MODEL REDUCTION

AND THE

ADEQUACY OF SIMPLE MODELS

AS APPLIED TO THE

DESIGN OF CONTROL SYSTEMS

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DECLARATION

ABSTRACT

Accurate models of complex systems are, of necessity of a higher order. It is often convenient to use a simpler model when investigating characteristics of such a system. For this purpose a method of model reduction is required. Much work has been carried out in recent years on model reduction techniques. A comprehensive survey of these has been carried out and is presented here.

The reviewed works do not deal with first order plus time delay models which can in certain cases be used for controller design. For this purpose the simple model must be less stable than the system itself. Many models do not fulfil this requirement. An investigation into the adequacy of first order plus time delay models obtained using the Padé technique has been carried out and its findings are given in section 3.

Finally an examination is carried out into the performance of closed loop systems using controllers obtained from first order plus time delay models. It is shown that adequate Padé models can be used to give suitable controllers and that if no adequate Padé model exists then an alternative method can be used to give acceptable results.

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I. INTRODUCTION.

The study and analysis of any process or system is commonly carried out by engineers using mathematical models. Accurate and precise models of complex systems are themselves by definition, very complex and complicated, consisting usually of either a large number of simultaneous first order differential equations or of one or more differential equations of a higher order.

To facilitate the study of a system and to design a suitable controller for it, it is often advantageous to produce a simplified model of the system, the object being to produce a model which is computationally simpler than the original system but which nevertheless represents the system sufficiently adequately for investigation of certain of its properties to be possible.

Much work has recently been done on the reduction of higher order mathematical models of complex systems to ones of lower order and a survey of this work appears as the first of the three major sections of this thesis.

The second major section is devoted to the investigation of the adequacy, in terms of open-loop stability, of simple models containing only a single time constant and a pure time delay. Although many practical systems, particularly in the process industry, can be represented by such a model, very little work has been done in this field. It is a fact, however, that if an adequate first order model of a system can be produced, the design of a simple contoller for the system is greatly facilitated since published data exists which enables Proportional and Integral (PI) and Proportional, Integral and Differential (PID) controller parameters for specified performance criteria to be calculated from first order model parameters.

The third major section of this thesis examines the application of model reduction techniques to controller design. Incorporating information derived from the adequacy investigation mathematical models of vaporisers are reduced to first order models which are then used in the design of appropriate controllers. The performance of these controllers is reviewed and the effects on them of using various reduced models for

their design is thoroughly explored. Additionally, the potential dangers of designing such controllers using unsuitable reduced models are also considered.

This thesis, therefore, undertakes a positive contribution to the existing knowledge on the adequacy of very simple reduced models for use in controller design, whilst, at the same time incorporating a comprehensive review of existing model reduction techniques.

2. A SURVEY OF EXISTING MODEL REDUCTION TECHNIQUES.

Since the early 1960s much interest has been shown in model reduction and many techniques have been developed for dealing with this subject.

The majority of these techniques can be broadly sub-divided into ten groups and in this survey they have been so dealt with, any techniques which do not fall into one of the ten groups being included in an additional eleventh group.

The survey refers, in the main, to those papers which deal with the reduction of single input-single output models. In the case of those systems specified in state variable form the reduction technique is identical whether a single input-single output system is being considered or whether the system under consideration is multivariable, therefore all such systems have been covered in this survey.

However, in those cases where the system is described by a single transfer function, or, in the case of a multivariable system, by a number of transfer functions in matrix form, the reduction of the multivariable system is a more complicated procedure, but since this procedure is effectively an extension into matrix form of the single input-single output method, no additional investigation has been done into the finer points of divergence.

It must be noted at this stage that all previous comments refer essentially to linear systems whereas, in reality, a high proportion of systems are non-linear. A possible method of reduction of such systems is to linearize the system about a definite operating point and then to apply a linear reduction technique to the resulting equations, this technique being the same as that normally used for linear systems. Since such a reduced model, being suitable only around the point about which the system was linearized, requires the incorporation of a function generator if it is to be used to represent the whole operating range and this is felt to be outside the scope of this thesis, the reduction of non-linear systems has not been dealt with as a separate issue.

2.1 MODEL REDUCTION BY MINIMISING THE ERROR OF THE TIME RESPONSE.

This method of model reduction endeavours to obtain a reduced model whose response approaches that of the original system, by minimising the mean square error between the two responses over a given finite time interval.

The first investigation of this method appears to have been undertaken in 1967 by ANDERSON (1), who describes the method as one based on the evaluation of the projection of a specified vector onto a linear subspace.

Initially, the original continuous system, described in state variable form is converted to an equivalent set of n first order difference equations describing the discrete-time form of the system, thus :-

$$\underline{x}\left[(k+1)T\right] = \phi(T)\underline{x}\left[kT\right] + \Delta(T)\underline{u}\left[kT\right] \qquad .....(1)$$

where

T = the sampling period
$\phi(T)$ = n x n discrete transition matrix
$\Delta(T)$ = n x n discrete driving matrix

Assuming 'r' to be the order of the simplified system the above equations may then be reduced to a set of r equations as follows :-

$$\overline{\underline{x}}\left[(k+1)T\right] = \overline{\phi}(T)\overline{\underline{x}}\left[kT\right] + \overline{\Delta}(T)\underline{u}\left[kT\right] \qquad .....(2)$$

where

$\overline{\phi}(T)$ = r x r reduced transition matrix
$\overline{\Delta}(T)$ = r x m reduced driving matrix

To achieve this, values for $\overline{\phi}(T)$ and $\overline{\Delta}(T)$ are selected which will minimise the differences occuring at each sampling point between $\overline{\underline{X}}$ and the first r elements of $\underline{X}$, referred to as $\underline{X}^{(r)}$.

Under ideal conditions, these selected values would satisfy the equation -

$$\underline{x}^{(r)}\left[(k+1)T\right] = \overline{\emptyset}(T)\underline{x}^{(r)}\left[kT\right] + \overline{\Delta}(T)\underline{u}\left[kT\right] \qquad .....(3)$$

which when expanded will give -

$$x_q^{(r)}\left[iT\right] = \sum_{j=1}^{r} \overline{\emptyset}_{qj} x_j^{(r)}\left[(i-1)T\right] + \sum_{l=1}^{m} \overline{\Delta}_{ql} u_l \left[(i-1)T\right] \qquad (4)$$

for

$i = 1, 2, \ldots\ldots\ldots k+1$

and $q = 1, 2, \ldots\ldots\ldots r$

assuming $\underline{x}^{(r)} = \underline{x}^{r}(0)$ at $t = 0$

This can be expressed in matrix form as

$$\underline{b}_q = M\underline{c}_q \qquad .....(5)$$

where

$\underline{b}_q$ and $\underline{c}_q$ are vectors given by

$$\underline{b}_q = \begin{bmatrix} x_q^{(r)}(T) \\ x_q^{(r)}(2T) \\ \vdots \\ x_q^{(r)}((k+1)T) \end{bmatrix} \qquad \underline{c}_q = \begin{bmatrix} \overline{\emptyset}_{q1} \\ \overline{\emptyset}_{q2} \\ \vdots \\ \overline{\emptyset}_{qr} \\ \overline{\Delta}_{q1} \\ \overline{\Delta}_{q2} \\ \vdots \\ \overline{\Delta}_{qm} \end{bmatrix} \qquad \ldots(6)$$

and M is a (k+1) x (r+m) matrix whose (i+1)th row is

given by

$$M_{i+1} = \begin{bmatrix} x_1^{(r)}(iT) & x_2^{(r)}(iT) & \cdots & x_r^{(r)}(iT) \\ u_1(iT) & u_2(iT) & \cdots & u_m(iT) \end{bmatrix} \quad \ldots.(7)$$

Equation 5 can now be solved by putting -

$$\underline{c}_q = (M'M)^{-1}M'\underline{b}_q \quad \ldots.(8)$$

which according to the Theory of Linear Spaces minimises the inner product-

$$<(\underline{b}_q - \overline{\underline{b}}_q)(\underline{b}_q - \overline{\underline{b}}_q)>$$

where

$$\overline{\underline{b}}_q = \begin{bmatrix} \overline{x}_q(T) \\ \overline{x}_q(2T) \\ \vdots \\ \overline{x}_q((k+1)T) \end{bmatrix} \quad \ldots.(9)$$

Since this also satisfies the equation -

$$\overline{\underline{b}}_q = M\underline{c}_q \quad \ldots.(10)$$

B and C can now be defined as

$$B = \begin{bmatrix} \underline{b}_1 & \underline{b}_2 & \cdots & \underline{b}_q & \cdots & \underline{b}_r \end{bmatrix}$$

$$C = \begin{bmatrix} \underline{c}_1 & \underline{c}_2 & \cdots & \underline{c}_q & \cdots & \underline{c}_r \end{bmatrix}$$

$$= \begin{bmatrix} \overline{\Phi}(T) & \overline{\Delta}(T) \end{bmatrix}' \quad \ldots.(11)$$

giving

$$B = MC = M\left[\bar{\Phi}(T) \vdots \bar{\Delta}(T)\right]' \qquad .....(12)$$

and hence

$$\left[\bar{\Phi}(T) \vdots \bar{\Delta}(T)\right] = C' = B'M(M'M)^{-1} \qquad .....(13)$$

In this way, $\bar{\Phi}(T)$ and $\bar{\Delta}(T)$ can be evaluated to give a least squares fit between the response of the reduced model and that of the full system bearing in mind that the time interval $(k+1)T$, over which the model is to be evaluated must be chosen with care so as to be longer than the largest time constant but not so large as to give rise to singularity of $M'M$.

The main limitations of this method are that it provides a reduced model which is valid only for the particular input disturbance used to evaluate it, and is susceptible to steady state error. Additionally, it is relatively complex from a computational viewpoint.

However, some of these computational problems, together with certain other difficulties associated with this reduction technique are tackled by ANDERSON in further papers published in 1967 and 1968 (2 , 3 , 4 , 5), in which he observes that for a system which takes a comparatively long time to settle, the matrices M and B can become extremely large, thus requiring a considerable amount of computer store, (M having a dimension of $(k+1) \times (r+m)$ and B having a dimension of $(k+1) \times r$.).

To overcome this difficulty, ANDERSON suggests that equation 12 be partitioned into l blocks so that -

$$\begin{bmatrix} B_1 \\ B_2 \\ \cdot \\ \cdot \\ \cdot \\ B_1 \end{bmatrix} = \begin{bmatrix} M_1 \\ M_2 \\ \cdot \\ \cdot \\ \cdot \\ M_1 \end{bmatrix} C \qquad .....(14)$$

The least squares solution for C' is now given by -

$$C' = \left[\sum_{i=1}^{l} B_i' M_i \right] \left[\sum_{i=1}^{l} M_i' M_i \right]^{-1} \qquad \ldots\ldots(15)$$

$B_i' M_i$ and $M_i' M_i$ can now be evaluated immediately after the evaluation of B_i and M_i, thus eliminating the necessity of storing all the values of B and M and consequently reducing the maximum size of matrix to be stored from (k+1) x (r+m) to (r+m) x (r+m).

In a further publication (3), ANDERSON considers the case where certain elements of the reduced system matrices are known.

Thus, expressing equation 10 in the format -

$$\underline{b}_q = \left[m_1 m_2 \ldots\ldots m_j \ldots\ldots m_{r+m} \right] \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_j \\ \vdots \\ C_{r+m} \end{bmatrix} \qquad \ldots\ldots(16)$$

and then assuming C_j to be the known element, and re-expressing

$$\underline{b}_q - m_j c_j = \left[m_1 m_2 \ldots m_{j-1}, m_{j+1} \ldots m_{r+m} \right] \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_{j-1} \\ C_{j+1} \\ \vdots \\ C_{r+m} \end{bmatrix} \qquad (17a)$$

or

$$\underline{b}_q^j = M^j . C_q^j \qquad \ldots\ldots(17b)$$

where

$$\underline{b}_q^j = \underline{b}_q - m_j c_j$$

$$M^j = \left[m_1 m_2 \cdots m_{j-1}, m_{j+1} \cdots m_{r+m} \right]$$

and $$C_q^j = \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_{j-1} \\ C_{j+1} \\ \vdots \\ C_{r+m} \end{bmatrix}$$

The unknown elements of the qth row of $\overline{\phi}(T)$ and $\overline{\Delta}(T)$ can now be obtained by solving equation 17 putting

$$(c_q^j)' = (b_q^j)' M^j \left[(M^j)' M^j \right]^{-1} \qquad \ldots\ldots(18)$$

A fourth paper(5) by ANDERSON deals with those cases where the response of the reduced model varies widely from that of the full model at specific points,for example - in the steady state or during the initial transient ($t = 0$).

The author suggests the use of weighting factors to overcome this problem and recommends the multiplication of those rows of M and B,which correspond to the time at which greater accuracy is required, by a weighting factor of N, which must be greater than 1.

A similar approach is used by SINHA and PILLE (6), whose work on the discrete time form of the model, published in 1971, can be considered as a refinement of ANDERSON's technique.

Their method uses the matrix pseudoinverse to estimate the parameters of the model.This minimises the sum of the squares of the errors between the response of the actual

system and that of the model at sampling intervals.

This approach gives a satisfactory fit on the transient but still does not eliminate the steady state error which occurs with ANDERSON's technique.

Later in 1971 another paper was published by SINHA and BEREZNA (7) putting forward an optimisation technique for model reduction which uses the pattern search algorithm of Hooke and Jeeves to minimise the error according to a given criterion between the reduced model and the original system.

Work on similar lines by other authors includes an optimisation technique for model reduction published by WILSON (8) in 1970 and followed up in 1974 (9) with an algorithm for this method,and also papers published in 1973 by APLEVICH (10) and GALIANA (11) whose techniques minimise quadratic functions of the error of the time response.

Thereafter, in 1974, RIGGS and EDGAR (12) published a paper on the reduction of linear systems using a least squares fit of the impulse response, in which they concluded that

(i) Reduced models can be rapidly computed using optimisation techniques

(ii) Optimal reduced models can be distinctly superior to reduced models obtained by non-optimal methods as measured by the integral square error

(iii) More flexibility and control over development of the reduced model is available using optimisation techniques.

EDGAR followed this up, in 1975, with a publication (13) in which, after quoting from his previous work, he detailed a reduction method which, by using the step response, minimises the function

$$J = \int_0^\infty (y(t) - \bar{y}(t))^2 \, dt. \qquad \ldots\ldots(19)$$

He then produced an example which he compared with results obtained by CHEN and SHIEH (14). However, although both methods guarantee an exact steady state fit, EDGAR's method necessarily gives the better result since the comparison is based on the integral square error.

Further work on model reduction using optimisation in the time domain was published in 1975 by HIRZINGER and KREISSELMEIER (15), who, working in state variable notation, reduced the model

$$\dot{x} = Ax + Bu \qquad \text{.....(20)}$$

$$y = Cx \qquad \text{.....(21)}$$

to

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}u \qquad \text{.....(22)}$$

$$\tilde{y} = \tilde{C}\tilde{x} \qquad \text{.....(23)}$$

where x is of order n

$\tilde{x}$ is of order $r < n$

by minimising the function

$$J = \int_0^{\infty} (y - \tilde{y})^T Q (y - \tilde{y})\, dt. \qquad \text{.....(24)}$$

and selecting suitable $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ matrices.

In the above function, Q is a weighting matrix which can be selected as a diagonal matrix whose elements are q_1 to q_p where p is the dimension of y and $\tilde{y}$.

q_i is given by -

$$q_i = \eta_i \left(\int_0^{\infty} y_i^2 dt \right)^{-1} \qquad \text{for } i=1,2.....p \qquad \text{.....(25)}$$

where

η_i is a weighting coefficient whose value is ascertained by relating the importance of output y_i to the other outputs and the expression in the brackets is the integral square of the output y_i.

The value thus considered represents the percentage square error since, when expanded within function J, the integral square error of output y_i is multiplied by the inverse of the bracketed expression present in q_i.

The effect of selecting J in this manner is to eliminate any unintentional weighting of output errors.

The authors suggest that this method can be used to minimise

the error of the response obtained from a variety of inputs including unit impulse functions, unit step functions and unit impulse response of linear shaping filters.

However, as with all optimisation reduction techniques, the reduced model obtained in this way is dependent on the specific input function used and cannot be applied to give a general model.

The final method examined was that proposed by OBINATA and INOOKA (16) in 1976. In this case the authors reduced

$$\dot{x} = Ax + Bu \qquad .....(26)$$

$$y = Cx \qquad .....(27)$$

to

$$\dot{z} = Fz + Gu \qquad .....(28)$$

where the error e is given by

$$e = y - z. \qquad .....(29)$$

As previously shown, if an error criterion of the form -

$$J = \int_0^T eQe \, dt. \qquad .....(30)$$

is considered, the reduced model can be obtained by minimising J.

However, this is somewhat difficult to achieve due to the non-linear manner in which the reduced model parameters appear in e.

To facilitate this optimisation OBINATA and INOOKA have suggested an error criterion based on the derivative of the error e.

Thus -

$$\dot{e} = \dot{y} - \dot{z} \qquad .....(31)$$

$$= C\dot{x} - \dot{z} \qquad .....(32)$$

$$= CAx + CBu - Fz - Gu \qquad .....(33)$$

$$= Fe + (CA - FC)x + (CB - G)u \qquad .....(34)$$

This can be re-specified as

$$\dot{e} = Fe + d \qquad .....(35)$$

where

$$d = (CA - FC)x + (CB - G)u. \qquad .....(36)$$

Then, since e = 0 at time t = 0, an ideal reduced model can be obtained if d = 0, i.e. if CA = FC and CB = G.

If an error function based on d is then minimised, an optimum model will result.

Since it can be further shown that

$$d = \dot{y} - (Fe + \dot{z}) \qquad .....(37)$$

$$= \dot{y} - (Fy + Gu) \qquad .....(38)$$

it can then be seen that d is a linear function of the reduced model parameters F and G.

To obtain optimum values for these parameters it is suggested that the error criterion to be minimised should be -

$$J = \int_0^T d'.Q.d.dt \qquad .....(39)$$

where Q is a positive definite m x m matrix.

If G = CB and CW_xC' is non singular

where

$$W_x = \int_0^T x(t).x'(t)dt. \qquad .....(40)$$

then F which minimises J is given by -

$$F = CAW_xC'(CW_xC')^{-1}. \qquad .....(41)$$

If, however, G ≠ CB

J can be minimised if

$$(FC - CA)W_xC' + (G - CB)W_{xu}'C' = 0 \qquad .....(42)$$

and

$$(FC - CA)W_{xu} + (G - CB)W_u = 0 \qquad .....(43)$$

where

$$W_{xu} = \int_0^T x(t)u'(t)dt \qquad .....(44)$$

$$W_u = \int_0^T u(t)u'(t)dt. \qquad .....(45)$$

Then, if W_u is non singular

$$W = W_x - W_{xu}(W_u)^{-1}W_{xu}' \qquad .....(46)$$

and

$$F = CAWC'(CWC')^{-1} \qquad .....(47)$$

and

$$G = CA(I - WC'(CWC')^{-1}C)W_{xu}(W_u)^{-1}. \qquad .....(48)$$

As can be seen from the above equations, explicit expressions can be obtained for F and G which minimise J, given the parameters of the original system together with its time response to a given input.

This is an advantage of this particular reduction technique over those others which also minimise error functions in the time domain. However, as with all previously described methods this final example of this particular reduction technique still results in a reduced model which is dependent on the input used for the reduction calculations.

2.2 MODEL REDUCTION BY MODAL ELIMINATION.

The second method of model reduction considered is that of modal elimination. This was introduced in the early 1960s by NICHOLSON, who, in his paper 'Dynamic Optimisation of a Boiler' (August 1964 (17), suggests that the reduction of the system order can be achieved analytically by neglecting higher order modes.

With the system equations in state variable form, his proposed reduction method is as follows :-

Original System : $\dot{\underline{x}} = A\underline{x} + B\underline{u}$ .....(49)

where $\underline{x}$ = n - vector of state variable
$\underline{u}$ = m - vector of system inputs
A = n x n - system matrix describing a continuous time system
B = n x m - forcing matrix driving a continuous time system.

This is reduced to a rth order system

$$\dot{\underline{x}}^* = A^*\underline{x}^* + B^*\underline{u} \qquad .....(50)$$

where $\underline{x}^*$ = r - vector of state variables consisting of selected elements from the original vector $\underline{x}$
A^* = r x r - system matrix
B^* = r x m - forcing matrix.

The system equations can be rewritten using Λ instead of A, where Λ is a diagonal matrix containing the n eigen values of A, arranged in increasing modulus order.

Each eigen value, λ_i, has an associated eigen vector, u_i, which satisfies the equation

$$(\lambda_i I - A)u_i = 0 . \qquad .....(51)$$

The square modal matrix, U, consists of the n eigen

vector, u_1 , and satisfies the equation

$$AU = U\Lambda \quad . \qquad .....(52)$$

By introducing a new state variable vector, z, given by -

$$\underline{x} = U\underline{z} \qquad .....(53)$$

the original system equation (49) can be rewritten as

$$U\dot{\underline{z}} = AU\underline{z} + B\underline{u} \qquad .....(54)$$

or, using equation 52 as

$$U\dot{\underline{z}} = U\Lambda\underline{z} + B\underline{u} \qquad .....(55)$$

and

$$\dot{\underline{z}} = \Lambda\underline{z} + U^{-1}Bu. \qquad .....(56)$$

To obtain the reduced model, the equation is then partitioned, giving

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} U_1 & U_2 \\ U_3 & U_4 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \qquad .....(57)$$

where

x_1 is the vector of r state variables to be retained
x_2 is the vector of n - r remaining state variables
z_1 is the vector of r varables corresponding to the r dominant modes
z_2 is the vector of n - r remaining variables.

U_1 is a r x r)
U_3 is a (n - r) x r) matrix of modal vectors associated with the r dominant modes

U_2 is a r x (n - r))
U_4 is a (n - r) x (n - r)) matrix of modal vectors associated with the remaining modes.

The higher modes are then eliminated by allocating a value of zero to z_2, thus giving

$$x_1 = U_1 z_1 \qquad .....(58)$$

and

$$x_2 = U_3 z_1 \qquad .....(59)$$

and hence

$$x_2 = U_3 . U_1^{-1} x_1 \quad . \qquad .....(60)$$

The original system equation (49) is then similarly partitioned giving

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \underline{u} \qquad .....(61)$$

The reduced rth order system then becomes

$$\dot{x}_1 = A_1 x_1 + A_2 x_2 + B^* \underline{u} \qquad .....(62)$$

or

$$\dot{x}_1 = (A_1 + A_2 . U_3 . U_1^{-1}) x_1 + B^* \underline{u} \quad . \qquad .....(63)$$

B^* can then be evaluated seperately by equating the responses due to forcing and is given by

$$B^* = \overline{U}_1 . \overline{Q}_A . B \qquad .....(64)$$

where

$\overline{U}_1$ is the normalised form of U_1

Q_A is a r x n matrix associated with the first r rows of U^{-1}

and $\overline{Q}_A$ is obtained from Q_A using row multiplying factors.

Next to tackle the model reduction problem using modal elimination , in an almost identical manner, was DAVISON (18) (1966), whose approach differs from that of NICHOLSON only in that DAVISON partitions the equation in the following

manner -

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + U^{-1}B\underline{u} \qquad \text{.....(65)}$$

where

Λ_1 is a diagonal matrix containing the r dominant eigen values

Λ_2 is a diagonal matrix containing the other n - r eigen values.

By equating z_2 to zero DAVISON then obtains

$$\dot{z}_1 = \Lambda_1 z_1 + U^{-1}B\underline{u} \qquad \text{.....(66)}$$

giving, as the reduced system -

$$\dot{x}_1 = U_1\Lambda_1 U_1^{-1}x_1 + U_1(U^{-1}B)\underline{u} \qquad \text{.....(67)}$$

with

$$A^* = U_1\Lambda_1 U_1^{-1} = A_1 + A_2U_3U_1^{-1} \qquad \text{.....(68)}$$

(as obtained by NICHOLSON)

and $B^* = U_1(U^{-1}B)$ .....(69)

where

$(U^{-1}B)$ are the first r rows of the n x m matrix $U^{-1}B$.

However, neither of the reduced models developed by NICHOLSON and DAVISON, has eliminated the steady state error which tends to zero as r tends to n.

In late 1966, a method which <u>does</u> include such an elimination was published by MARSHALL (19), who obtained his simple model by putting $\dot{z}_2$ (rather than z_2) equal to zero.

By putting $U^{-1} = V$, MARSHALL represents equation 65 in the following format -

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} V_1 & V_2 \\ V_3 & V_4 \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \qquad .....(70)$$

If $\dot{z}_2$ is then equated to zero, this gives

$$\dot{z}_1 = \Lambda_1 z_1 + (V_1 B_1 + V_2 B_2)u \qquad .....(71)$$

and

$$0 = \Lambda_2 z_2 + (V_3 B_1 + V_4 B_2)u \qquad .....(72)$$

which, in turn gives

$$z_2 = -\Lambda_2^{-1}(V_3 B_1 + V_4 B_2)u \quad . \qquad .....(73)$$

The reduced model can then be obtained from equations 57, 71, and 73 giving

$$A^* = U_1 \Lambda_1 U_1^{-1} \qquad .....(74)$$

and

$$B^* = B_1 - A_2 V_4^{-1} \Lambda_2^{-1}(V_3 B_1 + V_4 B_2) \quad . \qquad .....(75)$$

A similar suggestion was also made by CHIDAMBARA (20), who, in correspondence with DAVISON, entered into after the publication of DAVISON's paper, proposed that z_2 would be more correctly eliminated by putting $z_2 = Fu$ . .....(76)

CHIDAMBARA's initial communication unfortunately contained an algebraic error. However, in his second note (21) to DAVISON he corrected this error and produced two models C1 and C2.

To show the advantage of his methods over that of DAVISON, CHIDAMBARA included in this second note an example which he gave in state variable form, but is here shown in transfer function form.

For

$$G(s) = \frac{1}{(1 + s)(1 + 0.25s)} \qquad .....(77)$$

CHIDAMBARA's two methods give the following -

For C1

$$\hat{G}(s) = \frac{1 - \frac{1}{3}s}{1 + s} \qquad .....(78)$$

For C2

$$\hat{G}(s) = \frac{1}{1 + s} \qquad .....(79)$$

while DAVISON's method gives

$$\hat{G}(s) = \frac{1.333}{1 + s} \qquad .....(80)$$

(As can be seen, CHIDAMBARA's second model, C2, is identical to that obtained by MARSHALL, while his first model, C1, varies in that it is obtained by evaluating F, (in equation 76), separately.)

DAVISON, however, criticised CHIDAMBARA's models specifying that C2, although giving a correct sready state response to a step function disturbance, would give transient errors, since it would not excite the same proportion of modes of the model's response as were excited in the original system.

He was, however, prepared to acknowledge that this criticism did not apply to model C1 and possibly as a result of this, he suggested, in 1967, (22) the following correction for the elimination of steady state error to his own model (see eqns.50, 68 and 69)

$$\dot{x}_1 = A^* x_1 + B^* u \qquad .....(81)$$

$$x^* = x_1 + \left(A^{*-1} B^* - (A^{-1}B)\right) u \qquad .....(82)$$

where

x^* is the output

and $A^{-1}B$ is a vector whose elements correspond to the variables retained in the reduced model.

Summarising the various models discussed so far, it can be seen that NICHOLSON's model and DAVISON's original model have a steady state error and tend to contain non-minimum

phase properties which do not exist in the full model.

DAVISON's modified model of April 1967 and CHIDAMBARA's model, C1, which produce identical results, eliminate steady state error but still retain the non-minimum phase tendencies.

MARSHALL's model and CHIDAMBARA's model, C2, which are also identical, do not have this tendency.

Although both these models were developed in the state variable form, they are more easily understood if presented in the transfer function form, thus -

If the original model is presented as follows -

$$G(s) = \frac{1}{(1 + 10s)(1 + 5s)(1 + 2s)(1 + s)} \qquad .....(83)$$

reduced models become

$$\hat{G}_1(s) = \frac{1}{1 + 10s} \qquad .....(84)$$

$$\hat{G}_2(s) = \frac{1}{(1 + 10s)(1 + 5s)} \qquad .....(85)$$

and $$\hat{G}_3(s) = \frac{1}{(1 + 10s)(1 + 5s)(1 + 2s)} \qquad .....(86)$$

This can be considered as modal elimination in its simplest form.

It is worth mentioning at this stage, that all the models considered thus far i.e. those obtined by modal elimination, have a system matrix, A^*, given by -

$$A^* = U_1 \Lambda_1 U_1^{-1} = A_1 + A_2 U_3 U_1^{-1} \qquad .....(87)$$

However, in a further paper published in 1968, DAVISON (23) introduced a new modification where this does not apply.

For this new model

$$x^* = Dx_1 \qquad .....(88)$$

where

x_i is obtained as shown in equation 68

and $D = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_r \end{bmatrix}$

where

$$d_i = \frac{(A^{-1}B)_i^*}{(A^{*-1}B^*)_i} \qquad \text{.....(89)}$$

$(A^{*-1}B^*)_i$ being the ith element of the r vector $A^{*-1}B^*$

and $(A^{-1}B)_i^*$ being the element of the r vector $A^{-1}B$ which corresponds to the ith state retained in the reduced system.

This system can also be presented in the format -

$$x^* = DA^*D^{-1}x^* + DB^*u. \qquad \text{.....(90)}$$

Model reduction by modal elimination has further been considered by several other authors, including KUPPURAJULU and ELANGOVAN (24), GOLD (25), TOWILL AND MEHDI (26), FOSSARD (27) and WILSON, FISHER and SEBORG (28).

KUPPURAJULU and ELANGOVAN (1970) (24) recommend using different models to represent the system at different points in the transient response. These different models are obtained by eliminating differing modes. Models containing poles further from the imaginary axis are used to represent the initial transient responses, whereas poles nearer to this imaginary axis are used to represent the final transients.

GOLD (1970) (25) examines the effect of eliminating various types of modes from the system.
He concludes that real or complex poles much more remote from

the s-plane origin than the dominant poles have little effect on the resultant time response. However, poles, zeroes or dipoles located closer to the origin than the dominant poles must be included in the transfer function.

To illustrate this point, GOLD produces the following example -

$$G(s) = \frac{(1+0.5s)(1+0.25s)}{(1+0.51s)(1+0.21s)(1+(0.013\pm0.16j)s)(1+(0.31\pm1.98j)\times10^{-3}s)(1+0.001s)} \quad .....(91)$$

Eliminating $(1+(0.31\pm1.98j)\times10^{-3}s)$ and/or $(1+0.001s)$ has little effect.

However, eliminating $\frac{1+0.5s}{1+0.51s}$ and/or $\frac{1+0.25s}{1+0.21s}$ would have a greater effect.

Evidence for this example can be obtained using the transient response to a unit step input.

In January 1970, TOWILL and MEHDI (26) published a paper which included a description of three different model reduction techniques, one of which, referred to by the authors as the s-plane model, is , in fact a modal elimination technique.

The original system is represented thus -

$$G(s) = K. \frac{1 + b_1 s + b_2 s^2 + ... + b_m s^m}{1 + a_1 s + a_2 s^2 + ... + a_n s^n} \quad .....(92)$$

$$= K. \frac{\prod_{j=1}^{j=m} (1 + T_j s)}{\prod_{i=1}^{i=n} (1 + T_i s)} \quad .....(93)$$

When reduced, this becomes

$$\hat{G}(s) = K.\frac{\prod_{j=1}^{j=q}(1 + T_j s)}{\prod_{i=1}^{i=r}(1 + T_i s)} \cdot e^{-\tau s} \qquad .....(94)$$

where τ is given by the MATSUBARA (1965) (29) equivalent time theorem as

$$\tau = \sum_{i=r+1}^{i=n} T_i - \sum_{j=q+1}^{j=m} T_j \qquad .....(95)$$

This theorem was derived by MATSUBARA from

$$\tau = \int_0^{\infty} (u(t) - \frac{i(t)}{K}) \, dt \qquad .....(96)$$

where $u(t)$ is a unit step function
$i(t)$ is the indicial response of $G_1(s)$
and K is the final value of the response.

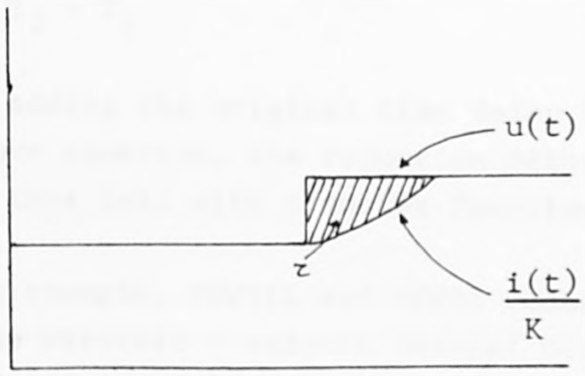


Figure 2.1 Basic definition of τ

$$= \frac{1}{s}\int \left[\int_0^{t} (u(t) - \frac{i(t)}{K}) \, dt \right] \qquad .....(97)$$

$$= \int \left[u(t) - \frac{i(t)}{K} \right] \qquad .....(98)$$

$$= \frac{1}{s}\left(1 - \frac{G_1(s)}{K}\right) \qquad \text{.....(99)}$$

Then, if -

$$G_1(s) = K\,\frac{1 + a_1 s + a_2 s^2 + \ldots\ldots}{1 + b_1 s + b_2 s^2 + \ldots\ldots}$$

$$\tau = \frac{(b_1 - a_1) + (b_2 - a_2)s + \ldots.}{1 + b_1 s + b_2 s^2 + \ldots\ldots} \qquad \text{.....(100)}$$

Applying the Final Value Theorem, this gives -

$$\tau = b_1 - a_1 \qquad \text{.....(101)}$$

Thus, if -

$$G_1 = \frac{1 + T_1 s}{1 + T_2 s} \qquad \text{.....(102)}$$

then

$$\tau = T_2 - T_1 \qquad \text{.....(103)}$$

By simply adding the original time delay to that obtained by the above equation, the reduction method of TOWILL and MEHDI can thus deal with transfer functions having pure time delay.

As an example, TOWILL and MEHDI reduce a sixth order model of an aircraft - asimuth channel blind landing system to a third order model.

The transient response of the model thus obtained is extremely close to that of the full sixth order model and is, in fact, closer than the response of those models obtained using CHEN and SHIEN's Continued Fraction method and the Updated Open Loop Bode Plot method, also described in this paper. (See Section 2.4)

The work of FOSSARD (27) also published in 1970, consists

of a comparison of the 1967 models of DAVISON andCHIDAMBARA. FOSSARD firstly suggests a correction that can be applied to DAVISON's original model eliminating the steady state error. This corrected model is as follows -

$$\dot{x}_1 = A^* x_1 + B^* u \quad \text{(as in equation 81)} \qquad .....(104)$$

$$x^* = x_1 - U_2 \Lambda_2^{-1} (V_3 B_1 + V_4 B_2) \qquad .....(105)$$

Then, using as an example, the evaluation of the time response to a single input of a chosen fourth order system, FOSSARD compares the two models obtained from it using DAVISON's method (incorporating his own correction) and CHIDAMBARA's method, and goes on to claim that the DAVISON model as corrected by himself gives a closer fit to the original time response than does the model obtined using CHIDAMBARA's method.

WILSON, FISHER and SEBORG (1972) (28) extend MARSHALL's and DAVISON's methods to linear descrete-time models and produce two approaches for reducing a high order continuous time model to a low order discrete-time model.

2.3 MODEL REDUCTION USING AGGREGATED MODELS.

A certain quantity of work on model reduction has been directed towards aggregated models.

These models are expressed in state variable form and the reduced model state variables are related to those of the original model by a matrix known as the aggregation matrix.

Hence, the system -

$$\dot{x} = Ax + Bu \qquad .....(106)$$

(where x is of dimension n)

is reduced to

$$\dot{x}^* = A^*x^* + B^*u \qquad .....(107)$$

(where x^* is of dimension l
and $l << n$)

and

$$x^* = Cx \qquad .....(108)$$

(where C is the aggregation matrix
with the dimensions l x n)

From these relationships it follows that -

$$A^*C = CA \qquad .....(109)$$

and

$$B^* = CB \qquad .....(110)$$

Now, if C were a square matrix, A^* would be related to A by the equation -

$$A^* = CAC^{-1} \qquad .....(111)$$

However, since C is not a square matrix and therefore has no

simple inverse, a pseudo inverse must be used instead. This pseudo inverse is equal to -

$$C'(CC')^{-1} \qquad \text{.....(112)}$$

and defines the relationship between A^* and A as -

$$A^* = CAC'(CC')^{-1} \qquad \text{.....(113)}$$

Now, from

$$A^*C = CA \qquad \text{(eqn. 109)}$$

$$CA\nu_i = A^*C\nu_i \qquad \text{.....(114)}$$

$$= \lambda_i C\nu_i \qquad \text{.....(115)}$$

where λ_i is the ith eigen value of A

and ν_i is the ith eigen vector of A

Then. if $C\nu_i \neq 0$

λ_i is also an eigen value of A^*

and $C\nu_i$ is the equivalent eigen vector of A^*

$C\nu_i \neq 0$ for $1 \leqslant i \leqslant 1$

$C\nu_i = 0$ for $d+1 \leqslant i \leqslant n$.

From the above it can be seen that this method of model reduction is effectively a form of modal elimination where the main problem is the selection of a suitable value for C i.e. the aggregation matrix.

Work along these lines has been carried out by a number of authors including MITRA (30, 31) who tackled the problem over the years 1967 - 1969, and AOKI (32) who published work on this method of model reduction in 1968. Neither of these authors was, however, able to define a suitable method for finding C.

Then, in 1970, CHIDAMBARA and SCHAINKER (33) published a paper, in which they suggested an alternative reduction method,

simplifying

$$\dot{x} = Ax + Bu \qquad .....(116)$$

$$y = Hx \qquad .....(117)$$

to

$$\dot{x}^* = A^*x^* + B^*u \qquad .....(118)$$

$$y^* = H^*x^* \qquad .....(119)$$

However, the reduced model thus obtained would, include an error e in the output y.

To overcome this problem, the authors suggest that the reduced model be amended to -

$$\dot{x}^* = A^*.x^* + f.e., + B^*.u \qquad .....(120)$$

$$y^* = H^*x^* \qquad .....(121)$$

where

$$e = y - y^* = Hx - H^*x^* \qquad .....(122)$$

This then gives

$$\dot{x}^* = \left[A^* - f.H^*\right] x^* + f.Hx + B^*u \qquad .....(123)$$

$$= Dx^* + f.H.x. + B^*u \qquad .....(124)$$

where

$$D = A^* - f.H^* \qquad .....(125)$$

and

$$x^* = Cx \qquad .....(126)$$

where

C is again the aggregation matrix.

By relating the states of the original system to those of the model, the authors obtain the relationship

$$CA - DC = f.H \qquad .....(127)$$

$$CB = B^* \qquad .....(128)$$

which corresponds to

$$CA = A^{*}C \qquad .....(129)$$

and

$$CB = B^{*} \qquad .....(130)$$

as given by AOKI.

The ouput equations then give

$$H = H^{*}C \qquad .....(131)$$

CHIDAMBARA and SCHAINKER further suggest a number of optimisation techniques to minimise f and thus to give an optimum aggregation matrix and hence a much improved reduced model.

Approaching this method of model reduction from another viewpoint, LAMBA and VITTAL RAO (34,35,36,37) chose, over the years 1973 - 1975, to examine the models obtained by DAVISON and CHIDAMBARA with aggregation in mind.

Reducing

$$\dot{x} = Ax + Bu \qquad .....(132)$$

to

$$\dot{z} = Fz + Gu \qquad .....(133)$$

gives

$$z = Cx \qquad .....(134)$$

$$F = CAC'(CC')^{-1} \qquad .....(135)$$

and $$G = CB \qquad .....(136)$$

where

C is the aggregation matrix.

Now,

$$x = Mv \qquad .....(137)$$

where

M is the modal matrix of A

and $$\dot{v} = \Lambda v + \Gamma u \qquad .....(138)$$

where

$$\Lambda = M^{-1}AM \qquad .....(139)$$

$$\Gamma = M^{-1}B \qquad .....(140)$$

Similarly

$$z = M_o w \qquad .....(141)$$

where

M_o is the modal matrix of F.

Retaining the first l modes, this then gives

$$w = Tv \qquad .....(142)$$

where

$$T = \begin{bmatrix} I_l & 0 \end{bmatrix} \qquad .....(143)$$

and hence

$$\dot{w} = T\Lambda T'w + T\Gamma u \qquad .....(144)$$

From the above, therefore

$$\dot{z} = M_o T\Lambda T' M_o^{-1} z + M_o T M^{-1} Bu \qquad .....(145)$$

and

$$z = M_o w = M_o T M^{-1} x \qquad .....(146)$$

and therefore

$$C = M_o T M^{-1} \qquad .....(147)$$

Under these conditions, M_o can be selected at will to give the desired form of F, so that, if, for example, it is decided to retain certain state variables of the original system, an appropriate value to achieve this can be allocated to M_o.

Further work on aggregated models was published in 1975 by MICHAILESCO, SIRET and BERTRAND (38) and by HICKIN and SINHA (39), who had previously published a number of shorter works on aggregated models.

In December 1975, these authors put forward a method for

selecting a suitable aggregation matrix. Working along similar lines to LAMBA and VITTAL RAO, they reduce

$$\dot{x} = Ax + Bu \qquad .....(148)$$

$$y = Cx \qquad .....(149)$$

to

$$\dot{z} = Fz + Gu \qquad .....(150)$$

$$y_1 = Hz \qquad .....(151)$$

where x is of the nth order
u is of the qth order
z is of the rth order
y & y_1 are of the pth order

and $r < n$

However, at this point the various authors choose to differ, for, whereas MITRA and MICHAILESCO, SIRET and BERTRAND have chosen to take $r = p$ in their work, HICKIN and SINHA have decided to dispense with this restriction.

Now, taking

$$z = Kx \qquad .....(152)$$

the aggregation matrix, K, is then defined as

$$K = M.K_o \qquad .....(153)$$

where

$$\left[M\right] \neq 0 \qquad .....(154)$$

and $$K_o = \left[I_r \vdots 0\right] V^{-1} \qquad .....(155)$$

where

V is the modal matrix of A

and V^{-1} is the modal matrix of A'

The state equations then give -

$$FK = KA \qquad .....(156)$$

$$G = KB \qquad .....(157)$$

$$HK = C \qquad .....(158)$$

(as do those of AOKI)
(see eqns.109-111)

These equations are then re-written as

$$F = KAK^{+} \qquad .....(159)$$

$$G = KB \qquad .....(160)$$

$$H = CK^{+} \qquad .....(161)$$

where

K^{+} is a pseudo inverse of K.

Here again, the authors differ in their ideas, since MITRA chooses to use K^{n} instead of K^{+}

where K^{n} is the right weak generalised inverse of K

and $KAK^{+} - KAK^{n} = 0 \qquad .....(162)$

BUT $CK^{+} \neq CK^{n} \qquad .....(163)$

HICKIN and SINHA, however, state that $K^{+}$is to be preferred as it minimises the output error

i.e. $\| C(I_o - K^{+}K) \|$

An example offered by HICKIN and SINHA to illustrate their reduction technique is included at this point.This example, although trivial, serves to demonstrate the similarity of this technique to others dealing with modal elimination.

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \qquad ..(164)$$

which is equivalent to

$$G(s) = \frac{(s + 1)}{(s + 1)(s + 2)(s + 3)} \qquad .....(165)$$

For a second order model, K_o is obtained by selecting 2 out of 3 rows from

$$V^{-1} = \begin{bmatrix} 3 & 5/2 & 1/2 \\ -3 & -4 & -1 \\ 1 & 3/2 & 1/2 \end{bmatrix} \qquad .....(166)$$

The choice

$$K_o = \begin{bmatrix} -3 & -4 & -1 \\ 1 & 3/2 & 1/2 \end{bmatrix} \qquad .....(167)$$

gives

$$J_{min}\ (\emptyset\ ,\ \tau\ ,\ u) = 0$$

for all τ and u values as the original system was not of minimal order.

The reduced system is

$$F = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix} \qquad G = \begin{bmatrix} -1 \\ 1/2 \end{bmatrix} \qquad H = (-1 \quad -2) \qquad .....(168)$$

which is equivalent to

$$\hat{G}(s) = \frac{1}{(s + 2)(s + 3)} \qquad .....(169)$$

as would be expected.

A further contribution to model reduction by aggregation was published by TSE, MEDANIC and PERKINS (40) in 1973.

The system representation

$$\dot{x} = Ax + Bu \qquad \text{(of order n)} \qquad .....(170)$$

$$y = Cx \qquad \text{(of order r)} \qquad .....(171)$$

is transformed into the generalised HESSENBERG representation

$$\dot{z} = Fz + Gu \qquad \text{.....(172)}$$

$$y = Dx \qquad \text{.....(173)}$$

where

$$F = \begin{bmatrix} F_{11} & F_{12} & 0 & \cdots & 0 \\ F_{21} & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & F_{k-1k} \\ F_{k_1} & \cdots & \cdots & & F_{k,k} \end{bmatrix} \qquad G = \begin{bmatrix} C_1 \\ \vdots \\ \vdots \\ \vdots \\ G_k \end{bmatrix} \qquad \text{.....(174)}$$

$$\text{and} \quad D = \begin{bmatrix} i & 0 & \cdots & 0 \end{bmatrix} \qquad \text{.....(175)}$$

This transformation is achieved in a number of steps which are referred to as chained aggregation.

The first step involves replacing the first r states of the state vector x by the r output y, i.e.

$$z' = T'x = \begin{bmatrix} y \\ \cdots \\ x'' \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_r \\ x_{r+1} \\ \vdots \\ x_n \end{bmatrix} \qquad \text{.....(176)}$$

where

$$T' = \begin{bmatrix} C_1 & \vdots & C_2 \\ \cdots & \vdots & \cdots \\ 0 & \vdots & I \end{bmatrix} \begin{matrix} \wedge \\ r \\ \vee \\ \wedge \\ n-r \\ \vee \end{matrix} \qquad \text{.....(177)}$$

$$< r \quad > < n-r >$$

$$C = \begin{bmatrix} C_1 & \vdots & C_2 \end{bmatrix} \qquad \text{.....(178)}$$

giving

$$\dot{z}' = F'z' + G'u \qquad \text{.....(179)}$$

or

$$\begin{bmatrix} \dot{y} \\ \cdots \\ \dot{x}'' \end{bmatrix} = \left[\begin{array}{c:c} F'_{11} & F'_{12} \\ \hdashline F'_{21} & F_{22} \end{array}\right] \begin{bmatrix} y \\ \cdots \\ x'' \end{bmatrix} + \begin{bmatrix} G'_1 \\ \cdots \\ G'_2 \end{bmatrix} u \qquad \text{.....(180)}$$

where

$$F' = T'AT'^{-1} \qquad \text{.....(181)}$$

and

$$G' = T'B \qquad \text{.....(182)}$$

The system will be completely aggregable if $F'_{12} = 0$

giving

$$\dot{y} = F'_{11}y + G'_1u.$$

More often than not, however, $F'_{12} \neq 0$ and a further transformation is required to be carried out in order to zeroise as many elements of F'_{12} as possible.

Thus

$$z'' = T''z' = \begin{bmatrix} y \\ \cdots \\ y'' \\ \cdots \\ x''' \end{bmatrix} \qquad \text{.....(183)}$$

where

$$T'' = \left[\begin{array}{c:c:c} I & 0 & 0 \\ \hdashline 0 & C'_1 & C'_2 \\ \hdashline 0 & 0 & I \end{array}\right] \qquad \text{.....(184)}$$

$$C' = \left[\begin{array}{c:c} C'_1 & C'_2 \end{array}\right] \qquad \text{.....(185)}$$

and

$$y'' = C'x'' = E'F'_{12}x'' \qquad .....(186)$$

where

E' is the product of elementary GAUSSIAN matrices, each differing from the unit matrix only in specific rows.

This transformation now gives

$$\dot{z}'' = F''z'' + G''u \qquad .....(187)$$

or

$$\begin{bmatrix} \dot{y} \\ \dots \\ \dot{y}'' \\ \dots \\ x''' \end{bmatrix} = \left[\begin{array}{c:c:c} F'_{11} & F''_{12} & 0 \\ \hdashline F''_{21} & F''_{22} & F''_{23} \\ \hdashline F''_{31} & F''_{31} & F''_{33} \end{array}\right] \begin{bmatrix} y \\ \dots \\ y'' \\ \dots \\ x''' \end{bmatrix} + \begin{bmatrix} G'_1 \\ \dots \\ G''_2 \\ \dots \\ G''_3 \end{bmatrix} u \qquad .....(188)$$

where

$$F'' = T''F'T''^{-1} = T''T'AT'^{-1}T''^{-1} \qquad .....(189)$$

and

$$G'' = T''G' = T''T'B \qquad .....(190)$$

If $F''_{23} = 0$, x''' can then be eliminated without any loss of accuracy.

Otherwise , the transformation must be continued until the generalised HESSENBERG representation is reached. This can then be partitioned to give a reduced order state vector of ith order.

However, this involves disregarding a term in F, namely $F_{i,i+1}$, which must approach zero to give a good approximation, consequently i must be selected appropriately.

Although this technique is considerably more complicated than the other aggregation techniques described, nevertheless, on examples produced by the authors i.e. a ninth order system reduced to fifth order, it appears to give satisfactory results.

2.4 MODEL REDUCTION BY CONTINUOUS FRACTION EXPANSION.

Another method of model reduction was proposed by CHEN and SHIEH (19) in 1968.

Working in transfer function notation and using basic systems theory, these two authors proposed a method of reduction which consists of expanding the transfer function into a continued fraction and then truncating it at the point required to give a reduced transfer function of the desired order.

For example:-

$$G(s) = \frac{1 + b_1 s + b_2 s^2}{1 + a_1 s + a_2 s^2 + a_3 s^2} \qquad \ldots\ldots(191)$$

$$= \frac{1}{1 + s\,\dfrac{c_1 + c_2 s + c_3 s^2}{1 + b_1 s + b_2 s^2}} \qquad \ldots\ldots(192)$$

where $c_1 = a_1 - b_1$

$c_2 = a_2 - b_2$

$c_3 = a_3$

$$= \cfrac{1}{1 + \cfrac{s}{\cfrac{1}{c_1} + \cfrac{s}{\cfrac{c_1}{d_1} + \cfrac{s}{\cfrac{d_1}{e_1} + \cfrac{s}{\cfrac{e_1}{f_1} + \cfrac{s}{\cfrac{f_1}{g_1}}}}}}} \qquad \ldots\ldots(193)$$

where $d_1 = b_1 - \dfrac{c_2}{c_1}$

$$d_2 = b_2 - \frac{c_3}{c_1}$$

$$e_1 = c_2 - \frac{c_1 d_2}{d_1}$$

$$f_1 = d_2 - \frac{d_1 c_3}{e_1}$$

$$g_1 = c_3$$

Reduced models are obtained by truncating the continued fraction at any given point.

$$\hat{G}_1(s) = \frac{1}{1 + \frac{s}{\frac{1}{c_1}}} = \frac{1}{1 + (a_1 - b_1)s} \qquad \ldots\ldots(194)$$

$$\hat{G}_2(s) = \frac{1}{1 + \frac{s}{\frac{1}{c_1} + \frac{s}{\frac{c_1}{d_1}}}} = \frac{1 + (b_1 - \frac{a_2 - b_2}{a_1 - b_1})\, s}{1 + (a_1 - \frac{a_2 - b_2}{a_1 - b_1})\, s} \qquad \ldots\ldots(195)$$

The advantage of this method is that, although none of of the original poles are retained, the time response of models obtained in this way is generally much closer to the response of the original system than that of models obtained by modal elimination.

The main disadvantage, however, is that if the original system is unstable, this method can give a stable reduced model and vice versa.

Nevertheless, this method has become quite popular and since CHEN and SHIEH's original paper in 1968 much work has been done along these lines.

In 1970, CHEN and SHIEH themselves published a second paper (41) which gave further details of their method and also included several examples.

Also in 1970, CHUANG (42) published some observations on CHEN and SHIEH's method and proposed the following refinement. After pointing out that this method is equivalent to a Taylor series expansion about s = 0, thus ensuring that the model gives the correct steady state response, but, not giving a particularly good approximation to the initial transient response, CHUANG goes on to suggest that this latter factor, namely the transient response of the reduced model, can be improved by obtaining this model from a Taylor series expansion of the original system transfer function not only about s = 0, but also about s = ∞.

He proposes that this be achieved by forming the continuous fraction expansion, starting alternatively from the constant term and the highest order, as follows:-

$$G(s) = \frac{1 + b_1 s + b_2 s^2}{1 + a_1 s + a_2 s^2 + a_3 s^3} \qquad .....(196)$$

$$= \cfrac{1}{1 + \cfrac{s}{\cfrac{b_2}{c_3} + \cfrac{1}{\cfrac{c_1}{d_1} + \cfrac{s}{\cfrac{d_2}{e_2} + \cfrac{1}{\cfrac{e_1}{f_1} + \cfrac{s}{\cfrac{f_1}{g_1}}}}}}} \qquad .....(197)$$

where $c_1 = a_1 - b_1$, $c_2 = a_2 - b_2$, $g_1 = e_2 = c_3 = a_3$,

$$d_1 = 1 - \frac{b_2 c_1}{c_2} \quad , \quad d_2 = b_1 - \frac{b_2 c_2}{c_3} \quad ,$$

$$e_1 = c_2 - \frac{c_1 d_2}{d_1} \quad , \quad f_1 = d_1 - \frac{d_2 e_1}{e_2} \quad .$$

In common with CHEN and SHIEH's method, reduced models are then obtained by truncating the above fraction at any given point.

Hence

$$\hat{G}_1(s) = \frac{1}{1 + \dfrac{s}{\dfrac{b_2}{c_3} + \dfrac{d_1}{c_1}}} = \frac{1}{1 + (a_1 - b_1)s} \qquad .....(198)$$

$$\hat{G}_2(s) = \frac{1}{1 + \dfrac{s}{\dfrac{b_2}{c_3} + \dfrac{1}{\dfrac{c_1}{d_1} + \dfrac{s}{\dfrac{d_2}{e_2} + \dfrac{f_1}{e_1}}}}} = \frac{1 + T_1 s}{1 + T_2 s + T_3 s^2} \qquad ..(199)$$

where

$$T_1 = \frac{(a_2 - b_2 - b_1(a_1 - b_1))a_3}{a_3^2 + (a_3 - 2b_2)(a_1 - b_1) + (b_2^2 - b_1)(a_1 - b_1)^2}$$

$$T_2 = \frac{(b_2 - a_2 + (a_3 - b_1)(a_1 - b_1) - b_1(a_1 - b_1)^2)a_3}{a_3^2 + (a_3 - 2b_2)(a_1 - b_1) + (b_2^2 - b_1)(a_1 - b_1)^2}$$

$$T_3 = \frac{(a_2 - b_2 - b_1(a_1 - b_1))a_3^2}{a_3^2 + (a_3 - 2b_2)(a_1 - b_1) + (b_2^2 - b_1)(a_1 - b_1)^2}$$

This was followed in 1971 by work published by AKIN (43) in which the author suggests mathematical refinements to the computer implementation of CHEN and SHIEH's method.

Two further papers were published in 1971 and 1972 (44,45) by CHEN himself, in which he extends his reduction technique to the multivariable case by replacing the continuous fraction coefficients by matrices.

Having indicated that even the simplest model obtained using this method involves all elements of the state equations, CHEN then demonstrates that the matrix continued fraction can be represented by a block diagram where n closed loops are formed by feedforward links H_1 , H_2 , ... H_{2n-1}

and feedback links H_2/s , H_4/s , ... H_{2n}/s

where

H_1 ..... H_{2n} are the coefficients in the matrix continued fraction expansion:-

$$G(s) = (H_1 + s(H_2 + s(H_3 + \ldots s(H_{2n})^{-1} \ldots)^{-1})^{-1})^{-1} \quad \ldots(200)$$

Truncating the continued fraction is then equivalent to disregarding the innermost loops.

These two papers also include suggestions by CHEN on how this reduction technique could be used for the design of multivariable systems.

A further variation of this reduction technique is due to D.J. WRIGHT (46) who, in September 1973, issued an analysis of the continued fraction method which included some of its shortcomings, his main criticism of CHEN and SHIEH's method being its lack of generality in that not all transfer functions can be expanded in this way.

WRIGHT, himself, then proposes another form of continued fraction expansion to overcome this drawback. Initially, he multiplies both the numerator and the denominator of the original transfer function by s^{-n} (where n is the order of the system) thereby obtaining G(s) as a ratio of two polynomials in t (where $t = s^{-1}$), each of degree n.

Hence $$G(s) = \frac{R_1(t)}{R_0(t)} \quad \ldots\ldots(201)$$

This is then expanded as follows:-

$$G(s) = \frac{R_1(t)}{R_0(t)} = \frac{1}{R_0/R_1} \quad \ldots\ldots(202)$$

$$= \frac{1}{Q_1 + \frac{1}{R_1/R_2}} \qquad \text{.....(203)}$$

$$= \frac{1}{Q_1 + \frac{1}{Q_2 + \frac{1}{R_2/R_3}}} \qquad \text{.....(204)}$$

where Q_1 is determined by the recursive equation

$$R_{i-1} = R_i Q_i + R_{i+1} \qquad \text{.....(205)}$$

At this point, CHEN and SHIEH's method of expansion would give

$$Q_i = \begin{cases} t.h_i & \text{for i even} \\ h_i & \text{for i odd} \end{cases}$$

where h_i is a constant.

WRIGHT, however, suggests the use of a more general form of h_i, making it a polynomial in t and thus overcoming the problem that results if, during the expansion, two functions R_i and R_{i+1} arise, and the terms of these functions are related in such a way that R_{i+2} is p degrees less than R_i where $p > 1$.

Under these circumstances, it becomes impossible to use CHEN and SHIEH's method to expand the continued fraction any further without resorting to functions R_{i+3} and R_{i+4} etc. which progressively increase in degree and are consequently not suitable for truncation.

This point can be illustrated thus -

$$G(s) = \frac{1 + s - s^2}{1 + s + s^2 + as^3} \qquad \text{.....(206)}$$

$$= \cfrac{1}{1 + \cfrac{s}{h_1 + \cfrac{s}{h_2 + \cfrac{s}{h_3}}}} \qquad \text{.....(207)}$$

where $h_1 = (\frac{1}{2} - \frac{a}{4}) + \frac{1}{2s}$

$$h_2 = \frac{8}{a^2 - 2a - 4}$$

$$h_3 = (\frac{a}{4} - \frac{1}{2} - \frac{1}{a})$$

At this stage, if h_1 is not taken as a function of $\frac{1}{s}$, i.e. if it is considered as a constant, as in CHEN and SHIEH's method, this transfer function cannot be expanded as a continuous fraction of this form and so no simplification is possible.

However, if WRIGHT's method is used, then -

$$G_1(s) = \frac{1}{1 + (\frac{4}{2 - a})\, s} \qquad \text{.....(208)}$$

$$G_2(s) = \frac{1 + (1 - \frac{a}{2})\, s}{1 + (1 - \frac{a}{2})\, s + 2s^2} \qquad \text{.....(209)}$$

It is worth mentioning that the above transfer function was also expanded by CHUANG, who obtained two models, neither of which , unfortunately was satisfactory, since both were unstable where the original system was stable.

Several papers were published in 1973 and 1974, in which the various authors commented on, compared and criticised existing continued fraction expansion techniques.

Thus in September 197?, BOSLEY, KROPHOLLER and LEES (47) compared CHEN and SHIEH's method with the moment matching

method of GILIBARO and LEES (48) and showed that the two methods always give identical results and are therefore equivalent to each other.

Some later papers do not differentiate between the two methods & in 1974, DAVIDSON and LUCAS (49) proposed a reduction technique which they referred to as a continued fraction method but which, in fact, involves matching moments rather than a continued fraction expansion.(See Section 2.5)

Then, in May 1974, CALFE and HEALEY (50) published a work which considered some of the difficulties experienced when applying matrix continued fraction reduction techniques to multivariable systems, as described by CHEN in 1971 and 1972. These two authors point out that if

$$G(s) = \frac{N(s)}{D(s)} \qquad .....(210)$$

then N(s) must be nonsingular for expansion to be possible yet under most circumstances this is simply not the case.

CALFE and HEALEY further indicate that if the order of D(s) is reduced by the cancellation of a common factor also occuring in N(s), then an attempt at further reduction using the continued fraction technique may give a denominator polynomial of higher order than D(s).

Consequently the authors conclude that the matrix continued fraction method is totally unsuitable as a basis for a method of reduction of the order of a multivariable system.

However, it is only fair to say that although the argument concerning the unsuitability of this particular method is valid for certain specific systems, the criticism levelled by these authors appears to be somewhat severe, since the evidence tends to suggest that all the reduction techniques examined are considerably more suited to certain systems than to others and CHEN has, at least, been able to demonstrate that for certain systems, the reduced models obtained by his method are very close in frequency and time response to the original system.

In August 1974, a further contribution to the work on continued fraction truncation as a method of model reduction was published by SHAMASH (51), who produced a paper extending the use of the method to discrete time systems.

A method of continued fraction expansion and inversion for multivariable systems using a generalised form of matrix Routh algorithm was published by SHIEH and GAUDIANO in two papers issued in 1974 and 1975 (52,53). These papers also included the laying out of three types of matrix continued fraction expansions referred to as the 1st., 2nd., and 3rd. Cauer forms.

The 1st. Cauer form is -

$$G(s) = (H_1 s + (H_2 + (H_3 s + (H_4 + (....)^{-1})^{-1})^{-1})^{-1})^{-1} \quad .....(211)$$

and is an expansion about $s = \infty$.

A model based on this expansion gives a good fit to the transient response but results in a steady state error.

The 2nd. Cauer form is -

$$G(s) = (H_1 + (H_2\frac{1}{s} + (H_3 + (H_4\frac{1}{s} + (....)^{-1})^{-1})^{-1})^{-1})^{-1}$$

$$= (H_1 + s(H_2 + s(H_3 + s(H_4 + s(....)^{-1})^{-1})^{-1})^{-1})^{-1} \quad .....(212)$$

This is the form used by CHEN in his matrix continued fraction expansion and is an expansion about $s = 0$.

Hence it eliminates steady state error but is very susceptible to error in transient response.

The 3rd. Cauer matrix form is -

$$G(s) = (H_1 + H_1's + (H_2\frac{1}{s} + H_2' + (H_3 + H_3's + (H_4\frac{1}{s} + H_4' + (..$$

$$..)^{-1})^{-1})^{-1})^{-1} \quad .....(213)$$

This form of the matrix is a simultaneous expansion about $s = 0$ and $s = \infty$.

A model obtained by truncating this expansion has no steady state error and tends to give a good fit to the initial transient response.

SHIEH and GAUDIANO then go on to suggest a remedy for what is considered by CALFE and HEALEY to be one of the greatest faults of matrix continued fractions, namely the fact that if the determinant of any matrix to be inverted is zero then any matrix continued fraction expansion undertaken will be prematurely terminated.

The authors suggest that this difficulty can be overcome by introducing a constant matrix Q_0, which can act on the original transfer function either by multiplication of this transfer function by $(sI + Q_0)$ or by the addition of $\frac{1}{s - x} Q_0$ to it.

Meanwhile. in November 1974, a paper was published by SHIEH and GOLDMAN (54) in which the authors propose a continued fraction technique which bears a striking similarity to that previously published by CHUANG in 1970. However, no reference is made by the authors to this fact.

The method consists of expanding the original transfer function into a continuous fraction, namely the 2nd. Cauer or mixed Cauer form, and then, as in other continuous fraction reduction techniques, truncating this at a given point.

The form of the expansion, which differs slightly from that used by CHUANG, is as follows:-

$$G(s) = \frac{A_{21} + A_{22}s + A_{23}s^2 + \ldots + A_{2,n-1}s^{n-2} + A_{2,n}s^{n-1}}{A_{11} + A_{12}s + A_{13}s^2 + \ldots + A_{1,n}s^{n-1} + A_{1,n+1}s^n} \quad (214)$$

$$= \frac{1}{\frac{A_{11}}{A_{21}} + \frac{A_{1,n+1}}{A_{2,n}}s + \frac{A_{31}s + A_{32}s^2 + \ldots + A_{3,n-1}s^{n-1}}{A_{21} + A_{22}s + \ldots + A_{2,n}s^{n-1}}} \quad (215)$$

where

$$A_{31} = A_{12} - \frac{A_{11}A_{22}}{A_{21}} - \frac{A_{1,n+1} \cdot A_{21}}{A_{2n}} \quad \ldots\ldots(216)$$

$$A_{32} = A_{13} - \frac{A_{11}A_{23}}{A_{21}} - \frac{A_{1,n+1} \cdot A_{22}}{A_{2n}} \quad \ldots\ldots(217)$$

and, in general

$$A_{3i} = A_{1,i+1} - \frac{A_{11}A_{2,i+1}}{A_{21}} - \frac{A_{1,n+1}\cdot A_{2,i}}{A_{2n}} \qquad \text{.....(218)}$$

for i = 1 to n-1.

Further division gives:-

$$G(s) = \cfrac{1}{h_1 + h_1's + \cfrac{1}{\cfrac{h_2}{s} + h_2' + \cfrac{1}{h_3 + h_3's + \cfrac{1}{\cfrac{h_4}{s} + h_4' + \cfrac{1}{\ddots}}}}} \qquad \text{.(219)}$$

where

$$h_i = \frac{A_{i,1}}{A_{i+1,1}} \qquad h_i' = \frac{A_{i,n+2-i}}{A_{i+1,n+1-i}}$$

for i = 1 , 2 , 3 ,....., n

and

$$A_{j,k} = A_{j-2,k+1} - h_{j-2}\cdot A_{j-1,k+1} - h_{j-2}'\cdot A_{j-1,k}$$

for j = 3 , 4 ,....., n+1

and k = 1 , 2 , 3 ,.....

Using this method, each step in the expansion produces 2 coefficients, one of which is for an expansion about s = 0 and the other for an expansion about s = ∞.

CHUANG's method, which produces an almost identical reduced model requires 2 separate steps to obtain these coefficients and consequently the values of these coefficients are different from those obtained by the method proposed by SHIEH and GOLDMAN.

Nevertheless, since both methods use the principle of simultaneously expanding the transfer function about s = 0 and s = ∞, both are subject to the same basic advantages and disadvantages.

In 1977, SHIEH made yet another contribution to the work on this method of model reduction by publishing, this time in combination with PATEL and CHOW, a further paper on the application of this method to multivariable systems, a topic previously raised by him in the paper published in co-operation with GAUDIANO.

SHIEH, PATEL and CHOW (55) extend CHEN's matrix continued fraction method to multivariable systems having a different number of inputs and outputs. Since such systems have non-square transfer function matrices, which consequently cannot be inverted in the normal way, a method using pseudo inverses is proposed and described.

Then, in 1978, two papers were published by PARTHASARTHY and SARASU JOHN (56,57), in which the authors present an algorithm for CHUANG's method of model reduction, which they subsequently extend into a matrix continued fraction reduction technique giving the expansion -

$$G(s) = (H_1 + s(H_1{}' + (H_2 + s(H_2{}' + (\ldots.)^{-1})^{-1})^{-1})^{-1})^{-1} \qquad \ldots..(220)$$

These papers also include an algorithm based on the matrix Routh array for expanding and inverting transfer function equations in this way.

Also, in 1978, FIELD and OWENS(58) presented a technique which although based on a continued fraction expansion can incorporate other methods of model reduction.

This technique bears some similarity to that of WRIGHT (to whom the authors do not refer), in that both methods expand the transfer function in terms of polynomials rather than simple constants.

Thus

$$G(s) = \frac{B_1 \prod_{j=1}^{m} (s - z_j)}{\prod_{j=1}^{n} (s - p_j)} \qquad \ldots..(221)$$

where

$$B_1 \neq 0$$

is expanded to

$$G(s) = \frac{g_1(s)}{1 - g_1(s)\frac{1}{B_1}.h_1(s)} \qquad .....(222)$$

where

$h_1(s)$ is a strictly proper transfer function of order m with poles at z_1 to z_m.

and

$g_1(s)$ is a transfer function of order $k_1 = n - m$

i.e.

$$g_1(s) = \frac{B_1}{s^{k_1} + \alpha_{1,1}s^{k_1-1} + ... + \alpha_{1,k_1}} \qquad .....(223)$$

$h_1(s)$ to $h_{r-1}(s)$ can be expanded similarly to G(s)

i.e.

$$h_j = \frac{g_{j+1}(s)}{1 - g_{j+1}(s)\frac{1}{B_{j+1}} . h_{j+1}(s)} \qquad .....(224)$$

giving

$$G(s) = (g_1^{-1}(s) - \frac{1}{B_1}(g_2^{-1}(s) - \frac{1}{B_2}(g_3^{-1} ..$$

$$.. \frac{1}{B_{r-1}}(g_r^{-1} - \frac{1}{B_r}(g_{r+1}^{-1})^{-1})^{-1} ....)^{-1})^{-1})^{-1} \qquad .....(225)$$

where r = m if , for all h_1 to h_r, the order of the numerator is 1 less than that of the denominator

otherwise r is less than m.

This expansion is equivalent to the arrangement shown in Figure 2.2 .

The authors suggest that the system can now be reduced

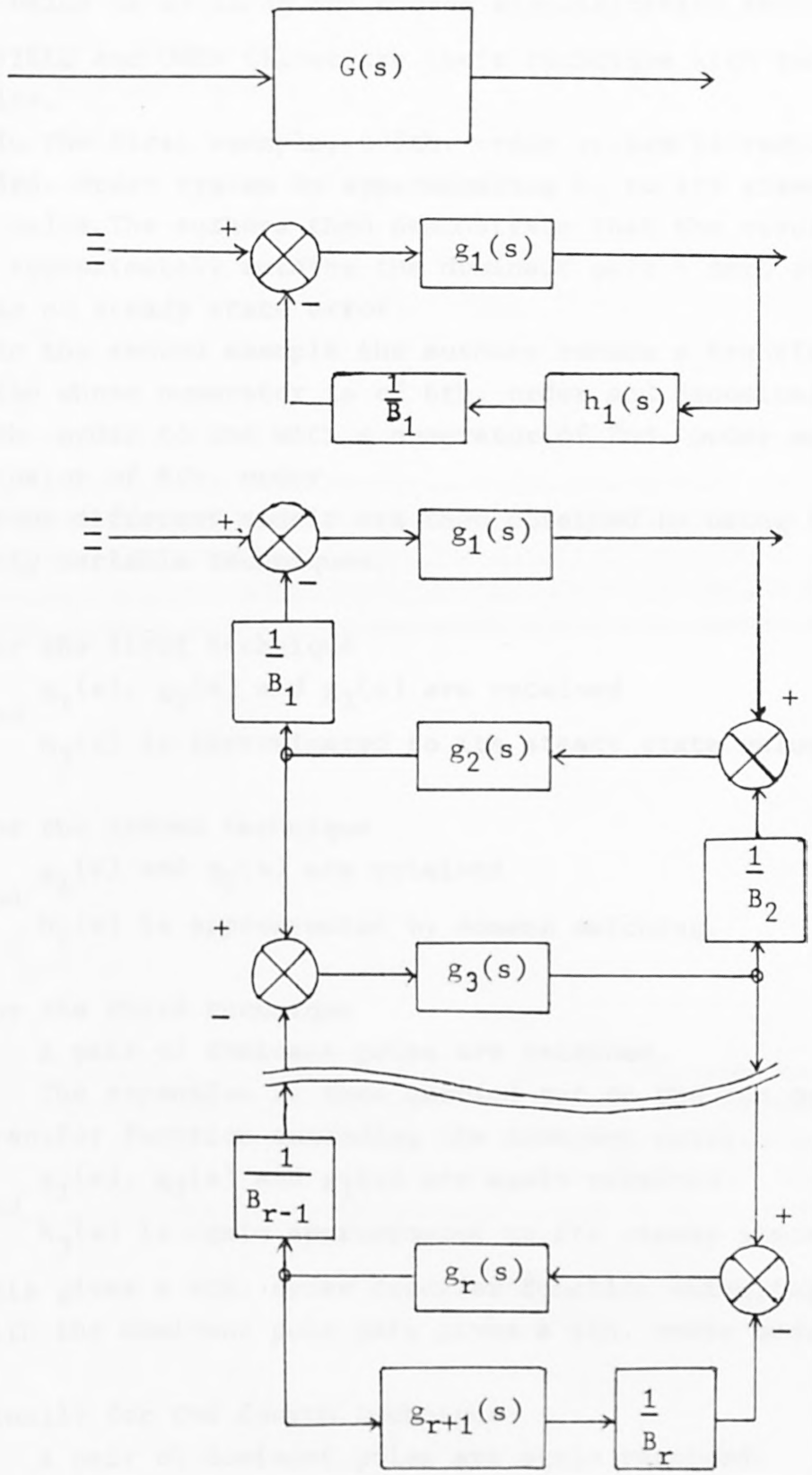


Figure 2.2

Diagrammatic representation of Field and Owens' expansion.

by approximating h_j at any stage, either simply to its steady state value or by using any chosen simplification technique.

FIELD and OWEN illustrate their technique with two examples.

In the first example, a 5th. order system is reduced to to a 3rd. order system by approximating h_2 to its steady state value.The authors then demonstrate that the resulting model approximately retains the dominant pole - zero structure and has no steady state error.

In the second example the authors reduce a transfer function whose numerator is of 6th. order and denominator is of 10th. order to one with a numerator of 2nd. order and a denominator of 6th. order.

Four different models are then obtained by using four slightly variable techniques.

1) For the first technique
   $g_1(s)$, $g_2(s)$ and $g_3(s)$ are retained
   and
   $h_3(s)$ is approximated to its steady state value.

2) For the second technique
   $g_1(s)$ and $g_2(s)$ are retained
   and
   $h_2(s)$ is approximated by moment matching.

3) For the third technique
   A pair of dominant poles are retained.
   The expansion is then carried out on the 3th.order transfer function excluding the dominant poles.
   $g_1(s)$, $g_2(s)$ and $g_3(s)$ are again retained
   and
   $h_3(s)$ is again approximated to its steady state value.
   This gives a 4th. order transfer function which,together with the dominant pole pair gives a 6th. order model.

4) Finally for the fourth technique
   A pair of dominant poles are again retained.
   The expansion is then carried out as in the third technique except that
   $g_1(s)$ and $g_2(s)$ only are retained
   and
   $h_2(s)$ is again approximated by moment matching as in

the second technique.

None of the models resulting from the above variations has a steady state error and the fourth model, in particular, gives a very good step response.

The method proves itself to be very versatile and its ability to incorporate other techniques is easily discernable.

Despite the fact that continued fraction techniques can be seen to have certain disadvantages, overall it would appear that, for many cases, very satisfactory models can be obtained.

This method of model reduction and its various techniques has been shown by certain authors to be related to model reduction techniques using moment matching. This latter method of model reduction is dealt with in the following section.

2.5 MODEL REDUCTION BY MOMENT MATCHING

The first full scale introduction of moment matching as a method of model reduction was published in 1969 by GILIBARO and LEES (48).

However, work on this method had been previously undertaken, in 1965, by GUSTAFSON (59) (See Section 2.7) who, in turn, referred to earlier work carried out in Germany by PAYNTER in 1956. (60).

The central principle of this method of model reduction is exemplified by taking the nth. moment of G(s) about the origin, M_n, as :-

$$M_n = \frac{\alpha_n}{\alpha_0} \qquad \text{.....(226)}$$

where $$\alpha_n = \int_0^{\infty} t^n g(t)dt \qquad \text{.....(227)}$$

$$= (-1)^n \left[\frac{d^n G(s)}{ds^n}\right]_{s=0} \qquad \text{.....(228)}$$

GUSTAFSON then takes the time moments as the values α_0 , α_1 , ... α_n etc. and states that α_0 is the area of the impulse response of G(s), as defined above.

When this area is normalised to unity, the first moment gives the mean time delay of the impulse response measured from the time axis.

At this point, GUSTAFSON is in complete agreement with GILIBARO and LEES who say that the first moment about the origin, M_1, localises the response on the time axis.

Subsequently, however, GUSTAFSON states that the second moment gives the mean square time delay of the impulse response which he considers as a measure of the rise time, whereas GILIBARO and LEES state that higher moments taken about M_1 characterise the shape of the curve.

Based on this latter assertion, GILIBARO and LEES suggest that it is generally advantageous to work in terms of the

mean, M_1 , and moments about the mean which are unaffected by pure time delays.

The nth. moment about the mean, T_n, is then given by:-

$$T_n = \frac{\int_0^{\infty} (t - M_1)^n \; g(t)dt}{\alpha_0} \qquad .....(229)$$

which gives

$$T_1 = 0$$
$$T_2 = M_2 - M_1^2$$
$$T_3 = M_3 - 3M_2M_1 + 2M_1^3$$
$$\text{etc.}$$

A reduced model of the required form and order can now be obtained by equating the same moments of the full system equation and the simple model and solving for the unknown reduced model coefficients.

If, however, G(s) is in the form of a single polynomial in s , then the reduced model can be obtained by equating the coefficients of s.

Following this work, papers were published in 1970 and 1972 by T.C. HSIA (61,62) who puts forward a reduction method which, although ultimately very similar in its method of evaluation to that of GILIBARO and LEES, is arrived at in the following very different manner.

Putting

$$G(s) = K \frac{a(s)}{b(s)} \qquad .....(230)$$

and

$$\hat{G}(s) = K \frac{c(s)}{d(s)} \qquad .....(231)$$

and writing

$$M(s) = a(s).d(s) \qquad .....(232a)$$

$$N(s) = b(s).c(s) \qquad .....(232b)$$

HSIA continues thus:-

Let

$$E(w) = \left| \frac{G(jw)}{\hat{G}(jw)} \right|^2 = \frac{M(jw)M(-jw)}{N(jw)N(-jw)} \qquad .....(233)$$

$$= \frac{M_0 + M_2 w^2 + M_4 w^4 + ...}{N_0 + N_2 w^2 + N_4 w^4 + ...} \qquad .....(234)$$

where

M_{21} (and similarly N_{21}) is given by

$$M_{21} = \frac{1}{(21)!} \cdot \frac{d^{21}(M(jw).M(-jw))}{dw^{21}} \Bigg|_{w=0} \qquad .....(235)$$

for $1 = 0 , 1 , 2 , ...$

If jw is then replaced by s this then gives

$$M_{21} = \sum_{k=0}^{21} (-1)^{k+1} \frac{M^{(k)}(s).M^{(21-k)}(s)}{k!\ (21-k)!} \Bigg|_{s=0} \qquad .....(236)$$

where

$$M^{(k)}(s) = \frac{d^k(M(s))}{ds^k} \qquad .....(237)$$

N(s) can also be expressed by similar equations.

Since for a perfect fit E(w) must equal 1, it follows that M_{21} must equal N_{21} for all values of 1. In reality, of course, one can only hope for an optimum model of a required order and in this case M_{21} must equal N_{21} for all values of 1 as far as this is possible.

The above equations then give -

$$M_0 = 1 \qquad .....(238)$$

$$M_2 = (M'(s))^2 - M^0(s).M^2(s)) \Big|_{s=0} \quad .....(239)$$

$$M_4 = \frac{1}{12}(3(M^2(s))^2 - 4.M'(s).M^3(s) + M^0(s).M^4(s)) \Big|_{s=0} \quad .....(240)$$

$$M_6 = \frac{1}{360}(10(M^3(s))^2 - 15.M^2(s).M^4(s) + 6M'(s).M^5(s) - M^0(s).M^6(s)) \Big|_{s=0}$$

etc. .....(241)

As can be seen, the terms obtained using HSIA's method are similar to those given by GILIBARO and LEES for the moments about the mean, the difference being that whereas HSIA deals in terms of -

$$M(s) = a(s).d(s) \qquad \text{(eqn. 232a)}$$

$$N(s) = b(s).c(s) \qquad \text{(eqn. 232b)}$$

GILIBARO and LEES work in terms of -

$$M(s) = a(s).c^{-1}(s) \quad .....(242)$$

$$N(s) = b(s).d^{-1}(s) \quad .....(243)$$

HSIA does not appear to have been aware of the work of GILIBARO and LEES and refers only to the reduction techniques of DAVISON and CHIDAMBARA (See Section 2.2), giving examples to demonstrate what he considers to be the superiority of his method over any techniques produced by these two authors. His criterion for this comparison is the integral square error (ISE) of the unit step response.

HSIA claims that pure time delays can be adequately dealt with using his method.

This, however, is not so since if either $G(s)$ or $\hat{G}(s)$ contains a pure time delay then $a(s)$ or $c(s)$ and consequently $M(s)$ or $N(s)$ contain a factor of the form $e^{-\tau s}$.

The resulting functions M_0 , M_2 , ... M_{21} or N_0 , N_2 , ... N_{21} are, however, completely independent of this factor.

Hence, if G(s) contains a time delay, this will be completely ignored using this method of reduction. Similarly, no pure time delay can be included in $\hat{G}(s)$ since no equations exist which could contain τ, which can, therefore, not be evaluated. (See Appendix).

Further papers on this method were published in 1971 by BROWN, who describes the use of moment matching techniques for modelling discrete time systems.(63,64)

Then, in a paper published in 1973 BOSLEY, KROPHOLLER and LEES (47) show that the moment matching method of GILIBARO and LEES and the continued fraction method of CHEN and SHIEH are equivalent and will give the same reduced models in all cases.

Subsequently, in 1974, DAVIDSON and LUCAS (49) published a reduction method under the description of " a continued fraction expansion about a general point ". However, on investigation, the work appears to follow moment matching lines rather than continued fraction expansion.

Thus, whereas CHUANG's continued fraction method matches alternatively about s=0 and s= ∞, and the methods of CHEN and SHIEH match about s=0, the method of DAVIDSON and LUCAS matches about s=a, where a is selected according to the system characteristics.

This method, then, matches the integrals of the impulse response weighted by $e^{-at}t^i$ for i = 0 , 1 , 2 , ... 2m-1

where m is the order of the reduced model.

Thus

$$G(s) = \int_0^{\infty} e^{-st}\, g(t)dt \qquad .....(244)$$

Then, by putting $s = a + z$

$$F(z) = G(a + z) = \int_0^{\infty} e^{-(a + z)t}\, g(t)dt \qquad .....(245)$$

$$= \sum_{i=0}^{\infty} \frac{(-1)^i z^i}{i!} \int_0^{\infty} e^{-at}\, t^i\, g(t)dt \qquad .....(246)$$

The reduced model is subsequently obtained by equating

$$\int_0^{\infty} e^{-at}\, t^i\, g(t)dt = \int_0^{\infty} e^{-at}\, t^i\, \hat{g}(t)dt \qquad .....(247)$$

for $i = 0 , 1 , 2 , \ldots 2m-1$

and $\hat{G}(s) = \hat{F}(S - a)$ .....(248)

In this paper, DAVIDSON and LUCAS also demonstrate that the CHEN and SHIEH continued fraction method is equivalent to a $(m , m-1)$ Padé approximation.

In October 1974, LAL and MITRA published two papers (65, 66). In the first of these, the authors present a moment evaluation algorithm for model reduction.The second paper presents a comparison of transfer function simplification methods and again shows that the moment matching method of model reduction is equivalent to the continued fraction method of CHEN and SHIEH.

This is demonstrated in the following manner -

Taking

$$G(s) = \sum_{k=0}^{\infty} C_k s^k \qquad .....(249)$$

and

$$\hat{G}(s) = \sum_{k=0}^{\infty} D_k s^k \qquad .....(250)$$

Moments are given by

$$M_k = (-1)^k\, k!\, C_k \qquad .....(251)$$

and

$$\hat{M}_k = (-1)^k \, k! \, D_k \qquad \text{.....(252)}$$

for k = 0 , 1 , 2 , ...

Matching moments then gives

$$M_k = \hat{M}_k \qquad \text{.....(253)}$$

and hence

$$C_k = D_k \qquad \text{.....(254)}$$

for k = 0 , 1 , 2 , ... 2r-1

where r is the order of the reduced model

Using CHEN and SHIEH's continued fraction expansion -

$$G(s) = \cfrac{1}{h_1 + \cfrac{1}{\cfrac{h_2}{s} + \cfrac{1}{h_3 + \ldots}}} \qquad \text{.....(255)}$$

the following cefficients are obtained -

$$h_1 = \frac{1}{C_0} \qquad h_2 = -\frac{C_0^{\,2}}{C_1}$$

$$h_3 = -\frac{C_1^{\,2}}{C_0(C_1^{\,2} - C_0C_2)}$$

$$h_4 = -\frac{(C_1^{\,2} - C_0C_2)^2}{C_1(C_0C_2^{\,2} - C_1C_0C_3)}$$

Model reduction by truncating the continued fraction gives similar equations for these coefficients in terms of the coefficients of the reduced model.

These two sets of equations simplify to

$$C_k = D_k \qquad \text{.....(256)}$$

for k = 0 , 1 , 2 , ... 2r-1 as before.

LAL and MITRA also comment on HSIA's method which they describe as a least square fit of frequency response.

