Citation: Muir, A. (1982). Symmetries of neural systems. (Unpublished Doctoral thesis, The City University)

Symmetries of Neural Systems

by

Allan Muir

A thesis submitted for the degree of
Doctor of Philosophy in Mathematics

Department of Mathematics
The City University
London

July 1982

Table of Contents

Chapter 2 Symmetries of Net Structure

Chapter 3 Symmetries of Net Dynamics

Chapter 4 Some Special Cases

Chapter 5 Tolerance Automata

Chapter 6 Lattice-Valued Relations

Table of Figures

Acknowledgement

I should like to express my gratitude to Mary Warner for continuous dialogue and collaboration on much of this work and for urging me to write it up as a thesis. Acknowledgements are also due to [redacted] for her excellent typing and for her patience in the face of my innumerable corrections.

Declaration

I grant powers of discretion to the University Librarian to allow this thesis to be copied in whole or in part without further reference to me. This permission covers only single copies made for study purposes, subject to the normal conditions of acknowledgement.

Abstract

The role of symmetry in simplifying the dynamics of finite systems is implicit in work on neural networks. It is to be expected that such systems will be more amenable to analysis whenever regularities are discovered experimentally or postulated within a theoretical model.

This thesis characterises the symmetry of a network by its group of isomorphisms and examines some of the dynamical consequences of a network possessing a non-trivial symmetry group. The framework of the discussion, while motivated by the special instance of neural systems, is the idea of a lattice-valued relation. Neural networks and automata are developed as exemplary cases of this general concept and a number of specific models discussed.

CHAPTER I

INTRODUCTION

The triumph of the neuron doctrine - the idea that the basic structural and functional unit, in terms of which the nervous system might be understood, was a nerve cell - is a landmark in the history of science. From the inferences of Sherrington, arguing the existence of the synaptic barrier from reflex studies, to the electron micrographs of Palay, providing "visual" evidence of the actual cell boundary, there was a growth of acceptance that neurons and their interconnections were an appropriate level to which the properties of the nervous system might be reduced.

It may be said that a nascent science becomes fully fledged when it has an agreed object of investigation. This has been provided by the neuron and its concomitant synaptic structures for interconnection, together with its functional role of signalling by impulse propagation.

Of course, things have moved on in neural science, shattering this simple model, perhaps even before it fully ossified. A concern with the complexities of the synaptic mechanism, focussing on the inter-relation of manifold transmitter substances: the possible importance in the brain of graded electrical phenomena: specificity or otherwise of neuronal connections and their realization during neurogenesis; these are just a few of the current concerns of neurobiologists which go far beyond the highly stylized formal-neuron networks.

However, in historical terms, such networks have played a considerable role, taking on a mathematical life of their own and generating spin-off. Most profoundly, the initial formulation by McCullogh and Pitts (1943), deriving in part from Turing's work on automated computation, was, in its turn, a direct spur to the creation of abstract automata theory.

(The extent to which ideas clustering around logic, networks and computation cross-fertilized since their inception, is evident from the introduction to Wiener's book "Cybernetics" (1948) and from a recent joint biography of Wiener and Von Neumann (Heims, 1980).)

It is a remarkable fact that networks of entities as simple as 2-state threshold devices can perform any logical computation (see e.g. Arbib, 1969) and, faced with this universality, two major directions of work emerged. One continues to pursue the network idea with its neural interpretation but attempts to make the theoretical "neurons" more realistic. The extra variety and complexity thus introduced makes a general mathematical theory yet more remote and attention is directed, rather, to the simulation of particular neural structures which are known to exhibit a good deal of regularity; the retina, cerebellum, reticular system and so on. A good summary of such approaches is to be found in (Szenthagothai and Arbib, 1974).

The second direction studies automata theory per se forgetting its origins in neural networks. This affords all the advantages of abstraction and the various sets employed in the automaton definition (see Chapter 5 of this thesis) are not necessarily interpreted in terms of a network sub-structure.

However, the network idea - the general conception of an interconnected set of similar entities - is pervasive enough to reappear from many sources. Cellular automata, for example, consist of regular meshes of identical automata. Also, cortical cell assemblies have been modelled as relatively uniform local regions bound together by excitatory, long-range connections (e.g. Palm, 1981). Yet again, organismic development, modelled by formal rules for the growth of components of a system in terms of their neighbouring components, forms the subject matter of so-called L-systems (Rozenberg and Salomaa, 1974). Indeed, any of the innumerable models which employ graph theory may be viewed in this light.

This somewhat partial and abrupt collapsing of many ideas into a few short sentences has but one purpose; to situate the contents of the present thesis.

It deals with networks which at the outset are clearly motivated by the standard neural interpretation. As the thesis proceeds, however, things become more abstract, passing from neural networks through automata and thence to what I have called "lattice valued relations", of which automata are seen to be just an example.

Loosely speaking, lattice valued relations arise from attempting to characterise in general how systems might be specified in terms of their parts. We begin with those parts considered as elements of a given set X and wish to state how they are to be connected together. So to every pair of elements of X we assign an element from the set L of possibilities for connection; that is, we define a function $\lambda: X \times X \to L$. Assuming we wish to compare strengths of connection, L should possess a (partial) ordering and, as will be seen in the sequel, it is desirable for our purposes that L should actually be a lattice.

I have preferred to leave this somewhat abstract treatment till last, choosing rather to prove all the theorems in the more intuitive context of formal neural networks. Actually that particular interpretation is still held slightly at a distance by employing the rather uncommitted word "net". This is intended to emphasize the fact that, although many of the examples are indeed taken from a neural context, we are really treating a more abstract systemic concept, of which neural networks and their dynamics are just one, albeit a very fruitfully suggestive, exemplar. In fact, even the neural interpretation switches pretty freely between the network nodes being sometimes single formal neurons and, at other times, averaged aggregates of neurons.

We begin then with nets, looking first at their static structure in Chapter 2. Here is introduced the key idea which pervades the whole thesis - the group of symmetry of the net. The treatment follows that in (Muir, 1981a) and is concerned with making a fruitful definition of homogeneity, from which the construction of all such homogeneous nets may be achieved. The intent is to make precise, and to generalise, various notions of homogeneity occuring, usually implicitly, elsewhere. These have variously appeared in terms of regular arrays in space with identical connections between neighbours, as some more intrinsic specification postulating an equal number of connections between objects in a network, or even as an all-to-all identical connection between any pair of objects.

The definition here proposed is general enough to accommodate any desired finite group of symmetry and, moreover, it is intrinsic; this permits the inherent symmetry to be conceptually detached from how the net might be embedded in space.

In Chapter 3 dynamics is introduced in discrete time by postulating a map from the set of states of the net, suitably defined, to itself. Now we no longer demand total homogeneity but are concerned to explore the dynamical implications of <u>any</u> symmetry group. The underlying idea is that symmetry should imply simplicity, in the sense that regularities in the structure of a net ought to be reflected in its dynamical behaviour. Obviously, this demands that some relation between the dynamics and the symmetry group must prevail. Necessary conditions for this are proposed, in an abstract way but guided by the particular dynamics of formal neuronal networks.

Chapter 4 is concerned with specific, neuronal networks. It begins by reminding us that knowing the feasibility of a particular dynamical trajectory is not enough to declare it to be physically significant. Consideration of stability for such trajectories must be undertaken. Here we encounter a difficulty which be-devils all discussion of stability in finite systems; stability pre-supposes that some meaning be given to the notion of nearness, so that perturbation makes sense.

In our discussion we temporarily dodge this issue by treating neuronal states probabilistically. Thus we deal with the firing probability of each network node so, assuming these variables are stochastically independent, the set of states of the system can be regarded as a unit cube with edges, indexed by the nodes. This device gives us a topology and even a differential structure on the state set so we can, moreover, presume the dynamics to be differentiable.

This naive, semi-probabilistic modelling is not really satisfactory, but I know of no good treatment of neuronal aggregation which would properly legitimate the transition from complex, deterministic network models to the desired, more simple, averaged models. The objective here is merely to examine the stability of some examples of the latter and we show explicitly how the non-linearities of the dynamics lead to catastrophe-theoretic behaviour of the equilibria. We can note also at this point, an alternative stochastic model, set out in Appendix 1, which exhibits qualitatively similar features.

In Chapter 5 we turn to more abstract considerations based on automata theory. We try to solve the problem, noted earlier, of how to treat "nearness" in finite sets, by imposing on the state set of the automaton a so-called tolerance. Most of the chapter is then concerned with an attempt to re-work some of the classical decomposition theory of automata in a way which respects the tolerance structure. This is followed by a brief sketch of joint work with Dr Mary Warner towards applying homology theory to the classification of automata (Warner and Muir, 1981). Indeed, I hope it is abundantly clear, from the references throughout this chapter, that I owe a great debt to this collaboration.

With Chapter 6 we arrive at the lattice-valued relations described earlier. We now re-work the homogeneity discussion of Chapter 2 in this more general setting. In effect, we have come almost full circle but have arrived at a higher level, in the following sense. The earlier treatment of nets is, from the vantage point of Chapter 6, a special kind of lattice-valued relation. But if a net is regarded neuronally, its dynamics, or more exactly, its equivalent automaton representation, now also appears as a lattice valued relation. A good deal of the chapter concerns the automaton case and we show how the concepts of the abstract theory translate into automaton terms, linking the results back to the subject matter of Chapter 5.

Finally, a second Appendix based on (Muir, 1979) describes a particular neural model which, while not being concerned with our central theme of symmetry, sheds some light on other parts of the text, and is referred to at various points therein.

I have adopted a style of presentation which demarcates the strict mathematical content from the more informal, discursive sections. The former consists of numbered sections, each of which is a definition, a lemma, a theorem or an example; numbering is in a decimal manner, where the pre-decimal digit is the chapter number, and each such section is terminated by the mark # . The latter consists of commentary, interpretations of the abstract theory or more tentative suggestions for further development.

CHAPTER 2.

SYMMETRIES OF NET STRUCTURE

Why Symmetry?

In considering the dynamics of a neural network with a large number of neurons, simplifying assumptions must be made to render the mathematics tractable.

Griffith (1971) introduced a concept of uniformity in which every pair of neurons was connected in an identical way. He assumed the connections were random but with the same probability of connection between any pair of neurons. Then, by rather loose but plausible probabilistic arguments, he obtained discrete time dynamical equations for the firing probability of any neuron and looked at some of the qualitative behaviour of this dynamics. A fuller discussion, going beyond Griffith's work, is given in Appendix 1. We need merely note here the essential limitation of his very strong concept of structural uniformity - that it admits of no dynamic patterns of activity other than uniform.

For even if a non-uniform activity is initiated, at the next time-instant uniformity appears, because every neuron has an identical relation to the whole network.

A more sophisticated concept of uniformity is implicit in the work of Wilson and Cowan (1972;1973).

In setting up their equations for activity of the cortical surface, considered as a two-dimensional continuum of points, they express inter-neuronal connection strengths between points with positions $\underset{\sim}{X}$, $\underset{\sim}{X}'$ by functions of the form $\beta(\underset{\sim}{X}-\underset{\sim}{X}')$. This dependence only on the difference $\underset{\sim}{X}-\underset{\sim}{X}'$ expresses very nicely the independence of cortical inter-connections under translations and, indeed, the vector notation also carries with it the implication of isotropy.

