}{P}\left[\frac{-P}{k_0}UW_2 + \frac{1}{2}U_2W - U_4W\right],$$

$$\phi_8 = \frac{ik_0}{P}\left[PU_1W_2 + \frac{1}{2}U_2W_1 - U_4W_1\right],$$

$$\phi_9 = \left[\frac{1}{2}VW_2' - \frac{1}{2P}V_2W' + \frac{1}{P}V_4W' - \frac{\pi}{2}WW_2\right],$$

$$\phi_{10} = \left[\frac{1}{2}VW_3' - \frac{1}{2P}V_3W' - \frac{\pi}{2}WW_3\right],$$

$$\phi_{11} = i\left[\frac{1}{2}V_1W_2' - \frac{1}{2P}V_2W_1' + \frac{1}{P}V_4W_1' - \frac{\pi}{2}W_1W_2\right],$$

$$\phi_{12} = i\left[\frac{1}{2}V_1W_3' - \frac{1}{2P}V_3W_1' - \frac{\pi}{2}W_1W_3\right],$$

$$\phi_{14} = \frac{i}{P}\left[-UU_2 - 2UU_4 + \frac{1}{2k_0}V_2U' + \frac{1}{k_0}V_4U' - \frac{\pi P}{2k_0}W_2U\right],$$

$$\phi_{15} = \frac{1}{P}\left[-k_0U_1U_2 - 2k_0U_1U_4 + \frac{1}{2}V_2U_1' + V_4U_1' - \frac{\pi P}{2}W_2U_1\right],$$

$$\phi_{16} = \frac{i}{P}\left[-\frac{1}{2}U_2U - U_4U + \frac{1}{2}VU_2' + VU_4' - \pi WU_2\right],$$

$$\phi_{17} = \frac{1}{P}\left[\frac{-k_0}{2}U_2U_1 - k_0U_4U_1 - \frac{1}{2}V_1U_2' - V_1U_4' + \pi W_1U_2\right],$$

$$\phi_{18} = \frac{i}{k_0}\left[\frac{1}{2P}V_3U' - \frac{\pi}{2}W_3U\right],$$

$$\phi_{19} = \left[\frac{1}{2P}V_3U_1' - \frac{\pi}{2}W_3U_1\right],$$

$$\phi_{20} = \frac{1}{p}[-UV_2 - 2UV_4 - \frac{k_0}{2}U_2V - k_0U_4V],$$

$$\phi_{21} = \frac{ik_0}{p}[U_1V_2 + 2U_1V_4 - \frac{1}{2}U_2V_1 - U_4V_1],$$

$$\phi_{22} = \frac{1}{p}[\frac{1}{2}VV_2' + VV_4' + \frac{1}{2}V_2V' + V_4V' - \pi WV_2 - \frac{\pi p}{2}W_2V],$$

$$\phi_{23} = \frac{1}{p}[\frac{1}{2}VV_3' + \frac{1}{2}V_3V' - \pi WV_3 - \frac{\pi p}{2}W_3V],$$

$$\phi_{24} = \frac{i}{p}[\frac{1}{2}V_1V_2' + V_1V_4' + \frac{1}{2}V_2V_1' + V_4V_1' - \pi W_1V_2 - \frac{\pi p}{2}W_2V_1],$$

$$\phi_{25} = \frac{i}{p}[\frac{1}{2}V_1V_3' + \frac{1}{2}V_3V_1' - \pi W_1V_3 - \frac{\pi p}{2}W_3V_1],$$

$$\phi_{26} = \frac{1}{p}[U\Theta_{10} + \frac{1}{2}V\Theta_{10}' + \frac{1}{2}V\Theta_{11}' - \pi W\Theta_{10} - \pi W\Theta_{11}],$$

$$\phi_{27} = \frac{1}{p}[U\Theta_{12} + \frac{1}{2}V\Theta_{12}' + \frac{1}{2}V\Theta_{13}' - \pi W\Theta_{12} - \pi W\Theta_{13}],$$

$$\phi_{28} = \frac{1}{p}[\frac{1}{2}V\Theta_{13}' - \pi W\Theta_{13}],$$

$$\phi_{29} = \frac{1}{p}[-\frac{1}{2}V\Theta_{11}' + \pi W\Theta_{11}],$$

$$\phi_{30} = [UW_{10} + \frac{1}{2}VW_{10}' + \frac{1}{2}VW_{11}' - \pi WW_{10} - \pi WW_{11}],$$

$$\phi_{31} = [UW_{12} + \frac{1}{2}VW_{12}' + \frac{1}{2}VW_{13}' - \pi WW_{12} - \pi WW_{13}],$$

$$\phi_{32} = [\frac{1}{2}VW_{13}' - \pi WW_{13}],$$

$$\phi_{33} = [-\frac{1}{2}VW_{11}' + \pi WW_{11}],$$

$$\phi_{35} = \frac{1}{p}[UU_{10} + 2UU_{17} + \frac{1}{2}VU_{10}' + \frac{1}{2}VU_{11}' + VU_{17}' + VU_{18}' - \pi WU_{10} - \pi WU_{11}],$$

$$\phi_{36} = \frac{1}{p}[UU_{12} + 2iUU_{16} + \frac{1}{2}VU_{12}' + \frac{1}{2}VU_{13}' + iVU_{16}' - \pi WU_{12} - \pi WU_{13}],$$

$$\phi_{37} = \frac{1}{p}[-\frac{1}{2}VU_{13}' + \pi WU_{13}],$$

$$\phi_{38} = \frac{1}{p}[\frac{1}{2}VU_{11}' + VU_{18}' - \pi WU_{11}],$$

$$\phi_{39} = \frac{1}{p}[UV_{10} + 2iUV_{17} + \frac{1}{2}VV_{10}' + \frac{1}{2}VV_{11}' + iVV_{17}' - \pi WV_{10} - \pi WV_{11}],$$