Here the authors demonstrate that the method proposed by HSIA will effectively fit the following equations -

$$C_0^2 = D_0^2 \qquad \text{.....(257)}$$

$$C_1^2 - 2C_0C_2 = D_1^2 - 2D_0D_2 \qquad \text{.....(258)}$$

$$C_2^2 - 2C_1C_3 + 2C_0C_4 = D_2^2 - 2D_1D_3 + 2D_0D_4 \qquad \text{.(259)}$$

etc.

Another technique which can best be described in this section was put forward in July 1979 by DALY and COLEBOURN (67). This technique considers the system equations in state variable form and suggests a method for reducing the order of the system which matches either time moments, Markov parameters or a combination of both.

The basis of this technique can be represented in the following manner.

If the original system is expressed as -

$$\dot{x} = Ax + Bu \qquad \text{.....(260)}$$

$$y = Cx \qquad \text{.....(261)}$$

where x is of the order n

and this is then reduced to

$$\dot{x}^+ = A^+x^+ + B^+u \qquad \text{.....(262)}$$

$$y = C^+x^+ \qquad \text{.....(263)}$$

where x^+ is of the order m

and

$$A = \begin{bmatrix} A^* & A^c & 0 \\ \hline A^r & & \\ \hline 0 & & A^+ \end{bmatrix} \begin{matrix} \wedge \\ n-m \\ \vee \\ \wedge \\ 1 \\ \vee \\ m-1 \end{matrix} \qquad B = \begin{bmatrix} 0 \\ \hline \\ B^+ \end{bmatrix} \begin{matrix} \wedge \\ n-m \\ \vee \\ \wedge \\ 1 \\ \vee \\ m \\ \vee \end{matrix}$$

$$< \quad n-m \quad >< \quad 1 \quad m-1 \quad > \qquad \ldots.(264)$$

$$C = \begin{bmatrix} 0 & C^+ \end{bmatrix}$$

$$< \quad n-m \quad >< \quad m \quad > \qquad \ldots.(265)$$

where A^* , A^r , A^c are of any form but of appropriate dimension

and

$$A^+ = \begin{bmatrix} \alpha_m & 1 & 0 & \cdots & 0 \\ \beta_m & \alpha_{m-1} & 1 & \ddots & \vdots \\ 0 & \beta_{m-1} & \ddots & \ddots & 0 \\ \vdots & 0 & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & \beta_2 & \alpha_1 \end{bmatrix} \qquad B^+ = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \qquad \ldots.(266)$$

$$C^+ = \begin{bmatrix} 0 & \cdots & 0 & \beta_1 \end{bmatrix} \qquad \ldots.(267)$$

then

$$J_i = J_i^+ \quad \text{for} \quad i=0 \quad \text{to} \quad 2m-1 \qquad \ldots.(268)$$

where

$$J_i = CA^iB \ , \ i = 0 \ , \ 1 \ , \ 2 \ , \ .... \qquad .....(269)$$

are the Markov parameters of the system.

Consequently the technique is essentially a method for transforming A, B and C into the required form, so that they can be partitioned to give the reduced model.

The transformation is carried out in 2m steps and can be terminated after any even number of steps if the model obtained is considered to be sufficiently accurate.

For odd numbered steps (step number 2i - 1)

$$\hat{A}i = Fi.Ai\text{-}1.Fi^{-1} \qquad .....(270)$$

$$\hat{B}i = Fi.Bi\text{-}1. \qquad .....(271)$$

$$\hat{C}i = Ci\text{-}1.Fi^{-1} \qquad .....(272)$$

For even numbered steps (step number 2i)

$$Ai = Gi^{-1}\hat{A}iGi \qquad .....(273)$$

$$Bi = Gi^{-1}\hat{B}i \qquad .....(274)$$

$$Ci = \hat{C}iGi \qquad .....(275)$$

where

$$F_1 = \begin{bmatrix} & \vdots & -\frac{b_1}{b_n} \\ I_{n-1} & \vdots & \vdots \\ & \vdots & -\frac{b_{n-1}}{b_n} \\ \cdots\cdots\cdots & \vdots & \cdots\cdots\cdots \\ & \vdots & \\ 0 & \vdots & \frac{1}{b_n} \end{bmatrix} \qquad .....(276)$$

$$G_1 = \left[\begin{array}{c|c} I_{n-1} & 0 \\ \hline -\frac{\hat{C}_1}{\hat{C}_n} \cdots\cdots -\frac{\hat{C}_{n-1}}{\hat{C}_n} & 1 \end{array}\right] \qquad \text{.....(277)}$$

for i = 2 to n

$$Fi = \left[\begin{array}{c|c|c} I_{n-1} & \begin{matrix} f_1 \\ \vdots \\ f_{n-i} \end{matrix} & 0 \\ \hline 0 & f_{n-i+1} & 0 \\ \hline 0 & 0 & I_{i-1} \end{array}\right] \qquad \text{.....(278)}$$

$$f_j = -\frac{a_{j,n-i+2}}{a_{n-i+1,n-i+2}} \qquad \text{for } j = 1 \text{ to } n-i$$

$$f_{n-i+1} = \frac{1}{a_{n-i+1,n-i+2}}$$

$$G_i = \left[\begin{array}{c|c|c} I_{n-i} & 0 & 0 \\ \hline g_1 \cdots\cdots g_{n-i} & 1 & 0 \\ \hline 0 & 0 & I_{i-1} \end{array}\right] \qquad \text{.....(279)}$$

$$g_j = - \frac{a_{n-i+2,j}}{a_{n-i+2,n-i+1}}$$

If the transformation is carried out on $(A^{-1}, A^{-1}B, C)$ or (A^{-1}, B, CA^{-1}) instead of (A, B, C) the reduction will give $((A^+)^{-1}, (A^+)^{-1}B^+, C^+)$ or $((A^+)^{-1}, B^+, C^+(A^+)^{-1})$ matching time moments.

If the transformation is applied to $(A, A^{-q}B, C)$, then q time moments and a variable number of Markov parameters can be matched.

Alternatively, if the transformation is applied to $(A^{-1}, A^{(q-1)}B, C)$, then q Markov parameters and a variable number of time moments can be matched.

This reduction technique is essentially another mathematical algorithm for matching time moments and/or Markov parameters and is undoubtedly extremely convenient for a system expressed in state variable form.

A further method of model reduction which involves time moments and/or Markov parameters is that which makes use of partial realisation techniques.

This is described in the next section.

2.6 MODEL REDUCTION USING PARTIAL REALISATION TECHNIQUES

A number of authors, namely HO and KALMAN (1965) (68), SILVERMAN (1971) (69) and ROZSA and SINHA (1974) (70) have developed methods for the minimal realisation of a system model, given as a transfer function, in terms of state equations of the form -

$$\dot{x} = Ax + Bu \qquad .....(280)$$

and $$y = Cx \qquad .....(281)$$

Assuming a transfer function matrix of order q x p
where q is the dimension of the output vector y
and p is the dimension of the input vector u,
A, B and C are found of order n x n, n x p and q x n respectively, so as to minimise n.

A modified form of these techniques has since been used as a method of model reduction by SHAMASH (1975) (71) and HICKIN and SINHA (1976) (72).

The resulting model reduction method employs a matrix consisting of the Markov parameters or time moments of the original system, which are equivalent to a Taylor series expansion of the transfer function about $s = \infty$ and $s = 0$ respectively.

i.e. $$G(s) = \sum_{i=1}^{\infty} C_i s^{i-1} \qquad .....(282)$$

Time moments about $s = 0$

or

$$G(s) = \sum_{i=1}^{\infty} D_i s^{-i} \qquad .....(283)$$

Markov parameters about $s = \infty$

As such, this method can be said to be based on principles

similar to those of the moment matching or continued fraction truncation techniques.

The Hankel matrix for the system is set up as follows -

$$H_{ij} = \begin{bmatrix} D_1 & D_2 & \cdots\cdots & D_{i-1} & D_i \\ D_2 & D_3 & \cdots\cdots & D_i & D_{i+1} \\ \vdots & \vdots & & \vdots & \vdots \\ D_{j-1} & D_j & \cdots\cdots & D_{i+j-2} & D_{i+j-1} \\ D_j & D_{j+1} & \cdots\cdots & D_{i+j-1} & D_{i+j} \end{bmatrix} \qquad \ldots\ldots(284)$$

The order of the minimal realisation n , is equal to the rank of the matrix H_{ij} provided that $i \geq \alpha$ (the controllability index of the system) and $j \geq \beta$ (the observability index of the system). (See Kalman (73).)

The Hankel matrix is then transformed into the Hermite normal form which is a square matrix of upper triangular form with elements on the main diagonal of 1 or 0. Thus, if a certain diagonal element is 1, all other elements in the column are 0, whereas if the diagonal element is 0, all elements of the row are also 0.

When the Hankel matrix is not square, the Hermite form produced will not be square either, although it will effectively be a square matrix truncated.

This Hermite form is obtained using an algorithm proposed by ROZSA and SINHA (70) which is based on outer products. The transformation is completed after n steps and the matrices A, B and C which give a minimal realisation of order n are now determined by selecting appropriate elements of the Hankel matrix in the original and Hermite forms.

Thus C is derived from the top left-hand corner of the original Hankel matrix by selecting the appropriate number of rows and columns (q x n)

B is derived from the top left-hand corner of the Hermite form of the Hankel matrix taking the first p columns and the first n rows.

A is similarly derived from the top of the Hermite form of the Hankel matrix taking n columns after those selected for the B matrix, and the 1st n rows.

The procedure for obtaining a reduced order model from the Hankel matrix follows similar lines in that for a model of order r, the transformation is discontinued after r steps.

Using this method of model reduction the first m Markov parameters of the reduced model agree exactly with the first Markov parameters of the original system when -

$$m = \left[\frac{r}{p}\right] + \left[\frac{r}{q}\right] \qquad \text{.....(285)}$$

where $\left[\frac{r}{p}\right]$ is the integer division of r by p

The following example taken from ROZSA and SINHA illustrates this method -

$$G(s) = \frac{\begin{bmatrix} s^2 + 6s + 11 \\ 6s + 12 \end{bmatrix}}{s^3 + 6s^2 + 11s + 6} \qquad \text{.....(286)}$$

The first five Markov parameters of this system are -

$$D_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad D_2 = \begin{bmatrix} 0 \\ 6 \end{bmatrix} \quad D_3 = \begin{bmatrix} 0 \\ -24 \end{bmatrix} \quad D_4 = \begin{bmatrix} -6 \\ +78 \end{bmatrix} \quad D_5 = \begin{bmatrix} 36 \\ -240 \end{bmatrix}$$

and the four by two Hankel matrix is -

$$H_{42} = \left[\begin{array}{c:c:c:c} 1 & 0 & 0 & -6 \\ 0 & 6 & -24 & 78 \\ \hdashline 0 & 0 & -6 & 36 \\ 6 & -24 & 78 & -240 \end{array}\right] \qquad \text{.....(287)}$$

Since the system being dealt with is of 3rd. order, this can now be transformed into the Hermite form in 3 steps, these being -

1) $$H_{42}^{(1)} = H_{42} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 6 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & -6 \\ 0 & 6 & -24 & 78 \\ 0 & 0 & -6 & 36 \\ 0 & -24 & 78 & -204 \end{bmatrix} \qquad .....(288)$$

2) $$H_{42}^{(2)} = H_{42}^{(1)} - \frac{1}{6} \begin{bmatrix} 0 \\ 5 \\ 0 \\ -24 \end{bmatrix} \begin{bmatrix} 0 & 6 & -24 & 78 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & -6 \\ 0 & 1 & -4 & 13 \\ 0 & 0 & -6 & 36 \\ 0 & 0 & -18 & 108 \end{bmatrix} \qquad .....(289)$$

3) $$H_{42}^{(3)} = H_{42}^{(2)} + \frac{1}{6} \begin{bmatrix} 0 \\ -4 \\ -7 \\ -18 \end{bmatrix} \begin{bmatrix} 0 & 0 & -6 & 36 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & -11 \\ 0 & 0 & 1 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad .....(290)$$

Thus, in general, the kth. step of the transformation can be defined as -

$$H^{(k)} = H^{(k-1)} - \frac{1}{h_{k,k}} \begin{bmatrix} h_{1,k} \\ h_{2,k} \\ \vdots \\ h_{k-1,k} \\ \vdots \\ (h_{k,k}-1) \\ \vdots \\ h_{k+1,k} \\ \vdots \end{bmatrix} \begin{bmatrix} h_{k,1} & h_{k,2} \cdots & h_{k,k} \cdots \end{bmatrix} \quad \ldots\ldots(291)$$

The minimal realisation of this 3rd. order system is now obtained from H_{42} and $H_{42}^{(3)}$.

C is made up of the first 2 rows of the first 3 columns of H_{42}

$$\text{i.e.} \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 6 & -24 \end{bmatrix} \quad \ldots\ldots(292)$$

B is taken from the first 3 rows of the 1st. column of $H_{42}^{(3)}$

$$\text{i.e.} \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \ldots\ldots(293)$$

A is taken from the next 3 columns of the first 3 rows of $H_{42}^{(3)}$

$$\text{i.e.} \quad A = \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix} \quad \ldots\ldots(294)$$

At this stage, it can easily be verified that the system represented by the state equations whose coefficients are A, B and C (as evaluated above) has the transfer function shown at the beginning of this example.

A 2nd. order reduced model can be obtained in a similar manner - from H_{42} and $H_{42}^{(2)}$.

Thus

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix} \qquad \text{.....(295)}$$

$$B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 0 \\ 1 & -4 \end{bmatrix} \qquad \text{.....(296)}$$

The transfer function of this reduced model is -

$$G(s) = \frac{\begin{bmatrix} s + 4 \\ 6 \end{bmatrix}}{s(s + 4)} = \begin{bmatrix} \frac{1}{s} \\ \frac{6}{s^2 + 4s} \end{bmatrix} \qquad \text{.....(297)}$$

Evaluation of the Markov parameters of this reduced model will show that the first three coincide with those of the original system.

System reduction can also be achieved using the same technique applied to a matrix made up of the coefficients Ci of the expansion about s = 0, providing that the system has no pole at the origin of the s-plane.

Under these circumstances, B, A^{-1} and C will be obtained.

This method of model reduction is in effect a technique of moment matching or Markov parameter matching for multi-variable systems. As such it has the same advantages and disadvantages as these methods.

However, this particular method of calculation does have one specific advantage in that if the reduced model of order

r is not considered to be sufficiently accurate, the transformation can be carried further thereby giving a reduced model of higher order.

A problem that this technique has in common with certain other reduction methods is that numerical difficulties arise if a pole or zero occurs at the origin in the original system. A possible way of overcoming this feature is to retain this pole or zero, that is to reduce the system (ignoring the pole or zero) and then to add the retained element to the reduced model after reduction.

2.7 MODEL REDUCTION BY MINIMISING ERROR FUNCTIONS IN THE FREQUENCY DOMAIN

Minimising error functions in the frequency domain is a method of model reduction which was first presented in 1966 and 1967 by MEIER and LUENBERGER (74,75), who initially formulated the problem in the time domain, but subsequently solved it in the frequency domain.

Taking the system transfer function as G(s) and the reduced model as $\hat{G}(s)$, with respective outputs of y(t) and $\hat{y}(t)$ and applying a stationary random input u(t), the reduced model is obtained by minimising the function:-

$$J = \int_0^{\infty} (y(t) - \hat{y}(t))^2 \, dt \qquad .....(298)$$

Using Parseval's theorem, this can be transformed into -

$$J = \frac{1}{2\pi j} \int_{-j\infty}^{+j\infty} \left| G(s) - \hat{G}(s) \right|^2 \phi_u(s) \, ds \qquad .....(299)$$

where $\phi_u(s)$ is the power spectral density of the input signal u(t).

Taking

$$\hat{G}(s) \text{ as } \sum_{i=1}^{m} \frac{\hat{r}_i}{s - \hat{p}_i} \qquad .....(300)$$

J is a function of $\hat{r}_1 \ldots \hat{r}_m$ and $\hat{p}_1 \ldots \hat{p}_m$ and can be minimised by differentiating with respect to these parameters giving

$$\frac{\partial J}{\partial \alpha} = \frac{2}{2\pi j} \int_{-j\infty}^{j\infty} (G(s) - \hat{G}(s)) \frac{\partial \hat{G}(s)}{\partial \alpha} . \phi_u(s) . \, ds \qquad .....(301)$$

where α is any real parameter of $\hat{G}(s)$.

Then if

$$\frac{\partial \hat{G}(s)}{\partial \hat{r}_i} = \frac{1}{s - \hat{p}_i} \qquad \text{.....(302)}$$

this gives

$$\frac{\partial J}{\partial \hat{r}_i} = \frac{2}{2\pi j} \int_{-j\infty}^{j\infty} (G(s) - \hat{G}(s)) \frac{1}{s - \hat{p}_i} . \phi_u(s).ds \qquad \text{.....(303)}$$

and similarly if

$$\frac{\partial \hat{G}(s)}{\partial \hat{p}_i} = \frac{1}{(s - \hat{p}_i)^2} \qquad \text{.....(304)}$$

this then gives

$$\frac{\partial J}{\partial p_i} = \frac{2}{2\pi j} \int_{-j\infty}^{j\infty} (G(s) - \hat{G}(s)) \frac{1}{(s - \hat{p}_i)^2} . \phi_u(s).ds \qquad \text{.....(305)}$$

To obtain the simple model parameters, J is now partially differentiated with respect to each of these parameters, and each resulting differential is then set equal to zero, giving 2m simultaneous equations:-

$$\int_{-j\infty}^{j\infty} (G(s) - \hat{G}(s)) \frac{1}{s - \hat{p}_i} . \phi_u(s).ds = 0 \qquad \text{.....(306)}$$

for i = 1, 2, ... m

$$\int_{-j\infty}^{j\infty} (G(s) - \hat{G}(s)) \frac{1}{(s - \hat{p}_i)^2} . \phi_u(s).ds = 0 \qquad \text{.....(307)}$$

for i = 1, 2, ... m

The authors recommend that these equations be solved using Newton's method and subsequently cite the following examples to illustrate their technique -

	$G(s)$	$\hat{G}(s)$
1)	$\frac{1}{(s + 0.1)(s + 10)}$	$\frac{-1}{s + 0.1}$
2)	$\frac{s + 0.9}{(s + 0.1)(s + 10)}$	$\frac{0.9}{s + 7}$
3)	$\frac{s + 0.2}{(s + 0.1)(s + 10)}$	$\frac{1}{s + 10}$
4)	$\frac{s - 0.9}{(s + 0.1)(s + 10)}$	$\frac{0.9}{s + 13.3}$
5)	$\frac{s + 4}{(s + 1)(s + 3)(s + 5)(s + 10)}$	$\frac{0.9}{(s +1)(s + 3)}$

calculated taking ϕ_u as 1

As can be seen, each of these reduced models has a large steady state error and this factor must be considered to be one of the greatest disadvantages of this particular technique.

In 1974, VITTAL RAO and LAMBA (76) published a paper, in which they proposed a model reduction technique (not unlike that of MEIER and LUENBERGER) which is based on the minimisation of the integral square error between the frequency response of the original system and that of the proposed reduced model over a chosen frequency range.

Thus

$$G(s) = \frac{K(1 + a_1 s + a_2 s^2 + \ldots\ldots + a_m s^m)}{1 + b_1 s + b_2 s^2 + \ldots\ldots + b_n s^n} \qquad \ldots\ldots(308)$$

$m < n$

is reduced to

$$\hat{G}(s) = \frac{K(1 + c_1 s + c_2 s^2 + ... + c_p s^p)}{1 + d_1 s + d_2 s^2 + ... + d_q s^q} \qquad p < q < n \qquad .....(309)$$

By substituting jw for s :-

$$G(jw) = \frac{K(R + jwI)}{F + jwL} = \frac{K.P(w)}{Q(w)} \qquad .....(310)$$

where

$$R = 1 - a_2 w^2 + a_4 w^4 - .....$$

$$I = a_1 - a_3 w^2 + a_5 w^4 - .....$$

$$F = 1 - b_2 w^2 + b_4 w^4 - .....$$

$$L = b_1 - b_3 w^2 + b_5 w^4 - .....$$

and

$$\hat{G}(jw) = \frac{K(A + jwB)}{C + jwD} = \frac{K.N(w)}{M(w)} \qquad .....(311)$$

where

$$A = 1 - c_2 w^2 + c_4 w^4 - .....$$

$$B = c_1 - c_3 w^2 + c_5 w^4 - .....$$

$$C = 1 - d_2 w^2 + d_4 w^4 - .....$$

$$D = d_1 - d_3 w^2 + d_5 w^4 - .....$$

The error in frequency response is then given by

$$e(w) = G(jw) - \hat{G}(jw) \qquad .....(312)$$

$$= \frac{K(P(w).M(w) - Q.(w).N(w))}{Q(w).M(w)} \qquad .....(313)$$

$$= \frac{K(S(w) + jT(w))}{Q(w).M(w)} \qquad .....(314)$$

where

$$S(w) + jT(w) = P(w).M(w) - Q(w).N(w) \quad \ldots\ldots (315)$$

$$= (R + jwI)(C + jwD) - (F + jwL)(A + jwB) \quad \ldots\ldots (316)$$

$$= RC - FA - w^2ID + w^2LB + jw(IC + RD - LA - FB) \quad \ldots\ldots (317)$$

therefore

$$S(w) = RC - FA - w^2ID + w^2LB \quad \ldots\ldots (318)$$

$$T(w) = (IC + RD - LA - FB)w \quad \ldots\ldots (319)$$

By putting

$$e_1(w) = \frac{e(w)}{K} \quad \ldots\ldots (320)$$

$$\left|Q(w).M(w).e_1(w)\right| = \left|S(w) + jT(w)\right| \quad \ldots\ldots (321)$$

therefore

$$\left|Q(w).M(w).e_1(w)\right|^2 = S^2(w) + T^2(w) \quad \ldots\ldots (322)$$

A weighted integral square error, E, is now obtained by integrating between chosen frequencies given by w_1 and w_2

Thus

$$E = \int_{w_1}^{w_2} \left|Q(w).M(w).e_1(w)\right|^2 dw \quad \ldots\ldots (323)$$

$$= \int_{w_1}^{w_2} (S^2(w) + T^2(w))\, dw \quad \ldots\ldots (324)$$

This can be re-expressed as

$$E = \int_{w_1}^{w_2} \left[(RC - FA - w^2ID + w^2LB)^2 + w^2(IC + RD - LA - FB)^2\right] dw \quad \ldots\ldots (325)$$

By minimising E, the reduced model can now be optimised. This is done by partially differentiating E with respect to each of the unknown coefficients of G(s) in turn and then setting these partial differentials to zero.

After performing these differentiations, VITTAL RAO and LAMBA produce a matrix equation, the solution of which gives values for c_1 to c_p and d_1 to d_q.

As an illustration of their technique, these two authors published the following example which, unfortunately, contained an error in its numerical evaluation -

$$G(s) = \frac{1}{6} \cdot \frac{1}{1 + \frac{11}{6}s + s^2 + \frac{1}{6}s^3}$$

$$= \frac{0.1667}{1 + 1.8333s + s^2 + 0.1667s^3} \qquad .....(326)$$

The reduced model thus obtained gives

$$\hat{G}(s) = \frac{0.1667(1 - 0.1584s)}{1 + 1.6550s + 0.7850s^2} \qquad .....(327)$$

[VITTAL RAO and LAMBA numerically inaccurate version is -

$$\hat{G}(s) = \frac{0.1667(1 - 0.1698s)}{1 + 1.6691s + 0.7710s^2}$$]

By comparison, using CHEN and SHIEH's method, this would be -

$$\hat{G}(s) = \frac{0.1667(1 - 0.1667s)}{1 + 1.6667s + 0.6944s^2} \qquad .....(328)$$

and using HSIA's method -

$$\hat{G}(s) = \frac{0.1667}{1 + 1.6150s + 0.6236s^2} \qquad .....(329)$$

It is, however, worth noting that since the method proposed by HSIA gives an imaginary value for c_1 , no realisable model of the form

$$\frac{K(1 + c_1 s)}{1 + d_1 s + d_2 s^2}$$ could be

obtained in this case using this method.

As the examples given prove, the model obtained using VITTAL RAO and LAMBA's method (in either of its forms) does not differ greatly from that obtained using CHEN and SHIEH's method, the main distinction being that the frequency response of CHEN and SHIEH's model is closer to the original model at lower frequencies. However, VITTAL RAO and LAMBA's model is less stable and hence is safer for controller design.

Unfortunately, if taken as published, this method of model reduction cannot cope with full systems or reduced models which contain time delays. There is, however, no apparent reason why the technique could not be extended to do so, although the calculus involved would be extremely complicated and tedious.

Nevertheless, being an analytical optimisation technique, the computation is not as complicated as that of a numerical optimisation method although it is still far more complex than that of other analytical reduction techniques, since, being a finite optimisation method, boundaries must be decided upon and the frequency range of interest must first be found.

A very similar approach to model reduction was produced in 1976 by REDDY (77), who expresses the full and reduced models in exactly the same terms as VITTAL RAO and LAMBA, but uses a different error criterion to obtain the reduced model parameters. Instead of minimising one error function with respect to each of the reduced model parameters, REDDY uses four seperate error functions.

Using

$$G(jw) = \frac{K(R + jwI)}{F + jwL} \qquad \ldots\ldots(330)$$

and

$$\hat{G}(jw) = \frac{K(A + jwB)}{C + jwD} \qquad .....(331)$$

(as did VITTAL RAO and LAMBA), REDDY minimises the integral square error between two chosen frequencies of the real and imaginary parts of the numerator and the denominator seperately to produce the following four error functions -

$$E_1 = \int_{w_1}^{w_2} (R - A)^2 \, dw \qquad .....(332)$$

$$E_2 = \int_{w_1}^{w_2} (I - B)^2 \, w^2 \, dw \qquad .....(333)$$

$$E_3 = \int_{w_1}^{w_2} (F - C)^2 \, dw \qquad .....(334)$$

$$E_4 = \int_{w_1}^{w_2} (L - D)^2 \, w^2 \, dw \qquad .....(335)$$

These error functions are then minimised by partially differentiating each function with respect to the relevant parameters of the reduced model and equating to zero, i.e.

$$\frac{\delta E_1}{\delta c_i} = \frac{\delta E_3}{\delta d_i} = 0 \qquad \text{for } i = 2, 4, 6 \qquad .....(336)$$

$$\frac{\delta E_2}{\delta c_i} = \frac{\delta E_4}{\delta d_i} = 0 \qquad \text{for } i = 1, 3, 5 \qquad .....(337)$$

These conditions give the following equations -

$$\frac{\delta E_1}{\delta c_i} = \int_{w_1}^{w_2} -2(R - A)\left(\frac{dA}{dc_i}\right) dw = 0 \qquad .....(338)$$

$$\text{for } i = 2, 4, 6$$

$$\frac{\delta E_3}{\delta d_1} = \int_{w_1}^{w_2} - 2(F - C)\left(\frac{dC}{dd_i}\right) dw = 0 \qquad .....(339)$$

for i = 2, 4, 6

$$\frac{\delta E_2}{\delta c_i} = \int_{w_1}^{w_2} - 2(I - B)w^2\left(\frac{dB}{dc_i}\right) dw = 0 \qquad .....(340)$$

for i = 1, 3, 5

$$\frac{\delta E_4}{\delta d_i} = \int_{w_1}^{w_2} - 2(L - D)w^2\left(\frac{dD}{dd_i}\right) dw = 0 \qquad .....(341)$$

for i = 1, 3, 5

The parameters of the reduced model are now obtained by solving these equations.

REDDY then cites the same example as that used by VITTAL RAO and LAMBA to illustrate his technique:-

$$G(s) = \frac{1}{6} \cdot \frac{1}{1 + \frac{11}{6}s + s^2 + \frac{1}{6}s^3} \qquad .....(342)$$

$$\hat{G}(s) = \frac{1}{6} \cdot \frac{1 + c_1 s}{1 + d_1 s + d_2 s^2} \qquad .....(343)$$

$\therefore$ R = 1 I = 0 $F = 1 - w^2$ $L = \frac{1}{6}(11 - w^2)$

A = 1 $B = c_1$ $C = 1 - d_2 w^2$ $D = d_1$

This gives $E_1 = 0$ .....(344)

$$E_2 = \int_0^2 c_1^2 w^2 dw \qquad .....(345)$$

$$E_3 = \int_0^2 (d_2 - 1)^2 \, w^4 \, dw \qquad \text{.....(346)}$$

$$E_4 = \int_0^2 \left(\frac{11}{6} - \frac{1}{6}.w^2 - d_1\right)^2 w^2 \, dw \qquad \text{.....(347)}$$

Now since -

$$\frac{\partial E_2}{\partial c_1} = 2 \int_0^2 c_1 \, w^2 \, dw = \left[c_1 . \frac{w^3}{3} \right]_0^2 = 0 \qquad \text{.....(348)}$$

therefore

$$c_1 = 0 \qquad \text{.....(349)}$$

and since -

$$\frac{\partial E_3}{\partial d_2} = 2 \int_0^2 (d_2 - 1) \, w^4 \, dw = 2 \left[(d_2 - 1) . \frac{w^5}{5} \right]_0^2 = 0 \qquad \text{.....(350)}$$

therefore

$$d_2 = 1 \qquad \text{.....(351)}$$

Similarly, since

$$\frac{\partial E_4}{\partial d_1} = -2 \int_0^2 \left(\frac{11}{6} - \frac{1}{6}.w^2 - d_1\right) w^2 \, dw$$

$$= -2 \left[\frac{11}{6}. \frac{w^3}{3} - \frac{1}{6}.\frac{w^5}{5} - d_1 . \frac{w^3}{3} \right]_0^2 = 0 \qquad \text{.....(352)}$$

therefore

$$d_1 = 1.4333 \qquad \text{.....(353)}$$

and thus

$$\hat{G}(s) = \frac{0.1667}{1 + 1.4333s + s^2} \qquad \text{.....(354)}$$

This model gives a greater error in frequency response than that obtained by VITTAL RAO and LAMBA as observed on a Nyquist plot.

In common with the models of VITTAL RAO and LAMBA, and CHEN and SHIEH, among others, the model is not necessarily stable if the original model is stable.

However, as the author points out, the calculations invol ved in this method of reduction are far simpler than those involved in the method of VITTAL RAO and LAMBA, since each of the four functions, A, B, C and D which make up the simplified model is evaluated seperately using distinctive error functions. In this way, it can be claimed that the errors in both the phase angle and the magnitude of both the numerator and the denominator of the transfer function are minimised seperately.This, however, does not guarantee that the errors in magnitude and phase angle of the transfer function, as a whole, are minimised.
i.e.

$$E(s) = \hat{G} - G = \frac{K(A + jwB)}{C + jwD} - \frac{K(R + jwI)}{F + jwL} \qquad .....(355)$$

Let

$$A = R + r \qquad B = I + i \qquad C = F + f \qquad D = L + l$$

where r, i, f, and l are the errors in the four terms of the reduced model

then

$$E(s) = \frac{K(rF - Rf - w^2(iL - Il) + jw(iF - If + Lr - lR))}{(F + f + jw(L + l))(F + jwL)} \qquad ....(356)$$

As shown by the above equation, minimal values of r, i, f and l do not guarantee a minimal value of E(s).

Due to this fact, this technique for model reduction is unlikely to be satisfactory unless each of the four errors is extremely small as in the case where the degree of reduction is limited or when E(s) is coincidentally minimised.

2.8 MODEL REDUCTION USING THE SCHWARZ CANONICAL FORM.

The Schwarz Canonical form is a form of the system equation in which the A matrix is replaced by the Schwarz or B matrix, whose elements are derived from the Routh array of the system.

Details of the Schwarz form were first given by SARMA, PAI and VISWANATHAN in 1968 (78) as follows:-

The system given as

$$\dot{x} = Ax + Cu \qquad \text{.....(357)}$$

where x is of order n

can be transformed into the Schwarz Canonical form

$$\dot{z} = Bz + fu \qquad \text{.....(358)}$$

where z is also of order n

by using the transformation matrix H, where

$$x = Hz. \qquad \text{.....(359)}$$

Then

$$B = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ -b_1 & 0 & 1 & \cdots & 0 & 0 \\ 0 & -b_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -b_3 & \cdots & 0 & 0 \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & -b_{n-1} & -b_n \end{bmatrix} \qquad \text{.....(360)}$$

where $$b_i = \frac{r_{n,i+2}}{r_{n,i}} \qquad \text{for } i = 1, 2, 3 \ldots n-1$$

and

$$b_n = \frac{r_2}{r_1}$$

where r_i are the elements of the first column of the Routh array, determined from the characteristic equation of A. (See Section 2.9)

and

$$f = (0 \quad 0 \quad 0 \ldots. 0 \quad 1)^T \qquad .....(361)$$

for u as a single variable.

If, however, u is a vector, then C and f become matrices and the standard form of f, as given above, no longer applies. Under these circumstances, the transformation into the Schwarz Canonical form becomes rather difficult since it can no longer be expressed in explicit equations as in the single variable case.

Now, given

$$x = Hz \qquad \text{(eqn.359)}$$

then

$$H\dot{z} = AHz + Cu \qquad .....(362)$$

and from

$$\dot{z} = Bz + fu \qquad .....(363)$$

it follows that

$$HB = AH \qquad .....(364)$$

and

$$Hf = C \qquad .....(365)$$

If B, f, A and C are known, H can be evaluated.

Equations 364 and 365 can be re-expressed in column vector form as

$$Ah_1 : Ah_2 : \ldots. : Ah_{n-1} : Ah_n =$$

$$(-b_1h_2 : h_1-b_2h_3 : \ldots. : h_{n-2}-b_{n-1}h_n : h_{n-1}-b_nh_n)$$

$$.....(366)$$

and

$$h_n = c \qquad \text{.....(367)}$$

where

h_1 is the first column of H

h_2 is the second column of H etc.

and b_1 to b_n are the elements of B as defined earlier.

The transformation matrix, H, is now obtained by rewriting the last two equations as -

$$h_n = c \qquad \text{(eqn.367)}$$

$$h_{n-1} = A.c + b_n.c \qquad \text{.....(368)}$$

$$h_{n-k} = A.h_{n-k+1} + b_{n-k+1}.h_{n-k+2} \qquad \text{.....(369)}$$

$$\text{for } k = 2, 3, \ldots\ldots, n-1$$

Once the system equations are in the Schwarz Canonical form, the system order can be reduced by eliminating some of the states from the Schwarz equation.

This method of model reduction was first proposed by ARUMUGAN in a Ph.D.thesis in India in 1971,(79) and was then more widely published by ARUMUGAN and RAMAMOORTY in 1973 (80).

The reduction is carried out by partitioning the Schwarz equation (equation 358),which gives

$$\begin{bmatrix} \dot{z}_m \\ \dot{z}_i \end{bmatrix} = \begin{bmatrix} B_{mm} B_{mi} \\ B_{im} B_{ii} \end{bmatrix} \begin{bmatrix} z_m \\ z_i \end{bmatrix} + \begin{bmatrix} f_m \\ f_i \end{bmatrix} u \qquad \text{.....(370)}$$

$\dot{z}_i$ is now taken as zero, giving

$$B_{im} z_m + B_{ii} z_i + f_i u = 0 \qquad \text{.....(371)}$$

and hence

$$z_i = - B_{ii}^{-1} (B_{im} z_m + f_i u) \qquad \text{.....(372)}$$

which gives

$$\dot{z}_m = B_{mm}z_m - B_{mi}B_{ii}^{-1}(B_{im}z_m + f_i u) + f_m u \quad .....(373)$$

Since $f_m = 0$, this becomes

$$\dot{z}_m = (B_{mm} - B_{mi}B_{ii}^{-1}B_{im})z_m - B_{mi}B_{ii}^{-1}f_i u \quad .....(374)$$

This is the reduced order Schwarz equation -

$$\dot{z}^* = B^* z^* + f^* u \quad .....(375)$$

where

$$B^* = B_{mm} - B_{mi}B_{ii}^{-1}B_{im} \quad .....(376)$$

and

$$f^* = - B_{mi}B_{ii}^{-1}f_i \quad .....(377)$$

This system equation can now be transformed back into the terms of the original system variables by using the transformation matrix, H, which, for this purpose, is also partitioned and reduced as follows:-

$$\begin{bmatrix} x_m \\ x_i \end{bmatrix} = \begin{bmatrix} H_{mm} H_{mi} \\ H_{im} H_{ii} \end{bmatrix} \begin{bmatrix} z_m \\ z_i \end{bmatrix} \quad .....(378)$$

Substituting for z_i from equation 372, this then gives

$$x_m = H_{mm}z_m - H_{mi}B_{ii}^{-1}(B_{im}z_m + f_i u)$$

$$= (H_{mm} - H_{mi}B_{ii}^{-1}B_{im})z_m - H_{mi}B_{ii}^{-1}f_i u \quad .....(379)$$

Now, if $\dot{z}_i$ is again taken as zero, and on this occasion, is applied to equation 378 giving

$$\dot{x}_m = H_{mm}\dot{z}_m \quad .....(380)$$

and $\dot{z}^* = \dot{z}_m$ and $z^* = z_m$ are substituted into equation 375 giving

$$H_{mm}^{-1}\dot{x}_m = B^* (H_{mm} - H_{mi}B_{ii}^{-1}B_{im})^{-1} (x_m + H_{mi}B_{ii}^{-1}f_i u) + f^* u \quad .....(381)$$

Then

$$\dot{x}_m = H_{mm}B^* (H_{mm} - H_{mi}B_{ii}^{-1}B_{im})^{-1}x_m + \left[H_{mm}B^* (H_{mm} - H_{mi}B_{ii}^{-1}B_{im})^{-1}H_{mi} - B_{mi}\right] B_{ii}^{-1}f_i u \quad .....(382)$$

The reduced system in terms of the original system variables now becomes -

$$\dot{x}^* = A^* x^* + C^* u \quad .....(383)$$

where

$$A^* = H_{mm}B^* (H_{mm} - H_{mi}B_{ii}^{-1}B_{im})^{-1} \quad .....(384)$$

and

$$C^* = \left[H_{mm}B^* (H_{mm} - H_{mi}B_{ii}^{-1}B_{im})^{-1} - B_{mi}\right] B_{ii}^{-1}f_i \quad (385)$$

(ARUMUGAN and RAMAMOORTY do not consider a forced system in their derivation and hence do not obtain an expression for C^*.)

There is a marked similarity between the modal elimination technique of MARSHALL and CHIDAMBARA (See Section 2.2), in which the least dominant modes of the system are directly eliminated and this technique in which the last states of the Schwarz equation are eliminated in that these states are related to the least dominant poles of the system. The main advantage of this method , as claimed by the authors, is that it does not require the evaluation of eigen values or eigen vectors.

A further aspect of this technique concerns the salient fact that the coefficients of the Schwarz matrix, which are derived from the Routh array of the characteristic equation are always positive for a stable system. Elimination of some of these coefficients still leaves the remaining coefficients all positive and, consequently, the reduced Schwarz Canonical equation represents a stable system.

The degree of reduction is determined by evaluating the ratio between successive elements b_i of the B Schwarz matrix. The evaluation of these ratios i.e.

$\frac{b_{n-1}}{b_{n-2}}, \frac{b_{n-2}}{b_{n-3}}, \ldots, \frac{b_{n-i}}{b_{n-i-1}}$ is continued until the ratio

$\frac{b_2}{b_1}$ has been calculated. However, if the ratio $\frac{b_{n-i}}{b_{n-i-1}} < 1$

then the calculations must be terminated.

Likewise, no reduction is possible if none of the ratios is greater than K, where K is normally taken as 10.(However, this value depends on the variables which are to be retained.)

When no ratio is greater than 10, then the system eigen values are of the same order of magnitude.

If, however, some of the ratios are greater than K, and the ith. ratio is the final one to be greater than K, then the system can be reduced to an order of $m = n-i$

An example, which is given by ARUMUGAN and RAMAMOORTY for the reduction of a seventh order system to one of fourth order is shown to give a model which has a time response closer to that of the original seventh order model than that of the models obtained using the modal elimination method of KUPPURAJULU and ELANGOVAN.(See Section 2.2)

LAL, MITRA and JAIN (1975) (81) also exemplify this technique by taking a sixth order system and reducing it progressively to a fifth, fourth, third and second order model. They then compare the time moments of these models and arrive at the following conclusions:-

1) The first moment (M_0) is the same in each case -

$$M_0 = \frac{1}{b_5 b_3 b_1}$$

This ensures identical steady state behaviour irrespective of the order of reduction and hence showa that models obtained in this way have no steady state error.

2) The second moment (M_1) is the same only if a system of even order is reduced to the next lower order.

3) The criterion for reduction to the next lower order

$$\frac{b_{n-1}}{b_{n-2}} \geq K \quad (\text{where } K = 10)$$

is valid if n is odd, but if n is even then

$$\frac{b_{n-1}}{b_{n-2} + b_{n-3}} > K \quad (\text{where } K = 10)$$

is more appropriate.

4) When reducing an nth. order system to an mth. order system all ratios i.e.

$$\frac{b_{n-1}}{b_{n-2}}, \frac{b_{n-2}}{b_{n-3}}, \ldots, \frac{b_{n-i}}{b_{n-i-1}} \quad \text{must be greater than K}$$

and not only the last ratio as had been previously suggested by ARUMUGAN and RAMAMOORTY.

LAL, MITRA and JAIN then claim that if K = 10 and the above condition is met, the resulting model will have a frequency response which is a good approximation of the original.

Several observations on this reduction method were published in June 1976 by DAVIDSON and LUCAS (82) who start by pointing out that if the system in Schwarz form is reduced, the resultant system can be given as

$$\dot{z}_r = B_r z_r + k f_i u \qquad \ldots\ldots(386)$$

where

B_r is the r x r Schwarz matrix formed from the last r+1 rows of the Routh array associated with the full system

and

$$k = \frac{1}{\left| \det B_{ii} \right|} \qquad \text{.....(387)}$$

The transfer function $\frac{z_1(s)}{u(s)}$ can then be written as

$$\frac{z_1(s)}{u(s)} = \frac{k}{D_r(s)} \qquad \text{.....(388)}$$

where

$D_r(s)$ is the characteristic polynomial of B_r.

The next comment concerns the fact that when reducing from an n to an n-1 order, the Schwarz reduction simply eliminates the 1st. row of the Routh array.

Thus, if

$$D_n(s) = s^n + a_{n-1}s^{n-1} + ... + a_1 s + a_0 \qquad \text{.....(389)}$$

then for n even

$$D_{n-1}(s) = s^{n-1} + ... + (\frac{a_1}{a_{n-1}})s + \frac{a_0}{a_{n-1}} \qquad \text{.....(390)}$$

and for n odd

$$D_{n-1}(s) = s^{n-1} + ... + (\frac{a_1}{a_{n-1}} - \frac{a_0}{a^2_{n-1}})s + \frac{a_0}{a_{n-1}} \qquad (391)$$

In this case

$$k = \frac{1}{b_n} = \frac{1}{a_{n-1}} \qquad \text{.....(392)}$$

Finally, DAVIDSON and LUCAS consider the fact that since the reduced system is based on the Routh array of the full system, the reduced system will be stable if the original system is stable, whereas, if the original system is unstable,

the reduced system could be either stable or unstable.

However, LAL, MITRA and JAIN (81) and also DAVIDSON and LUCAS (82) all consider the system in the Schwarz form only. Because of this fact, they do not encounter the difficulties and disadvantages of this reduction method.

Thus, using DAVIDSON and LUCAS's observation

$$\frac{z_1(s)}{u(s)} = \frac{k}{D(s)} \quad \text{and} \quad \frac{z_1^*(s)}{u(s)} = \frac{k}{D^*(s)} \qquad .....(393)$$

and similarly

$$\frac{z_2(s)}{u(s)} = \frac{ks}{D(s)} \quad \text{and} \quad \frac{z_2^*(s)}{u(s)} = \frac{k_s}{D^*(s)} \qquad .....(394)$$

Since the same applies in all other cases, it can be seen that the reduced transfer function of z_1, z_2 etc. differs from the original only in the denominator.

This is the characteristic equation of the system and, for the reduced system, is based on the Routh array of the unreduced system.

Under these conditions, all the conclusions drawn by DAVIDSON and LUCAS apply.

If, however, the original system is not in the Schwarz form, the situation becomes very different in that, in this situation, the transformation matrix plays a very important role.

In this case, for the unreduced system -

$$A = HBH^{-1} \qquad \text{(See equation 364)}$$

and the characteristic equation of the original system will remain the same on transformation into the Schwarz form.

However, for the reduced system -

$$A^* = H_{mm}B^* \left(H_{mm} - H_{mi}B_{ii}^{-1}B_{im}\right)^{-1}$$

(See equation 389)

and the transformation from the Schwarz form back into the original form produces a change in the characteristic equation unless

$$H_{mi} \cdot B_{ii}^{-1} \cdot B_{im} = 0 \qquad \text{.....(395)}$$

Since this occurs only in the special case where

$$H_{mi} = 0$$

it can be clearly seen that the characteristic equation of the reduced system in the original form is now not related to the original characteristic equation in the same manner as is the characteristic equation of the reduced system in the Schwarz form. Hence, the reduced system no longer possesses the properties attributed to it by DAVIDSON and LUCAS.

To illustrate this point more clearly, the following reduction has been detailed step by step:-

A third order system, as defined below, is reduced to a second order system.

$$\dot{x} = Ax + Cu \qquad \text{.....(396)}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \qquad C = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$$

The transfer functions are therefore:

$$\frac{x_1}{u}(s) = \frac{s^2 + 4s + 2}{s^3 + 6s^2 + 11s + 6} = \frac{\frac{1}{3}(1 + 2s + 0.5s^2)}{1 + \frac{11}{6}s + s^2 + \frac{1}{6}s^3} \qquad \text{.....(397)}$$

$$\frac{x_2}{u}(s) = \frac{-(2s^2 + 9s + 6)}{s^3 + 6s^2 + 11s + 6} = \frac{-(1 + 1.5s + \frac{1}{3}s^2)}{1 + \frac{11}{6}s + s^2 + \frac{1}{6}s^3} \qquad \text{.....(398)}$$

$$\frac{x_3}{u}(s) = \frac{3s^2 + 16s + 12}{s^3 + 6s^2 + 11s + 6} = \frac{2(1 + \frac{4}{3}s + 0.25s^2)}{1 + \frac{11}{6}s + s^2 + \frac{1}{6}s^3} \qquad (399)$$

The poles of the original system are at s = -1, -2 and -3 and the Routh array is:-

$A_1 = 0$	$R_{1,1} = a_n = \frac{1}{6}$	$R_{1,2} = a_{n-2} = \frac{11}{6}$
$A_2 = 0$	$R_{2,1} = a_{n-1} = 1$	$R_{2,2} = a_{n-3} = a_0 = 1$
$A_3 = \frac{R_{1,1}}{R_{2,1}}$ $= \frac{1}{6}$	$R_{3,1} = R_{1,2} - A_3 \cdot R_{2,2}$ $= \frac{11}{6} - \frac{1}{6}.1 = \frac{10}{6}$	$R_{3,2} = R_{1,3} - A_3 \cdot R_{2,3}$ $= 0$
$A_4 = \frac{R_{1,1}}{R_{3,1}}$ $= \frac{6}{10}$	$R_{4,1} = R_{2,2} - A_4 \cdot R_{3,2}$ $= 1$	

(See Section 2.9 for a detailed explanation of Routh array.)

The coefficients of the Schwarz B matrix are -

$$b_1 = \frac{R_{4,1}}{R_{2,1}} = \frac{1}{1} = 1 \quad : \quad b_2 = \frac{R_{3,1}}{R_{1,1}} = \frac{\frac{10}{6}}{\frac{1}{6}} = 10$$

$$b_3 = \frac{R_{2,1}}{R_{1,1}} = \frac{1}{\frac{1}{6}} = 6$$

giving the Schwarz form

$$\dot{z} = Bz + f.u \qquad \text{(eqn.358)}$$

where

$$B = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -10 & -6 \end{bmatrix} \qquad f = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The relationship between x and z is given by -

$$x = Hz \qquad \text{(eqn.359)}$$

where the transformation matrix, H, is

$$H = \begin{bmatrix} 1 & 4 & 1 \\ -4 & -9 & -2 \\ 9 & 16 & 3 \end{bmatrix}$$

The transfer function between the Schwarz variable vector z and the input u is now -

$$\frac{z}{u}(s) = \frac{\begin{bmatrix} 1 \\ s \\ 1 + s^2 \end{bmatrix}}{6 + 11s + 6s^2 + s^3} \qquad \text{.....(400)}$$

To test for the reduciblity of a third order system, the ratio $\frac{b_2}{b_1}$ must be examined. In this case, $\frac{b_2}{b_1} = 10$, and hence, according to the criteria given by ARUMUGAN and RAMAMOORTY, this system can be reduced.

Now, by putting

$$\dot{z}_3 = 0 \qquad \text{.....(401)}$$

which gives

$$z_3 = -\frac{5}{3}z_2 + \frac{1}{6}u \qquad \text{.....(402)}$$

The reduced system in Schwarz form can be given, at this stage, as -

$$\dot{z}^* = \begin{bmatrix} 0 & 1 \\ -1 & -\frac{5}{3} \end{bmatrix} z^* + \begin{bmatrix} 0 \\ \frac{1}{6} \end{bmatrix} u \qquad \ldots\ldots(403)$$

and

$$\frac{z^*}{u}(s) = \frac{\begin{bmatrix} 1 \\ s \end{bmatrix}}{6 + 10s + 6s^2} \qquad \ldots\ldots(404)$$

Partitioning the transformation matrix, H, now gives H_{mm} and H_{mi} in

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 1 \\ -4 & -9 & -2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \qquad \ldots\ldots(405)$$

Next, by substituting for z_3, we obtain

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & \frac{7}{3} \\ -4 & -\frac{17}{3} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{6} \\ -\frac{1}{3} \end{bmatrix} u \qquad \ldots\ldots(406)$$

which gives

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} -\frac{17}{11} & -\frac{7}{11} \\ \frac{12}{11} & \frac{3}{11} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{22} \\ -\frac{1}{11} \end{bmatrix} u \qquad \ldots\ldots(407)$$

Substituting this into the reduced Schwarz system equation then gives -

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} \frac{12}{11} & \frac{3}{11} \\ -\frac{3}{11} & \frac{2}{11} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -\frac{1}{11} \\ \frac{3}{11} \end{bmatrix} u \qquad \text{.....(408)}$$

Partitioning H, as above, and taking $\dot{z}_3 = 0$, as before, we now obtain

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ -4 & -9 \end{bmatrix} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} \qquad \text{....(409)}$$

and, hence

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{21}{11} & -\frac{30}{11} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ -\frac{23}{11} \end{bmatrix} u \qquad \text{.....(410)}$$

which is the reduced system in the original form i.e.

$$\dot{x}^* = A^* x^* + C^* u \qquad \text{(eqn.383)}$$

with poles at $-0.715 \pm 0.187j$

These values give

$$\frac{x_1^*}{u}(s) = \frac{\frac{1}{3}(1 + 1.57s)}{1 + 1.43s + 0.52s^2} \qquad \text{.....(411)}$$

$$\frac{x_2^*}{u}(s) = \frac{-(1 + 1.095s)}{1 + 1.43s + 0.52s^2} \qquad \text{.....(412)}$$

An alternative approach for the last step of the transformation would be to take -

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & \frac{7}{3} \\ -4 & -\frac{17}{3} \end{bmatrix} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} + \begin{bmatrix} -\frac{1}{6} \\ -\frac{1}{3} \end{bmatrix} \dot{u} \tag{413}$$

which is obtained by differentiating the expression for x^*.

This would then give

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \frac{5}{11} & \frac{23}{33} \\ -\frac{31}{11} & -\frac{70}{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{6}{11} \\ -\frac{13}{11} \end{bmatrix} u + \begin{bmatrix} \frac{1}{6} \\ -\frac{1}{3} \end{bmatrix} \dot{u} \tag{414}$$

This system equation is, however, not of the desired form and gives the following transfer functions:-

$$\frac{x_1^*}{u}(s) = \frac{\frac{1}{3}(1 + 2s + 0.5s^2)}{1 + \frac{5}{3}s + s^2} \tag{415}$$

$$\frac{x_2^*}{u}(s) = \frac{-(1 + \frac{3}{2}s + \frac{1}{3}s^2)}{1 + \frac{5}{3}s + s^2} \tag{416}$$

with poles at $-0.8333 \pm 1.1056j$

The characteristic equation is now the same as that of the reduced Schwarz equation and hence the properties, as described by DAVIDSON and LUCAS, do apply. However, these transfer functions contain an element of pure transfer between the input and the output, and, therefore, cannot be considered as satisfactory reductions of systems which contain no such transfer.

Examining x_1 more carefully shows that the moments of the reduced system obtained using the first method, are a good deal closer to those of the original system than are those

obtained using the second method. i.e.

$$\frac{x_1}{u}(s) \quad \text{(Original System)} =$$

$$\frac{1}{3}(1 + 0.1667s - 0.8056s^2 + 1.1435s^3 - 1.3188s^4 \ldots\ldots) \tag{417}$$

$$\frac{x_1^*}{u}(s) \quad \text{(1st. Reduced Model)} =$$

$$\frac{1}{3}(1 + 0.1429s - 0.7279s^2 + 0.9650s^3 - 0.9973s^4 \ldots\ldots) \tag{418}$$

$$\frac{x_1^*}{u}(s) \quad \text{(2nd. Reduced Model)} =$$

$$\frac{1}{3}(1 + 0.3333s - 1.0556s^2 + 1.4259s^3 - 1.3210s^4 \ldots\ldots) \tag{419}$$

Comparing the Markov parameters of the original system with those of the two reduced models in a similar manner, we discover that whereas the first three Markov parameters of the first reduced model are quite close to those of the original system, those of the second reduced model are not. i.e.

$$\frac{x_1}{u}(s) \quad \text{(Original System)} =$$

$$s^{-1} - 2s^{-2} + 3s^{-3} - 2s^{-4} - 9s^{-5} + \ldots\ldots \tag{420}$$

$$\frac{x_1}{u}(s) \quad \text{(1st. Reduced System)} =$$

$$s^{-1} - 2.0909s^{-2} + 3.7934s^{-3} - 6.3539s^{-4} \ldots\ldots \tag{421}$$

$$\frac{x_1}{u}(s) \quad \text{(2nd. Reduced System)} =$$

$$0.1667 + 0.3889s^{-1} - 0.4815s^{-2} + 0.4135s^{-3} - \ldots\ldots \tag{422}$$

From this comparison of the moments and also the Markov parameters of the three systems, it can be seen that the first model is considerably closer to the original system than is the second model, even though, as previously mentioned, it does not possess the properties attributed to this reduction technique by DAVIDSON and LUCAS.

If a system represented by a given single input, single output transfer function is to be reduced using this method, it must first be represented in state variable form. Since this involves choosing a suitable A matrix, and since this choice is not unique, the final reduced model is heavily dependent on the astuteness of this choice.

To expound on this point -

A close examination has already been made of the reduction of the transfer function -

$$G(s) = \frac{x_1}{u}(s) = \frac{s^2 + 4s + 2}{s^3 + 6s^2 + 11s + 6} \qquad (423)$$

using a state variable form where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$$

This system can be realised using other A matrices and corresponding C vectors.

In each case, of course, the Schwarz form will be the same (since this is dependent on the denominator of the transfer function), but the transformation matrix, H, will vary.

Thus if, for example, the following values are used:-

$$A = \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix} \qquad C = \begin{bmatrix} 1 \\ 1.5 \\ \frac{1}{3} \end{bmatrix}$$

then, the reduced model obtained is

$$G^*(s) = \frac{\frac{1}{3}(1 + 1.68s)}{1 + 1.52s + 0.55s^2} \qquad (424)$$

$$= \frac{1}{3}(1 + 0.1667s - 0.7980s^2 + 1.118s^3 - 1.2589s^4 \ldots\ldots)$$

$$= 1.0278s^{-1} - 2.2438s^{-2} + 4.3486s^{-3} - 7.9657s^{-4} + \ldots\ldots$$

If we now compare the moments of this reduced model with those of the first reduced model and the original system, we discover that this model does, in fact, give better agreement with the original system.

However, if we compare the Markov parameters of these two models we see that the first reduced model gives better agreement with the original system than does this final version.

According to these criteria, therefore both these models are reasonably close to the original system and neither can be said to be definitely superior to the other. Nevertheless, the A matrix must always be chosen with extreme care since the above situation is unlikely to be typical for all cases.

As a general assessment of this method of model reduction it can be said that, with the exception of models obtained using moment matching techniques, the first few moments of models obtained using this method have a better agreement with those of the original system than those of models obtained using, for example, either MARSHALL's or DAVISON's modal elimination techniques or VITTAL RAO and LAMBA's or REDDY's techniques for the minimisation of error functions in the frequency domain.

Much the same can be said for the Markov parameters, although here it can be definitely stated that the agreement achieved using this method of model reduction is considerably superior to that obtained using CHEN and SHIEH's continued fraction expansion method.

2.9 OTHER MODEL REDUCTION TECHNIQUES USING THE ROUTH ARRAY.

A number of authors have published papers describing methods of model reduction which use the Routh array without using the Schwarz Canonical form.

The oldest of these papers, which dates back to 1965, is that published by GUSTAFSON, (59) in which the author recommends the reduction of the original system to two different models, using two distinct methods of reduction, and, subsequently utilising both models, together, for controller design.

Taking the original system as

$$G(s) = \frac{K}{1 + a_1 s + a_2 s^2 + ... + a_n s^n} \qquad .....(425)$$

the first suggested reduction is

$$\hat{G}(s) = \frac{K}{1 + a_1 s + a_2 s^2} \qquad .....(426)$$

At this point, GUSTAFSON observes that for this simple model, the first three time moments, m_0, m_1 and m_2, as shown below, agree.

$$m_0 = \lim_{s \to 0} \left[G(s) \right] = K \qquad .....(427)$$

$$m_1 = \lim_{s \to 0} \left[(-1) \frac{d}{ds} G(s) \right] = K a_1 \qquad .....(428)$$

$$m_2 = \lim_{s \to 0} \left[\frac{1}{2} \frac{d^2}{ds^2} G(s) \right] = K(a_1^2 - a_2) \qquad .....(429)$$

The second model is given by

$$G^*(s) = \frac{K}{R_{n+1,1} + R_{n,1} s + R_{n-1,1} s^2} = \frac{K}{1 + R_{n,1} s + R_{n-1,1} s^2} \qquad .....(430)$$

where the Rs are the last 3 elements of the first column of the Routh array.

This array is constructed using the coefficients of the denominator of the system equation, and is then utilized for determining the stability of the system - thus, if the elements in the first column of the array are all positive, the system is stable.

The array construction is as follows -

	$R_{1,1}$	$R_{1,2}$	.....	R_{1,z_1-2}	R_{1,z_1-1}	R_{1,z_1}
	$R_{2,1}$	$R_{2,2}$	.....	R_{2,z_2-1}	R_{2,z_2}	
α_1	$R_{3,1}$	$R_{3,2}$	.....	R_{3,z_3-1}	R_{3,z_3}	
α_2	$R_{4,1}$	$R_{4,2}$	.....	R_{4,z_4}		
α_3	$R_{5,1}$	$R_{5,2}$	.....	R_{5,z_5}		
	⋮	⋮				
α_{n-3}	$R_{n-1,1}$	$R_{n-1,2}$				
α_{n-2}	$R_{n,1}$					
α_{n-1}	$R_{n+1,1}$					

shown for even value of n

The first two rows of the array are made up of the coefficients of the denominator of the system equation. i.e.

$$R_{1,i} = a_{n+2-2i} \qquad \text{for } i = 1, 2, \ldots z_1 \quad ..(431)$$

$$R_{2,i} = a_{n+1-2i} \qquad \text{for } i = 1, 2, \ldots z_2 \quad ..(432)$$

Where, if n is even

$$z_1 = \frac{n+2}{2} \quad \text{and} \quad z_2 = \frac{n}{2}$$

but, if n is odd $\qquad z_1 = z_2 = \frac{n+1}{2}$

Other elements in the array are evaluated from the elements of the preceding rows using the equations -

$$R_{j,i} = R_{j-2,i+1} - \alpha_{j-2} R_{j-1,i+1} \qquad \text{.....(433)}$$

for $i = 1, 2, 3, \ldots z_j$

where

$$\alpha_j = \frac{R_{j,1}}{R_{j+1,1}} \qquad \text{.....(434)}$$

and $z_j = z_{j-2} - 1$

both for $j = 3, 4, \ldots n+1$

.....(435)

$R_{n+1,1} = 1$. since $a_0 = 1$

The model obtained by GUSTAFSON using the Routh array has the same integral square impulse response (ISIR) as that possessed by the original system, and the first three frequency moments are matched, thus -

$$\text{ISIR of } G(s) = \frac{1}{2} \frac{K^2}{(R_{n,1})(R_{n+1,1})} = \frac{1}{2} \frac{K^2}{R_{n,1}} \qquad \text{.....(436)}$$

$$\text{ISIR of } G^*(s) = \frac{1}{2} \frac{K^2}{(R_{n,1})(R_{n+1,1})} = \frac{1}{2} \frac{K^2}{R_{n,1}} \qquad \text{.....(437)}$$

$$\text{ISIR of } \hat{G}(s) = \frac{1}{2} \frac{K^2}{a_1} \qquad \text{.....(438)}$$

Both the models produced by GUSTAFSON are accurate in the steady state and are always stable if the full system is stable (although this latter fact could be true even if the full system were unstable.). The first model has close agreement with the full system at low frequencies, since both systems possess the same first three time moments. The second model, whose integral square impulse response is the same as that of the full model is dependent on all the coefficients

of the full model.

GUSTAFSON now recommends that these two reduced models be considered as boundaries between which the actual system can be found and further advocates their combined use for the design of a controller for the system.

Another method using the Routh array was put forward in March 1975, by SHAMASH (83) who included, in his method, the use of the Padé approximation technique.

Taking the original system as -

$$G(s) = \frac{b_0 + b_1 s + ... + b_{n-1} s^{n-1}}{a_0 + a_1 s + ... + a_{n-1} s^{n-1} + a_n s^n} \qquad .....(439)$$

$$= c_0 + c_1 s + c_2 s^2 + c_3 s^3 + ...$$

where

$$c_0 = \frac{b_0}{a_0} \quad \text{and} \quad c_i = \frac{1}{a_0} (b_i - \sum_{j=1}^{i} a_j c_{i-j}) \quad \text{for } i > 0$$

with

$$b_i = 0 \quad \text{for } i > n-1$$

($c_i = \frac{1}{(i!)}$ x ith. time moment of the system)

Representing the reduced model $\hat{G}(s)$ of order k as -

$$\hat{G}(s) = \frac{d_0 + d_1 s + ... + d_{k-1} s^{k-1}}{e_0 + e_1 s + ... + e_{k-1} s^{k-1} + e_k s^k} \qquad ....(440)$$

the reduction proceeds as follows.

The denominator coefficients e_0 to e_k are obtained by using the first k α parameters of G(s).

These are produced by performing a continued fraction expansion on the elements of the denominator of the system equation as shown -

$$Q(s) = \frac{1}{1 + \dfrac{a_0 + a_2 s^2 + \ldots\ldots + a_n s^n}{a_1 s + a_3 s^3 + \ldots\ldots + a_{n-1} s^{n-1}}}$$

taking n as even

$$= \cfrac{1}{1 + \alpha_1 \frac{1}{s} + \cfrac{1}{\alpha_2 \frac{1}{s} + \cfrac{1}{\alpha_3 \frac{1}{s} + \cfrac{1}{\ddots + \cfrac{1}{\alpha_n \frac{1}{s}}}}}} \qquad \ldots\ldots(441)$$

giving

$$\alpha_1 = \frac{a_0}{a_1}$$

$$\alpha_2 = \frac{a_1}{a_2 - \alpha_1 a_3}$$

$$\alpha_3 = \frac{a_2 - \alpha_1 a_3}{a_3 - \alpha_2 (a_4 - \alpha_1 a_5)}$$

It will be noticed that these are the α parameters (as defined on page (122) for the Routh array of

$$a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots\ldots a_n$$

which is the characteristic equation of $G(\frac{1}{s})$.

The denominator of the reduced model is now obtained by truncating the expansion after α_k and reconstituting Q, whereas the numerator is obtained by evaluating the coefficients c_0 to c_{k-1} and using the first k Padé equations to give d_0 to d_{k-1}.

i.e.

$$d_0 = e_0 c_0$$

$$d_1 = e_0c_1 + e_1c_2$$
$$\vdots$$
$$d_{k-1} = e_0c_{k-1} + e_1c_{k-2} + \ldots + e_{k-2}c_1 + e_{k-1}c_0 \qquad \ldots\ldots(442)$$

SHAMASH illustrates this method using an example taken from HUTTON (1971) (84) (See overleaf)

$$G(s) = \frac{2400 + 1800s + 496s^2 + 28s^3}{240 + 360s + 204s^2 + 36s^3 + 2s^4} \qquad \ldots\ldots(443)$$

$$= 10 - 7.5s + 3.5667s^2 + \ldots\ldots$$

$$Q(s) = \cfrac{1}{1 + \frac{2}{3}.\frac{1}{s} + \cfrac{1}{2.\frac{1}{s} + \cfrac{1}{\frac{45}{8}.\frac{1}{s} + \cfrac{1}{16.\frac{1}{s}}}}} \qquad \ldots\ldots(444)$$

for $k = 2$

$$\hat{Q}(s) = \cfrac{1}{1 + \frac{2}{3}.\frac{1}{s} + \cfrac{1}{2.\frac{1}{s}}} = \frac{6s}{4 + 6s + 3s^2} \qquad \ldots\ldots(445)$$

since

$$d_0 = e_0c_0 = 4 \times 10 = 40$$

and $$d_1 = e_0c_1 + e_1c_0 = 4.(-7.5) + 6 \times 10 = 30$$

this gives

$$\hat{G}(s) = \frac{d_0 + d_1s}{4 + 6s + 3s^2} = \frac{40 + 30s}{4 + 6s + 3s^2} \qquad \ldots\ldots(446)$$

The main advantage of the above method is the resultant stability of the reduced model if the original transfer function is stable.

This is further exemplified in the work done by HUTTON (1971,1974) (84,85) and subsequently by HUTTON and FRIEDLAND in 1975 (86), who, despite using a different algorithm to obtain their reduced model, produce a model which is identical to that obtained by SHAMASH. The technique used is, however, considerably more complicated, as is shown below.

Taking the original transfer function as

$$G(s) = \frac{b_1 s^{n-1} + \ldots + b_{n-1}s + b_n}{a_0 s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n} \qquad \ldots\ldots(447)$$

a reciprocal transformation is first performed on G(s) giving

$$H(s) = \frac{1}{s} G\left(\frac{1}{s}\right) = \frac{b_n s^{n-1} + \ldots + b_2 s + b_1}{a_n s^n + a_n s^{n-1} + \ldots + a_1 s + a_0} \qquad \ldots\ldots(448)$$

Next, an alpha - beta expansion of H(s) is computed, in which H(s) is represented as a function of s and n α coefficients and n β coefficients. i.e.

$$H(s) = f(s, \alpha_1, \ldots\ldots, \alpha_n, \beta_1, \ldots\ldots, \beta_n) \qquad \ldots\ldots(449)$$

This expansion is defined as -

$$H(s) = \beta_1 F_1(s) + \beta_2 F_1(s)F_2(s) + \ldots + \beta_n F_1(s)F_2(s) \ldots F_n(s)$$

$$= \sum_{l=1}^{n} \beta_l \prod_{j=1}^{l} F_j(s) \qquad \ldots\ldots(450)$$

where

$$F_i = \cfrac{1}{\alpha_1 s + \cfrac{1}{\alpha_{i+1} s + \cfrac{1}{\alpha_{i+2} s + \ddots + \cfrac{1}{\alpha_{n-1} s + \cfrac{1}{\alpha_n s}}}}}$$

for i = 2 to n $\qquad \ldots\ldots(451)$

and $$F_1 = \frac{1}{1 + \alpha_1 s} \qquad \text{.....(452)}$$

The values of the alpha coefficients can now be obtained from the Routh array for the denominator of H(s) and the beta coefficients from a similar table for the numerator of H(s).

The reduced model is obtained by truncating H(s) to give

$$\hat{H}(s) = f(s, \alpha_1, \ldots\ldots, \alpha_k, \beta_1, \ldots\ldots, \beta_k) \qquad \text{.....(453)}$$

$$= \frac{d_k s^{k-1} + \ldots\ldots + d_2 s + d_1}{c_k s^k + c_{k-1} s^{k-1} + \ldots\ldots + c_1 s + c_0} \qquad \text{.....(454)}$$

$\hat{H}(s)$ can now be reconstituted in the polynomial form from the alpha - beta coefficients using the recursive equations:

$$C_k(s) = \alpha_k . s . C_{k-1}(s) + C_{k-2}(s) \qquad \text{.....(455)}$$

and

$$D_k(s) = \alpha_k . s . D_{k-1}(s) + D_{k-2}(s) + \beta_k \qquad \text{.....(456)}$$

where $C_k(s)$ and $D_k(s)$ are the denominator and the numerator respectively of the kth. order reduced version of H(s).

$$C_{-1}(s) = 1, \quad D_{-1}(s) = 0, \quad C_0(s) = 1, \quad D_0(s) = 0$$

The final reduced model, $\hat{G}(s)$, can now be obtained by once more performing a reciprocal transformation on H(s).

The same example as that used by SHAMASH is now cited by HUTTON and FRIEDLAND to illustrate this technique. i.e.

$$G(s) = \frac{2400 + 1800s + 496s^2 + 28s^3}{240 + 360s + 204s^2 + 36s^3 + 2s^4} \qquad \text{.....(457)}$$

which gives

$$H(s) = \frac{2400s^3 + 1800s^2 + 496s + 28}{240s^4 + 360s^3 + 204s^2 + 36s + 2} \qquad \text{.....(458)}$$

Routh (Alpha) Array of H(s)

	$a_0^0 = 240$	$a_2^0 = 204$	$a_4^0 = 2$
	$a_0^1 = 360$	$a_2^1 = 36$	
$\alpha_1 = \frac{a_0^0}{a_0^1} = \frac{240}{360}$	$a_0^2 = 180$	$a_2^2 = 2$	
$\alpha_2 = \frac{a_0^1}{a_0^2} = \frac{360}{180}$	$a_0^3 = 32$		
$\alpha_3 = \frac{a_0^2}{a_0^3} = \frac{180}{32}$	$a_0^4 = 2$		
$\alpha_4 = \frac{a_0^3}{a_0^4} = \frac{32}{2}$			

Beta Array

	$b_0^1 = 2400$	$b_2^1 = 496$
	$b_0^2 = 1800$	$b_2^2 = 28$
$\beta_1 = \frac{b_0^1}{a_0^1} = \frac{2400}{360}$	$b_0^3 = b_2^1 - \beta_1 a_2^1 = 256$	$b_2^3 = b_4^1 - \beta a_4^1 = 0$
$\beta_2 = \frac{b_0^2}{a_0^2} = \frac{1800}{180}$	$b_0^4 = b_2^2 - \beta_2 a_2^2 = 8$	
$\beta_3 = \frac{b_0^3}{a_0^3} = \frac{256}{32}$		
$\beta_4 = \frac{b_0^4}{a_0^4} = \frac{8}{2}$		

Therefore

$$\alpha_1 = \frac{2}{3} \qquad \alpha_2 = 2 \qquad \alpha_3 = \frac{45}{8} \qquad \alpha_4 = 16$$

$$\beta_1 = \frac{20}{3} \qquad \beta_2 = 10 \qquad \beta_3 = 8 \qquad \beta_4 = 4$$

and

$$\hat{H}(s) = \frac{40s + 30}{4s^2 + 6s + 3} \qquad .....(459)$$

therefore

$$\hat{G}(s) = \frac{30s + 40}{3s^2 + 6s + 4} \qquad .....(460)$$

On comparison, it can be seen that these results are identical to those obtained by SHAMASH using a far simpler method.

As stated previously, the main advantage of this method of model reduction lies in the fact that when the original system is stable, the reduced model is also stable. This is due to the fact that the model is based on the Routh array whose elements, a_0^0 to a_0^n, are entirely positive in the case of a stable system, as are the resultant alpha coefficients. Elimination of any number of these coefficients will, therefore, leave the remainder positive, indicating that the system represented by them is also stable.

In 1976, KRISHNAMURTHY and SESHADRI (87) presented an algorithm for carrying out model reduction using what amounts to a simplified version of the principles put forward by HUTTON and FRIEDLAND and thus tending towards the technique of SHAMASH.

For their algorithm, these two authors use the same alpha and beta arrays as those used by HUTTON and FRIEDLAND, but then, instead of using reciprocal transformation to give H(s) and hence the first two rows of the two arrays, the array coefficients are derived directly from G(s).

If G (s) is taken as -

$$G(s) = \frac{b_n s^{n-1} + \ldots\ldots + b_2 s + b_1}{a_n s^n + \ldots\ldots + a_1 s^1 + a_0} \qquad \ldots\ldots(461)$$

then the alpha and beta arrays are set up according to the proposals of HUTTON and FRIEDLAND.

However, if G(s) is taken as

$$G(s) = \frac{b_1 s^{n-1} + \ldots\ldots + b_{n-1} s + b_n}{a_0 s^n + a_1 s^{n-1} + \ldots\ldots + a_{n-1} s + a_n} \qquad \ldots\ldots(462)$$

then the first row of the alpha array is given as

$$a_i^0 = a_{n-i} \qquad \text{for } i = 0, 2, 4, \ldots \frac{n}{2} \text{ (or } \frac{n-1}{2}) \qquad \ldots\ldots(463)$$

and the second row as

$$a_i^1 = a_{n-1-i} \qquad \text{for } i = 0, 2, 4, \ldots \frac{n}{2} \text{ (or } \frac{n-1}{2}) \qquad \ldots\ldots(464)$$

Similarly, the first two rows of the beta table are given as

$$\left.\begin{array}{l} b_i^1 = b_{n-i} \\ \\ b_i^2 = b_{n-1-i} \end{array}\right\} \text{ for } i = 0, 2, 4, \ldots \qquad \ldots\ldots(465), \ldots\ldots(466)$$

Having obtained the alpha and beta coefficients, the reduced model is then calculated directly using the following algorithm:-

$$B_k(s) = \beta_k . s^{k-1} + s^2 . B_{k-2}(s) + \alpha_k . B_{k-1}(s) \qquad \ldots\ldots(467)$$

$$A_k(s) = s^2 . A_{k-2}(s) + \alpha_k . A_{k-1}(s) \qquad \ldots\ldots(468)$$

where

$$B_{-1}(s) = B_0(s) = 0$$

and $A_{-1}(s) = A_{-1}(s) = 1$

giving

$$G_k(s) = \frac{B_k(s)}{A_k(s)} \qquad \text{.....(469)}$$

This work was followed by the presentation of a method for extending this reduction technique which was published in 1977 by HUTTON (88).

Then, in 1978, a number of papers were published in which this method of model reduction was either derived in a different manner or was further extended.

The first of these, presented in May 1978, was that of LANGHOLZ and FEINMESSER (89), who reviewed the use of this method for reducing transfer functions whose numerator has an order more than one less than the order of the denominator.

Thereafter, in September 1978, an equivalent reduction technique using Hurwitz polynomials was proposed by APPIAH (90).

Taking

$$G(s) = \frac{b_0 + b_1 s + \ldots + b_{n-1} s^{n-1}}{a_0 + a_1 s + \ldots + a_{n-1} s^{n-1} + a_n s^n} \qquad \text{.....(470)}$$

$$= c_0 + c_1 s + c_2 s^2 + c_3 s^3 + \ldots$$

APPIAH derives the numerator using the Padé approximation technique and matching the first k C parameters, after having determined the denominator coefficients. This he does using a procedure similar to that employed by SHAMASH.

From the denominator coefficients, he then produces a function of $z = s^2$

giving

$$f(z) = \frac{a_1 + a_3 z + \ldots + a_{n-1} z^{n/2-1}}{a_0 + a_2 z + \ldots + a_n z^{n/2}} \qquad \text{(shown for n even)}$$

$$= \gamma_0 + \gamma_1 z + \gamma_2 z + \ldots \qquad \text{.....(471)}$$

The denominator parameters of the reduced transfer function can now be obtained from the parameters of a similar function $f_r(z)$, by matching γ_0 to γ_k using Padé approximation once again.

It can then be shown that this technique is equivalent to those of SHAMASH, and HUTTON and FRIEDLAND. The same example applied to all three techniques will result in the same reduced model in all cases.

Two further papers extending this reduction principle to state space representation were published in October 1978 by RAO, LAMBA and VITTAL RAO (91) and SHAMASH (92).

The first of these assumes a system represented by

$$\dot{x} = Ax + Bu \qquad \ldots\ldots(472)$$

$$y = Cx \qquad \ldots\ldots(473)$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} b_0 & b_1 & b_2 & \cdots & b_{n-2} & b_{n-1} \end{bmatrix}$$

giving

$$G(s) = \frac{b_0 + b_1 s + b_2 s^2 + \ldots\ldots + b_{n-2} s^{n-2} + b_{n-1} s^{n-1}}{a_0 + a_1 s + \ldots\ldots + a_{n-2} s^{n-2} + a_{n-1} s^{n-1} + s^n} \qquad \ldots\ldots(474)$$

This is then transformed into

$$\dot{v} = Rv + Mu \qquad \ldots\ldots(475)$$

$$y = Ev \qquad \ldots\ldots(476)$$

where

for n odd

$$R = \begin{bmatrix} -\alpha_1 & 0 & -\alpha_3 & 0 & -\alpha_5 & \cdots & -\alpha_n \\ 0 & 0 & \alpha_3 & 0 & \alpha_5 & \cdots & \alpha_n \\ -\alpha_1 & -\alpha_2 & -\alpha_3 & 0 & -\alpha_5 & \cdots & -\alpha_n \\ 0 & 0 & 0 & 0 & \alpha_5 & \cdots & \alpha_n \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 & -\alpha_5 & \cdots & -\alpha_n \end{bmatrix} \qquad M = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

whereas

for n even

$$R = \begin{bmatrix} 0 & \alpha_2 & 0 & \alpha_4 & 0 & \cdots & \alpha_n \\ -\alpha_1 & -\alpha_2 & 0 & -\alpha_4 & 0 & \cdots & -\alpha_n \\ 0 & 0 & 0 & \alpha_4 & 0 & \cdots & \alpha_n \\ -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 & 0 & \cdots & -\alpha_n \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 & -\alpha_5 & \cdots & -\alpha_n \end{bmatrix} \qquad M = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

and where

$$E = \begin{bmatrix} \beta_1 & \beta_2 & \beta_3 & \cdots & \beta_n \end{bmatrix}$$

(The coefficients α_1 to α_n and β_1 to β_n used are those given by HUTTON and FRIEDLAND.)

Now -

$$v = Px \qquad \text{.....(477)}$$

where

$$P = \begin{bmatrix} a_0^1 & 0 & a_2^1 & \cdots & 0 & 1 \\ 0 & a_0^2 & 0 & \cdots & a_{n-3}^2 & 0 \\ \vdots & & a_0^3 & \cdots & 0 & 1 \\ & & & a_{n-5}^4 & 0 & \vdots \\ & & & \vdots & \vdots & \\ & & & a_0^{n-1} & 0 & \\ 0 & \cdots & \cdots & 0 & 1 & \end{bmatrix} \quad \text{for n odd}$$

and

$$P = \begin{bmatrix} a_0^1 & 0 & a_2^1 & \cdots & a_{n-2}^1 & 0 \\ 0 & a_0^2 & 0 & \cdots & 0 & 1 \\ \vdots & & a_0^3 & \cdots & a_{n-4}^3 & 0 \\ & & & 0 & 1 & \vdots \\ & & & \vdots & \vdots & \\ & & & a_0^{n-1} & 0 & \\ 0 & \cdots & \cdots & 0 & 1 & \end{bmatrix} \quad \text{for n even}$$

thus giving

$$R = PAP^{-1} \qquad \ldots\ldots(478)$$

$$M = PB \qquad \ldots\ldots(479)$$

$$E = CP^{-1} \qquad \ldots\ldots(480)$$

The system is then reduced by reducing the order of R,M and E.

In the case of a reduction by an even number all three matrices are simply truncated, whereas for a reduction by an odd number, R and M must first be re-expressed in a suitable form although E can be readily truncated.

RAO, LAMBA and VITTAL RAO illustrate their technique by using the same example as that used by HUTTON and FRIEDLAND, and obtaining equivalent results. Nevertheless, as proposed, this particular technique is rather limited since it restricts the original form of A,B and C. However, the possibility of further expansion of this technique appears to exist.

SHAMASH, in the second of the papers which propose the extension of this model reduction method to state space representation, uses the Schwarz Canonical form.

The system is again taken as

$$\dot{x} = Ax + Bu \qquad .....(481)$$

$$y = Cx \qquad .....(482)$$

In this case, A,B and C are not limited in form as in the previous technique. However, the system itself is limited to a single input and a single output.

As in HUTTON and FRIEDLAND's technique, the first step taken is an inverse transformation giving

$$\dot{\bar{x}} = \bar{A}\,\bar{x} + \bar{B}u \qquad .....(483)$$

$$\bar{y} = C\,\bar{x} \qquad .....(484)$$

where

$$\bar{A} = A^{-1}$$

and $\bar{B} = A^{-1}B$

These equations are then transformed into the Schwarz Canonical form giving

$$\dot{\bar{z}} = \bar{F}\,\bar{z} + \bar{G}u \qquad .....(485)$$

where

$$\bar{x} = T\,\bar{z} \qquad .....(486)$$

(See Section 2.8)

The system equation is then reduced (as previously described)

to give

$$\dot{\bar{z}}_r = \bar{F}_r \bar{z}_r + \bar{G}_r u \qquad \ldots\ldots(487)$$

A second inverse transformation is now carried out giving

$$\dot{z}_r = F_r z_r + G_r u \qquad \ldots\ldots(488)$$

where

$$F_r = \bar{F}_r^{-1}$$

and

$$G_r = \bar{F}_r^{-1} \bar{G}_r$$

The output matrix, H, in the output equation

$$y = Hz_r \qquad \ldots\ldots(489)$$

is now carefully selected to match the r initial time moments of the original system. i.e.

$$H.F_r^{-j}.G_r = C.A^{-j}.B \qquad \text{for } j = 1, 2, \ldots\ldots, r \quad \ldots(490)$$

Finally, if any of the initial states are to be retained, the reduced system equations can be transformed using the inverse of a suitably partitioned form of T.

Meanwhile, in August 1978, KRISHNAMURTHY and SESHADRI (93) expanded their work on this method of model reduction by proposing another, somewhat different, reduction technique based on the Routh array.

Taking the system transfer function as

$$G(s) = \frac{b_{11}s^m + b_{21}s^{m-1} + b_{12}s^{m-2} + b_{22}s^{m-3} + \ldots\ldots}{a_{11}s^n + a_{21}s^{n-1} + a_{12}s^{n-2} + a_{22}s^{n-3} + \ldots\ldots} \qquad \ldots(491)$$

and using the usual Routh algorithm, separate Routh arrays are formed from the denominator and numerator polynomials.

Thus, the denominator array is given as

$$
\begin{array}{llll}
a_{11} & a_{12} & a_{13} & a_{14} \quad \cdots \\
a_{21} & a_{22} & a_{23} & a_{24} \quad \cdots \\
a_{31} & a_{32} & a_{33} & \cdots \\
a_{41} & a_{42} & a_{43} & \cdots \\
\vdots & & & \\
a_{n,1} & & & \\
a_{n+1,1} & & &
\end{array}
$$

and the numerator array is given as

$$
\begin{array}{llll}
b_{11} & b_{12} & b_{13} & b_{14} \quad \cdots \\
b_{21} & b_{22} & b_{23} & b_{24} \quad \cdots \\
b_{31} & b_{32} & b_{33} & \cdots \\
b_{41} & b_{42} & b_{43} & \cdots \\
\vdots & & & \\
b_{m,1} & & & \\
b_{m+1,1} & & &
\end{array}
$$

The reduced order model is then obtained by using different rows of the array to form the numerator and denominator polynomials.

As an example , if the orders of both the denominator and the numerator are to be reduced by 1, the 2nd and 3rd row of each array is used, giving

$$G_{n-1}(s) = \frac{b_{21}s^{m-1} + b_{31}s^{m-2} + b_{22}s^{m-3} + \cdots}{a_{21}s^{n-1} + a_{31}s^{n-2} + a_{22}s^{n-3} + \cdots} \quad \cdots\cdots(492)$$

Thus, for a reduced order model of order k, the rows used for the numerator are the $m+2-k$ and the $m+3-k$ rows, and the rows used for the denominator are the $n+1-k$ and the $n+2-k$ rows.