In the light of our ensuing discussion of homogeneity, this is conveniently seen as a demand for invariance of connections with respect to the Euclidean group of plane motions. This is, of course, a local approximation for, unless one is really brainy, one's cortical surface is unlikely to be a Euclidean plane. Still, it motivates our definition of homogeneity, although we find it more convenient to work with discrete neural networks in the first instance.

Nets and their Symmetry Groups

Definition 2.1 A *net* (X,ϕ) is a set X and a real-valued function $\phi : X \times X \rightarrow R$. #

X is intended to be a set of neurons, neural masses, cell-assemblies or what have you in the sequel and ϕ a corresponding measure of synaptic strength.

However, as defined, we have nothing more than a complete graph with a real-valued weighting on each edge, hence the neutral term "net". We will employ the equally uncommitted term node for any $x \in X$ and call ϕ the connection function of the net.

Our objective is to define homogeneity by expressing the fact that a net looks the same when viewed from any of its nodes. Same-ness is, in mathematical theories, best thought of as isomorphism, which we now define.

Definition 2.2 A homomorphism from a net (X,ϕ) to a net (X',ϕ') is a function $g: X \to X'$ such that $\phi'(g(x),g(x')) \geq \phi(x,x')$ for all $x,x' \in X$. #

Definition 2.3. An isomorphism of nets is a bijective homomorphism whose inverse is a homomorphism. #

Note that Definition 2.3 replaces the inequality in Definition 2.2 by equality because

$$\phi'(g(x),g(x')) \geq \phi(x,x') = \phi(g^{-1}g(x),g^{-1}g(x'))$$

$$\geq \phi'(g(x), g(x')).$$

If f,g are two homomorphisms from (X,ϕ) to (X',ϕ') then we define

$$\alpha(f,g) = \min_{x \in X} \phi'(f(x),g(x))$$

which allows us to consider the set A of all such homomorphisms as a net with connection function α. In particular, if $(X',\phi') = (X,\phi)$ we have

<u>Lemma 2.4.</u> Let $x \varepsilon X$ and define the function $\psi_x : A \to X$ by $\psi_x(g) = g(x)$. Then ψ_x is a homomorphism from (A,α) to (X,ϕ).

<u>Proof.</u> $\phi[\psi_x(g), \psi_x(g')] = \phi[g(x), g'(x)]$

$$\geq \min_{\bar{x}\varepsilon X} \phi[g(\bar{x}), g'(\bar{x})] = \alpha(g,g'). \#$$

Until further notice we will regard $x \varepsilon X$ as chosen and fixed. Call it the <u>base-point.</u>

<u>Lemma 2.5.</u> The set G of all isomorphisms of a net (X,ϕ) to itself forms a group under composition of functions.

<u>Proof.</u> (i) Closure: If $f, g \varepsilon G$ then for all $x, x' \varepsilon X$, $\phi[f\cdot g(x), f\cdot g(x')] = \phi[g(x),g(x')]$

$= \phi(x,x')$ so $f\cdot g \varepsilon G$.

(ii) Identity: The identity function $id: X \to X$ is trivially ϕ-preserving.

(iii) Associativity: This is automatic for composition of functions.

(iv) Inverse: $f \varepsilon G \Longrightarrow \phi[f(x),f(x')] = \phi(x,x')$ for all $x, x' \varepsilon X \Longrightarrow \phi[f^{-1}(x),f^{-1}(x')] = \phi(x,x')$ $\Longrightarrow f^{-1} \varepsilon G$, since f is bijective and so $f^{-1}(x)$ ranges over X when x does. #

Lemma 2.6. If x is the base-point, then

$H_x = \{g \in G: g(x) = x\}$ is a subgroup of G.

Proof. If $f(x) = x$ and $g(x) = x$, then $f \cdot g(x) = x$.

Also $f^{-1}(x) = f^{-1}(f(x)) = x$ and $\mathrm{id}\,(x) = x$. #

Definition 2.7 F_x denotes the set of left cosets of H_x in G. The coset containing $g \in G$ will be denoted by $[g]$. #

Lemma 2.8. $\hat{\psi}_x : F_x \to X$ defined by

$\hat{\psi}_x([g]) = \psi_x(g)$ is a well-defined function.

Proof. $[g] = [g_1] \Longrightarrow g$ and g_1 lie in the same coset of $H_x \Longrightarrow g_1 = g \cdot h$ for some $h \in H_x$.

$$\text{Then } \hat{\psi}_x([g_1]) = \psi_x(g_1) = g_1(x)$$

$$= g \cdot h(x) = g(x) = \psi_x(g) = \hat{\psi}_x([g]). \quad \#$$

Our aim is to show that when (X,ϕ) is homogeneous, in a suitable sense, then it is identical with F_x equipped with an appropriate connection function. To do this we will prove that $\hat{\psi}_x$ is an isomorphism but we need a connection function on F_x. First we prove:

Lemma 2.9. α is invariant under left and right translations of its arguments by elements of G.

Proof (i) Right translation: let $g, g', \bar{g} \in G$. As $\bar{x}$ ranges over X so does $\bar{x}' = \bar{g}(\bar{x})$ since $\bar{g}$ is bijective. So we have

$$\alpha(g\bar{g},g'\bar{g}) = \min_{\bar{x}\in X} \phi[g\bar{g}(\bar{x}),g'\bar{g}(\bar{x})]$$

$$= \min_{\bar{x}'\in X} \phi[g(\bar{x}'),\, g'(\bar{x}')] = \alpha(g,g')$$

(ii) Left translation: again let $g, g', \bar{g} \in G$. Since $\bar{g}$ preserves ϕ values $\phi[\bar{g}g(\bar{x}),\bar{g}g'(\bar{x})] = \phi[g(\bar{x}),g'(\bar{x})]$.

This equality then holds for minimization over $\bar{x} \in X$

so $$\alpha(\bar{g}g,\bar{g}g') = \alpha(g,g'). \quad \#$$

Lemma 2.10. $\hat{\alpha}_X([g],[g']) = \max_{h\in H_X} \alpha(g,g'h)$

gives a well-defined connection function

$$\hat{\alpha}_X : F_X \times F_X \to R.$$

Proof. If $[g_1] = [g]$ and $[g_1'] = [g']$ then for some $h_1, h_1' \in H_X$ we have $g_1 = gh_1,\ g_1' = g'h_1'$.

Thus $\alpha(g_1,g_1'h) = \alpha(gh_1,g'h_1'h)$

$= \alpha(g,g'h_1'hh_1^{-1})$ using right invariance of α.

Now as h ranges over H_X so does $\bar{h} = h_1'hh_1^{-1}$

so $\max_{h\in H_X} \alpha(g_1,g_1'h) = \max_{\bar{h}\in H_X} \alpha(g,g'\bar{h})$. #

Lemma 2.11. $\hat{\psi}_X$ is an injective homomorphism from $(F_X, \hat{\alpha}_X)$ to (X,ϕ).

Proof. For $[g]$, $[g'] \varepsilon F_X$, there exists $h \varepsilon H_X$ such that $\hat{\alpha}_X([g], [g']) = \alpha(g,g'h)$.

Now $\phi[\hat{\psi}_X([g]), \hat{\psi}_X([g'])] = \phi[g(x),g'(x)]$

$$= \phi[g(x),g'h(x)] \geq \min_{\bar{x}\varepsilon X} \phi[g(\bar{x}),g'h(\bar{x})]$$

$$= \alpha(g,g'h) = \hat{\alpha}_X([g],[g'])$$

so $\hat{\psi}_X$ is a homomorphism.

Also $\hat{\psi}_X([g]) = \hat{\psi}_X([g'])$

$$\implies g(x) = g'(x) \implies g^{-1}g'(x) = x$$

$$\implies g^{-1}g' \varepsilon H_X \implies [g] = [g']$$

so $\hat{\psi}_X$ is injective. #

Homogeneity

We still have not introduced the idea of homogeneity. This is because, as will be seen in Chapter 3, the symmetry group G of the net has implications for the dynamics, irrespective of homogeneity. Homogeneity is, however, essential to complete our proof that $\hat{\psi}_X$ is an isomorphism.

Definition 2.12. (X,ϕ) is said to be homogeneous if for all $x', x'' \in X$, there exists $g \in G$ such that $x'' = g(x')$. #

This is the neatest way to express the desired equivalence of all nodes within the net. Viewed from either x' or x'' the incoming connections from other nodes look the same. However, those same connection strengths need not emerge from the same nodes; rather if x' sees itself as affected by node $\bar{x}$, then x'' sees itself as identically affected by node $g(\bar{x})$.

Now Griffith's idea of homogeneity is seen to be ours, with G the full symmetry group on X; Wilson and Cowan's group, as previously noted, is the Euclidean group. Another attempt to simplify neural dynamics by imposing symmetry is to be found in (Kobuchi, 1976) where the dynamics of a discrete 2-dimensional cellular space with nearest neighbour connections is discussed. The group there is $Z \times Z$. Yet again, an explicit concept of homogeneity is found in (Palm, 1981) in which it is required that the summed connection strengths, both to and from, each node be the same for all nodes; obviously this is a consequence of our definition.

Lemma 2.13. If (X,ϕ) is homogeneous, then $\hat{\psi}_x$ is surjective.

Proof. For $x' \in X$ choose $g \in G$ such that $x' = g(x)$. Then $\hat{\psi}_x([g]) = x'$. #

Unfortunately, our definition of homogeneity is too weak to ensure that $\hat{\psi}_X^{-1}$ is a homomorphism, the last condition needed for $\hat{\psi}_X$ to be an isomorphism. This is essentially because the element g of Definition 2.12, while it maps x' to x'', is not sufficiently controlled with respect to its effect on the rest of the net. It turns out that the following, more stringent homogeneity condition is required.

<u>Definition 2.14</u>. (X,ϕ) is said to be <u>very homogeneous</u> (abbreviated v.h.) if for all $x', x'' \varepsilon X$ there exists $g \varepsilon G$ such that $x'' = g(x')$ and $\alpha(e,g) = \phi(x',x'')$. #

We can now establish one of the main theorems of this chapter.

<u>Theorem 2.15</u>. If (X,ϕ) is v.h. then $\hat{\psi}_X$ is an isomorphism.

<u>Proof</u>. By lemmas 2.11 and 2.13, $\hat{\psi}_X$ is a bijective homomorphism and it remains to show that $\hat{\psi}_X^{-1}$ is a homomorphism.

$\hat{\psi}_X^{-1} : X \to F_X$ is defined thus: for $x' \varepsilon X$ choose $g' \varepsilon G$ s.t. $g'(x) = x'$ and write $\hat{\psi}_X^{-1}(x') = [g']$. This is well-defined since if $g''(x) = x'$ then $g'' = g'h$ for some $h \varepsilon H_X$ and so $[g'] = [g'']$.