$$\phi_{40} = \frac{1}{p}[UV_{12} + 2VV_{16} + \frac{1}{2}VV_{12}' + \frac{1}{2}VV_{13}' + VV_{16}' - \pi WV_{12} - \pi WV_{13}],$$

$$\phi_{41} = \frac{1}{p}\left[\frac{1}{2} V V_{13}' - \pi W V_{13} \right],$$

$$\phi_{42} = \frac{1}{p}\left[-\frac{1}{2} V V_{11}' + \pi W V_{11} \right],$$

$$\phi_{43} = \frac{1}{p}\left[\frac{ik_0}{2} \Theta U_{10} - \frac{ik_0}{2} \Theta U_{11} - ik_0 \Theta U_{17} + ik_0 \Theta U_{18} - \frac{1}{2}\Theta' V_{10} - \frac{1}{2}\Theta' V_{11} + i\Theta' V_{17} + \frac{\pi p}{2}\Theta (W_{10} + W_{11}) \right]$$

$$\phi_{44} = \frac{1}{p}\left[\frac{ik_0}{2} \Theta U_{12} - \frac{ik_0}{2} \Theta U_{13} + k_0 \Theta U_{16} - \frac{1}{2}\Theta' V_{12} - \frac{1}{2}\Theta' V_{13} + \Theta' V_{16} + \frac{\pi p}{2}\Theta (W_{12} + W_{13}) \right],$$

$$\phi_{45} = \frac{1}{p}\left[\frac{ik_0}{2} \Theta U_{13} - \frac{1}{2}\Theta' V_{13} + \frac{\pi p}{2}\Theta W_{13} \right],$$

$$\phi_{46} = \frac{1}{p}\left[-\frac{ik_0}{2} \Theta U_{11} + ik_0 \Theta U_{18} + \frac{1}{2}\Theta' V_{11} - \frac{\pi}{2}\Theta W_{11} \right],$$

$$\phi_{47} = \frac{1}{p}\left[\frac{ik_0}{2} W U_{10} - \frac{ik_0}{2} W U_{11} - ik_0 W U_{17} + ik_0 W U_{18} - \frac{1}{2} W' V_{10} - \frac{1}{2} W' V_{11} + i W' V_{17} + \frac{\pi p}{2} W (W_{10} + W_{11}) \right],$$

$$\phi_{48} = \frac{1}{p}\left[\frac{ik_0}{2} W U_{12} - \frac{ik_0}{2} W U_{13} + k_0 W U_{16} - \frac{1}{2} W' V_{12} - \frac{1}{2} W' V_{13} + W' V_{16} + \frac{\pi p}{2} W (W_{12} + W_{13}) \right],$$

$$\phi_{49} = \frac{1}{p}\left[\frac{ik_0}{2} W U_{13} - \frac{1}{2} W' V_{13} + \frac{\pi p}{2} W W_{13} \right],$$

$$\phi_{50} = \frac{1}{p}\left[-\frac{ik_0}{2} W U_{11} + ik_0 W U_{18} + \frac{1}{2} W' V_{11} - \frac{\pi}{2} W W_{11} \right],$$

$$\phi_{51} = \frac{1}{p}\left[-\frac{1}{2} U U_{10} - \frac{1}{2} U U_{11} - U U_{17} - U U_{18} - \frac{i}{2k_0} U' V_{10} + \frac{1}{k_0} U' V_{17} + \frac{i\pi p}{2k_0} U (W_{10} - W_{11}) \right],$$

$$\phi_{52} = \frac{1}{p}\left[-\frac{1}{2} U U_{12} - \frac{1}{2} U U_{13} - i U U_{16} - \frac{i}{2k_0} U' V_{12} + \frac{i}{2k_0} U' V_{13} - \frac{i}{k_0} U V_{16} + \frac{i\pi p}{2k_0} U (W_{12} - W_{13}) \right],$$

$$\phi_{53} = \frac{1}{p}\left[\frac{1}{2} U U_{13} + \frac{i}{2k_0} U' V_{13} - \frac{i\pi p}{2k_0} U W_{13} \right],$$

$$\phi_{54} = \frac{1}{p}\left[-\frac{1}{2} U U_{11} - U U_{16} - \frac{i}{2k_0} U' V_{11} + \frac{i\pi p}{2k_0} U W_{11} \right],$$

$$\phi_{55} = \frac{1}{p}\left[-\frac{ik_0}{2} V U_{10} + \frac{ik_0}{2} V V_{11} - ik_0 V U_{17} + ik_0 V U_{18} + \frac{1}{2} V' V_{10} + \frac{1}{2} V' V_{11} + i V' V_{17} - \frac{\pi p}{2} V (W_{10} + W_{11}) \right],$$

$$\phi_{57} = \frac{1}{p}\left[-\frac{ik_0}{2} V U_{13} + \frac{1}{2} V' V_{13} - \frac{\pi}{2} V W_{13} \right],$$

$$\phi_{56} = \frac{1}{p}\left[-\frac{ik_0}{2} V U_{12} + \frac{ik_0}{2} V U_{13} + k_0 V U_{16} + \frac{1}{2} V' V_{12} + \frac{1}{2} V' V_{13} + V' V_{16} - \frac{\pi p}{2} V (W_{12} + W_{13}) \right],$$

$$\phi_{58} = \frac{1}{p}\left[\frac{ik_0}{2} V U_{11} + ik_0 V U_{18} - \frac{1}{2} V' V_{11} + \frac{\pi}{2} V W_{11} \right], \qquad \text{(F1)}$$

where $'$ denotes $\frac{d}{dy}$.

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