The authors themselves then present an eighth order example to illustrate their technique, but for ease of comparison, the example of HUTTON and FRIEDLAND is cited below -

$$G(s) = \frac{2400 + 1800s + 496s^2 + 28s^3}{240 + 360s + 204s^2 + 36s^3 + 2s^4} \qquad .....(493)$$

Denominator array

	2	204	240
	36	360	
1/18	184	240	
9/46	313.05		
1.304	240		

Numerator array

	28	1800
	496	2400
0.056	1664.51	
0.694	2400	

Hence

$$G_3(s) = \frac{2400 + 1664.51s + 496s^2}{240 + 360s + 184s^2 + 36s^3} \qquad .....(494)$$

$$G_2(s) = \frac{2400 + 1664.51s}{240 + 313.05s + 184s^2}$$

$$= \frac{40 + 27.742s}{4 + 5.297s + 3.067s^2}$$

$$= 10 - 6.307s + 0.6845s^2 + \ldots\ldots \qquad \ldots\ldots(495)$$

As can be seen, the model obtained from this example is comparatively similar to that obtained using the technique of HUTTON and FRIEDLAND, and can be said to give a consistent steady state match although it does not succeed in matching time moments.

In April 1979, a further technique based on those referred to in this section was published by PAL (94), who obtains the denominator of his reduced model in the same manner as that employed by KRISHNAMURTHY and SESHADRI but produces the numerator using a Pade approximation.

For the example of HUTTON and FRIEDLAND, previously quoted, this technique gives -

$$G_2(s) = \frac{40 + 22.97s}{4 + 5.297s + 3.067s^2} \qquad \ldots\ldots(496)$$

This last reduction technique was subsequently criticised by MARSHALL (95), as having a bad transient fit. PAL replied to this criticism by stating that the technique was aimed at giving stable Pade approximation for stable systems and that it is an established fact that Pade approximation leads to unstable models if the step response has a large overshoot. However, if some Markov parameters are fitted, a good transient response can be obtained using this technique.

2.10 MODEL REDUCTION BY COMBINED METHODS INVOLVING MODAL ELIMINATION.

In the last few years several methods of model reduction have been published which combine modal elimination with other techniques in some way. Two examples of this are reduction techniques proposed by CHIDAMBARA (96) in 1969, and NAGARAJAN (97) in 1971 in which modal elimination was followed by a form of optimisation technique.

The first of these, CHIDAMBARA, initially proceeds along the lines of DAVISON's original reduction method, reducing

$$\dot{x} = Ax + Bu \qquad \text{.....(497)}$$

$$y = Cx \qquad \text{where x is of order n} \qquad \text{.....(498)}$$

to

$$\dot{x}^* = A^*x^* + B^*u \qquad \text{.....(499)}$$

$$y = C^*x^* \qquad \text{where } x^* \text{ is of order } 1 << n \qquad \text{..(500)}$$

The original system can be rewritten as

$$\dot{z} = \Lambda z + Gu \qquad \text{.....(501)}$$

$$y = Kz \qquad \text{.....(502)}$$

where Λ is a diagonal matrix containing the n eigen values of A, arranged in increasing order.

The two sets of state variables x and z are related to each other by the equation

$$x = Uz \qquad \text{.....(503)}$$

where

U is the square modal matrix consisting of the n eigen vectors associated with A.

The system equations can now be written as

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_r \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} u \qquad \text{.....(504)}$$

and

$$y = \begin{bmatrix} K_1 & K_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \qquad .....(505)$$

where

Λ_1 contains the l dominant poles

and Λ_r contains the remaining (n-l) poles

If z_2 is then taken as 0 and only the l dominant poles are retained, the system equations become

$$\dot{z}_1 = \Lambda_1 z_1 + G_1 u \qquad .....(506)$$

and

$$y_1 = K_1 z_1 \qquad .....(507)$$

As observed by CHIDAMBARA in his first paper on model reduction (20), this reduced system has a steady state error. To eliminate this, the author now recommends that the output equation be replaced by

$$y = (K_1 + Q) z_1 \qquad .....(508)$$

where

Q is a constant chosen to eliminate the steady state error and to minimise the integral square error in y.

The error, E_{ij}, in the ith output, y_i, due to the jth input, u_j, is given by the equation

$$E_{ij} = \left[- \sum_{m=1}^{l} \frac{q_{im}\, g_{mj}}{s - \lambda m} + \sum_{m=l+1}^{n} \frac{k_{im}\, g_{mj}}{s - \lambda m} \right] u_j \qquad .....(509)$$

It can therefore be seen that for $E_{ij} = 0$

$$\sum_{m=1}^{l} \frac{q_{im}\, g_{mj}}{s - \lambda m} = \sum_{m=l+1}^{n} \frac{k_{im}\, g_{mj}}{s - \lambda m} \qquad .....(510)$$

With the exception of the elements of Q, this equation consists

only of system constants, and, hence, appropriate selection of Q will minimise the error.

The ith column of Q, q_i, is obtained by minimising

$$\int_0^{\infty} \underline{E_i}' \, \underline{E_i} \, dt$$

where

$$E_i = \begin{bmatrix} E_{i1} \\ E_{i2} \\ \vdots \\ E_{ir} \end{bmatrix} \qquad \text{.....(511)}$$

where r is the order of the input vector.

With this method CHIDAMBARA overcomes some of the disadvantages of model reduction using modal elimination.

Unlike CHIDAMBARA, NAGARAJAN adopts a purely numerical minimisation for his technique and reduces the transfer function

$$G(s) = \frac{1}{1 + a_1 s + \ldots + a_n s_n} = \frac{1}{F(s)} \qquad \text{.....(512)}$$

to

$$\hat{G}(s) = \frac{1}{1 + b_1 s + \ldots + b_m s^m} = \frac{1}{\hat{F}(s)} \qquad \text{.....(513)}$$

$$m < n$$

in the following two steps:-

1) Using modal elimination and supressing the modes associated with large eigen values ,which NAGARAJAN describes as those that fall outside a circle of radius K multiplied by the magnitude of the smallest eigen value.

K, which then determines the accuracy of the approximation is taken as 25, after GUSTAFSON (1968).

2) Obtaining optimum parameters, based on the Feedback Error Correlation (FEC) performance index, considering the original system as the reference. The FEC is maximised using Rosenbrock's optimisation technique (1968), thus -

$$F.E.C. = \frac{|(e_1, e_2)|}{\sqrt{\left[(e_1, e_1)(e_2, e_2)\right]}} \leqslant 1 \qquad \ldots\ldots(514)$$

where

$$(e_i, e_k) = \int_0^{\infty} e_i(t) e_k(t)\, dt$$

$$= \frac{1}{2\pi j} \int_{-jx}^{+jx} E_i(s).E_k(-s)\, ds \qquad \ldots\ldots(515)$$

where $e_1(t)$ and $e_2(t)$ are the feedback errors of the original and reduced systems due to inputs.

When $e_1(t) = e_2(t)$, both systems respond identically and the F.E.C. = 1.

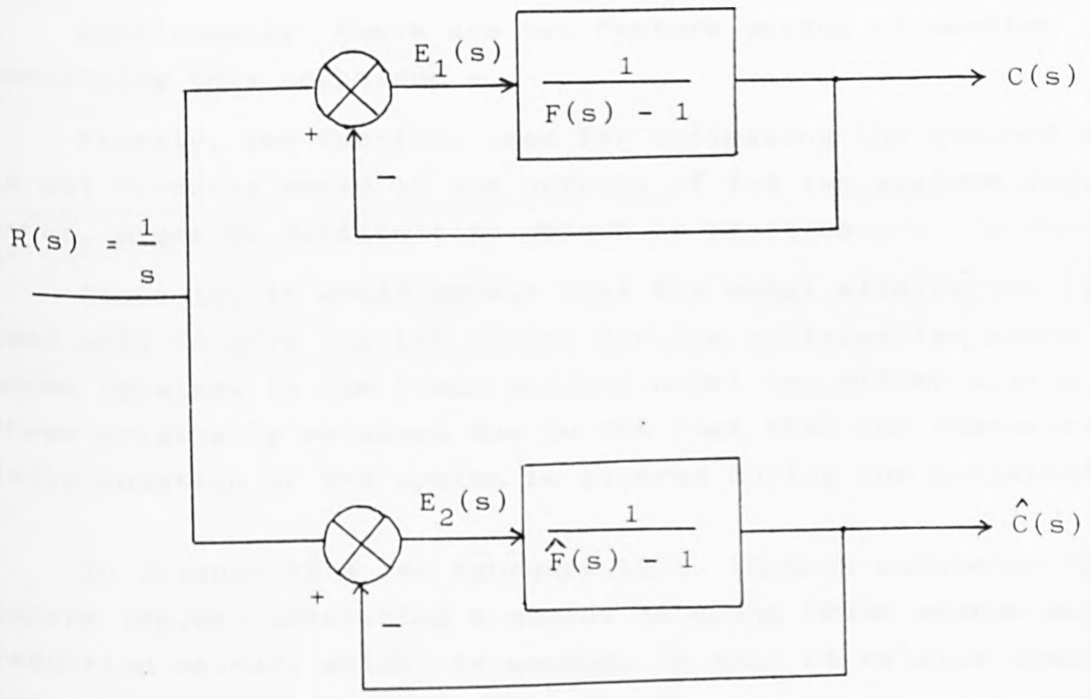


Figure 2.3 Comparison of feedback errors of original and reduced systems.

To illustrate this technique, NAGARAJAN offers the following example:-

An original system

$$G(s) = \frac{1}{1 + 11.03s + 41.33s^2 + 55.17s^3 + 15.91s^4 + 1.50s^5 + 0.4s^6} \quad .....(516)$$

is reduced by modal elimination to

$$\hat{G}(s) = \frac{1}{1 + 10.91s + 33.42s^2 + 39.62s^3} \quad .....(517)$$

This is optimised by maximising the Feedback Error Correlation (FEC) index to give

$$\hat{G}(s) = \frac{1}{1 + 4.85s + 31.91s^2 + 36.54s^3} \quad .....(518)$$

Despite the fact that the technique is illustrated using an example which contains no zeros, there seems to be no apparent reason why the method should not be extended to systems containing zero s.

Additionally, there are two factors worthy of mention concerning this technique -

Firstly, the function used for optimising the reduced model is not directly based on the outputs of the two systems and, hence, might be difficult to obtain in practice.

Secondly, it would appear that the modal elimination is used only to give initial values for the optimisation since the modes retained in the final reduced nodel can differ widely from those originally obtained due to the fact that the characteristic equation of the system is altered during the optimisation.

In October 1974 and February 1975, SHAMASH published two papers (98,99) describing a moment matching (Pade approximation) reduction method, which is unusual in that it retains dominant poles.

Employing the same notation as that subsequently used in his paper of March 1975,(83),(See section 2.9), including the

identical forms of the system equation and the reduced model, SHAMASH proceeds in the following manner -

After initially evaluating the coefficients c_0, c_1 to c_{2k-1-j}, where j is the number of poles to be retained, he next evaluates the denominator coefficients using the Padé equations -

$$e_0c_k + e_1c_{k-1} + \cdots + e_{k-1}c_1 + e_kc_0 = 0 \qquad \cdots\cdots(519)$$

$$e_0c_{k+1} + e_1c_k + \cdots + e_{k-1}c_2 + e_kc_1 = 0 \qquad \cdots\cdots(520)$$

$$\vdots$$

$$e_0c_{2k-1-j} + e_1c_{2k-2-j} + \cdots + e_{k-1}c_{k-j} + e_kc_{k-1-j} = 0 \qquad \cdots\cdots(521)$$

and

$$e_0 - e_1s_1 + e_2s_1^2 - \cdots + (-1)^k s_1^k = 0 \qquad \cdots\cdots(522)$$

$$\vdots$$

$$e_0 - e_1s_j + e_2s_j^2 - \cdots + (-1)^k s_j^k = 0 \qquad \cdots\cdots(523)$$

where

$-s_1, -s_2, \cdots -s_j$ are the known poles which are to be retained.

Taking e_0 as 1, there are then k unknown denominator coefficients and k equations and, hence, the denominator can be calculated.

The evaluation of the numerator is then carried out (using the same method as described under SHAMASH's work in Section 2.9) from the first k Padé equations

$$d_0 = e_0c_0 \qquad \cdots\cdots(524)$$

$$d_1 = e_0c_1 + e_1c_2 \qquad \cdots\cdots(525)$$

$$\vdots$$

$$d_{k-1} = e_0c_{k-1} + e_1c_{k-2} + \cdots + e_{k-2}c_1 + e_{k-1}c_0 \qquad \cdots(526)$$

This technique is then illustrated with the following two

examples:-

1)

$$G(s) = \frac{(s + 1.5)(s + 4)}{(s + 1)(s + 2)(s + 6)} \qquad .....(527)$$

$$= 0.5(1.0 - 0.75s + 0.6667s^2 - 0.63194s^3 + 0.615741s^4 - 0.607832s^5 + 0.603910s^6 - .....)$$

Retaining poles at s = 1 and s = -2 then gives

$$\hat{G}(s) = \frac{1 + 0.75s}{2 + 3s + s^2} \qquad .....(528)$$

2) The second example is taken from CHIDAMBARA 1969.

$$G(s) = \frac{1}{1 + s} + \frac{2}{2 + s} - \frac{6}{3 + s} + \frac{4}{4 + s} \qquad .....(529)$$

Using the above technique of SHAMASH, this gives

$$\hat{G}(s) = \frac{1.166}{1 + s} - \frac{0.332}{2 + s} \qquad .....(530)$$

whereas, CHIDAMBARA's method C2 gives

$$\hat{G}(s) = \frac{0.8333}{1 + s} + \frac{0.0833}{2 + s} \qquad .....(531)$$

and DAVISON's method D2 gives

$$\hat{G}(s) = \frac{0.6111}{1 + s} + \frac{0.1944}{2 + s} \qquad .....(532)$$

SHAMASH thus demonstrates that the model due to his combined technique gives a better fit in the frequency domain than either of the two other methods, both of which have a steady state error.

In June and October 1975, two further papers on model reduction methods for multivariable systems were published by SHIEH and WEI (100,101) whose technique is a combination of the matrix continued fraction methods and dominant pole retention.

This technique is, in effect, equivalent to that of SHAMASH (Feb.'75 and Oct. '74), displaying the same advantages and disadvantages. However, when extended to the multivariable case, it appears somewhat over-complicated.

Another reduction technique which is very similar to that of SHAMASH (Oct.'74, Feb.'75) is that due to HICKIN and SINHA (September 1976 and 1978)(102,103). In this technique, the authors consider a transfer function -

$$G(s) = \frac{b_0 + b_1 s + \ldots\ldots + b_{n-1} s^{n-1}}{a_0 + a_1 s + \ldots\ldots + a_{n-1} s^{n-1} + s^n} \qquad \ldots\ldots(533)$$

$$= -j_{-1} s - j_{-2} s - j_{-3} s^2 - \ldots\ldots$$

$$= j_0 s^{-1} + j_1 s^{-2} + j_2 s^{-3} + \ldots\ldots$$

A minimal realisation of the system is then formed in the following terms:-

$$\dot{x} = \begin{bmatrix} 0 & 0 \ldots\ldots\ldots 0 & -a_0 \\ 1 & 0 \ldots\ldots\ldots 0 & -a_1 \\ 0 & 1 \ldots\ldots\ldots 0 & -a_2 \\ \vdots & \vdots \quad \ddots \quad \vdots & \vdots \\ 0 & 0 \ldots\ldots\ldots 1 & -a_{n-1} \end{bmatrix} x + \underset{\sim}{e}_k u \qquad \ldots\ldots(534)$$

$$y = \begin{bmatrix} j_{-k+1} & j_{-k+2} & \ldots\ldots & j_{-k+n-1} & j_{-k+n} \end{bmatrix} x \qquad \ldots\ldots(535)$$

where

$1 \leqslant k \leqslant n$ and $\underset{\sim}{e}_k$ denotes the unit vector having a 1 in the kth row and zeroes elsewhere.

A reduced model can now be produced in one of two ways, the first being a method of modal elimination. The denominator of the reduced system is obtained by eliminating less dominant poles from the denominator of the original system transfer function. If k dominant poles are retained, the reduced system can be represented as -

$$\hat{G}(s) = \frac{d_0 + d_1 s + \ldots\ldots + d_{k-1} s^{k-1}}{e_0 + e_1 s + \ldots\ldots + e_{k-1} s^{k-1} + s^k} \qquad \ldots\ldots(536)$$

and realised as -

$$\dot{\hat{x}} = \begin{bmatrix} 0 & 0 \ldots\ldots\ldots 0 & -e_0 \\ 1 & 0 \ldots\ldots\ldots 0 & -e_1 \\ 0 & 1 \ldots\ldots\ldots 0 & -e_2 \\ \vdots & \vdots \quad \ddots \quad \vdots & \vdots \\ 0 & 0 \ldots\ldots\ldots 1 & -e_{k-1} \end{bmatrix} \hat{x} + \begin{bmatrix} \\ \\ \\ \\ \\ \end{bmatrix} u \qquad \ldots\ldots(537)$$

$$y = \begin{bmatrix} c_1, & c_2, & c_3, & \ldots\ldots & c_{k-1}, & c_k \end{bmatrix} \hat{x} \qquad \ldots\ldots(538)$$

e_0 to e_{k-1} are then obtained as described in SHAMASH's method, whereas d_0 to d_{k-1} depend on c_1 to c_k and i which can be chosen to minimise the output error.

Alternatively, the output vector, c, can be obtained using the same method as that used for the original realisation. This results in moment matching with retention of dominant poles.

This reduction technique can be illustrated using the system -

$$G(s) = \frac{1/3(1 + 2s + 0.5s^2)}{1 + 11/6s + s^2 + 1/6s^3} \qquad \ldots\ldots(539)$$

$$= \frac{s^2 + 4s + 2}{(s + 1)(s + 2)(s + 3)}$$

$$= 1/3 + 1/18s - 0.2685s^2 + ...$$

$$= s^{-1} - 2s^{-2} + 3s^{-3} - .....$$

This can then be reduced to a second order model retaining the poles at $s = -1$ and $s = -2$, and giving the reduced model as -

$$\dot{\hat{x}} = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix} \hat{x} + \tilde{e}_i u \qquad .....(540)$$

$$y = \begin{bmatrix} j_{-i+1} & j_{-i+2} \end{bmatrix} \hat{x} \qquad .....(541)$$

If $i = 2$

$$y = \begin{bmatrix} -1/3 & 1 \end{bmatrix} \hat{x} \qquad .....(542)$$

and the reduced transfer function becomes

$$\hat{G}(s) = \frac{1/3(1 + 1.5s)}{1 + 1.5s + 0.5s^2} \qquad .....(543)$$

Using further examples, however, it has been found that not all values of i give good reduced models in that values of less than 2 tend to give steady state error, which is not present for $i \geq 2$.

Comparison of this model with those obtained using the Schwarz Canonical form and also CHEN and SHIEH's method (14) indicate that in all three cases results obtained are very similar.

In February 1977, OBINATA and INOOKA published a paper (103) in which they put forward a reduction technique similar, in certain respects, to that of CHIDAMBARA 1969, but which the authors describe as a mixed method of aggregation and first

criterion approach.

Reducing

$$\dot{x} = Ax + Bu \qquad \ldots\ldots(544)$$

$$y = Ex \qquad \text{where x is of order n} \qquad \ldots(545)$$

to

$$\dot{z} = Fz + Gu \qquad \ldots\ldots(546)$$

$$\hat{y} = Hz \qquad \text{where z is of order l} \qquad \ldots(547)$$

OBINATA and INOOKA then relate the two state vectors by the aggregation matrix, C. i.e.

$$z = Cx \qquad \ldots\ldots(548)$$

where C is chosen so as to retain the l dominant poles giving

$$C = VTU^{-1} \qquad \ldots\ldots(549)$$

where

U is the modal matrix of A

V is the modal matrix of F

$$T = \left[I : 0 \right]$$

where

I is the l x l unit matrix.

V can then be chosen at will either to retain certain state variables from the original system or to be set to the unit matrix.

F and G are then given by -

$$F = CAC'(CC')^{-1} \qquad \ldots\ldots(550)$$

and $$G = CB \qquad \ldots\ldots(551)$$

where

$C'(CC')^{-1}$ is the pseudo inverse of C.

The authors then recommend that H be determined so as to minimise the integral square error between y and $\hat{y}$ by

minimising J where

$$J = \int_0^T (y - \hat{y})'Q(y - \hat{y})\, dt \qquad .....(552)$$

This is minimised by putting

$$H = EWC'(CWC')^{-1} \qquad .....(553)$$

where

$$W = \int_0^T xx'\, dt \qquad .....(554)$$

If W is then put equal to 1

$$H = EC'(CC')^{-1} \qquad .....(555)$$

and the model becomes identical to the aggregated model of HICKIN and SINHA (May 1975).

OBINATA and INOOKA illustrate their technique using a comparatively simple example. They reduce

$$\dot{x} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -10 \end{bmatrix} x + \begin{bmatrix} 20 \\ -5 \\ 5 \end{bmatrix} u \qquad .....(556)$$

$$y = \begin{bmatrix} \frac{1}{9} & \frac{1}{2} & \frac{1}{18} \end{bmatrix} x \qquad .....(557)$$

to

$$\dot{z} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} x + \begin{bmatrix} 20 \\ -5 \end{bmatrix} u \qquad .....(558)$$

$$y = \begin{bmatrix} h_1 & h_2 \end{bmatrix} x \qquad .....(559)$$

having taken

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad .....(560)$$

Assuming that the input factor is given by -

$$u = e^{-\alpha t} \qquad \ldots\ldots(561)$$

(where α is a constant)

and minimising

$$J = \int_0^{\infty} (y - \hat{y})^2 \, dt \qquad \ldots\ldots(562)$$

the authors obtain

$$h_1 = \frac{14\alpha^2 + 185\alpha + 435}{132(\alpha + 3)(\alpha + 10)} \qquad \ldots\ldots(563)$$

and

$$h_2 = \frac{5\alpha^2 + 67\alpha + 158}{11(\alpha + 3)(\alpha + 10)} \qquad \ldots\ldots(564)$$

Then, for $\alpha = 0$, which corresponds to the step response,

$$h_1 = 0.10985 \quad \text{and} \quad h_2 = 0.47879$$

giving

$$\hat{G}(s) = \frac{1 - 0.09848s}{1 + 1.5s + 0.5s^2} \qquad \ldots\ldots(565)$$

$$= \frac{-0.197(s - 10.154)}{(s + 1)(s + 2)}$$

$$= 1 - 1.59848s + 1.8977s^2 - 2.0473s^3 + \ldots\ldots$$

while

$$G(s) = \frac{1}{1 + 1.6s + 0.65s^2 + 0.05s^3} \qquad \ldots\ldots(566)$$

$$= \frac{20}{(s + 1)(s + 2)(s + 10)}$$

$$= 1 - 1.6s + 1.91s - 2.066s + \ldots\ldots$$

For $\alpha = \infty$, which corresponds to the impulse response

$$h_1 = 0.106 \quad \text{and} \quad h_2 = 0.455$$

giving

$$\hat{G} = \frac{0.985(1 - 0.077s)}{1 + 1.5s + 0.5s^2} \qquad \text{.....(567)}$$

$$= \frac{-0.1515(s - 13)}{(s + 1)(s + 2)}$$

$$= 0.985 - 1.5533s + 1.8375s^2 - 1.9796s^3 + ...$$

As shown above, this model has a steady state error. Using this form of input, the steady state is only matched if

$$\alpha = 0$$

since, for all other values of α, the final value of y is equal to zero and no steady state error can be detected in the minimisation.

The authors subsequently compare the integral square impulse response error of this model with that of four other models. They are able to demonstrate that their model gives a better result than that obtained from three of these other models but worse than that of WILSON's model obtained by minimising the ISE of the impulse response without retaining any particular poles.

This result is not surprising since the two models designed by minimising the ISE of the impulse response give the lowest ISE values of all.

It can also be seen that the initial time moments of the step response model are closer to those of the full system than are those of the impulse response model. This is also to be expected since the former model matches the steady state.

For comparison, CHEN and SHIEH's method gives -

$$\hat{G}(s) = \frac{1 - 0.077s}{1 + 1.523s + 0.528s^2} \qquad \text{.....(568)}$$

$$= \frac{-0.146(s - 13)}{(s + 1.882)(s + 1.008)}$$

This closely resembles the impulse response model but does not contain the steady state error.

A technique similar to that of OBINATA and INOOKA was published by GRUCA and BERTRAND in December 1978.(104).

For this technique, the authors obtain the F and G matrices in the same manner as OBINATA and INOOKA, but then, instead of taking -

$$\hat{y} = Hz \qquad \text{(eqn.597)}$$

GRUCA and BERTRAND allow for a time delay giving

$$v = Hz \qquad .....(569)$$

and

$$\hat{y} = v(t - \theta) \qquad .....(570)$$

and then proceed to evaluate H and θ by minimising the integral square error of the time response.

The examples subsequently given indicate that by introducing this time delay, the authors have succeeded in reducing the least square error.

2.11 OTHER MODEL REDUCTION TECHNIQUES.

In addition to the reduction methods already covered in the preceding sections, various methods of model reduction have been proposed which do not fall into the categories with which these sections deal.

The first of these is the updated Bode Plot method published by TOWILL and MEHDI (26) in January 1970, which is based on the Kan Chen Curve Fitting Technique of 1957, (105) and utilises the Open Loop Bode Plot.

For the purpose of model reduction, Chen recommends that the Open Loop Bode Plot be divided into three regions:-

$$|G(jw)|_{dB} > 15dB \quad , \quad 15dB > |G(jw)|_{dB} > -15dB$$

and

$$-15dB > |G(jw)|_{dB}$$

These regions correspond approximately to the presence of dipoles near the origin; dominant poles and zeroes; and far-off poles and zeroes respectively.

The model is then based in that section of the Bode Plot which falls between $\pm$ 15dB, and thus takes into account the dominant poles and zeroes which this area represents. Although these boundaries are, to a certain extent, arbitrary, they have apparently, proved to be satisfactory. However, inaccuracies can and do appear when break points occur just outside the boundaries.

In order to reduce these inaccuracies, TOWILL and MEHDI propose that a first order lag be substituted in place of the eliminated poles and zeroes.

Thus -

$$T_{eq} = \sum T_p - \sum T_z \qquad .....(571)$$

They exemplify this by reducing -

$$G(s) = \frac{K(1 + T_r.s)}{s^2(1 + T_n.s)^3 (1 + T_a.s)} \qquad .....(572)$$

to

$$\hat{G}(s) = \frac{K(1 + T_r.s)}{s^2(1 + T_{eq}.s)} \qquad .....(573)$$

Now, T_r is a dominant zero which falls between $\pm$ 15dB on the Bode Plot and, consequently, it is retained.

T_a and T_n, however, fall outside these boundaries, and are, therefore, replaced by an equivalent time lag -

$$T_{eq} = 3T_n + T_a \qquad .....(574)$$

The closed loop model obtained using this method of model reduction proves to be almost identical to that obtained using the continued fraction technique of CHEN and SHIEH.

In 1972 DE SARKAR and DHARMA RAO (106) put forward a significant reduction technique which is based on the geometric properties of the Lyapunov Function.

An unforced system, represented by the equation -

$$\dot{x} = Ax \qquad .....(575)$$

which has unique eigen_values, i.e.

$$\lambda_i(A) - \lambda_j(A) \neq 0 \quad \text{(for all i and j)} \qquad .....(576)$$

is examined and is found to be asymptotically stable if its Lyapunov Function ,V, is positive definite.

This Lyapunov Function is given by:

$$V = x' Px \qquad .....(577)$$

where P is a symmetric matrix satisfying the equation

$$PA + A'P = Q \qquad .....(578)$$

Q, here, is any positive definite matrix.

If these conditions are satisfied, then -

$$\dot{v} = -x' Qx \qquad \ldots\ldots (579)$$

The reduced model is then derived to give a Lyapunov Function which will satisfy the equation -

$$\frac{\dot{V}_m}{V_m} \simeq \frac{\dot{V}}{V} \qquad \ldots\ldots (580)$$

For the sake of convenience V_m is taken as equal to V and the reduction is achieved by the elimination of some of the states of the system.

After deciding which states are to be retained, Q, is selected as follows:-

$$Q = \text{diag}\ (q_1, q_2, \ldots\ldots q_i, \ldots\ldots q_n) \qquad \ldots\ldots (581)$$

where

$q_i = 1$ for all retained states

and $q_i = 0$ for eliminated states.

Using this Q matrix, the P matrix can now be found from the equation

$$PA + A'P = Q \qquad \ldots\ldots (582)$$

This involves the solution of $\frac{n}{2}(n + 1)$ linear simultaneous equations and can be achieved by employing an iterative method of solving Lyapunov matrix equations.

The elements of P associated with the eliminated modes should be 20% or less than those for retained modes.

Once Q, P and A are known, it is then possible to evaluate S, which is defined as:-

$$S = PA + \frac{Q}{2} = -A'P - \frac{Q}{2} \qquad \ldots\ldots (583)$$

P_m and S_m for the reduced model can then be evaluated by elim-

inating, from P and S, those rows and columns which are associated with the eliminated states.

Then, when P_m and Q_m are known, the system matrix, A_m, can be calculated, by applying the equation:-

$$A_m = P_m^{-1}(S_m - \frac{Q_m}{2}) \qquad \text{.....(584)}$$

DE SARKAR and DHARMA RAO use an example taken from KUPPURAJULU and ELANGOVAN (24) to illustrate this technique. Thus

$$(\dot{x}) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -0.21 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & -20 & -12 \end{bmatrix} (x) \qquad \text{.....(585)}$$

To eliminate x_4 -

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{.....(586)}$$

which gives

$$P = \begin{bmatrix} 3.063 & 2.381 & 1.266 & 0.103 \\ 2.381 & 2.759 & 1.489 & 0.122 \\ 1.266 & 1.489 & 1.237 & 0.099 \\ 0.103 & 0.122 & 0.099 & 0.008 \end{bmatrix} \qquad \text{.....(587)}$$

P_3 is then obtained by eliminating the last row and last column of P and is subsequently used to calculate A_3 which becomes -

$$A_3 = \begin{bmatrix} 0 & 1.003 & 0.00623 \\ -0.21 & -1.0025 & 0.9495 \\ 0 & -0.0776 & -1.5532 \end{bmatrix} \quad \text{.....(588)}$$

The poles of the original system were

-0.264, -0.847, -1.875, -10.014.

Those of the reduced system are

-0.264, -0.869, -1.423.

Thus, the pole farthest from the origin has been eliminated and the others have been retained with slight modification to take into account the eliminated mode.

This technique can therefore be described as a modified method of modal elimination.

The authors also produce time response curves which are used to demonstrate that this model has a closer agreement with the original system than those models produced by KUPPURAJULU and ELANGOVAN.

Although designed around an unforced system, this technique can be extended to forced systems with little difficulty.

A_m is obtained as before and B_m is selected to eliminate steady state error between the states in the reduced model and the corresponding states of the original system.

Thus, if

$$\dot{x} = Ax + Bu \quad \text{.....(589)}$$

and

$$\dot{x}_m = A_m x_m + B_m u \quad \text{.....(590)}$$

then, in the steady state

$$x = -A^{-1}Bu \quad \text{.....(591)}$$

and

$$x_m = -A_m^{-1}B_m u \quad \text{.....(592)}$$

Putting $C = A^{-1}$ and partitioning then gives -

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = - \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \qquad \ldots\ldots(593)$$

where

x_1 are the retained states

and x_2 are the eliminated states

This gives:

$$x_1 = -(C_{11}B_1 + C_{12}B_2)u \qquad \ldots\ldots(594)$$

Now, putting $x_m = x_1$, this gives

$$A_m^{-1}B_m = C_{11}B_1 + C_{12}B_2 \qquad \ldots\ldots(595)$$

$$B_m = A_m(C_{11}B_1 + C_{12}B_2) \qquad \ldots\ldots(596)$$

Hence, given A_m, which has already been found, and also the original system parameters, B_m can be calculated with relative ease.

The main advantage of this technique lies in the fact that if the original system is asymptotically stable then the reduced model obtained will also be asymptotically stable.

The main disadvantage, however, is the relative complexity of the method with regard to the calculation of P. This is especially true for higher order systems.

Other work concerning 'non-standard' reduction methods includes a paper published in 1974 by SCHWARTZ and EDGAR (107). These two authors examine second order models, obtained by using continuous fraction techniques, whose parameters have a particular physical relevance.

In such cases, there is a requirement for the relationship of certain specified parameters of the reduced model to the physical parameters of the system to be noted, thus enabling any change in one of these physical parameters to be easily reflected as a corresponding change in the reduced model.

An original but somewhat complicated method of model reduction is that put forward by SHAKED and KARCANIAS (108), and published in 1976.

This method uses state feedback, which is applied to the original system in such a way that the dimension of the observable subspace of the modified system is reduced and thus a lower minimal realisation of the system can be found.

Another original reduction technique is that of BISTRITZ and LANGHOLZ (109), published in August 1979, which the authors describe as model reduction by best Chebyshev rational approximations in the complex plane.

Thus G(s) is reduced to $\hat{G}(s)$ so as to minimise the absolute impulse response error.

$$e(t) = |g(t) - \hat{g}(t)|$$

$$= \frac{1}{2\pi} \int_{-j\alpha}^{j\alpha} \left| (G(s) - \hat{G}(s)) \exp(st)\, ds \right| \qquad \ldots\ldots(597)$$

BISTRITZ and LANGHOLZ recommend that the approximation be carried out in the z domain where

$$s = \frac{1 - z}{1 + z} \qquad \ldots\ldots(598)$$

To achieve this, G(s) is transformed into F(z)

and $\hat{G}(s)$ is transformed into $\hat{F}(z)$

giving

$$e(t) = \frac{1}{\pi} \left| \oint_{\gamma} (F(z) - \hat{F}(z)) \exp\left(\frac{1 - z}{1 + z} t\right) \frac{dz}{(1 + z)^2} \right| \qquad \ldots(599)$$

where the contour γ is the unit circle centred on the origin of the complex z plane, corresponding to the imaginary axis in the s plane, with the interior of the circle matching the right hand side of the s plane.

An approximation to F(z) in the z plane is then obtained

by using least square fitting techniques, and $\hat{G}(s)$ can then be derived by inverse transformation.

As described by the authors, this technique appears to be over-complicated and an investigation of its effects has not, therefore been carried out.

2.12 CONCLUSIONS.

As can be seen from the whole of section 2 many methods of model reduction have been developed since the early 1960's. Each method has its own particular advantages and disadvantages. Some methods give a good fit in the time domain for a given input, others give a relatively good fit at certain frequencies in the frequency domain. Still others guarantee that a model obtained from a stable system will also be stable. If a reduced model is obtained using one given reduction technique, according to one particular criterion it might or might not be a good model as tested according to another criterion. Because of this if a given condition is to be satisfied by a reduced model developed for any given purpose this condition should be included in the criteria which are used in obtaining the reduced model. It is for this reason that much work has recently been carried out on combined reduction techniques which attempt to satisfy more than one criterion, giving models which, for example, are stable for stable systems and also have a close fit in the frequency domain.

An alternative method for guaranteeing that a model satisfies certain conditions is to test the model to see if this is so. If a series of models are to be obtained, all of which satisfy given criteria, tests can be carried out to find under what conditions a given reduction technique produces models which satisfy the desired criteria. This approach has been used in the next section.

The reduction techniques reviewed in this section can be divided into optimisation techniques or analytical techniques. Optimisation techniques in general give very good fits in the range over which the optimisation was carried out and, in the case of time domain optimisation, for the particular input used during the optimisation. However for other inputs or at a different frequency the fit can be very poor indeed. On the other hand analytical methods are not so much tied to the input signal, as normally the analysis is carried out in the frequency domain but is very dependent on the particular criteria chosen. Hence model reduction using moment matching matches the frequency response about $s = 0$ while matching Markov parameters fits the frequency response about $s = \infty$. Any combination of the two gives a better match between these two points but a worse match at these points themselves.

It can easily be seen that no reduced model can match perfectly the original system although in many cases the fit can be very close. For this reason, great care must be taken in model reduction to ensure that the reduced model is obtained using the best criteria for the purpose for which the model is intended. In the case of controller design the closed loop response of the reduced model is very important, as it is the closed loop response of the system that is ultimately the point of interest. Because of this the effect of the model reduction technique on the closed loop response must be taken into account.

3.0 THE ADEQUACY OF FIRST ORDER MODELS OF HIGHER ORDER SYSTEMS

As shown by the prece ding review, much work has been done on reducing higher order mathematical models to ones of lower order, which can then be used either to represent the system in the investigation of certain system characteristics (which have been carefully retained in the reduction) or for other specific purposes. General reduction techniques have been considered, with emphasis being applied to the reduction of models to second, third, fourth and fifth order models and little attention being paid either to first order models or to models containing time delays. However, a combination of these two features giving a first order model with time delay can conveniently be used for the design of controllers for the system under consideration. If an adequate representation of this type can be produced for the system, the parameters can then be used to evaluate parameters for proportional and integral (PI) and proportional, integral and derivative (PID) controllers. This is further faciliated by the fact that published data is available for this purpose.

Controller design using this method requires a model of the form:-

$$\hat{G}(s) = \frac{K\, e^{-\hat{\tau}s}}{1+Ts} \qquad (A1)$$

This section is devoted to the investigation of the adequacy of such models for controller design.

When designing a controller for a system, one of the prime considerations must be that the closed loop system including the controller be stable. For this reason when considering the adequacy of the simple model for controller design, the greatest attention must be paid to the stability of the simple model. It is desirable that the simple model of the plant be less stable than, or equally stable to the plant itself, so that any controller based on the model should give stable control when used with the plant itself. Although these measures do not

guarantee stable control they make it far more probable.

The question that now arises, is how to obtain suitable parameters for this simple model; this is dealt with in the next sub-section.

3.1 EVALUATION OF SIMPLE MODEL PARAMETERS

The preceding review indicates that simple parameters can be selected in a number of different ways. Some of these, however, cannot be considered suitable for controller design. Optimisation techniques, for example, eliminate almost entirely the advantages of designing a controller using model reduction since the controller parameters can be optimised equally as well as the model parameters with a probability of better results. Alternatively, if the model parameters were obtained by modal elimination no account would be taken of any modes other than the one retained.

The Padé approximation method as used by Gilibaro and Lees in their moment matching technique, and by Chen and Shieh in their continued fraction technique would seem to be the most suitable choice for controller design. Simple model parameters can be obtained by expanding both the first order and higher order models into polynomial series and equating the coefficients to obtain expressions for $\hat{\tau}$ and T.

Thus when considering a general higher order system represented by the transfer function:-

$$G(s) = \frac{Ke^{-\tau s}(1+b_1 s+b_2 s^2+\ldots+b_m s^m)}{1+a_1 s+a_2 s^2+\ldots \quad +a_n s^n} \qquad \ldots\ldots(A2)$$

for a real process n>m

The expansion can be given as:-

$$G(s) = K(1+b_1 s+b_2 s^2+b_3 s^3+\ldots)(1-\tau s+ \frac{\tau^2 s^2}{2} - \frac{\tau^3 s^3}{6} +\ldots)$$

$$(1+a_1 s+a_2 s^2+a_3 s^3+\ldots)^{-1}$$

$$= K(1+(b_1-\tau-a_1)s+(b_2+ \frac{\tau^2}{2} -b_1\tau+a_1^2-a_2-a_1 b_1+a_1\tau)s^2$$

$$+(b_3-\frac{\tau^3}{6} -b_2\tau+ \frac{b_1\tau^2}{2} +2a_1a_2-a_3-a_1^3+b_1a_1^2-b_1a_2$$

$$-\tau a_1^2+\tau a_2-a_1 b_2- \frac{a_1\tau^2}{2} +a_1 b_1\tau)s^3+\ldots.) \qquad \ldots\ldots(A3)$$

Similarly, the first order plus time delay model:-

$$\hat{G}(s) = \frac{Ke^{-\hat{\tau}s}}{1+Ts}$$

can be expanded to give:

$$\hat{G}(s) = K(1-(\hat{\tau}+T)s+(\frac{\hat{\tau}^2}{2}+T^2+\hat{\tau}T)s^2-$$
$$(\frac{\hat{\tau}^3}{6}+T^3+\frac{\hat{\tau}^2T}{2}+\hat{\tau}T^2)s^3+....) \qquad .....(A4)$$

Equating the coefficients of s in these two expansions gives:-

$$-(\hat{\tau}+T) = b_1-\tau-a_1$$

Similarly, equating the coefficients of s^2 produces:-

$$\frac{\hat{\tau}}{2}+T^2+\hat{\tau}T = b_2+\frac{\tau^2}{2}-b_1\tau+a_1^2-a_2-a_1b_1+a_1\tau$$

Solving these two equations simultaneously expressions for T and $\hat{\tau}$ can be obtained. Thus

$$T = (a_1^2-b_1^2-2a_2+2b_2)^{\frac{1}{2}} \qquad .....(A5)$$

$$\hat{\tau} = \tau+a_1-b_1-T \qquad .....(A6)$$

Explicit expressions for the simple model parameters are thus obtained and these can now be evaluated by substituting appropriate system parameters.

By equating the coefficients of s and s^2 a good fit can be guaranteed at lower frequencies. At higher frequencies however the error can be quite substantial. A possible course of action in these circumstances would be to equate the coefficients of higher orders of s in the two expansions above, and take any two of these equations thus obtaining different expressions for $\hat{\tau}$ and T. However these expressions will not then be explicit but implicit functions of $\hat{\tau}$ and T. Thus by equating coefficients of s and also s^3 expressions for $\hat{\tau}$ and T are obtained which include a third order polynomial which must be solved to give $\hat{\tau}$ and T. A model obtained in this way would give a better fit at higher frequencies at the expense of lower frequencies. However, since no explicit solution exists for the simple model parameters when coefficients of

higher orders of s are used, it is in general more practical to use the Padé technique of solving the first two equations.

3.2 REALIZABILITY AND STABILITY OF FIRST ORDER MODELS

For the simple model to be useable it is necessary for it to be both realizable and stable. Such models cannot be obtained from higher order models of all systems.

The conditions which must be satisfied by the simple model parameters for the model to be realizable are that the time delay τ must not be negative and that T must be real.

i.e., $\hat{\tau} \geq 0$

$T^2 \geq 0$

For the model to be open-loop stable T must also be positive

i.e., $T \geq 0$

These conditions indicate that for realizability and stability of the simple model, the higher order model parameters must satisfy the following conditions:-

$$a_1^2 - b_1^2 - 2a_2 + 2b_2 \geq 0 \qquad \text{.....(A7)}$$

(as $T^2 = a_1^2 - b_1^2 - 2a_2 + 2b_2$)

and

$$\tau + a_1 - b_1 - (a_1^2 - b_1^2 + 2a_2 + 2b_2)^{\frac{1}{2}} \geq 0$$

(as $\hat{\tau} = \tau + a_1 - b_1 - T$) .....(A8)

For given values of τ, a_1, b_1, b_2 these inequalities place upper and lower limits on the value of a_2 which can be used to give a realizable and stable simple model.

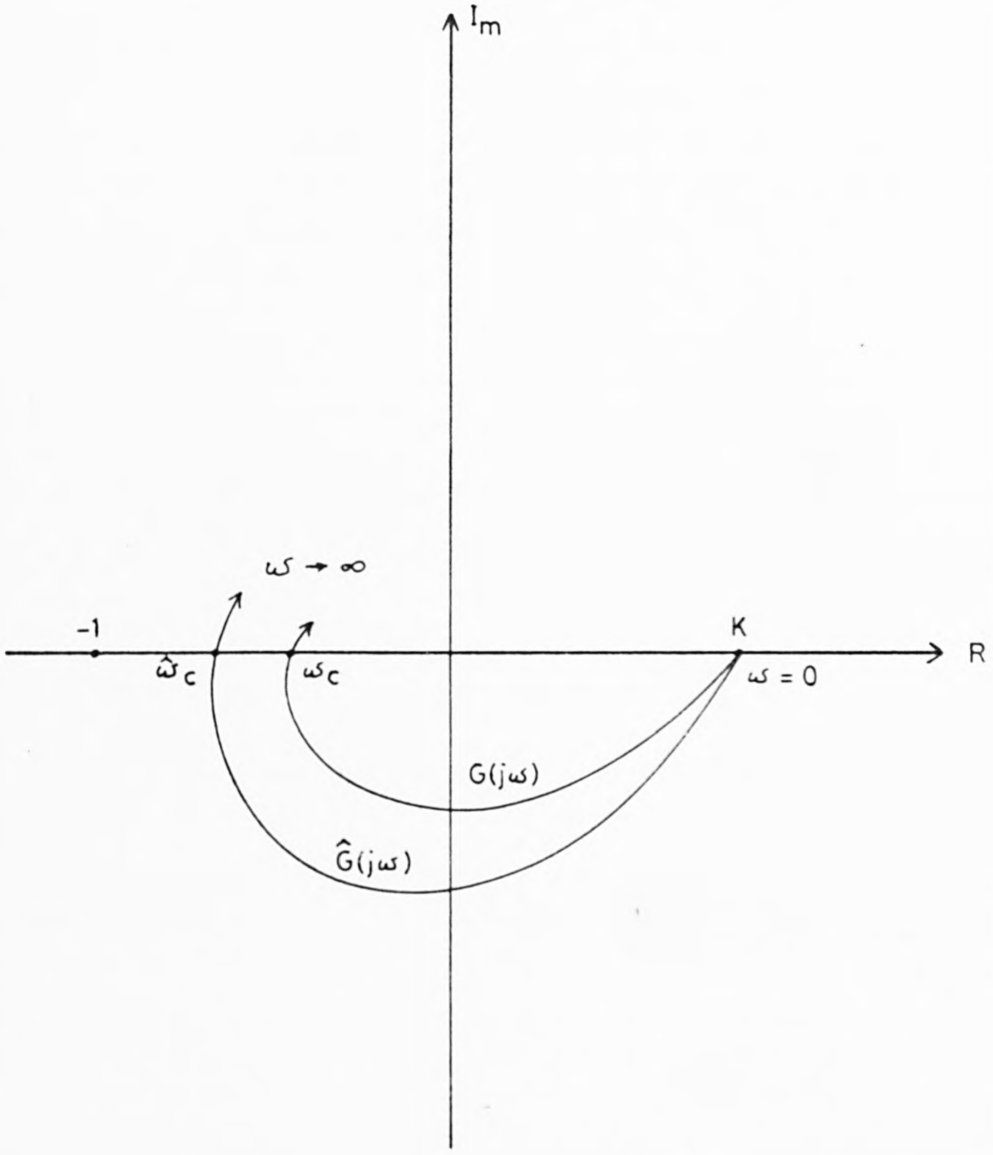
i.e., $$a_2 \geq b_2 + b_1\tau + a_1 b_1 - \tau a_1 - b_1^2 - \frac{\tau^2}{2} \qquad \text{.....(A9)}$$

$$a_2 \leq b_2 + \tfrac{1}{2}(a_1^2 - b_1^2) \qquad \text{.....(A10)}$$

Hence, any higher order model whose parameters do not satisfy these inequalities will not give a realizable and stable first order plus time delay model. This limitation greatly reduces the number of higher order models which can be simplified in this way. Furthermore, not all of the simple models obtained from these higher order models, will satisfy the adequacy conditions for controller design.

3.3 ADEQUACY

When designing a closed-loop control system, the minimum requirement is that it be stable in all circumstances. Consequently when using a simple model to design a controller for such a system the model must be one that will result in stable control. To ensure this the closed-loop system containing the simple model must be less stable than the closed-loop system containing the process itself. This requirement is fulfilled if the open loop frequency response plots of the process and simple model are such that the locus of the first order model passes to the left of the locus of the process when crossing the negative real axis. Alternatively we may say that the model is considered adequate if the gain margin of the closed loop system containing the model is not greater than the gain margin of the closed loop system containing the process.



$$\left|\hat{G}(j\hat{\omega}_c)\right| \geqslant \left|G(j\omega_c)\right|$$

where $\hat{\omega}_c$ does not necessarily equal ω_c.

FIGURE 3.1

EXAMPLE OF OPENLOOP FREQUENCY RESPONSES OF A HIGHER ORDER SYSTEM AND ITS ADEQUATE LOWER ORDER MODEL

3.4 SIMPLE MODEL CHARACTERISTICS

Prior to considering the detailed adequacy boundaries of higher order systems it is relevant to examine the characteristics of the first order plus time delay model paying particular attention to the model at the critical frequency $\hat{\omega}_{c1}$, where the frequency response plot first crosses the negative real axis.

For the model

$$\hat{G}(s) = \frac{Ke^{-\hat{\tau}s}}{1+Ts} \qquad .....(A11)$$

putting $s=j\omega$ gives

$$\hat{G}(j\omega) = \frac{K\ (\cos\omega\hat{\tau}-\omega T\sin\omega\hat{\tau})-j(\omega T\cos\omega\hat{\tau}+\sin\omega\tau)}{1+\omega^2T^2} \qquad .....(A12)$$

At $\hat{\omega}c$ $\quad Im(\hat{G}(j\hat{\omega}_c)) = 0 \qquad .....(A13)$

$$\therefore \quad \tan\hat{\omega}_c\hat{\tau} = -\hat{\omega}_cT \qquad .....(A14)$$

$$\hat{G}_c = Re(\hat{G}(j\hat{\omega}_c)) = \frac{K}{\sqrt{1+\hat{\omega}c^2T^2}} = K\cos\hat{\omega}_c\hat{\tau} \qquad .....(A15)$$

Solving these equations simultaneously shows that $\frac{\hat{G}(j\hat{\omega}_c)}{K}$ is a function of the ratio τ:T.

These relationships are shown in Figures 3.2 and 3.3.

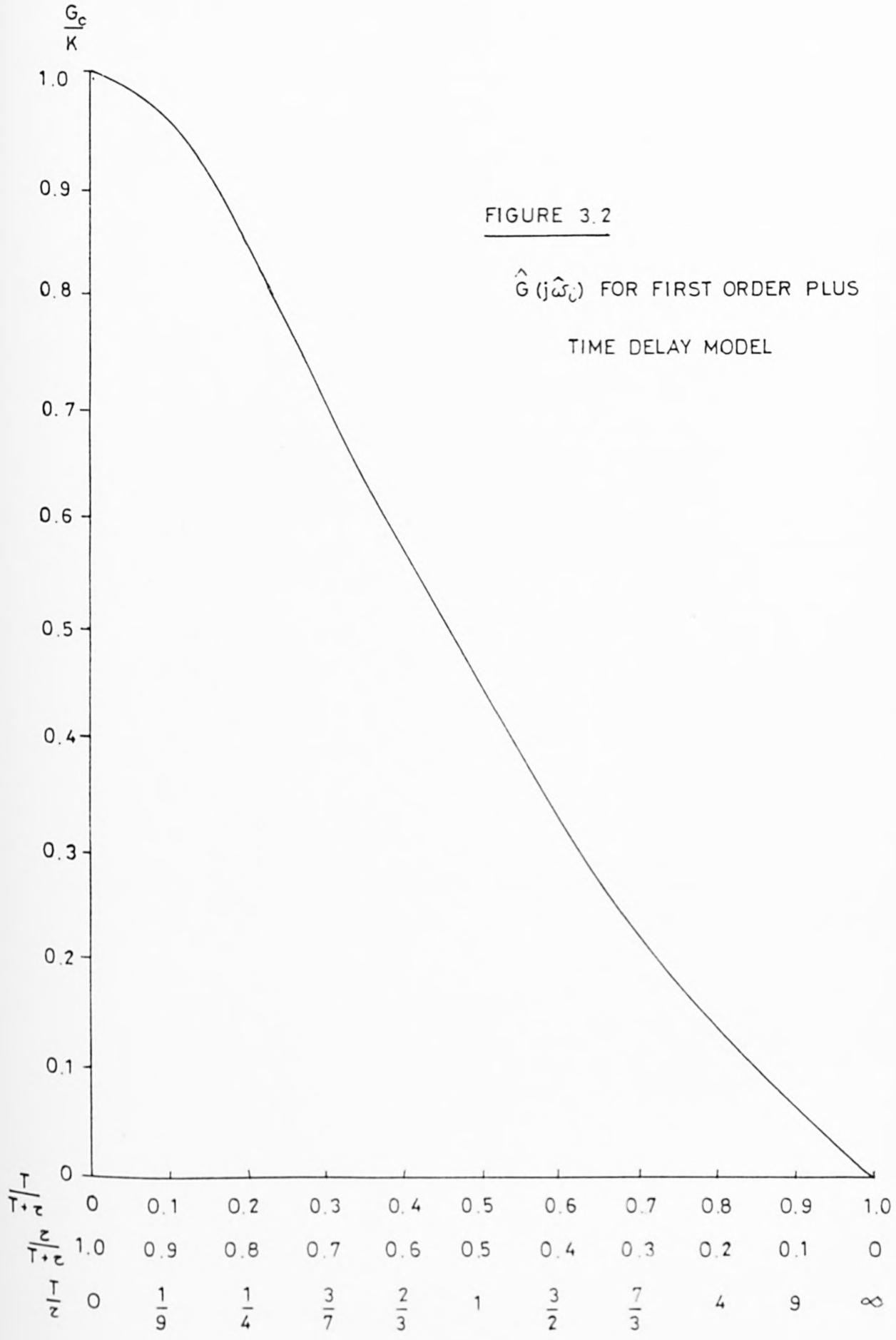


FIGURE 3.2

$\hat{G}(j\hat{\omega}_i)$ FOR FIRST ORDER PLUS TIME DELAY MODEL

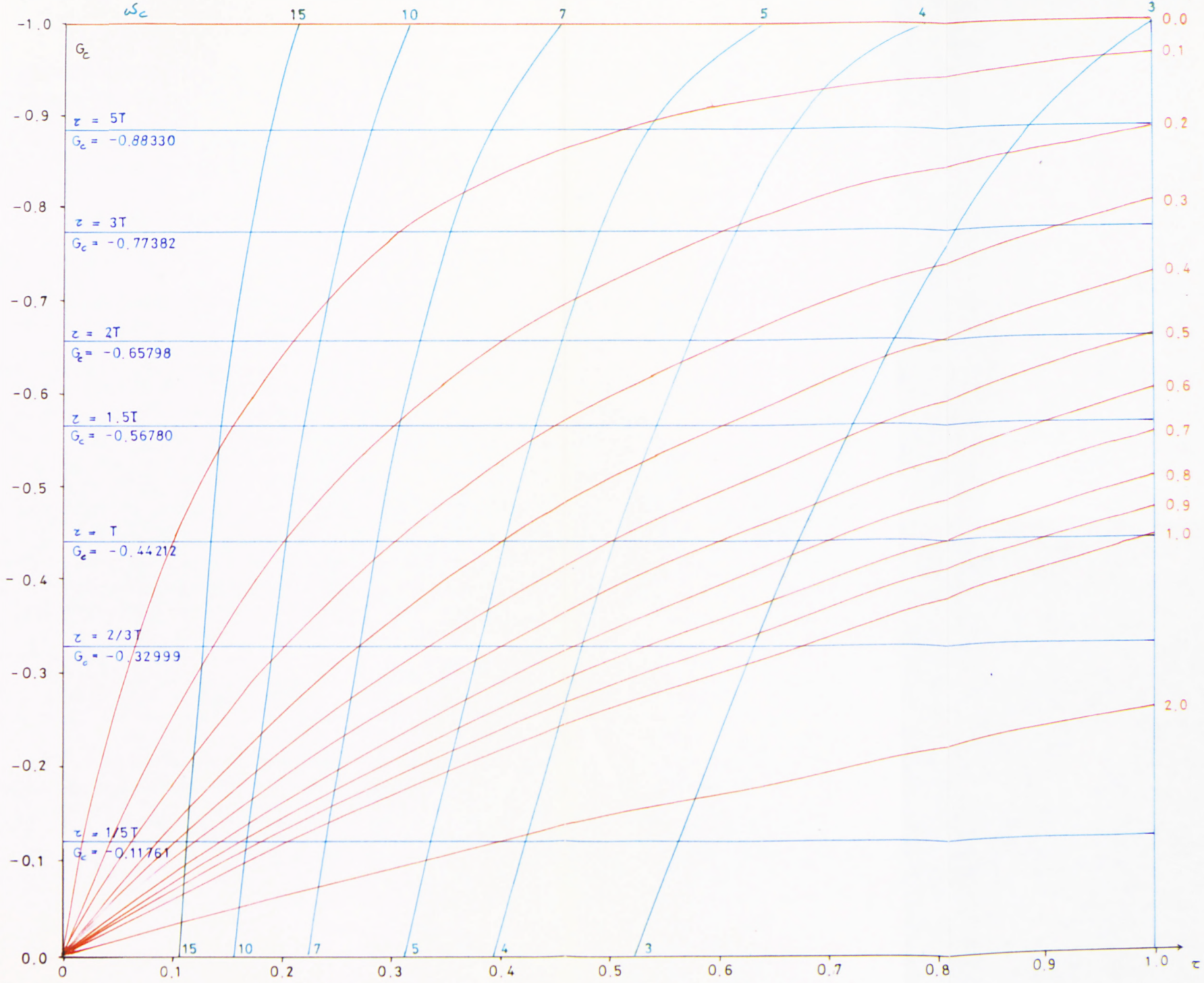


FIGURE 3.3 RELATIONSHIP BETWEEN T, τ, G_c AND ω_c FOR THE SIMPLE MODEL $G(j\omega) = \frac{e^{-\tau j\omega}}{1 + Tj\omega}$

3.5 ADEQUACY BOUNDARY FOR 2ND ORDER SYSTEMS WITHOUT TIME DELAY

The simplest system to which reduction techniques can be applied is a second order system without time delay.

Consider the second order system

$$G(s) = \frac{K(1+b_1 s)}{1+a_1 s+a_2 s^2} \qquad .....(A16)$$

The realizability and stability boundaries (eqns.A7-A10) require that

$$(a_1-b_1)b_1 \leqslant a_2 \leqslant (a_1-b_1)\left(\frac{a_1+b_1}{2}\right) \qquad .....(A17)$$

since a_2 must be positive for a stable system, $-a_1 \leqslant b_1 \leqslant a_1$ for a realizable and stable model. Analysis of $G(j\omega)$ shows that for such a system

$$\omega_c = \sqrt{\frac{b_1-a_1}{a_2 b_1}} \qquad .....(A18)$$

For $0 \leqslant b_1 \leqslant a_1$ ω_c is imaginary i.e., the frequency response does not cross the negative real axis. As a consequence all realizable stable first order plus time delay models obtained from stable second order models without time delay and with positive values for b_1 will satisfy the adequacy conditions.

For negative values of b_1

$$G(j\omega_c) = \frac{Kb_1}{a_1} \qquad .....(A19)$$

and an adequacy boundary exists which occurs when $G(j\omega_c)=\hat{G}(j\hat{\omega}_c)$. Although this boundary cannot be positioned analytically it has been found numerically to be as represented in Fig.3.4.

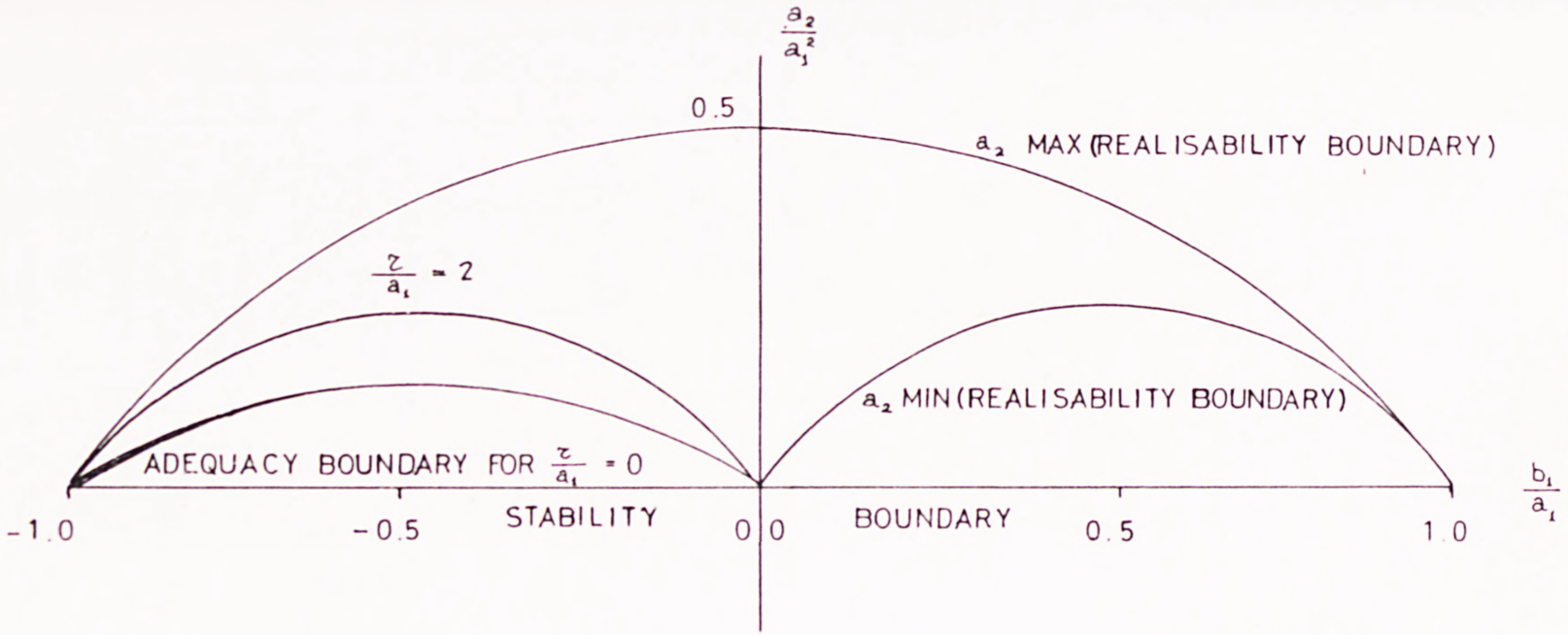


FIGURE 3.4

STABILITY, REALISABILITY AND ADEQUACY BOUNDARIES FOR SECOND ORDER SYSTEMS. a_2 MIN IS THE REALISABILITY BOUNDARY FOR SYSTEMS WITHOUT TIME DELAY. IT IS ALSO AN ADEQUACY BOUNDARY FOR SYSTEMS WITH TIME DELAY. THESE HAVE A REALISABILITY BOUNDARY BELOW THIS ONE.

3.6 ADEQUACY BOUNDARIES FOR HIGHER ORDER SYSTEMS WITH TIME DELAY

Let us now consider higher order systems with time delay.

For such systems there is a point on the adequacy boundary where the simple model is exactly equivalent to the higher order model, that is to say when the higher order model contains factors which can be cancelled to give a first order model. This can be illustrated with the following general fourth order system:-

$$G(s) = \frac{Ke^{-\tau s}(1+b_1 s+b_2 s^2+b_3 s^3)}{1+a_1 s+a_2 s^2+a_3 s^3+a_4 s^4} \qquad .....(A20)$$

If the numerator polynomial is divided into the denominator a quotient of $1+(a_1-b_1)s$ is obtained leaving a remainder of

$$\left[a_2-b_2-b_1(a_1-b_1)\right] s^2+\left[a_3-b_3-b_2(a_1-b_1)\right] s^3+\left[a_4-b_3(a_1-b_1)\right] s^4 \quad .(A21)$$

If this remainder is equal to zero, then

$$G(s) = \frac{Ke^{-\tau s}}{1+(a_1-b_1)s} \qquad .....(A22)$$

and we have a simple first order plus time delay model in which

$\hat{\tau} = \tau$

and $T=a_1-b_1$

For this however we must have

$$a_2-b_2-b_1(a_1-b_1)=0$$

$$a_3-b_3-b_2(a_1-b_1)=0$$

$$a_4 \quad -b_3(a_1-b_1)=0 \qquad .....(A23)$$

Therefore, the adequacy criteria for the model should be expressed in terms of conditions which must be satisfied by these coefficients. These conditions also depend on the order of the full model and can only be generalised for a full model of a given order. For this reason different orders of full models are considered separately in the following sub-sections.

A computer programme has been written which for an nth order system (given τ, b_1 to b_m, a_1, a_3 to a_{j-1} and a_{j+1} to a_n) will calculate the limits of a_2 according to the stability and relizability criteria and evaluate simple models at various values between a_{2min} and a_{2max}. It then evaluates $\hat{G}(j\hat{\omega}_c)$ for each simple model and subsequently proceeds to evaluate $G(j\omega_c)$ for various processes, for each value of a_2, by varying a_j between a_{jmin} and a_{jmax}, these having been previously determined, j being an input parameter and for a third order system always equal to 3, for a fourth order model 3 or 4 etc. Boundary values are then recorded between processes which give adequate models and those that give inadequate models. The programme also gives boundaries between real and imaginary and positive and negative roots of the characteristic equation (see Appendix II). This programme has been used to find adequacy boundaries for 3rd and higher order systems and the results obtained are presented in the individual sections dealing with each order of full model.

3.7 ADEQUACY BOUNDARIES FOR 2ND ORDER SYSTEMS WITH TIME DELAY

The second order system is a special case. As can be shown all the coefficients of the full model effect the parameters of the simple model. Thus the second order system

$$G(s) = \frac{Ke^{-\tau s}(1+b_1 s)}{1+a_1 s+a_2 s^2} \qquad \text{.....(A24)}$$

can be reduced to the simple model:-

$$\hat{G}(s) = \frac{Ke^{-\hat{\tau} s}}{1+Ts} \qquad \text{.....(A25)}$$

where $T=(a_1^2-b_1^2-2a_2)^{\frac{1}{2}}$ .....(A26)

and $\hat{\tau}=\tau+a_1-b_1-T$ .....(A27)

From the conditions for realizability and stability stated previously it can be deduced that for a second order model to give a realizable and stable model a_2 must satisfy the following inequalities:-

$$a_2 \geq b_1\tau+a_1b_1-\tau a_1-\frac{\tau^2}{2}-b_1^2 \qquad \text{.....(A28)}$$

$$a_2 \leq \tfrac{1}{2}(a_1^2-b_1^2) \qquad \text{.....(A29)}$$

For positive values of b_1, the only adequacy boundary is that given by the cancellation conditions. Hence the model is adequate if

$$a_2 \geq b_1a_1-b_1^2 \qquad \text{.....(A30)}$$

This is the lower realizability boundary for a second order model without time delay. The time delay does not affect the adequacy area for systems with positive values of b_1 which give stable, realizable and adequate models using this technique only if

$$b_1a_1-b_1^2 \leq a_2 \leq \tfrac{1}{2}(a_1^2-b_1^2) \qquad \text{.....(A31)}$$

giving a range of values

$$a_{2max}-a_{2min}=\tfrac{1}{2}(a_1-b_1)^2 \qquad \text{.....(A32)}$$

(see Fig.3.5)

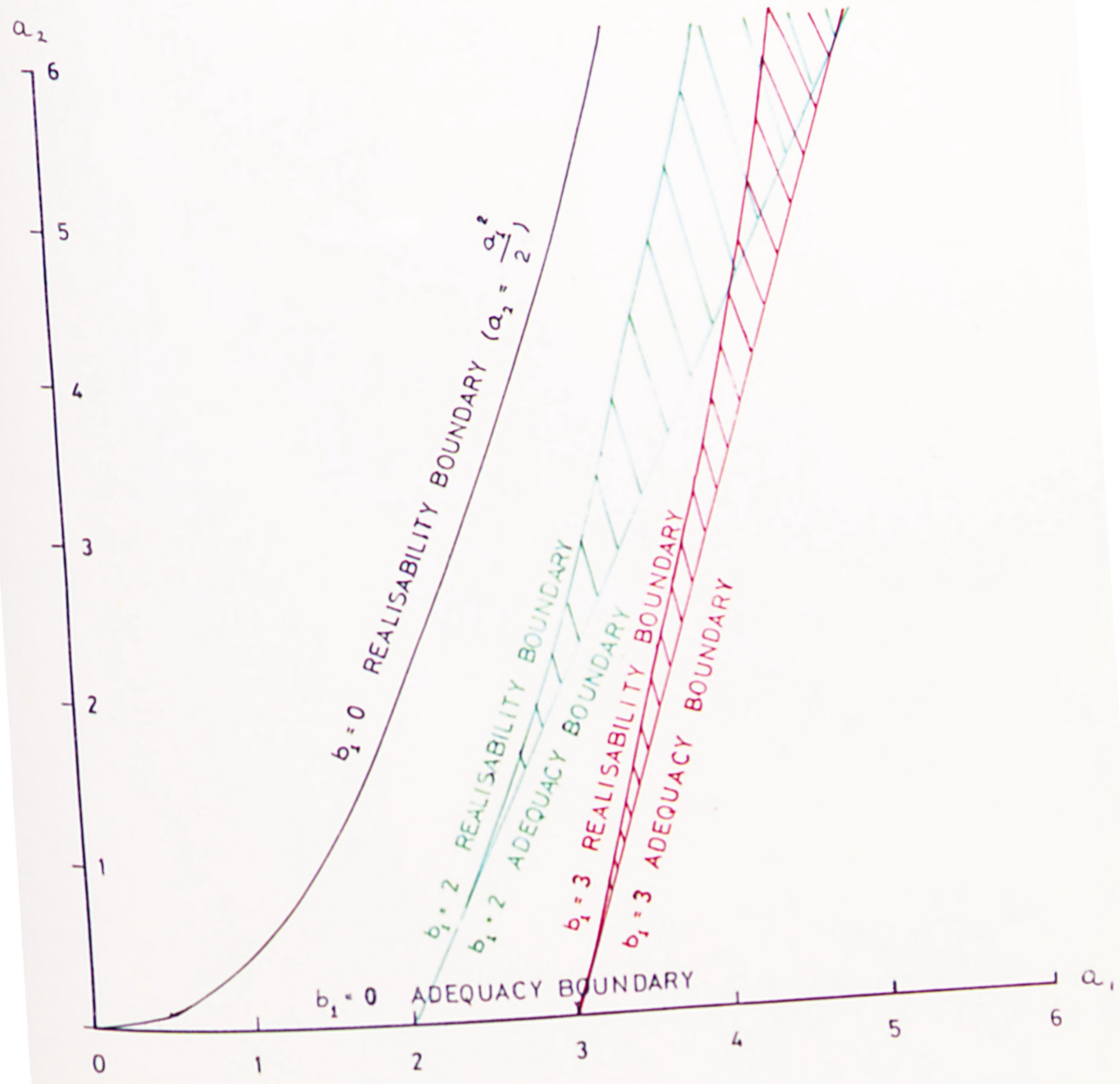


FIGURE 3.5

REALISABILITY AND ADEQUACY BOUNDARIES FOR

2ND ORDER PLUS TIME DELAY SYSTEMS IN a_2, a_1 PLANE

However for negative values of b_1, the adequacy boundary can only be found using a search technique. This boundary is of similar shape to that for positive values of b_1 but for $\tau=2$ gives a slightly larger adequacy area (see Fig.3.4)

3.8 NORMALISATION WITH RESPECT TO a_1

For full models of 3rd or higher order the complete adequacy boundary can only be found using a systematic search. In order to simplify this search the process equation can be normalised with respect to a_1 thus effectively reducing the number of parameters of the full model and consequently reducing the number of searches that have to be carried out. This normalisation can be illustrated using the following third order model -

$$G(s) = \frac{Ke^{-\tau s}(1+b_1 s+b_2 s^2)}{1+a_1 s+a_2 s^2+a_3 s^3} \qquad .....(A33)$$

Putting $\overset{*}{s}=a_1 s$ the transfer function can be re-written as

$$G(\overset{*}{s}) = \frac{Ke^{-(\frac{\tau}{a_1})\overset{*}{s}}\left[1+(\frac{b_1}{a_1})\overset{*}{s}+(\frac{b_2}{a_1^2})\overset{*}{s}^2\right]}{1+\overset{*}{s}+(\frac{a_2}{a_1^2})\overset{*}{s}+(\frac{a_3}{a_1^3})\overset{*}{s}^3} \qquad .....(A34)$$

This is a dimensionless form of G(s), since a_1 has dimensions of time, whereas s has dimensions of (time $^{-1}$).

The simple model now becomes:-

$$\hat{G}(\overset{*}{s}) = \frac{Ke^{-(\frac{\hat{\tau}}{a_1})\overset{*}{s}}}{1+(\frac{T}{a_1})\overset{*}{s}} \qquad .....(A35)$$

with

$$(\frac{T}{a_1}) = \left[1+2(\frac{b_2}{a_1^2}) - (\frac{b_1}{a_1})^2 - 2(\frac{a_2}{a_1^2})\right]^{\frac{1}{2}} \qquad .....(A36)$$

and

$$(\frac{\hat{\tau}}{a_1}) = (\frac{\tau}{a_1}) +1-(\frac{b_1}{a_1})-(\frac{\hat{T}}{a_1}) \qquad .....(A37)$$

It will be noticed that normalisation in no way affects the relationship between the full and simple models and can therefore be used to reduce the required number of searches. These need now only be carried out for a_1=1.0; results for other values of a_1 being obtained by multiplying out.

In normalised terms the realizability conditions become

$$\frac{a_2}{a_1^2} \leqslant \frac{b_2}{a_1^2} + \frac{1}{2} - \frac{1}{2}\left(\frac{b_1}{a_1}\right)^2 \qquad \text{.....(A38)}$$

and

$$\frac{a_2}{a_1^2} \geqslant \frac{b_2}{a_1^2} + \frac{b_1}{a_1} - \left(\frac{b_1}{a_1}\right)^2 - \frac{\tau}{a_1}\left[1 - \frac{b_1}{a_1} + \tfrac{1}{2}\left(\frac{\tau}{a_1}\right)\right] \qquad \text{.....(A39)}$$

The second of these conditions can also be expressed in the form:-

$$\frac{a_2}{a_1^2} \geqslant \frac{b_2}{a_1^2} + \frac{1}{2}\left[1-\left(\frac{b_1}{a_1}\right)^2\right] - \frac{1}{2}\left[\frac{\tau}{a_1} - \frac{b_1}{a_1} + 1\right]^2 \qquad \text{.....(A40)}$$

From the above form it can easily be seen that if $\frac{\tau}{a_1} - \frac{b_1}{a_1} + 1 = 0$ then no simple models are realizable since the minimum realizable value of $\frac{a_2}{a_1^2}$ then becomes equal to the maximum value of $\frac{a_2}{a_1^2}$ which would give a realizable simple model.

3.9 ADEQUACY BOUNDARIES FOR 3RD ORDER SYSTEMS WITH TIME DELAY

In order to obtain the adequacy boundaries for a third order system it is first necessary to determine the stability boundary. This can be obtained using the Routh array.

The characteristic equation of a third order system can be given in the form:-

$$1 + a_1 s + a_2 s^2 + a_3 s^3 = 0$$

From the resulting Routh array the following conditions for stability can be obtained:-

$$a_3 \geq 0 \qquad \text{.....(A41)}$$

$$a_2 \geq 0 \qquad \text{.....(A42)}$$

$$a_1 - \frac{a_3}{a_2} \geq 0 \quad \text{which can be rewritten as } a_1 \geq \frac{a_3}{a_2} \qquad \text{.....(A43)}$$

In normalised terms these become:-

$$\frac{a_3}{a_1^3} \geq 0 \qquad \text{.....(A44)}$$

$$\frac{a_2}{a_1^2} \geq 0 \qquad \text{.....(A45)}$$

$$\frac{a_2}{a_1^2} - \frac{a_3}{a_1^3} \geq 0 \quad \text{or} \quad \frac{a_2}{a_1^2} \geq \frac{a_3}{a_1^3} \qquad \text{.....(A46)}$$

These stability conditions limit the area for adequate models to:-

$$\frac{a_2}{a_1^2} \geq \frac{a_3}{a_1^3} \geq 0 \qquad \text{.....(A47)}$$

The requirement that the simple model be stable and realizable further limits this area as it requires that:-

$$\frac{a_2}{a_1^2} \geq \left(\frac{a_2}{a_1^2}\right)_{min} = \frac{b_2}{a_1^2} + \frac{b_1}{a_1} - \left(\frac{b_1}{a_1}\right)^2 - \frac{\tau}{a_1}\left[1 - \frac{b_1}{a_1} + \frac{1}{2}\frac{\tau}{a_1}\right] \quad ..(A48)$$

and

$$\frac{a_2}{a_1^2} \leq \left(\frac{a_2}{a_1^2}\right)_{max} = \frac{b_2}{a_1^2} + \frac{1}{2}\left[1 - \left(\frac{b_1}{a_1}\right)^2\right] \quad .....(A49)$$

(see Section 3.8).

This gives minimum and maximum limits for $\frac{a_2}{a_1^2}$.

If however the minimum limit given by eqn. A48 is less than zero, it must be taken as zero to satisfy the stability criterion for the full model. Likewise $\left(\frac{a_2}{a_1^2}\right)_{max} \geq 0$. To ensure this

$$\left(\frac{b_1}{a_1}\right)^2 \leq 2\frac{b_2}{a_1^2} + 1 \quad .....(A50)$$

and consequently, as $\frac{b_1}{a_1}$ must be real $\frac{b_2}{a_1^2} \geq -0.5$.

Some points on the adequacy boundaries are given by the cancellation conditions (see eqn. A23) i.e.,

$$\frac{a_2}{a_1^2} - \frac{b_2}{a_1^2} - \frac{b_1}{a_1}\left(1 - \frac{b_1}{a_1}\right) = 0 \quad .....(A51)$$

$$\frac{a_3}{a_1^3} - \frac{b_2}{a_1^2}\left(1 - \frac{b_1}{a_1}\right) = 0 \quad .....(A52)$$

giving as the equation of lines of constant $\frac{b_1}{a_1}$ in the

$(\frac{a_3}{a_1^3})$, $(\frac{a_2}{a_1^2})$ plane,

$$\frac{a_3}{a_1^3} = (1 - \frac{b_1}{a_1}) \left[\frac{a_2}{a_1^2} - \frac{b_1}{a_1} (1 - \frac{b_1}{a_1}) \right] \qquad \text{.....(A53)}$$

and

$$\frac{a_2}{a_1^2} = \frac{b_2}{a_1^2} + \frac{(\frac{a_3}{a_1^3})}{(\frac{b_2}{a_1^2})} - \left[\frac{(\frac{a_3}{a_1^3})}{(\frac{b_2}{a_1^2})} \right]^2 \qquad \text{.....(A54)}$$

as the equation of curves of constant $\frac{b_2}{a_1^2}$ in the same place.

The intersection of these curves for given values of $\frac{b_1}{a_1}$ and $\frac{b_2}{a_1^2}$ gives one point on the adequacy boundary for these parameters (see Fig.3.6). Thus cancellation conditions only give boundary values for positive values of $\frac{b_2}{b_1^2}$ and for values of $\frac{b_1}{a_1}$ between 0 and 1. This is to be expected as the system is open-loop stable and has no positive poles.

Other points on these boundaries and boundaries for other values of $\frac{b_1}{a_1}$ and $\frac{b_2}{a_1^2}$ can only be found using a computer search. Such a search has been carried out in the following manner.

For each run $\frac{\tau}{a_1}$, $\frac{b_1}{a_1}$ and $\frac{b_2}{a_1^2}$ were pre-determined thus giving maximum and minimum values for $\frac{a_2}{a_1^2}$.

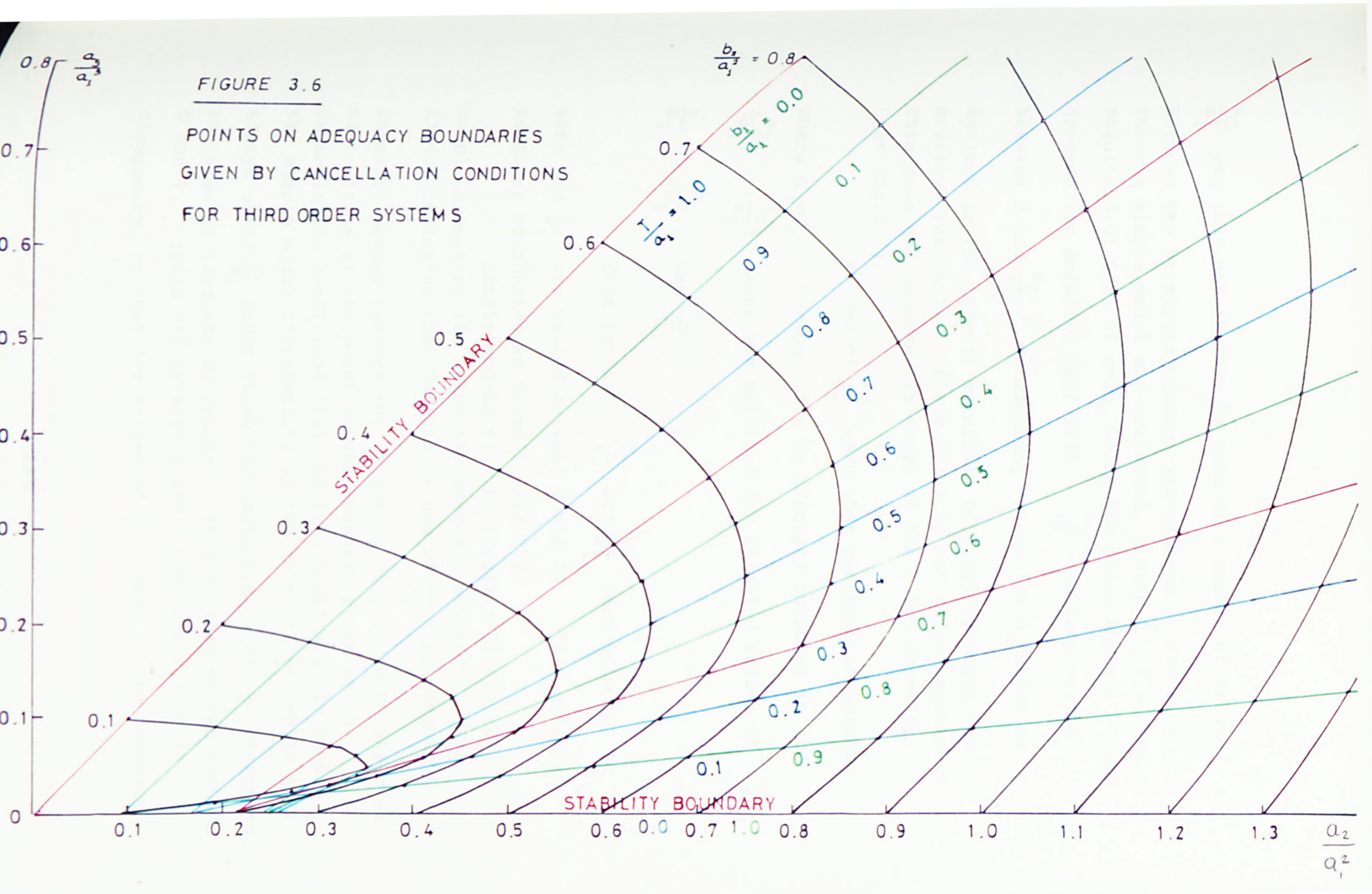


FIGURE 3.6
POINTS ON ADEQUACY BOUNDARIES
GIVEN BY CANCELLATION CONDITIONS
FOR THIRD ORDER SYSTEMS
$\frac{a_3}{a_1^3}$
$\frac{b_2}{a_1^2} = 0.8$
$\frac{b_2}{a_1^2} = 0.0$
$\frac{T}{a_1} = 1.0$
STABILITY BOUNDARY
STABILITY BOUNDARY
0.0
1.0
$\frac{a_2}{a_1^2}$
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
1.1
1.2
1.3

$\frac{a_2}{a_1^2}$ was then set in turn to each of a number of values between the permitted minimum and maximum. For each of these a simple model was evaluated, giving $\hat{G}_c$, i.e. the magnitude of $\hat{G}(s)$ at $s=j\hat{\omega}_c$ being the point where it first crosses the negative real axis. $\frac{a_3}{a_1^3}$ was then varied between 0 and $\frac{a_2}{a_1^2}$, determining values for G_c which is defined as the largest magnitude of $G(j\omega)$ crossing the negative real axis. Since $G(j\omega)$ is not always monotonic this does not necessarily occur at the first crossing. (see Fig.3.7).

Values of $\frac{a_3}{a_1^3}$ were then recorded at points where $\hat{G}_c=G_c$. Consequently an adequacy boundary in the $(\frac{a_2}{a_1^2})$, $(\frac{a_3}{a_1^3})$ plane was obtained for selected values of $\frac{\tau}{a_1}$, $\frac{b_1}{a_1}$, and $\frac{b_2}{a_1^2}$.

Thus initially $\frac{b_1}{a_1}$ and $\frac{b_2}{a_1^2}$ were both set to zero and $\frac{\tau}{a_1}$ was varied between 0 and 10. This produced adequacy boundaries as shown in Fig.3.8.