Now for any $x', x'' \varepsilon X$ choose $g' \varepsilon G$ as above and g to satisfy the v.h. conditions. gg' maps x to x'' so we can write $\hat{\psi}_x^{-1}(x'') = [gg']$ and get

$$\hat{\alpha}_x[\hat{\psi}_x^{-1}(x'), \hat{\psi}_x^{-1}(x'')] = \hat{\alpha}_x([g'],[gg'])$$

$$= \max_{h \varepsilon H_x} \alpha(g',gg'h) \geq \alpha(g',gg')$$

$$= \alpha(e,g) = \phi(x',x''). \quad \#$$

It is curious that the above theorem is framed in terms of a fixed base-point x, whereas the theorem itself states that no particular point differs from any other. The contradiction is resolved by the following Theorem which shows how the constructions for different base points are related.

<u>Theorem 2.16</u>. Let $\bar{g}(x) = y$. Then:

(i) $H_y = \bar{g}H_x\bar{g}^{-1}$

(ii) $[g] \varepsilon F_y$ iff $[g\bar{g}] \varepsilon F_x$

(iii) $\hat{\alpha}_y([g],[g']) = \hat{\alpha}_x([g\bar{g}],[g'\bar{g}])$

(iv) $\hat{\psi}_y([g]) = \hat{\psi}_x([g\bar{g}])$

<u>Proof</u>. (i) $h \varepsilon H_x \Longleftrightarrow h(x) = x \Longleftrightarrow h\bar{g}^{-1}(y) = \bar{g}^{-1}(y)$

$\Longleftrightarrow \bar{g}h\bar{g}^{-1}(y) = y \Longleftrightarrow \bar{g}h\bar{g}^{-1} \varepsilon H_y.$

(ii) $[g] \in F_y \Longleftrightarrow [g] = gH_y \Longleftrightarrow [g] = g\cdot\bar{g}H_x\bar{g}^{-1}$

$$\Longleftrightarrow [g]\bar{g} = g\bar{g}H_x \Longleftrightarrow [g\bar{g}] \in F_x .$$

(iii) $\hat{\alpha}_y([g],\ [g']) = \max_{h \in H_y} \alpha(g,g'h)$

$$= \max_{h' \in H_x} \alpha(g,g'\cdot\bar{g}h'\bar{g}^{-1})$$

$$= \max_{h' \in H_x} \alpha(g\bar{g},g'\bar{g}h') = \hat{\alpha}_x([g\bar{g}],\ [g'\bar{g}])$$

(iv) $\hat{\psi}_y([g]) = g(y) = g\bar{g}(x) = \hat{\psi}_x([g\bar{g}])$. #

The v.h. definition is rather obscure and it is difficult to see its meaning intuitively. Later on, when we discuss in Chapter 6 a special case allied to the topological category and, again in that chapter, when we re-vamp automata theory in an analogous way, we will hopefully make the idea clearer. For the moment, though, we will merely illustrate its necessity for the establishment of Theorem 2.15 by detailing an example of a net which is homogeneous but not v.h., showing explicitly that the theorem's conclusions do not then hold.

Example 2.17. $X = \{x_i : i = 1,2,\dots,12\}$

$\phi(x_i,x_j) = 0$ except for

$\phi(x_i,x_j) = 1$ iff

$$j = i-1,\ i,\ i+1 \ \text{ or } \begin{cases} i+7 & \text{if } i \text{ is odd} \\ i+5 & \text{if } i \text{ is even} \end{cases}$$

all statements about indices being interpreted modulo 12.

$f : X \to X$ given by $x_i \mapsto x_j$, $j = 3-i$

is an isomorphism, for

$$\phi(f(x_i),\ f(x_k)) = 1 \ \text{ iff } \ \phi(x_{3-i},\ x_{3-k}) = 1$$

$$\text{iff } \ 3-k = 2-i,\ 3-i,\ 4-i \ \text{ or } \begin{cases} 10-i & 3-i \ \text{odd} \\ 8-i & 3-i \ \text{even} \end{cases}$$

$$\text{iff } \ k = i-1,\ i,\ i+1 \ \text{ or } \begin{cases} -7+i & 3-i \ \text{odd} \\ -5+i & 3-i \ \text{even} \end{cases}$$

$$\text{iff } \ k = i-1,\ i,\ i+1 \ \text{ or } \begin{cases} i+5 & i \ \text{even} \\ i+7 & i \ \text{odd} \end{cases} \quad (\text{mod } 12)$$

$g : X \to X$ given by $x_i \mapsto x_j$, $j = i+2$

is similarly an isomorphism.

Note that x_1 is mapped to $x_2, x_3, \dots, x_{12}$ by

$f, g, gf, g^2, g^2f, g^3, \dots, g^5f$ all of which are

isomorphisms.

It follows that an isomorphism exists mapping any x_i to any x_j (the inverse of that mapping x_1 to x_i followed by that mapping x_1 to x_j) so (X,ϕ) is homogeneous.

To show it is not v.h. consider any isomorphism h which maps x_2 to x_3 and ask, what is $h(x_1)$? Since $1 = \phi(x_1,x_2) = \phi(h(x_1), x_3), h(x_1) = x_2, x_4$ or x_{10} (since it cannot be x_3 as h is bijective).

If $h(x_1) = x_2$ we can observe that $\phi(x_2,x_7) = 1$ implies $\phi(x_3,h(x_7)) = 1$. Now $h(x_7)$ cannot be x_2 or x_3 since both are images of other x_i, so $h(x_7)$ equals x_4 or x_{10}. But $\phi(x_7,x_4) = \phi(x_7,x_{10}) = 0$.

Conversely, if $h(x_1) = x_4$ or x_{10}, note that $\phi(x_1,x_4) = \phi(x_1,x_{10}) = 0$, so in all cases there is an $\bar{x}$ such that $\phi(\bar{x},h(\bar{x})) = 0$.

But $\alpha(e,h) = \min\limits_{\bar{x}} \phi(\bar{x},h(\bar{x})) = 0$ whereas $\phi(x_2,x_3) = 1$ so (X,ϕ) is not v.h.

Actually, the 12 group elements formed from f,g and listed above are, with the identity, the whole isomorphism group G which is just the dihedral group

$$\{f,g : f^2 = g^6 = id, fg = g^{-1}f\}.$$

Now $H_x = \{id\}$ so $\hat{\alpha} = \alpha$ for any base point.

$\alpha(id,f) = \min_i \phi(x_i, x_{3-i})$. Since $\phi(x_3, x_{12}) = 0$, $\alpha(id,f) = 0$. But $\hat{\psi} = \psi$ and with base-point x_1, $\psi(id) = x_1$ $\psi(f) = x_2$. Since $\phi(x_1, x_2) = 1$ we therefore find $\alpha(id,f) \neq \phi(\hat{\psi}(id), \hat{\psi}(f))$ so $\hat{\psi}$ is not an isomorphism. #

Group Nets.

Theorem 2.15 shows how to re-construct a net from a knowledge of its symmetry group, but this is evidently somewhat circular for the group cannot be known except through a knowledge of the net's structure.

The remaining results of this chapter show, however, that any group suitably endowed with a connection function can be used to construct v.h. nets. Altogether we then will have a complete classification and means for constructing all v.h. nets.

Definition 2.18. A group net (G,α) is a group G with a real-valued function $\alpha : G \times G \to R$ which is left and right invariant under translation by elements of G. #

α can clearly be reconstructed by left translation from the function $\beta(g) = \alpha(e,g)$ since

$\alpha(g,g') = \alpha(e,g^{-1}g') = \beta(g^{-1}g')$.

Recall that for any group G we can define an equivalence relation by $g_1 \equiv g_2$ iff there is $g \in G$ such that $g_2 = g^{-1}g_1g$. It is easy to prove that this relation is an equivalence; the classes are called *conjugacy classes* and a function which is constant on conjugacy classes is a *class function*. In these terms we can state:

Lemma 2.19. $\beta : G \to R$ is a class function and, conversely, every class function defines a left and right translation invariant connection function.

Proof.
$$\begin{aligned}\beta(g^{-1}g'g) &= \alpha(e,g^{-1}g'g) = \alpha(g,g'g)\\ &= \alpha(e,g') = \beta(g')\end{aligned}$$

using left and right translation invariance of α.

Conversely, given $\beta : G \to R$ constant on conjugacy classes we define $\alpha(g,g') = \beta(g^{-1}g')$. Then

$$\alpha(\bar{g}g,\bar{g}g') = \beta(g^{-1}\bar{g}^{-1}\bar{g}g') = \beta(g^{-1}g') = \alpha(g,g')$$

$$\alpha(g\bar{g},g'\bar{g}) = \beta(\bar{g}^{-1}g^{-1}g'\bar{g}) = \beta(g^{-1}g') = \alpha(g,g') \text{ . \#}$$

Lemma 2.20. Let H be a subgroup of a group G and $F = G/H$. Then $\hat{\alpha}([g],[g']) = \max_{h \in H} \beta(g^{-1}g'h)$ is a well-defined function $\hat{\alpha} : F \times F \to R$.

Proof. If $[g_1] = [g], [g_1'] = [g']$ then $g_1 = gh_1$ and $g_1' = g'h_1'$ for $h_1, h_1' \in H$. Then

$$\hat{\alpha}\Big([g_1], [g_1']\Big) = \max_{h\in H} \beta(h_1^{-1}g^{-1}g'h_1'h)$$

$$= \max_{h\in H} \beta(g^{-1}g'h_1'hh_1^{-1})$$

by the conjugacy invariance of β.

As h ranges over H so does $h_1'hh_1^{-1} = h'$ and we get $\max\limits_{h'\in H} \beta(g^{-1}g'h') = \hat{\alpha}([g],[g'])$. #

Lemma 2.21. $\hat{\alpha}$ is left translation invariant.

Proof. $$\hat{\alpha}(\bar{g}[g], \bar{g}[g']) = \max_{h\in H} \beta(g^{-1}\bar{g}^{-1}\bar{g}g'h)$$

$$= \max_{h\in H} \beta(g^{-1}g'h) = \hat{\alpha}([g],[g'])$$ #

Theorem 2.22 $(F,\hat{\alpha})$ is v.h.

Proof Let $[g], [g'] \in F$ and choose representatives $g, g' \in G$. Let $h \in H$ be such that $\beta(g^{-1}g'h)$ is maximum and write $\bar{g} = g'hg^{-1}$.

By the previous Lemma left translation by an element of G is an isomorphism of $(F, \hat{\alpha})$, so G is a subgroup of the isomorphism group of $(F,\hat{\alpha})$. In particular, left translation by $\bar{g}$ maps $[g]$ to $[\bar{g}g] = [g'h] = [g']$ so

$(F,\hat{\alpha})$ is homogeneous.