Simple models will be adequate if the full model parameters lie below the adequacy boundary. The figure indicates that for all values of $\frac{\tau}{a_1}$ the adequacy boundary passes through the origin. This is to be expected since it is at the point on the boundary given by the cancellation conditions that the full model exactly equals the simple model irrespective of the value of $\frac{\tau}{a_1}$ which will always equal $\frac{\hat{\tau}}{a_1}$ under these circumstances. As $\frac{\tau}{a_1}$ increases, the area of adequacy decreases. This is very marked between 0 and 1, falls off between 1 and 2 and then virtually disappears, so that for values of $\frac{\tau}{a_1}$ above 2, the adequacy

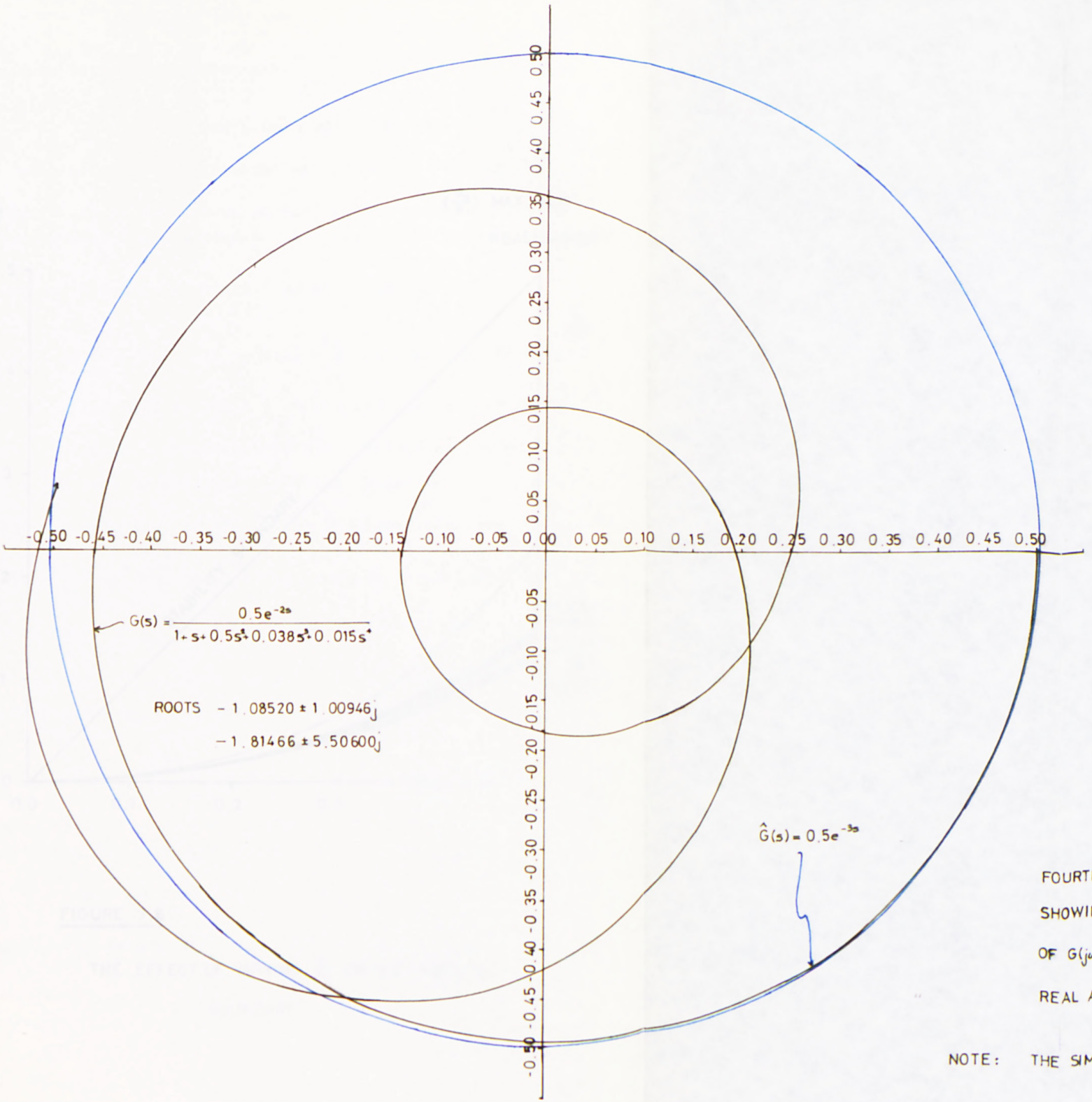


FIGURE 3.7

FOURTH ORDER PLUS TIME DELAY SYSTEM SHOWING THAT THE LARGEST MAGNITUDE OF $G(j\omega)$ WHEN IT CROSSES THE NEGATIVE REAL AXIS OCCURS AT THE THIRD CROSSING.

NOTE: THE SIMPLE MODEL OBTAINED IS INADEQUATE

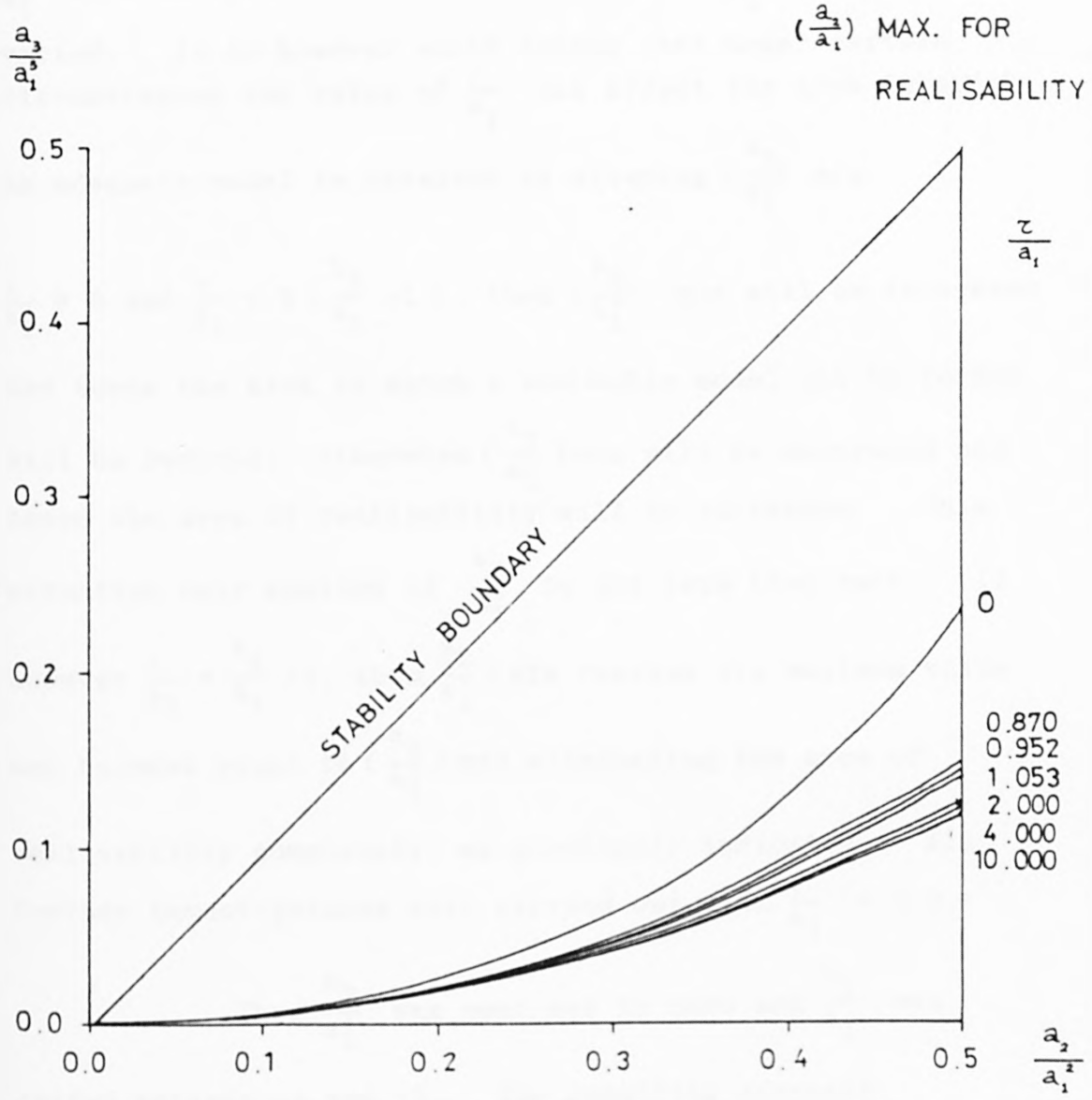


FIGURE 3.8

THE EFFECT OF VARYING $\frac{\tau}{a_1}$ ON THE ADEQUACY BOUNDARY

boundary stays almost constant.

Similar conclusions were reached when $\frac{b_1}{a_1}$ and $\frac{b_2}{a_1^2}$ were next set to different values and $\frac{\tau}{a_1}$ was again varied. It is however worth noting that under certain circumstances the value of $\frac{\tau}{a_1}$ can affect the area in which an adequate model is obtained by altering $(\frac{a_2}{a_1^2})$ min. If $\frac{\tau}{a_1} \neq 0$ and $\frac{\tau}{a_1} < 2\,(\frac{b_1}{a_1} - 1)$, then $(\frac{a_2}{a_1^2})$ min will be increased and hence the area in which a realizable model can be formed will be reduced; otherwise $(\frac{a_2}{a_1^2})$ min will be decreased and hence the area of realizability will be increased. This situation only applies if $\frac{a_2}{a_1^2}$ is not less than zero. If however $\frac{\tau}{a_1} = \frac{b_1}{a_1} - 1$, then $(\frac{a_2}{a_1^2})$ min reaches its maximum value and becomes equal to $(\frac{a_2}{a_1^2})$ max eliminating the area of realizability completely, as previously indicated. All further investigations were carried out with $\frac{\tau}{a_1} = 2.0$.

Thus $\frac{b_2}{a_1^2}$ was next set to zero and $\frac{b_1}{a_1}$ was varied between -1 and +1. The resulting adequacy boundaries for positive values are shown in Fig.3.9. As $\frac{b_1}{a_1}$ increases from zero, so the area of adequacy decreases rapidly, since not only does $(\frac{a_2}{a_1^2})$ max decrease, but simultaneously the minimum value of $\frac{a_2}{a_1^2}$ for adequacy

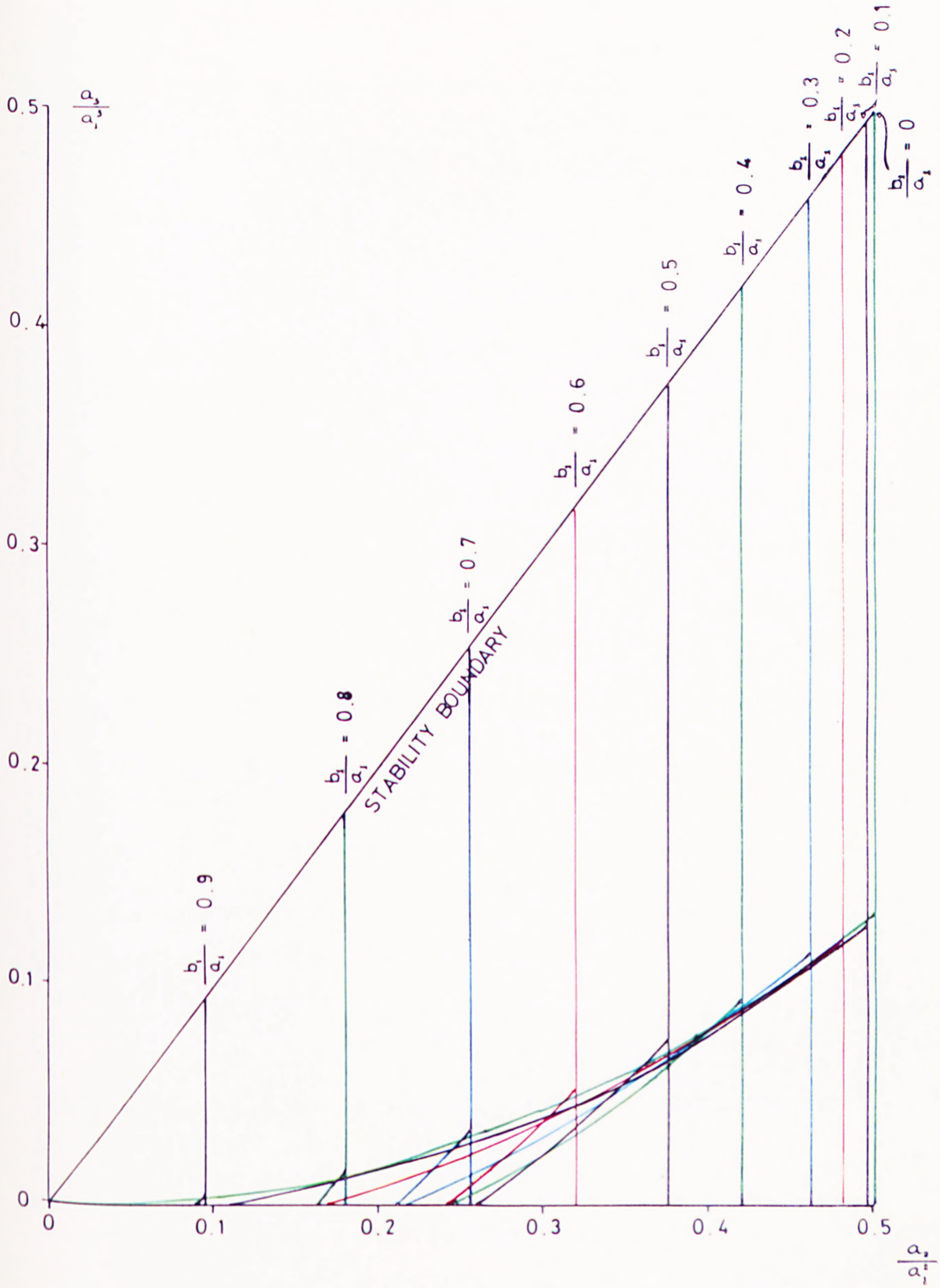


FIGURE 3.9

ADEQUACY BOUNDARIES FOR THIRD ORDER SYSTEMS WITH $\frac{\tau}{a_1} = 2.0$ AND $\frac{b_2}{a_1^2} = 0$ AND POSITIVE VALUES BETWEEN 0 AND 1

increases, this latter value being given by the cancellation conditions. In fact, the adequacy boundary in the $(\frac{a_2}{a_1^2})$, $(\frac{b_1}{a_1})$ plane at $\frac{a_3}{a_1^3} = 0$ is, as expected, that of a second order system as shown in Fig.3.4. With the exception of the fact that for -ve values of $\frac{b_1}{a_1}$ the area of adequacy is slightly larger (as for second order systems) the adequacy boundaries are very similar, meeting for each value of $\pm \frac{b_1}{a_1}$ at $\frac{a_2}{a_1^2}$ max and the maximum value of $\frac{a_3}{a_1^3}$ for adequacy.

$\frac{b_2}{a_1^2}$ was subsequently set to 0.5 and $\frac{b_1}{a_1}$ was again varied between -1 and ± 1, giving the adequacy boundaries as shown in Fig.3.10. Again the greatest adequacy area occurs for $\frac{b_1}{a_1} = 0$ and decreases as $\frac{b_1}{a_1}$ is increased or decreased, until $\frac{b_1}{a_1} = \pm 1$ when the system only gives an adequate model for $\frac{a_2}{a_1^2} = 0.5$ and $\frac{a_3}{a_1^3} = 0$. This is the case for all values of $\frac{b_2}{a_1^2}$; the area of adequacy is the $(\frac{a_2}{a_1^2})$, $(\frac{a_3}{a_1^3})$ plane always reduces to a single point at

$$\frac{a_2}{a_1^2} = \frac{b_2}{b_2^2} \quad , \quad \frac{a_3}{a_1^3} = 0 \text{ at } \frac{b_1}{a_1} = \pm 1 \qquad \text{.....(A55)}$$

Points given by cancellation conditions have also been shown in Fig.3.10.

To examine the effect of varying $\frac{b_1}{a_1}$ when the area of realizability was kept constant, $\frac{a_2}{a_1^2}$ max was chosen as 1.0 and $\frac{b_1}{a_1}$ was varied between -1 and +1, $\frac{b_2}{a_1^2}$ being varied appropriately. The resulting adequacy boundaries are shown

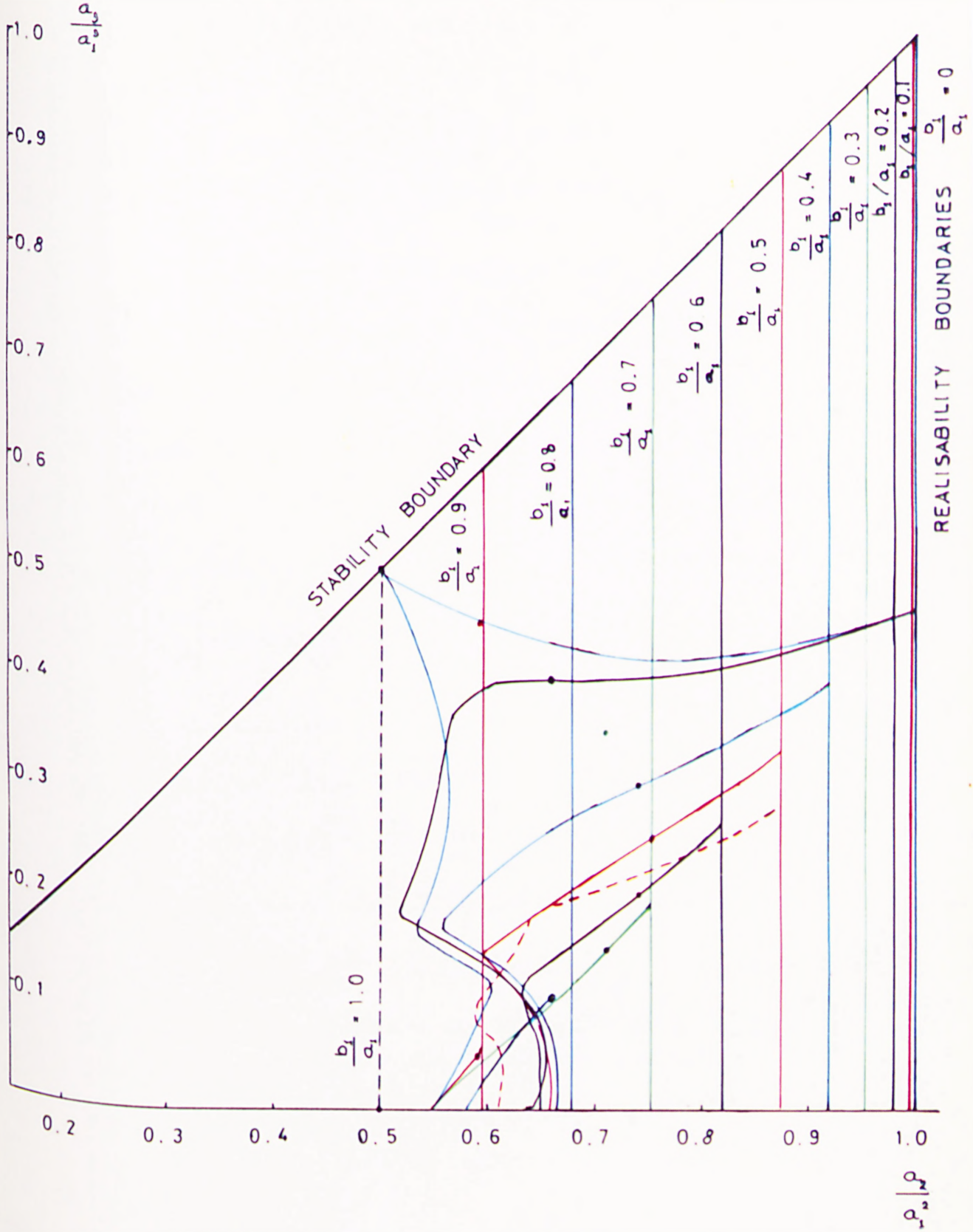


FIGURE 3.10

ADEQUACY BOUNDARIES FOR THIRD ORDER SYSTEMS WITH $\frac{\tau}{a_1} = 2.0$ AND $\frac{b_2}{a_1^2} = 0.5$ AND $\frac{b_1}{a_1} = -0.5, 0, 0.2, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0$.

in Fig.3.11. Values of $\frac{b_2}{a_1^2}$ for -ve values of $\frac{b_1}{a_1}$ are the same as for corresponding positive values.

From Fig.3.11 it can be seen that varying $\frac{b_1}{a_1}$ and $\frac{b_2}{a_1^2}$ in this way gives a very regular family of curves which could be interpolated between with reasonable accuracy. It will also be noticed that systems with negative values of $\frac{b_1}{a_1}$ have a greater tendency to retain the shape of the adequacy boundary at $\frac{b_1}{a_1} = 0$ than those with positive values, whose boundary tends to retreat towards $(\frac{a_2}{a_1^2})$ max at a greater rate. Finally $\frac{b_1}{a_1}$ was kept constant while $\frac{b_2}{a_1^2}$ was varied. Initially $\frac{b_1}{a_1}$ was set to 0 and $\frac{b_2}{a_1^2}$ was varied between -0.2 and 3.5 (see Figs. 3.12 and 3.13). The adequacy boundary seemed to retain its basic shape but as $\frac{b_2}{a_1^2}$ was increased the adequacy area became elongated in the $\frac{a_3}{a_1^3}$ direction. The adequacy area increased slowly as $\frac{b_2}{a_1^2}$ was increased but the realizability area increased much faster and hence the proportion of realizable models which were adequate fell rapidly. The process was repeated with other values of $\frac{b_1}{a_1}$ giving similar results (see Figs. 3.14 and 3.15).

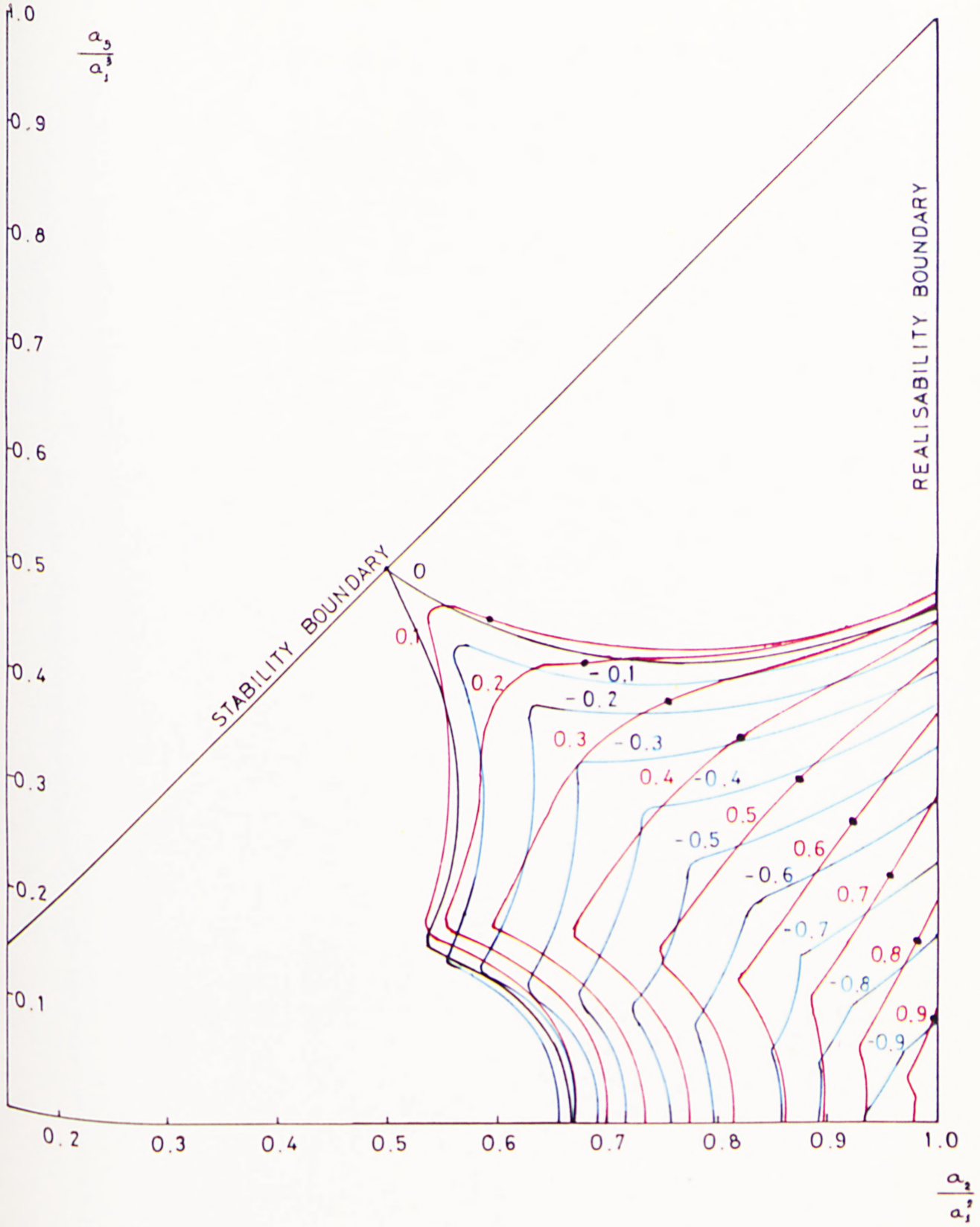


FIGURE 3.11

ADEQUACY BOUNDARIES FOR THIRD ORDER SYSTEMS WITH VARIOUS NUMERATOR PARAMETERS WITHIN A CONSTANT REALISABILITY AREA

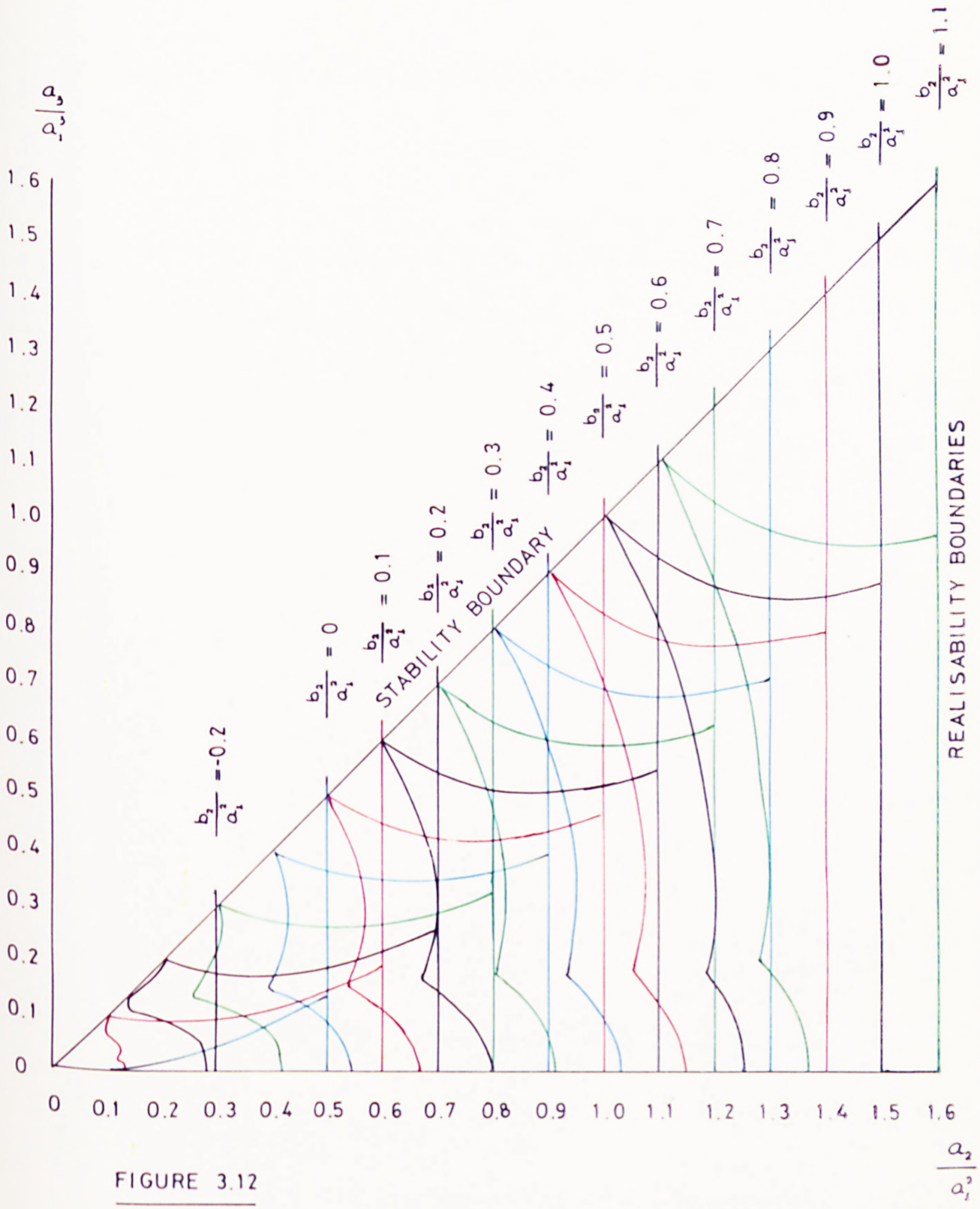


FIGURE 3.12

ADEQUACY BOUNDARIES FOR THIRD ORDER SYSTEMS WITH $\frac{\tau}{a_1} = 2.0$, $\frac{b_1}{a_1} = 0$ AND $\frac{b_2}{a_1^2}$ VARIED BETWEEN -0.2 AND 1.1

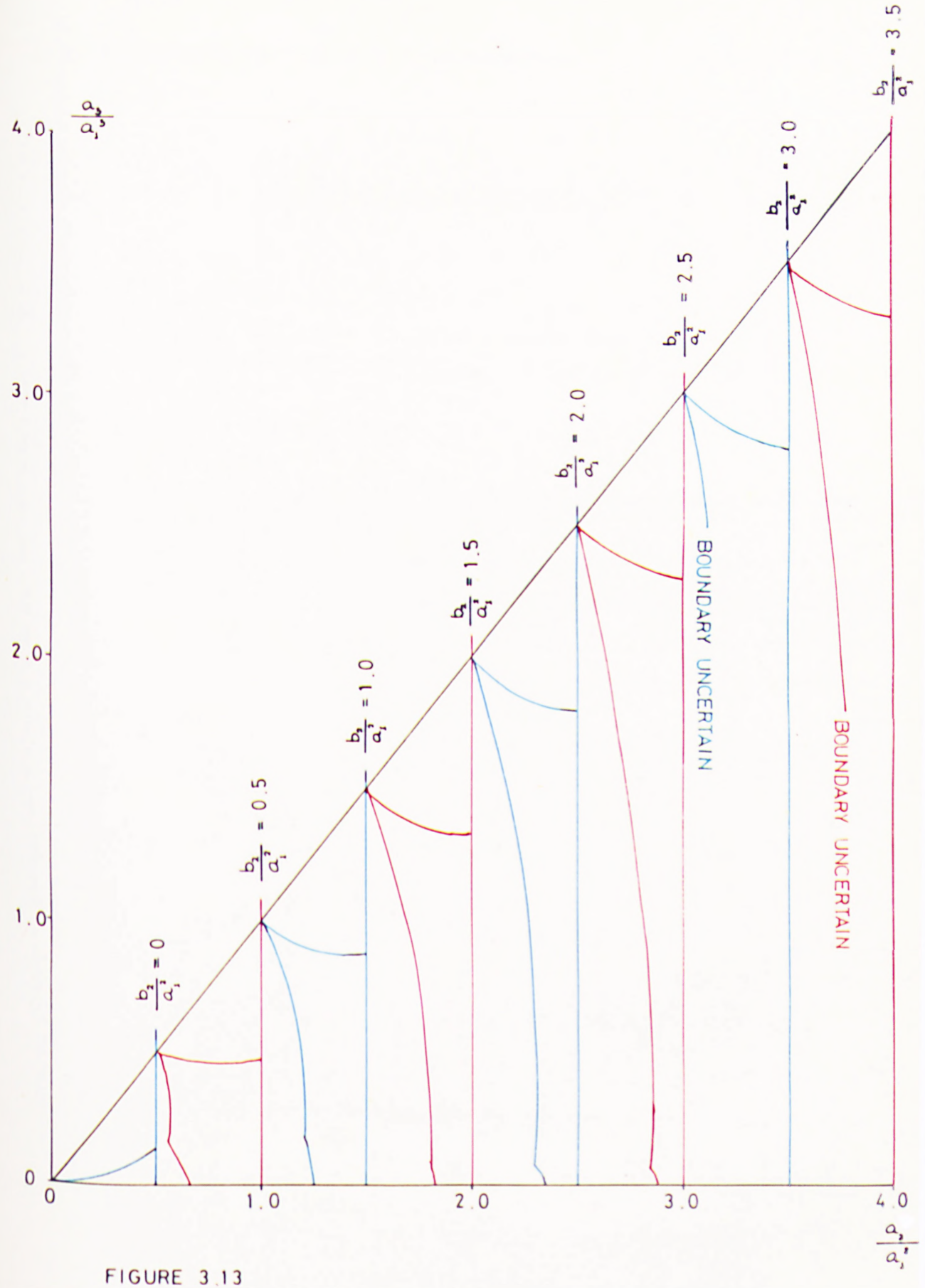


FIGURE 3.13

ADEQUACY BOUNDARIES FOR THIRD ORDER SYSTEMS WITH $\frac{\tau}{a_1} = 2.0$, $\frac{b_1}{a_1} = 0$ AND $\frac{b_2}{a_1^2} = 0, 0.5, 1.0, 1.5, 2.0, 2.5, 3.0$ AND 3.5

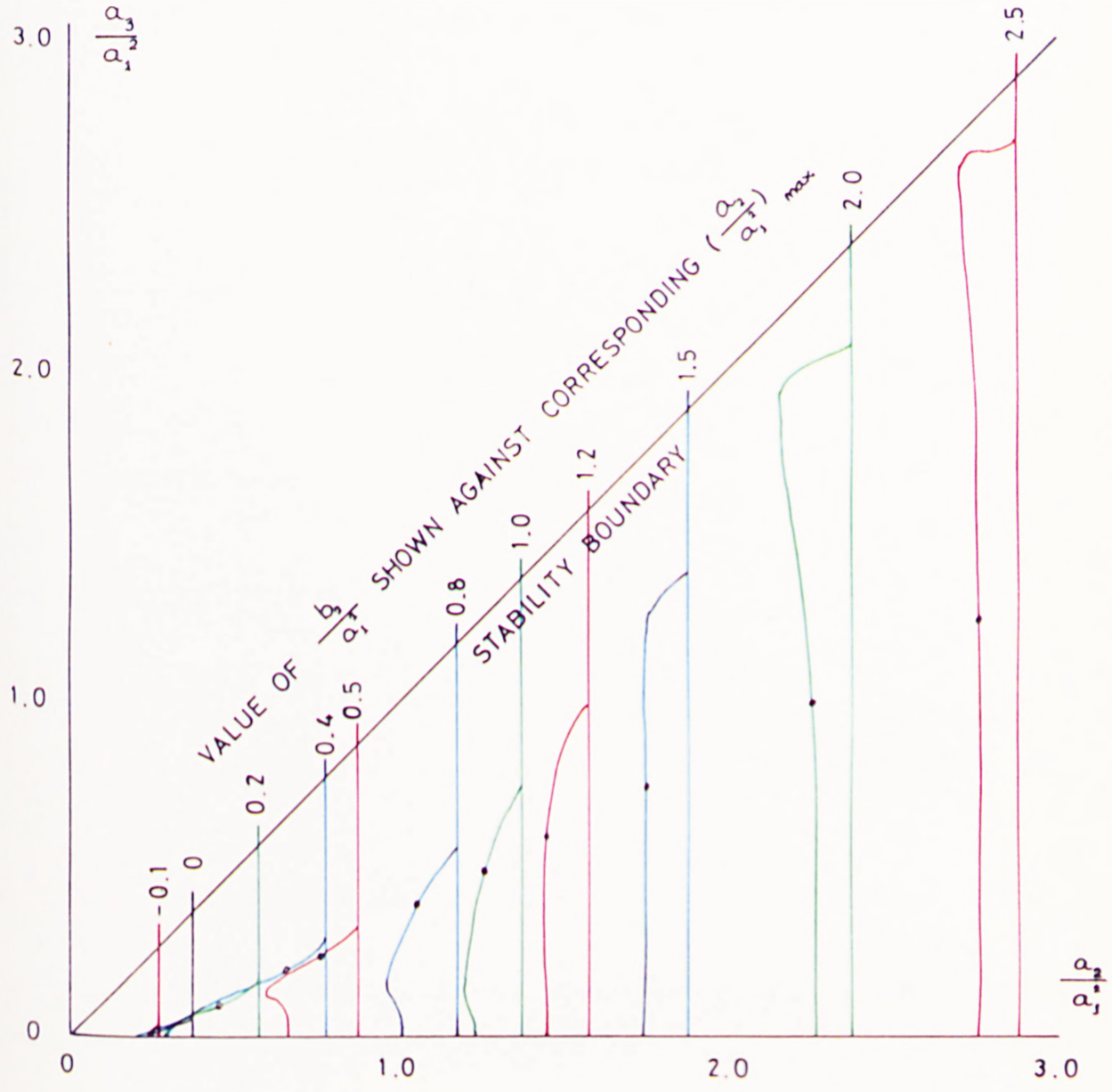


FIGURE 3.14

ADEQUACY BOUNDARIES FOR THIRD ORDER SYSTEMS WITH $\frac{\tau}{a_1} = 2.0$, $\frac{b_1}{a_1} = 0.5$ AND $\frac{b_2}{a_1^2} = -0.1, 0, 0.2, 0.4, 0.5, 0.8, 1.0, 1.2, 1.5, 2.0, 2.5$

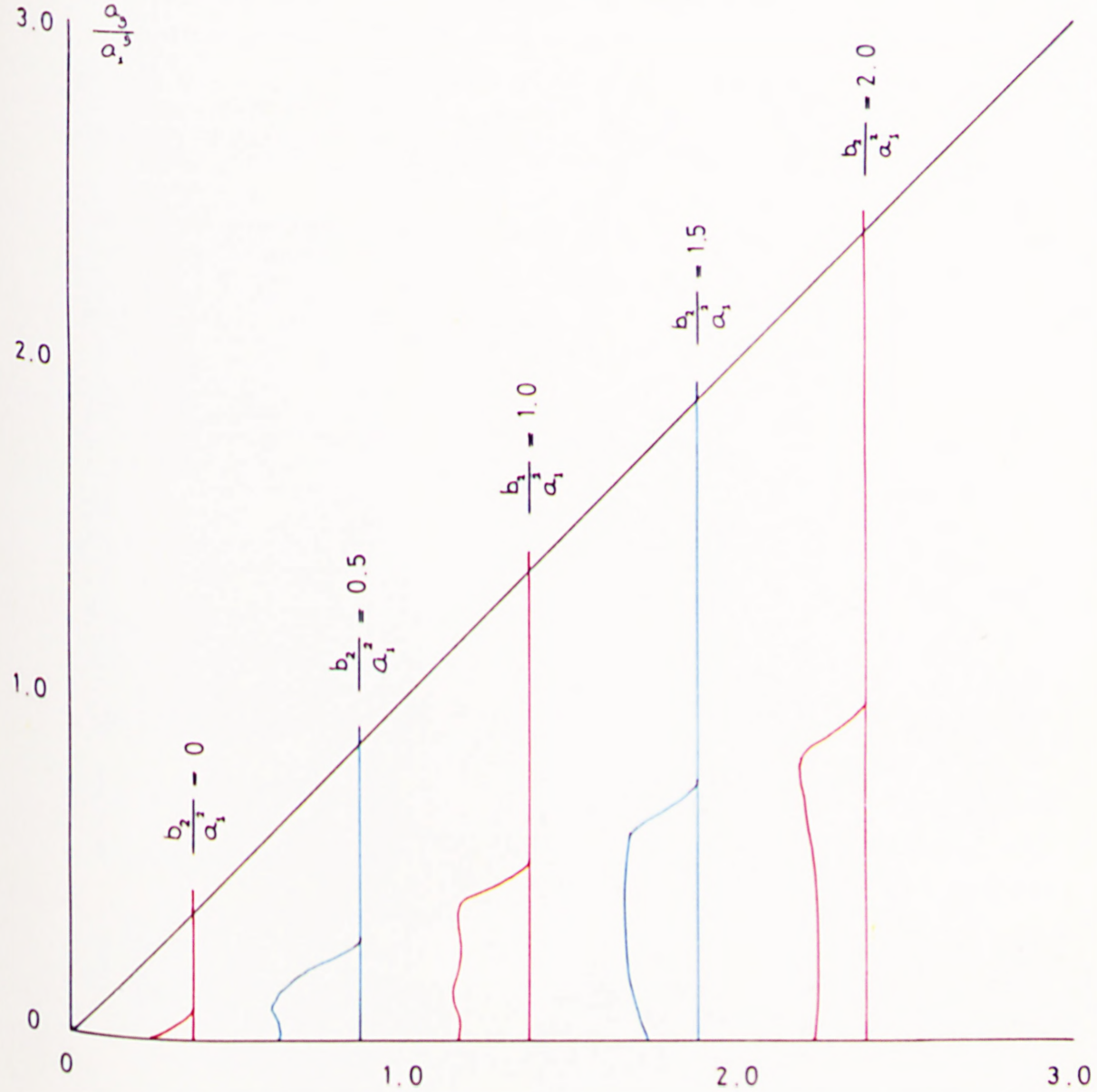


FIGURE 3.15

ADEQUACY BOUNDARIES FOR THIRD ORDER SYSTEMS WITH

$\frac{\tau}{a_1} = 2.0$, $\frac{b_1}{a_1} = -0.5$ AND $\frac{b_2}{a_1^2} = 0, 0.5, 1.0, 1.5, 2.0$

The boundary between real and imaginary zeroes is given by

$$\frac{b_2}{a_1^2} = \frac{1}{4}\left(\frac{b_1}{a_1}\right)^2 \qquad \text{.....(A56)}$$

However this does not appear to be relevant as far as adequacy boundaries are concerned. The same can be said of the boundary between real and imaginary poles. Fig.3.16 shows lines of constant $\frac{b_1}{a_1}$ and $\frac{b_2}{a_1^2}$ on an argand diagram.

It does not however indicate any further correlation between system zeroes and adequacy boundaries.

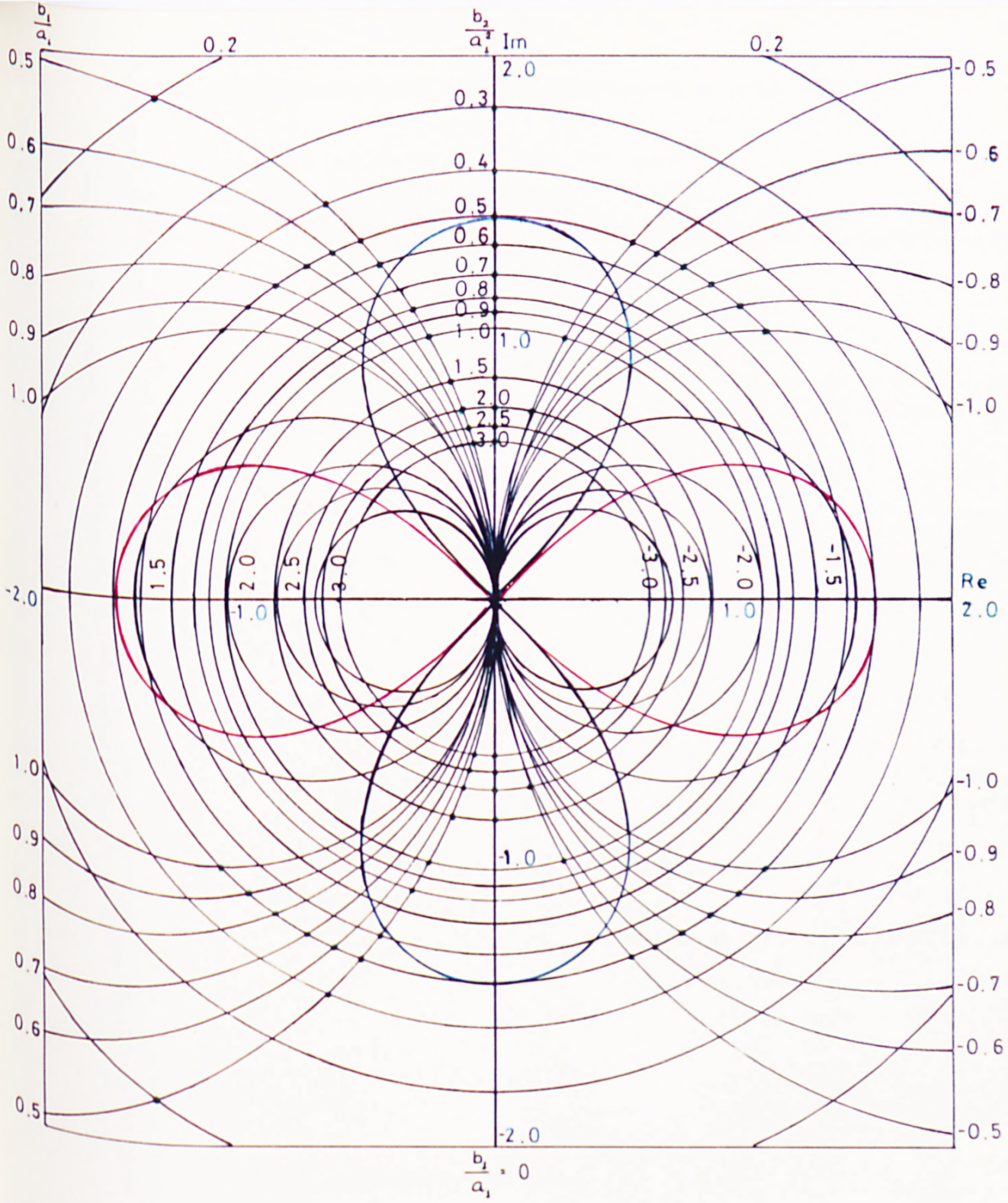


FIGURE 3.16

SYSTEM ZEROES AND LINES OF CONSTANT $\frac{b_1}{a_1}$ AND $\frac{b_2}{a_1^2}$ AND REALISABILITY BOUNDARY FOR THIRD ORDER SYSTEMS

— $\left(\frac{a_2}{a_1^2}\right)_{max} = 0$ SYSTEMS WITHIN THIS BOUNDARY CANNOT PRODUCE REALISABLE SIMPLE MODELS

— $\left(\frac{a_2}{a_1^2}\right)_{max} = 1$ ADEQUACY BOUNDARIES FOUND ALONG THIS LOCUS

• ZEROES OF SOME OF THE SYSTEMS ANALYSED

3.10 ADEQUACY BOUNDARIES FOR FOURTH ORDER SYSTEMS WITH TIME DELAY

As with third order systems the first factor to be considered is the stability boundary.

Writing the characteristic equation as

$$1 + a_1 s + a_2 s^2 + a_3 s^3 + a_4 s^4 \qquad \ldots\ldots(A57)$$

the Routh array gives the following stability conditions:-

$$a_4 \geq 0 \qquad \ldots\ldots(A58)$$

$$a_3 \geq 0 \qquad \ldots\ldots(A59)$$

$$a_2 - \frac{a_4}{a_3} a_1 \geq 0 \quad \text{or} \quad a_2 a_3 \geq a_4 a_1 \qquad \ldots\ldots(A60)$$

$$a_1 - \frac{a_3}{a_2 - \frac{a_4}{a_3} a_1} \geq 0 \quad \text{or} \quad a_1 a_2 a_3 \geq a_4 a_1^2 + a_3^2 \qquad \ldots\ldots(A61)$$

If these conditions are normalised with respect to a_1 we obtain:-

$$\frac{a_4}{a_1^4} \geq 0 \qquad \ldots\ldots(A62)$$

$$\frac{a_3}{a_1^3} \geq 0 \qquad \ldots\ldots(A63)$$

$$\frac{a_2}{a_1^2} \cdot \frac{a_3}{a_1^3} \geq \frac{a_4}{a_1^4} \qquad \ldots\ldots(A64)$$

and

$$\frac{a_2}{a_1^2} \cdot \frac{a_3}{a_1^3} \geq \frac{a_4}{a_1^4} + \left(\frac{a_3}{a_1^3}\right)^2 \qquad \ldots\ldots(A65)$$

As $\frac{a_4}{a_1^4}$ and $\frac{a_3}{a_1^3}$ must always be positive for the system to be

stable the third condition will always be satisfied if the final condition is satisfied. This final condition can be rewritten as

$$\frac{a_2}{a_1^2} \geq \frac{a_4}{a_1^4} \Big/ \frac{a_3}{a_1^3} + \frac{a_3}{a_1^3} \qquad \text{.....(A66)}$$

or to give an explicit boundary condition for $\frac{a_4}{a_1^4}$

as

$$\frac{a_4}{a_1^4} \leq \left(\frac{a_2}{a_1^2} - \frac{a_3}{a_1^3}\right) \frac{a_3}{a_1^3} \qquad \text{.....(A67)}$$

which since $\frac{a_4}{a_1^4} \geq 0$

shows that

$$\frac{a_3}{a_1^3} \leq \frac{a_2}{a_1^2} \qquad \text{.....(A68)}$$

This condition can also be rewritten explicitly for $\frac{a_3}{a_1^3}$ giving:-

$$\frac{1}{2}\left(\frac{a_2}{a_1^2}\right) - \sqrt{\left(\frac{1}{2} \cdot \frac{a_2}{a_1^2}\right)^2 - \frac{a_4}{a_1^4}} \leq \frac{a_3}{a_1^3} \leq \frac{1}{2}\left(\frac{a_2}{a_1^2}\right) + \sqrt{\left(\frac{1}{2} \cdot \frac{a_2}{a_1^2}\right)^2 - \frac{a_4}{a_1^4}} \qquad \text{.(A69)}$$

which requires

$$\left(\frac{1}{2} \cdot \frac{a_2}{a_1^2}\right)^2 \geq \frac{a_4}{a_1^4} \qquad \text{.....(A70)}$$

or

$$\frac{a_2}{a_1^2} \geq 2\sqrt{\frac{a_4}{a_1^4}} \qquad \text{.....(A71)}$$

These conditions give a stability boundary which is shown in Fig.3.17 in the $\frac{a_3}{a_1^3}$, $\frac{a_2}{a_1^2}$ plane for various values

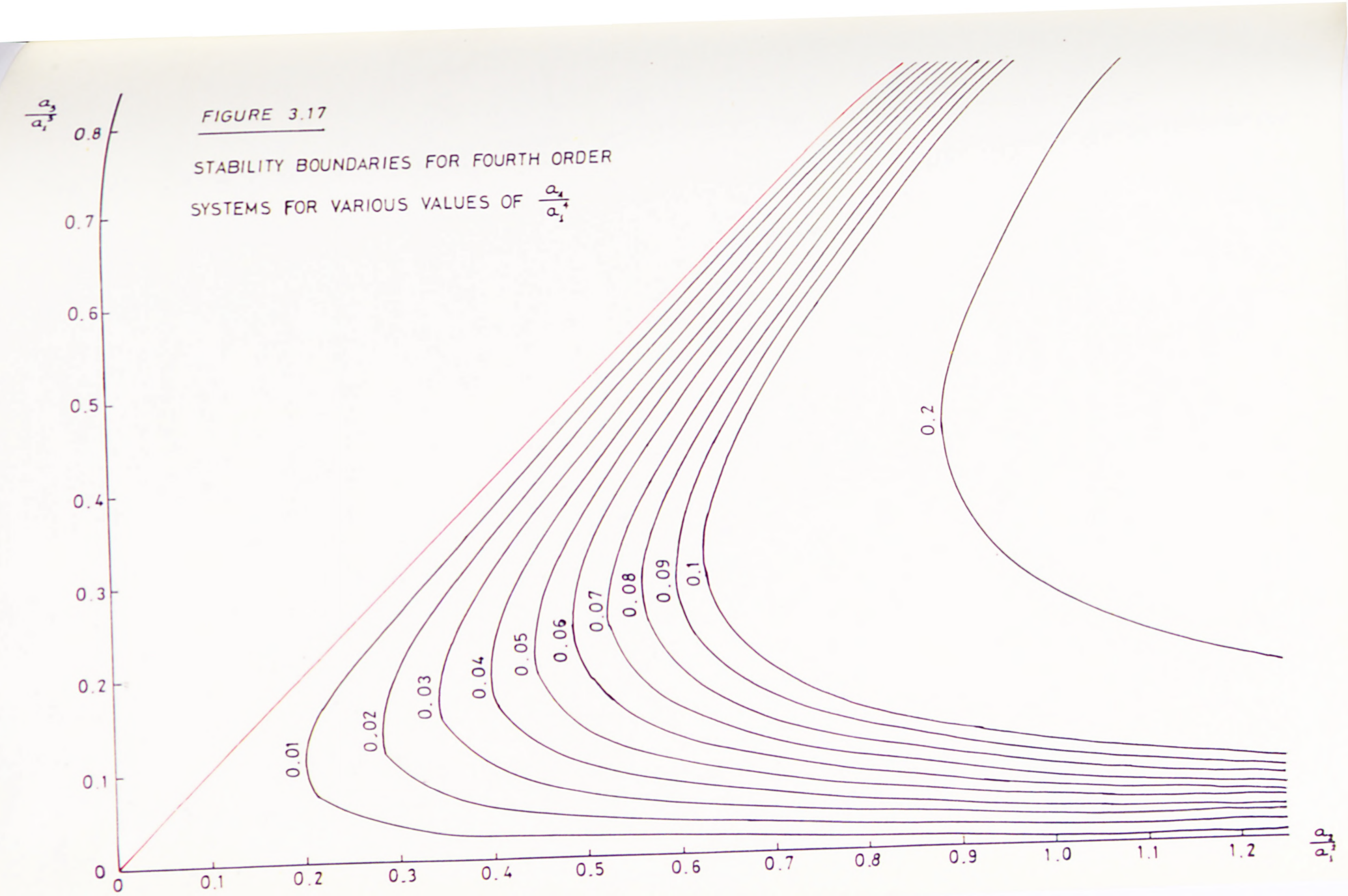


FIGURE 3.17
STABILITY BOUNDARIES FOR FOURTH ORDER SYSTEMS FOR VARIOUS VALUES OF $\frac{a_4}{a_1^4}$
$\frac{a_3}{a_1^3}$
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08
0.09
0.1
0.2
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
1.1
1.2
$\frac{a_2}{a_1^2}$

of $\frac{a_4}{a_1^4}$. The boundary in the $\frac{a_4}{a_1^4}$, $\frac{a_2}{a_1^2}$ plane is much simpler consisting of straight lines and has not been shown.

Other conditions, including those given by stability and realisability requirements of the simple model, are the same as those for third order models (see Section 3.9).

Again certain points on the adequacy boundaries are given by the cancellation conditions (see eqn. A23):-

$$\frac{a_2}{a_1^2} - \frac{b_2}{a_1^2} - \frac{b_1}{a_1}\left(1 - \frac{b_1}{a_1}\right) = 0 \qquad \text{.....(A72)}$$

$$\frac{a_3}{a_1^3} - \frac{b_3}{a_1^3} - \frac{b_2}{a_1^2}\left(1 - \frac{b_1}{a_1}\right) = 0 \qquad \text{.....(A73)}$$

$$\frac{a_4}{a_1^4} - \frac{b_3}{a_1^3}\left(1 - \frac{b_1}{a_1}\right) = 0 \qquad \text{.....(A74)}$$

Fig.3.18 shows points given by the cancellation conditions for values of $\frac{b_1}{a_1}$ and $\frac{b_2}{a_1^2}$ between 0 and 1, for $\frac{b_3}{a_1^3} = 0.1$. It is worth noting that there are no cancellation points for values of $\frac{a_3}{a_1^3}$ less than $\frac{b_3}{a_1^3}$ for stable systems. In this area cancellation points would apply to unstable systems resulting in unstable simple models. For values of $\frac{a_3}{a_1^3} > \frac{a_2}{a_1^2}$ cancellation points exist which would result in a stable model but although this would imply that the system is stable, the fact is that positive poles are exactly cancelled by zeroes and thus the stable system is of first order whereas fourth order systems of similar but not identical parameters are in fact unstable.

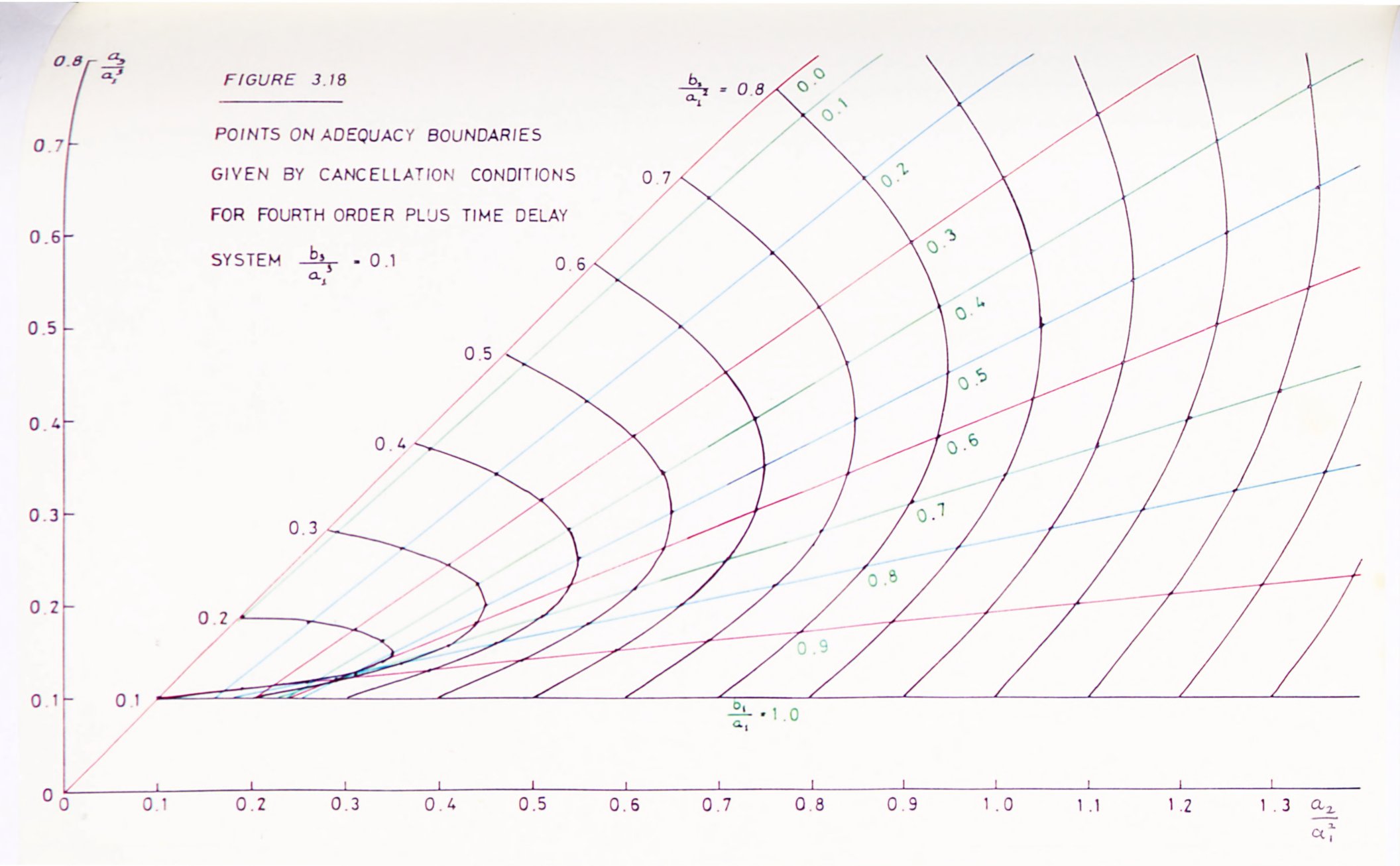


FIGURE 3.18
POINTS ON ADEQUACY BOUNDARIES
GIVEN BY CANCELLATION CONDITIONS
FOR FOURTH ORDER PLUS TIME DELAY
SYSTEM $\frac{b_3}{a_1^3} = 0.1$
$\frac{a_3}{a_1^3}$
$\frac{a_2}{a_1^2}$
$\frac{b_2}{a_1^2} = 0.8$
$\frac{b_1}{a_1} = 1.0$
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9

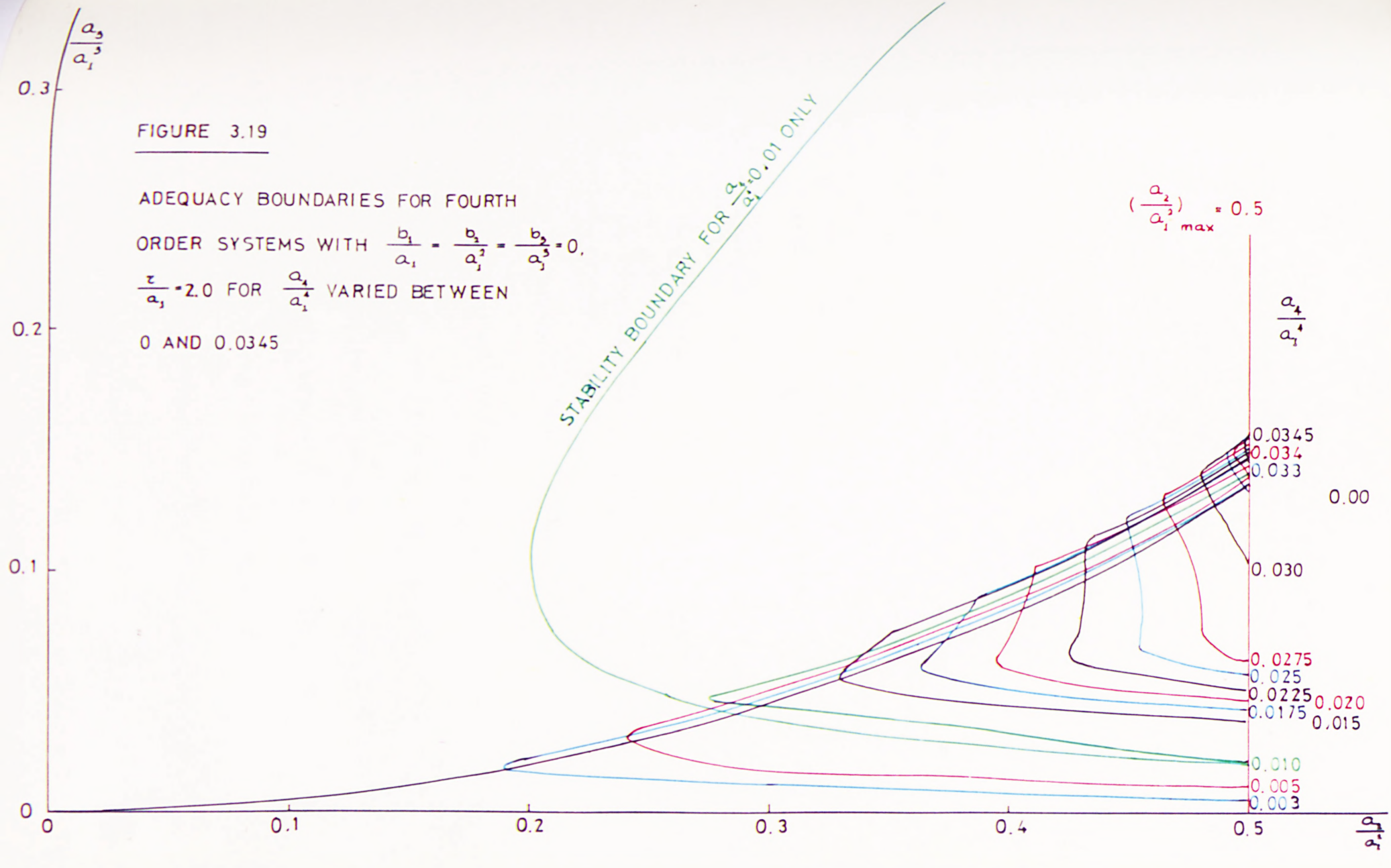
To obtain the full adequacy boundaries, computer searches similar to those used for third order systems were carried out, however, in this case two sets of searches were required:- one in the $\frac{a_3}{a_1^3}$, $\frac{a_2}{a_1^2}$ plane (as in the case of third order systems) and one in the $\frac{a_4}{a_1^4}$, $\frac{b_2}{a_1^2}$ plane. For each run $\frac{\tau}{a_1}$, $\frac{b_1}{a_1}$, $\frac{b_2}{a_1^2}$, $\frac{b_3}{a_1^3}$ were predetermined, as was either $\frac{a_3}{a_1^3}$ or $\frac{a_4}{a_1^4}$ depending on the plane in which the search was to be carried out. In the case of the search in the $\frac{a_3}{a_1^3}$, $\frac{a_2}{a_1^2}$ plane a maximum and minimum value of $\frac{a_3}{a_1^3}$ was calculated for each value of $\frac{a_2}{a_1^2}$, according to the full model stability criteria (see eqn.A69). For the search in the $\frac{a_4}{a_1^4}$, $\frac{a_2}{a_1^2}$ plane only a maximum value was calculated for $\frac{a_4}{a_1^4}$ and the minimum value was taken as 0. (see eqn. A.67).

The first fourth order searches were carried out in the $\frac{a_3}{a_1^3}$, $\frac{a_2}{a_1^2}$ plane with $\frac{b_1}{a_1}$, $\frac{b_2}{a_1^2}$ and $\frac{b_3}{a_1^3}$ all set to zero, for values of $\frac{a_4}{a_1^4}$ between 0 and 0.0345. (see Fig.3.19).

As can be seen the area of adequacy decreases as $\frac{a_4}{a_1^4}$ increases until it disappears altogether at $\frac{a_4}{a_1^4} = 0.035$. One of the main reasons for this is the stability boundary, which cuts off areas which were adequate for lower values of $\frac{a_4}{a_1^4}$.

FIGURE 3.19

ADEQUACY BOUNDARIES FOR FOURTH ORDER SYSTEMS WITH $\frac{b_1}{a_1} = \frac{b_3}{a_1^2} = \frac{b_2}{a_1^3} = 0$, $\frac{\tau}{a_1} = 2.0$ FOR $\frac{a_4}{a_1^4}$ VARIED BETWEEN 0 AND 0.0345



A similar search was next carried out in the $\frac{a_4}{a_1^4}$, $\frac{a_2}{a_1^2}$ plane (see Fig.3.20). Here again as $\frac{a_3}{a_1^3}$ was increased so the area of adequacy decreased (as would be expected from the previous results). It can be seen that in certain cases the adequacy boundary coincides with the stability boundary. In terms of the area in which a realizable stable first order plus time delay model can be obtained from a stable fourth order model, the area of adequacy decreases from 100% at $\frac{a_3}{a_1^3} = 0$ through 92% at 0.01, 72% at 0.05, 25% at 0.1 to 0 at 0.156. This decrease is non linear and in certain cases there appear to be slight increases.

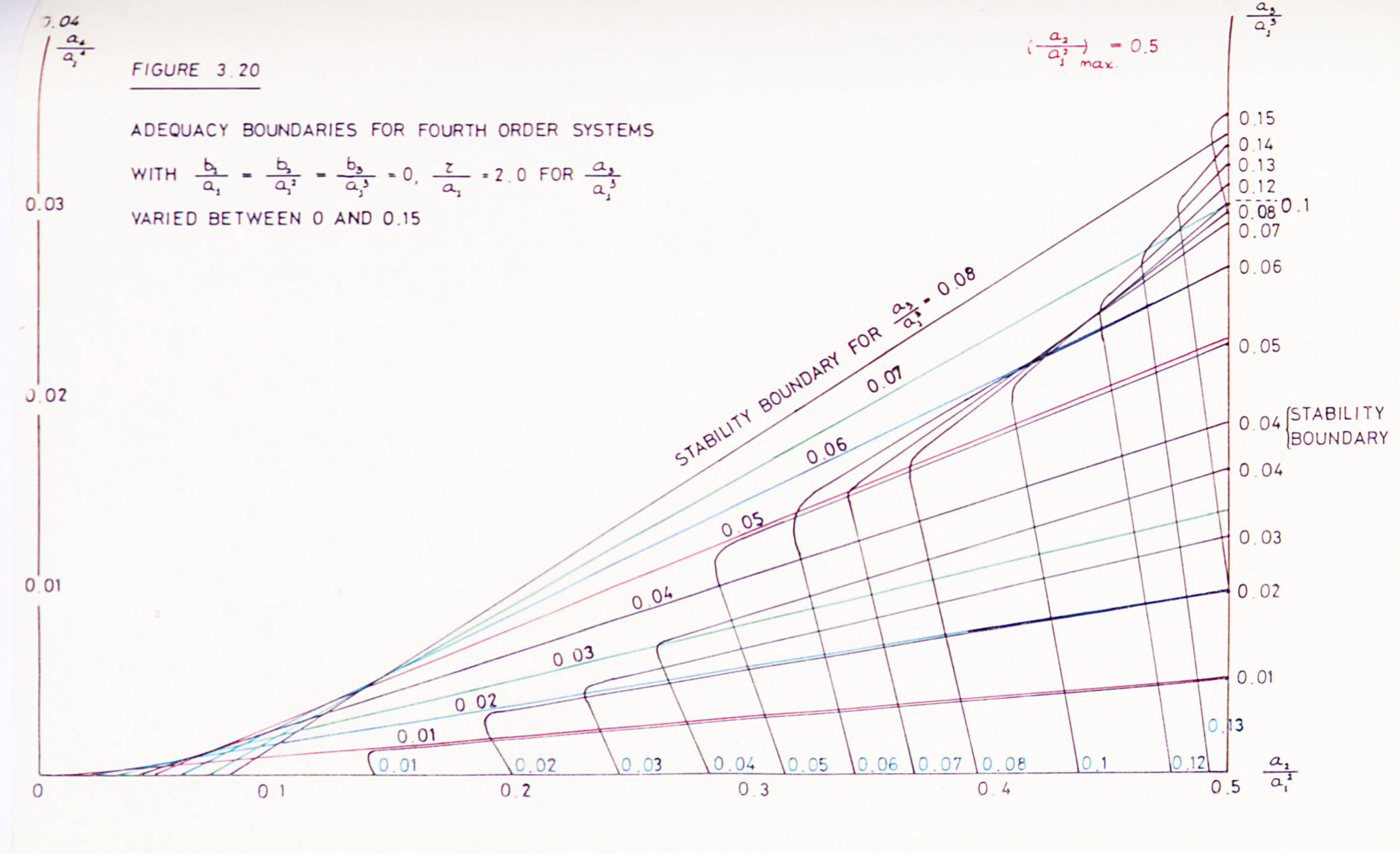
Returning to the $\frac{a_3}{a_1^3}$, $\frac{a_2}{a_1^2}$ plane, the effect of varying $\frac{b_1}{a_1}$ while keeping $\frac{a_4}{a_1^4}$ constant was next investigated for three different values of $\frac{a_4}{a_1^4}$ (0.01, 0.02, 0.025) (see Figs. 3.21, 3.22, 3.23). Again as $|\frac{b_1}{a_1}|$ is increased so the adequacy area decreases, largely due to the fact that the realizability boundary moves closer to zero. The adequacy area is also decreased as $\frac{a_4}{a_1^4}$ is increased.

The effect of varying $\frac{b_1}{a_1}$ on the adequacy boundary in the $\frac{a_4}{a_1^4}$, $\frac{a_2}{a_1^2}$ plane can be seen from Fig.3.24. As in Fig.3.20, the boundaries are made up of straight lines and for $\frac{b_1}{a_1} = 0$ the adequacy boundary coincides with the stability boundary until it reaches the minimum value of $\frac{a_2}{a_1^2}$ for adequacy. The adequacy area for $\frac{b_1}{a_1} = 0$ is relatively large (about 71% of the realizability area) and as $|\frac{b_1}{a_1}|$ is increased

FIGURE 3.20

ADEQUACY BOUNDARIES FOR FOURTH ORDER SYSTEMS

WITH $\frac{b_1}{a_1} = \frac{b_2}{a_1^2} = \frac{b_3}{a_1^3} = 0$, $\frac{\tau}{a_1} = 2.0$ FOR $\frac{a_3}{a_1^3}$ VARIED BETWEEN 0 AND 0.15



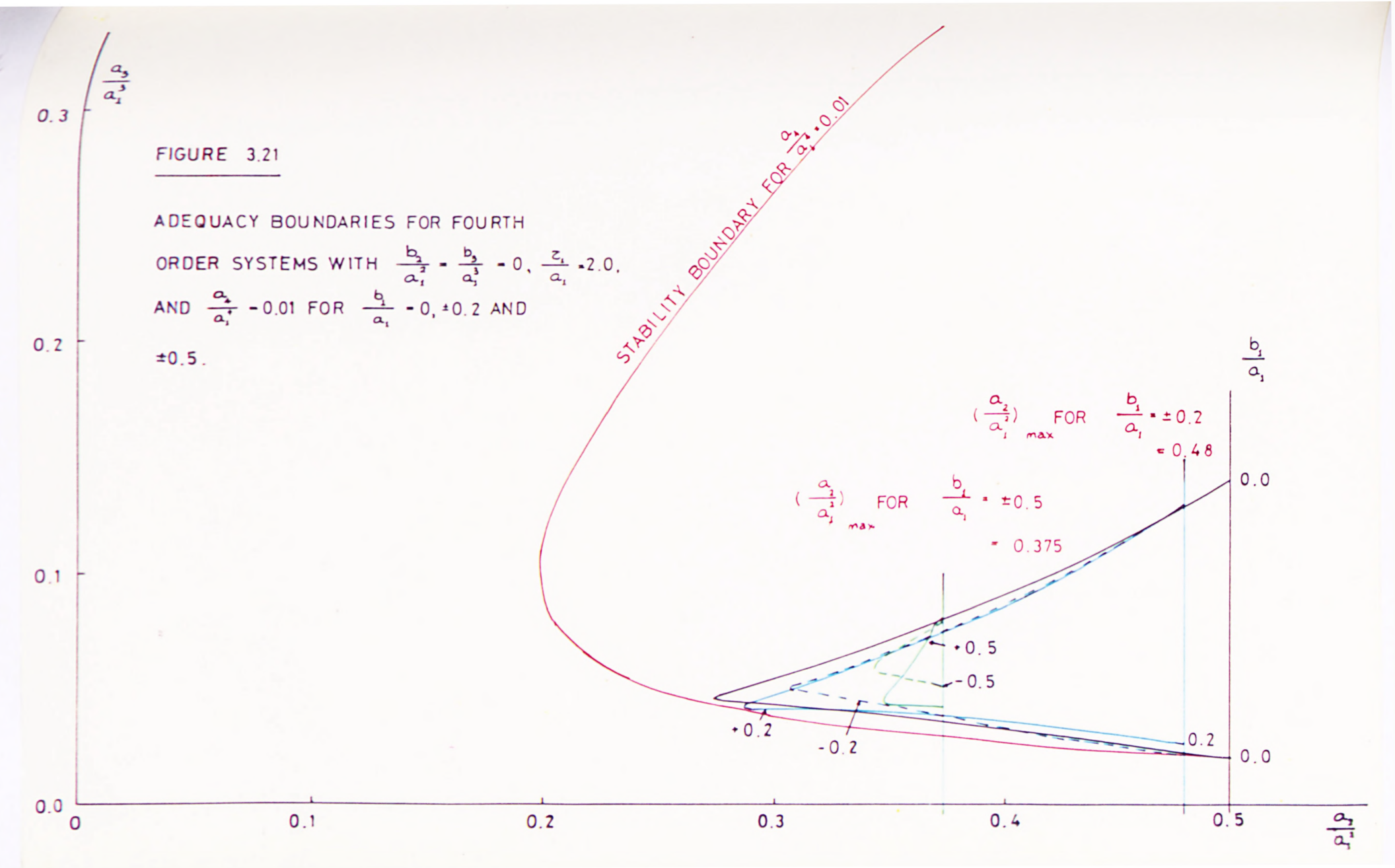


FIGURE 3.21

ADEQUACY BOUNDARIES FOR FOURTH ORDER SYSTEMS WITH $\frac{b_2}{a_1^2} = \frac{b_2}{a_1^3} = 0$, $\frac{z_1}{a_1} = 2.0$, AND $\frac{a_4}{a_1^4} = 0.01$ FOR $\frac{b_1}{a_1} = 0, \pm 0.2$ AND ± 0.5.

FIGURE 3.22

ADEQUACY BOUNDARIES FOR FOURTH ORDER SYSTEMS

WITH $\frac{b_2}{a_1^2} = \frac{b_3}{a_1^3} = 0$, $\frac{a_4}{a_1^4} = 0.02$ AND $\frac{\tau}{a_1} = 2.0$

FOR $\frac{b_1}{a_1} = 0, \pm 0.01, \pm 0.02, \pm 0.03, -0.4$

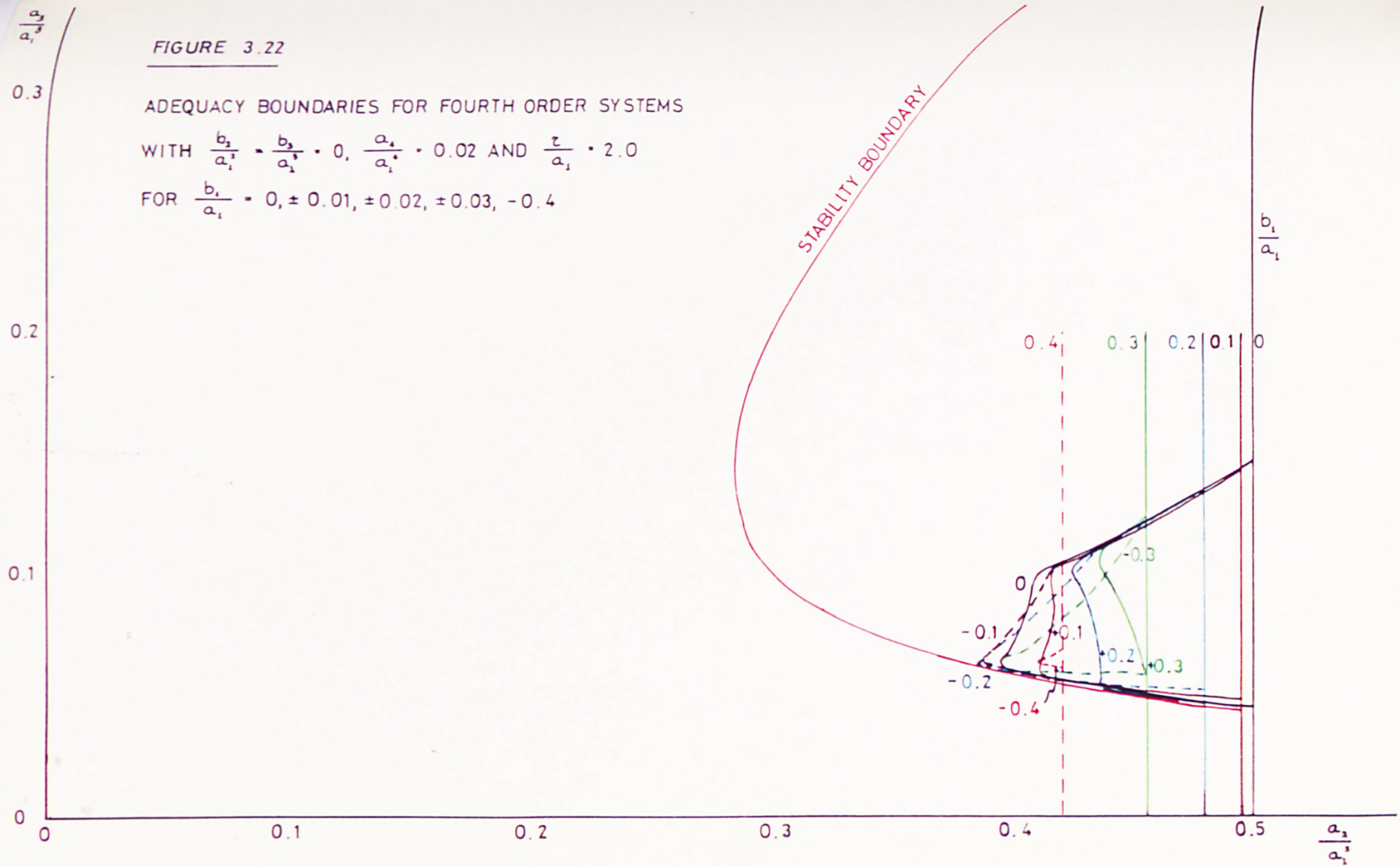


FIGURE 3.23

ADEQUACY BOUNDARIES FOR FOURTH ORDER SYSTEMS

WITH $\frac{b_2}{a_1^2} = \frac{b_3}{a_1^3} = 0$, $\frac{a_4}{a_1^4} = 0.025$ AND $\frac{\tau}{a_1} = 2.0$

FOR $\frac{b_1}{a_1} = 0, \pm 0.05, \pm 0.1, \pm 0.15, \pm 0.2, \pm 0.25, -0.3$

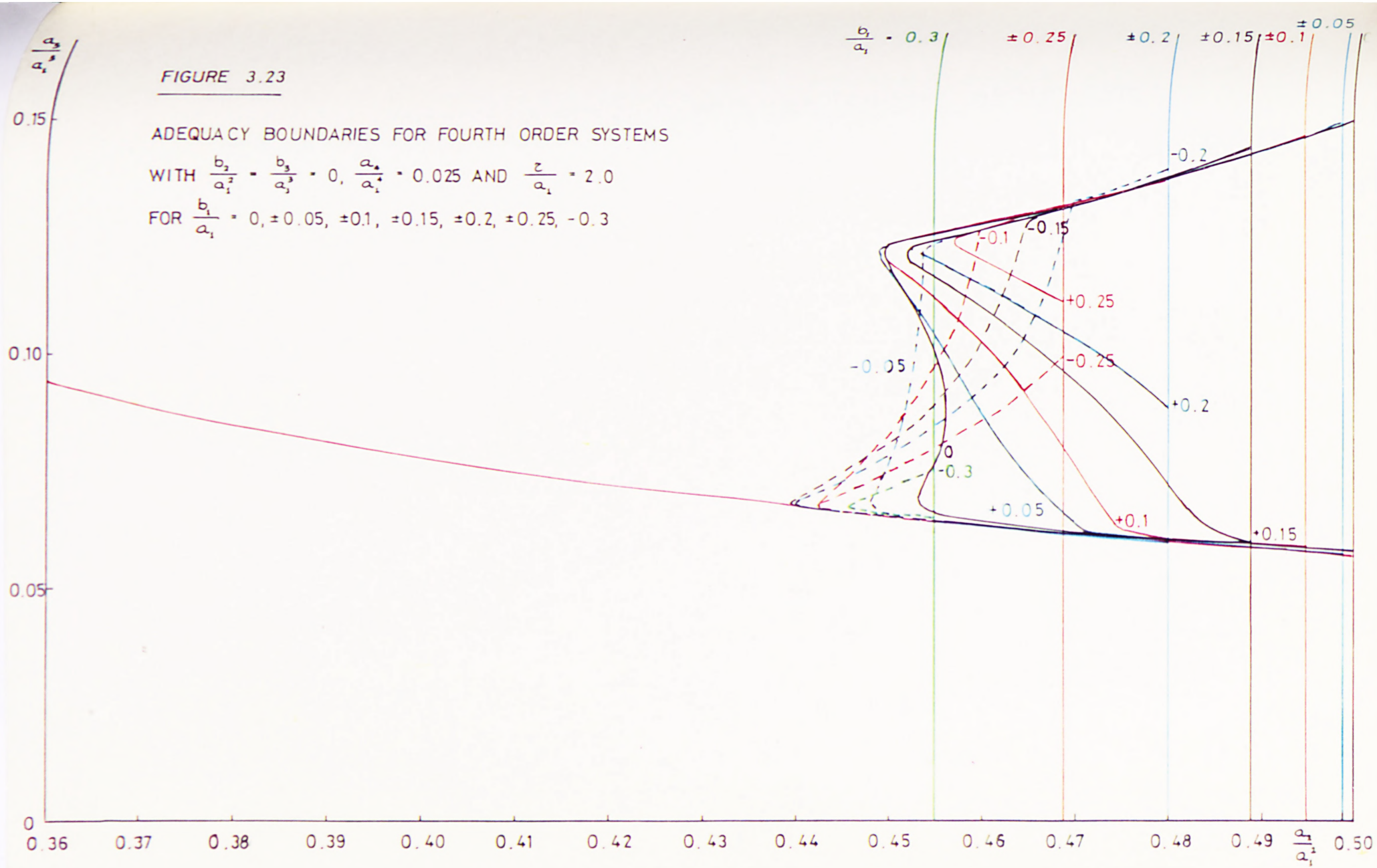
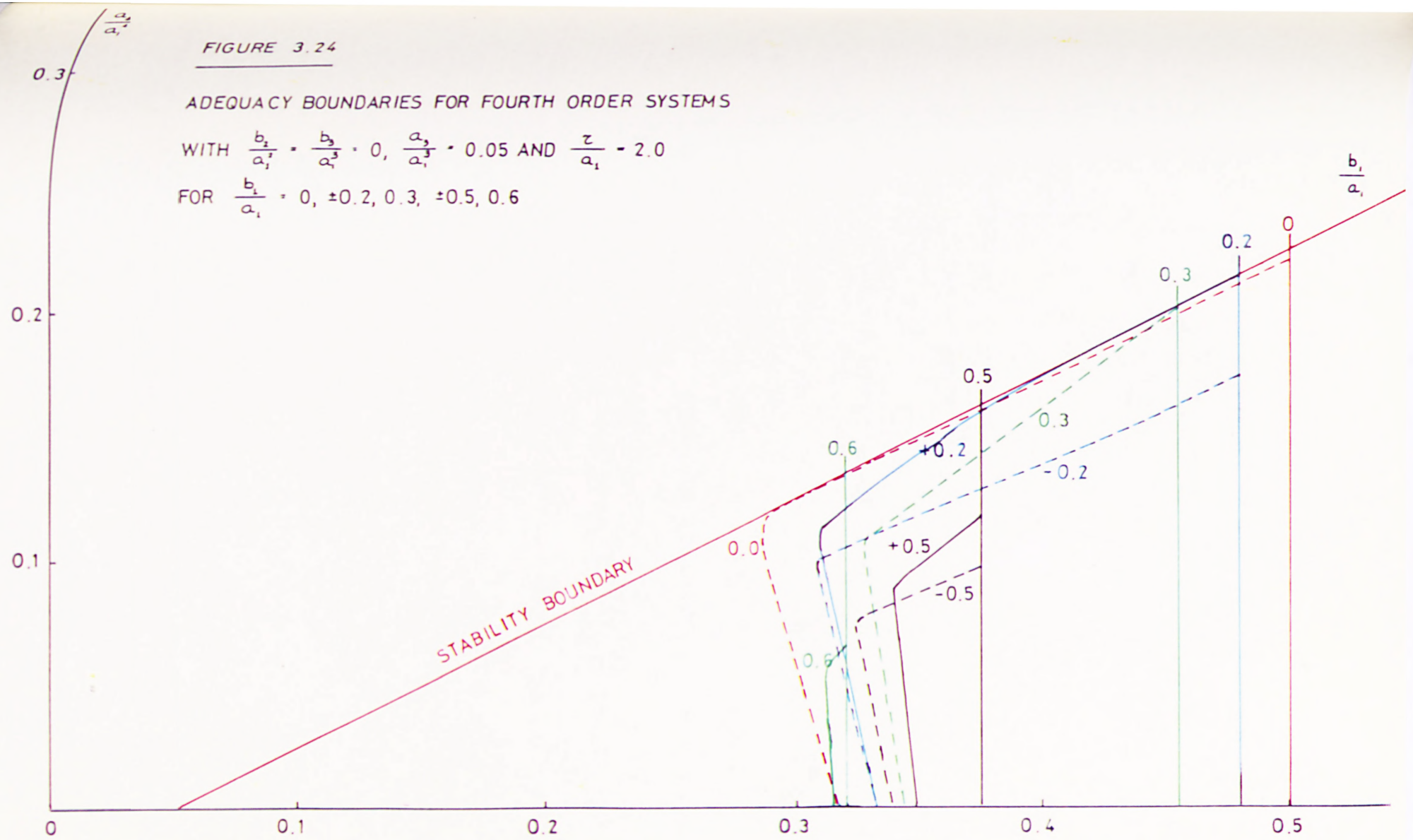


FIGURE 3.24

ADEQUACY BOUNDARIES FOR FOURTH ORDER SYSTEMS

WITH $\frac{b_2}{a_1^2} = \frac{b_3}{a_1^3} = 0$, $\frac{a_3}{a_1^3} = 0.05$ AND $\frac{\tau}{a_1} = 2.0$

FOR $\frac{b_1}{a_1} = 0, \pm 0.2, 0.3, \pm 0.5, 0.6$



the adequacy area decreases until it becomes zero slightly above $|\frac{b_1}{a_1}| = 0.6$. Another point of interest is that the minimum value of $\frac{a_2}{a_1^2}$ for adequacy varies relatively little with $\frac{b_1}{a_1}$. It varies between 0.28 and 0.34 while $(\frac{a_2}{a_1^2})_{max}$ varies between 0.32 and 0.5.

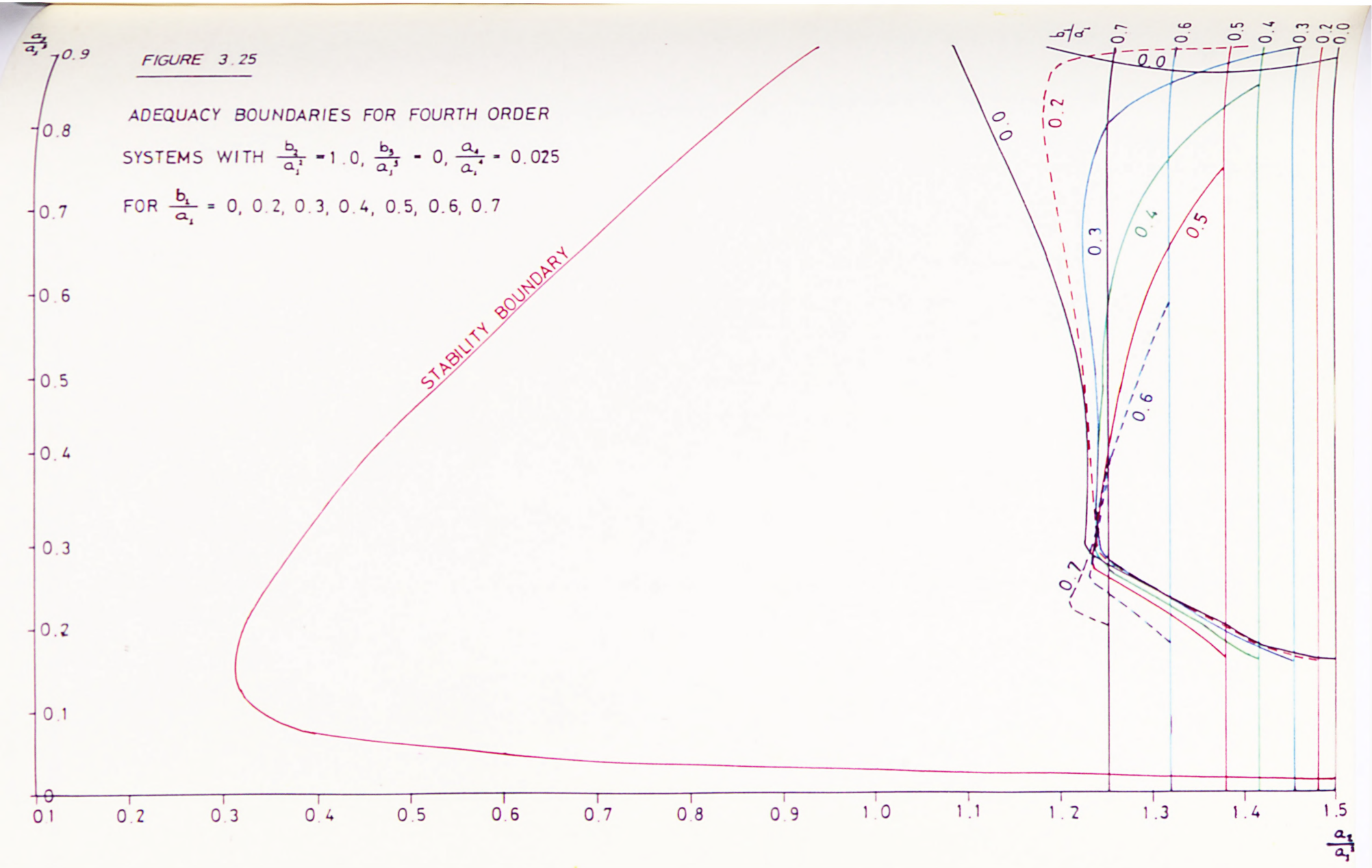
The effect of varying $\frac{b_1}{a_1}$ while keeping $\frac{b_2}{a_1^2} = 1.0$, $\frac{b_3}{a_1^3} = 0$ and $\frac{a_4}{a_1^4} = 0.025$ is shown in Fig.3.25. Figs. 3.26, 3.27 and 3.28 show the effect of varying $\frac{b_2}{a_1^2}$ while keeping other parameters constant. All these give boundaries in the $\frac{a_3}{a_1^3}$, $\frac{a_2}{a_1^2}$ plane. These results correspond to previous 3rd and fourth order results.

Adequacy boundaries in the $\frac{a_4}{a_1^4}$, $\frac{a_2}{a_1^2}$ plane were obtained by retaining $\frac{b_1}{a_1} = \frac{b_3}{a_1^3} = 0$ and $\frac{a_3}{a_1^3} = 0.05$ while varying $\frac{b_2}{a_1^2}$ between -0.1 and 5.0. Although the general trend of decreasing adequacy area is present, some surprising results were obtained around $\frac{b_2}{a_1^2} = 0.2$, 0.25 and 0.3 (see Fig.3.29). These unusual boundaries appear to be a transient stage between the boundaries for lower and higher values of $\frac{b_2}{a_1^2}$.

How such an adequacy boundary arises can be seen in Fig.3.30 where G_c is shown plotted against $\frac{a_4}{a_1^4}$. Up to six values of G_c are given for various values of $\frac{a_4}{a_1^4}$ and it is the fourth and fifth values of G_c that cause the reduced

FIGURE 3.25

ADEQUACY BOUNDARIES FOR FOURTH ORDER SYSTEMS WITH $\frac{b_2}{a_1^2} = 1.0$, $\frac{b_3}{a_1^3} = 0$, $\frac{a_4}{a_1^4} = 0.025$ FOR $\frac{b_1}{a_1} = 0, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7$



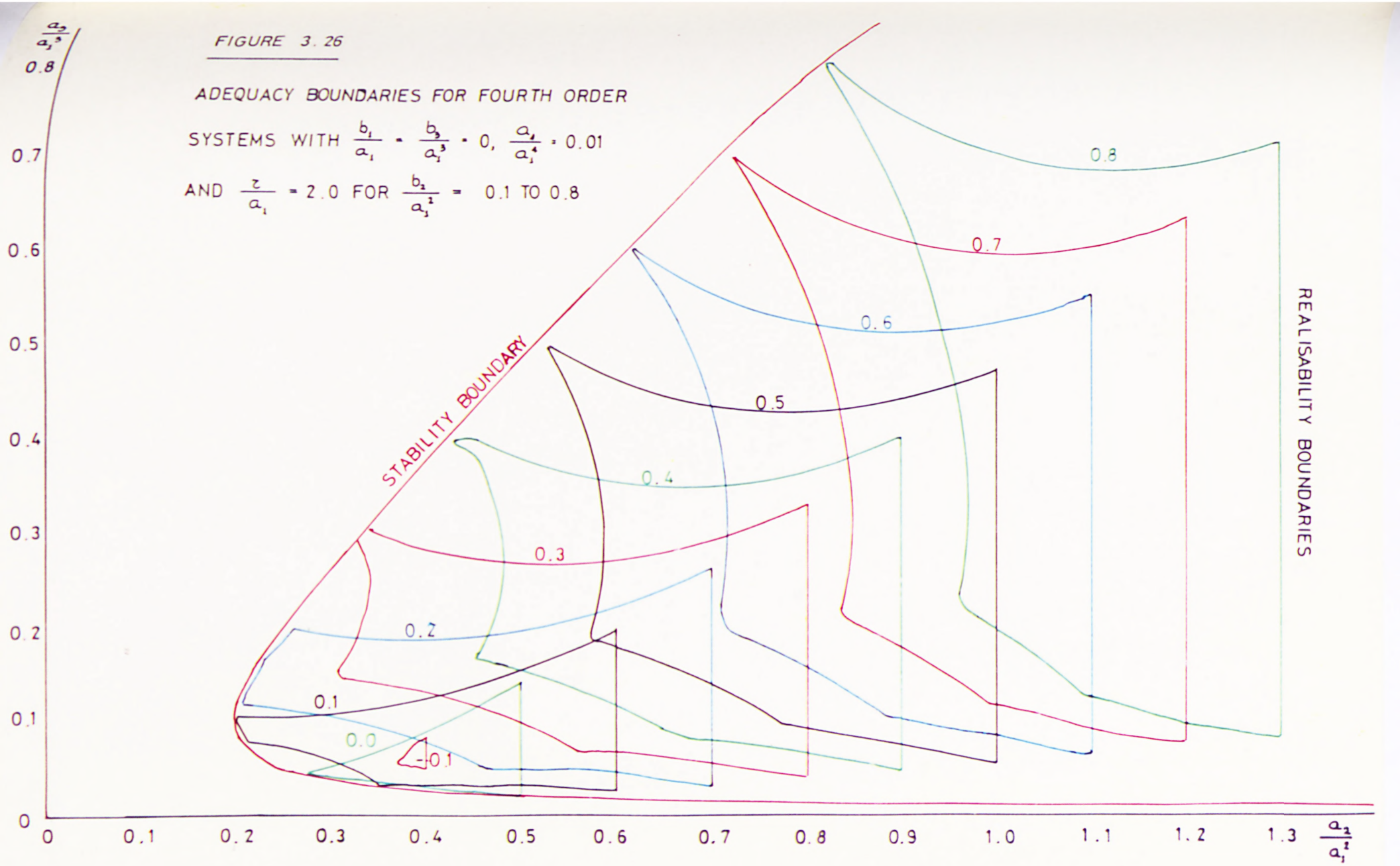


FIGURE 3.26
ADEQUACY BOUNDARIES FOR FOURTH ORDER
SYSTEMS WITH $\frac{b_1}{a_1} = \frac{b_3}{a_1^3} = 0$, $\frac{a_4}{a_1^4} = 0.01$
AND $\frac{z}{a_1} = 2.0$ FOR $\frac{b_2}{a_1^2}$ = 0.1 TO 0.8
$\frac{a_3}{a_1^3}$
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
STABILITY BOUNDARY
REALISABILITY BOUNDARIES
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
-0.1
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0 1.1 1.2 1.3
$\frac{a_2}{a_1^2}$

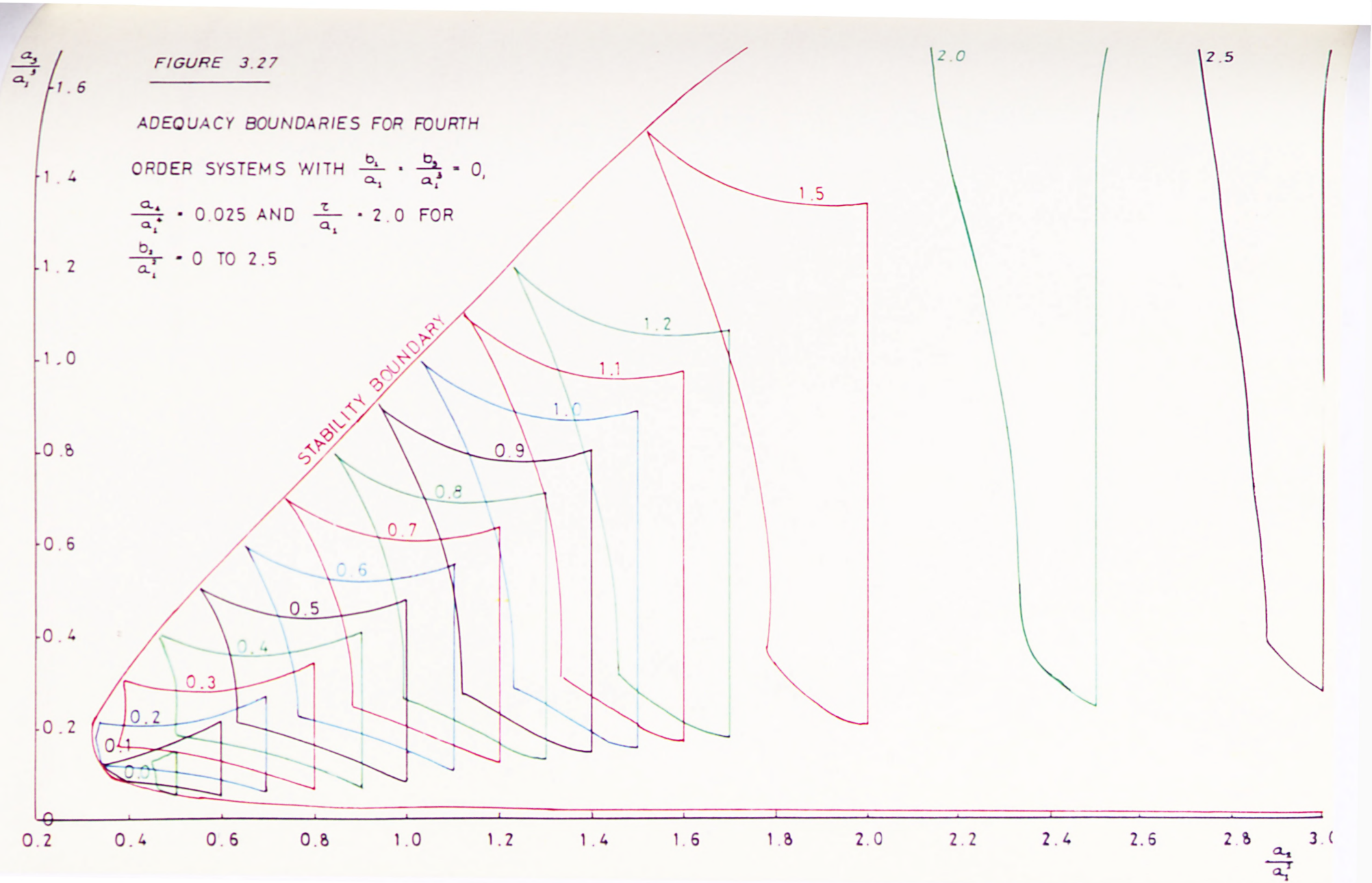


FIGURE 3.27
ADEQUACY BOUNDARIES FOR FOURTH ORDER SYSTEMS WITH $\frac{b_1}{a_1} = \frac{b_2}{a_1^2} = 0$, $\frac{a_4}{a_1^4} = 0.025$ AND $\frac{\tau}{a_1} = 2.0$ FOR $\frac{b_3}{a_1^3} = 0$ TO 2.5
STABILITY BOUNDARY
$\frac{a_3}{a_1^3}$
$\frac{a_2}{a_1^2}$
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
1.1
1.2
1.5
2.0
2.5
0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
2.0
2.2
2.4
2.6
2.8
3.0

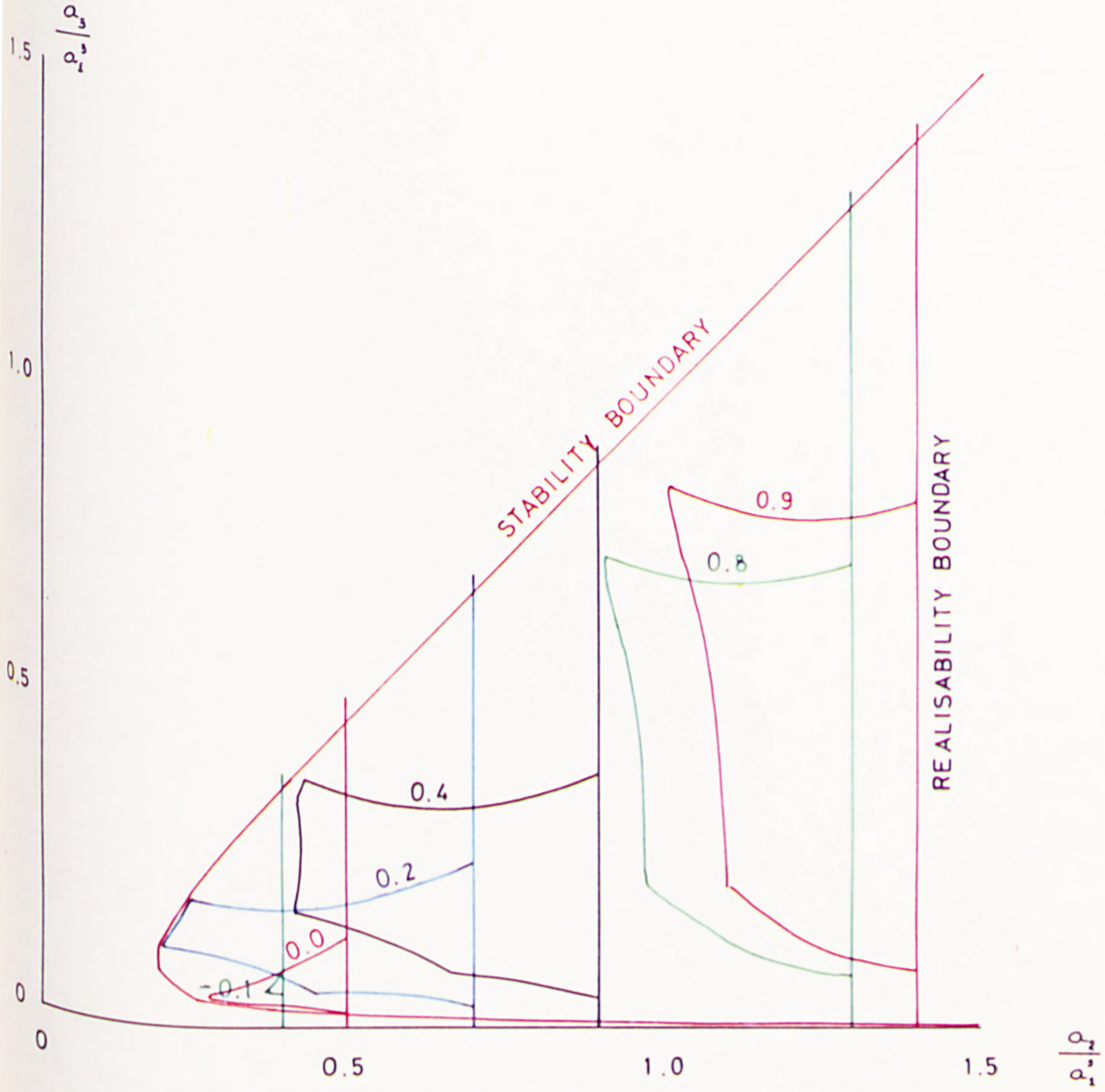


FIGURE 3.28

ADEQUACY BOUNDARIES FOR FOURTH ORDER SYSTEMS WITH $\frac{b_2}{a_1} = 0.1$, $\frac{b_3}{a_1^3} = 0.0$, $\frac{a_4}{a_1^4} = 0.01$ FOR $\frac{b_3}{a_1^2} = -0.1, 0.0, 0.2, 0.4, 0.8, 0.9$

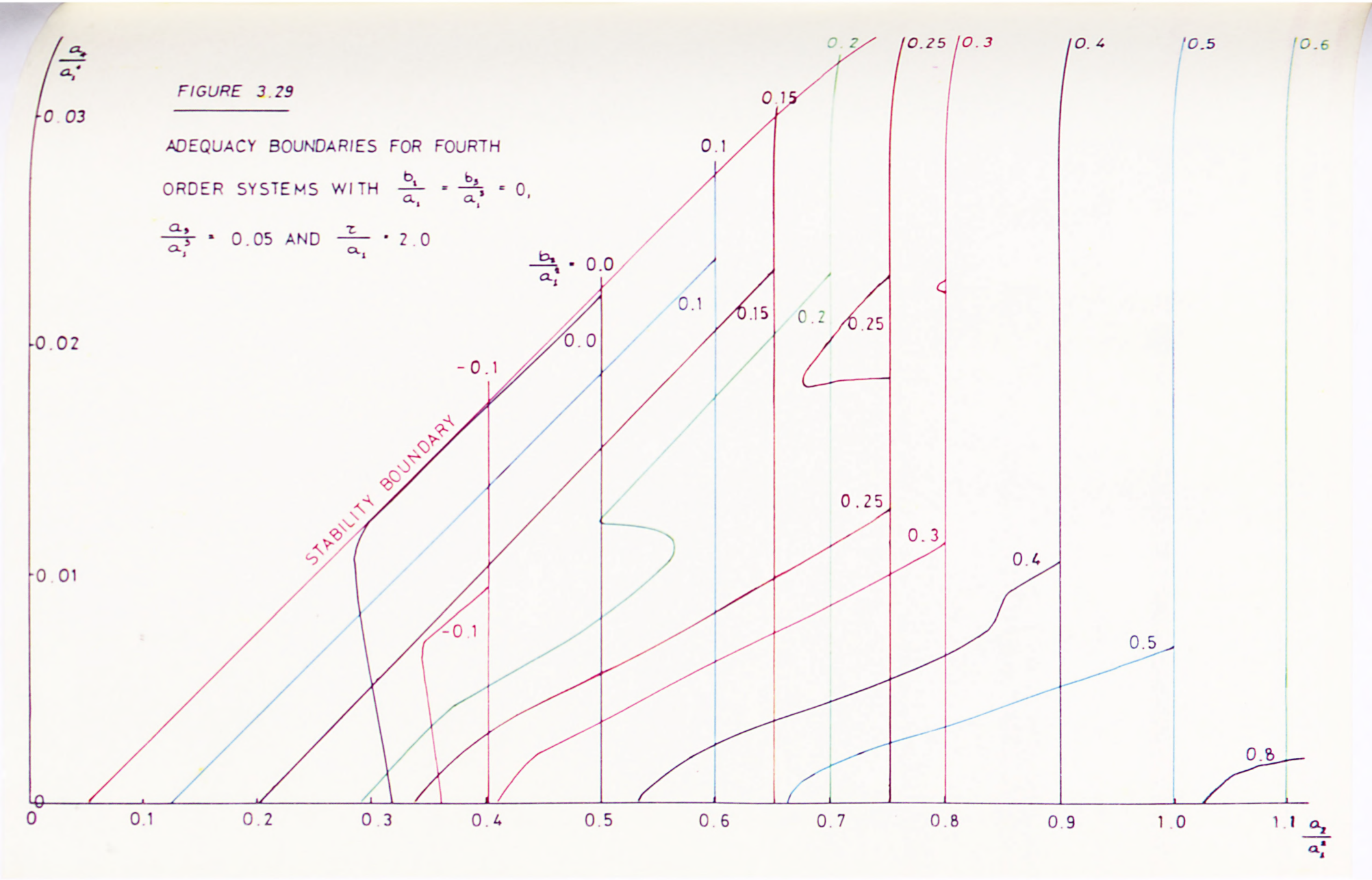


FIGURE 3.29
ADEQUACY BOUNDARIES FOR FOURTH ORDER SYSTEMS WITH $\frac{b_1}{a_1} = \frac{b_3}{a_1^3} = 0$, $\frac{a_3}{a_1^3} = 0.05$ AND $\frac{\tau}{a_1} = 2.0$
$\frac{a_2}{a_1^2}$
$\frac{a_2}{a_1^2}$
$\frac{b_3}{a_1^3} = 0.0$
STABILITY BOUNDARY
0.03
0.02
0.01
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
1.1
0.15
0.1
0.2
0.25
0.3
0.4
0.5
0.6
0.0
-0.1
0.8

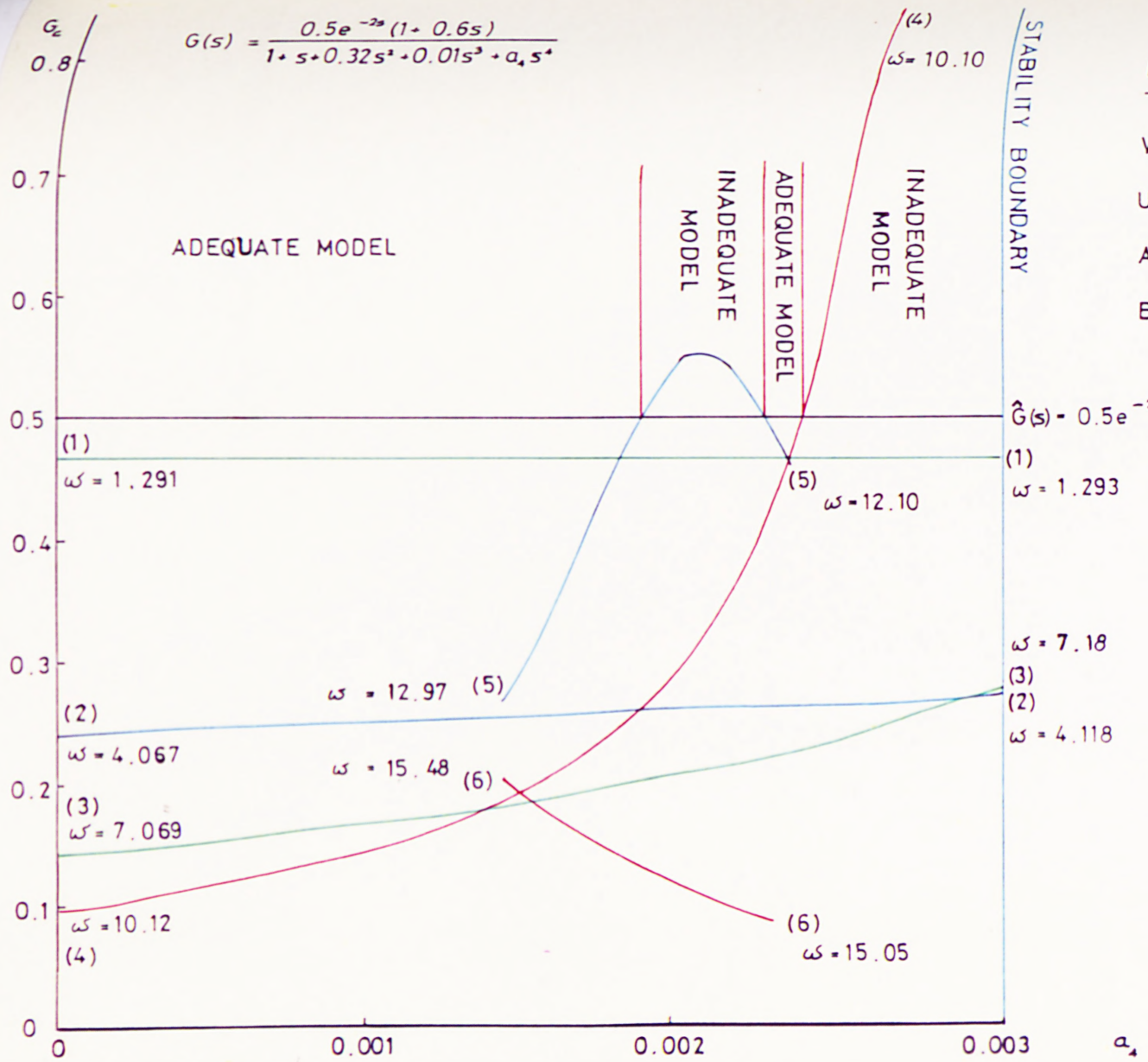


FIGURE 3.30

VARIATION OF $G_{(j\omega_c)}$ WITH a_4 SHOWN FOR UPTO SIX CROSSINGS OF THE NEGATIVE REAL AXIS, ALSO SHOWING THE ADEQUACY BOUNDARIES

N.B. NUMBERED ARE SUCCESSIVE CROSSINGS OF THE -VE REAL AXIS

model to be inadequate for certain of the systems represented. Using the previous example, for values of $\frac{b_2}{a_1^2} = 0.1$ and 0.15 the adequacy boundary is given by the fourth value of G_c (see Fig.3.31). For values of $\frac{b_2}{a_1^2} = 0.2$ to 0.3 the adequacy boundary is of the form shown in Fig.3.30 whereas for higher values of $\frac{b_2}{a_1^2}$ the higher adequacy area completely disappears as the fourth and fifth G_c curves cross above $\hat{G}_c$.

Varying $\frac{b_3}{a_1^3}$ while keeping other parameters constant gives the boundaries shown in Fig.3.32. These are similar to those in Fig.3.29 but more complex.

Returning to the $\frac{a_3}{a_1^3}$, $\frac{a_2}{a_1^2}$ plane Figs. 3.33 and 3.34 show the effect of varying $\frac{a_4}{a_1^4}$ and $\frac{b_2}{a_1^2}$ for fourth order systems with three zeroes (none of which is at infinity). The previouslyestablished pattern continues to hold under these circumstances.

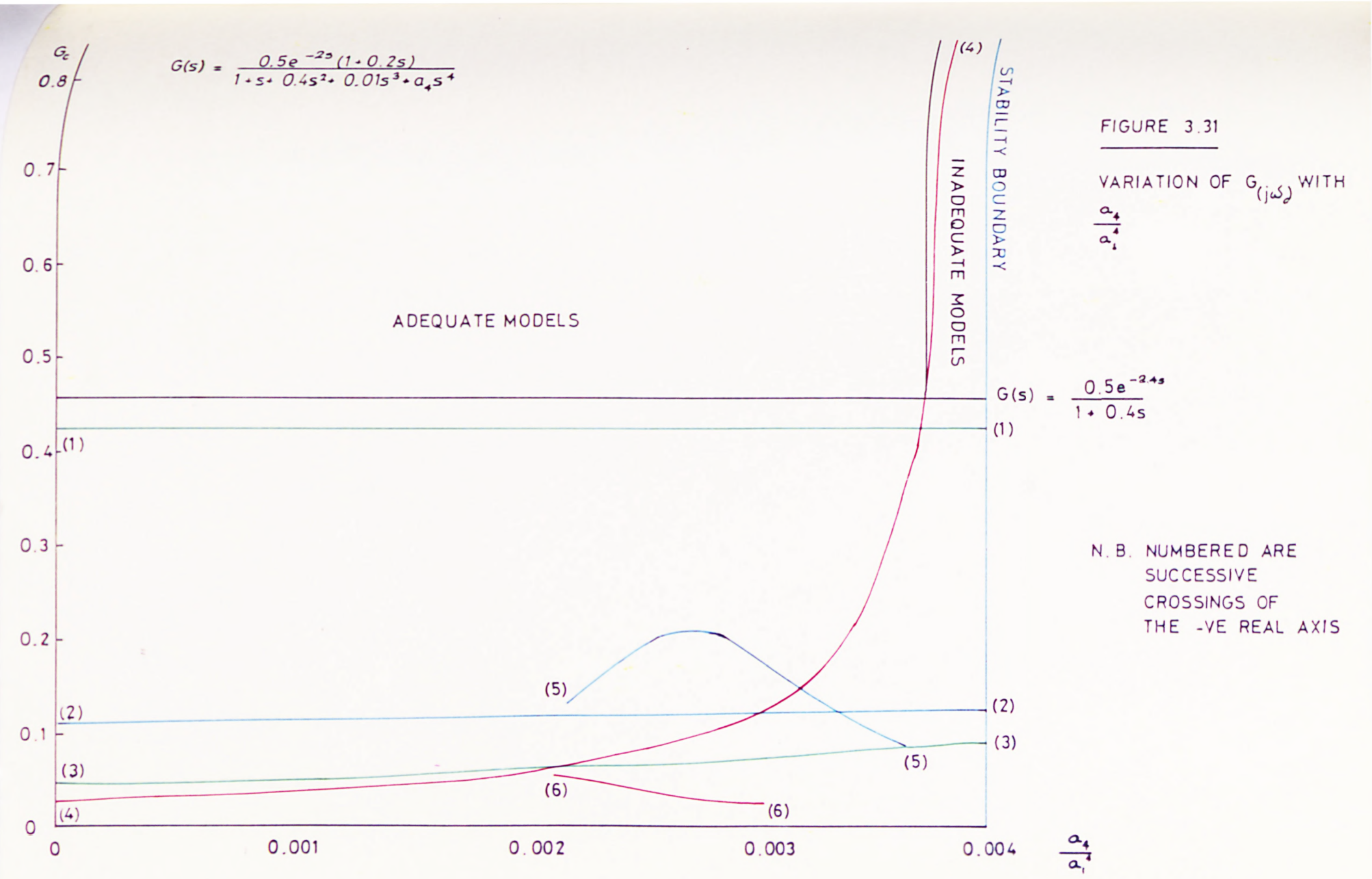


FIGURE 3.31

VARIATION OF $G_{(j\omega)}$ WITH $\frac{a_4}{a_1^4}$

N.B. NUMBERED ARE SUCCESSIVE CROSSINGS OF THE -VE REAL AXIS

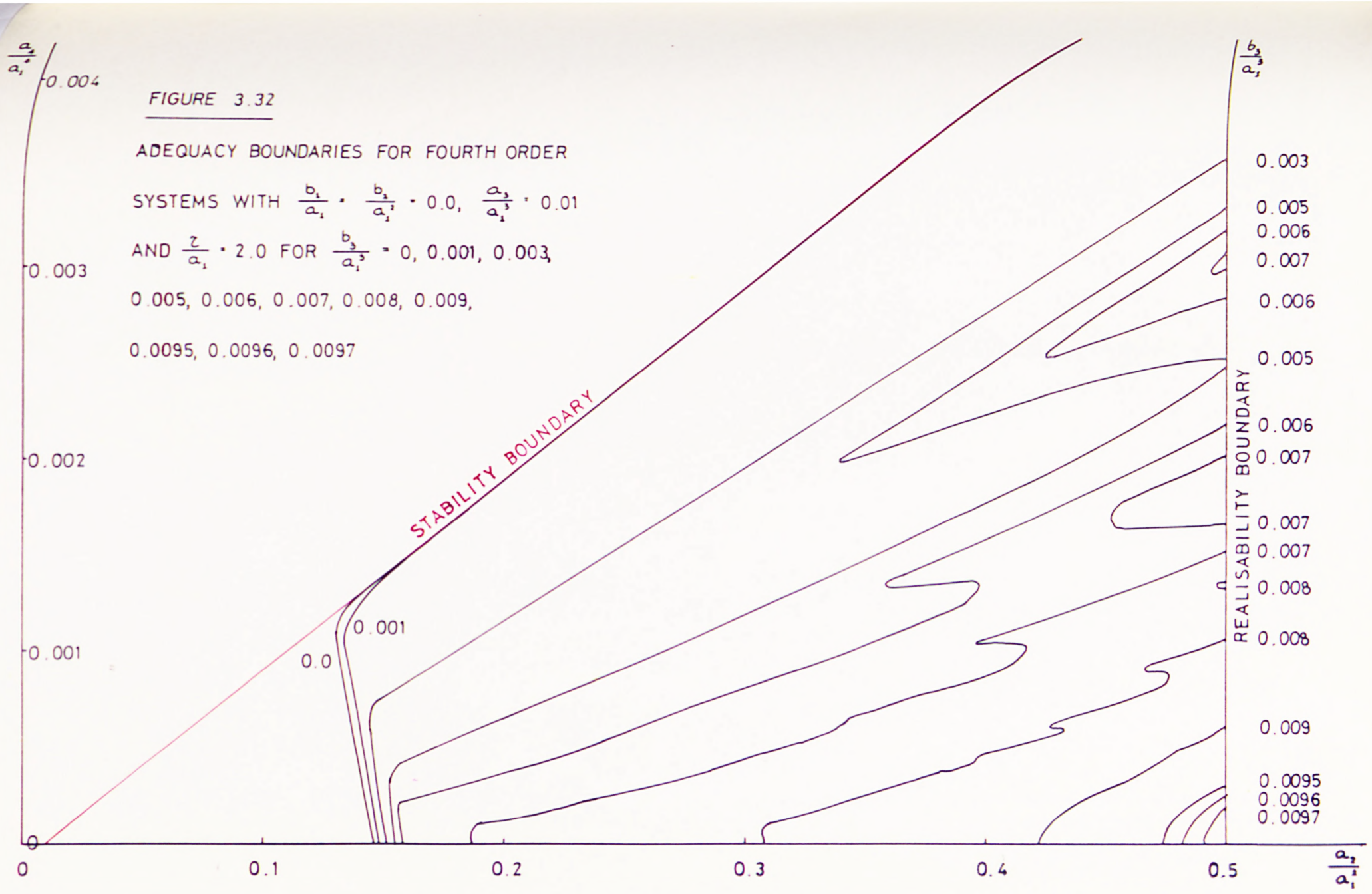


FIGURE 3.32
ADEQUACY BOUNDARIES FOR FOURTH ORDER
SYSTEMS WITH $\frac{b_1}{a_1} = \frac{b_2}{a_1^2} = 0.0$, $\frac{a_3}{a_1^3} = 0.01$
AND $\frac{\tau}{a_1} = 2.0$ FOR $\frac{b_3}{a_1^3} = 0, 0.001, 0.003,$
0.005, 0.006, 0.007, 0.008, 0.009,
0.0095, 0.0096, 0.0097
STABILITY BOUNDARY
REALISABILITY BOUNDARY
0.0
0.001
0.003
0.005
0.006
0.007
0.006
0.005
0.006
0.007
0.007
0.007
0.008
0.008
0.009
0.0095
0.0096
0.0097
0.004
0.003
0.002
0.001
0
0
0.1
0.2
0.3
0.4
0.5

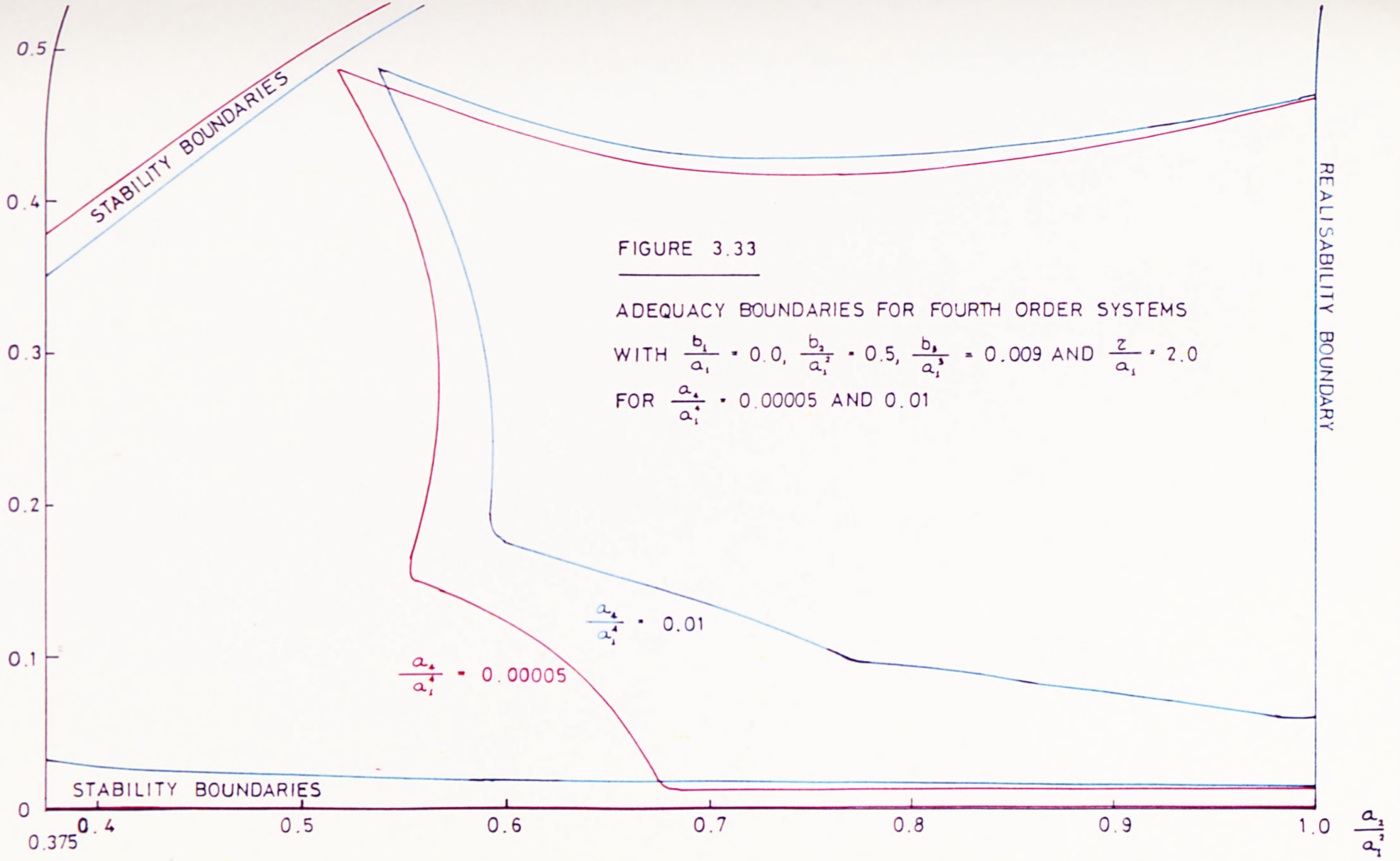


FIGURE 3.33

ADEQUACY BOUNDARIES FOR FOURTH ORDER SYSTEMS WITH $\frac{b_1}{a_1} = 0.0$, $\frac{b_2}{a_1^2} = 0.5$, $\frac{b_3}{a_1^3} = 0.009$ AND $\frac{z}{a_1} = 2.0$ FOR $\frac{a_4}{a_1^4} = 0.00005$ AND 0.01

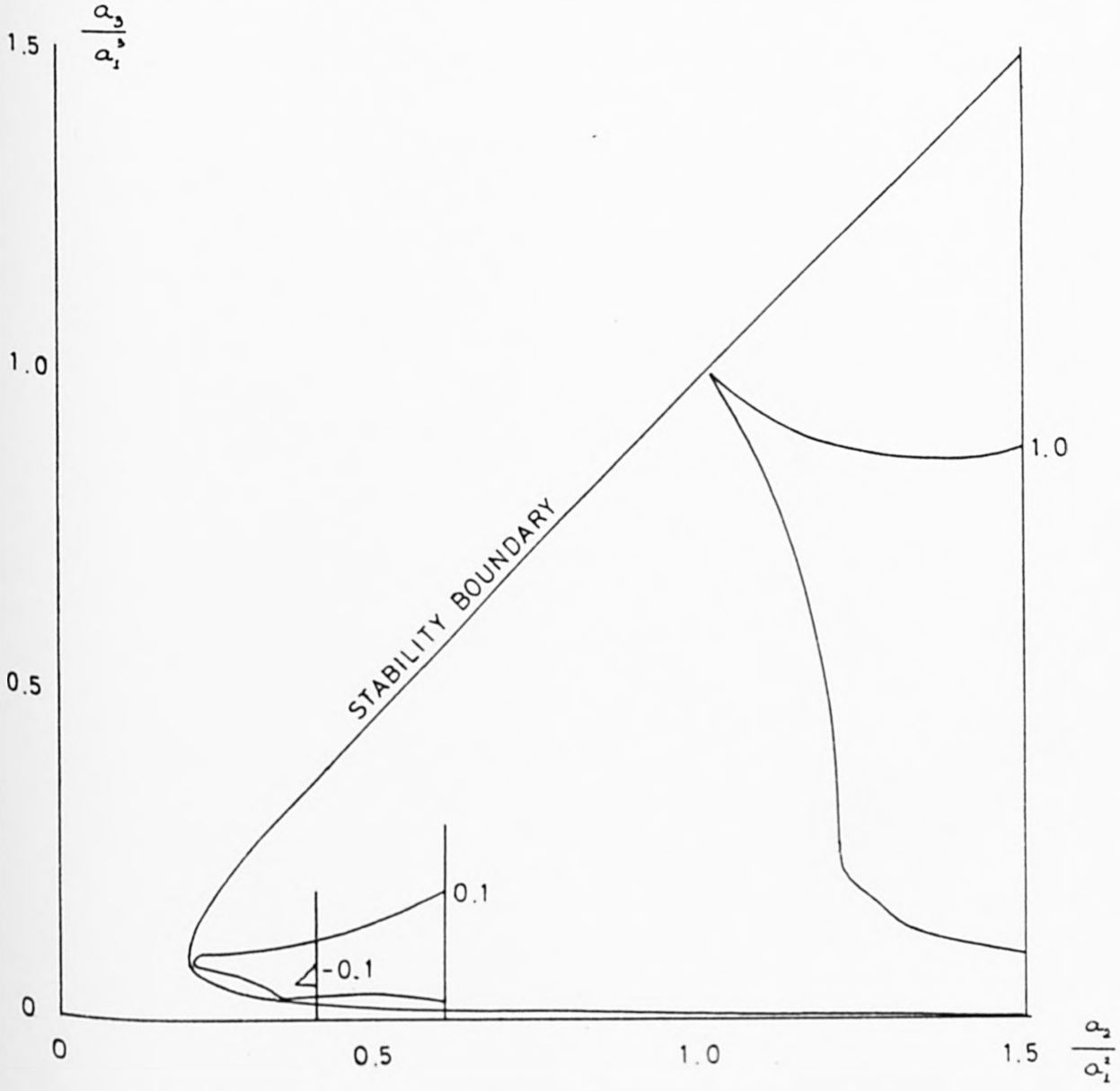


FIGURE 3.34

ADEQUACY BOUNDARIES FOR FOURTH ORDER SYSTEMS WITH $\frac{b_1}{a_1} = 0.0$, $\frac{b_3}{a_1^3} = -0.003$, $\frac{a_4}{a_1^4} = 0.01$ FOR $\frac{b_2}{a_1^2} = -0.1, 0.1, 1.0$

3.11 ADEQUACY BOUNDARIES FOR FIFTH ORDER SYSTEMS WITH TIME DELAY

Starting again with the stability boundary we form the Routh array from the characteristic equation:-

$$1 + a_1 s + a_2 s^2 + a_3 s^3 + a_4 s^4 + a_5 s^5 \qquad \text{.....(A75)}$$

and the following conditions for stability are obtained from this array.

$$a_5 \geq 0 \qquad \text{.....(A76)}$$

$$a_4 \geq 0 \qquad \text{.....(A77)}$$

$$a_3 - \frac{a_5}{a_4} a_2 \geq 0 \quad \text{or} \quad a_3 a_4 \geq a_5 a_2 \qquad \text{.....(A78)}$$

$$a_2 - \frac{a_4 (a_1 - \frac{a_5}{a_4})}{a_3 - \frac{a_5}{a_4} a_2} \geq 0 \qquad \text{.....(A79)}$$

$$\text{or} \quad a_2 a_3 - \frac{a_5}{a_4} a_2^2 \geq a_1 a_4 - a_5 \qquad \text{.....(A80)}$$

and

$$a_1 - \frac{a_5}{a_4} - \frac{a_3 - \frac{a_5}{a_4} a_2}{a_2 - \frac{a_4 a_1 - a_5}{a_3 - \frac{a_5}{a_4} . a_2}} \geq 0 \qquad \text{.....(A81)}$$

or

$$a_1 a_2 a_3 a_4 - a_3^2 a_4 - a_1 a_5 a_2^2 + a_5 a_2 a_3 - a_1^2 a_4^2 + 2 a_1 a_4 a_5 - a_5^2 \geq 0 \qquad \text{.....(A82)}$$

Normalized these conditions become:-

$$\frac{a_5}{a_1^5} \geq 0 \qquad \text{.....(A83)}$$

$$\frac{a_4}{a_1^4} \geq 0 \qquad \text{.....(A84)}$$

$$\frac{a_3}{a_1^3} \cdot \frac{a_4}{a_1^4} \geq \frac{a_5}{a_1^5} \cdot \frac{a_2}{a_1^2} \qquad \text{.....(A85)}$$

$$\frac{a_2}{a_1^2} \cdot \frac{a_3}{a_1^3} \cdot \frac{a_4}{a_1^4} - \frac{a_5}{a_1^5}\left(\frac{a_2}{a_1^2}\right)^2 \geq \left(\frac{a_4}{a_1^4}\right)^2 - \frac{a_4}{a_1^4} \cdot \frac{a_5}{a_1^5} \qquad \text{.....(A86)}$$

and

$$\frac{a_2}{a_1^2}\cdot\frac{a_3}{a_1^3}\cdot\frac{a_4}{a_1^4} - \left(\frac{a_3}{a_1^3}\right)^2\frac{a_4}{a_1^4} - \frac{a_5}{a_1^5}\cdot\left(\frac{a_2}{a_1^2}\right)^2 + \frac{a_5}{a_1^5}\cdot\frac{a_2}{a_1^2}\cdot\frac{a_3}{a_1^3} - \left(\frac{a_4}{a_1^4}\right)^2 + 2\frac{a_4}{a_1^4}\cdot\frac{a_5}{a_1^5} - \left(\frac{a_5}{a_1^5}\right)^2$$

$$\geq 0 \qquad \text{.....(A87)}$$

This last condition can be rewritten to give an explicit expression for the boundary condition for

$$\frac{a_3}{a_1^3} :-$$

$$\frac{1}{2}\cdot\frac{a_2}{a_1^2}\left(1+\frac{a_5}{a_1^5}\Big/\frac{a_4}{a_1^4}\right) - \sqrt{z} \leq \frac{a_3}{a_1^3} \leq \frac{1}{2}\cdot\frac{a_2}{a_1^2}\left(1+\frac{a_5}{a_1^5}\Big/\frac{a_4}{a_1^4}\right) + \sqrt{z} \qquad \text{..(A88)}$$

where

$$z = \left[\frac{1}{2}\cdot\frac{a_2}{a_1^2}\left(1-\frac{a_5}{a_1^5}\Big/\frac{a_4}{a_1^4}\right)\right]^2 - \left(\frac{a_5}{a_1^5}\right)^2\Big/\left(\frac{a_4}{a_1^4}\right) - \frac{a_4}{a_1^4} + 2\,\frac{a_5}{a_1^5}$$

$$= \left(\frac{a_4}{a_1^4} - \frac{a_5}{a_1^5}\right)^2\left[\left(\frac{1}{2}\cdot\frac{a_2}{a_1^2}\Big/\frac{a_4}{a_1^4}\right)^2 - 1\Big/\frac{a_4}{a_1^4}\right] \qquad \text{.....(A89)}$$

As z must be ≥ 0, $\left(\frac{a_2}{a_1^2}\right)^2 \geq 4\,\frac{a_4}{a_1^4}$ as for fourth order systems unless $\frac{a_4}{a_1^4} = \frac{a_5}{a_1^5}$. This condition for $\frac{a_3}{a_1^3}$ gives a stability

boundary in the $\frac{a_2}{a_1^2}$, $\frac{a_3}{a_1^3}$ plane which is asymptotic to the lines

$$\frac{a_3}{a_1^3} = \frac{a_2}{a_1^2} \quad \text{and} \quad \frac{a_3}{a_1^3} = \frac{a_2}{a_1^2}.\frac{a_5}{a_1^5}\Big/\frac{a_4}{a_1^4} \qquad \text{.....(A90)}$$

Consequently if $\frac{a_5}{a_1^5} = \frac{a_4}{a_1^4}$ then $\frac{a_3}{a_1^3} = \frac{a_2}{a_1^2}$ as the two arms of the stability boundary become coincident.

By rewriting this condition for $\frac{a_4}{a_1^4}$ a stability boundary can be obtained in the $\frac{a_4}{a_1^4}$, $\frac{a_2}{a_1^2}$ plane:-

$$\frac{a_5}{a_1^5} - \frac{1}{2}\left(\frac{a_3}{a_1^3} - \frac{a_2}{a_1^2}\right)\left[\frac{a_3}{a_1^3} - \sqrt{\left(\frac{a_3}{a_1^3}\right)^2 - 4\left(\frac{a_5}{a_1^5}\right)}\right] \leqslant \frac{a_4}{a_1^4} \leqslant \frac{a_5}{a_1^5} - \frac{1}{2}\left(\frac{a_3}{a_1^3} - \frac{a_2}{a_1^2}\right).$$

$$\left[\frac{a_3}{a_1^3} + \sqrt{\left(\frac{a_3}{a_1^3}\right)^2 - 4\left(\frac{a_5}{a_1^5}\right)}\right] \qquad \text{.....(A91)}$$

This gives a stability boundary consisting of two straight lines which meet at $\frac{a_2}{a_1^2} = \frac{a_3}{a_1^3}$

$\left(\frac{a_3}{a_1^3}\right)^2$ must be $\geqslant 4\,\frac{a_5}{a_1^5}$ unless $\frac{a_2}{a_1^2} = \frac{a_3}{a_1^3}$

If $\left(\frac{a_3}{a_1^3}\right)^2 = 4\,\frac{a_5}{a_1^5}$ then the area of stability reduces to a single line $\frac{a_4}{a_1^4} = \frac{1}{2}.\frac{a_3}{a_1^3}.\frac{a_2}{a_1^2} - \frac{a_5}{a_1^5}$ .....(A92)

Similarly this condition can be rewritten to give an explicit boundary for $\frac{a_5}{a_1^5}$:-

$$\frac{a_4}{a_1^4} - \frac{1}{2}\left(\frac{a_2}{a_1^2} - \frac{a_3}{a_1^3}\right)\left[\frac{a_2}{a_1^2} - \sqrt{\left(\frac{a_2}{a_1^2}\right)^2 - 4\left(\frac{a_4}{a_1^4}\right)}\right] \leq \frac{a_5}{a_1^5}$$

$$\frac{a_5}{a_1^5} \leq \frac{a_4}{a_1^4} - \frac{1}{2}\left(\frac{a_2}{a_1^2} - \frac{a_3}{a_1^3}\right)\left[\frac{a_2}{a_1^2} + \sqrt{\left(\frac{a_2}{a_1^2}\right)^2 - 4\left(\frac{a_4}{a_1^4}\right)}\right] \qquad .....(A93)$$

The stability boundary in the $\frac{a_5}{a_1^5}$, $\frac{a_2}{a_1^2}$ plane is asymptotic to the curve:-

$$\frac{a_5}{a_1^5} \cdot \frac{a_2}{a_1^2} = \frac{a_3}{a_1^3} \cdot \frac{a_4}{a_1^4} \qquad .....(A94)$$

and is enclosed by the lines $\frac{a_5}{a_1^5} = \frac{1}{4}\left(\frac{a_3}{a_1^3}\right)^2$ .....(A95)

which gives the maximum value of $\frac{a_5}{a_1^5}$ for stability for any given value of $\frac{a_3}{a_1^3}$

and $\frac{a_2}{a_1^2} = 2\sqrt{\frac{a_4}{a_1^4}}$ .....(A96)

which gives the minimum value of $\frac{a_2}{a_1^2}$ for stability for any given value of $\frac{a_4}{a_1^4}$. (see Fig.3.35)

Simple model realizability and stability boundaries are, of course, identical for all systems above second order. Cancellation conditions are also similar (see Sections 3.9 and 3.10).

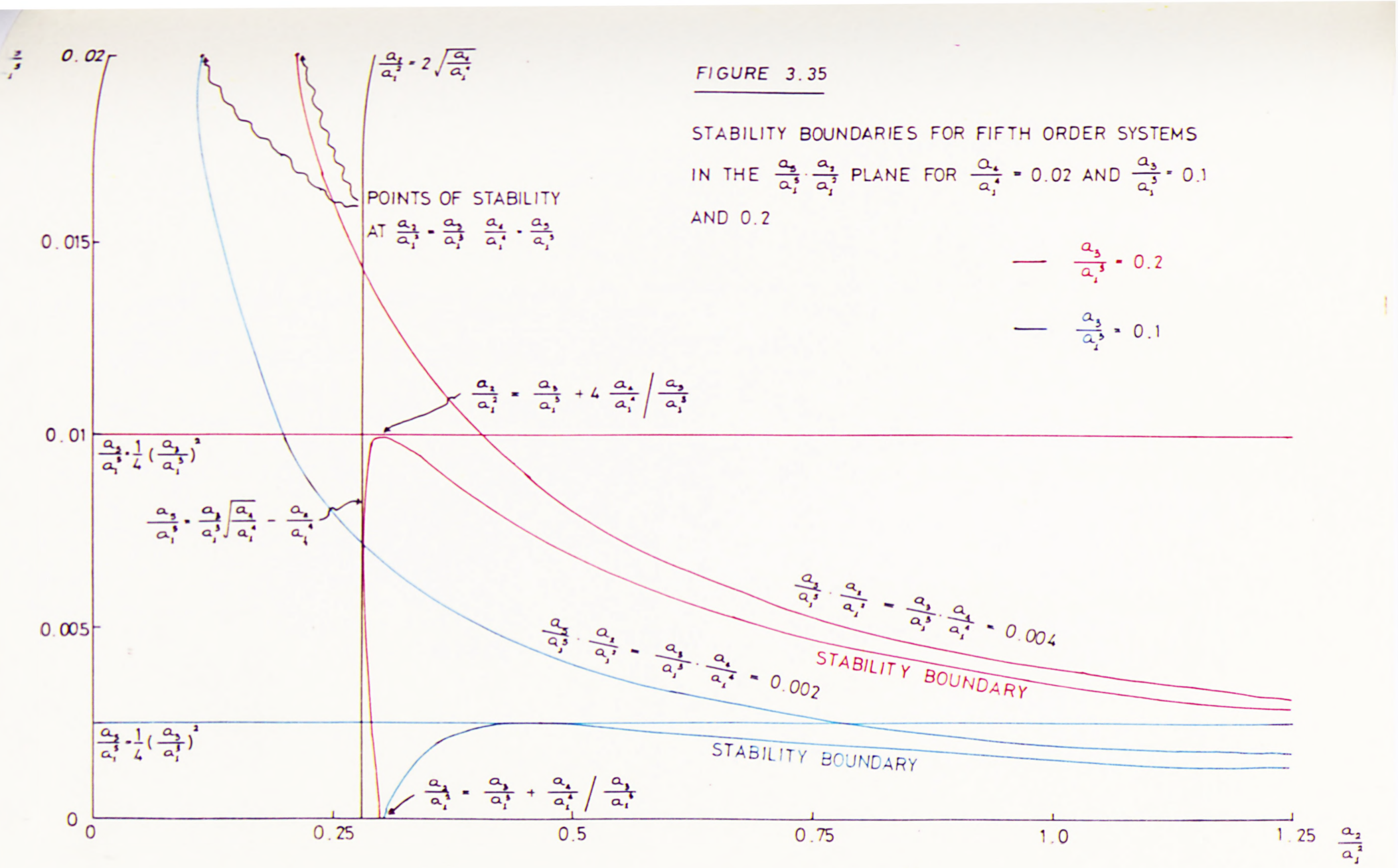


FIGURE 3.35
STABILITY BOUNDARIES FOR FIFTH ORDER SYSTEMS IN THE $\frac{a_0}{a_1^5}$ - $\frac{a_2}{a_1^3}$ PLANE FOR $\frac{a_4}{a_1^4}$ = 0.02 AND $\frac{a_3}{a_1^5}$ = 0.1 AND 0.2
$\frac{a_3}{a_1^5}$ = 0.2
$\frac{a_3}{a_1^5}$ = 0.1
$\frac{a_2}{a_1^3} = 2\sqrt{\frac{a_0}{a_1^5}}$
POINTS OF STABILITY AT $\frac{a_2}{a_1^3} = \frac{a_0}{a_1^5}$ $\frac{a_4}{a_1^4} = \frac{a_3}{a_1^5}$
$\frac{a_2}{a_1^3} = \frac{a_0}{a_1^5} + 4\frac{a_4}{a_1^4} \Big/ \frac{a_3}{a_1^5}$
$\frac{a_0}{a_1^5} = \frac{1}{4}\left(\frac{a_3}{a_1^5}\right)^2$
$\frac{a_0}{a_1^5} = \frac{a_2}{a_1^3}\sqrt{\frac{a_4}{a_1^4}} - \frac{a_4}{a_1^4}$
$\frac{a_0}{a_1^5} \cdot \frac{a_4}{a_1^4} - \frac{a_2}{a_1^3} \cdot \frac{a_3}{a_1^5} = 0.004$
$\frac{a_0}{a_1^5} \cdot \frac{a_4}{a_1^4} - \frac{a_2}{a_1^3} \cdot \frac{a_3}{a_1^5} = 0.002$
STABILITY BOUNDARY
STABILITY BOUNDARY
$\frac{a_0}{a_1^5} = \frac{1}{4}\left(\frac{a_3}{a_1^5}\right)^2$
$\frac{a_2}{a_1^3} = \frac{a_0}{a_1^5} + \frac{a_4}{a_1^4} \Big/ \frac{a_3}{a_1^5}$
0.02
0.015
0.01
0.005
0
0
0.25
0.5
0.75
1.0
1.25
$\frac{a_2}{a_1^3}$

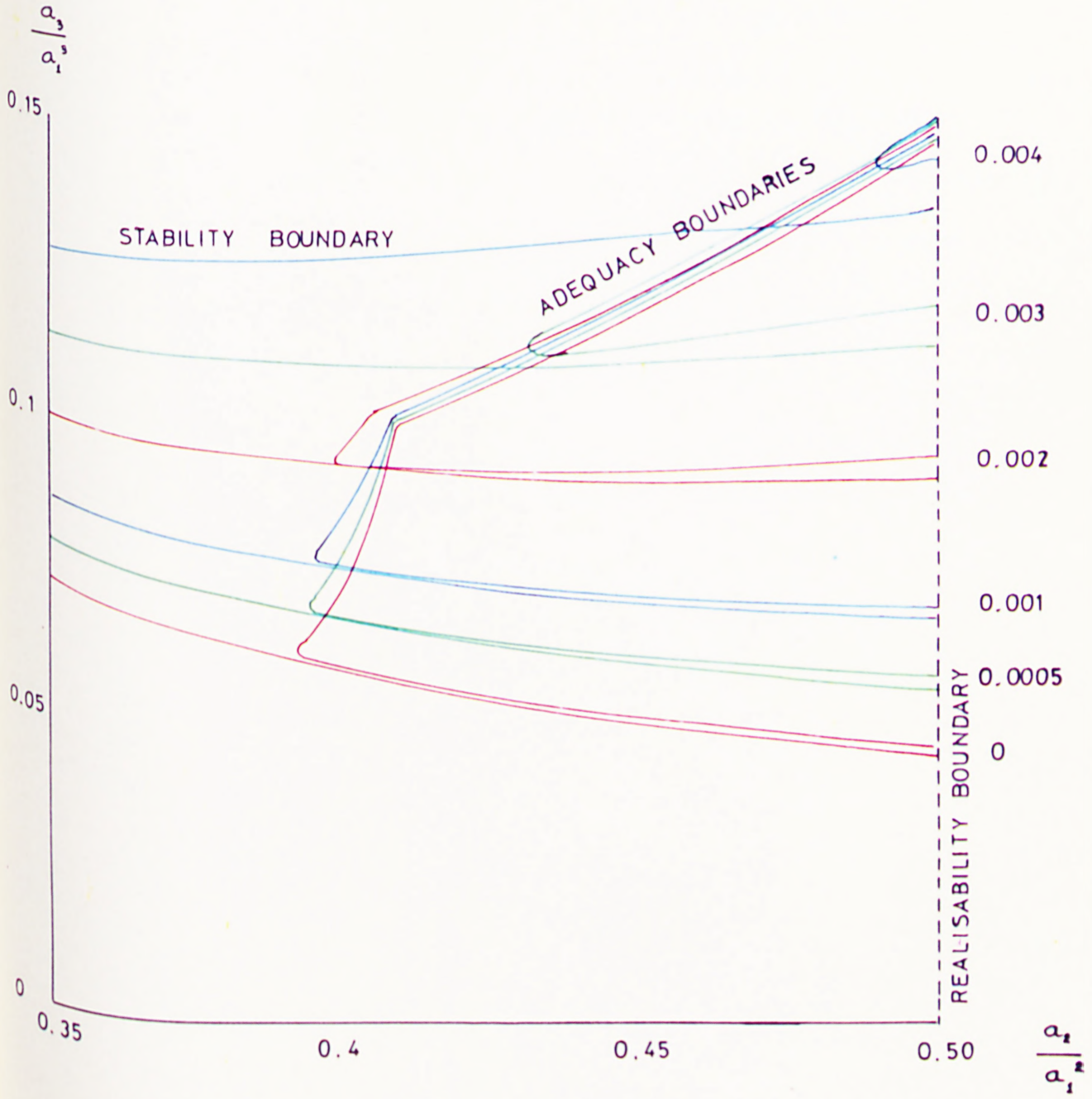


FIGURE 3.36

ADEQUACY BOUNDARIES FOR FIFTH ORDER SYSTEMS WITH $\frac{b_1}{a_1} = \frac{b_2}{a_1^2} = \frac{b_3}{a_1^3} = \frac{b_4}{a_1^4} = 0$, $\frac{\tau}{a_1} = 2.0$, $\frac{a_4}{a_1^4} = 0.02$ AND VALUES OF $\frac{a_5}{a_1^5}$ BETWEEN 0 AND 0.004.

Only a limited search was carried out for fifth order systems. Fig.3.36 shows the effect of increasing $\frac{a_5}{a_1^5}$ on the adequacy boundary in the $\frac{a_2}{a_1^2}$, $\frac{a_3}{a_1^3}$ plane. The area of adequacy is shown to be reduced by the stability boundary. A similar effect can be seen in Fig.3.37 where $\frac{a_5}{a_1^5}$ is kept constant and $\frac{a_4}{a_1^4}$ is varied. In this case, however the adequacy boundary also retreats towards the realizability boundary as $\frac{a_4}{a_1^4}$ is increased.

3.12 CONCLUSIONS ABOUT ADEQUACY BOUNDARIES FOR SIMPLE MODELS OBTAINED USING THE PADÉ APPROXIMATION TECHNIQUES

In the preceding sub sections, various adequacy boundaries for systems up to and including fifth order plus time delay systems have been examined. Higher order systems have not been investigated, although they could be analysed in the same way, since it can be expected that the trends displayed by the systems which have been examined will be maintained for systems of order greater than fifth.

As can be seen from the preceding pages, only a small percentage of systems which give stable, realizable simple models actually give adequate models. In fact as the order of the system increases so theprobability that the reduced model is adequate decreases.

In general the nearer $\frac{a_2}{a_1^2}$ is to $(\frac{a_2}{a_1^2})_{max}$, the upper realizability boundary, the more likely it is that the simple model will be adequate. This is to be expected as $\hat{G}_c$=K at $(\frac{a_2}{a_1^2})_{max}$ and 0 at $(\frac{a_2}{a_1^2})_{min}$ and in theory, at least, at $(\frac{a_2}{a_1^2})_{max}$, all systems which can give a realizable, stable, adequate first order plus time delay model using any technique will give one using this method. However, in fact all models obtained using this technique from systems with $\frac{a_2}{a_1^2} = (\frac{a_2}{a_1^2})_{max}$ have T=0 and consequently cannot be used for controller design.

From the systems examined, it also appears that few adequate models are obtained when $\frac{a_2}{a_1^2}$ is less than $(\frac{a_2}{a_1^2})_{max}$ -0.5, that is to say few adequate simple models have $\frac{T}{a_1} > 1$. $\hat{G}_c$ decreases as T increases so any increase

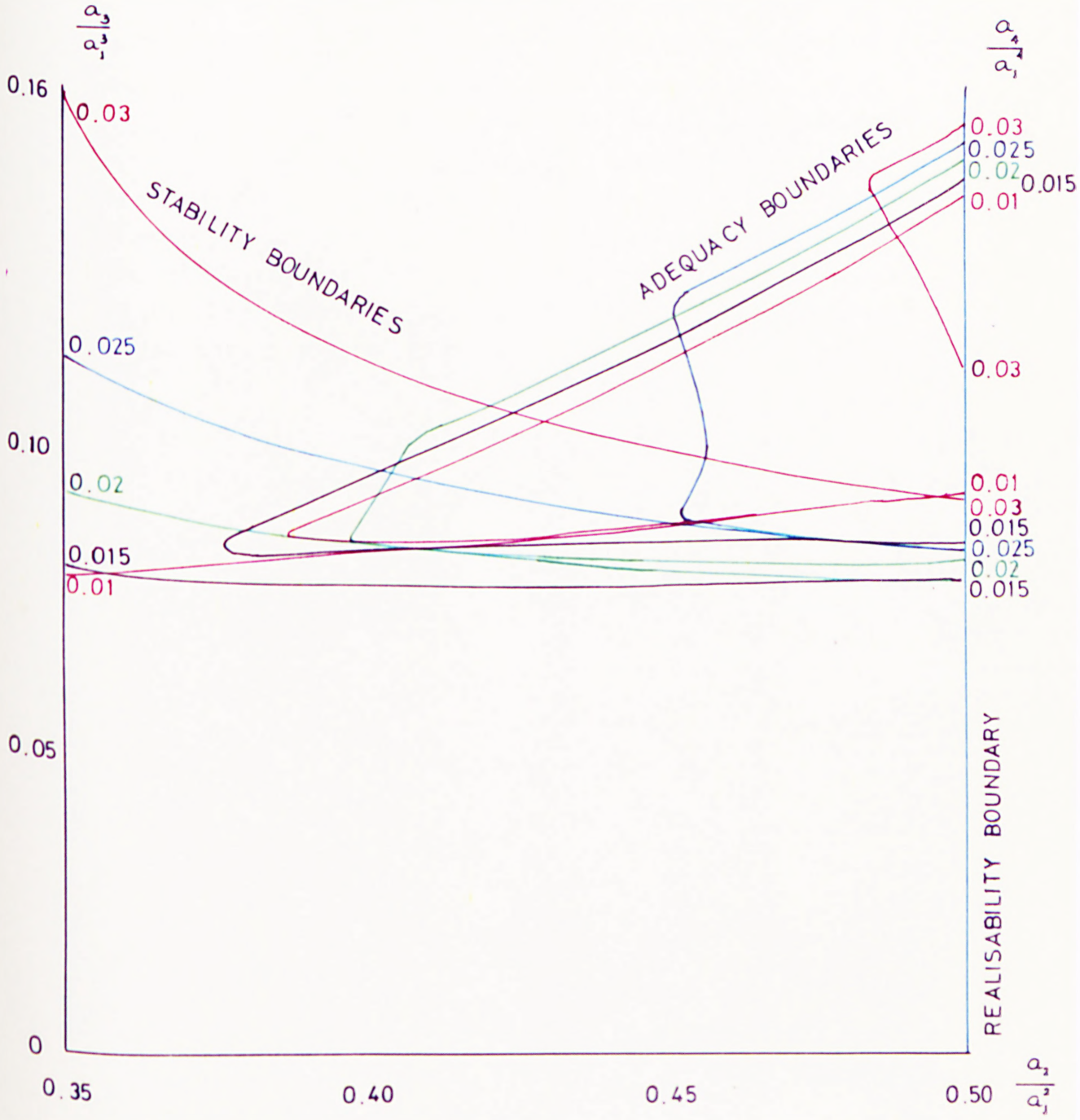


FIGURE 3.37

STABILITY AND ADEQUACY BOUNDARIES FOR FIFTH ORDER SYSTEMS WITH $\frac{\tau}{a_1} = 2.0$, $\frac{b_1}{a_1} = \frac{b_2}{a_1^2} = \frac{b_3}{a_1^3} = \frac{b_4}{a_1^4} = 0$, $\frac{a_5}{a_1^5} = 0.0015$ AND $\frac{a_4}{a_1^4}$ BETWEEN 0.01 AND 0.03

in T will reduce the probability that the model will be adequate

For other system parameters, it can, in general, be said that the nearer a normalised system parameter is to its permissible minimum for stability the more likely it is that the system will produce an adequate model.

Certain systems which cannot give an adequate model using the Padé approximation technique, will give an adequate model using other methods. These are systems for which $\hat{G}_c < G_c < K$, where $\hat{G}_c$ relates to the simple model obtained using the Padé approximation technique. An alternative method for these systems is given in the next subsection.

3.13 ADEQUATE SIMPLE MODELS FOR SYSTEMS WHICH CANNOT GIVE ADEQUATE MODELS USING THE PADÉ APPROXIMATION TECHNIQUE

Methods exist for the calculation of first order model parameters which will give a model which crosses the -ve real axis at a previously determined point. Hence we can produce a first order plus time delay model whose frequency plot crosses the negative real axis at the same point as the frequency plot of the system itself, given that $G_c<K$.

The equations for T and $\hat{\tau}$ are obtained from the theory of a first order plus time delay model (see Section 3.4). They are:-

$$T = \frac{1}{\hat{\omega}_c}\sqrt{\left(\frac{K}{G_c}\right)^2 - 1} \qquad .....(A97)$$

and

$$\hat{\tau} = \frac{1}{\hat{\omega}_c}\cos^{-1}\left(\frac{G_c}{K}\right) \qquad .....(A98)$$

It will be noticed that the terms for T and $\hat{\tau}$ include $\hat{\omega}_c$ which is the frequency at which we wish the plot to cross the negative real axis. Originally it was thought that setting this equal to the frequency at which the process plot crossed the axis at this point might give a suitable adequate model. It was however discovered that this often caused a model with a perfect fit at $\omega=0$ and $\omega=\hat{\omega}_c$ but with a very poor fit everywhere else. Consequently this idea was abandoned.

For the frequency plot of the model to cross the -ve real axis at a given point it is not the actual values of $\hat{\tau}$ and T that are important but the ratio between them:-

$$R = \frac{\hat{\tau}}{T} = \frac{\cos^{-1}\left(\frac{G_c}{K}\right)}{\sqrt{\left(\frac{K}{G_c}\right)^2 - 1}} \qquad \ldots\ldots(A99)$$

This ratio has been used in conjunction with the equations obtained earlier by equating coefficients of s and s^2 in the series expansion to give four further differencing but similar models.

Model 1 is the model obtained using Padé approximation.

Model 2 uses the value of T from Model 1 but adjusts $\hat{\tau}$ to give the desired ratio.

Model 3 uses the value of $\hat{\tau}$ from Model 1 and adjusts T to give the desired ratio. This model is the least similar of models 2, 3, 4 and 5.

Model 4 is obtained by simultaneously solving the ratio equation and the equation obtained by equating coefficients of s:-

$$\hat{\tau} = \tau + a_1 - b_1 - T \qquad \ldots\ldots(A100)$$

$$\frac{\hat{\tau}}{T} = R \qquad \text{(see A99)}$$

$$\therefore\ T = \frac{\tau + a_1 - b_1}{1+R} \qquad \ldots\ldots(A101)$$

and

$$\hat{\tau} = R.T \qquad \ldots\ldots(A102)$$

Model 5 is obtained by simultaneously solving the ratio equation and the equation obtained by equating coefficients of s^2:-

$$\frac{\hat{\tau}}{T} = R \qquad \text{(see A99)}$$

$$\frac{\hat{\tau}^2}{2} + T^2 + \hat{\tau}T = b_2 + \frac{\hat{\tau}^2}{2} - b_1\tau + a_1^2 - a_2 - a_1b_1 + a_1\tau = P \qquad \ldots\ldots(A103)$$

$$\therefore \quad T = \left(\frac{P}{\frac{R^2}{2} + 1 + R} \right)^{\frac{1}{2}} \qquad \text{.....(A104)}$$

and

$$\hat{\tau} = RT \qquad \text{(as A102)}$$

Fig.3.38 gives an example of how these model parameters occur for the fourth order system.

$$G(s) = \frac{0.5(1+2.5s+1.5s^2+0.5s^3)e^{-2s}}{1+3.5s+a_2s^2+2s^3+0.5s^4} \qquad \text{.....(A105)}$$

Lines of $\frac{\hat{\tau}}{T} = R$ are shown for various values of a_2 and model parameters obtained using the five different methods are indicated for four values of a_2 ($A \rightarrow a_2 = 3.4$, $B \rightarrow a_2 = 4.5$, $C \rightarrow a_2 = 6.0$, $D \rightarrow a_2 = 8.0$) It will be noticed that only $a_2 = 3.4$ and $a_2 = 4.5$ give a realizable model of type 1, and of these only the latter is adequate. There is an adequacy boundary at $a_4 = 4.0$, which, in fact, satisfies the cancellation conditions and for this value all five models coincide as they are equal to the full system. Model 1 gives adequate models for a_2 between 4 and 4.5. Since models 2 and 3 are dependent on the parameters of model 1 they do not exist if model 1 is not realizable and consequently for $a_2 = 6$ and $a_2 = 8$ only models 4 and 5 exist. For values of $a_2 \geq 9$ model 5 also disappears. Thus only model 4 gives a realizable model for all systems within the stability area, i.e. for all stable systems having $G_c \leq K$.

Models 2, 3, 4 and 5 all give $\hat{G}_c = G_c$ at the expense of the fit at lower frequencies. They are all "adequate" for controller design and a comparison of the controllers obtained using these various models is given in the next section in which the effect of using such first order plus time delay models in controller design is examined.

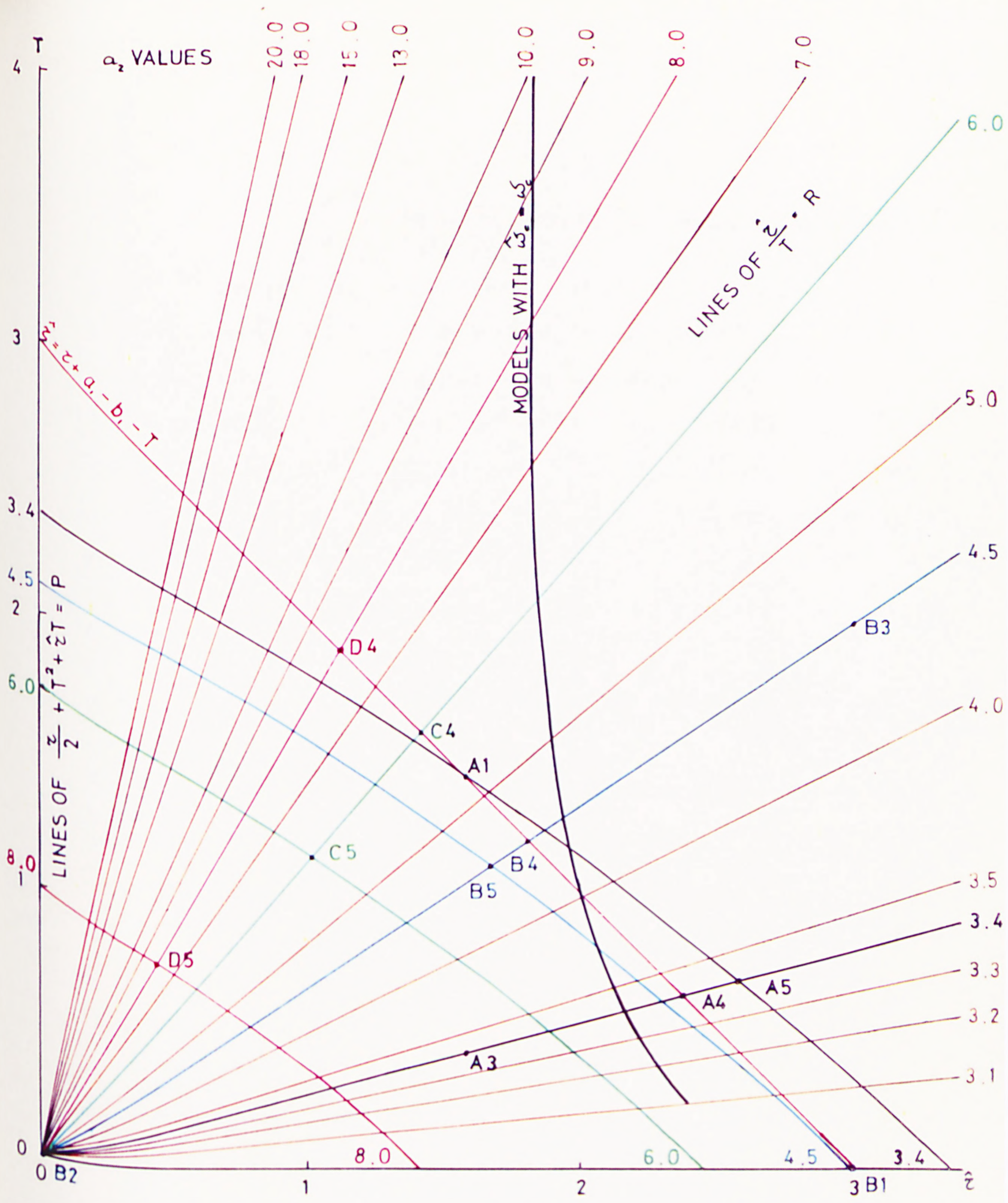


FIGURE 3.38

SIMPLE MODEL PARAMETERS OBTAINED USING METHODS 1 TO 5 FOR FOURTH ORDER SYSTEMS FOR VARIOUS VALUES OF a_2

4. THE USE OF FIRST ORDER PLUS TIME DELAY MODELS FOR CONTROLLER DESIGN

After obtaining a first order plus time delay model of the process under consideration, we can proceed to use this to design a controller for the process. In this section controller parameters are obtained using those models which are less stable or equally stable to the process itself. The effect of the resulting controller on the original system is then examined. Two methods are used for this purpose:-

1) open loop frequency responses of the system together with the controller, and
2) simulations of the closed loop system including the controller.

For this second method some form of error criterion must be used: Three such criteria, in common use, are:-

1) Integral Square Error ISE = $\int_0^T e(t)^2 dt$, which weights larger errors more than smaller ones. Minimising this error criterion in controller design results in a relatively small overshoot, with a somewhat drawn-out settling time.
2) Integral Absolute Error, IAE = $\int_0^T |e(t)| dt$, which weights all error equally. Minimising IAE gives larger initial errors but shorter settling time.
3) Integral Time Absolute Error, ITAE = $\int_0^T |e(t)| t \, dt$, which weights errors occurring later to a greater extent. Minimising ITAE will produce a short total response time with large initial errors.