Moreover, for any $[g_1] \varepsilon F$

$$\hat{\alpha}([g_1], \bar{g}[g_1]) = \hat{\alpha}([g_1], [g'hg^{-1}g_1])$$

$$= \max_{h_1} \beta(g_1^{-1}g'hg^{-1}g_1h_1) \geq \beta(g_1^{-1}g'hg^{-1}g_1)$$

$$= \beta(g'hg^{-1}) = \beta(g^{-1}g'h) = \max_{h_1 \varepsilon H} \beta(g^{-1}g'h_1)$$

$$= \hat{\alpha}([g], [g'])$$

It follows that $\hat{\alpha}([g], [g']) = \min\limits_{[g_1] \varepsilon F} \hat{\alpha}([g_1], \bar{g}[g_1])$

so $(F,\hat{\alpha})$ is v.h. #

Example 2.23. Consider the symmetric group A_4 which may be considered as generated by 2 generators a, b subject to the relations $a^3 = b^3 = 1$,

$bab = aba, \quad ba^2 = ab^2, \quad b^2a = a^2b.$

Explicitly we can write the different elements as

$$G = \{1,a,b,a^2,ba,ab,b^2,ba^2,aba,b^2a,aba^2,b^2a^2\}$$

The conjugacy class of 1 is $\{1\}$ since $g^{-1}1g = 1$ for all $g \varepsilon G$. The conjugacy class of a contains

$$b^{-1}ab = b^2ab = b{\cdot}aba = aba^2$$

$(ba)^{-1}a(ba) = a^2b^2aba = a^2\cdot a^2b\cdot ba = ab^2a = ba^3 = b$

$(ab)^{-1}a\ ab = b^2a^2\cdot aab = b^2ab = aba^2$

$(b^2)^{-1}a\ b^2 = bab^2 = b^2a^2.$

Continuing in this way we obtain the conjugacy classes

$$W = \{1\} \quad X = \{a,b,b^2a^2,aba^2\}$$

$$Y = \{a^2,ab,ba,b^2\} \quad Z = \{b^2a,aba,ba^2\}$$

which form, as expected, a partition of G.

We can choose our class function arbitrarily, so let us say $\beta|W = 0$, $\beta|X = 2$, $\beta|Y = 0$, $\beta|Z = -1$

Lastly, choose a subgroup H of G, say, that generated by b^2a; that is $H = \{1,b^2a\}$. The left cosets of H are then $[1] = H$, $[a] = \{a,b\}$, $[a^2] = \{a^2,ab\}$, $[aba] = \{aba,ba^2\}$, $[b^2a^2] = \{b^2a^2,aba^2\}$, $[ba] = \{ba,b^2\}$, which we will denote by I, A, B, C, D, E respectively.

The connection function on the set of cosets is found, according to Lemma 20. For example

$$\hat{\alpha}(B,D) = \max\ [\beta((a^2)^{-1}\cdot b^2a^2)\ ,\ \beta((a^2)^{-1}b^2a^2\cdot b^2a)]$$

$$= \max\ [\beta(ba),\ \beta(b^2)] = 0.$$

After a rather tedious calculation we arrive at

$\hat{\alpha}$	I	A	B	C	D	E
I	0	2	0	-1	2	0
A	0	0	2	0	-1	2
B	2	0	0	2	0	-1
C	-1	2	0	0	2	0
D	0	-1	2	0	0	2
E	2	0	-1	2	0	0

Each line of this table is the previous one shifted cyclically to the right which makes the homogeneity obvious, though the transparency of this result depends on our judicious labelling of the elements of A_4. #

At the end of (Muir, 1981a) it was suggested that the theory of homogeneous nets, might be carried over to the case when the set of nodes X was infinite, by requiring it to be a compact, topological space. For then, if the connection function ϕ were continuous in the product topology on $X \times X$, all the maxima and minima required in the theorems would exist and the theory would go through as before.

Such a structure with a neuronal interpretation would express the interaction between "nearness" of neurons in real physical space X, considered as a continuum of those neurons, and a "neuronal nearness" determined by the connection strength ϕ. Proof of a similar theorem where spatial nearness is expressed by tolerance has been carried through in (Warner, 1981).

CHAPTER 3

SYMMETRIES OF NET DYNAMICS

More arguments for symmetry

In the previous chapter we employed the isomorphism group of a net to define a concept of homogeneity. Of course, homogeneity is, in many instances, too stringent a requirement - even simplified discussion of neural networks might desire to have at least two types of cell, excitatory and inhibitory. Clearly a kind of generalized homogeneity could still prevail in such a network, consisting of say, an excitatory motor neuron pool with recurrent inhibitory collaterals. This suggests that the isomorphism group is of value in characterising any net with a large amount of symmetry, a feature we might expect to find mirrored in the net's behaviour. But why should we expect symmetry in nets; say, in particular, neural networks? The shortest answer is that it is mathematically convenient. This is an admission that a general theory of neural networks is just too difficult - their generality, their non-linearity, their tight interconnection of parts, condemns any generalization to triviality. In contrast, there is an honourable tradition in the theoretical physical sciences of recognizing the role of symmetries in the organization and simplification of models.

Experimental methodology in the neural sciences is such that we can hardly avoid some kind of homogeneity assumption. This will be covertly contained in a presumed typicality of single units from which microelectrode recordings are taken, or in any naive statistical discussion of the features of gross recording. In marked contrast to the difficulties of theorising about general neural networks, the fact that they can simulate any automaton (Arbib, 1969) presents us with an embarrassment of riches. If we can do <u>anything</u> neurally by cleverly-constructed networks, how are we going to control our modelling? We have the rather ludicrous situation that neural networks can model any behaviour ascribable to a finite-state machine, which looks like a really powerful principle; such universality, however, places no constraints on the modelling of any specific activity.

All of the above arguments concern reasons why our theorising might be aided by presuppositions of symmetry. Are there any good reasons for supposing that neural networks might actually <u>be</u> so arranged.

The most convincing argument I have derives from the simplicity of symmetric entities; less information need be supplied to build a symmetric object than an asymmetric one. If neural structures are indeed genetically specified, then there is a certain economy of coding allowed by requirements of symmetry in the finished product. This notion of simplicity is taken from (Zvonkin and Levin, 1970).

An immediate objection is that the more regular are the structures, the less interesting will be their dynamics - a homogeneous network will exhibit a tediously uniform, spatially constant behaviour. It is, however, becoming familiar from the physics of elementary particles that this need not be so. While uniformity is indeed a dynamical solution, it is not necessarily stable. Spontaneous symmetry-breaking will then occur and, as shown in the next chapter, this is precisely what can happen in network dynamics.

It is intriguing to me how this observation ties in with a philosophical position to which I incline. This argues for the primacy of function over structure and, if our neural dynamic theories are an apt metaphor, we have good reasons for believing that function can be more varied than the structure which supports it. The selection of which function is realised and must then lie outside the system considered - from within that system the external selecting agency appears as a symmetry-breaking perturbation.

Symmetry and Dynamics.

Definition 3.1. A *state* of a net $N = (X,\phi)$ is a function from X to a subset A of the real numbers. If ψ is a state, $\psi(x)$ will be called the *activity* of the node x. Ω will denote the set of states. #

Examples 3.2. (i) Let X be a set of neurons and $\phi(x,x')$ be a measure of the synaptic strength of the connection from neuron x' to neuron x. Considering the neurons as formal McCullogh-Pitts neurons, the activity set A is $\{0,1\}$ and any state ψ can be regarded as the characteristic function of the subset of active neurons.

(ii) Alternatively we might regard each element of X as a whole neuronal aggregate whose activity is the proportion of cells which are active in the aggregate, so $A = I = [0,1]$. Ω is the function set I^X, which when X is a finite set can be considered as the entity which Zeeman (1961) introduced under the name "the thought cube". #

Lemma 3.3. An isomorphism $g \varepsilon G$ induces a group action g^* on Ω which satisfies $(g_1 g_2)^* = g_2^* g_1^*$ and $(g^{-1})^* = (g^*)^{-1}$.

Proof. Define $g^*(\psi) = \psi \circ g$ for $g \varepsilon G$, $\psi \varepsilon \Omega$.

Then for any $x \varepsilon X$, $g_1, g_2 \varepsilon G$ we have

$$(g_1 g_2)^*(\psi)(x) = \psi \circ g_1 g_2(x) = g_2^*(\psi \circ g_1)(x)$$

$$= g_2^* g_1^*(\psi)(x) \Rightarrow (g_1 g_2)^*(\psi) = g_2^* g_1^*(\psi)$$

$$\Rightarrow (g_1 g_2)^* = g_2^* g_1^* .$$

Let $\bar{\psi} = (g^{-1})^*(\psi)$. Then $\bar{\psi}(x) = \psi \circ g^{-1}(x)$ for all $x \in X$. If $x' = g^{-1}(x)$, the bijectivity of g implies that x' ranges over X as x does, and we have $x = g(x')$ implying $\bar{\psi}(g(x')) = \psi(x')$ for all $x' \in X$. Thus $g^*(\bar{\psi}) = \psi$ so

$$\bar{\psi} = (g^*)^{-1}(\psi) \text{ and so } (g^{-1})^* = (g^*)^{-1}. \ \#$$

<u>Definition 3.4.</u> A <u>dynamics</u> for a net N is a function from Ω to itself. #

Note that we are here dealing with a dynamics in discrete time. A dynamics Δ is merely the function which gives the state at time $t + 1$ deterministically in terms of that at t, with Δ itself not explicitly time dependent.

<u>Definition 3.5.</u> Let K be some isomorphism group of a net N, Δ a dynamics for N. Δ is said to <u>respect K</u> if $\Delta \circ g^* = g^* \circ \Delta$ for every $g \in K$. #

<u>Example 3.6.</u> Following example 3.2(i), for $\psi \in \Omega$ we can write, when X is a finite set,

$$\psi'(x) = H\left\{ \sum_{x' \in X} \phi(x,x')\psi(x') - \theta \right\}$$

where H is Heaviside's step function and θ is a common <u>threshold</u> for each neuron of X.

If now $\Delta\psi = \psi'$ we define a dynamics on the net, which may be interpreted as follows. $\sum\phi\cdot\psi$ expresses the total activity of the net at any time, weighted by the connection strengths ϕ; if, for any neuron $x \varepsilon X$, this total effect exceeds the threshold θ then $\Delta\psi(x)$, which is the activity of x at the next temporal instant, is 1. That is, at time $t + 1$ a neuron is active if the weighted effect of all neurons at the previous instant exceeds threshold.

Let G be the isomorphism group of (X,ϕ). Then Δ respects G, because for $g \varepsilon G$ we have:

$$g^* \circ \Delta(\psi)(x) = g^*(\psi')(x) = \psi'(g(x))$$

$$= H\left[\sum_{x'\varepsilon X} \phi(g(x),x')\psi(x') - \theta\right]$$

$$= H\left[\sum_{x''\varepsilon X} \phi(g(x), g(x''))\psi(g(x'')) - \theta\right]$$

on writing $x' = g(x'')$ - since g is bijective, x'' ranges over X when x' does. Since g is also an isomorphism of the net we can re-write this as

$$H\left[\sum_{x''\varepsilon X} \phi(x,x'')\psi(g(x'')) - \theta\right]$$

$$= H\left[\sum_{x''\varepsilon X} \phi(x,x'')g^*(\psi)(x'') - \theta\right]$$

$$= \Delta \circ g^*(\psi)(x). \quad \#$$

This result clearly depends on the fact that ψ only occurs in the dynamical equations through the linear combination $\sigma(x) = \sum_{x'} \phi(x,x')\psi(x')$. But this is equally true for the more general activity set of example 3.2(ii); for any real-valued function $\mathcal{F}$ with codomain in A the expression $\Delta(\psi) = \mathcal{F} \circ \sigma$ gives a dynamics which respects $\mathbb{G}$.

To pursue the meaning of this dynamics we must interpret the function $\mathcal{F}$. Recall that $\psi(x)$ is now considered to be the proportion of cells of a neuronal aggregate x which are active at any time. With $\phi(x,x')$ interpreted as the synaptic strength of connections from aggregate x' to aggregate x, we may regard $\sigma(x)$ as the spatially summed average de-polarization, say in the initial segments of an aggregate x.

At the next time instant, the value of $\psi'(x)$ - that is, the proportion of cells in x which are active at time t + 1 - will be the proportion of cells whose threshold is less than $\sigma(x)$. The stochastic burden of this interpretation is therefore thrown into the distribution of thresholds in a typical aggregate. Most particularly, we will, in the sequel, make the assumption that this distribution is uni-modal. Our function $\mathcal{F}$, which is merely the cumulative distribution is then a sigmoidal function - that is, it satisfies the condition.

(i) $\mathcal{F}(0) = 0$ (ii) $\mathcal{F}(\sigma) \to 1$ as $\sigma \to \infty$

(iii) $\mathcal{F}'(\sigma) > 0$ for all $\sigma \geq 0$

(iv) $\mathcal{F}''(\sigma) = 0$ for only one positive σ.

(See Appendix 2 for the source of this argument.)

Definition 3.7. Let H be a subgroup of the isomorphism group G and x be a node of a net. Then $x \cdot H = \{h(x) : h \in H\}$ is called the orbit of x under H. #

This concept of orbit is well-known in the theory of dynamical systems where the following result is proved.

Lemma 3.8. The orbits of $x \in X$ form a partition of X.

Proof. Define a relation on X by $x \equiv x'$ iff x' lies in the orbit of x. Then this is an equivalence relation, for:

(i) $x \equiv x$ since $x = 1(x)$ with $1 \in H$

(ii) $x \equiv x' \Rightarrow x' = h(x)$ for some $h \in H$

$\Rightarrow x = h^{-1}(x')$ with $h^{-1} \in H \Rightarrow x' \equiv x$.

(iii) $x \equiv x', x' \equiv x'' \Rightarrow x'' = h'(x'), x' = h(x)$

for $h, h' \in H \Rightarrow x'' = h'h(x)$ with $h'h \in H \Rightarrow x \equiv x''$.

The equivalence classes are the orbits, which therefore partition X. #

Theorem 3.9. If a dynamics Δ respects a group H of isomorphisms of a net and if a state ψ is constant on the orbits of H, then $\psi' = \Delta\psi$ is also constant on those orbits.

Proof. Let X be the set of nodes of the net. For all $x \varepsilon X$, $h \varepsilon H$, $\psi(h(x)) = \psi(x)$

$\Rightarrow h^*(\psi)(x) = \psi(x)$ for all $x \varepsilon X \Rightarrow h^*(\psi) = \psi$.

Then $h^*(\psi') = h^* \circ \Delta(\psi) = \Delta \circ h^*(\psi) = \Delta(\psi) = \psi'$ so $\psi'(h(x)) = \psi'(x)$ for all $x \varepsilon X$, $h \varepsilon H$. #

Definition 3.10. A state ψ is a quasi-equilibrium for a dynamics Δ if there exists $g \varepsilon G$ such that $\Delta(\psi) = g^*(\psi)$. If $g = 1$ then ψ is an equilibrium for Δ. #

The motivation for this definition is the recognition that in a net with symmetry a pattern of activity might be shuttled around by the dynamics without an overall change of form, although the nodes implicated in that activity change from one instant to the next. The consistency of this view is implied by the following lemma.

Lemma 3.11. If ψ is the quasi-equilibrium of Definition 3.10 and Δ respects G then $\psi' = h^*(\psi)$ is a quasi-equilibrium for any $h \varepsilon G$.

Proof. $\Delta(\psi') = \Delta \circ h^*(\psi) = h^* \circ \Delta(\psi)$

$$= h^* \circ g^*(\psi) = h^* \circ g^* \circ (h^*)^{-1}(\psi')$$

$$= (h^{-1}gh)^*(\psi') \quad \text{by Lemma 3.3.}$$

We can now note that with $h = g$ we get $\Delta(\psi') = g^*(\psi')$, so the process may be iterated leading to a sequence $\{\psi_r\}$ of quasi-equilibria, according to $\psi_0 = \psi$, $\psi_{r+1} = g^*(\psi_r)$. Naturally since g has finite order this sequence is cyclic yielding a temporal periodicity of activity. #

An alternative interpretation would result if the monitoring of activity were undiscriminating with respect to the nodes which were actually active at any instant. A neuronal instance might occur in recording from the cortical surface where we may be failing to discriminate neural activity along the radial dimension. Dynamic shuffling of activity in cortical columns without change of overall pattern could make more explicit the device suggested by the present author (Muir, 1979) of time-sharing information along parallel pathways in order to overcome the limitations imposed on transmission rate by the refractory period following an action potential. This argument is a digression from the present theme so it is set out in more detail in Appendix 2.

To return to quasi-equilibria, their intimate connection with true equilibria is partly explained by the following.

<u>Theorem 3.12.</u> Let (X,ϕ) be a net with isomorphism group G, Δ a dynamics which respects G having an equilibrium ψ. For any $g \varepsilon G$ let

$$H_g = \{\bar{g} \varepsilon G : g\bar{g} = \bar{g}g\} \; ; \; \phi^g(x,x') = \phi(g(x),x')$$

for all $x, x' \varepsilon X$; $\Delta^g = \Delta \circ g^*$. Then H_g is an isomorphism group of (X,ϕ^g), Δ^g respects H_g and ψ is a quasi-equilibrium state of Δ^g.

<u>Proof</u>. H_g is a group since (i) $\bar{g},\bar{g}_1 \varepsilon H_g$

$\Rightarrow g\bar{g} = \bar{g}g, \; g\bar{g}_1 = \bar{g}_1 g \Rightarrow g\bar{g}\bar{g}_1 = \bar{g}g\bar{g}_1 = \bar{g}\bar{g}_1 g$

$\Rightarrow \bar{g}\bar{g} \varepsilon H_g$ and (ii) $g\bar{g} = \bar{g}g \Rightarrow \bar{g}^{-1} g = g\bar{g}^{-1}$

$\Rightarrow \bar{g}^{-1} \varepsilon H_g$.

Each $\bar{g} \varepsilon H_g$ is an isomorphism of ϕ^g since

$$\phi^g(\bar{g}(x), \bar{g}(x')) = \phi(g\bar{g}(x), \bar{g}(x'))$$
$$= \phi(\bar{g}(g(x)), \bar{g}(x')) = \phi(g(x), x') = \phi^g(x,x').$$
$$\Delta^g \circ \bar{g}^* = \Delta \circ g^* \circ \bar{g}^* = \Delta \circ (\bar{g}g)^* = \Delta \circ (g\bar{g})^*$$
$$= \Delta \circ \bar{g}^* \circ g^* = \bar{g}^* \circ \Delta \circ g^* = \bar{g}^* \circ \Delta^g.$$

Finally $\Delta^g(\psi) = \Delta \circ g^*(\psi) = g^* \circ \Delta(\psi) = g^*(\psi)$. #

CHAPTER 4.

SOME SPECIAL CASES

Homogeneous Nets. If a net (X,ϕ) is homogeneous there is only one orbit of its isomorphism group G, the whole of X.

Definition 4.1. A state ψ is uniform if $\psi(x)$ is the same for all $x \in X$.

By Theorem 3.9, in a homogeneous net a dynamics which respects G maps uniform states to uniform states; thus there is a feasible motion which has only one parameter, the common value ψ of $\psi(x)$.

Example 4.2. Following Example 3.2 (ii) and its dynamics described in Example 3.6, we have in the homogeneous case the possibility of dynamical equations $\psi' = \mathcal{F}(k\psi)$ where $k = \sum_{x'} \phi(x,x')$ is actually independent of x. Motion is along the diagonal of the cube I^X which, being a closed interval, contains an equilibrium ψ_0 if $\mathcal{F}$ is continuous.

Actually, by a standard fixed point theorem, there will be such an equilibrium for a general continuous dynamics on I^X, which is topologically a closed disc when X is finite.

However, even confining ourselves to the diagonal motion gives us quite a variety of qualitative possibilities depending upon the form of $\mathcal{F}$. The analysis of equilibria when $\mathcal{F}$ is differentiable and sigmoidal is qualitatively the same as in purely excitatory model in Appendix 1. In the present case the criteria for stability of the uniform equilibrium ψ_0 is that the modulus of $\alpha = k\,\mathcal{F}'(k\psi_0)$ be less than 1. Again, one can see the creation and annihilation of equilibria as the parameter k is varied, perhaps, in a neural interpretation, by homogeneously distributed inputs from an external system.

A more interesting possibility results from looking at the stability of the uniform motion by considering perturbations off the diagonal. On writing $\psi(x) = \psi + \varepsilon(x)$ $\psi'(x) = \psi' + \varepsilon'(x)$, the dynamical equations from Example 3.6 are $\psi'(x) = \mathcal{F}(\sigma(x))$ where

$$\sigma(x) = \sum_{x'} \phi(x,x')\left[\psi + \varepsilon(x')\right]$$

$$= k\psi + \sum_{x'} \phi(x,x')\varepsilon(x').$$

If $\mathcal{F}$ is differentiable we get the linear approximation

$$\psi' + \varepsilon'(x) = \mathcal{F}(k\psi) + \mathcal{F}'(k\psi)\cdot \sum_{x'} \phi(x,x')\varepsilon(x')$$

or $$\varepsilon'(x) = \mathcal{F}'(k\psi)\cdot\sum_{x'} \phi(x,x')\varepsilon(x').$$

Stability of uniform motion thus depends essentially on the eigenvalue structure of ϕ; intervals on the diagonal where $\mathcal{F}$ is monotonic are qualitatively the same with respect to stability. With ϕ suitably chosen symmetry-breaking will occur. That is, the overall structural symmetry of a homogeneous net does not demand that uniform motion be the physically significant one.

A Stochastic Model.

A more specific example, where the full dynamics of the symmetry-breaking may be explicitly seen, arises as follows. Consider a network of six neurons $X = \{N_1 \ldots N_6\}$ divided into two pools. $X_1 = \{N_1, N_3, N_5\}$, $X_2 = \{N_2, N_4, N_6\}$ with each neuron of X_1 synapsing inhibitorily with the same strength on every neuron of X_2 and vice-versa. This net is homogeneous with symmetry group $S_3 \times S_3$, where S_3 is the full symmetric group on 3 objects. This has a cyclic subgroup generated by the permutation

$$(N_1\,N_2\,N_3\,N_4\,N_5\,N_6) \longrightarrow (N_3\,N_4\,N_5\,N_6\,N_1\,N_2)$$

whose orbits are X_1 and X_2. By Theorem 3.9 there is a motion the states of which are (spatially) constant on X_1 and X_2 separately.