The ITAE criterion is used throughout this section for the comparison of the response of simulated systems, although the ISE and IAE are also evaluated on occasion.

Two different sets of comparisons are presented. Firstly a number of linear fourth order + time delay models are looked at, secondly a more practical example is examined. This latter is a third order non-linear macroscopic model of a vaporiser which must be linearised before reduction. Controllers obtained using first order plus time delay models are then used in a simulation of the system in order to evaluate their suitability.

4.1 THE EVALUATION OF CONTROLLER PARAMETERS

As previously mentioned, standard expressions exist which give controller parameters from first order plus time delay model parameters. (Full details are given by SMITH (110)) There are two sets of such expressions. The first give parameters for controller that deal with load changes and other disturbances, while the second set give paramters for set point changes. Parameters for both proportional-integral (PI) and proportional-integral-derivative (PID) controllers can be obtained in this way, but in this case only PI controllers are used. These are represented by transfer functions of the form:-

$$K(s) = K_c(1+ \frac{1}{T_i s}) \qquad .....(B1)$$

The parameters K_c and T_i are given by the following expressions:-

Load change parameters

$$K_c = \frac{a}{K} \left(\frac{\tau}{T}\right)^b \qquad .....(B2)$$

$$T_i = \frac{T}{c} \left(\frac{T}{\tau}\right)^d \qquad .....(B3)$$

where a, b, c, d are constants dependent on the criterion which the controller is to satisfy, as follows:-

Criterion	a	b	c	d
Min.ISE	1.305	-0.960	0.492	-0.739
Min.IAE	0.984	-0.986	0.608	-0.707
Min.ITAE	0.859	-0.977	0.674	-0.680

Set point parameters

$$K_c = \frac{a}{K}\left(\frac{\tau}{T}\right)^{b} \qquad \text{.....(B4)}$$

$$T_i = \frac{T}{e - f\left(\frac{\tau}{T}\right)} \qquad \text{.....(B5)}$$

where

Criterion	a	b	e	f
Min.IAE	0.758	-0.861	1.02	0.325
Min.ITAE	0.586	-0.916	1.03	0.165

As shown the value of K_c is a function of $\frac{\tau}{T}$ and is not dependent on the absolute value of τ or T but on the ratio between them. Consequently, if the simple model used for this purpose is one obtained using the system gain when the Nyquist plot crosses the negative real axis, (as suggested in section 3.13) then K_c becomes a function of K and G_c and is the same for all simple models obtained in this way i.e.,

$$K_c = \frac{a}{K}\left[\frac{\tan^{-1}\left(-\sqrt{\left(\frac{K}{G_c}\right)^2 - 1}\right)}{\sqrt{\left(\frac{K}{G_c}\right)^2 - 1}}\right]^{-b} \qquad \text{.....(B6)}$$

Hence controllers obtained from models 2, 3, 4 and 5 (see section 3.13) differ only in their value of T_i.

4.2 CONTROLLERS BASED ON SIMPLE MODELS OBTAINED FROM HIGHER ORDER PLUS TIME DELAY LINEAR MODELS - OPEN LOOP FREQUENCY RESPONSE

To examine the effect of designing controllers for higher order systems using first order plus time delay models a great number of systems were examined. From these, four fourth order plus time delay systems have been selected to demonstrate various points. In each case two different simple models were obtained: the first using the Padé reduction technique, the second by fitting a model through the point where the system frequency response crosses the negative real axis with $\hat{\omega}_c = \omega_c$. As mentioned in section 3.13, fixing $\hat{\omega}_c = \omega_c$ does not always give a very good fit, if ω_c does not apply to the first crossing of the negative real axis. The results for these four systems were, however, obtained before this conclusion was reached. Subsequent checks have, nevertheless, shown that in all these cases the model obtained in this way differs but little from that given by method 4 of section 3.13.

The controller parameters used have, in each case, been calculated using the expressions for a load change controller minimising the ITAE criterion.

The first system under consideration has the transfer function:-

$$G(s) = \frac{0.5\ e^{-2s}(1-0.2s)}{1+s+0.4s^2+0.015^3+0.0034s^4} \qquad .....(B7)$$

Reduction using the Padé technique gives an adequate model:

$$\hat{G}(s) = \frac{0.5\ e^{-2.8s}}{1+0.4s} \qquad .....(B8)$$

while the critical point calculations give:-

$$\hat{G}(s) = \frac{0.5\ e^{-2.7279s}}{1+0.5351s} \qquad .....(B9)$$

Polar plots of the frequency response of the system and the two models are shown in fig.4.1.

The controller obtained from the Padé model is

$$K(s) = 0.267\ (1+ \frac{1}{2.2288s}\ ) \qquad .....(B10)$$

The open loop frequency responses of both the system and the Padé model with this controller are shown in fig.4.2. It should be noted that, at lower frequencies, there is a perfect fit between the system and the model. Consequently at these frequencies the two closed loop systems will give very similar error index values. At higher frequencies, the model deviates from the system but moves to the left of it, indicating that the model is less stable than the system itself.

Fig.4.3 shows similar plots using the critical point model giving

$$K(s) = 0.3499\ (1+ \frac{1}{2.4033s}\ ) \qquad .....(B11)$$

as the controller. Here the two frequency responses do not agree at low frequencies but only have a slight

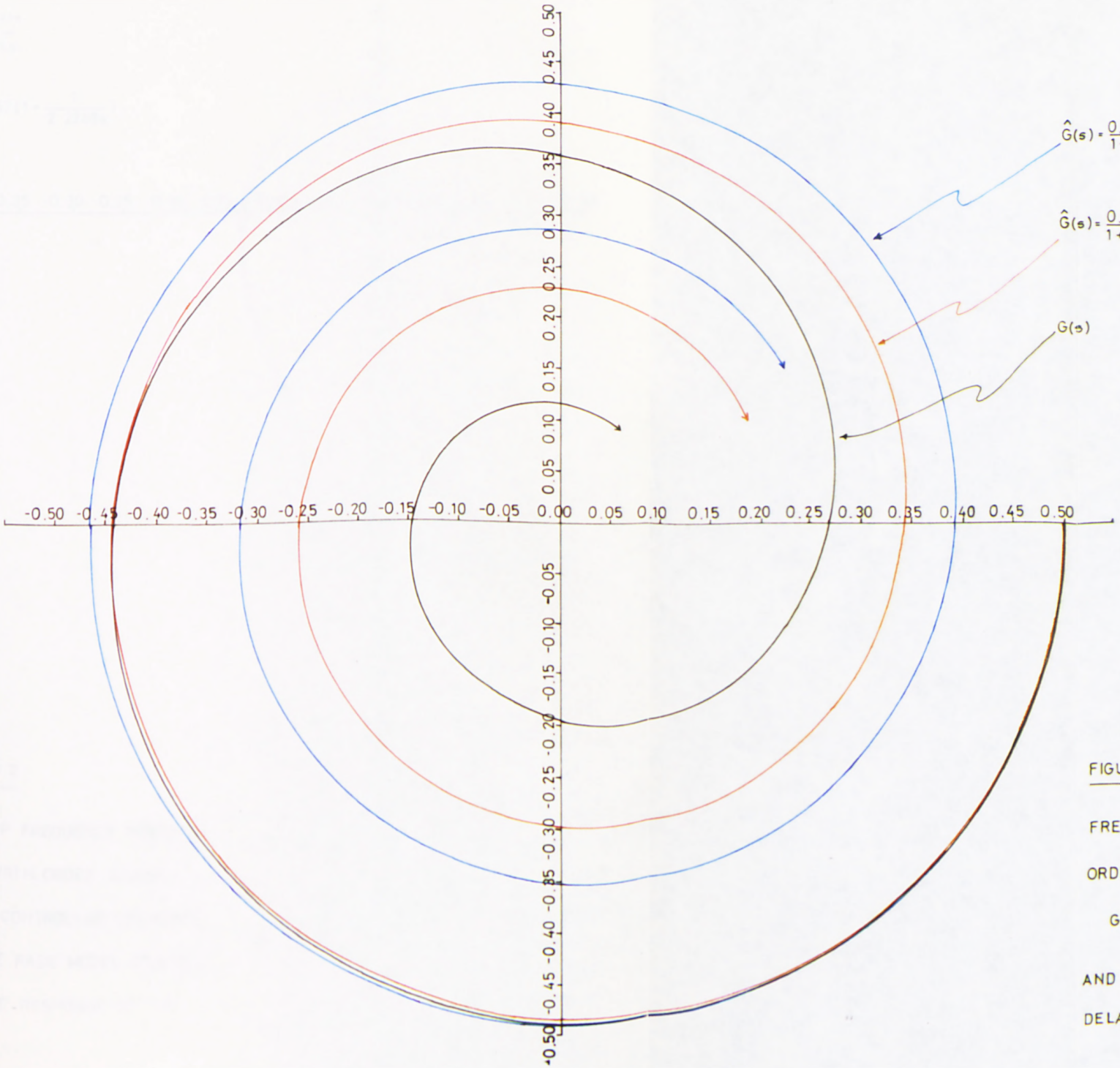


FIGURE 4.1

FREQUENCY RESPONSE OF A FOURTH ORDER SYSTEM

$$G(s) = \frac{0.5e^{-2s}(1-0.2s)}{1+s+0.4s^2+0.01s^3+0.0034s^4}$$

AND TWO FIRST ORDER PLUS TIME DELAY MODELS.

$$G(s) = \frac{0.5e^{-2s}(1-0.2s)}{1+s+0.4s^2+0.01s^3+0.0034s^4}$$

$$\hat{G}(s) = \frac{0.5e^{-2.8s}}{1+0.4s}$$

$$K(s) = 0.2567\left(1+\frac{1}{2.2288s}\right)$$

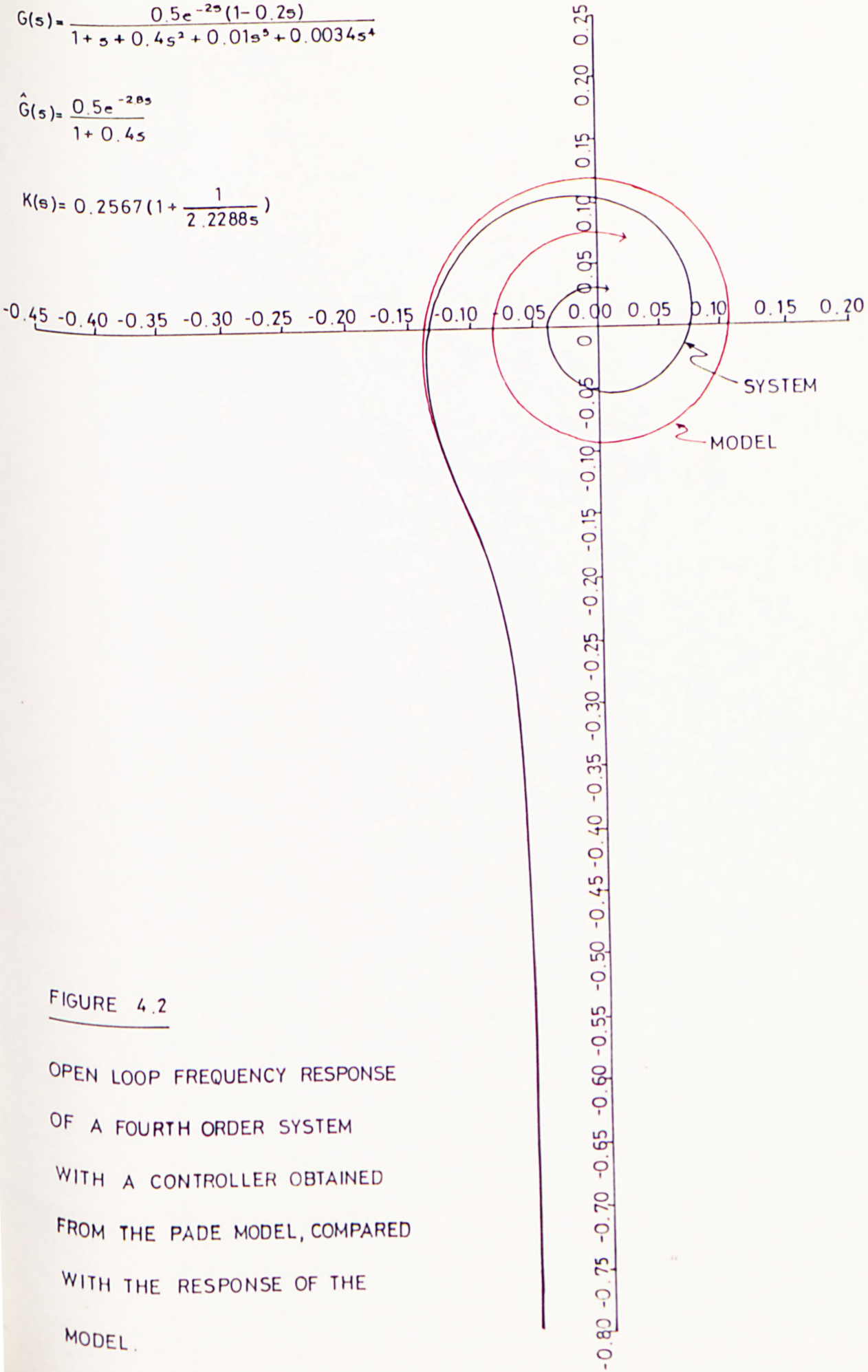


FIGURE 4.2

OPEN LOOP FREQUENCY RESPONSE OF A FOURTH ORDER SYSTEM WITH A CONTROLLER OBTAINED FROM THE PADE MODEL, COMPARED WITH THE RESPONSE OF THE MODEL.

$$G(s) = \frac{0.5e^{-2s}(1-0.2s)}{1+s+0.4s^2+0.01s^3+0.0034s^4}$$

$$\hat{G}(s) = \frac{0.5e^{-3.7279s}}{1+0.5351s}$$

$$K(s) = 0.3499\left(1+\frac{1}{2.4033s}\right)$$

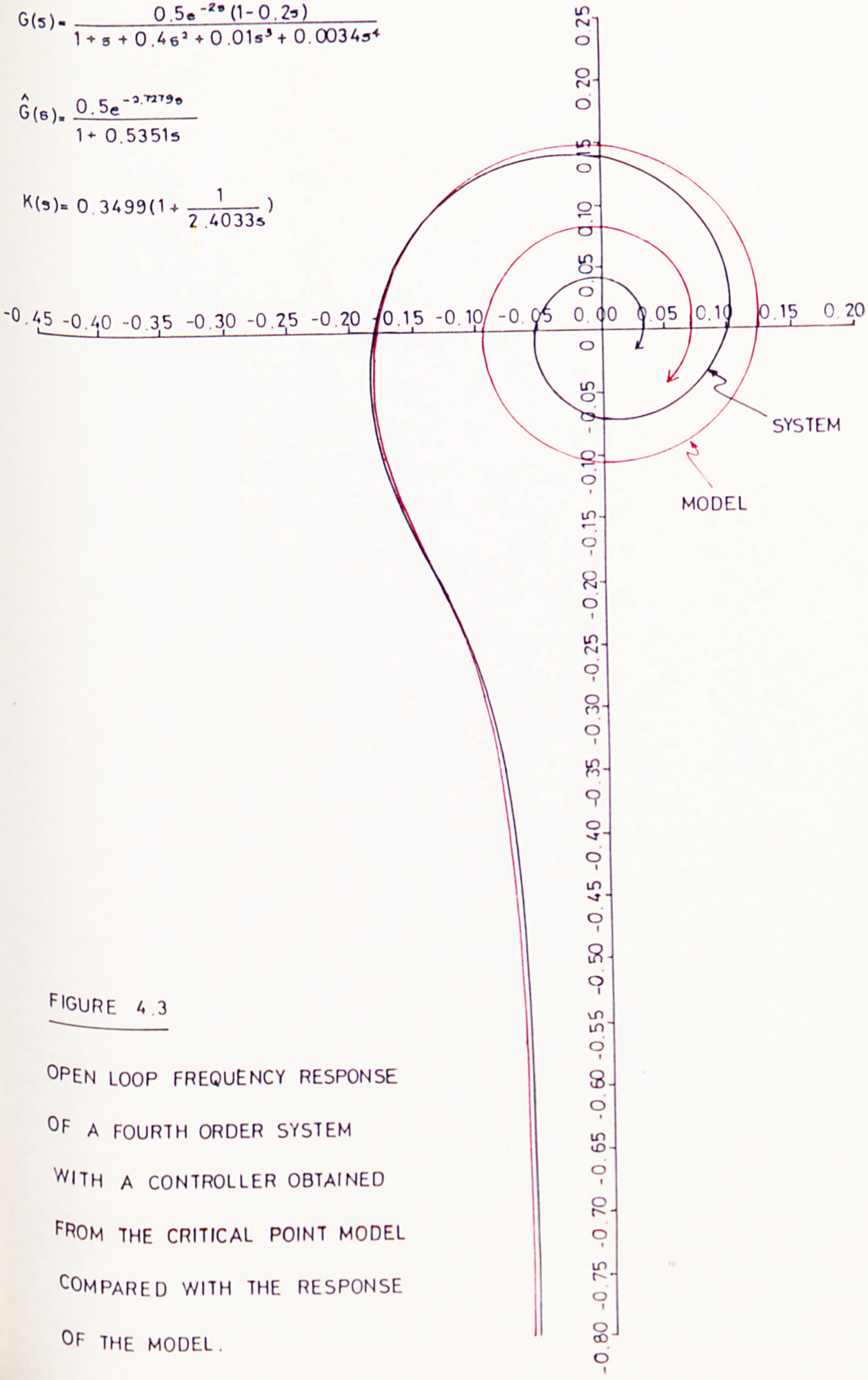


FIGURE 4.3

OPEN LOOP FREQUENCY RESPONSE OF A FOURTH ORDER SYSTEM WITH A CONTROLLER OBTAINED FROM THE CRITICAL POINT MODEL COMPARED WITH THE RESPONSE OF THE MODEL.

difference. Near the real axis the system plot crosses to the left of the model plot, indicating it to be less stable.

From these plots it would appear that the controller obtained from the ADEQUATE Padé model is better than that given by the critical point model. The agreement between system and model at lower frequencies, as stated above, indicates that similar values for error indicies will be obtained at these frequencies. Since the controller has been designed to minimise the error criterion for the model, the error criterion for the system will also be relatively small at these frequencies.

Although the fact that the model plus controller is more stable than the system plus controller has very little significance in this case (especially as the difference is very small) this could lead to instability in those cases where the difference is larger and the gain on the negative real axis is nearer -1. This is especially true for a non-linear system. (See section 4.4).

The next system to be considered is represented by

$$G(s) = \frac{0.5\ e^{-2s}}{1+5+0.1s^{2}+0.1s^{3}+0.033s^{4}} \qquad .....(B12)$$

and gives the Padé model:

$$\hat{G}(s) = \frac{0.5\ e^{-2.1056s}}{1+0.8944s} \qquad .....(B13)$$

This model is, however, inadequate according to our definition, (see fig.4.4) although, again, it gives a very good fit at lower frequencies.

The critical point model is:-

$$\hat{G}(s) = \frac{0.5\ e^{-2.0668s}}{1+0.8015s} \qquad .....(B14)$$

The controllers obtained from these two models so as to minimise the ITAE criterion, are

$$K(s) = 0.7443\ (1+ \frac{1}{2.3753s}\ ) \qquad .....(B15)$$

and

$$K(s) = 0.6809\ (1+ \frac{1}{2.2647s}\ ) \qquad .....(B16)$$

respectively.

As would be expected, the Padé model plus controller fit perfectly to the system plus controller at lower frequencies but at higher frequencies the two plots deviate (see fig.4.5). The frequency response polar plot of the model moves to the right of that for the system when crossing the negative real axis.

In the case of the critical point model the two plots agree in the vicinity of the negative real axis, but there is a larger difference between them at lower frequencies. (see fig.4.6).

Comparison of the frequency plots obtained for the system with each of the two controllers shows that at lower frequencies they are identical.

For this system, an inadequate Padé model gives a controller which could be considered adequate, but as the model has been rejected as inadequate, the critical point model becomes a useful substitute giving

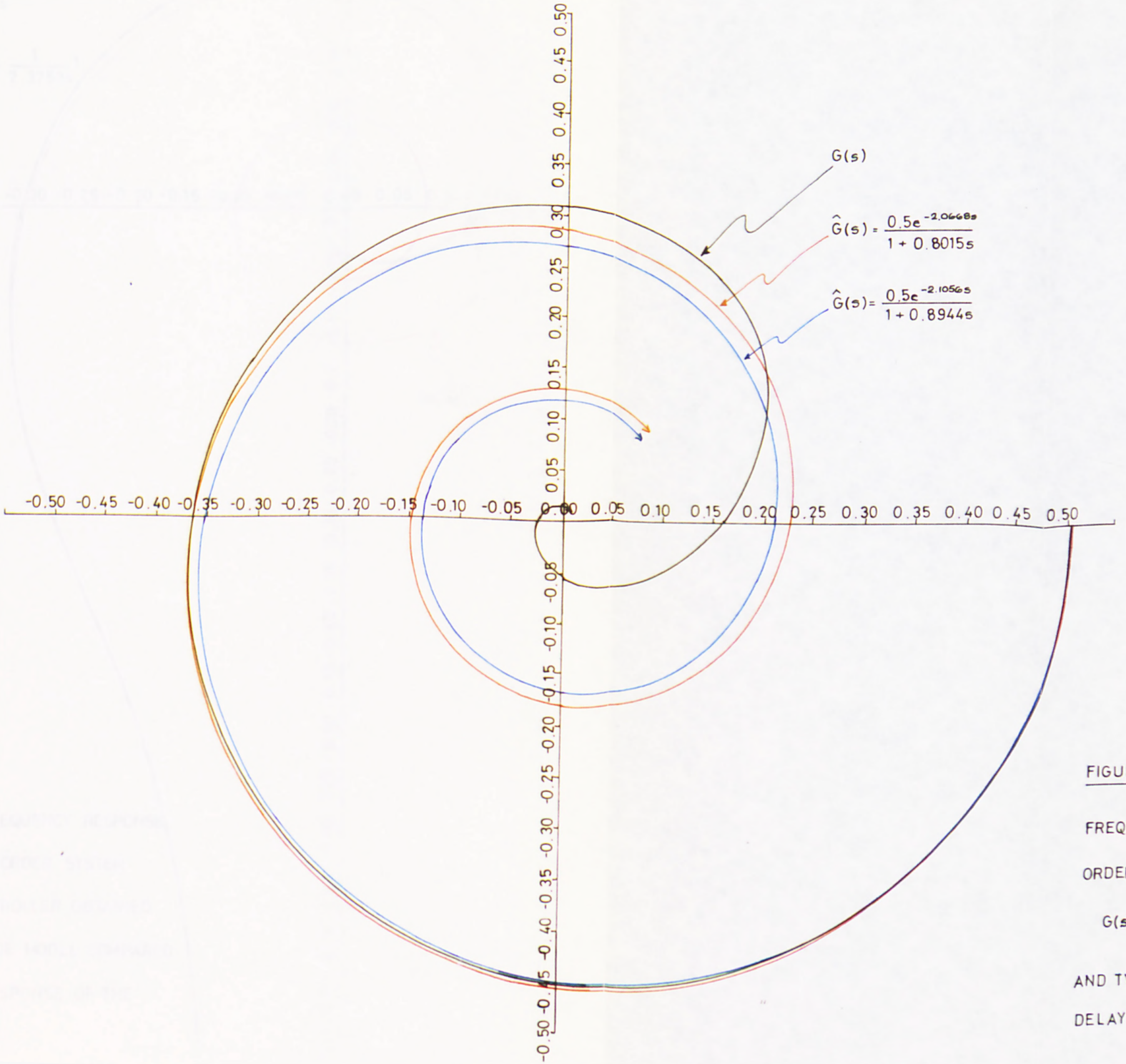


FIGURE 4.4

FREQUENCY RESPONSE OF A FOURTH ORDER SYSTEM

$$G(s) = \frac{0.5e^{-2s}}{1 + s + 0.1s^2 + 0.1s^3 + 0.033s^4}$$

AND TWO FIRST ORDER PLUS TIME DELAY MODELS.

$$G(s) = \frac{0.5^{-2s}}{1 + s + 0.1s^2 + 0.1s^3 + 0.033s^4}$$

$$\hat{G}(s) = \frac{0.5e^{-2.1056s}}{1 + 0.8944s}$$

$$K(s) = 0.7443\left(1 + \frac{1}{2.3753s}\right)$$

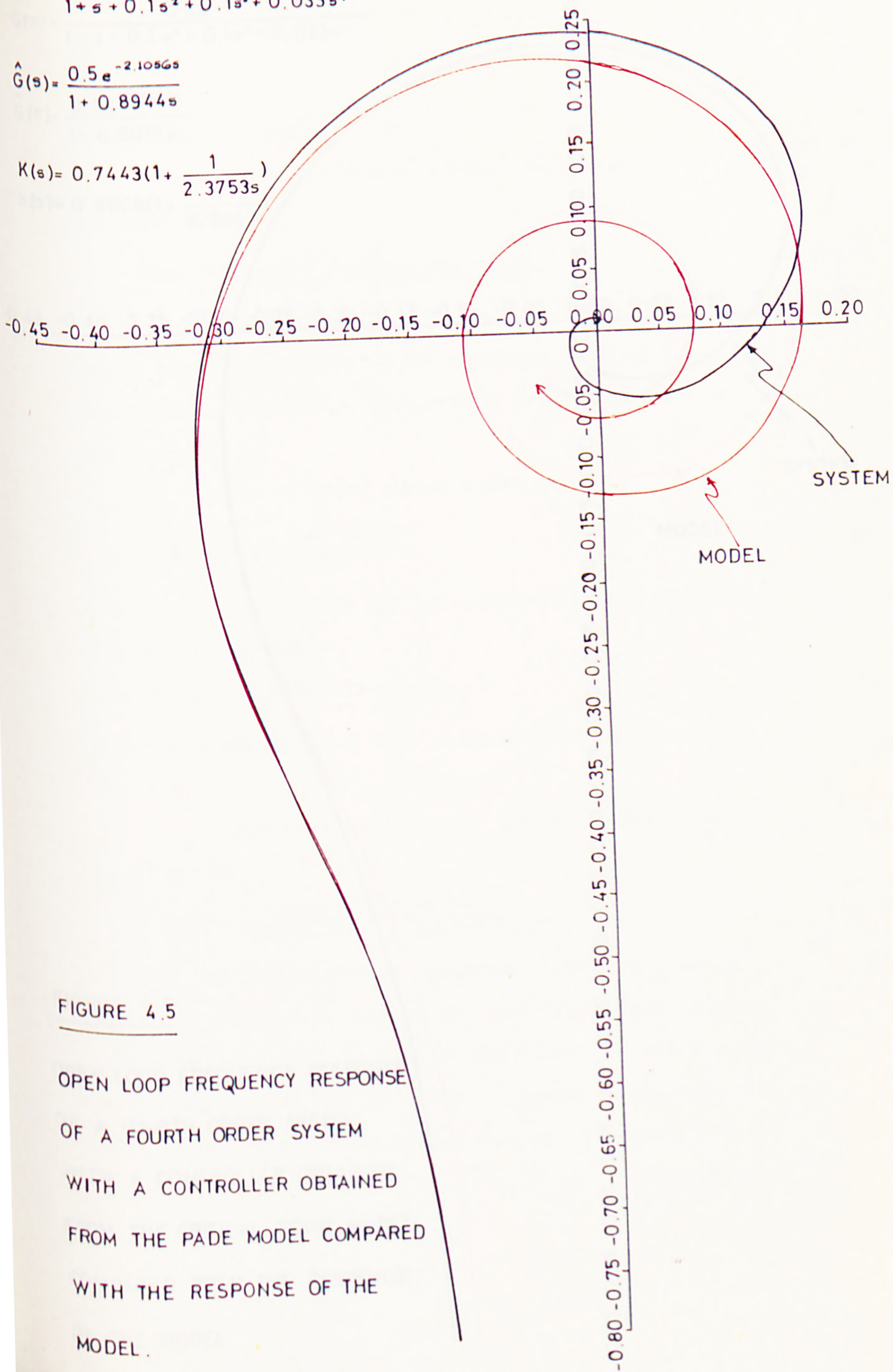


FIGURE 4.5

OPEN LOOP FREQUENCY RESPONSE OF A FOURTH ORDER SYSTEM WITH A CONTROLLER OBTAINED FROM THE PADE MODEL COMPARED WITH THE RESPONSE OF THE MODEL.

$$G(s) = \frac{0.5e^{-2s}}{1 + s + 0.1s^2 + 0.1s^3 + 0.033s^4}$$

$$\hat{G}(s) = \frac{0.5e^{-2.0668s}}{1 + 0.8015s}$$

$$K(s) = 0.6809\left(1 + \frac{1}{2.2647s}\right)$$

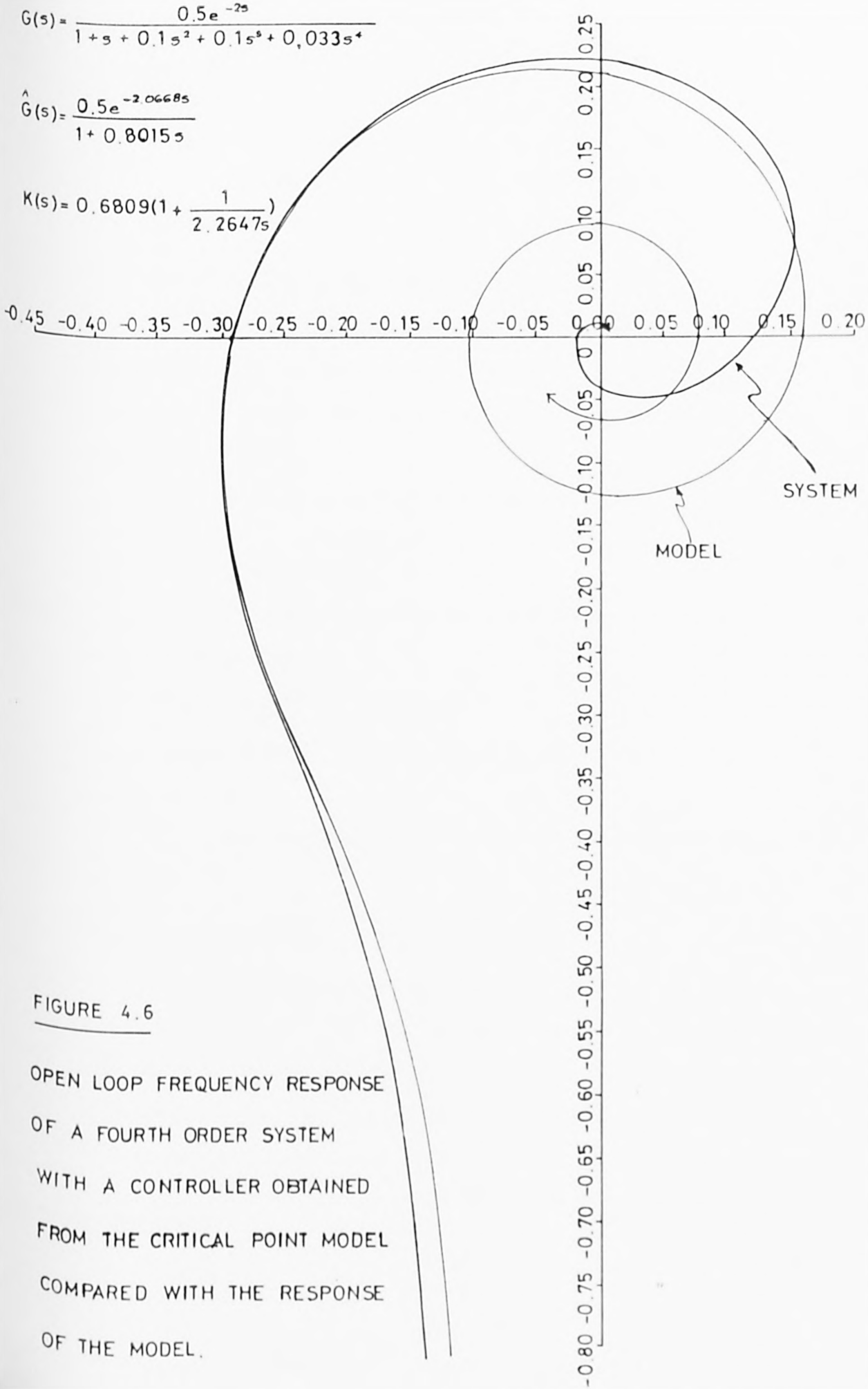


FIGURE 4.6

OPEN LOOP FREQUENCY RESPONSE OF A FOURTH ORDER SYSTEM WITH A CONTROLLER OBTAINED FROM THE CRITICAL POINT MODEL COMPARED WITH THE RESPONSE OF THE MODEL.

a suitable controller.

The third system to be considered is represented by

$$G(s) = \frac{0.5\ e^{-2s}}{1+s+0.5s^2+0.047s^3+0.015s^4} \qquad \ldots\ldots (B17)$$

This gives $\hat{G}(s) = 0.5\ e^{-3s}$ as the Padé model, which would theoretically be adequate, but as $\hat{\tau} = 0$, no meaningful controller parameters are obtainable (see fig.4.7).

The critical point model is

$$\hat{G}(s) = \frac{0.5\ e^{-2.6760s}}{1+0.4246s} \qquad \ldots\ldots (B18)$$

This again can be considered as a useful substitute giving:-

$$K(s) = 0.2844\ (1+\frac{1}{2.2029s}) \qquad \ldots\ldots (B19)$$

which appears to be a very suitable controller (see fig.4.8).

The last of the four systems being examined is given by:-

$$G(s) = \frac{0.5\ e^{-2s}(1+0.8s^2)}{1+s+0.81s^2+0.73s^3+0.01s^4} \qquad \ldots\ldots (B20)$$

The frequency response of this system is shown in fig.4.9. As can be seen, it passes through the origin. This is due to the numerator which contains a function of s^2 but no other s^n terms. Consequently the frequency response passes through the origin when $1+0.8(j\omega)^2 = 0$ or when $\omega = \sqrt{1.25}$. In practice such a

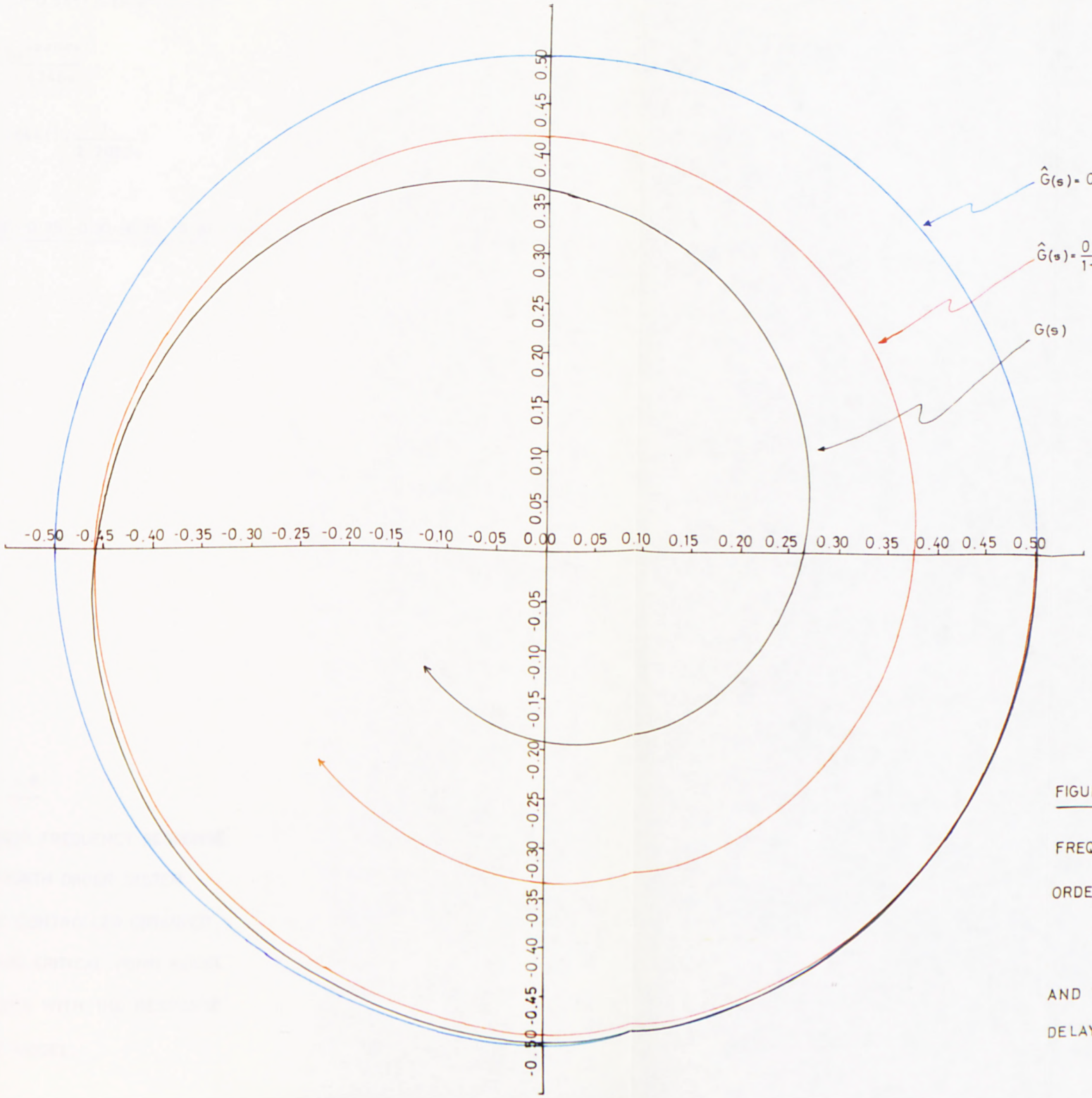


FIGURE 4.7

FREQUENCY RESPONSE OF A FOURTH ORDER SYSTEM

$$G(s) = \frac{0.5e^{-2s}}{1+s+0.5s^2+0.047s^3+0.015s^4}$$

AND TWO FIRST ORDER PLUS TIME DELAY MODELS.

$$G(s) = \frac{0.5e^{-2s}}{1 + s + 0.5s^2 + 0.047s^3 + 0.015s^4}$$

$$\hat{G}(s) = \frac{0.5e^{-2.6760s}}{1 + 0.4246s}$$

$$K(s) = 0.2844\left(1 + \frac{1}{2.2029s}\right)$$

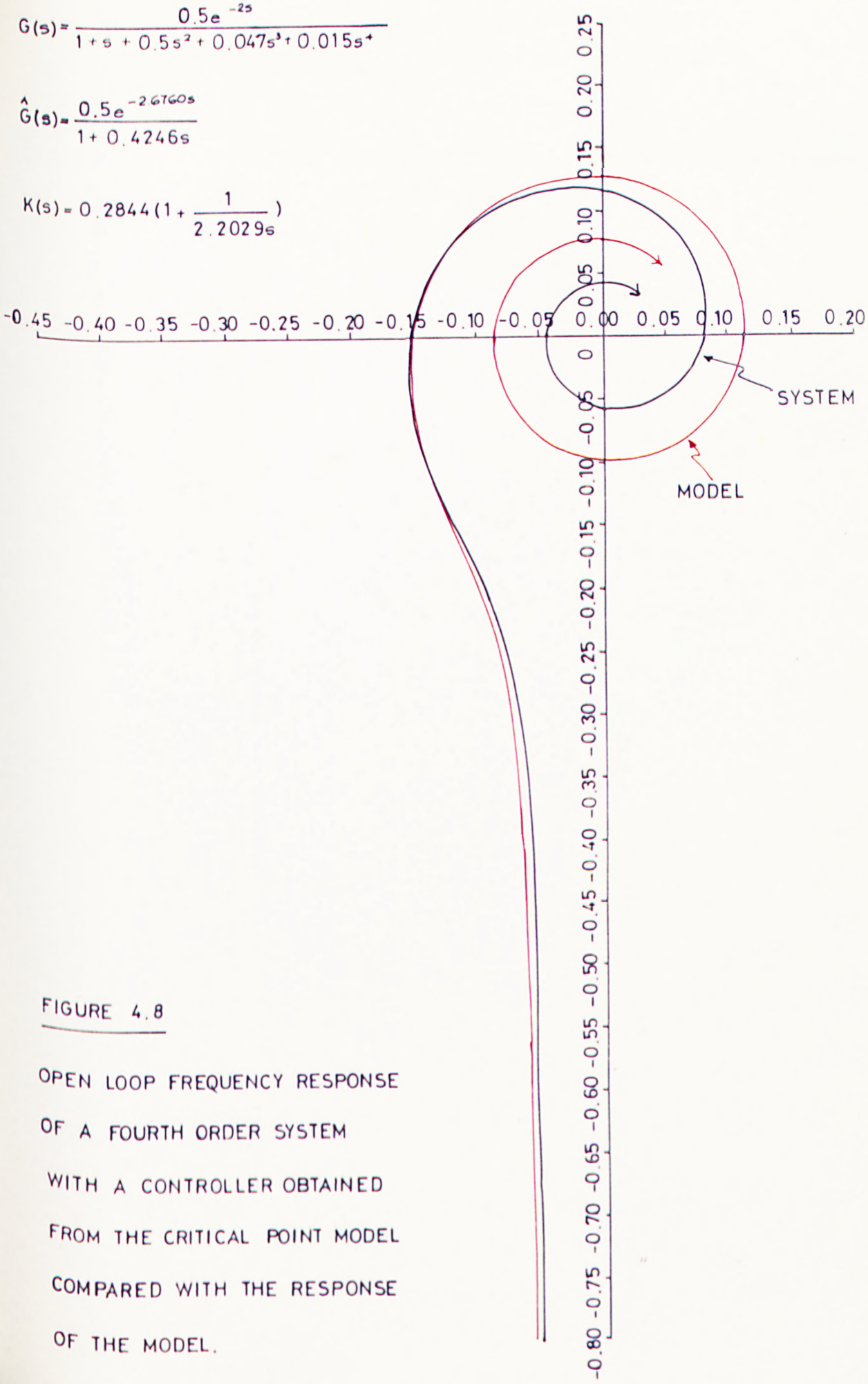


FIGURE 4.8

OPEN LOOP FREQUENCY RESPONSE OF A FOURTH ORDER SYSTEM WITH A CONTROLLER OBTAINED FROM THE CRITICAL POINT MODEL COMPARED WITH THE RESPONSE OF THE MODEL.

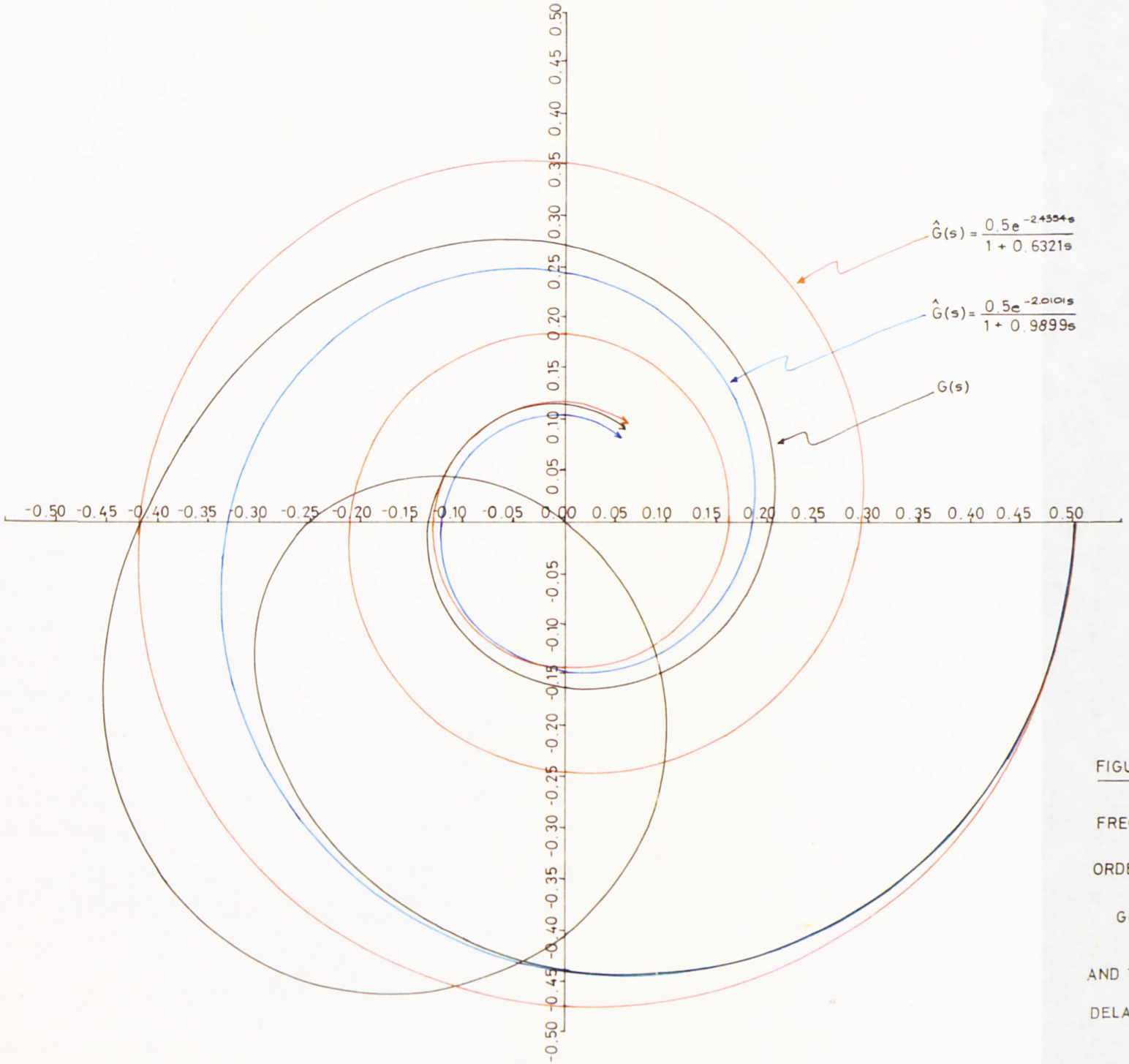


FIGURE 4.9

FREQUENCY RESPONSE OF A FOURTH ORDER SYSTEM

$$G(s) = \frac{0.5e^{-2s}(1 + 0.8s^2)}{1 + s + 0.81s^2 + 0.73s^3 + 0.01s^4}$$

AND TWO FIRST ORDER PLUS TIME DELAY MODELS.

frequency response is rather unlikely since the complete absence of an s term in the numerator is highly improbable. However, when this term is small, a similar frequency response will be given. This will pass close to the origin but not exactly through it.

Reduction of this system using the Padé technique gives:-

$$\hat{G}(s) = \frac{0.5\ e^{-2.0101s}}{1+0.9899s} \qquad .....(B21)$$

This model is plainly inadequate (see fig. 4.9). The controller given by this model is:-

$$K(s) = 0.86\ (1+ \frac{1}{2.3775s}\ ) \qquad .....(B22)$$

Again this model gives a good fit at lower frequencies but at higher frequencies the system and controller give a polar plot which lies far to the left of that given by the model and controller (see fig.4.10) but still lies well to the right of the (-1,0) point indicating that the closed loop system will be stable. For other systems, however, such an inadequate model could lead to instability in the closed loop system.

The critical point model is given by the second crossing of the negative real axis by the system polar frequency response plot. It is

$$\hat{G}(s) = \frac{0.5\ e^{-2.4354s}}{1+0.6321s} \qquad .....(B23)$$

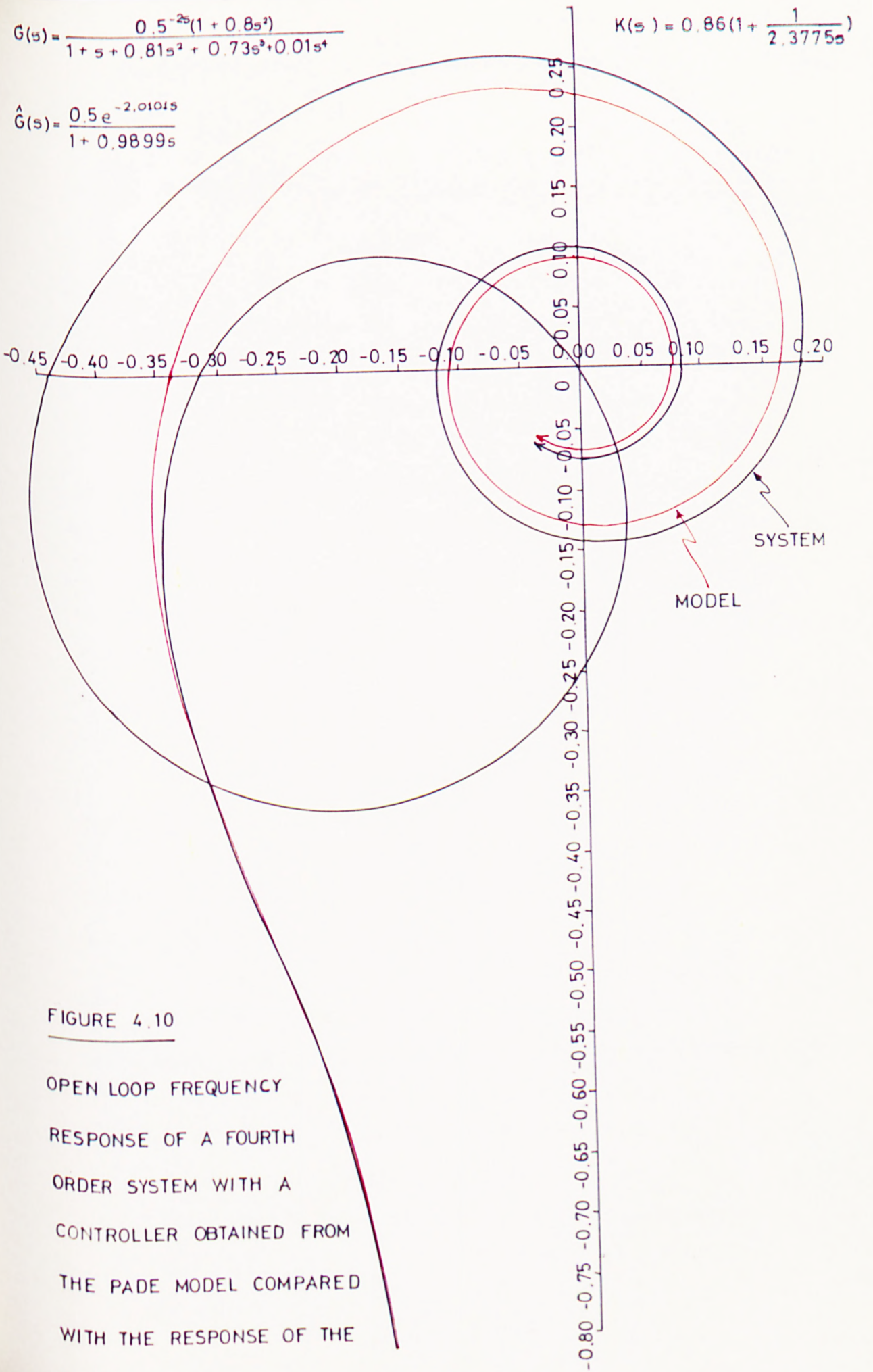


FIGURE 4.10

OPEN LOOP FREQUENCY RESPONSE OF A FOURTH ORDER SYSTEM WITH A CONTROLLER OBTAINED FROM THE PADE MODEL COMPARED WITH THE RESPONSE OF THE MODEL.

and gives the controller

$$K(s) = 6.4599 \quad 1+\frac{1}{2.3467s} \qquad .....(B24)$$

As shown in fig.4.11, this controller still does not guarantee that the model is less stable than the system. This is due to the fact that the original system has a higher gain than the model below the negative real axis. This is reflected on the axis when the controller is added as a result of the phase shift characteristics of the integral time.

It would appear that when the Padé reduction method gives an inadequate model and the system gain decreases as it crosses the negative real axis of the polar plot then the critical point model can also be considered inadequate. A better alternative might be considered taking this larger gain into account by, for example, fitting the model at the point below the negative real axis at which the real part reaches a minimum. Alternatively the point nearest to the -1 point might be considered.

From the foregoing examples it would appear that if the Padé reduction technique gives an"adequate" model then the resulting controller is quite likely to give almost optimal control at lower frequencies due to the good fit that this technique gives at these frequencies. If, however, such a model does not exist, the critical point model will, in most cases, act as an

$$G(s) = \frac{0.5e^{-2s}(1+0.8s^2)}{1+s+0.81s^2+0.73s^3+0.01s^4}$$

$$\hat{G}(s) = \frac{0.5e^{-2.4354s}}{1+0.6321s}$$

$$K(s) = 0.4599\left(1+\frac{1}{2.3467s}\right)$$

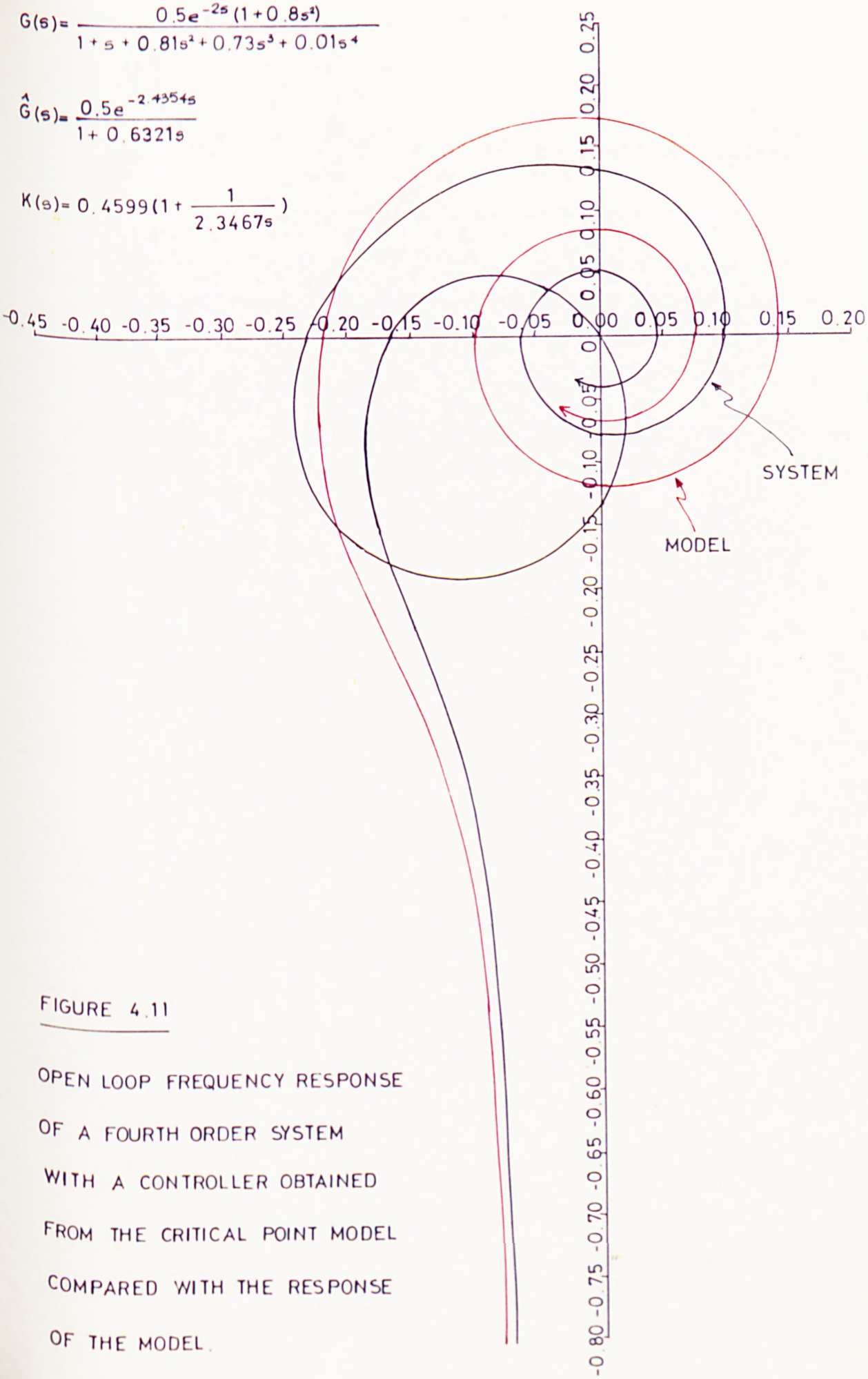


FIGURE 4.11

OPEN LOOP FREQUENCY RESPONSE OF A FOURTH ORDER SYSTEM WITH A CONTROLLER OBTAINED FROM THE CRITICAL POINT MODEL COMPARED WITH THE RESPONSE OF THE MODEL.

acceptable alternative giving suitable controller parameters. The exception is the case mentioned above.

The second example also suggests that maybe our criterion for adequacy is too strict and that an alternative definition should be considered, taking into account the expressions that are to be used for the design of the controller. This would, however, of necessity, be a more complicated criterion.

4.3 CONTROLLERS BASED ON SIMPLE MODELS OBTAINED FROM HIGHER ORDER PLUS TIME DELAY LINEAR MODELS - CLOSED LOOP SIMULATION

It has been shown in the previous section that controllers designed using simple models influence the open loop frequency response of the original system. Another and perhaps clearer way to examine the effect of such a controller is to simulate the closed loop system including the controller. Simulations of this type have been carried out on a large number of systems and a selection of the results is presented in this section.

By the very nature of model reduction many higher order systems will give one and the same reduced model. For models obtained using the Padé technique, the model will be adequate for some of the systems while for others it will be inadequate. A group of such systems is represented by the transfer function:-

$$G(s) = \frac{0.5\ e^{-2s}(1+0.5s^2)}{1+s+0.56s^2+a_3s^3} \qquad \text{.....(B25)}$$

where a_3 is different for each system. The system is open-loop stable if a_3 is between 0 and 0.56. Five of these systems have been examined, with a_3 = 0.05, 0.165, 0.3, 0.4, 0.5. For two of these, (a_3 = 0.165 and a_3 = 0.4) the model is adequate while for the other three it is inadequate. (see fig.3.13).

Each of these systems has been simulated with each of three controllers obtained from the simple model to minimise different error criteria, as has the model

itself. Table 4.1 gives error index values for these simulations.

	Min. ISE Controller	Min. IAE Controller	Min. ITAE Controller
K_c	1.2254	0.9053	0.7959
T_i	3.4122	2.6925	2.3777
	I T A E VALUES		
Model	25.008	25.708	24.965
a_3=0.05	25.005	25.705	24.962
a_3=0.165	25.005	25.705	24.962
a_3=0.3	25.006	25.705	24.962
a_3=0.4	25.133	25.748	25.024
a_3=0.5	26.189	27.552	27.203
	I A E VALUES		
Model	2.784	2.974	2.987
a_3=0.05	2.784	2.974	2.987
a_3=0.165	2.784	2.974	2.987
a_3=0.3	2.784	2.976	2.987
a_3=0.4	2.787	2.976	2.989
a_3=0.5	2.808	3.012	3.035
	I S E VALUES		
Model	0.6377	0.7272	0.7569
a_3=0.05	0.6111	0.7072	0.7384
a_3=0.165	0.6119	0.7083	0.7396
a_3=o.3	0.6187	0.7132	0.7440
a_3=0.4	0.6250	0.7177	0.7481
a_3=0.5	0.6420	0.7295	0.7588

$$G(s) = \frac{0.5\ e^{-2s}(1+0.5s^2)}{1+s+0.56s^2+a_3s^3} \qquad \text{Padé Model } \hat{G}(s) = \frac{0.5\ e^{-2.0619s}}{1+0.9381s}$$

For a_3 = 0.05, 0.3 and 0.5 the model is inadequate; for a_3 = 0.165, 0.4 it is adequate (see fig.3.13).

Table 4.1 Error index values over 75 secs for closed loop simulation of third order systems and their reduced model using controller parameters based on that model, with set point = 0, following a unit disturbance.

Frequency plots for $a_3 = 0.3$ and $a_3 = 0.4$ are shown in figs.4.12 and 4.13 respectively while figs.4.14, 4.15 and 4.16 show plots of the closed loop time response for $a_3 = 0.3$, 0.4 and 0.5 respectively, with the controller calculated to minimise ISE.

If we look at the ITAE and IAE values in table 4.1 we see that for $a_3 = 0.5$, 0.165 and 0.3, they are either identical or nearly identical to those obtained with the model, yet for two of these systems the model has been found to be inadequate. This would seem to support the premise, put forward in the previous sub-section that our criterion for adequacy is too strict.

It may be noticed that the controller parameters calculated to minimise the IAE values do not, in fact, give the lowest IAE values in these examples and that in the results presented these are often given by the ISE controller. This is due to the fact that the expressions used for calculating the controller parameters have been obtained by curve fitting using experimental results and consequently, for a particular simple model inaccuracies are possible.

As has been suggested in section 4.2, in the absence of an adequate Padé model, a model evaluated at the critical point might be used. The set of results presented in table 4.2 serves to compare controllers obtained from such models with controllers obtained from an adequate Padé model. The system considered is

$$G(s) = \frac{0.5\ e^{-3s}(1+2.5s+1.5s^2+0.5s^3)}{1+3.5s+3s^2+2s^3+0.5s^4} \qquad .....(B26)$$

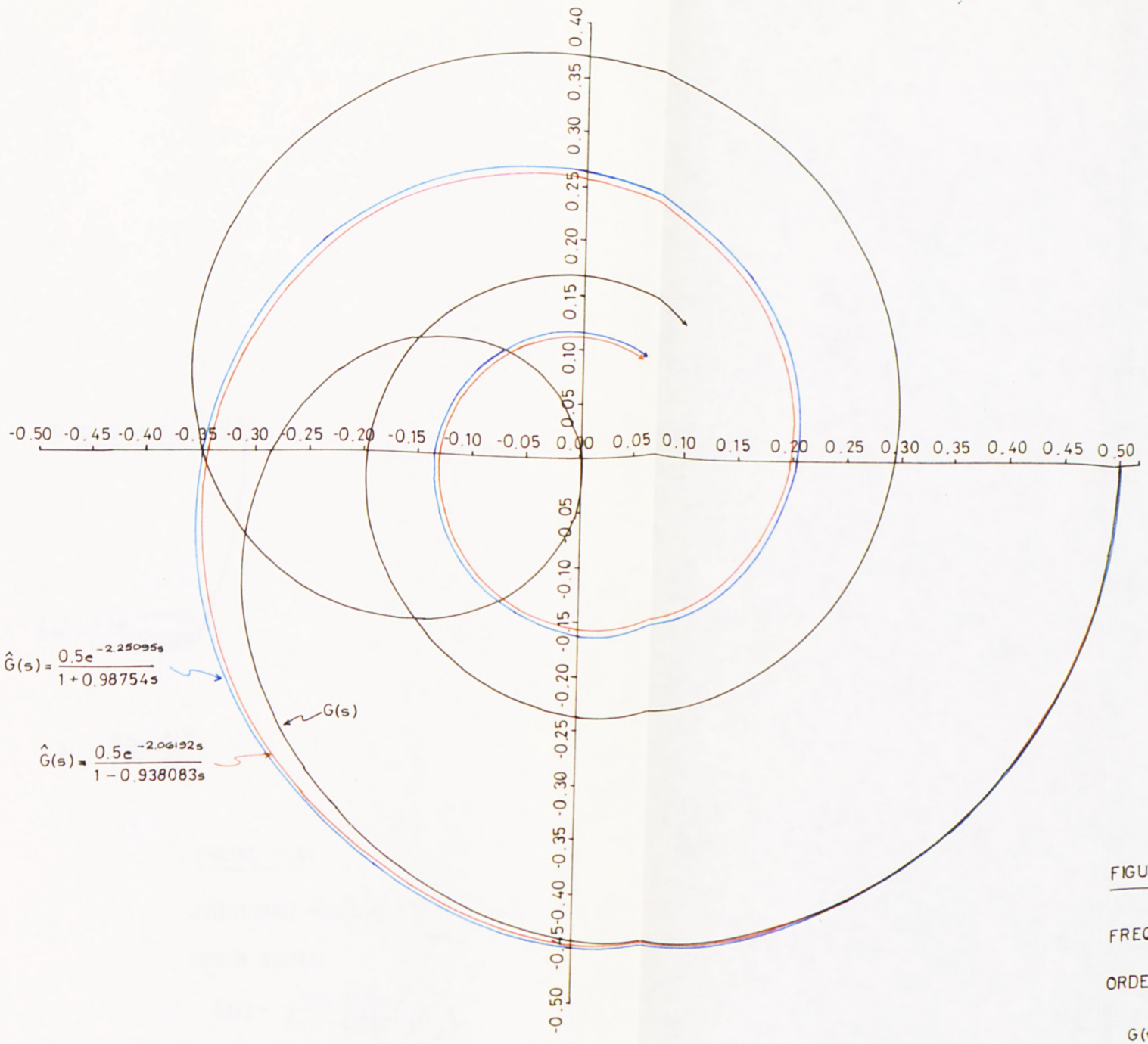


FIGURE 4.12

FREQUENCY RESPONSE OF A THIRD ORDER SYSTEM

$$G(s) = \frac{0.5e^{-2s}(1+0.5s^2)}{1+s+0.56s^2+0.30s^3}$$

AND ITS TWO MODELS.

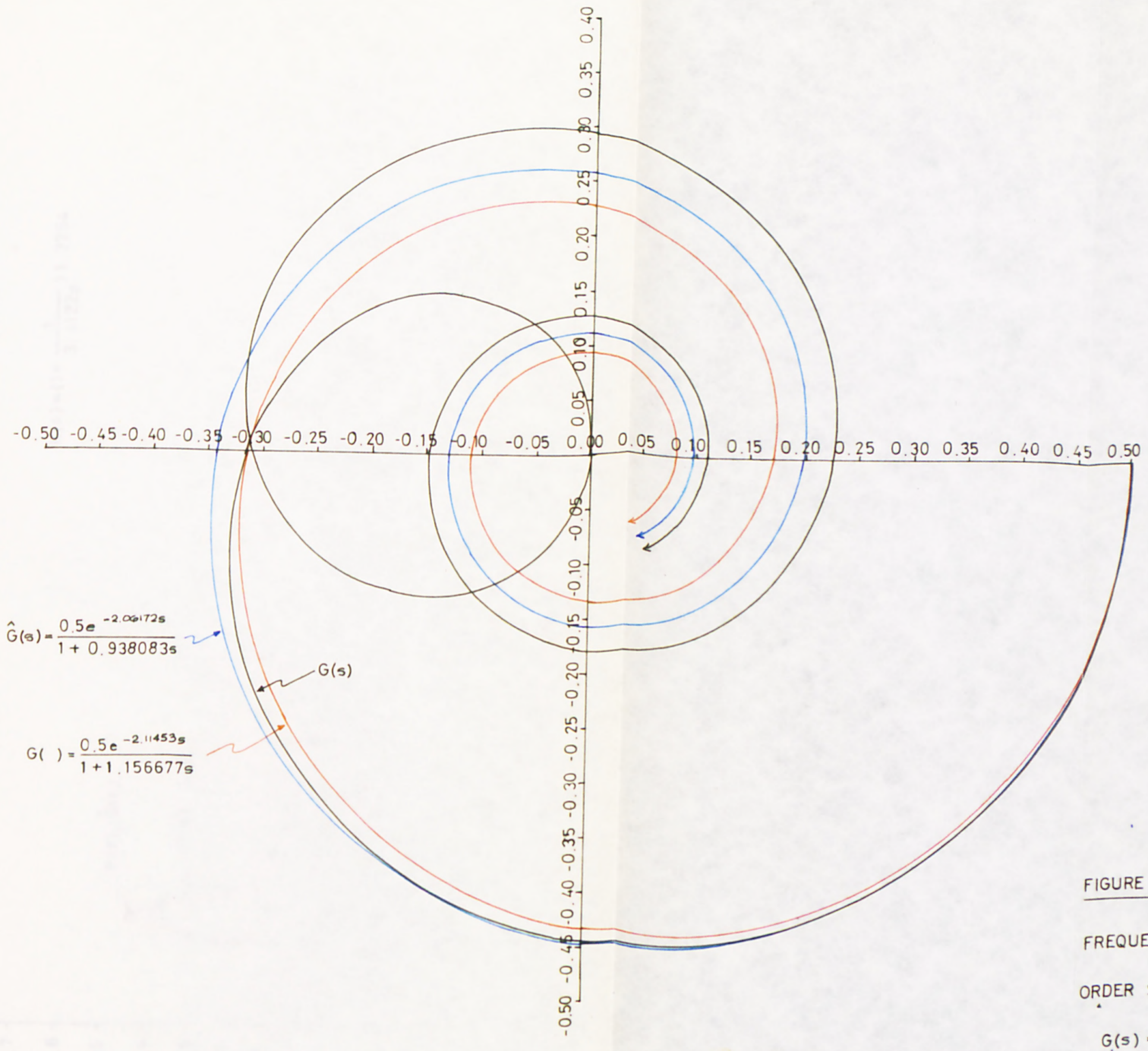


FIGURE 4.13

FREQUENCY RESPONSE OF A THIRD ORDER SYSTEM

$$G(s) = \frac{0.5e^{-2s}(1 + 0.5s^2)}{1 + s + 0.56s^2 + 0.40s^3}$$

AND ITS TWO MODELS.

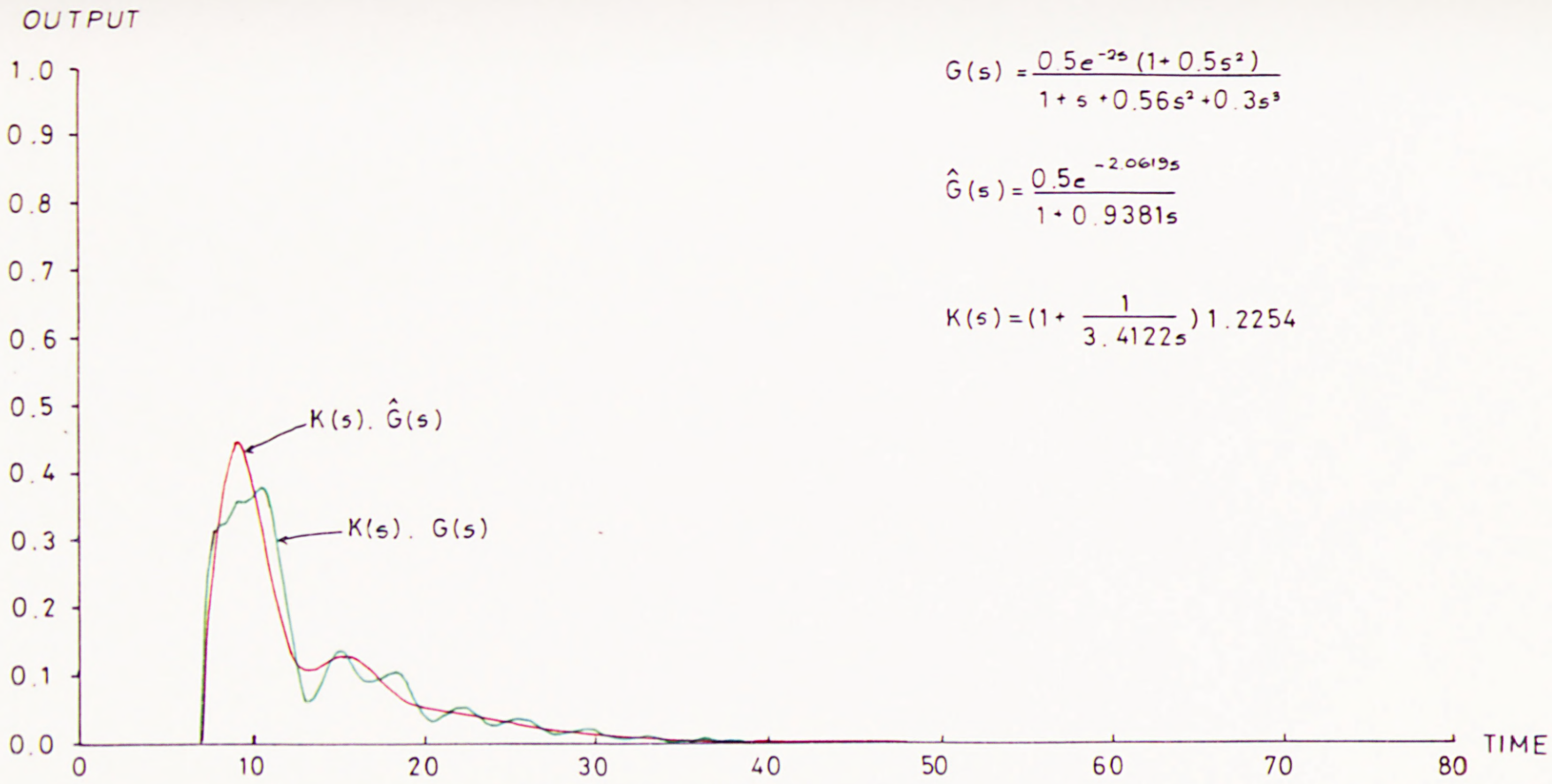


FIGURE 4.14

CLOSED LOOP RESPONSE TO A UNIT DISTURBANCE OF A THIRD ORDER SYSTEM (G(s)), AND ITS PADÉ REDUCED MODEL (Ĝ(s)) USING CONTROLLER PARAMETERS BASED ON THAT MODEL. DESIRED OUTPUT = 0.

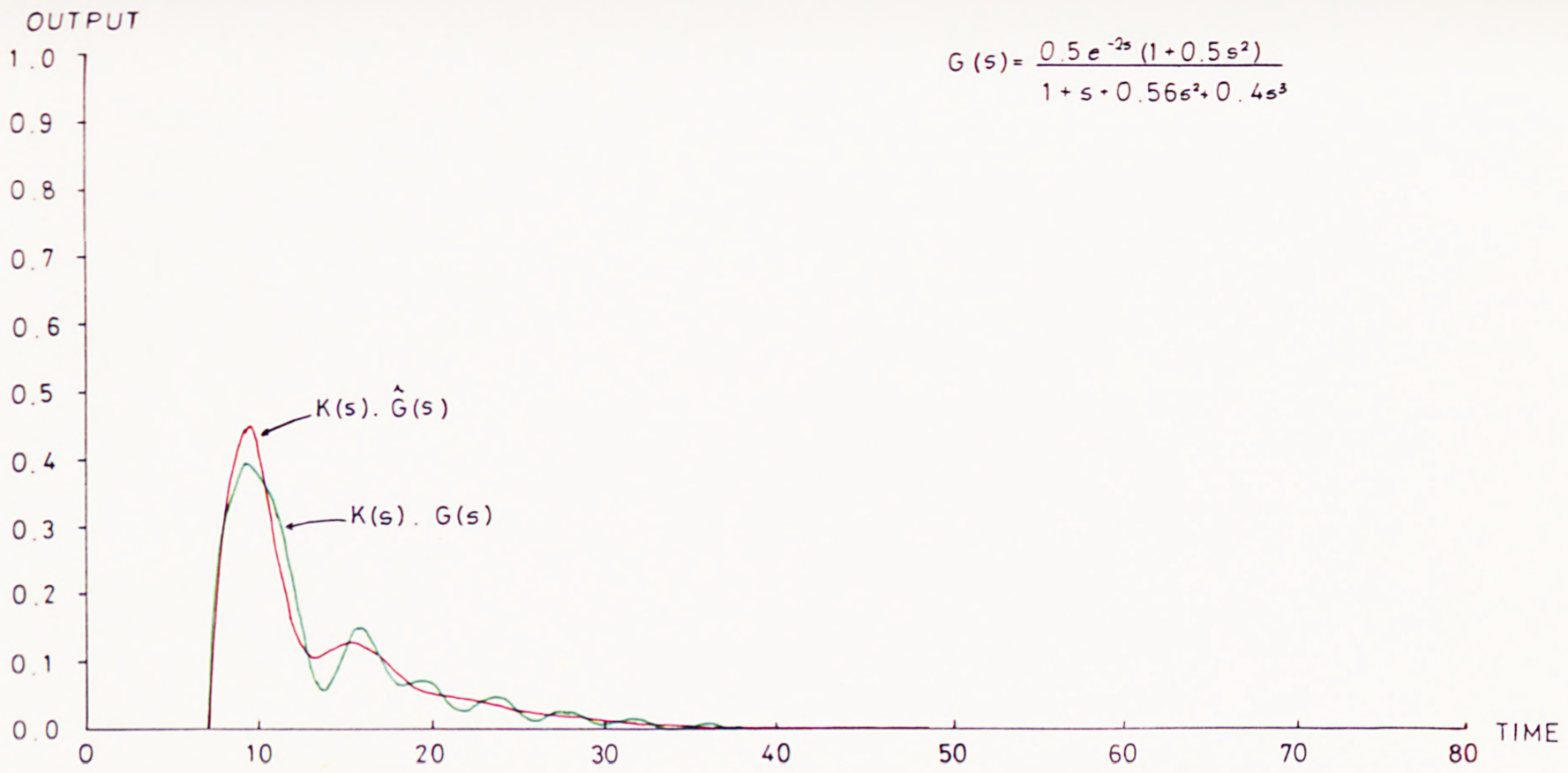


FIGURE 4.15

CLOSED LOOP RESPONSE TO A UNIT DISTURBANCE OF A THIRD ORDER SYSTEM (G(s)), AND ITS PADÉ REDUCED MODEL ($\hat{G}(s)$) USING CONTROLLER PARAMETERS BASED ON THAT MODEL. (MODEL AND CONTROLLER AS IN FIGURE 4.14)

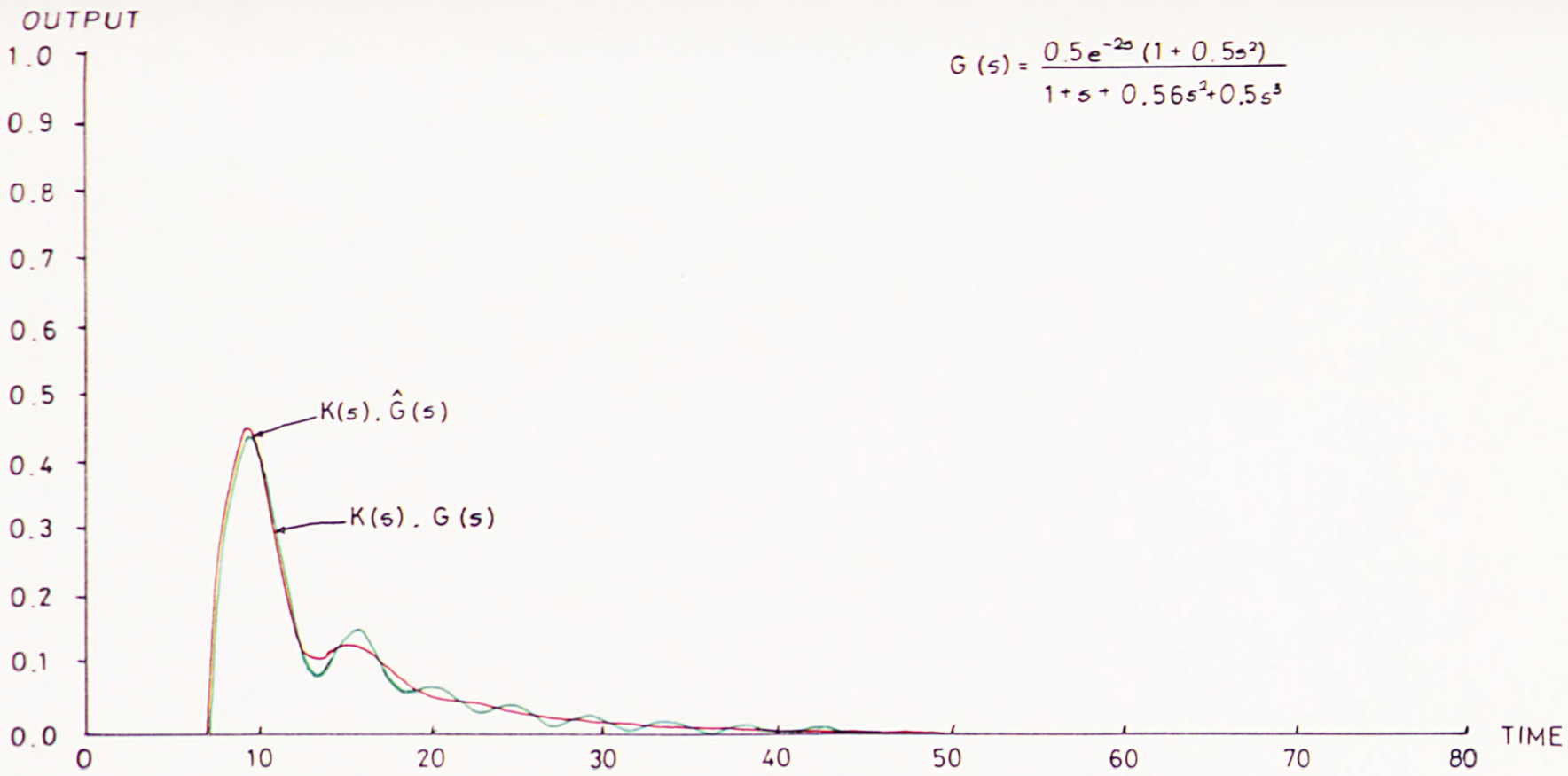


FIGURE 4.16

CLOSED LOOP RESPONSE TO A UNIT DISTURBANCE OF A THIRD ORDER SYSTEM (G(s)), AND ITS PADÉ REDUCED MODEL ($\hat{G}(s)$) USING CONTROLLER PARAMETERS BASED ON THAT MODEL, (MODEL AND CONTROLLER AS IN FIGURE 4.14)

	ITAE Values	IAE Values	ISE Values
Pade Model	T=1.7321	$\hat{\tau}$=2.2680	
Min.ISE Cont.			
K_c=2.0149) (P	319.91	10.522	2.345
T_i=4.2965) (M	18.26	2.135	0.464
Min.IAE Cont.			
K_c=1.5087) (P	34.82	3.101	0.842
T_i=3.4469) (M	18.33	2.288	0.528
Min.ITAE Cont.			
K_c=1.3202) (P	22.82	2.595	0.798
T_i=3.0868) (M	18.16	2.341	0.563
Critical Point Model	$\hat{\omega}_c=1.1w_c$	T=0.3760	$\hat{\tau}$=2.8347
Min.ISE Cont.			
K_c=0.3753) (P	175.92	8.836	2.240
T_i=3.4005) (M	173.22	8.829	2.233
Min.IAE Cont.			
K_c=0.2685) (P	188.59	9.363	2.462
T_i=2.5795) (M	185.85	9.352	2.455
Min.ITAE Cont.			
K_c=0.2387) (P	175.66	9.049	2.413
T_i=2.2034) (M	172.92	9.041	2.406
Critical Point Model	$\hat{\omega}_c=0.9w_c$	T=0.4595	$\hat{\tau}$=3.4646
Min.ISE Cont.			
K_c=0.3753) (P	236.75	10.479	2.652
T_i=4.1562) (M	236.16	10.473	2.723
Min.IAE Cont.			
K_c=0.2685) (P	253.26	11.094	2.919
T_i=3.1527) (M	252.60	11.087	2.993
Min.ITAE Cont.			
K_c=0.2387) (P	238.65	10.768	2.860
T_i=2.6930) (M	238.01	10.711	2.935

P - Plant $G(s) = \dfrac{0.5\ e^{-3s}(1+2.5s+1.5s^2+0.5s^3)}{1.3.5s+3s^2+2s^3+0.5s^4}$

M - Model $\hat{G}(s) = \dfrac{0.5\ e^{-\tau s}}{1+Ts}$

Table 4.2 Error index values over 75 secs for closed loop simulation of fourth order system and 3 reduced models using controller parameters based on these models, with set point =0, following a unit disturbance.

Reduced models of the system have been obtained using the Padé technique and by fitting through the critical point with $\hat{\omega}_c = 1.1\omega_c$ and $\hat{\omega}_c = 0.9\omega_c$. Three sets of controller parameters were then calculated from each of these models. The system was subsequently simulated with each of these models, resulting in the error index values given in table 4.2. Also given are values obtained by simulating each reduced model with the controllers derived from it. From these results, it can be seen that, with the exception of the controller designed to minimise ISE obtained from the Padé model, all the controllers, when simulated with the higher order system give error index values similar to those given by the model from which they are designed. This indicates that these controllers give results with the system which are as good as those given with the simple model. It should be noted that the best control is obtained using the ITAE and IAE controllers based on the Padé model. The results obtained from controllers based on the critical point model are, however, almost as good and are, in most cases, better than the results obtained from the Padé ISE controller.

Fitting a model through the critical point establishes the ratio $\frac{\hat{\tau}}{T}$ and hence the controller gain, as obtained using expressions such as those given in section 4.1 and used here. The exact values of $\hat{\tau}$ and T and, consequently, the integral time of the controller, depends on the frequency at which the reduced model

frequency plot passes through the critical point. This could be set directly as, for example, a ratio of ω_c (the frequency at which the system frequency response passes through this point) or alternatively it could be set so that $\hat{\tau}$ and T satisfy some other function as suggested in section 3.13. A series of simulations has been carried out to discover the effect of varying $\hat{\omega}_c$ on the final closed loop system.