To make an explicit dynamics let us assume that each neuron is spontaneously active and it requires at least two neurons firing in the other pool to inhibit it. Denoting the common probability of neuronal firing at some instant in X_1, X_2 by x_1, x_2 respectively, then the probability x'_1 of X_1 firing in the next instant is equal to the probability that only one or none of the neurons in X_2 is firing - with a corresponding equation for x'_2.

$$x'_2 = (1-x_1)^3 + 3x_1(1-x_1)^2$$

$$x'_1 = (1-x_2)^3 + 3x_2(1-x_2)^2.$$

Strictly, such equations need a more rigorous derivation, somewhat along the lines of the discussion in Appendix 1, but that is not the purpose here. Rather, we are concerned to present a model which is qualitatively transparent. What we have is a recurrence relation in the $x_1 - x_2$ plane. The equilibria, obtained by solving the equations $x'_1 = x_1$, $x'_2 = x_2$, are the uniform one $x_1 = x_2 = \frac{1}{2}$ and the asymptotic ones $x_1 = 0$, $x_2 = 1$; $x_1 = 1$, $x_2 = 0$. By a simple numerical check, it is easy to see that the uniform equilibrium is unstable and that a slight bias of one probability over the other causes escalation to the equilibrium where the weaker activation has died to zero.

I'll discuss stability more fully in the context of a generalized model where external controls are introduced. Rather than have the neurons spontaneously active consider an external input line to each, the excitation probability of the neurons of X_1, X_2 being a_1, a_2 respectively. This introduces factors a_1, a_2 into the right-hand sides of the equations for x'_1, x'_2 respectively.

The iteration in the $x_1 - x_2$ plane so obtained depends on the values of a_1 and a_2. In Figure 1 we show graphically that the iteration tends to a unique stable limit for certain choices of a_1 and a_2. In fact, for some values of a_1 there is precisely one stable limit for all a_2. It is easy to check that $x'_2 = a_2\left[(1-x_1)^3+3x_1(1-x_1)^2\right] = a_2(1-x_1)^2(1+2x_1)$ is a kind of decreasing sigmoidal function, like II and III of Figure 1, with maximum a_2 at $x_1 = 0$.

Figure 2, in contrast, shows that other a-values can yield 3 equilibria. However, the middle one β is unstable and will never be reached by the step-wise iteration which the activation probabilities go through in time. Variation of the external inputs a_1, a_2 can lead to elimination and creation of equilibria. Thus, if a_2 is steadily decreased the equilibria α and β will merge and disappear leaving a single stable point γ.

Fig. 1

with a_1 sufficiently small there is only one intersection for each a_2. I: $a_1 = 0.6$; II: $a_2 = 0.7$; III: $a_2 = 0.5$.

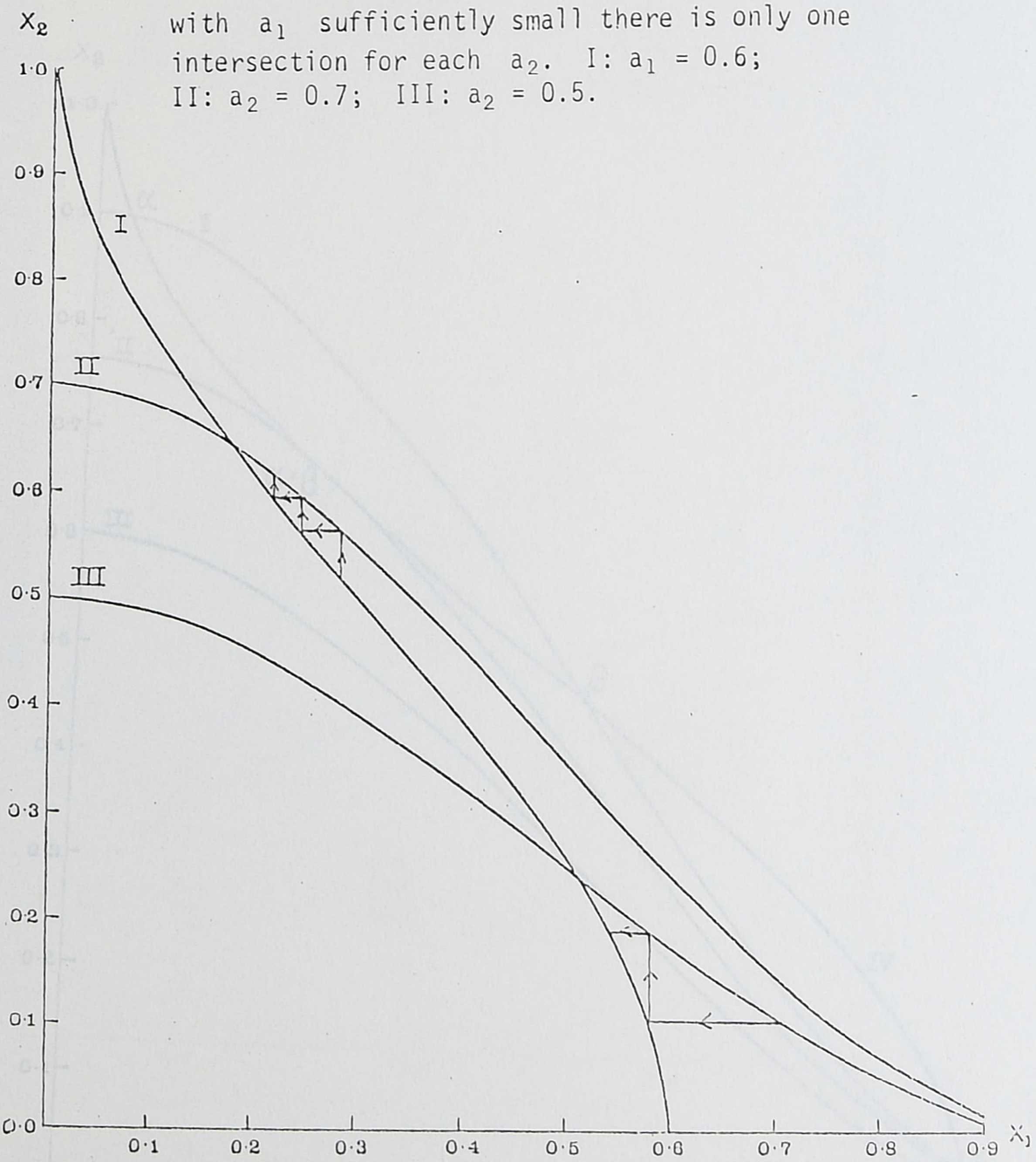


Fig. 2.

Merging of equilibria as the control point varies in a-space. I: $a_2 = 0.9$; II: $a_2 = 0.768$; III: $a_2 = 0.6$; IV: $a_1 = 0.879$.

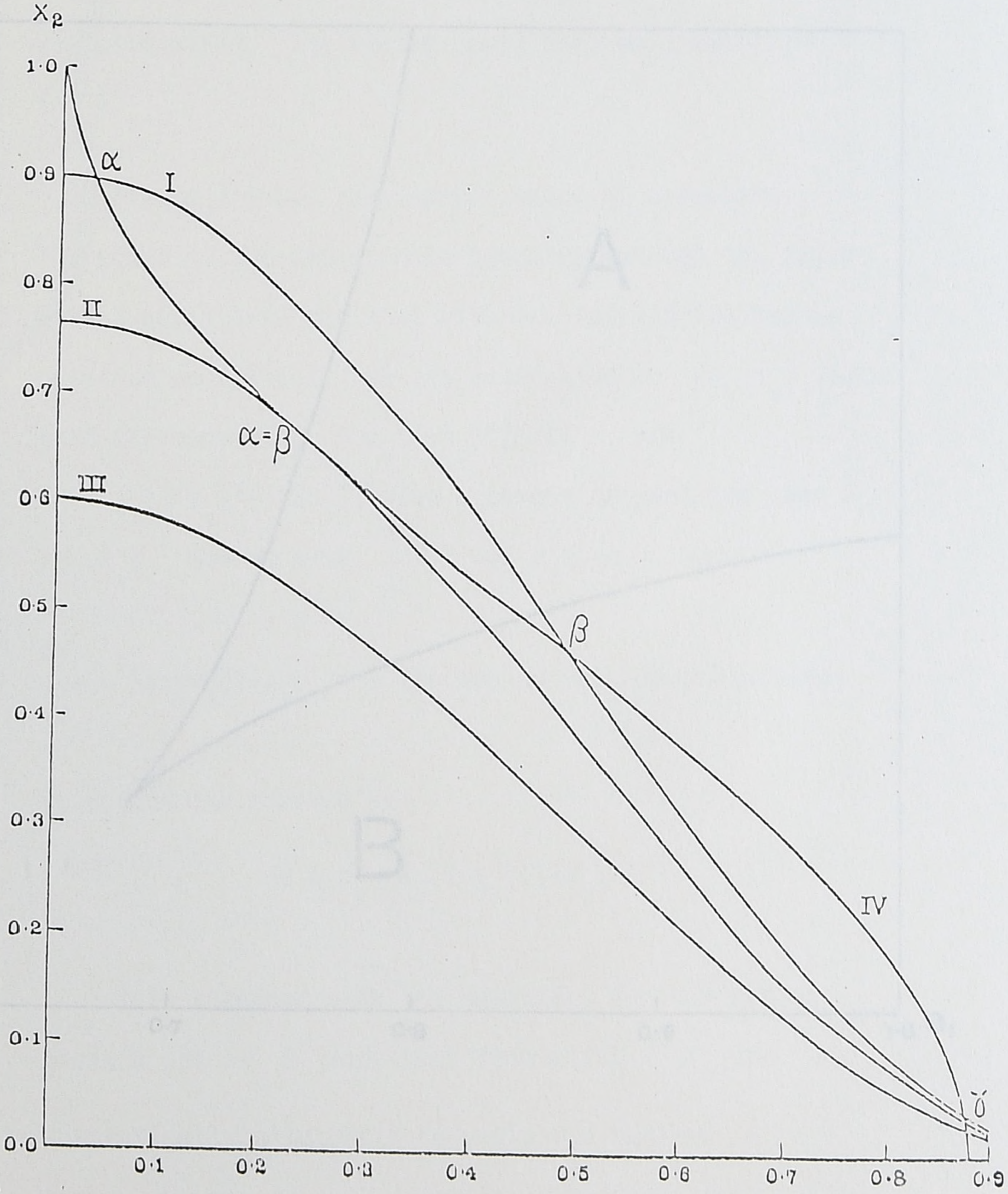
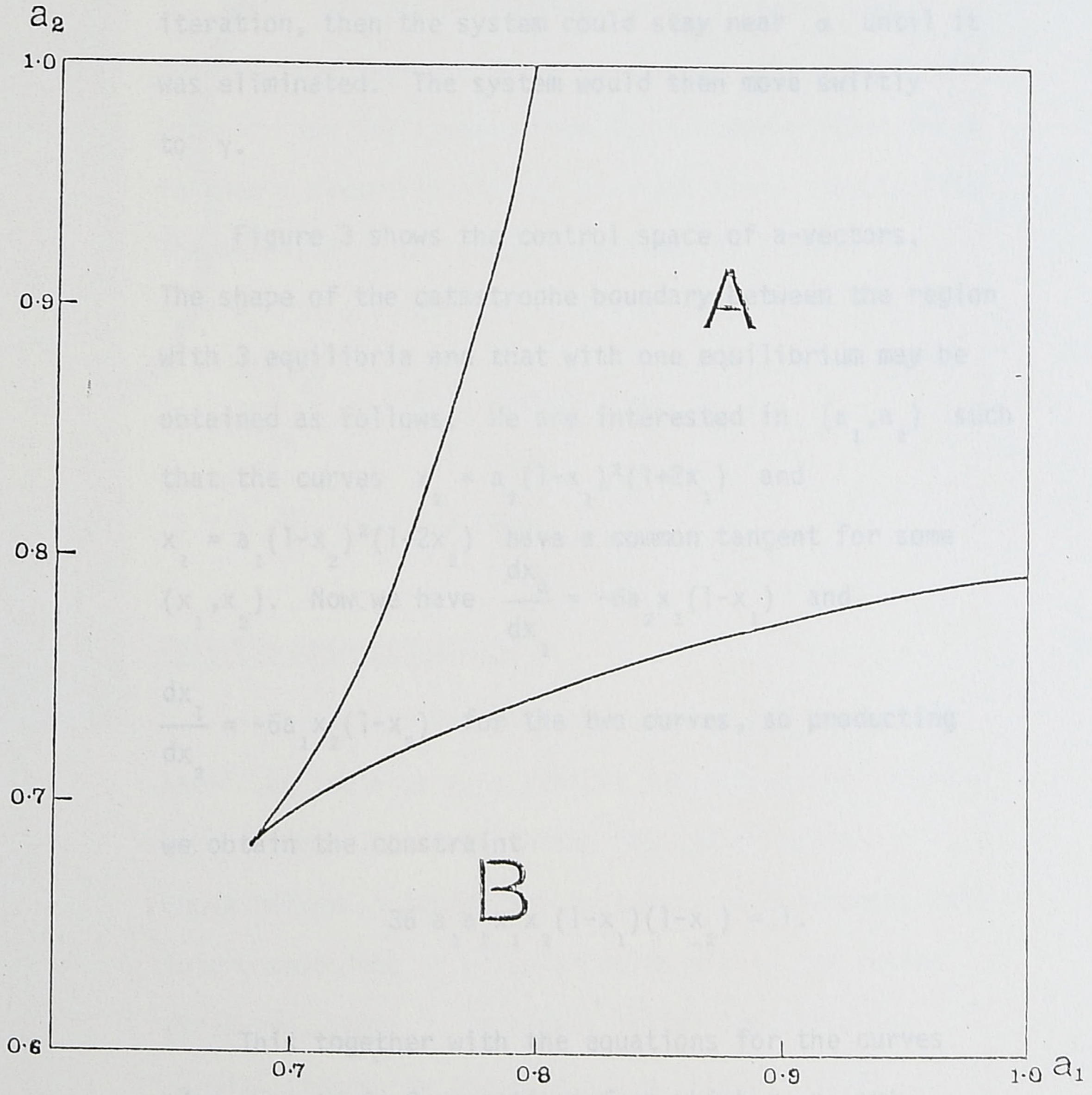


Fig. 3.

Catastrophe boundary in a - space. A = Region with 3 equilibria, B = Region with 1 equilibrium.



Paraphrasing Thom's catastrophe argument (Thom, 1975), if the system were initially located at α and if the variation of a_2 were quasi-static, that is, slow with respect to the internal dynamics of the stabilizing iteration, then the system could stay near α until it was eliminated. The system would then move swiftly to γ.

Figure 3 shows the control space of a-vectors. The shape of the catastrophe boundary between the region with 3 equilibria and that with one equilibrium may be obtained as follows. We are interested in (a_1,a_2) such that the curves $x_2 = a_2(1-x_1)^2(1+2x_1)$ and $x_1 = a_1(1-x_2)^2(1+2x_2)$ have a common tangent for some (x_1,x_2). Now we have $\frac{dx_2}{dx_1} = -6a_2x_1(1-x_1)$ and

$\frac{dx_1}{dx_2} = -6a_1x_2(1-x_2)$ for the two curves, so producting

we obtain the constraint

$$36\, a_1a_2x_1x_2(1-x_1)(1-x_2) = 1.$$

This together with the equations for the curves gives a set of 3 equations from which x_1,x_2 are theoretically eliminable to yield the desired $a_1 - a_2$ locus. We can proceed numerically by first eliminating a_1, a_2 to get

$$36\, x_1^2x_2^2 = (1-x_1)(1+2x_1)(1-x_2)(1+2x_2).$$

For any $0 \leq x_1 \leq 1$, this is a quadratic in x_2 and by rather tedious calculations involving completing the square a number of times, it can be seen that x_2 is real for all physical x_1. Figure 3 shows a plot of the locus obtained by finding (x_1, x_2) pairs and substituting into the curve equations. The catastrophe boundary has the cuspoid form reminiscent of that found in Thom's classification of gradient field singularities. The position of the cusp is obtained by putting $x_2 = x_1$ to get $8x_1^2 - x_1 - 1 = 0$ so

$x_1 = \dfrac{1+\sqrt{33}}{16} \approx 0.4215$ which gives $a_1 = a_2 =$

$(9 + 7\sqrt{33}) / 72 \approx 0.6835$.

Possible interpretations.

The model of the previous section was designed in the first instance as a vehicle for discussing lateral inhibition. This omnipresent feature of neural organization is found, for instance, in horizontal cell interconnections of bi-polar cells within the retina (Shepherd, 1979), in basket cells' inhibition of parallel fibres in the cerebellum (Eccles, 1977), and at the medullary level in the dorsal columns of the somatosensory system (Gordon and Jukes, 1964).

It is usually considered as a difference enhancer according to the following, oft-presented verbal argument. Consider two neurons A and B receiving excitatory inputs and having cross-inhibitory connections. If the input to A slightly exceeds that to B then A will suppress B's output more than vice-versa. The difference in excitation will thus be sharpened. Mathematical models have been presented by Ratcliff (1965) and others.

The simplest formal way to exhibit the effect uses the approach of the previous section; if x_1 and x_2 are firing probabilities of A and B, a_1 and a_2 the external input probabilities we can write, in a crude model

$$x'_1 = a_1(1-x_2) \qquad x'_2 = a_2(1-x_1).$$

Stability occurs when $x'_1, = x_1, x'_2 = x_2$ and gives

$$x_1 = a_1(1-a_2)/(1-a_1a_2), \quad x_2 = a_2(1-a_1)/(1-a_1a_2).$$

Then when $a_1 > a_2$ we have $1 - a_2 > 1 - a_1$ so $x_1/x_2 = a_1(1-a_2)/a_2(1-a_1) > a_1/a_2$ exhibiting a relative enhancement of outputs with respect to inputs.

Now this interpretation of our six neuron model is qualitatively the same in the region where there is just one equilibrium, although the enhancement is more pronounced. The three equilibria region, however, needs a different interpretation, since the system now possesses a rudimentary memory of its past experience in the sense that it can get trapped in the "wrong" stable state, stuck in a mode where, say x_1 is smaller than x_2 even when the external input a_2 is steadily raised above a_1.

A "hysteresis" effect has indeed been treated as a basis for short-term memory (Cragg and Temperley, 1955), a possibility given further discussion by Wilson and Cowan (1972) in the context of their own model of interacting excitatory and inhibitory neural populations. The merit of the present model is its simplicity and its amenability to exact calculation and proof.

A similar model with many neuronal pools could be constructed in which the uniform motion was unstable with symmetry-breaking by an external control causing transition to a stable state where one dominant pool suppressed the action of all others. Mode switching of this sort has been discussed using more ad-hoc models by Barnwell and Stafford (1977) and Shallice (1972) as a model for attention switching. A similar theoretical employment of mutually inhibiting sub-systems is Minsky and Papert's "Society Theory of Mind" for the establishment of dominance within a competitive aggregate (see e.g. Minsky, 1979).

Using our Theorem 3.12 we can convert any such dominance model into one in which the dominant mode shuffles dynamically around a loop. To explicate this consider a three node network $X = \{x_1, x_2, x_3\}$ with ϕ = unit matrix and in the notation of Example 3.6 $\Delta(\psi) = \mathcal{F} \circ \sigma$ where $\mathcal{F}$ is a sigmoidal function admitting three solutions to $\mathcal{F}(x) = x$. Then, as usual, the middle solution is unstable with the others stable. Since the neurons are de-coupled an equilibrium state ψ_0 can exist with $\psi_0(x_1)$ at the high stable value and $\psi_0(x_2)$, $\psi_0(x_3)$ at the low one.

If we take g to be the permutation $x_1 \longrightarrow x_2$, $x_2 \longrightarrow x_3$, $x_3 \longrightarrow x_1$ then H_g is the cyclic group on three objects, ϕ^g establishes unit excitatory connections from each neuron to the next in a cyclic way. Δ^g then maps the state ψ_0 round the cycle.

A similar 2-state system functions as a simple oscillator and is qualitatively like the system detailed in (Miller and Scott, 1977) forthe generation of locomotor movements within a limb.

An approach to hierarchies.

In the case of a group net, defined in 2.18, the set of nodes G is a subgroup of the isomorphism group. Any subgroup H of G breaks G into orbits which are cosets of H in G. Thus all orbits have the same number of elements and a solution ψ_0 which is constant on orbits can be considered as a dynamics on the quotient set G/H.

If the group net is homogeneous the cosets are also similarly connected, both internally and between cosets. This gives a strong suggestion of how to theorise about hierarchies. Each coset is "the same" and the whole set can be viewed either through the detailed dynamics ψ_0 of its individual nodes or through the factored dynamics on its cosets. Even if a dynamics ψ is not uniform on cosets we can still imagine ψ_0 to be stable on a fast time scale and write $\psi = \psi_0 + \psi^1$. ψ_0 will describe the large-scale dynamics, treating the cosets as aggregates; ψ' on each coset can be considered as a perturbation from the uniform state, being subject primarily to an internal dynamics.

In the case of group nets there are three levels of analysis involved - the net, the cosets, the nodes. Now we can consider the cosets, as networks of neurons, to be identical automata. This suggests that we might develop the theory of hierarchies into a way of building automata from homogeneously inter-connected, identical sub-automata.

This is the motivation for the greater abstractions in succeeding chapters.

CHAPTER 5.

TOLERANCE AUTOMATA

The problem of nearness in finite sets.

Definition 5.1. An *automaton* $A = (X,Y,Q,\delta,\lambda)$ is a triple of finite sets (X,Y,Q) and a pair of functions (δ,λ), $\delta : X \times Q \to Q$, $\lambda : X \times Q \to Y$. Q is called the *state set*, X the *input set*, Y the *output set*, δ the *next-state function* and λ the *output function* of A. #

It is a classic result (see Arbib, 1969) that a network of formal neurons with external input and output connections can be constructed to simulate any automaton. Essentially, the proof proceeds by coding the elements of the sets Q,X,Y by distinct binary vectors, so that δ,λ can be regarded together as a single binary vector-valued function of a binary vector. Each component of this function is then re-expressible as a Boolean function and any such is realizable by a combination of simple formal neurons which can act as negation, disjunction and conjunction units.