The system used for this is given by:-

$$G(s) = \frac{0.5\ e^{-2s}(1+2.5s+1.5s^2+0.5s^3)}{1+3.5s+4.5s^2+2s^3+0.5s^4} \qquad .....(B27)$$

This system gives a Padé model, $\hat{G}(s) = 0.5\ e^{-3s}$, which does not produce a usable controller. Various models were obtained by fitting through the critical point and controllers based on these models were then used to simulate the closed loop system. ITAE values were obtained for these simulations and these, together with the relevant parameters, are given in table 4.3. Plots of the time response of these closed loop systems are given in figs. 4.17 to 4.26.

The best time response was obtained using the controller based on the model calculated with $\hat{\omega}_c = 1.3\omega_c$, but this is just a feature of this particular example. As mentioned earlier, when the critical point does not occur on the first crossing of the negative real axis but on a later one, ω_c is relatively high. Under these circumstances $\hat{\omega}_c$ will be much lower than ω_c. For this

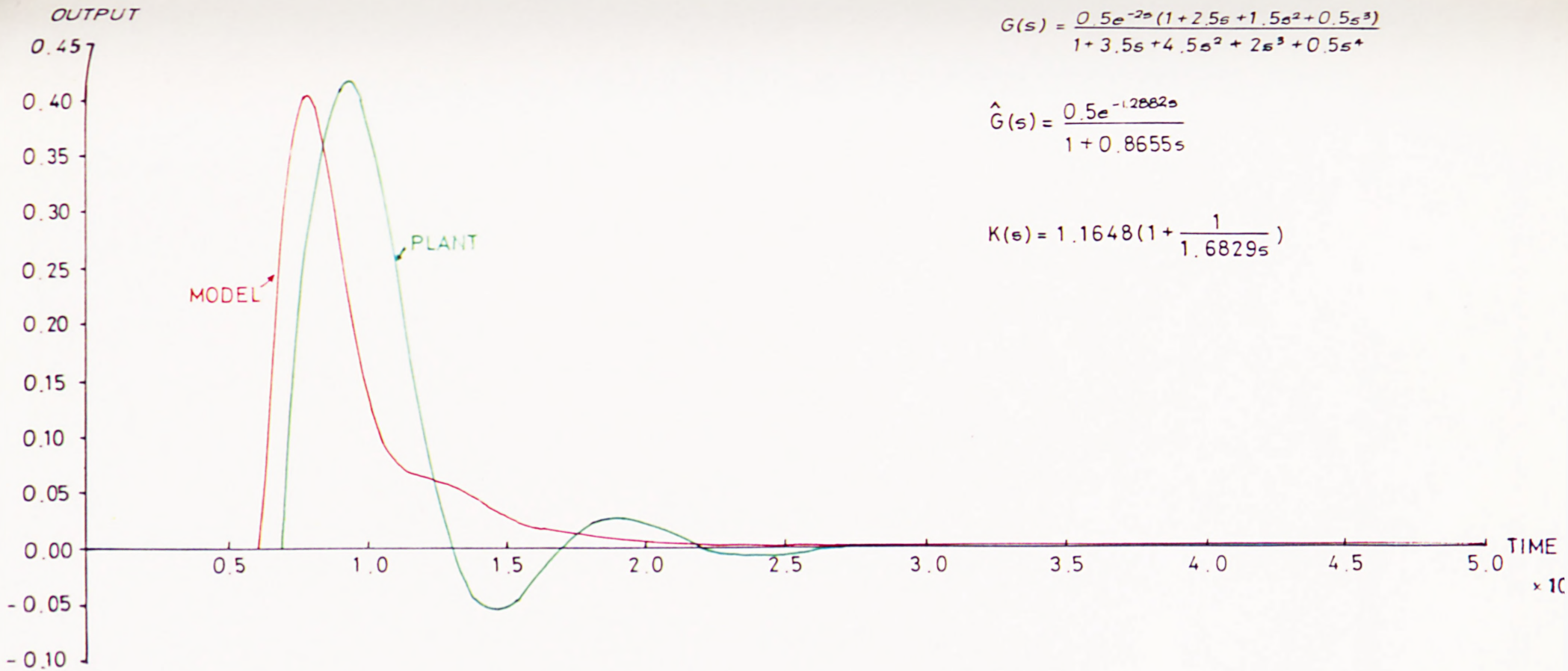


FIGURE 4.17

TIME RESPONSE OF A CLOSED LOOP SYSTEM CONSISTING OF A FOURTH ORDER PLANT AND A MIN. ITAE CONTROLLER DESIGNED USING A CRITICAL POINT MODEL FITTED AT $\hat{\omega}_c = 1.5\omega_c$ THE CLOSED LOOP SYSTEM INCLUDING THE MODEL IS ALSO SHOWN.

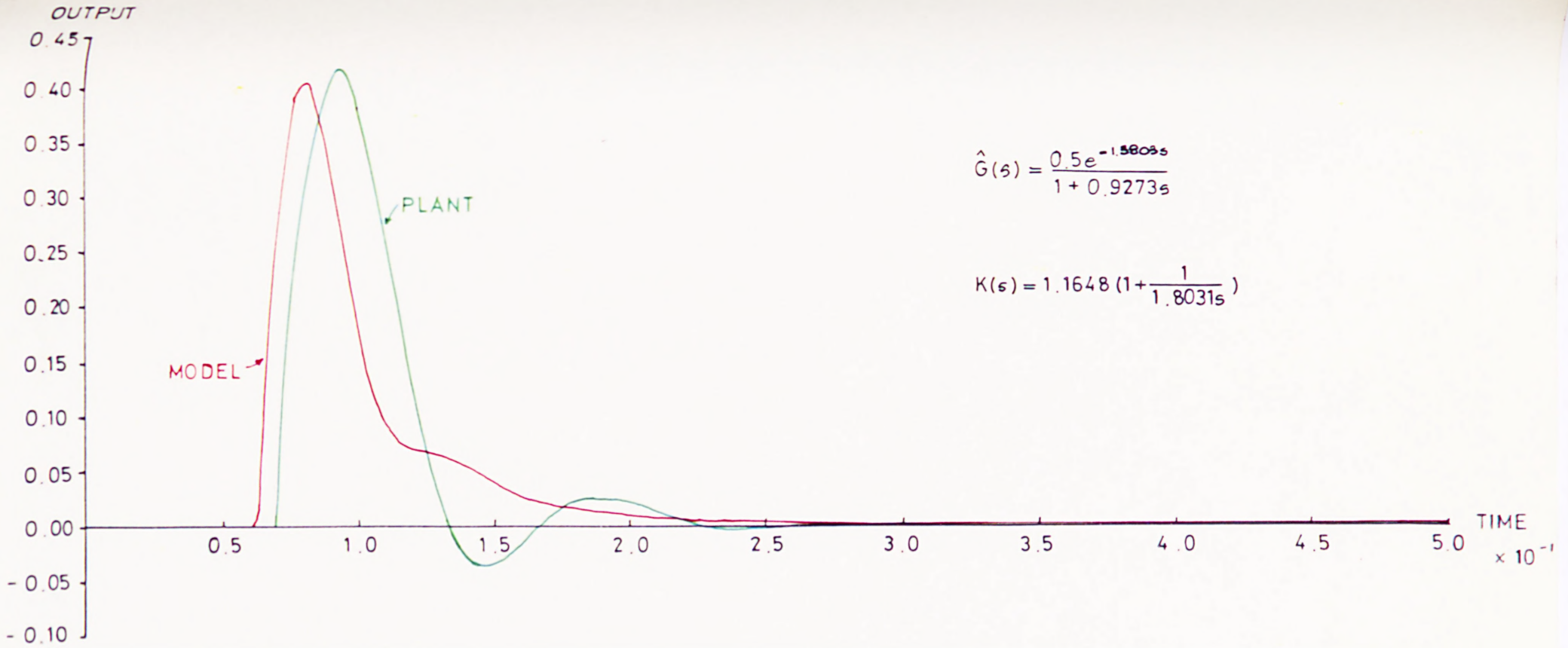


FIGURE 4.18

AS FIGURE 4.17 BUT FITTED AT $\hat{\omega}_c = 1.4\omega_c$ WITH VALUES FOR $\hat{G}(s)$ AND $K(s)$ AS SHOWN ABOVE.

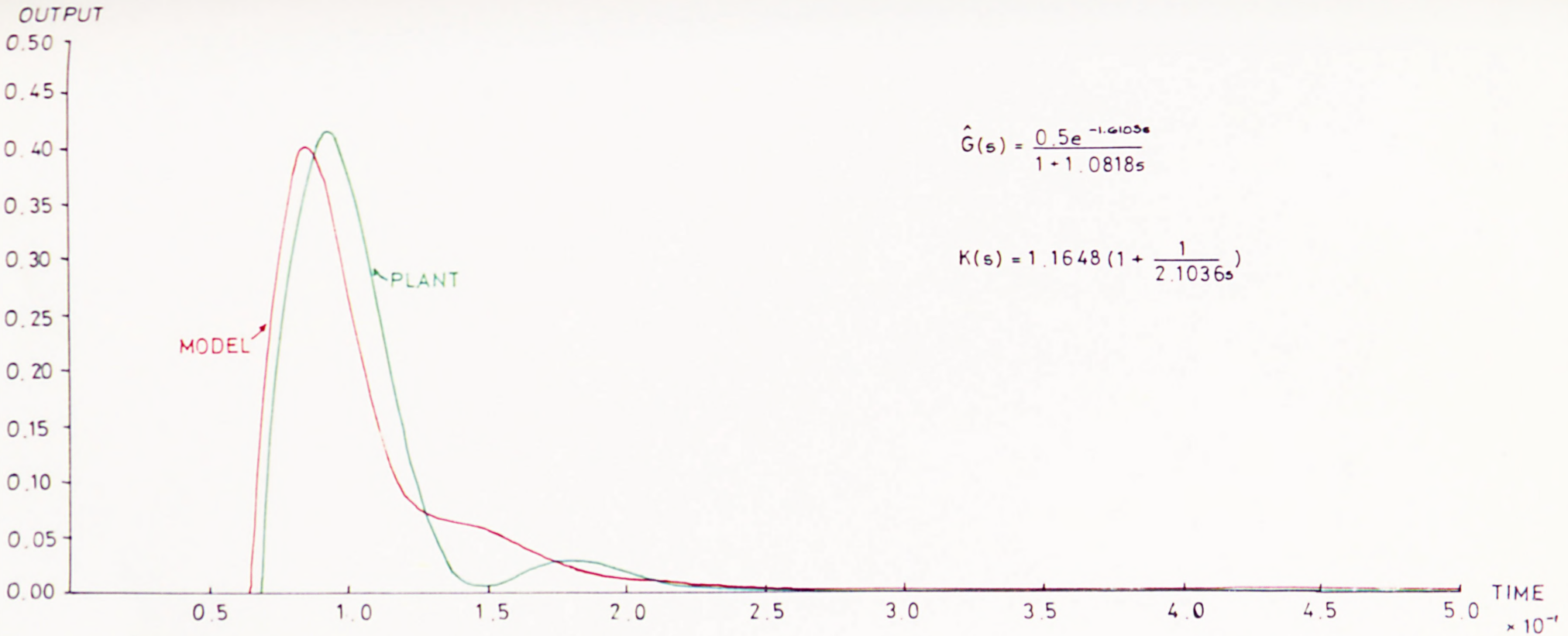


FIGURE 4.19

AS FIGURE 4.17 BUT FITTED AT $\hat{\omega}_c = 1.2\omega_c$ WITH VALUES FOR $\hat{G}(s)$ AND $K(s)$ AS SHOWN ABOVE.

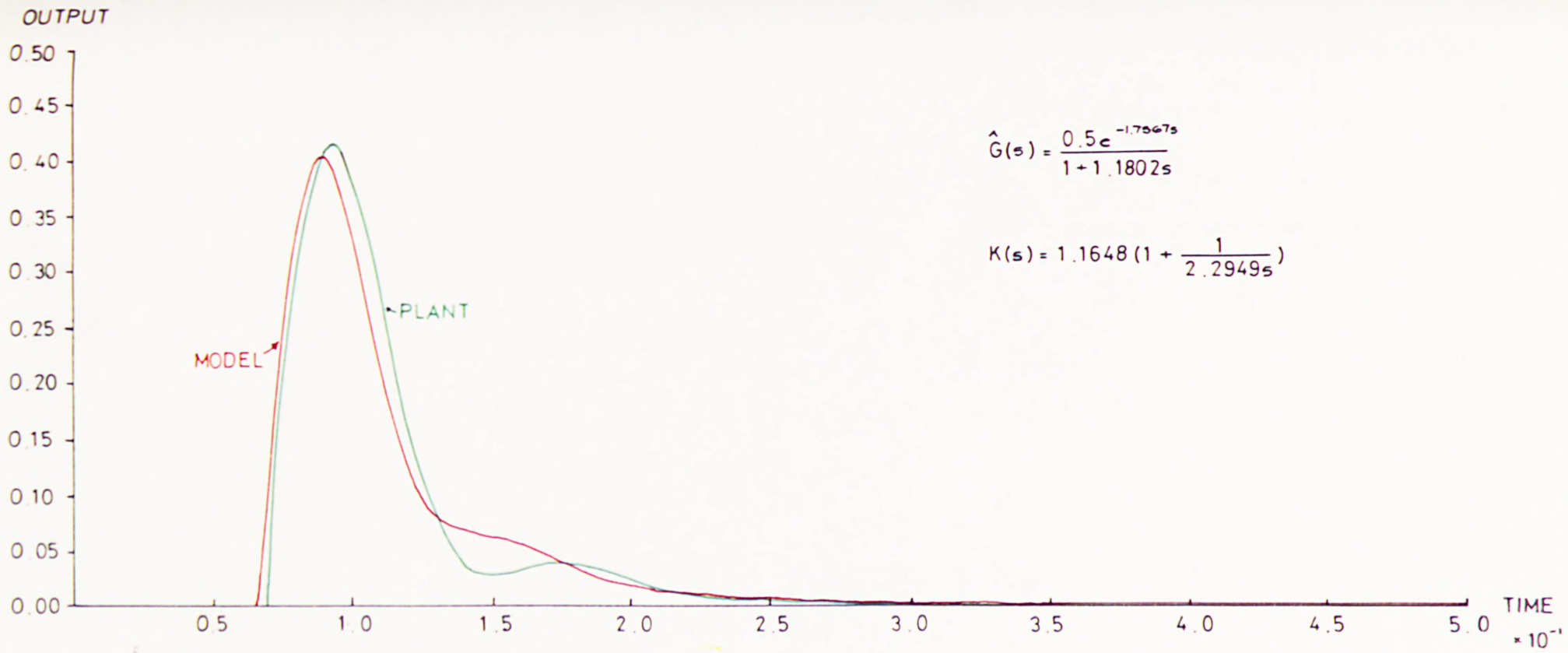


FIGURE 4.20

AS FIGURE 4.17 BUT FITTED AT $\hat{\omega}_c = 1.1\omega_c$ WITH VALUES FOR $\hat{G}(s)$ AND $K(s)$ AS SHOWN ABOVE.

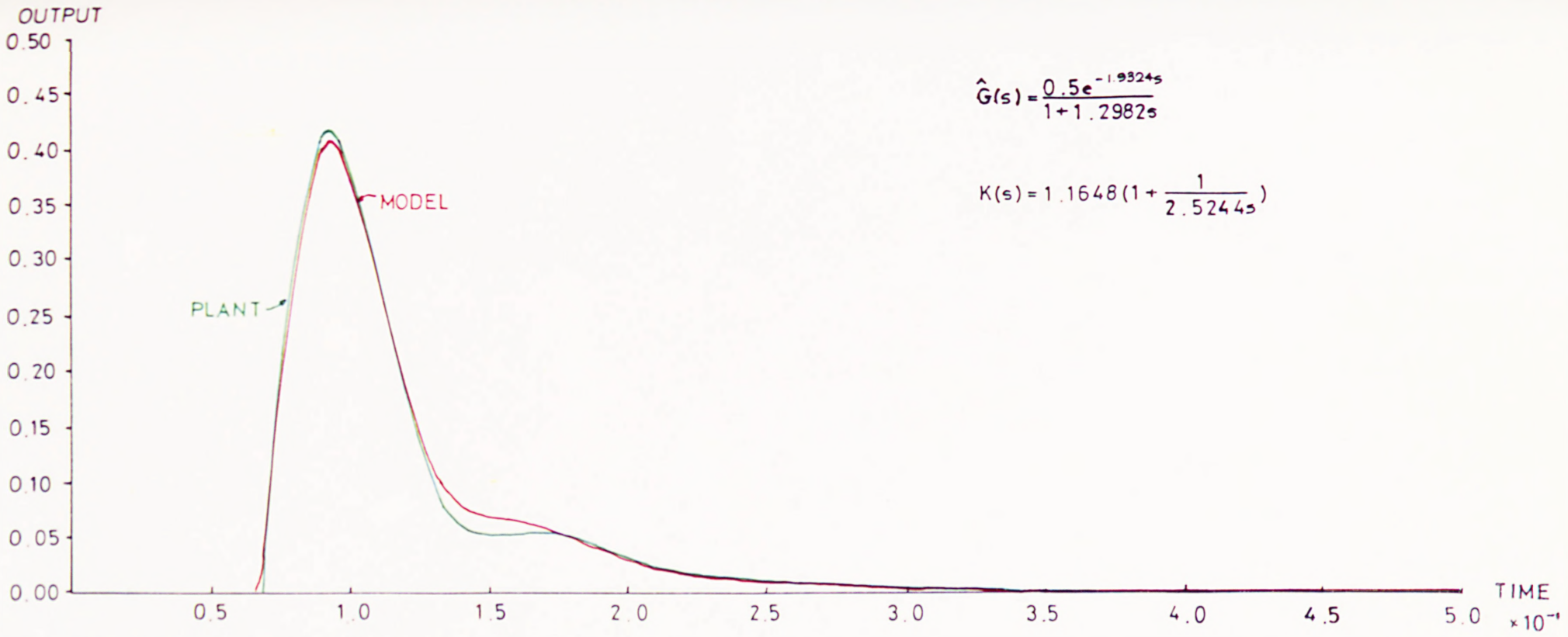


FIGURE 4.21

AS FIGURE 4.17 BUT FITTED AT $\hat{\omega}_c = 1.0\omega_c$ WITH VALUES FOR $\hat{G}(s)$ AND $K(s)$ AS SHOWN ABOVE.

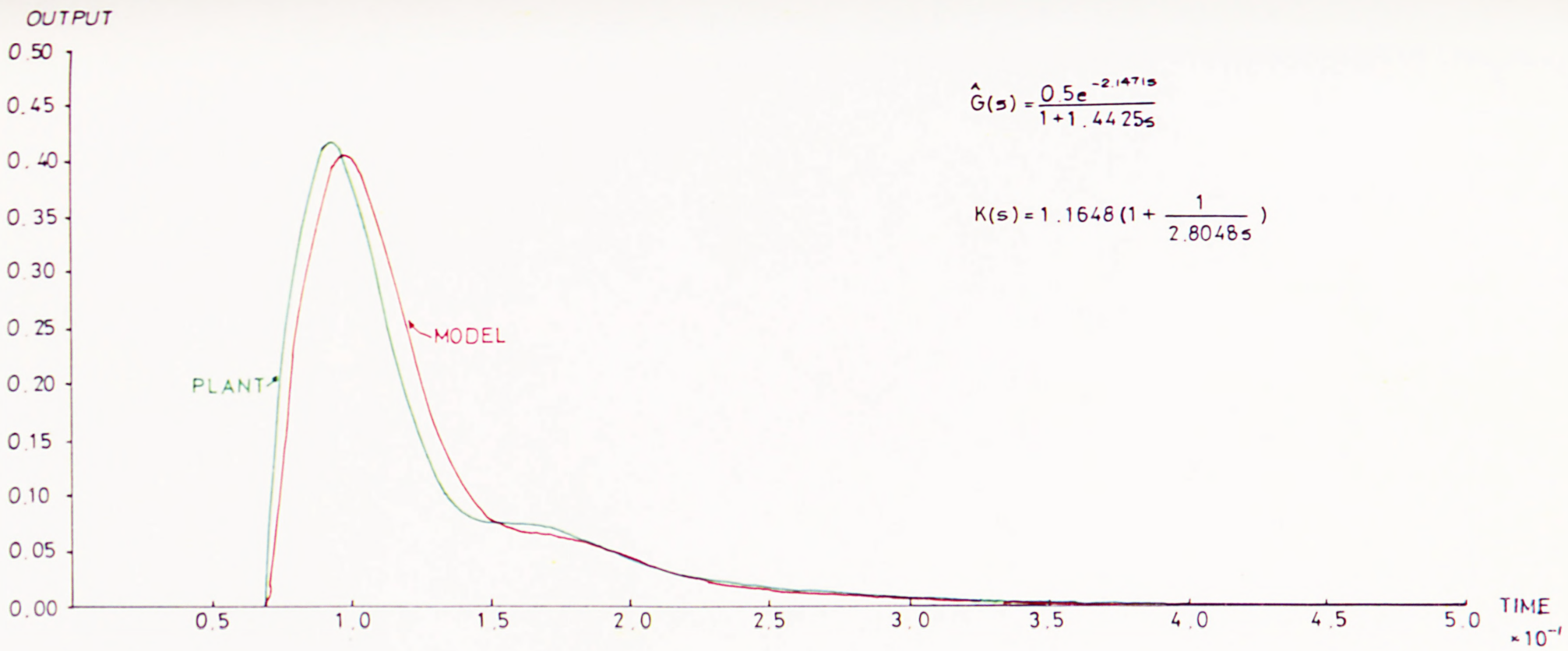


FIGURE 4.22

AS FIGURE 4.17 BUT FITTED AT $\hat{\omega}_c = 0.9\omega_c$ WITH VALUES FOR $\hat{G}(s)$ AND $K(s)$ AS SHOWN ABOVE.

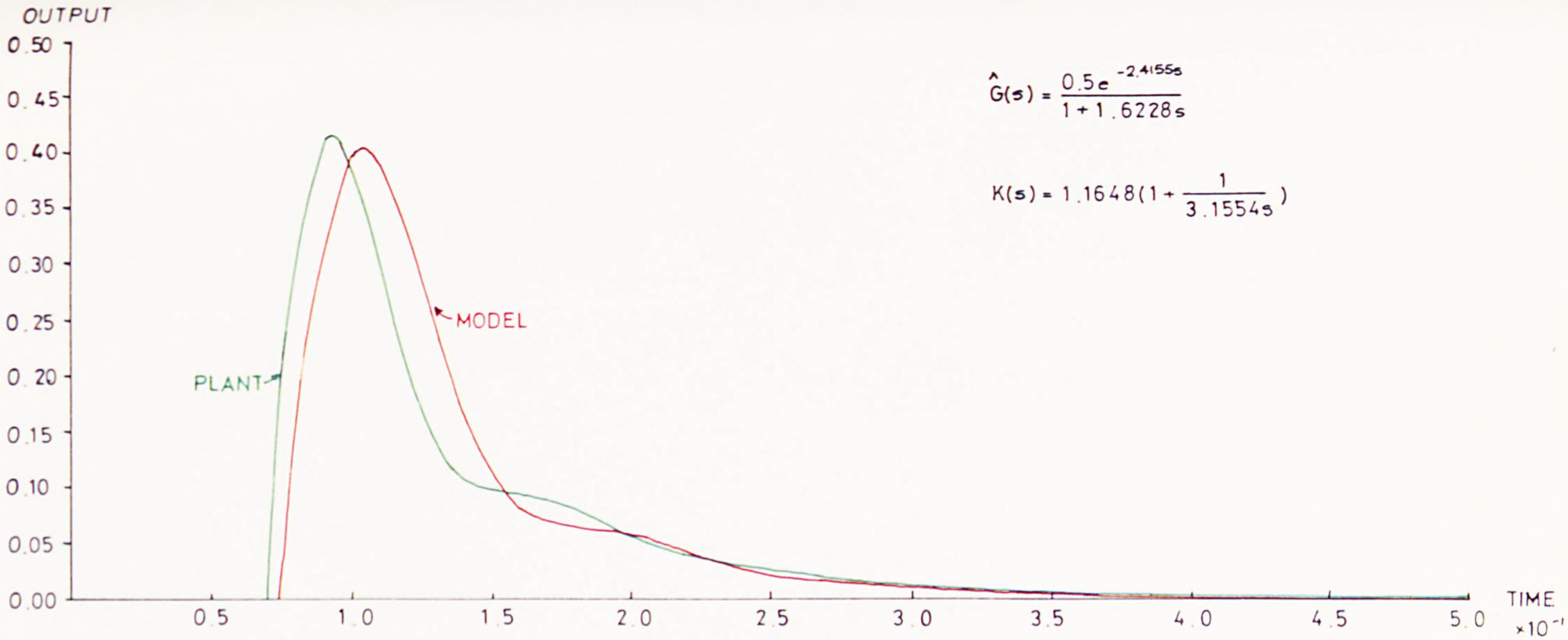


FIGURE 4.23

AS FIGURE 4.17 BUT FITTED AT $\hat{\omega}_c = 0.8\omega_c$ WITH VALUES FOR $\hat{G}(s)$ AND $K(s)$ AS SHOWN ABOVE.

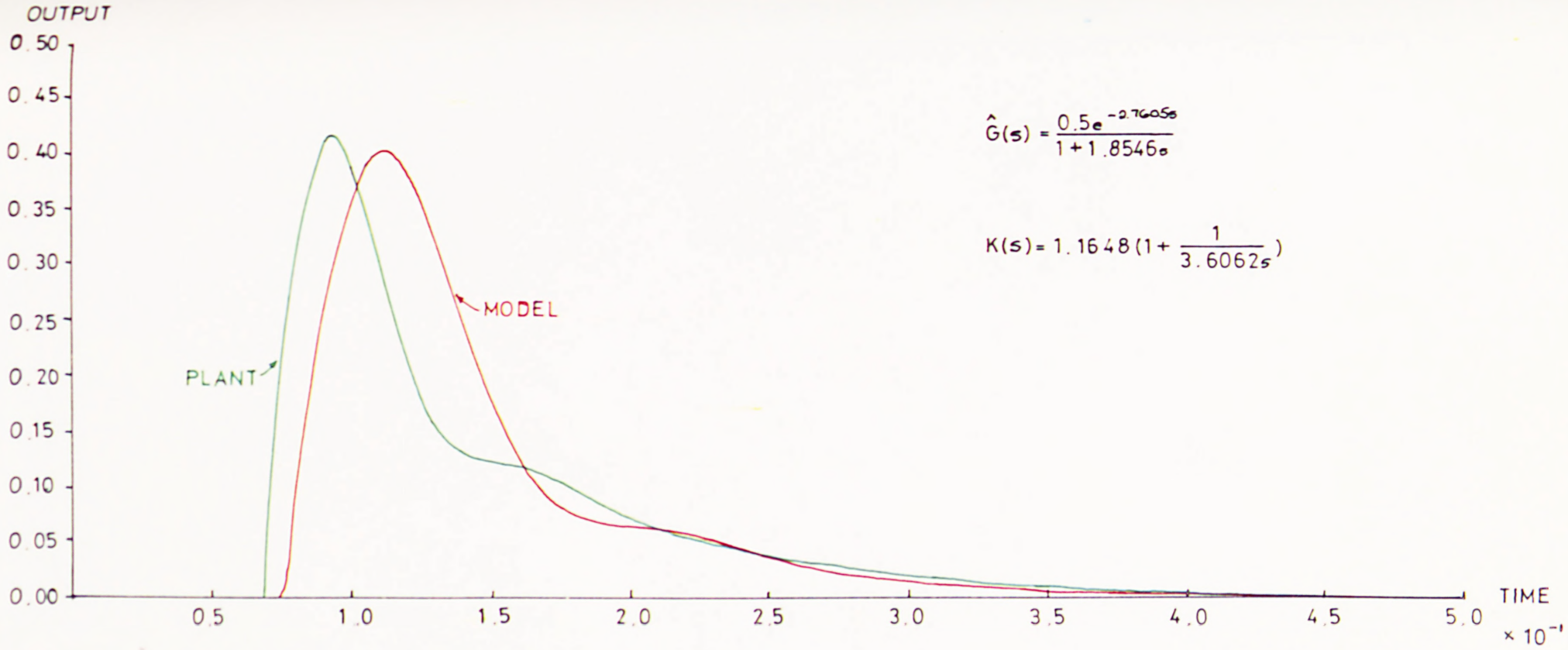


FIGURE 4.24

AS FIGURE 4.17 BUT FITTED AT $\hat{\omega}_c = 0.7\omega_c$ WITH VALUES FOR $\hat{G}(s)$ AND K(s) AS SHOWN ABOVE.

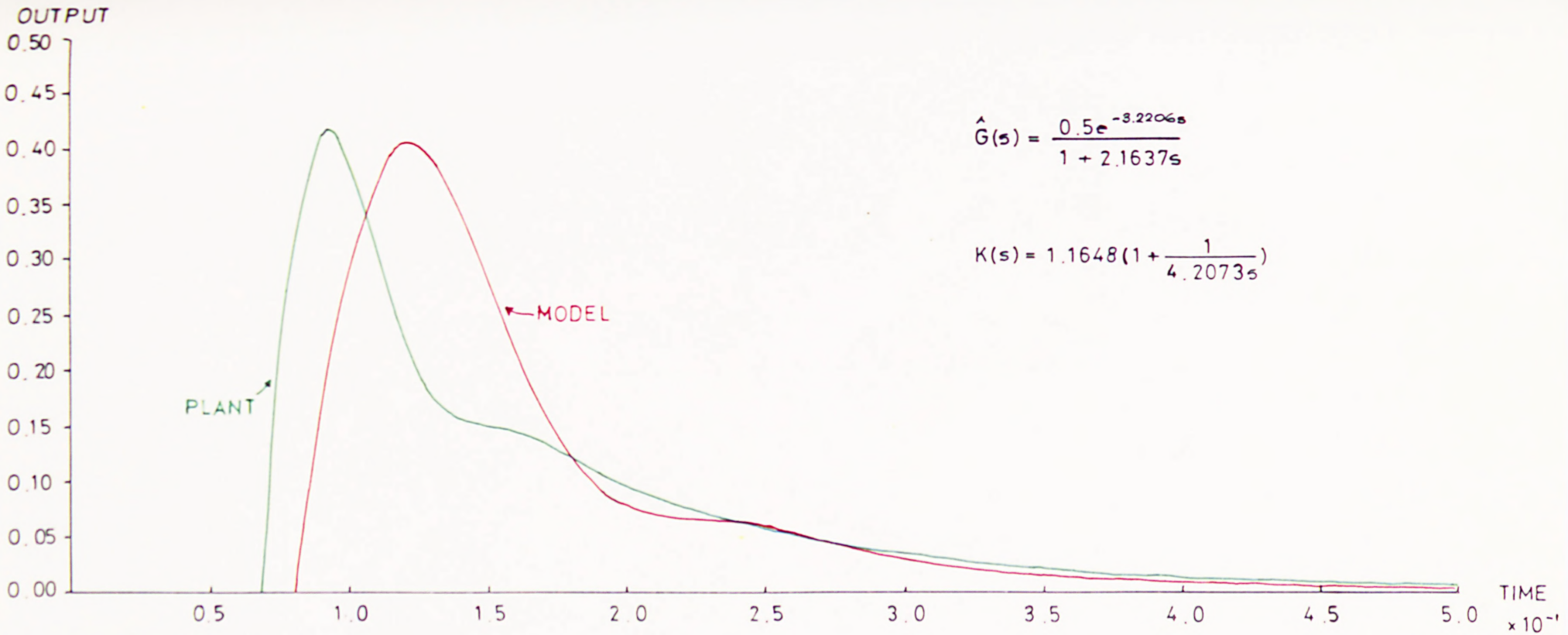


FIGURE 4.25

AS FIGURE 4.17 BUT FITTED AT $\hat{\omega}_c = 0.6\omega_c$ WITH VALUES FOR $\hat{G}(s)$ AND $K(s)$ AS SHOWN ABOVE.

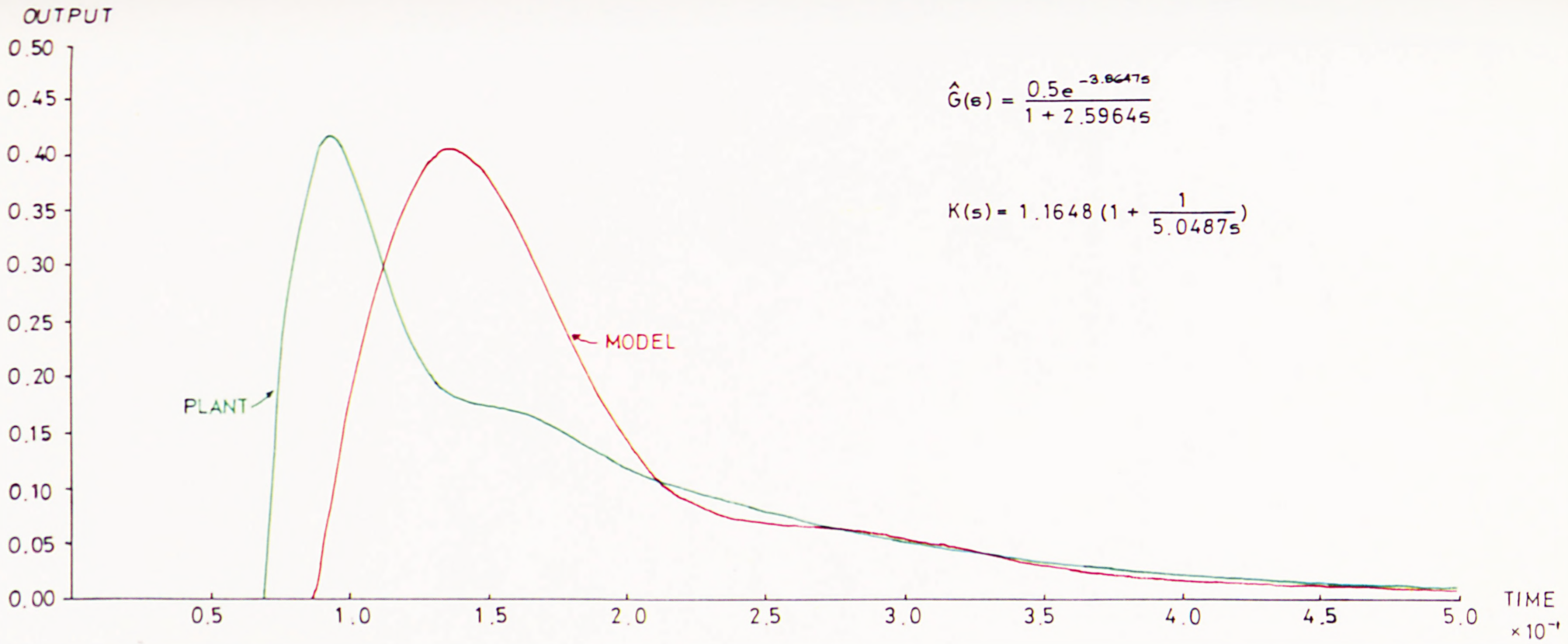


FIGURE 4.26

AS FIGURE 4.17 BUT FITTED AT $\hat{\omega}_c = 0.5\omega_c$ WITH VALUES FOR $\hat{G}(s)$ AND $K(s)$ AS SHOWN ABOVE.

reason, models 2, 3, 4 and 5 are suggested in section 3.13. In this case, models 4 and 5 occur at $\hat{\omega}_c = 1.08\omega_c$ and $\hat{\omega}_c = 1.16\omega_c$, relatively close to the optimum value.

Other examples have also shown these models to give promising results, and it is suggested that they be further investigated.

$\frac{\hat{\omega}_c}{\omega_c}$	T	$\hat{\tau}$	T_i	Plant ITAE	Model ITAE
0.5	2.5964	3.8647	5.0487	56.33	58.52
0.6	2.1637	3.2206	4.2073	40.46	40.97
0.7	1.8546	2.7605	3.6062	30.13	30.23
0.8	1.6228	2.4155	3.1554	23.18	23.22
0.9	1.4425	2.1471	2.8048	18.35	18.37
1.0	1.2982	1.9324	2.5244	14.87	14.89
1.1	1.1802	1.7567	2.2949	12.29	12.29
1.2	1.0818	1.6103	2.1036	10.33	10.34
1.3	0.9986	1.4864	1.9418	9.28	8.80
1.4	0.9273	1.3803	1.8031	9.47	7.57
1.5	0.8655	1.2882	1.6829	10.16	6.62

$$G(s) = \frac{0.5\ e^{-2s}(1+2.5s+1.5s^2+0.5s^3)}{1+3.5s+4.5s^2+2s^3+0.5s^4}$$

Pade Model $\hat{G}(s) = 0.5\ e^{-3s}$ does not give usable controller parameters.

K_c for all controllers obtained from critical point models is 1.1648.

Method 4 (section 13.3) gives T=1.2056 $\hat{\tau}$=1.7944 $\frac{\hat{\omega}_c}{\omega_c}$=1.078

Method 5 (section 13.3) gives T=1.180 $\hat{\tau}$=1.6650 $\frac{\hat{\omega}_c}{\omega_c}$=1.163

Table 4.3 Comparison of various critical point models using ITAE values obtained over 60 secs (following a unit disturbance with set point =0) from closed loop simulations of a fourth order system and these critical point models, using controller parameters, calculated to minimise ITAE values, from these models.

4.4 CONTROLLERS BASED ON SIMPLE MODELS OBTAINED FROM NON LINEAR MACROSCOPIC MODELS - OPEN LOOP FREQUENCY RESPONSE

To investigate further the usefulness of employing simple models to design controllers for higher order systems, a non-linear macroscopic model of a vaporiser was examined. (Full details of this model are given by ELLIS (111)) Two different versions of the model were considered:-

1. taking the model as a freon vaporiser
2. taking it as a chlorine vaporiser.

To obtain a reduced model from a non-linear one, the model must first be linearised about a suitable operating point. In the case of the freon vaporiser, the output value from the macroscopic model is the pressure in the vaporiser in terms of inches of water above atmospheric pressure. An output value of 10" was chosen (from a range 0"-20") as a suitable operating point and the model was linearised about this value. For comparison the model was also linearised about 9" and 11". The linearised models are as follows:-

9"

$$G(s) = \frac{5.62735(1-8.87928s)}{1+57.2162s+9.27025s^2+0.03522s^3} \quad \ldots\ldots(B28)$$

10"

$$G(s) = \frac{5.8159(1-8.9230s)}{1+56.8780s+9.6342s^2+0.0392s^3} \quad \ldots\ldots(B29)$$

11"

$$G(s) = \frac{5.9789(1-8.9667s)}{1+56.5219s+9.9617s^2+0.0431s^3} \quad \ldots\ldots(B30)$$

Five models have been obtained from each of these system equations, using the Pade technique and the four critical point methods described in sec. 3.13. The parameters resulting from the 10" linearised model are given below in table 4.4 together with the load change controller parameters obtained from them.

Normalisation of the 10" linearised model gives:-

$$G(\overset{*}{s}) = \frac{5.8159(1-0.1569\overset{*}{s})}{1+\overset{*}{s}+0.00298\overset{*}{s}^2+0.0000002\overset{*}{s}^3} \qquad \ldots\ldots(B31)$$

where $\overset{*}{s} = 56.8780s$

As $\frac{a_3}{a_1^3}$ is so small, the normalised system can be considered as a second order system. Referring to fig.3.4 we see that this system will give an inadequate Pade model. This is confirmed by the frequency response plot of the linearised model and its reduced models, fig.4.27.

Open loop frequency plots have also been produced for the linearised third order model with the ITAE and ISE controllers obtained from the Padé model and the critical point models (see figs. 4.28 and 4.29). From these it can be seen that all the critical point reduced models, each with its own controller, give the same open loop frequency plot although any point on the plot occurs at a different frequency for the various models.

	Model 1	Model 2	Model 3	Model 4	Model 5
T	56.002	56.002	35.676	51.622	53.333
$\hat{\tau}$	9.799	15.382	9.799	14.179	14.649
ISE (K_c	1.1960	0.758	0.7758	0.7758	0.7758
ISE (T_i	31.3907	43.8045	27.9054	40.3185	41.7170
IAE (K_c	0.9436	0.6049	0.6049	0.6049	0.6049
IAE (T_i	26.8588	36.9435	23.5347	34.0542	35.1830
ITAE (K_c	0.8109	0.5220	0.5220	0.5220	0.5220
ITAE (T_i	25.3963	34.5091	21.9839	31.8102	32.8646

Table 4.4 Reduced models and resulting controller parameters obtained from a third order linearised model of a freon vapouriser (linearised about an output value of 10)

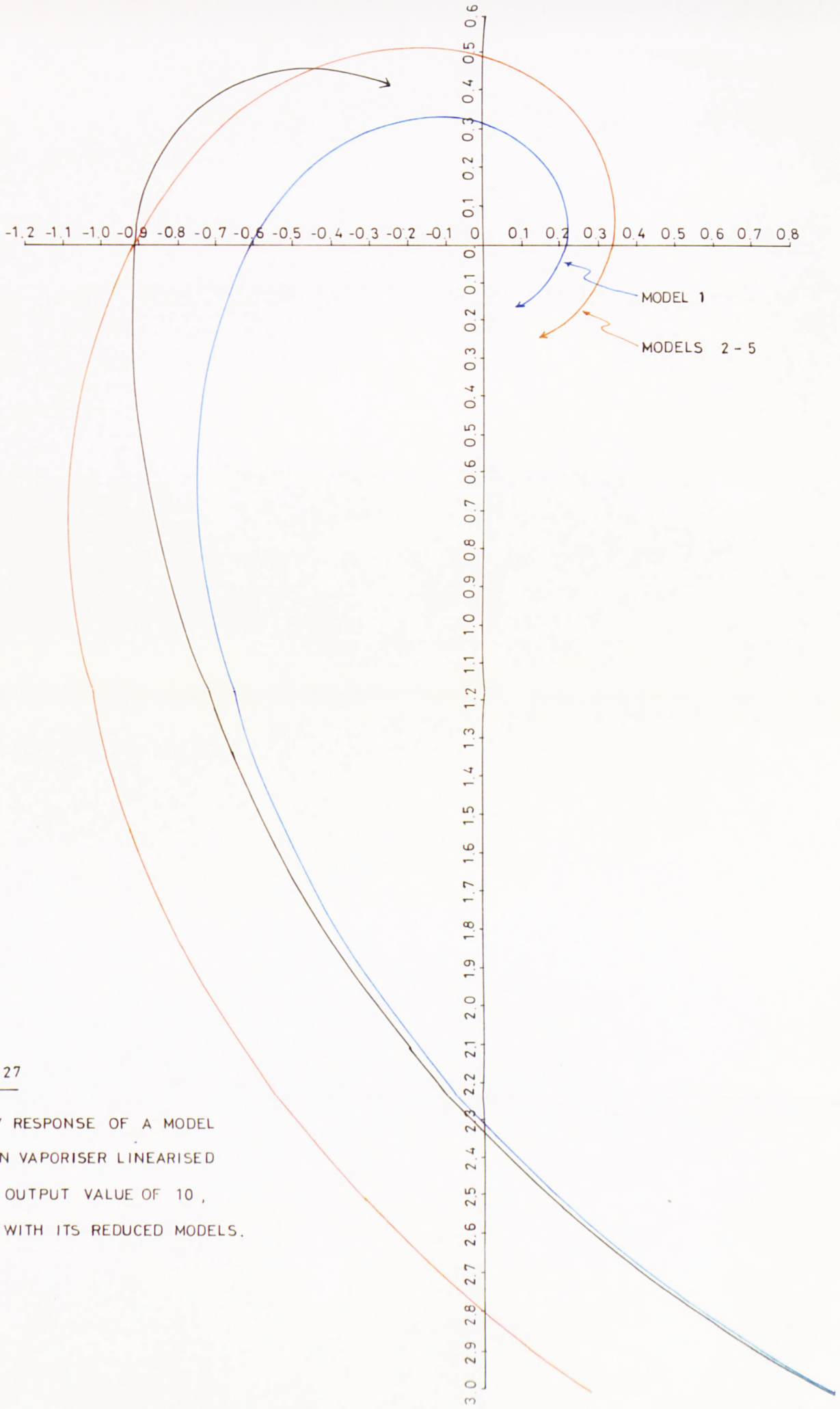


FIGURE 4.27

FREQUENCY RESPONSE OF A MODEL OF A FREON VAPORISER LINEARISED ABOUT AN OUTPUT VALUE OF 10, TOGETHER WITH ITS REDUCED MODELS.

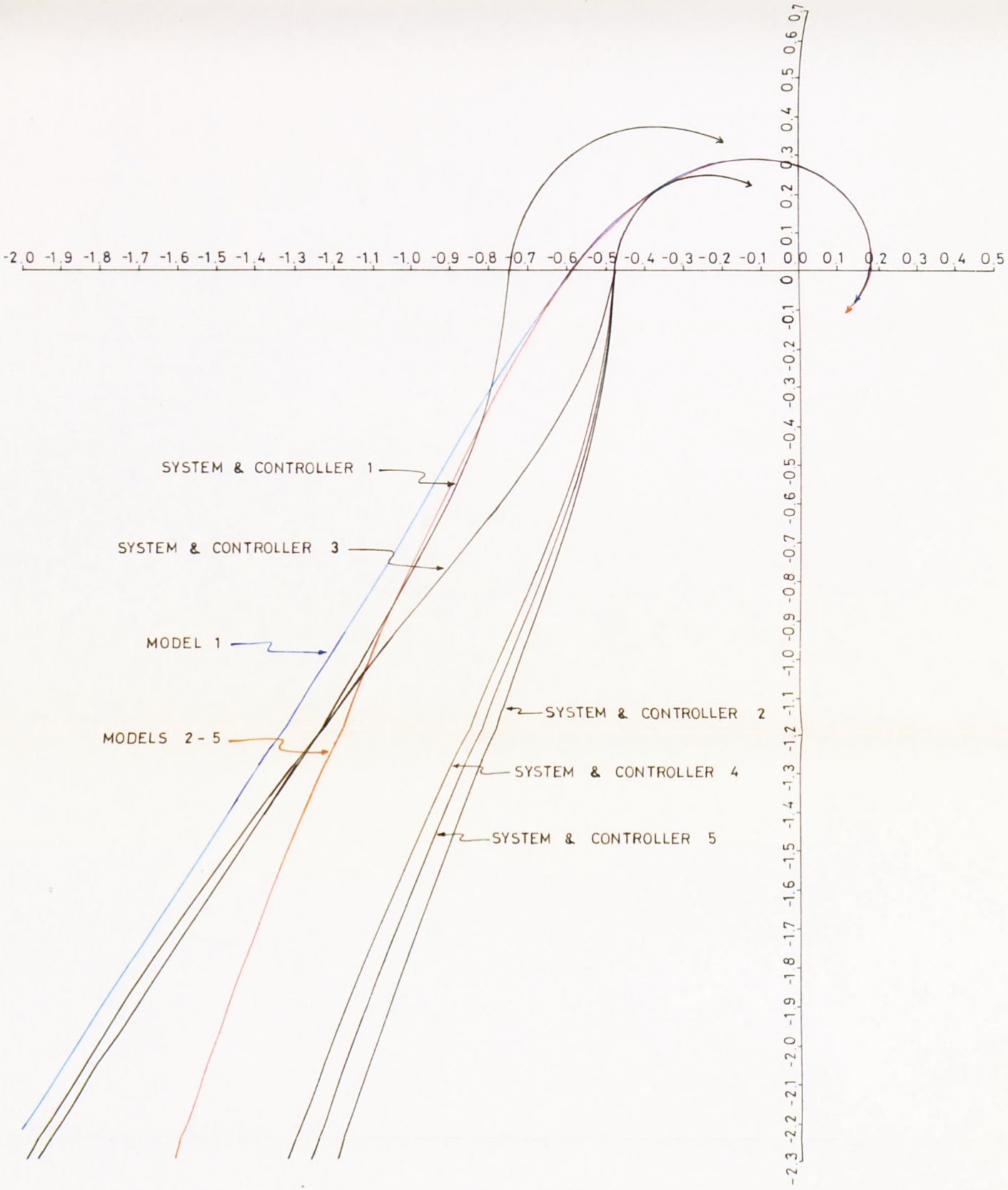


FIGURE 4.28

OPEN LOOP FREQUENCY PLOTS OF A LINEARISED THIRD ORDER MODEL OF A FREON VAPORISER LINEARISED ABOUT AN OUTPUT VALUE OF 10, WITH MIN. ITAE CONTROLLERS OBTAINED FROM REDUCED MODELS, THE REDUCED MODEL PLOTS ALSO BEING SHOWN.

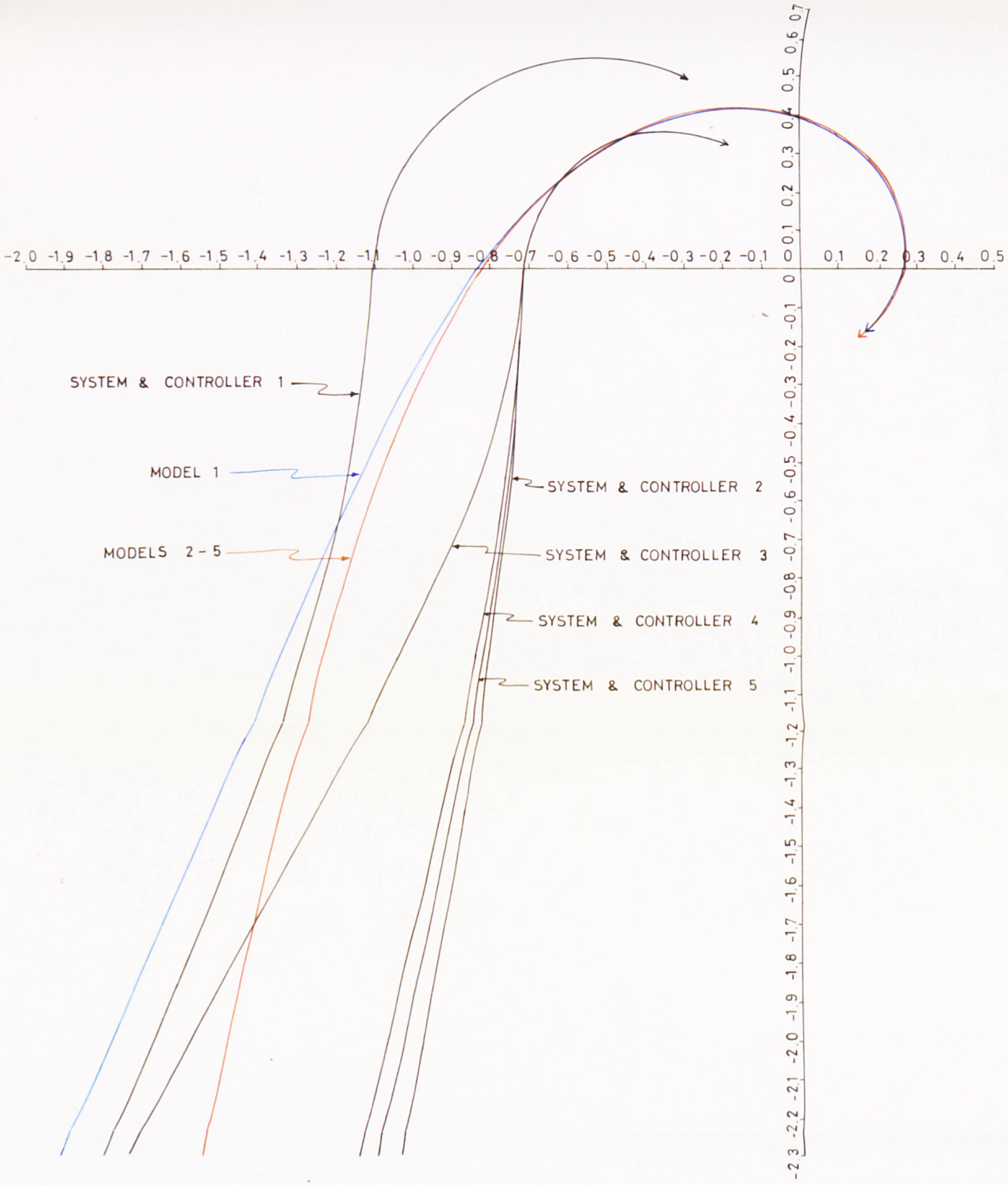


FIGURE 4.29

OPEN LOOP FREQUENCY PLOTS OF A LINEARISED THIRD ORDER MODEL OF A FREON VAPORISER LINEARISED ABOUT AN OUTPUT VALUE OF 10 WITH MIN.ISE LOAD CHANGE CONTROLLERS OBTAINED FROM REDUCED MODELS, THE REDUCED MODEL PLOTS ALSO BEING SHOWN.

From fig.4.28 it can be seen that the linearised model with the min ISE controller obtained from the Padé model is still stable, but far less so than the Padé model plus controller itself. Although this controller would give stable control for the linear third order system from which the reduced models were obtained, this will not necessarily be so with the original non-linear system, especially if a disturbance moves the system some distance from the operating point about which the linearisation was carried out.

Fig.4.29 shows that the ITAE controller obtained from the Padé model gives an unstable closed loop system (see also section 4.5). Note also that all the controllers obtained from the critical point models give closed loop systems with the linearised model, which are more stable than the closed loop models obtained with the corresponding critical point model itself.

For the chlorine version of the vaporiser model the output value is the pressure in the vaporiser in terms of inches of water above atmospheric pressure divided by 20, giving again a range 0 to 20. Linearisation gives:-

about 9.5

$$G(s) = \frac{0.6128(1-4.3909s)}{1+28.0316s+31.1875s^2+0.08465s^3} \qquad \ldots.(B32)$$

and about 10

$$G(s) = \frac{0.4800(1-4.6912s)}{1+24.4092s+27.5291s^2+0.0918s^3} \qquad \ldots.(B33)$$

Normalisation of these two models gives:-

9.5

$$G(\overset{*}{s}) = \frac{0.6128(1-0.1566\overset{*}{s})}{1+\overset{*}{s}+0.03969\overset{*}{s}^2+0.0000038\overset{*}{s}^3} \qquad \ldots.(B34)$$

where $\overset{*}{s}=28.0316s$

and 10.0

$$G(\overset{*}{s}) = \frac{0.4800(1-0.1922\overset{*}{s})}{1+\overset{*}{s}+0.0462\overset{*}{s}^2+0.0000063\overset{*}{s}^3} \qquad \ldots.(B35)$$

where $\overset{*}{s}=24.4092s$

Again referring to fig.3.4 it can be seen that Pade models obtained from these two models will be inadequate.

As in the case of the Freon Vaporiser, five reduced models have been obtained from each linearised model. Fig.4.30 shows the frequency plots for the linearised model at an output of 10, and its reduced models. Parameters of these models and the load change controllers obtained from them are given in table 4.5 below. Also given are the set point change controller parameters.

The open loop frequency response of these models with the min. ITAE load change controllers are plotted in fig.4.31. These plots are similar to those for the freon vaporiser. Again it can be seen that the inadequate Padé model gives the least suitable controller, as the plot of this controller with the linearised model passes very close to the (-1,0) point. The linearised system equations given earlier show the relatively rapid change in gain in the vicinity of the operating point and it can be assumed from these results that the controller will give unstable control.

Similar open loop frequency plots are shown in fig.4.32 using the min. IAE set point change controllers. In this case although the controller obtained from the Padé model gives a response which passes further to the left that the other plots, it

SEE TABLE 4.5

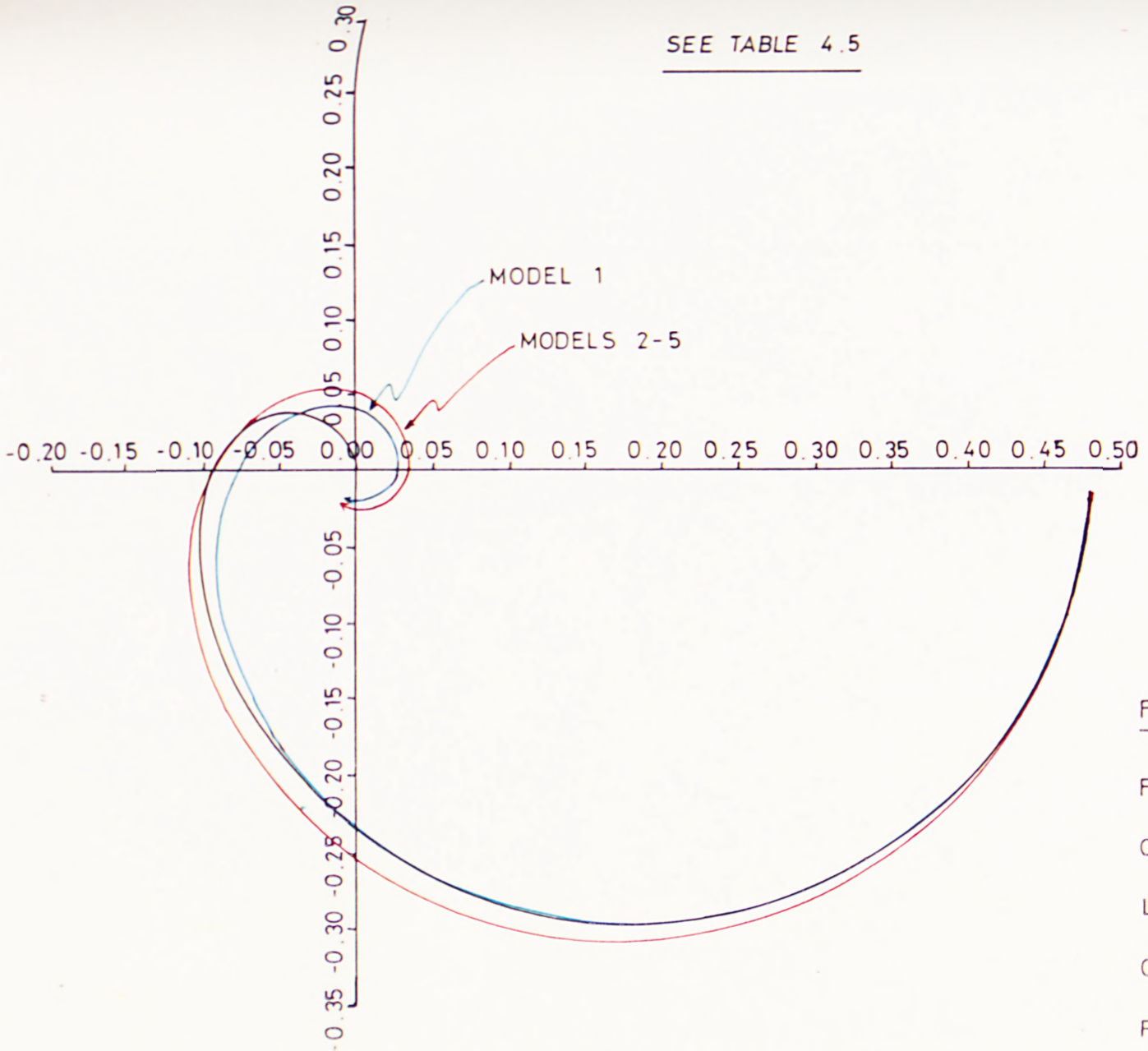


FIGURE 4.30

FREQUENCY RESPONSE OF A MODEL OF A CHLORINE VAPORISER LINEARISED ABOUT AN O.P. VALUE OF 10, TOGETHER WITH ITS REDUCED MODELS.

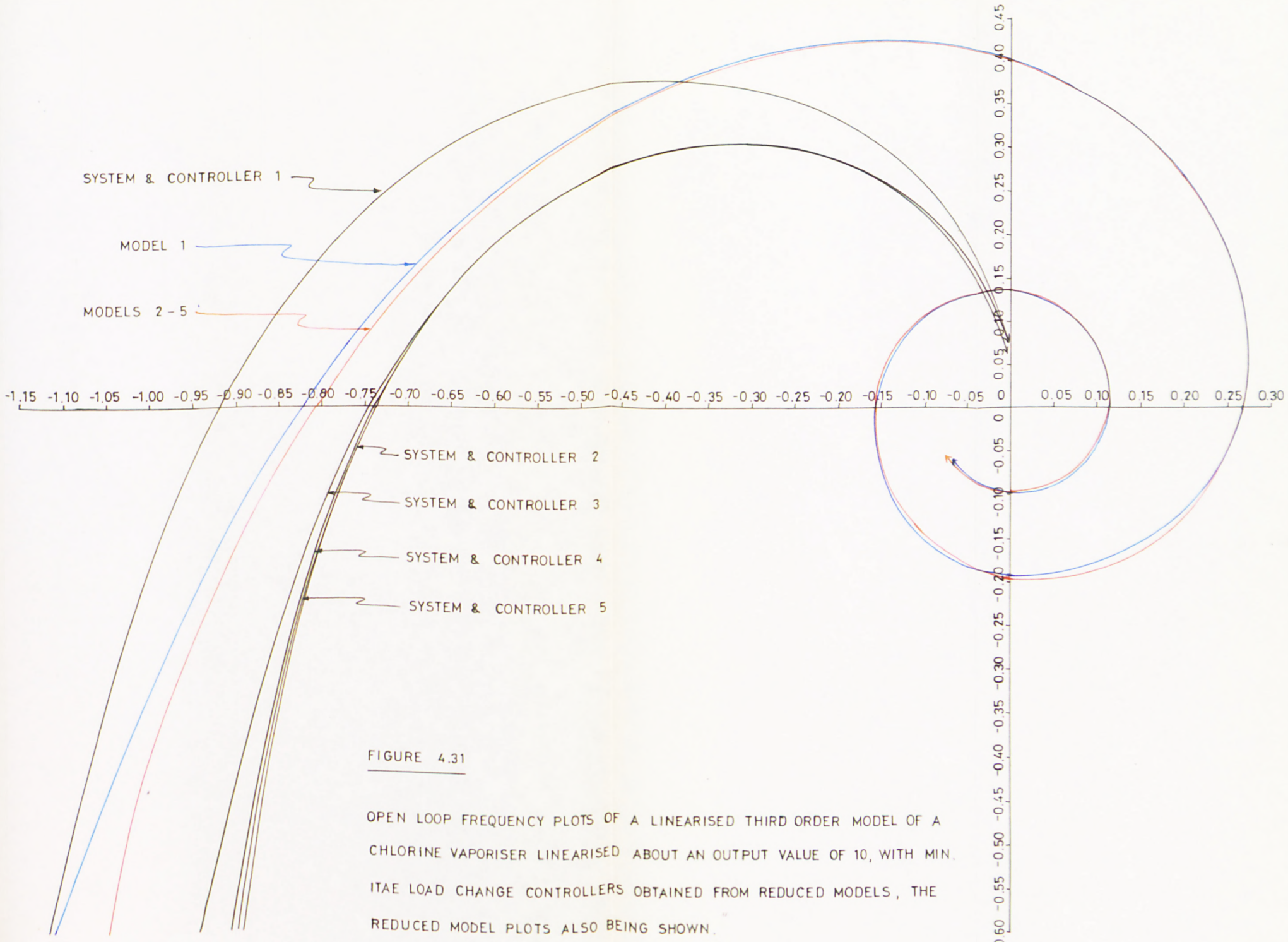


FIGURE 4.31

OPEN LOOP FREQUENCY PLOTS OF A LINEARISED THIRD ORDER MODEL OF A CHLORINE VAPORISER LINEARISED ABOUT AN OUTPUT VALUE OF 10, WITH MIN. ITAE LOAD CHANGE CONTROLLERS OBTAINED FROM REDUCED MODELS, THE REDUCED MODEL PLOTS ALSO BEING SHOWN.

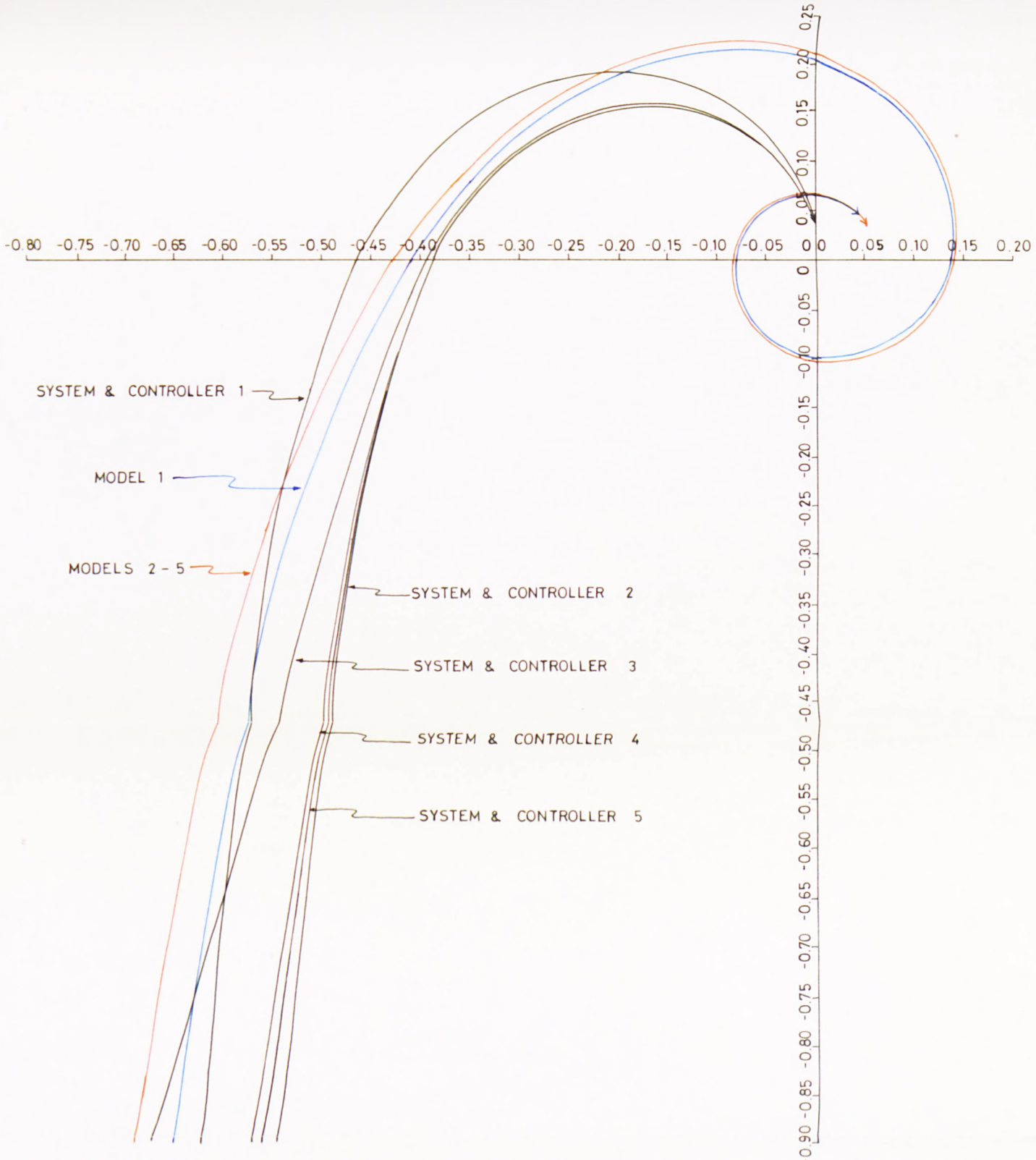


FIGURE 4.32

OPEN LOOP FREQUENCY PLOTS OF A LINEARISED THIRD ORDER MODEL OF A CHLORINE VAPORISER LINEARISED ABOUT AN OUTPUT VALUE OF 10, WITH MIN. IAE SET POINT CHANGE CONTROLLERS OBTAINED FROM REDUCED MODELS, THE REDUCED MODEL PLOTS ALSO BEING SHOWN.

	Model 1	Model 2	Model 3	Model 4	Model 5
T	22.7759	22.7759	18.2874	21.6225	22.0397
$\hat{\tau}$	6.3245	7.8768	6.3245	7.4779	7.6222
	Load Change Controller Parameters				
ISE (K_c	9.3025	7.5351	7.5351	7.5351	7.5351
(T_i	17.9595	21.1222	16.9596	20.0525	20.4394
IAE (K_c	7.2519	5.8407	5.8407	5.8407	5.8407
(T_i	15.1413	17.6830	14.1982	16.7875	17.1114
ITAE (K_c	6.2581	6.2581	5.0502	5.0502	5.0502
(T_i	14.1394	16.4154	13.1804	15.5841	15.8848
	Set Point Change Controller Parameters				
ISE (K_c	4.6457	3.8975	3.8975	3.8975	3.8975
(T_i	20.4990	20.0081	16.0650	18.9949	19.3614
IAE (K_c	4.7596	3.9400	3.9400	3.9400	3.9400
(T_i	20.5246	20.1253	16.1592	19.1061	19.4748
ITAE (K_c	3.9482	3.2291	3.2291	3.2291	3.2291
(T_i	21.1708	20.9518	16.8228	19.8908	20.2745

Table 4.5 Reduced models and resulting controller parameters obtained from a third order linearised model of a chlorine vaporiser (linearised about an output value of 10).

crosses the negative real axis at about -0.47 and will give quite adequate control (see section 4.5).

From these frequency plots we can see that although inadequate models can sometimes give adequate controllers they can also give inadequate ones. Controllers which have relatively higher gains are more likely to give rise to instability and consequently, in considering the adequacy of a model, it might be worth considering the type of controller that is to be designed. It is worth noting that all controllers obtained using the critical point models give open loop frequency response with the linearised system which are more stable than those given with the model from which they have been designed.

4.5 CONTROLLERS BASED ON SIMPLE MODELS OBTAINED FROM NON-LINEAR MACROSCOPIC MODELS - CLOSED LOOP SIMULATION

In section 4.4 simple models were obtained from non-linear macroscopic models and controller parameters were then calculated from these models. Having examined the open loop frequency response of the linearised models with these controllers, we can now proceed to simulate closed loop systems using these controllers with the non-linear macroscopic model. Table 4.6 gives ISE and ITAE values for simulations of the freon vaporiser model with various controllers over 300 seconds, following three different disturbances. All the values obtained using the minimum ISE controller calculated from the Padé model are extremely high, since, with this controller, the system enters a limit cycle following all three disturbances. This result is, however, to be expected as shown by the open loop frequency plot in fig.4.29. Since the disturbances can be seen to affect the system in different ways it would appear that for optimum control the source of the disturbance should be taken into account. Despite this the results obtained using controllers based on adequate models seem to be satisfactory.

Results from these simulations are also shown in figs.4.33 to 4.41. Figs.4.33 to 4.35 give responses to a down stream pressure change from 101825 to 101325 N/m^2.

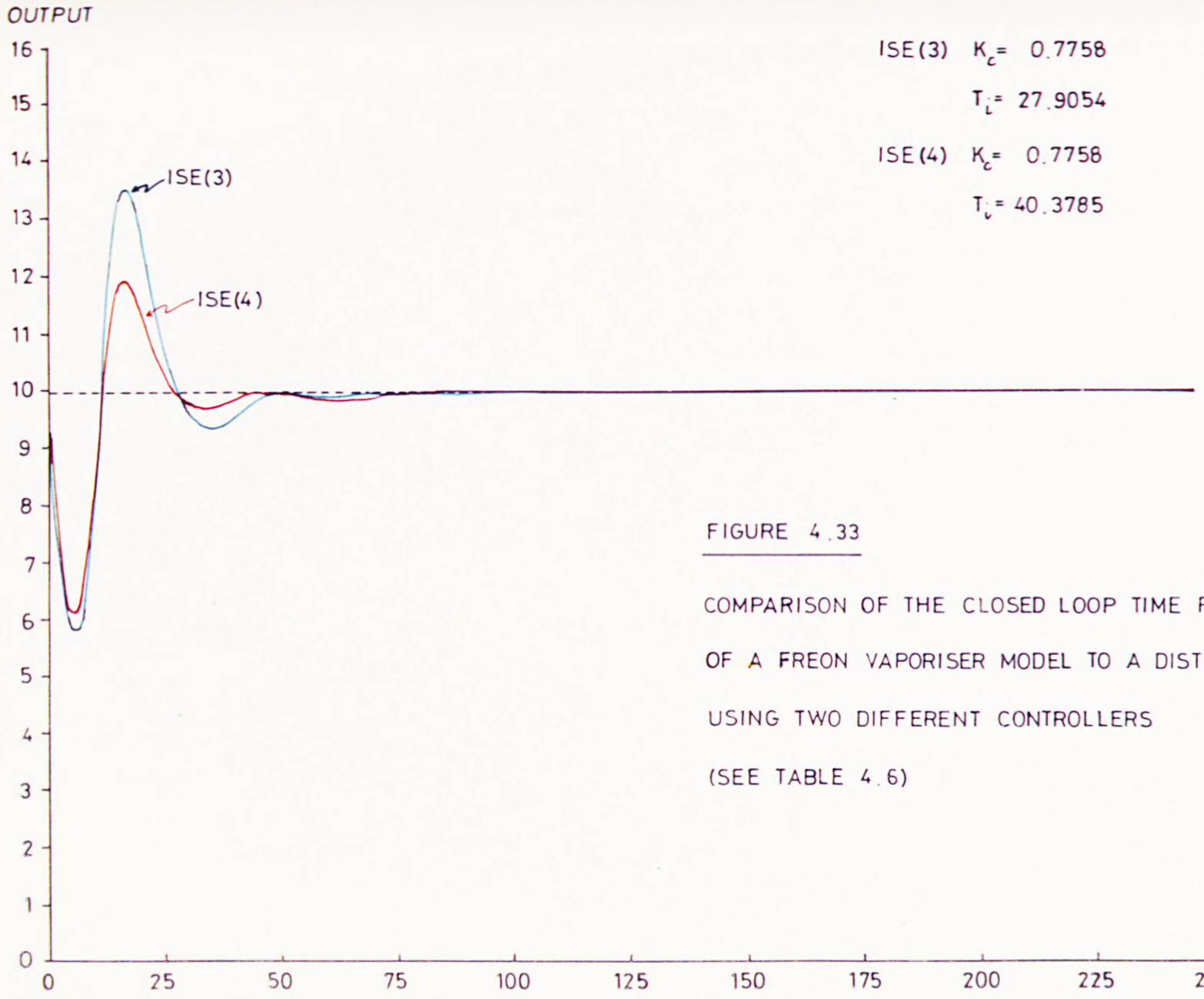
	Model No.	K_c	T_i	Disturbance 1	Disturbance 2	Disturbance 3
				I S E VALUES		
Min.ISE Controller	1	1.1960	31.3907	6131.2	6818.6	951.5
	3	0.7758	27.9054	209.3	6.5	110.6
	4	"	40.3785	122.3	7.8	89.6
	5	"	41.7170	118.5	8.0	88.5
	2	"	42.8045	114.3	8.3	87.0
				I T A E VALUES		
Min.ISE Controller	1	1.1960	31.3907	164818.0	167978.0	12030.7
	3	0.7758	27.9054	1443.1	666.2	1044.0
	4	"	40.3785	940.8	1330.9	1396.0
	5	"	41.7170	935.5	1414.5	1433.4
	2	"	43.8045	948.4	1549.2	1490.9
Min.ITAE Controller	1	0.8109	25.3963	3469.1	531.7	1200.6
		0.5220	15.0000	2135.3	978.4	4923.6
		"	21.0000	1073.4	883.7	2727.1
	3	"	21.9839	996.8	898.6	2594.4
	4	"	31.8102	605.1	1397.3	2143.8
	5	"	32.8646	578.8	1486.3	2146.8
	2	"	34.5091	532.5	1630.4	2164.7
		"	35.0000	517.4	1674.4	2173.1
		"	40.0000	334.1	2156.6	2327.7

Disturbance 1 - change of downstream pressure from 101825 to 101325 N/m^2.

Disturbance 2 - change of input pressure from 125325 to 120525 N/m^2.

Disturbance 3 - change of vaporiser wall temperature from 364 to 359 K.

Table 4.6 Comparison of various controllers using ISE and ITAE values over 300 sec following 3 different disturbances applied to a closed loop simulation of the macroscopic model of a freon vaporiser. Controller parameters are mostly based on simple models obtained from the 10" linearised model.



ISE(3) K_c = 0.7758

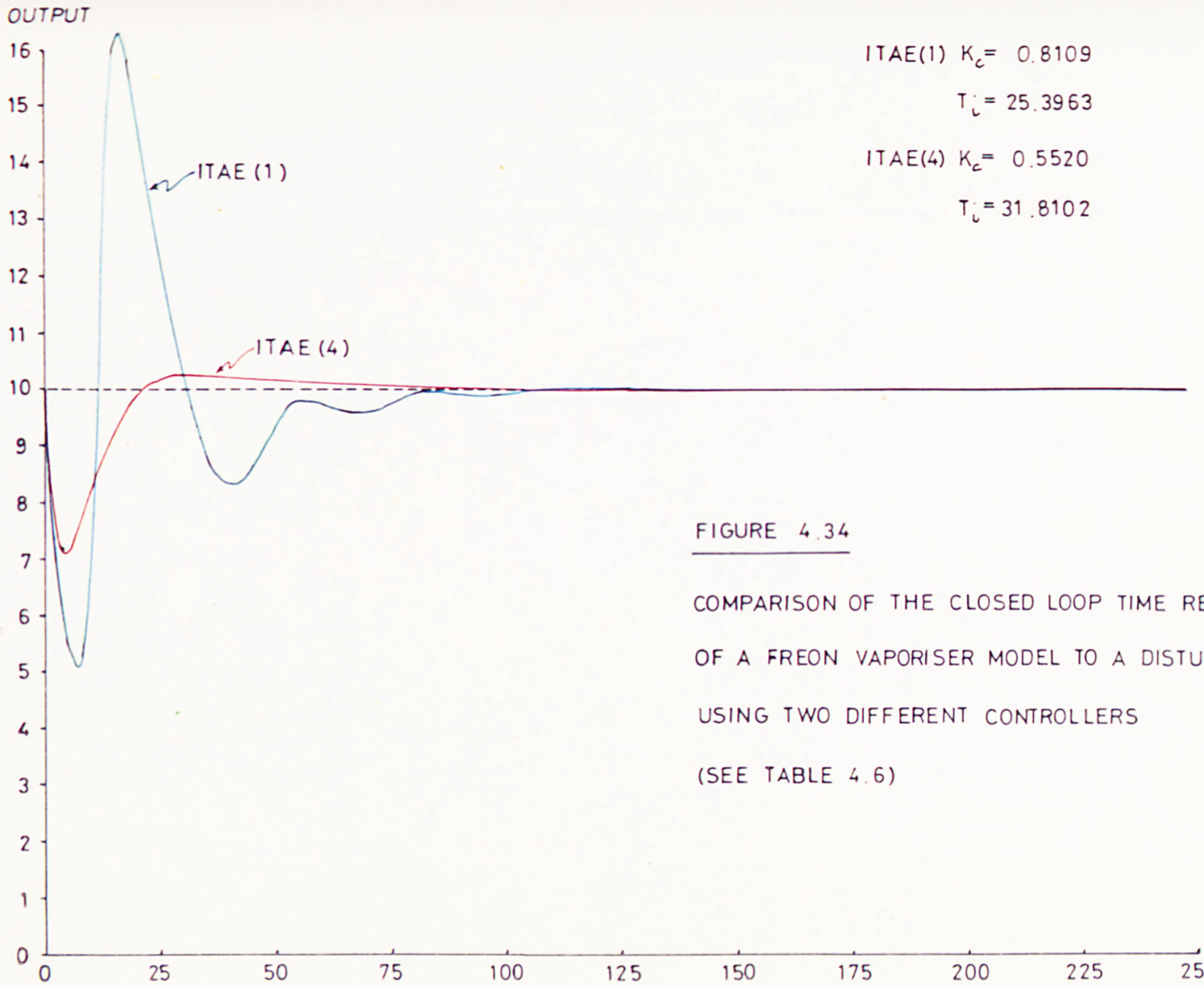
T_i = 27.9054

ISE(4) K_c = 0.7758

T_i = 40.3785

FIGURE 4.33

COMPARISON OF THE CLOSED LOOP TIME RESPONSE OF A FREON VAPORISER MODEL TO A DISTURBANCE (1) USING TWO DIFFERENT CONTROLLERS (SEE TABLE 4.6)



ITAE(1) K_c= 0.8109

T_i= 25.3963

ITAE(4) K_c= 0.5520

T_i= 31.8102

FIGURE 4.34

COMPARISON OF THE CLOSED LOOP TIME RESPONSE OF A FREON VAPORISER MODEL TO A DISTURBANCE (1) USING TWO DIFFERENT CONTROLLERS

(SEE TABLE 4.6)

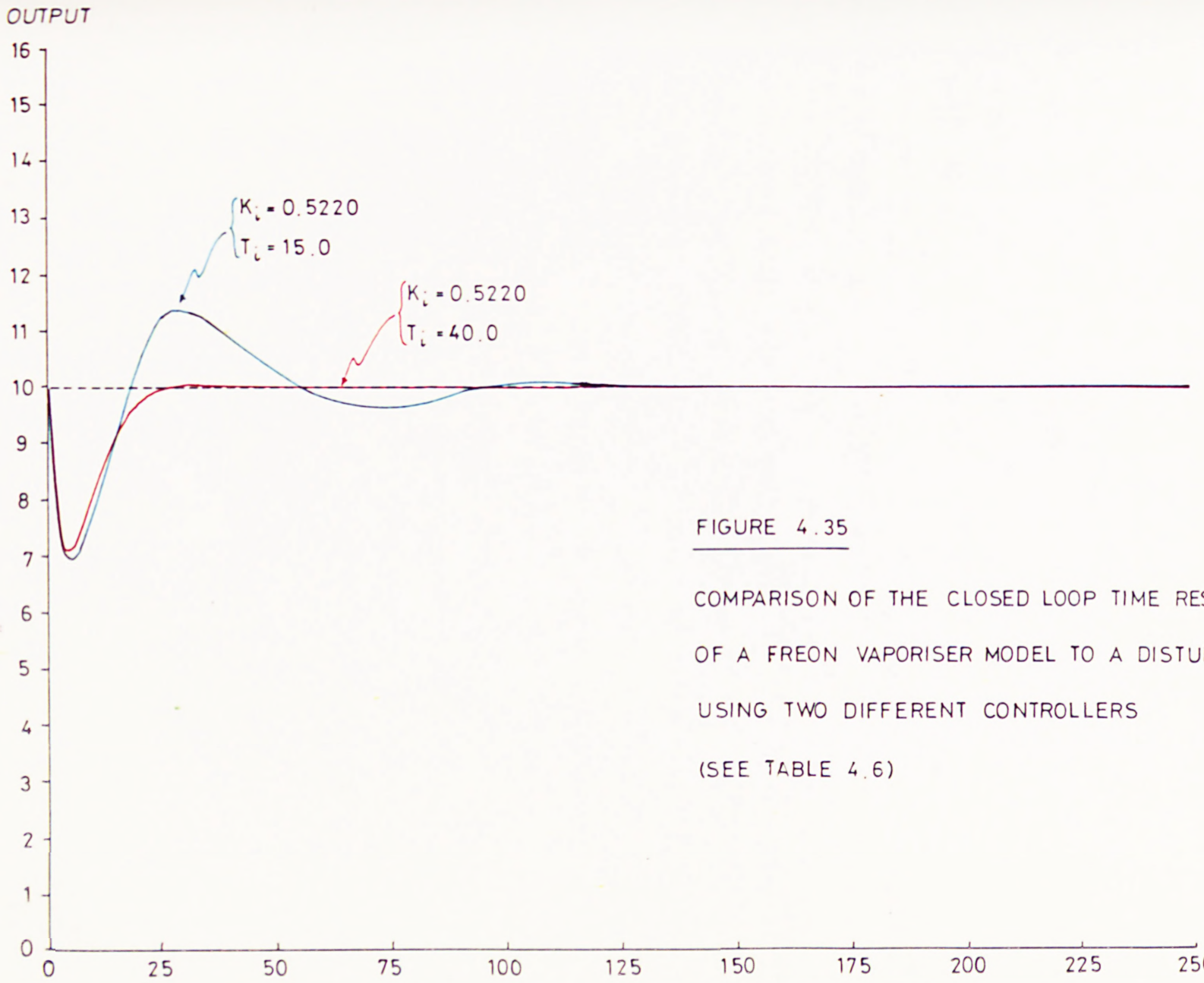


FIGURE 4.35

COMPARISON OF THE CLOSED LOOP TIME RESPONSE OF A FREON VAPORISER MODEL TO A DISTURBANCE (1) USING TWO DIFFERENT CONTROLLERS

(SEE TABLE 4.6)

Fig.4.33 shows the closed loop system output using controllers calculated to minimise the ISE criterion from simple models obtained using methods 3 and 4 at the critical point. Fig.4.34 shows results using min. ITAE controllers based on the Padé model (1), and on the critical point model 4. For comparison fig.4.35 gives the response using controllers having the gain obtained using the ITAE calculations with the critical point models and the integral time constant set in turn to a relatively small and relatively large value. From these output plots it can be seen that the last mentioned controller gives the best results for this disturbance. This is also evident from table 4.6.

Figs.4.36 to 4.38 show responses of these same six closed loop systems to the second disturbance which is a change of input pressure from 125325 to 120525 N/m^2. This disturbance has a much smaller effect on the output and adequate control is achieved with all six controllers, although even in the best case the settling time is longer than that following the first disturbance. For this disturbance the best control is obtained using the min. ITAE controller calculated from the Padé model. This is again confirmed by results in table 4.6.

Figs.4.39 to 4.41 give corresponding results for these same six closed loop systems following the third and final disturbance. In this case the vaporiser

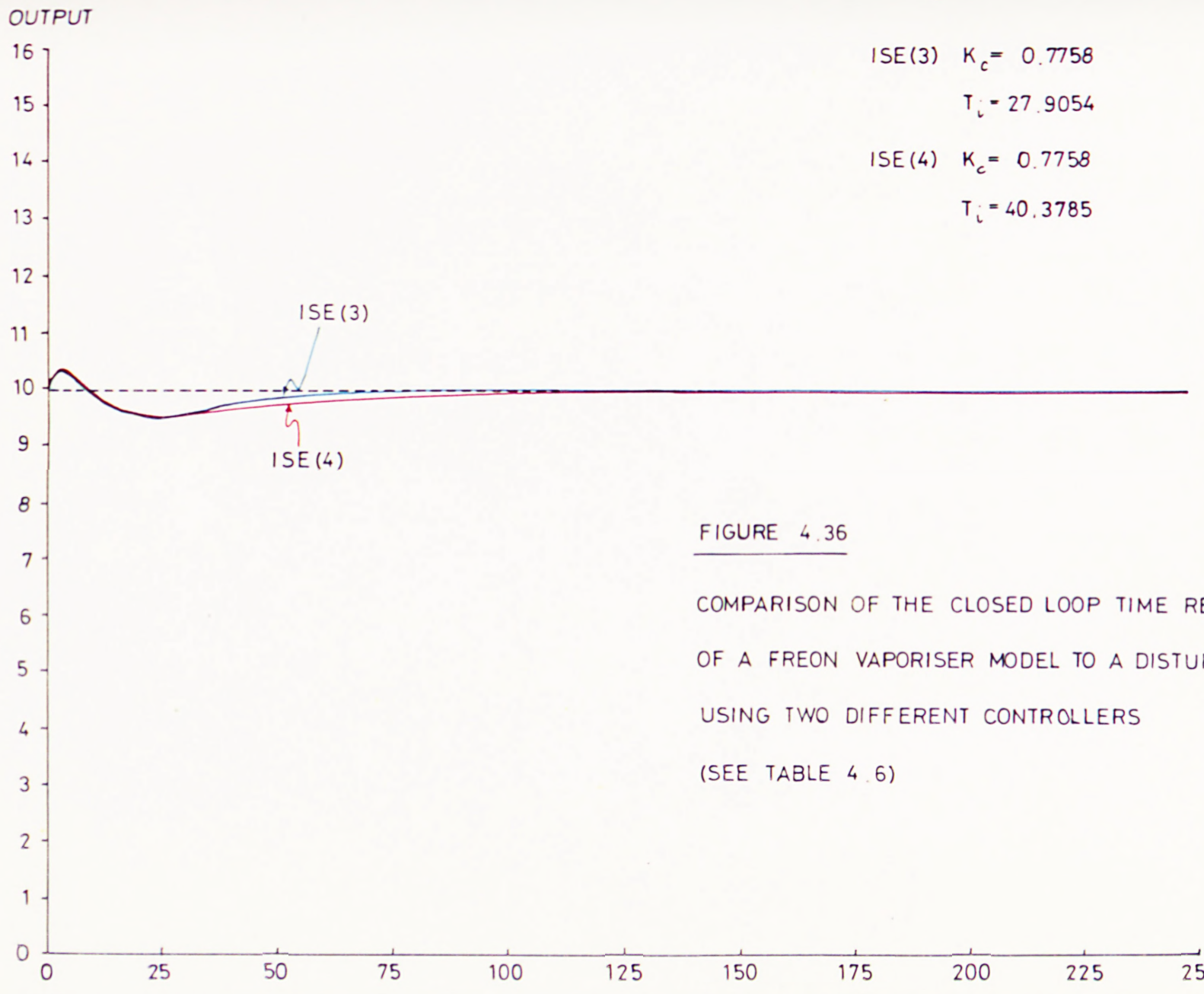


FIGURE 4.36

COMPARISON OF THE CLOSED LOOP TIME RESPONSE OF A FREON VAPORISER MODEL TO A DISTURBANCE (2) USING TWO DIFFERENT CONTROLLERS

(SEE TABLE 4.6)

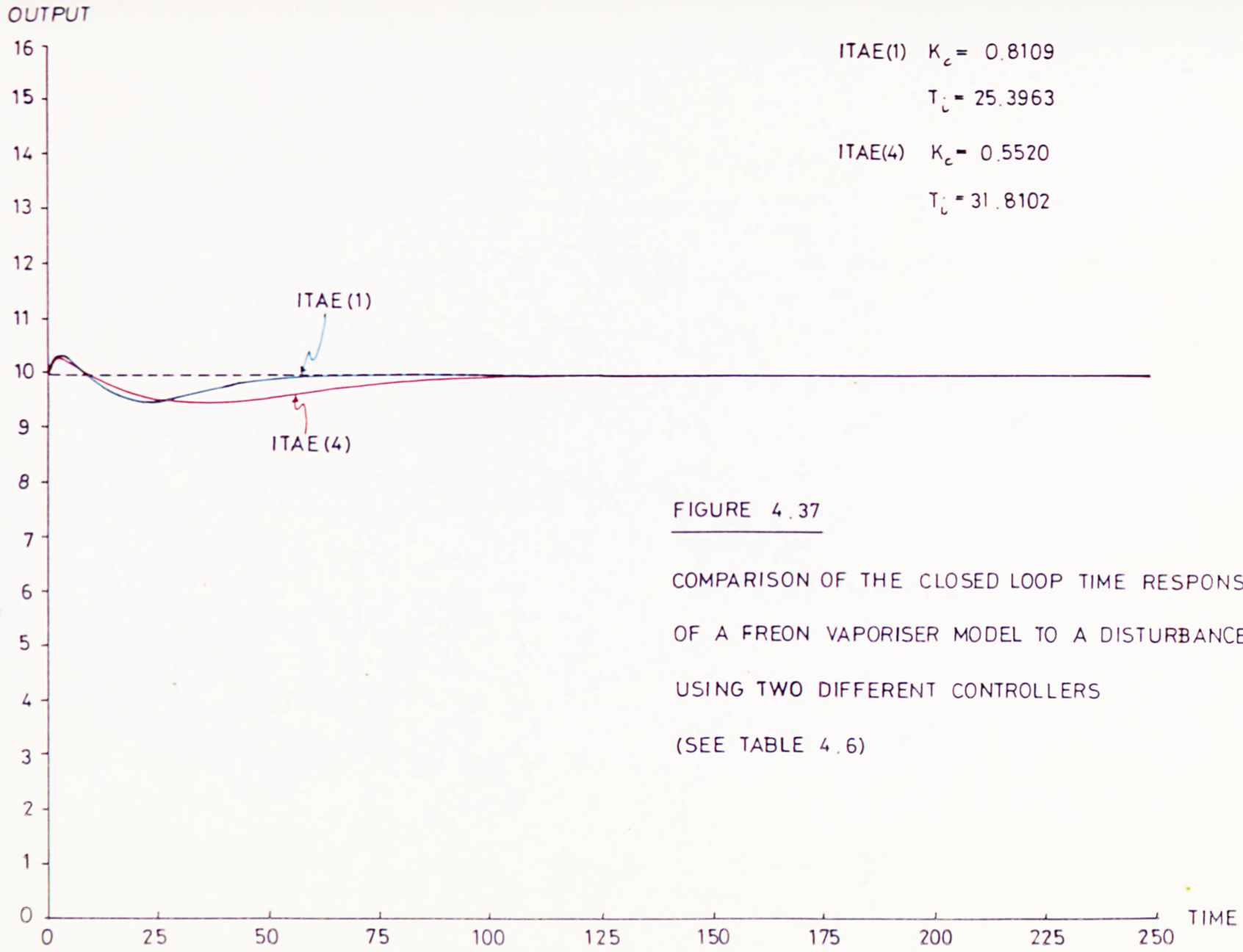


FIGURE 4.37

COMPARISON OF THE CLOSED LOOP TIME RESPONSE OF A FREON VAPORISER MODEL TO A DISTURBANCE (2) USING TWO DIFFERENT CONTROLLERS (SEE TABLE 4.6)

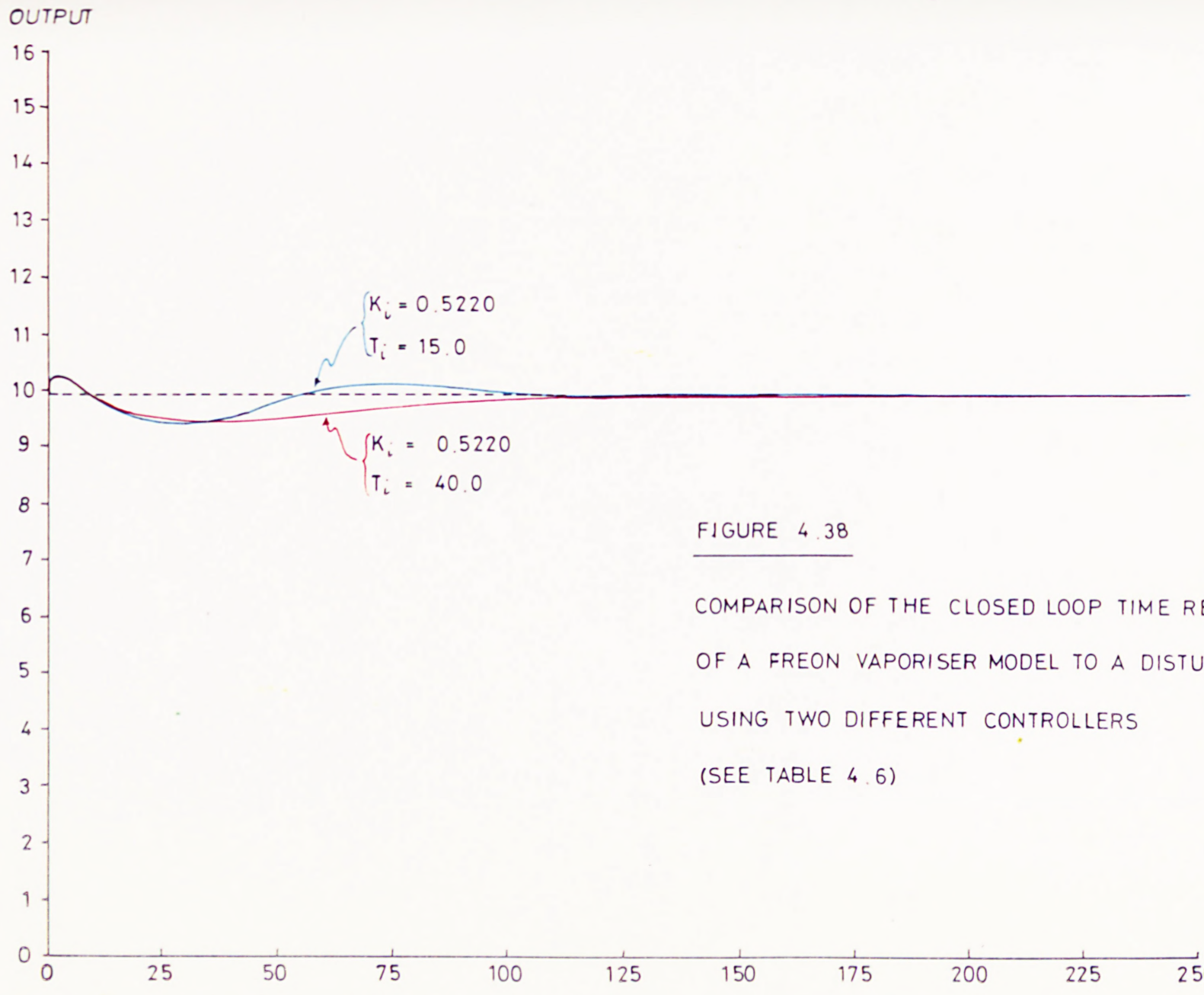
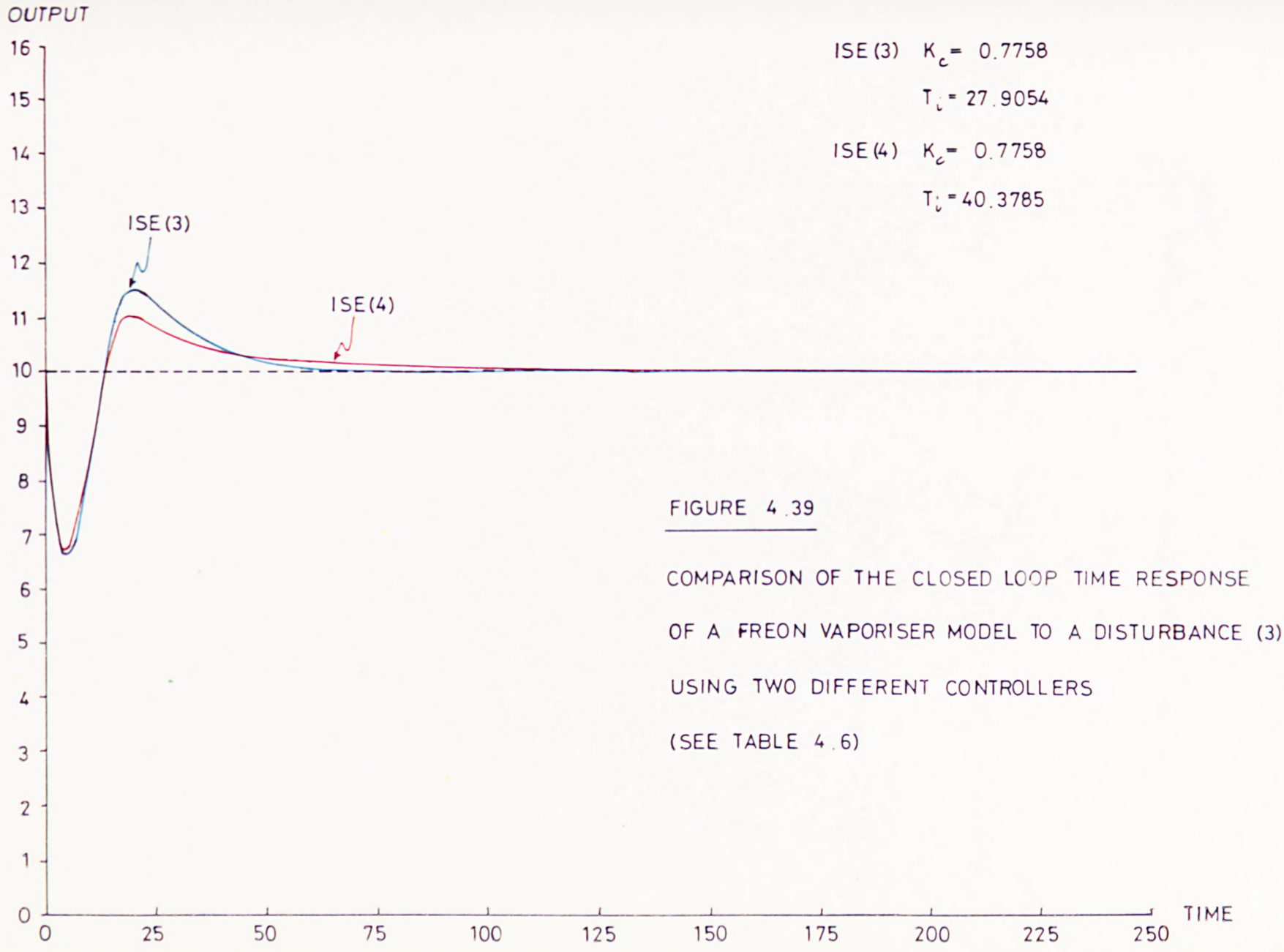


FIGURE 4.38

COMPARISON OF THE CLOSED LOOP TIME RESPONSE OF A FREON VAPORISER MODEL TO A DISTURBANCE (2) USING TWO DIFFERENT CONTROLLERS (SEE TABLE 4.6)



ISE (3) K_c = 0.7758

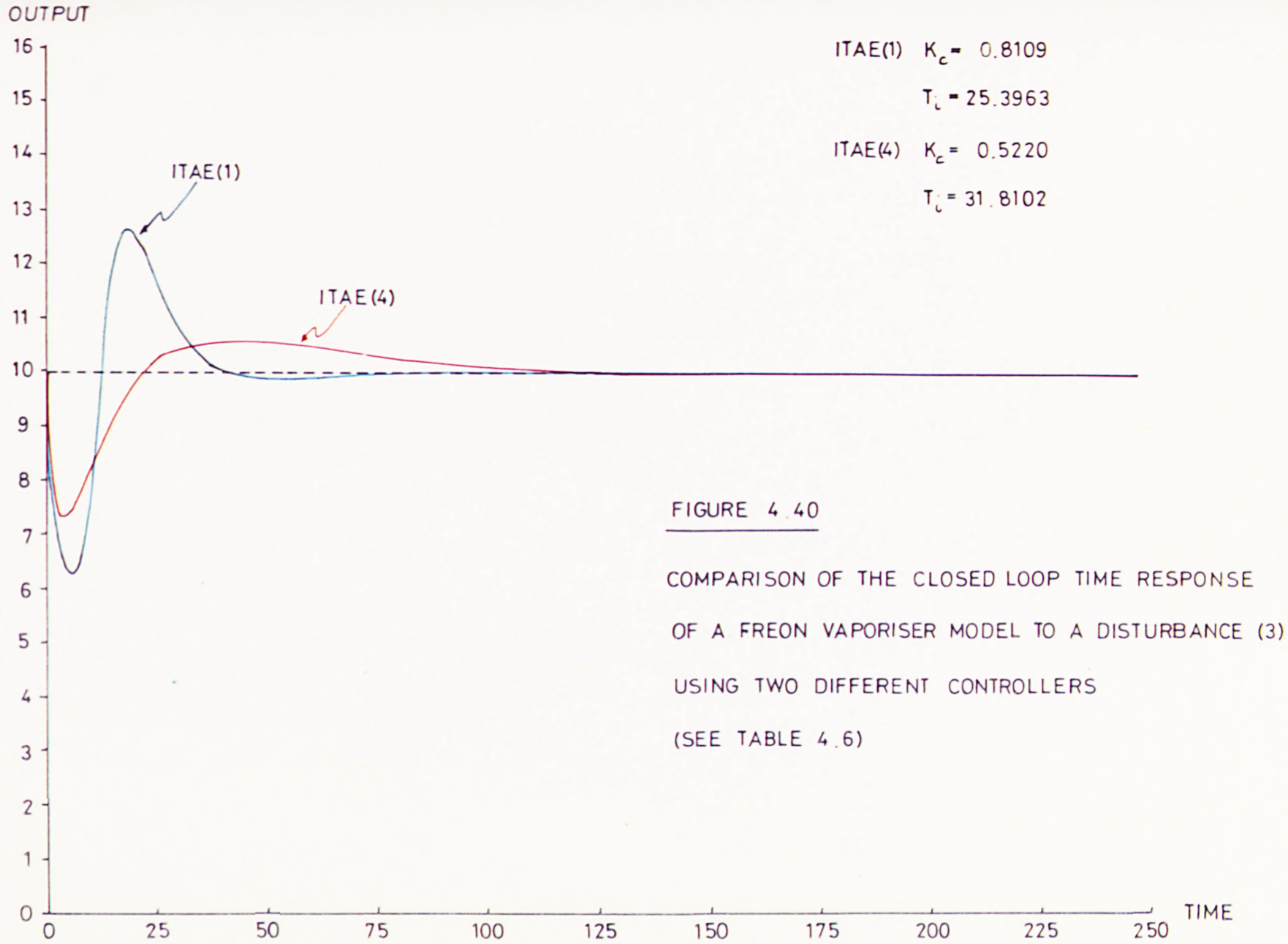
T_i = 27.9054

ISE (4) K_c = 0.7758

T_i = 40.3785

FIGURE 4.39

COMPARISON OF THE CLOSED LOOP TIME RESPONSE OF A FREON VAPORISER MODEL TO A DISTURBANCE (3) USING TWO DIFFERENT CONTROLLERS (SEE TABLE 4.6)



ITAE(1) K_c = 0.8109

T_i = 25.3963

ITAE(4) K_c = 0.5220

T_i = 31.8102

FIGURE 4.40

COMPARISON OF THE CLOSED LOOP TIME RESPONSE OF A FREON VAPORISER MODEL TO A DISTURBANCE (3) USING TWO DIFFERENT CONTROLLERS (SEE TABLE 4.6)

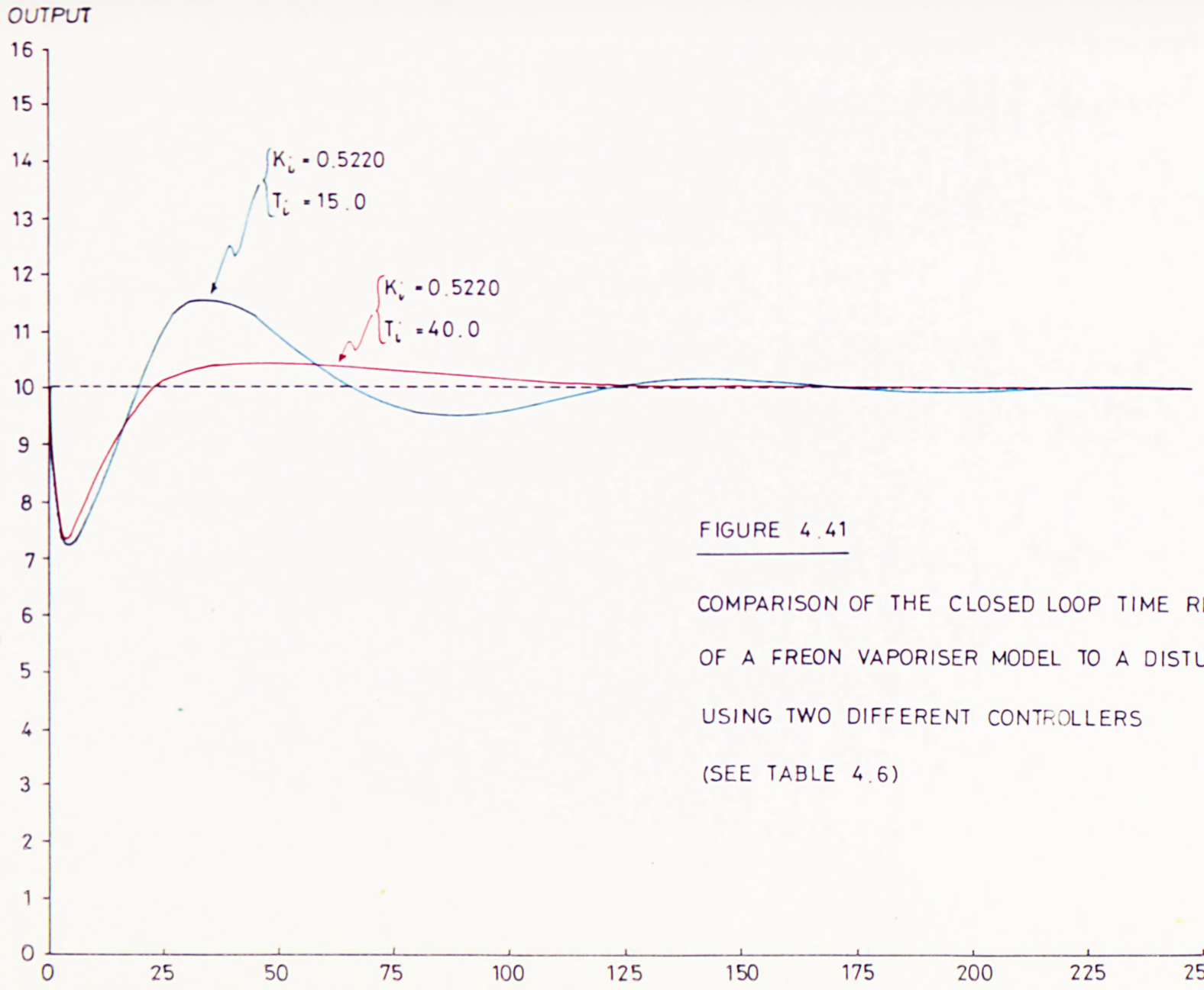


FIGURE 4.41

COMPARISON OF THE CLOSED LOOP TIME RESPONSE OF A FREON VAPORISER MODEL TO A DISTURBANCE (3) USING TWO DIFFERENT CONTROLLERS (SEE TABLE 4.6)

wall temperature was changed from 364 to 359 K. This disturbance had a greater effect on the output than the second and it is more difficult to determine which controller gives the best response. The lowest ITAE value is given using the controller calculated using the min.ISE formulae on the model parameters obtained using method 3. This controller gives the shortest settling time although it gives a relatively large maximum error. On the other hand, the smoothest response appears to be given by the last controller (K_c = 0.5220, T_i = 40.0) which gives the smallest maximum error of the six controllers, while giving a relatively long settling time. A good ISE value would probably be given but no such value was calculated.

From these plots and the error criterion values listed in table 4.6, the controllers which are shown to be suitable by the frequency plots in the previous subsection are confirmed to be so. All the controllers obtained from the critical point models give stable control, as does the ITAE controller obtained from the inadequate Padé model.

Similar closed loop simulations have also been carried out on the macroscopic model of the chlorine vaporiser. For these simulations controller parameters calculated using the min.ISE formulae with models 1 to 4 were used. (ISE and ITAE values are given in table 4.7), Again three different

Model No.	K_c	T_i	Disturbance 1	Disturbance 2	Disturbance 3
			I.S.E. VALUES		
1	9.0379	17.7264	82.45	1.19	41.18
3	7.3729	16.7718	1.11	1.26	28.13
4	"	19.8815	0.92	1.34	25.44
2	"	20.7345	0.89	1.37	24.94
			I.T.A.E. VALUES		
1	9.0379	17.7264	22451.5	123.5	5396.7
3	7.3729	16.7718	277.9	143.5	684.0
4	"	19.8815	194.9	202.2	555.6
2	"	20.7345	182.0	220.1	550.5

Disturbance 1 - Change of downstream pressure from 276,000 to 246,000 N/m^2.

Disturbance 2 - Change of input pressure from 760,000 to 680,000 N/m^2.

Disturbance 3 - Change of vaporiser wall temperature from 388 to 368 K.

Note that disturbance 1 produces a limit cycle in the closed loop system containing the controller based on the Padé model(1).

Table 4.7 Comparison of min. ISE controllers obtained using simple models calculated from a linearised third order model of a chlorine vaporiser, (linearised about an output value of 9.5), the comparison using ISE and ITAE values over 300 secs following 3 different disturbances.

disturbances were used. The output from two of these closed loop systems (including controllers 1 and 4) is plotted in figs.4.42 to 4.44. Fig.4.42 shows the response to a downstream pressure change from 276,000 to 246,000 N/m^2. This disturbance causes the system, including the controller based on the inadequate Padé model, to enter a limit cycle. The other controller (based on model 4) gives adequate, if slightly underdamped control. Error values in table 4.7 show that for this disturbance model 2 gives best results.

Fig.4.43 shows the response of these two closed loop systems to a change in input pressure from 760,000 to 680,000 N/M^2. As in the case of the freon vaporiser, a change in the input pressure does not greatly affect the vaporiser output and consequently all four controllers give quite adequate results. Error values indicate that model 1 gives best results.

Fig.4.44 shows the response to a change in the vaporiser wall temperature from 388 to 368 K.

As shown, the controller based on the Padé model gives stable control, but it is very underdamped and cannot be considered adequate. Model 4 gives much better results although the output from the closed loop system based on this model is still underdamped. Overall, for this disturbance, model 2 gave the best results.

Further closed loop simulations were carried out using controllers calculated using the IAE formulae

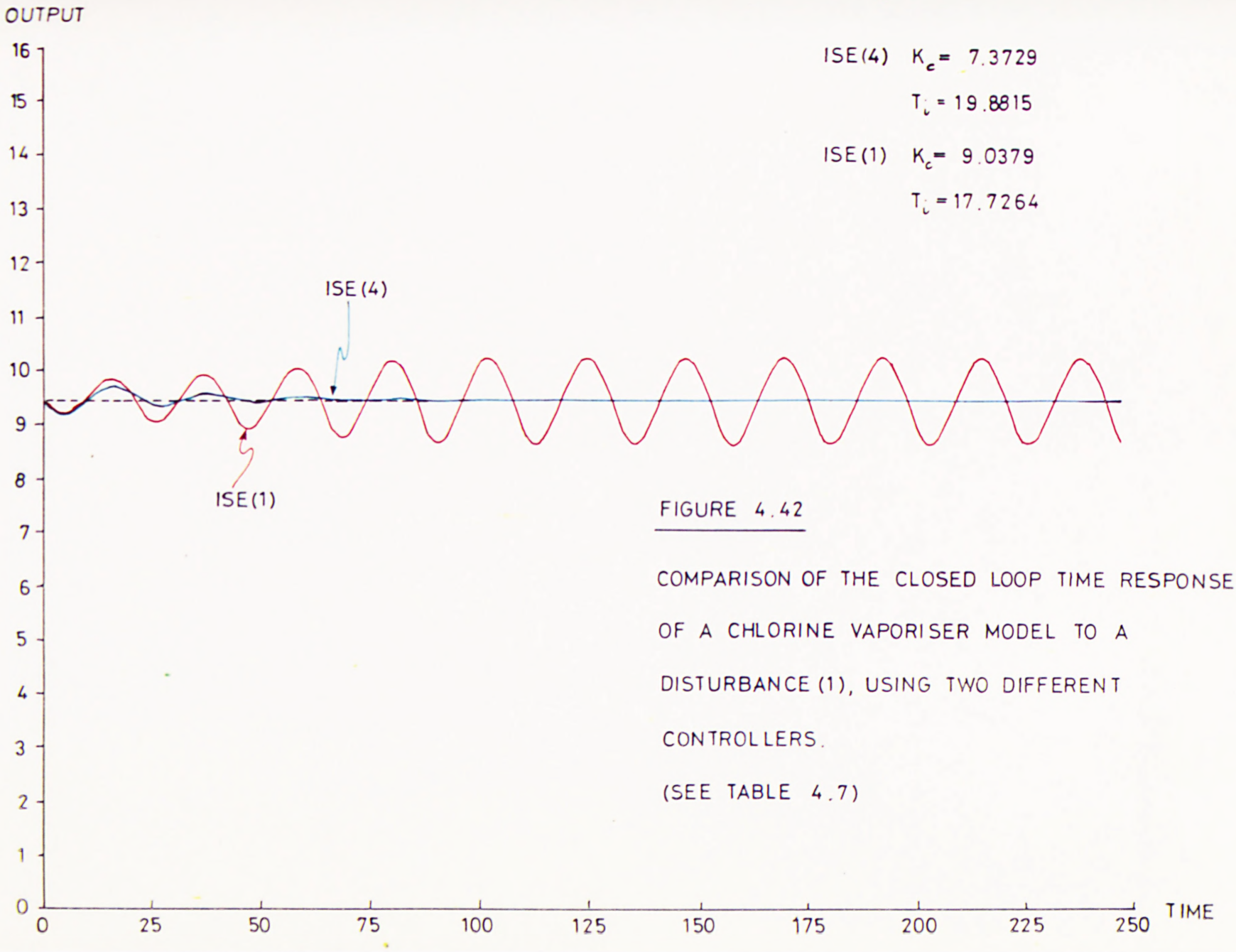
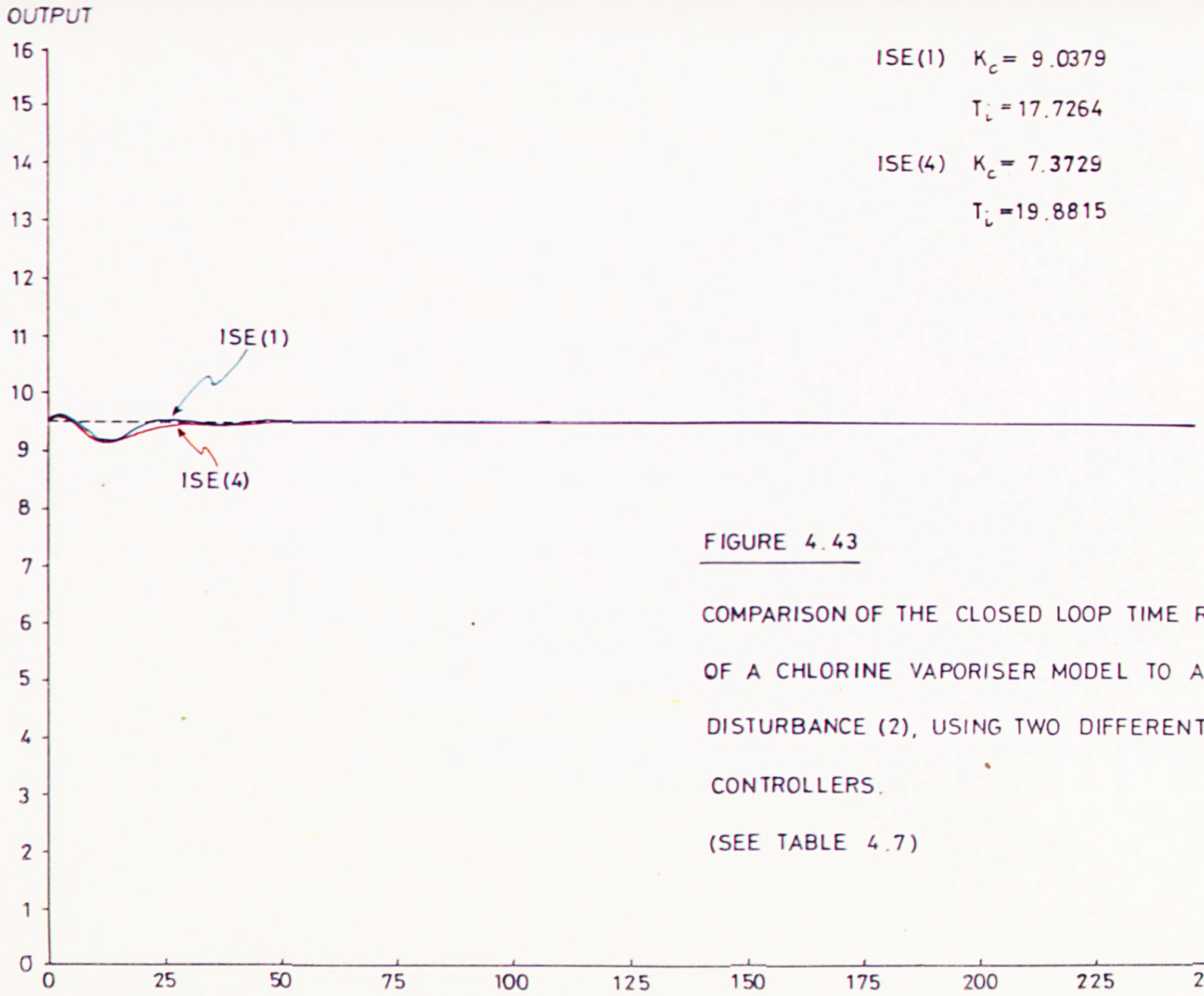


FIGURE 4.42

COMPARISON OF THE CLOSED LOOP TIME RESPONSE OF A CHLORINE VAPORISER MODEL TO A DISTURBANCE (1), USING TWO DIFFERENT CONTROLLERS.

(SEE TABLE 4.7)



ISE(1) K_c = 9.0379

T_i = 17.7264

ISE(4) K_c = 7.3729

T_i = 19.8815

FIGURE 4.43

COMPARISON OF THE CLOSED LOOP TIME RESPONSE OF A CHLORINE VAPORISER MODEL TO A DISTURBANCE (2), USING TWO DIFFERENT CONTROLLERS.

(SEE TABLE 4.7)

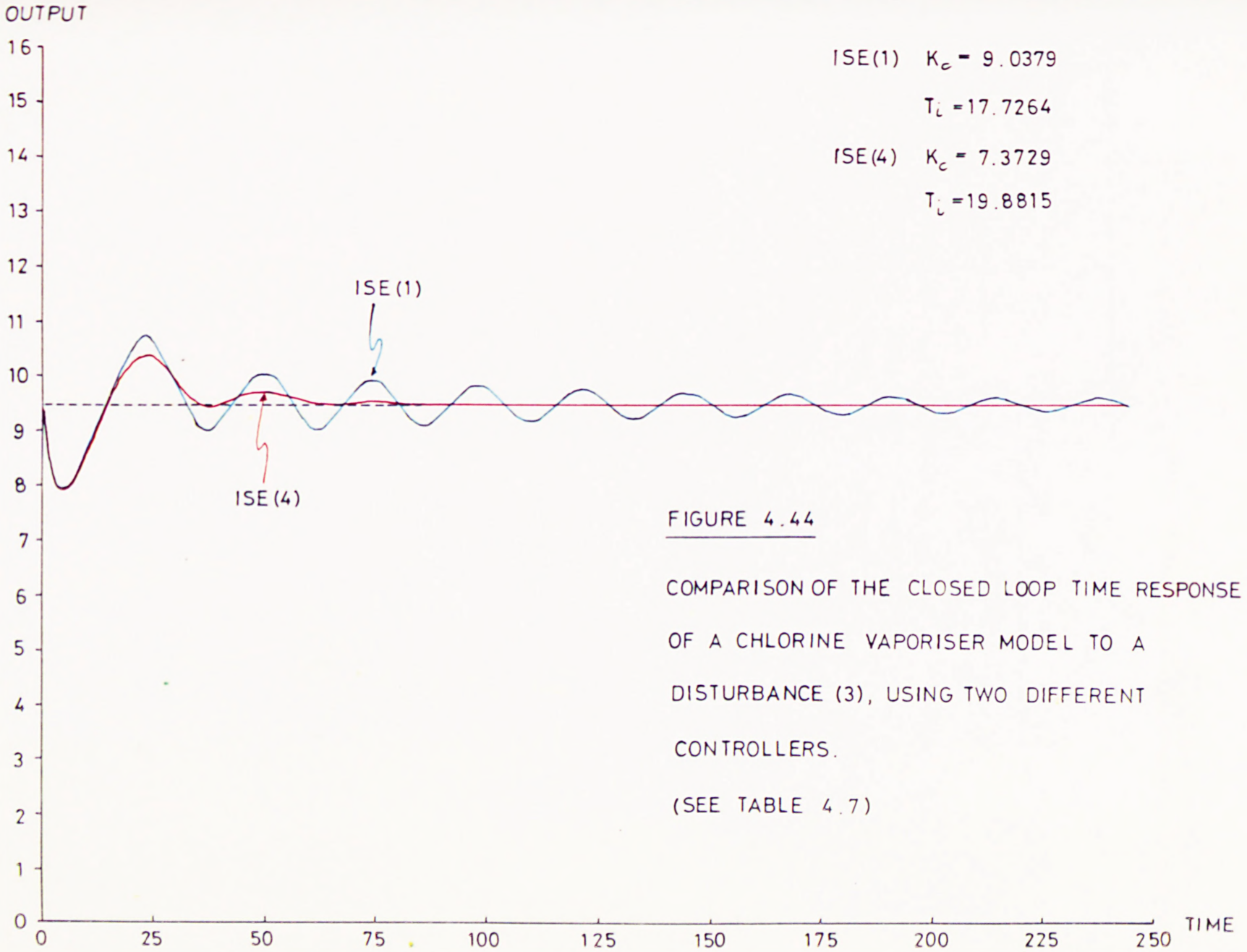


FIGURE 4.44

COMPARISON OF THE CLOSED LOOP TIME RESPONSE OF A CHLORINE VAPORISER MODEL TO A DISTURBANCE (3), USING TWO DIFFERENT CONTROLLERS.

(SEE TABLE 4.7)

Model No.	K_c	T_i	Set Point Changes		
			10→8	10→6	10→12
			I S E VALUES		
1	4.7596	20.5246	62.75	365.88	60.45
2	3.9400	20.1253	60.60	360.37	64.33
4	"	19.1061	61.62	364.80	63.35
3	"	16.1592	65.57	378.71	60.70
			I T A E VALUES		
1	4.7596	20.5246	930.73	6436.19	2911.41
2	3.9400	20.1253	818.68	6069.73	3854.92
4	"	19.1061	837.00	6135.63	3516.25
3	"	16.1592	883.62	6340.67	2530.77

Table 4.8 Comparison of min. IAE controllers based on simple models obtained from a linearised third order model of a chlorine vaporiser, linearised about an output value of 10, the comparison using ISE and ITAE values over 300 secs. following 3 different set point changes for closed loop simulation of the macroscopic model.

for set point change controllers from each of the four models previously evaluated. Each of these closed loop systems was simulated with three different set point changes: 10 to 8, 10 to 6 and 10 to 12. Error values from these sinulations are given in table 4.8 and responses to these three set point changes of two of the systems are shown in figs.4.45 to 4.47. The results show that similar responses are given by all four systems. It will be noticed, however, that for a set point change from 10→12 the response is over-damped and due to the non-linearity of the macroscopic model, the final value at 300 sec. is only approximately 11.97.

It would therefore appear that for complete optimum control the non-linearity of the macroscopic model should be catered for in the design of the controller.

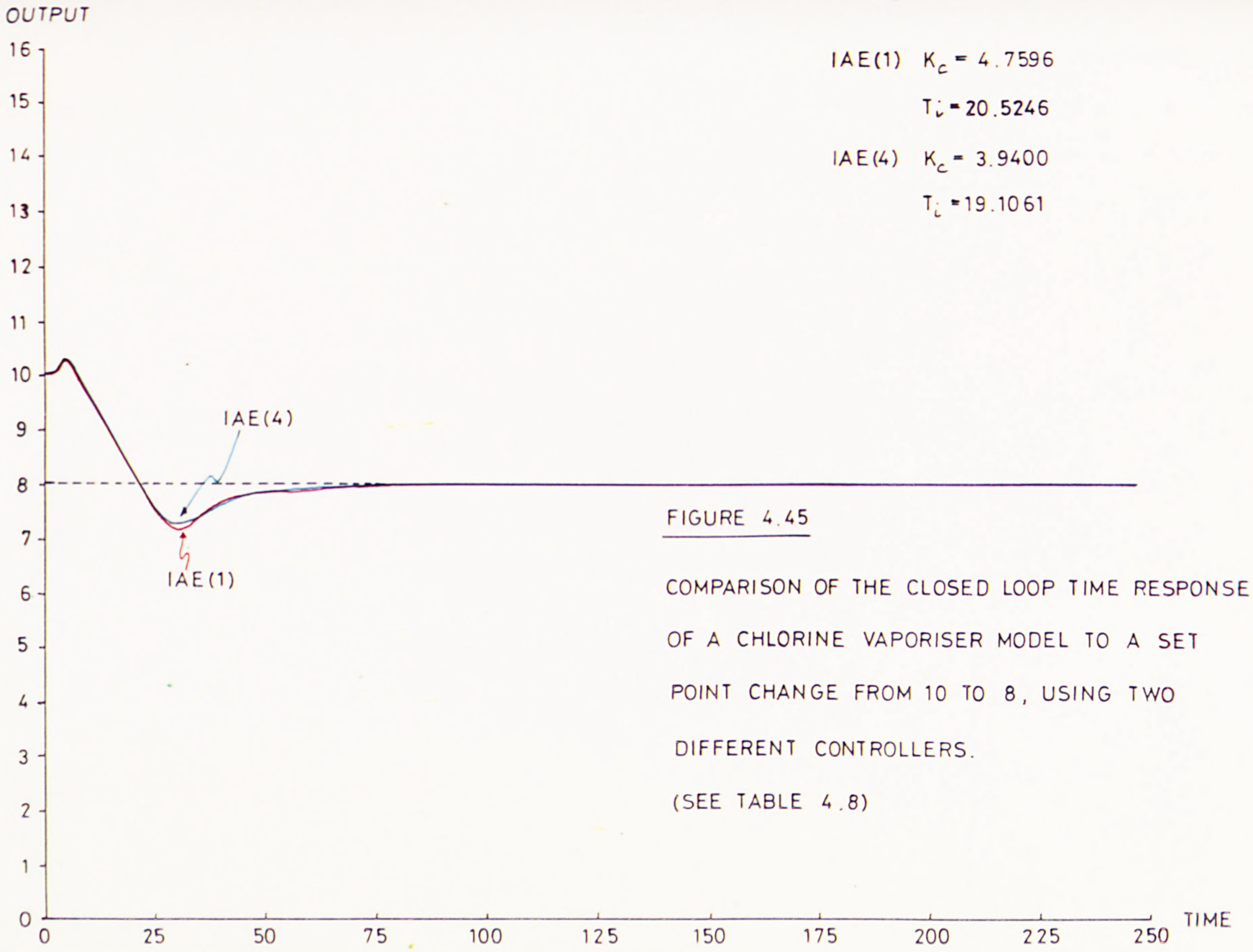


FIGURE 4.45

COMPARISON OF THE CLOSED LOOP TIME RESPONSE OF A CHLORINE VAPORISER MODEL TO A SET POINT CHANGE FROM 10 TO 8, USING TWO DIFFERENT CONTROLLERS.

(SEE TABLE 4.8)

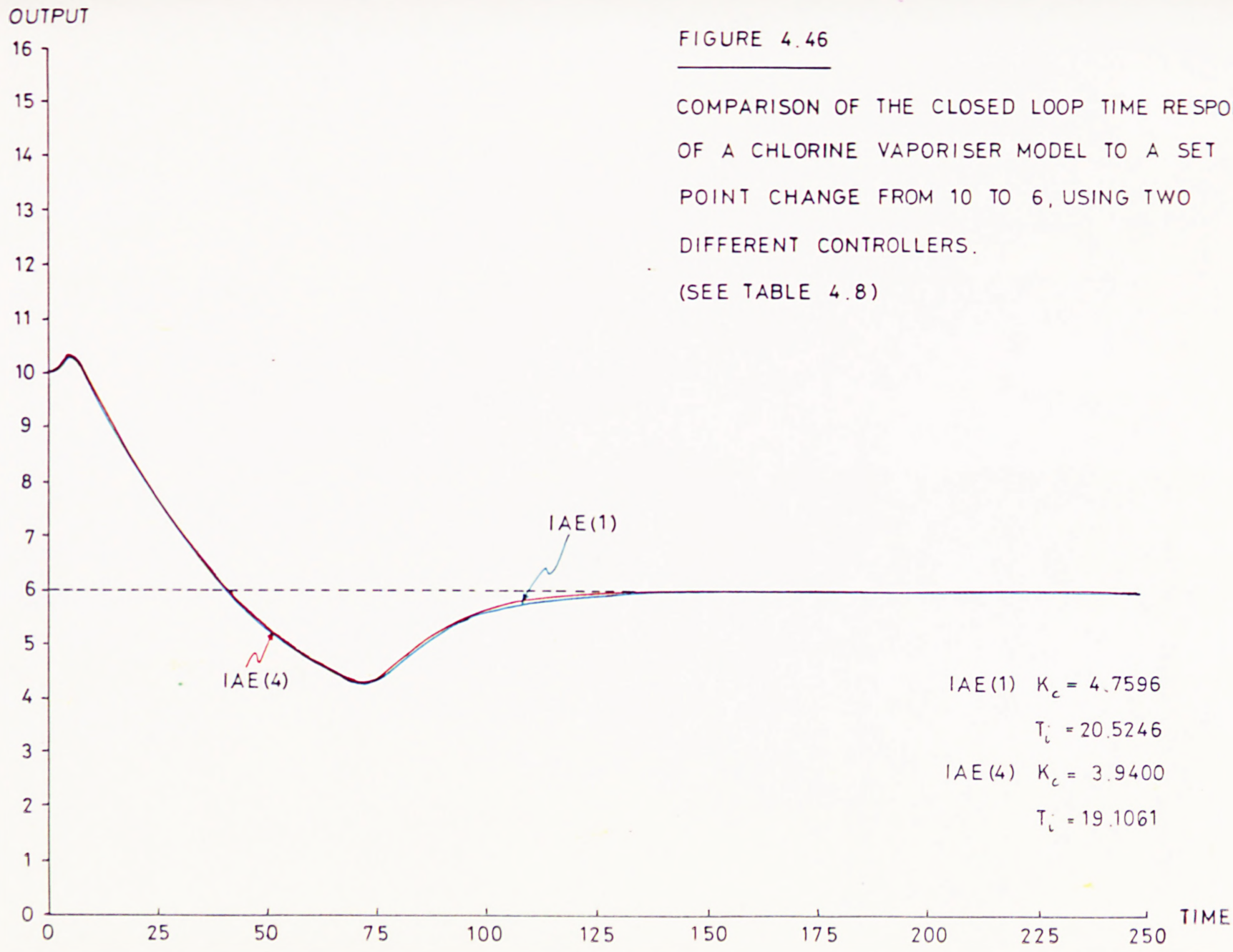


FIGURE 4.46

COMPARISON OF THE CLOSED LOOP TIME RESPONSE OF A CHLORINE VAPORISER MODEL TO A SET POINT CHANGE FROM 10 TO 6, USING TWO DIFFERENT CONTROLLERS.

(SEE TABLE 4.8)

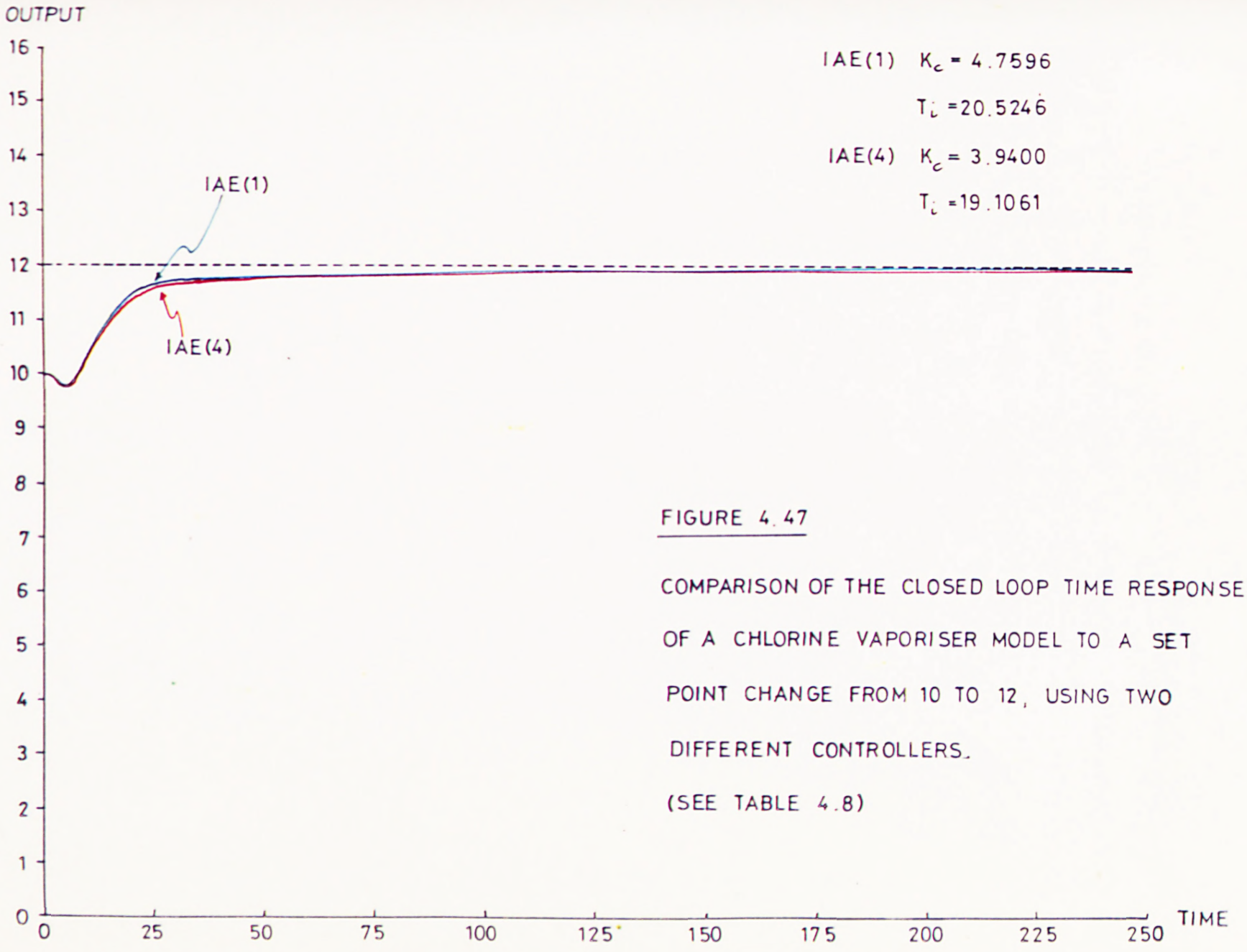


FIGURE 4.47

COMPARISON OF THE CLOSED LOOP TIME RESPONSE OF A CHLORINE VAPORISER MODEL TO A SET POINT CHANGE FROM 10 TO 12, USING TWO DIFFERENT CONTROLLERS.

(SEE TABLE 4.8)

5. CONCLUSIONS

As can be seen from section 2 of this thesis much work has been carried out in recent years on model reduction techniques. This work has been very varied and many different reduction methods have been put forward. Authors have viewed this subject from diverse angles. These works have been studied and a comprehensive review of those published up to 1979 is contained in section 2. First order models have, however, scarcely been mentioned in these works despite the fact that they are very suitable for controller design as can be seen in section 4. So as to obtain the best possible controller it is desirable to design it using a simple model which fits the system as closely as possible. For this reason the Padé reduction technique was initially chosen for the calculation of the first order plus time delay model, as at lower frequencies it gives a very good fit in terms of frequency response. It has been shown that not all first order plus time delay models can be used to design usable controllers, as models which are "too" stable will produce controllers which will result in unstable closed loop systems. For this reason a criterion is required for determining which models are suitable for controller design. Such a criterion has been suggested in section 3 and various systems up to and including fifth order systems, reduced using the Padé technique have been tested for adequacy according

to this criterion. From these tests adequacy boundaries have been obtained and these are given in section 3. For systems that are covered by these charts the adequacy of their Padé model can be checked by simply normalising the system transfer function and referring to the appropriate diagram. If the model is shown to be adequate no further investigation is required and the reduced model can safely be used for controller design. For higher order systems the adequacy of the Padé reduced model can be ascertained by using the computer program MALG, listed in appendix 2, which has been written for this purpose and which was used to determine the adequacy boundaries. From the charts, it can be seen that there are large areas of inadequacy. For these systems alternative reduction techniques have been suggested which guarantee adequacy according to our criterion. These however give a less accurate fit at lower frequencies. Results in section 4 suggest that the adequacy criterion used might be too strict, as models considered to be inadequate according to this criterion have given very suitable controllers. It is consequently suggested that the adaptation of this criterion be attempted so as to make the adequacy boundary correspond more closely with the boundary between satisfactory and unsatisfactory controllers obtained from the reduced models.

It can thus be concluded that, in this thesis, it has been established that suitable controllers can be designed using reduced models for single-input single-output systems, and that the Padé reduction technique can be used for this purpose with certain systems. For other systems alternative methods give acceptable results.

It has also been shown that the certain adequacy of a Padé model can easily be established.

APPENDIX I

HSIA'S REDUCTION TECHNIQUE AS APPLIED TO SYSTEMS CONTAINING PURE TIME DELAY

If we write $\frac{dm(s)}{ds}$ as m' and consider a function

$$m(s) = y(s)e^{-\tau s} \quad \text{then} \qquad .....(C1)$$

$$m' = (y'-\tau y)e^{-\tau s} \qquad .....(C2)$$

$$m'' = (y''-2\tau y'+\tau^2 y)e^{-\tau s} \qquad .....(C3)$$

$$m''' = (y''' -3\tau y''+3\tau^2 y'-\tau^3 y)e^{-\tau s} \qquad .....(C4)$$

$$m^{iv} = (y^{iv} -4\tau y''' +6\tau^2 y'-4\tau^3 y'+\tau^4 y)e^{-\tau s} \qquad .....(C5)$$

etc.

Substituting into eqn. 239 and taking $e^{-\tau s}/_{s=0}$ as 1 gives

$$M_2 \Big|_{s=o} = \Big[(m')^2-m^o m''\Big]_{s=o}$$

$$= \Big[(y'-\tau y)^2-y(y''-2\tau y'+\tau^2 y)\Big]_{s=o}$$

$$= \Big[(y')^2-yy''\Big]_{s=o} \qquad .....(C6)$$

similarly substituting into eqn.240

$$M_4 \Big|_{s=o} = \frac{1}{12}\Big[3(m'')^2-4m'm''' +m^o m^{iv}\Big]_{s=o}$$

$$= \frac{1}{12}\Big[3(y''-2\tau y'+\tau^2 y)^2-4(y'-\tau y)(y''' -3\tau y''+3\tau^2 y'-\tau^3 y) + y(y^{iv} -4\tau y''' +6\tau^2 y''-4\tau^3 y'+\tau^4 y)\Big]_{s=o}$$

$$= \frac{1}{12}\Big[3(y'')^2-4y'y''' +yy^{iv}\Big]_{s=o} \qquad .....(C7)$$

similarly

$$M_6 \Big|_{s=o} = \frac{1}{360}\Big[10(m''')^2-15m''m^{iv} +6m'm^{v}-m^o m^{vi}\Big]_{s=o}$$

$$= \frac{1}{360}\Big[10(y''')^2-15y''y^{iv} +6y'y^{v}-y^o y^{vi}\Big]_{s=o} \qquad .....(C8)$$

As can be seen, the $e^{-\tau s}$ term in m(s) completely disappears in the terms M_2, M_4, M_6 etc.

This result is self evident if we consider the basis of this reduction technique. As can be seen from equation 233, the technique is based on the minimisation of the error in magnitude between the original system and the reduced model without considering the phase difference and of course a pure time delay will only affect the phase difference. In the second half of the equation we find the term M(jw).M(-jw). If as previously $(M(jw)=y(jw)e^{-\tau jw}$ then

$$M(jw).M(-jw) = y(jw)e^{-\tau jw}\ y(-jw)e^{+\tau jw}$$

$$= y(jw)y(-jw) \qquad \ldots\ldots(C9)$$

APPENDIX II

COMPUTER PROGRAM LISTING

A listing follows of program MALG which was used to determine the adequacy boundaries for Pade reduction.

A model of order N1 given by

$$G(s) = \frac{GAIN(1+B(1)S+B(2)S^2+...+B(M1)S^{M1})e^{-TOR.S}}{1+A(1)S+A(2)S^2+A(3)S^3+\quad A(N1)S^{N1}}$$

is reduced to

$$\hat{G}(s) = \frac{GAIN.e^{-TOR1.S}}{1+T(1)S}$$

All the parameters of the full model are input, apart from A(2) and A(NX), where NX is also an input value between 3 and N1 inclusive. A2MAX and A2MIN are calculated according to the stability and realisability criteria. Simple models are then evaluated for values of A(2) between A2MIN and A2MAX giving $\hat{G}(j\hat{\omega}_c)$. ANXMIN and ANXMAX are then calculated from the stability criteria and full models are evaluated for the same values of A(2) and values of A(NX) between ANXMIN and ANXMAX giving $G(j\omega)$ for each model. From these calculations an adequacy boundary is obtained in the A(2), A(NX) plane. The program also evaluates roots of the characteristic equation and gives boundaries between real and imaginary and positive and negative roots.

```
PROGRAM MALG       76/76   OPT=2   TRACE                    UMRCC FTN 4.2.178     01

      PROGRAM MALG (INPUT,OUTPUT,TAPE1=INPUT,TAPE2=OUTPUT)
      DIMENSION A(10),B(10),R(10),G(4),T(10),V(6,100),U(4),Z(4,100)
      COMPLEX  R
      COMMON NX
C
C          M1                 =          ORDER OF  B (NUMERATOR)
C          N1                 =          ORDER OF  A (DENOMINATOR)
C          IPT1               =          NO. OF POINTS / DECADE
C          A(1)-A(N)          =          DENOMINATOR COEFFICIENTS
C          B(1)-B(N)          =          NUMERATOR   COEFFICIENTS
C          GAIN               =          GAIN OF PROCESS AND MODEL
C          TOR1               =          TIME DELAY OF PROCESS
C          TOR                =          TIME DELAY OF MODEL
C          W                  =          FREQUENCY (ANGULAR VELOCITY=RADS-SEC)
C          W1                 =          INITIAL FREQUENCY
C          T(1)               =          DENOMINATOR COEFFICIENT OF MODEL)

C         READ IN DATA

      B(1) = 0.0
      B(2) = 0.0
      READ (1,1) NX
    1 FORMAT (10I5)
      READ (1,1) M1,N1,IPT1
      IF (M1 .EQ. 0)  GO TO 40
      READ (1,2) (B(I),I=1,M1)
    2 FORMAT (8F10.6)
   40 READ (1,2) A(1)
      IF (N1 .GT.3) READ (1,2) (A(I),I=4,N1)
      IF (NX .NE. 3) A(3) = A(NX)
      READ (1,2) GAIN,TOR1,W1

C         EVALUATION OF MAXIMUM VALUE OF A(2) FOR REALISABILITY OF SIMPLE
C         MODEL   (T = 0.0)

      A2MAX = B(2)+(A(1)**2-B(1)**2)/2.0

C         EVALUATION OF MINIMUM VALUE OF A(2) FOR REALISABILITY OF SIMPLE
C         MODEL   (TOR = 0.0)

      A2MIN = B(2)-(TOR1**2)/2.0-B(1)**2+TOR1*(B(1)-A(1))+A(1)*B(1)
      IF (A2MIN .LE. 0.0) A2MIN = 0.0

      N = N1
      IF (N .NE. 4) GO TO 50
      IF (NX .EQ. 3) A2 = 2.0*SQRT(A(4))
      IF (NX .EQ. 4) A2 = A(3)/A(1)
      IF (A2 .GT. A2MIN)  A2MIN = A2
   50 CONTINUE
      WRITE (2,16) GAIN,TOR1
   16 FORMAT (1H1,10X,6H GAIN:,F14.6,10X,6H TOR1:,F14.6)
      WRITE (2,3) (A(I),I=1,N1)

C         EVALUATION OF ZEROES
```

```
PROGRAM MALG       76/76   OPT=2   TRACE                    UMRCC FTN 4.2.178    01.

                  IF (M1 .EQ. 0)  GO TO 41
                  WRITE (2,4) (B(I),I=1,M1)
                  IF (M1 .EQ. 1) GO TO 41
                  CALL ROOTZ (M1,B,M1,0.001,R,NR,L)
                  WRITE (2,5) (R(I),I=1,M1)
               41 WRITE (2,6) A2MAX
                  WRITE (2,7) A2MIN
                  IF (A2MIN .GE. A2MAX) STOP
                3 FORMAT (1H0,10X,3H A:,8F14.6)
                4 FORMAT (1H0,10X,3H B:,8F14.6)
                5 FORMAT (1H0,10X,23H THEREFORE ZEROS ARE:  ,10(/20X,2F10.6,1HJ))
                6 FORMAT (1H0,10X,8H A2MAX = ,F10.6)
                7 FORMAT (11X,8H A2MIN =,F10.6)
               63 A2 = A2MAX -A2MIN
                  A2DIF = 0.5
                  IF (A2 .LE. 15.0) A2DIF = 0.25
                  IF (A2 .LE. 12.0) A2DIF = 0.2
                  IF (A2 .LE.  6.0) A2DIF = 0.1
                  IF (A2 .LE.  3.0) A2DIF = 0.05
                  IF (A2 .LE.  1.5) A2DIF = 0.025
                  IF (A2 .LE.  1.2) A2DIF = 0.02
                  IF (A2 .LE.  0.6) A2DIF = 0.01
                  IF (A2 .LE. 0.30) A2DIF = 0.005
                  IF (A2 .LE. 0.15) A2DIF = 0.002
                  IF (A2 .LE. 0.06) A2DIF = 0.001
                  IF (A2 .LE. 0.03) A2DIF = 0.0005
                  I2MIN = A2MIN / A2DIF+ 1.0
                  I2MAX = A2MAX / A2DIF + 2.0

               C        EVALUATION OF SIMPLE MODEL PARAMETERS AND CRITICAL FREQUENCY
               C        RESPONCE OF SIMPLE MODEL

                  WRITE (2,8)
                8 FORMAT (46H0SIMPLIFIED MODEL VALUES AT CRITICAL FREQUENCY //7X,4HA
                 -(2),11X,4HREAL,11X,4HIMAG,8X,9HMAGNITUDE,8X,5HPHASE,12X,1HW,14X,
                 -1HT,13X,3HTOR /)
                  W2 =  1.0/IPT1
                  DO 21  I = I2MIN,I2MAX
                  A(2) = (I-1)*A2DIF
                  IF(A(2) .EQ. 0.0) A(2) = 1.0E-20
                  IF (A(2) .GT. A2MAX)  A(2) = A2MAX
                  IF (A(2) .LT. A2MIN) A(2) = A2MIN
                  T(1) = SQRT ((A2MAX - A(2))*2.0)
                  TOR  = TOR1+A(1)-B(1)-T(1)
                  W  =  W1
                  CALL CRIT (W2,W,0.1,T,B,G,GAIN,TOR,V,1)
                  WRITE (2,9) A(2),G,W,T(1),TOR
                9 FORMAT (8E15.6)
                  J = I +1- I2MIN
                  V(3,J) = -99.0
                  V(4,J) = -99.0
                  V(5,J) = -99.0
                  V(6,J) = -99.0
                  DO 30  JJ = 1,4
               30 Z(JJ,J) = -99.0
```

```
      V(1,J) = A(2)
   21 V(2,J) = G(1)
      A3MAX = A(1) * A2MAX
   51 IF (N .NE. 4) GO TO 52
      IF (NX .EQ. 4) A3MAX = (A2MAX - A(3)/A(1))*A(3)/A(1)
      IF (NX .NE. 3) GO TO 52
      A3X = (A2MAX/2.0)**2-A(4)
      IF (A3X .LT. 0.0) A3X = 0.0
      A3MAX = (A2MAX/2.0 + SQRT(A3X))*A(1)
   52 CONTINUE
      A3DIF = 1.0
      IF (A3MAX .LE. 18.0) A3DIF = 0.1
      IF (A3MAX .LE.  9.0) A3DIF = 0.05
      IF (A3MAX .LE.  4.5) A3DIF = 0.02
      IF (A3MAX .LE.  1.8) A3DIF = 0.01
      IF (A3MAX .LE.  0.9) A3DIF = 0.005
      IF (A3MAX .LE.  0.45) A3DIF = 0.002
      IF (A3MAX .LE.  0.18) A3DIF = 0.001
      IF (A3MAX .LE. 0.09)  A3DIF = 0.0005
      IF (A3MAX .LE. 0.045) A3DIF = 0.0002
      IF (A3MAX .LE. 0.018) A3DIF = 0.0001
      IF (A3MAX .LE. 0.009) A3DIF = 0.00005
      IF(A3MAX .LE. 0.0) STOP
      CK = 2.5
      I3MIN = 1
      IF (N .GE. 5) A3 = A(1)/A3DIF
      IF (N .EQ. 3) A3 = A(1)/A3DIF
      IF (N .NE. 4.OR. NX. NE. 4) GO TO 53
      A3 = A(3)/ (A(1)*A3DIF)
      CK = CK - A3*A(3)/A(1)
   53 CONTINUE
      WRITE(2,100) A3MAX,A3DIF
  100 FORMAT(8H A3MAX =,F10.5,8H A3DIF =,F10.5)
      WRITE (2,15)
      A3DIV = A3MAX/1000.0
      II = I2MAX - I2MIN + 1
      DO 24   J = 1,II
      A(2) = V(1,J)
      IF (N .EQ.4.AND. NX. EQ.3) GO TO 54
      I3MAX = A3 * A(2) + CK
      GO TO 55
   54 A3Y = (A(2)/2.0)**2-A(4)
      IF (A3Y .LT. 0.0) A3Y = 0.0
      A3 = SQRT(A3Y)
      I3MIN = (A(2)/2.0-A3)/(A(1)*A3DIF) + 0.9
      I3MAX = (A(2)/2.0+A3)/(A(1)*A3DIF) + 2.0
   55 DO 23 I3 = I3MIN,I3MAX
      A(NX) = (I3-1)* A3DIF
      CALL ANAL (N1,M1,J,A,B,V,Z,W2,W1,GAIN,TOR1,A3DIF)
   23 CONTINUE
   24 WRITE (2,12) (V(K,J),K=1,6),(Z(L,J),L=1,4)
      J5 = 1
      J6 = II
   15 FORMAT (1H0,4X,4HA(2),9X,4HG(1),20X,19HADEQUACY BOUNDARIES,12X,31H
     -+VE/-VE REALPART OF ROOT BOUND.,2X,24HREAL/IMAG, ROOT BOUNDARY   /
     -28X,102HINADQT/ADQT  ADQT/INADQT  INADQT/ADQT  ADQT/INADQT  SOME+V
     -E/ALL-VE/SOME+VE ALL REAL/SOME IMAG/ALL REAL          /)
```

```
PROGRAM MALG      76/76   OPT=2   TRACE                    UMRCC FTN 4.2.178    01

      12 FORMAT (10E13.6)
      31 CONTINUE
         K1 = 0
         I5 = J5-1
         I6 = J6
         DO 28 J = J5,J6
         K2 = K1
         K1 = 0
         DO 60 I = 3,6
         IF (V(I,J) .EQ. -99.0) GO TO 60
         V(I,J) = V(I,J) - A3DIF
         K1 = K1 + 1
      60 CONTINUE
         DO 61 I = 1,4
         IF (Z(I,J) .EQ. -99.0) GO TO 61
         Z(I,J) = Z(I,J) - A3DIF
         K1 = K1 + 1
      61 CONTINUE
         IF (K1 .EQ.0 .AND. K2 .EQ. 0 .AND. I5 .EQ. (J-1)) I5 = J
         IF (K1 .EQ. 0 .AND. K2 .GT. 0) I6 = J
         IF (K1 .GT. 0 .AND. K2 .EQ. 0 .AND. I6 .NE. J6) I6 = J6
      28 CONTINUE
         IF (I5 .EQ. 0) I5 = 1
         J5 = I5
         J6 = I6
         IF ((J6-J5) .GT. 30) GO TO 62
         A2MIN = V(1,J5)
         A2MAX = V(1,J6)
         GO TO 63
      62 CONTINUE

C     REDUCE STEPLENGTH (A3DIF) AND REPEAT

         A3DIF = A3DIF/10.0
         WRITE (2,14)  A3DIF
      14 FORMAT (1H0,10X,8HA3DIF = ,F10.6)
      10 FORMAT (49H1PROCESS PARAMETERS,POLES AND G(JW) AT WC VALUES //
        -7X,4HA(2),11X,4HA(3),11X,4HREAL,11X,4HIMAG,22X,10HROOTS-REAL,5X,
        -10HROOTS-IMAG)
         WRITE (2,15)
         DO 70 J = J5,J6
         A(2) = V(1,J)
         DO 34 K=1,4
         U(K) = V(3,J)
         L = 3
         DO 35  I = 4,6
         IF (V(I,J) .GT. ABS(U(K)) .OR. V(I,J) .EQ. -99.0) GO TO 35
         IF (V(I,J) .EQ. U(K))  V(I,J) = -99.0
         U(K) = V(I,J)
         L = I
      35 CONTINUE
      34 V(L,J) = -99.0
         DO 38   I = 1,4
         IF (Z(I,J) .EQ. -99.0)  GO TO 38
         DO 36   L = 1,4
         IF (Z(I,J) .EQ. U(L))  GO TO 38
         IF (I .GT. L .AND. Z(I,J) .EQ. Z(L,J)) GO TO 38
```

```
PROGRAM MALG       76/76   OPT=2   TRACE                    UMRCC FTN 4.2.178     0

                  36 CONTINUE
230               ZIJ = Z(I,J)
                  DO 37    I3 = 1,12
                  A(NX) = ZIJ + (I3-1)*A3DIF
                  CALL ANAL (N1,M1,J,A,B,V,Z,W2,W1,GAIN,TOR1,A3DIF)
                  37 CONTINUE
235               38 CONTINUE
                  DO 29    I = 1,4
                  IF ( U(I) .EQ. -99.0)     GO TO 29
                  DO 32    I3 = 1,12
                  A(NX) = U(I) + (I3-1)*A3DIF
240               CALL ANAL (N1,M1,J,A,B,V,Z,W2,W1,GAIN,TOR1,A3DIF)
                  32 CONTINUE
                  29 CONTINUE
                  70 WRITE (2,12) (V(K,J),K=1,6),(Z(L,J),L=1,4)
                  IF (A3DIF .LT. A3DIV) STOP
245               GO TO 31
                  END

GISTER ALLOCATION
EGISTERS ASSIGNED OVER THE LOOP BEGINNING AT LINE 113


SUBROUTINE CRIT       76/76   OPT=2   TRACE                    UMRCC FTN 4.2.178     0

                  SUBROUTINE CRIT (W3,W,M,N,A,B,G,GAIN,TOR,V,JJ)

                C         THIS SUBROUTINE EVALUATES THE FREQUENCY RESPONSE OF A GIVEN
                C         SYSTEM AT THE CRITICAL POINT (WHEN IT FIRST CROSSES THE
              5 C         NEGATIVE REAL AXIS)

                  DIMENSION A(10),B(10),G(4),V(6,100)
                  COMMON NX
                  J = 0
             10   GDIF  = 0.1
                  PDIF = 15.0
                  K = 0
                  IF (N .LT. 3 .AND. TOR .LT. 1.0E-5) GO TO12
                  W1 = 0.02
             15   IJ = 0
                  G1 = 0.0
                  G(3) = 0.5
                  P = 0
                  KX = 0
             20   GW = 0.01
                  WX = GAIN
                  IX = 0
                1 P1 = P
                  GDIF1 = GDIF
             25   G3 = G(3)
                  PDIF1 = PDIF
                  GW1 = GW
```

```
      IF (W .LT. 1.0E-10) W = 1.0E-10
      IF (W .EQ. 1.0E-10 .AND. W1 .LT. -10.0) GO TO 13
   10 CONTINUE
      CALL FREQ(W,M,N,A,B,G,GAIN,TOR)
      P = G(4)
      PDIF = P1-P
      IF(P1 .LT. -180.0 .AND.P.GT. -90.0.AND.W1.GT.0.0)PDIF=PDIF+360.0
   16 GD = G(3)-G3
      GDIF = ABS(GD)
      IF (GDIF .LE. 0.0001) GDIF = 0.0001
      GW = GD/W1
      IF (GW1 .EQ. 0.0) GO TO 9
      IY = 0
      IF ((GW/GW1) .GT.  1.2 .AND. GW .GT. WX) IY = 1
      IF ((GW/GW1) .GT.  1.5 .AND. GW .GT. WX) IY = 3
      IF (ABS(GDIF/GDIF1) .GT. 10.0  .AND. GDIF  .GT. 0.1 ) GO TO 6
      IX = IX + IY
    9 CONTINUE
      GX = GDIF/G(3)
      IF (G(3) .LE. 0.5 .AND. GDIF .NE. 0.0)  GTX = ABS(0.10/GDIF)
      IF(G(3) .GT. 0.5 .AND. GX  .NE. 0.0)  GTX = ABS(0.45/GX) - 0.125
      IF (PDIF .NE. 0.0) PX = 15.0/PDIF
      P2 = P+180
      K = K+1
      P3 = ABS(P2)
      IF (P2 .LE. 60.0 .AND. P2 .GE. 0.0) GTX = GTX*((P3*0.01)**2+0.35)
      IF (P3 .LE. 0.001) GO TO 13
      IF (N .LT. 3) GO TO 2
      IF (K .GE. 50 .OR. (ABS(W1) .LT. 0.0002 .AND. P3 .GE. 5.0))
     -WRITE (2,7) K,J,A(2),A(NX),W,G,W1,PDIF,GDIF,P4
```

```
SUBROUTINE CRIT       76/76   OPT=2   TRACE                UMRCC FTN 4.2.178     01
```

```
      IF (ABS(A(2) - 0.820) .LE. 0.001.AND.ABS(A(NX)-0.8060).LT.0.0001)
     -WRITE (2,7) K,J,A(2),A(NX),W,G,W1,PDIF,GDIF,P4
    2 CONTINUE
      IF (G(1) .LT. (-2.0*GAIN)) GOTO 13
      IF (ABS(W1) .LE. 0.1E-2 .AND. P3 .GT. 5.0) GO TO 8
      IF (ABS(W1) .LE. 0.1E-6) GO TO 13
    7 FORMAT(2I4,11E11.4)
      IF (K .GT. 59) GO TO 13
      IF (PDIF .LE. 0.0 .AND. W1 .GT. 0.0 .AND. PDIF1 .GT. 0.0)GO TO 6
      IF(P1.GT.-90.0.AND.P.LT.-180.0.AND.P.GT.-270.0.AND.(G3+G(3)).LT.
     -0.2) GO TO 13
      IF (P2 .LT. 0.0 .AND. KX .EQ. 1 .AND. P1 .LT. -175.0) GO TO 4
      IF (P2 .LE. 30.0 .AND. P2 .GE. -90.0) GO TO 3
      IF (P3 .LT. 90.0 .AND. P1 .LT. -180.0 .AND. P1 .GT. -270.0) GO TO3
      IF (P  .GT. -180.0) GO TO 4
      IF ((G(3) + G3).GT. 0.2 .AND. P1 .GT. -270.0) GO TO 3
      P4 = 1.1
      W1 = W1*P4
      W = W+W1
      J = J+1
      IF (J .LE. 15) GO TO 1
      GO TO 13
    3 P4 = P2/PDIF
      KX = 0
      IF(ABS(P4) .GT. GTX    ) P4 = SIGN(GTX,P4)
    5 IF(ABS(P4) .GT. ABS(PX)) P4 = SIGN(PX,P4)
      IF (P4 .LT. -1.0) P4 = -1.0
```

```
      IF (PDIF .LT. 0.0 .AND. W1 .GT. 0.0 .AND. P2 .GT.   0.0) P4=1.1
      IF(PDIF.LT.0.0.AND.W1.GT.0.0.AND.P2.LT.0.0.AND.KX.NE.1) P4 = -2.0
      IF (KX .EQ. 1 .AND. P4 .LT. 0.1) P4 = 0.1
      IF (PDIF .GT. 0.0 .AND. W1 .LT. 0.0 .AND. P2 .LT.   0.0) P4 = 1.1
      IF (P4 .GT. 2.0)  P4 = 2.0
      IF (P4 .LT. 0.0 .AND. P4 .GT. -0.1) P4 = -0.1
      W1 = W1*P4
      IF (KX .EQ. 1 .AND. W1 .LT. 0.001) W1 = 0.001
      IF (P3 .GT. 5.0 .AND. ABS(W1) .LT. 0.001) W1 = SIGN(0.001,W1)
      W = W+W1
      GO TO 1
    4 P4 = SIGN(GTX,W1)
      GO TO 5
    6 W1 = W1*0.5
      W = W-W1
      GO TO 10
    8 W1 = W1*10.0
      W = W+W1
      GO TO 1
   12 G(1) = 0.0
      G(2) = 0.0
      RETURN
   18 G(1) = -2.0
   13 IF (N .LE. 2) RETURN
      IJ = IJ + 1
      IF (G(1) .LT. V(2,JJ)) RETURN
      IF (G(1) .GT. G1 .AND. IJ .GE. 4) RETURN
      IF ((G1-G(1)) .LT. 0.001 .AND. IJ .GE. 4) RETURN
      G1 = G(1)
      K = 0
```

```
SUBROUTINE CRIT       76/76   OPT=2   TRACE                UMRCC FTN 4.2.178     01
      KX = 1
      W1 = 0.04
      W = W + W1
      GO TO 1
      END
```

```
SUBROUTINE FREQ        76/76   OPT=2   TRACE                    UMRCC FTN 4.2.178
                      SUBROUTINE FREQ (W,M,N,A,B,G,GAIN,TOR)
                C          THIS SUBROUTINE EVALUATES THE FREQUENCY RESPONSE OF A GIVEN
                C          SYSTEM AT A GIVEN FREQUENCY
 5                    DIMENSION A(10),B(10),G(4)
                      COMPLEX S,GS1,GS2,GS
                      S = CMPLX(0.0,W)
                      GS1 = GAIN * CEXP(-TOR*S)
10                    GS2 = CMPLX(1.0,0.0)
                      IF (M .EQ. 0) GO TO 2
                      DO 1 I= 1,M
                    1 GS2 = GS2 + B(I)*S**I
                      GS1 = GS1 * GS2
15                    GS2 = CMPLX(1.0,0.0)
                    2 DO 3  I = 1,N
                    3 GS2 = GS2 + A(I)*S**I
                      IF (GS2 .EQ. 0.0) RETURN
                      GS  = GS1 / GS2
20                    G(1) = REAL(GS)
                      G(2) = AIMAG(GS)
                      G(3) = CABS(GS)
                      G(4) = 57.29578 * ATAN2(G(2),G(1))
                      IF (G(4) .GE. 0.0) G(4) = G(4)-360.0
25                    RETURN
                      END

SUBROUTINE ROOTZ       76/76   OPT=2   TRACE                    UMRCC FTN 4.2.178
                      SUBROUTINE ROOTZ (N,D,J,E,R,NR,L)
                      DIMENSION A(10),D(N),X(10),Y(10),F(10)
                      COMPLEX R(N)
                      A(1) = 1.0
 5                    DO 1 I = 1,N
                    1 A(I+1) = D(I)
                      CALL POLRT(A,F,N,X,Y,IFAIL)
                      DO 2   I = 1,N
                    2 R(I) = CMPLX (X(I),Y(I))
10                    IF (IFAIL .NE. 0) L = 1
                      RETURN
                      END
```

```
SUBROUTINE ANAL        76/76   OPT=2   TRACE                    UMRCC FTN 4.2.178     01

      SUBROUTINE ANAL (N1,M1,J,A,B,V,Z,W2,W1,GAIN,TOR1,A3DIF)
      DIMENSION A(N1),B(N1),V(6,100),Z(4,100),G(4)
      COMPLEX R(10)
      COMMON NX
      W = W1
      IF (A(NX) .LT. 1.0E-7) A(NX) = 1.0E-7
      CALL ROOTZ (N1,A,N1,0.001,R,NR,L)
      K4 = 0
      IF (L .EQ. 1)  GO TO 3
      K = 0
      K2 = 0
      DO 1  I = 1,N1
      IF (REAL(R(I)) .GE. -0.1E-3) K4 = K4+1
      IF (REAL(R(I)) .GE. 0.0) K = K+1
      IF (ABS(AIMAG(R(I))) .GT. 0.0) K2 = K2+1
    1 CONTINUE
      IF (A(NX) .LT. 1.0E-6) GO TO 2
      IF (J1 .NE. J) GO TO 2
      IF (A(NX) .GT. (A3+A3DIF*1.1)) GO TO 2
      IF (A(NX) .LT. A3) GO TO 2
      IF (K .EQ. 0 .AND. K1 .GT. 0)  Z(1,J) = A(NX)
      IF (K .GT. 0 .AND. K1 .EQ. 0)  Z(2,J) = A(NX)
      IF (K3 .EQ. 0 .AND. K2 .GE. 2)  Z(3,J) = A(NX)
      IF (K3 .GE. 2 .AND. K2 .EQ. 0)  Z(4,J) = A(NX)
    2 K1 = K
      K3 = K2
    3 IF (K4 .EQ. 0) GO TO 4
      G(1) = -2.0
      GO TO 8
    4 CALL CRIT (W2,W,M1,N1,A,B,G,GAIN,TOR1,V,J)
    8 CONTINUE
      IF (A(NX) .LE. 0.0001) GO TO 11
      IF (J1 .NE. J) GO TO 11
      IF (A(NX) .LT. A3) GO TO 11
      IF (A(NX) .GT. (A3+A3DIF*1.1)) GO TO 11
      IF (G(1) .GT. V(2,J) .AND. G1 .GT. V(2,J) .OR. G(1) .LT. V(2,J)
     -.AND. G1 .LT. V(2,J)) GO TO 11
      IF(G(1) .GE. V(2,J) .AND. G1 .LE. V(2,J)) GO TO 6
      IF (V(4,J) .NE. -99.0) GO TO 5
      V(4,J) = A(NX)
      GO TO 11
    5 V(6,J) = A(NX)
      GO TO 11
    6 IF (V(3,J) .NE. -99.0)  GO TO 7
      V(3,J) = A(NX)
      GO TO 11
    7 V(5,J) = A(NX)
   11 G1 = G(1)
      J1 = J
      A3 = A(NX)
      RETURN
      END
```

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