Now as Arbib (1966) has pointed out, automata theory looks strikingly like control theory for dynamical systems, but transferred into a discrete setting. This strongly suggests we might ask questions of automata theory, and hence of neural networks, similar to those raised by the need to guide dynamical systems. For instance: are equilibria stable and in what way? What is the most efficient means of guiding a system to a desired state?

Unfortunately, such questions only make sense when the state space has some structure which allows us to speak of "nearness" of points. This is alright in dynamical systems where the manifold structure of the state space presupposes a topology. In contrast, finite sets admit only uninteresting topologies. Briefly, this is because the neighbourhood set of any point, being finite allows us to declare that the intersection of all neighbourhoods of a point is open. Any of the usual separation axioms stronger than T_0 then implies that singleton sets are open so the topology is trivial (see Stong, 1966, for details).

Arbib's "rapprochement" of automata and control theory rested on the suggestion that "nearness" in finite sets be axiomatized through the notion of *tolerance* introduced by Zeeman (1961).

The purpose of this chapter is to enquire into some aspects of the theory of such tolerance automata. As will be seen, the idea is not a good one; tolerance is too rigid to accurately express what is desired. However, the directions pursued here can introduce the concerns of the following chapter.

Tolerance Automata.

Definition 5.2. A *tolerance space* (X,ρ) is a set X with a symmetric, reflexive relation ρ. We write $x \rho x'$ and read this as "x is near x'". #

Definition 5.3. A *tolerance map* f from a tolerance space X to a tolerance space Y is a function $f : X \to Y$ such that $x \rho x' \Longrightarrow f(x) \rho f(x')$. (Note that we use ρ indifferently for all tolerance spaces and henceforth we will merely refer to "spaces".) #

Definition 5.4. A *homeomorphism* is a bijective tolerance map whose inverse is a tolerance map. #

Notice that these definitions are merely a special case of nets and morphisms when the connection function has values in $\{0,1\}$. The words we employ, however, are intended to invoke the similarity of this theory with topology.

Definition 5.5. (i) Let X,Y be two spaces. We define a product space $X \times Y$ by $(x,y)\,\rho\,(x',y')$ iff $x \rho x'$, $y = y'$ or $x = x'$, $y \rho y'$, for all $x, x' \in X$, $y, y' \in Y$.

(ii) Let π be an equivalence relation on a space X. We define the quotient space X/π by $[x]\,\rho\,[y]$ iff there exist $x \in [x]$, $y \in [y]$ with $x \rho y$.

(iii) If X is a set and Y a space, define the function space Y^X by $f \rho g$ iff $f(x)\,\rho\,g(x)$ for all $x \in X$. #

Actually, there are various definitions which may be adopted (see Zeeman, 1961; Poston, 1968; Muir and Warner, 1980a) but these are adequate for our purposes.

In Arbib's discussion a tolerance automaton is just an automaton $A = (X,Y,Q,\delta,\lambda)$ with state set Q a tolerance space. Of course, this can't yield a good theory unless the tolerance and automata structures are linked. He defines:

(i) A is stable if for each $x \in X$, $q \to \delta(x,q)$ is a tolerance map;

(ii) A is inertial if the tolerance on Q is defined by $q \, \rho \, \delta(x,q)$ for all $x \in X$, $q \in Q$.

Henceforth we will not consider outputs, confining ourselves to the function $\delta : X \times Q \to Q$ and will write $\delta(x,q)$ as $q \cdot x$. X, being a set of maps from Q to Q, may be considered as a subspace of Q^Q - then inertia is the requirement that $x \, \rho \, 1$ for all $x \in X$.

We can extend these considerations to X^*, the set of finite strings of elements of X. Again each element of X^* determines an element of Q^Q. If two elements of X^* are considered as equivalent when they give the same map from Q to Q, the set of equivalence classes $S = X^*/\pi$ forms a semi-group endowed with the quotient space tolerance. S is called the semi-group of A.

Lemma 5.6. If A is stable the semi-group operation is a tolerance map.

Proof. Let $x,y,x',y' \in X^*$ with $(x,y)\rho(x',y')$ By definition 5.5 (i) either $x = x'$, $y \rho y'$ which implies $q \cdot xy = q \cdot x'y \, \rho \, q \cdot x'y'$ by 5.5 (iii) or $x \rho x'$, $y = y'$ which implies $q \cdot x \rho q \cdot x'$ by 5.5 (iii) so $q \cdot xy \rho q \cdot x'y'$ by stability assumption. In either case $xy \rho x'y'$ so $(x,y) \to xy$ is a tolerance map $X^* \times X^* \to X^*$ which factors to a tolerance map $S \times S \to S$. #

Assume now that there exists a state $q_0 \in Q$ from which all states are *reachable* - that is, for all $q \in Q$, $\exists x \in X^*$ such that $q = q_0 \cdot x$. Define an equivalence relation π on S by $x \equiv x' \pmod{\pi}$ iff $q_0 \cdot x = q_0 \cdot x'$. Then $\eta : S \to Q$ defined by $\eta(x) = q_0 \cdot x$ induces $\tilde{\eta} : S/\pi \to Q$ and there follows:

Theorem 5.7. If A is a tolerance automaton, then $\tilde{\eta}$ is a tolerance bijection. If A is inertial, $\tilde{\eta}$ is a homeomorphism.

Proof. Bijectivity is trivial. Let $[x]$ denote a π-class of S. If $[x] \rho [x']$ in S/π then by 5.5 (ii) there exists $x_1 \equiv x \pmod{\pi}$, $x_1' \equiv x' \pmod{\pi}$ with $x_1 \rho x_1'$. Thus $q_0 \cdot x = q_0 \cdot x_1 \rho q_0 \cdot x_1' = q_0 \cdot x'$ so $\tilde{\eta}([x]) \rho \tilde{\eta}([x'])$ and $\tilde{\eta}$ is a tolerance map.

If A is inertial, consider $q, q' \in Q$ with $q \rho q'$. There exists $y \in X$ such that $y \rho 1$ and $q' = q \cdot y$; by reachability there is $x \in X^*$ such that

$q = q_0 \cdot x$. Then for any $q'' \varepsilon Q$, $q'' \cdot xy \rho q'' \cdot x$ so $xy \; \rho \; x$ which implies $[xy] \rho [x]$. Since $\tilde{\eta}([xy]) = q'$, $\tilde{\eta}([x]) = q$, $\tilde{\eta}^{-1}$ is a tolerance map. #

Decomposition Theory. Some of the most striking results of automata theory concern the decomposition of a given automaton into a collection of simpler types suitably connected together. More exactly, the given automaton is simulated by a possibly larger automaton which can be so decomposed. In this section we attempt to follow through a similar project for tolerance automata. Our approach follows closely that in (Muir and Warner, 1980b), which is modelled on two approaches in the purely algebraic case. The first, which occupies us until Theorem 5.13, is that of (Ginzburg, 1968); the second is found in (Arbib, 1969). As the problem is stated above, it requires: (i) a concept of simulation - this is to be found in the notion of "cover", which is transferred to the tolerance context in definition 5.8 below; (ii) a way of connecting automata together - this is the "cascade product" of 5.9 below; (iii) a collection of special automata as the basic building blocks - these are described in 5.10.

Definition 5.8. A tolerance automaton $A = (X,Q)$ covers a tolerance automaton $A' = (X',Q')$ if there exists a subspace $\bar{Q}$ of Q, an identification $\eta : \bar{Q} \to Q'$ and a function $\zeta : X' \to X$ such that $\eta(q \cdot \zeta(x')) = \eta(q) \cdot x'$ for all $q \in \bar{Q}$, $x' \in X'$. #

The requirement that η be an identification is the tolerance addition to the usual definition.

Definition 5.9. Let A, A' be as above and $\omega : X \times Q \to X'$. Then the cascade product $A \,\omega\, A'$ is the automaton $(X, Q \times Q')$ such that $(q,q') \cdot x = (q \cdot x, q' \cdot \omega(x,q))$. #

To extend this to tolerance automata we need a canonical way of putting a tolerance on $Q \times Q'$. If inertia is desired the tolerance is determined by the input action so there is no problem. Stability is more of a problem since A, A' stable does not generally mean $A \,\omega\, A'$ is stable. (Not, at least, using our present definition 5.5(i) for a product of spaces - an alternative product, which Muir and Warner (1980b) call the "usual" product, does ensure this but suffers other deficiencies (see Muir and Warner 1980a).)

In general, we are here encountering a recurrent problem of this theory, how to reconstruct the tolerance on a given space, whose underlying set is decomposed as part of a Cartesian product, from the given tolerances on the factors. Unlike the topological case, there seems to be no canonical answer.

<u>Definition 5.10.</u> (i) An automaton every input of which is a permutation of the state set or maps every element to a fixed element, is called a <u>permutation-reset</u> (p.r.) automaton.

(ii) An automaton whose state space is a group and for which every input is a right translation by some element of that group is called a <u>group</u> automaton.

(iii) A p.r. automaton in which the only permutation inputs are identity maps is called an <u>identity-reset</u> (i.r.) automaton. #

<u>Theorem 5.11.</u> Every p.r. tolerance automaton can be covered by a cascade of an i.r. tolerance automaton and a group tolerance automaton.

<u>Prpof.</u> Let $A = (X,Q)$ be the given automaton.

Let X_p = the set of permutation inputs, X_r = the set of constant (reset) inputs. Throughout let $x_1 \in X_p$ $x_2 \in X_r$. Then X_p^* is a group under concatenation of strings.

Define a group automata $B = (X, X_p^*)$ with input action $x \cdot x_1 = xx_1$, $x \cdot x_2 = x$ and an i.r. automaton $C = (X \times X_p^*, Q)$ with input action

$$q \cdot (x_1, x) = q \ , \ q \cdot (x_2, x) = q \cdot x_2 x^{-1} \ .$$

If A is stable give B the function space tolerance on X_p^* while C retains the tolerance on Q. Then B and C are stable and the cascade product $B \mathbin{1} C$ with product tolerance on $X_p^* \times Q$ covers A. This is because the input action on $B \mathbin{1} C$ is

$$(x,q) \cdot x_1 = (xx_1, q) \ , \ (x,q) \cdot x_2 = (x, q \cdot x_2 x^{-1})$$

both of which are tolerance maps.

$$\eta : X_p^* \times Q \to Q \quad \text{defined by} \quad (x,q) \longmapsto q \cdot x$$

is a tolerance map. It is also an identification since $q \rho q'$ comes from $\eta(\Lambda, q) = q$, $\eta(\Lambda, q') = q'$ with $(\Lambda, q) \rho (\Lambda, q')$.

The algebraic cover conditions of definition 5.8 are satisfied with ζ = identity since

$$\eta((x,q) \cdot x_1) = \eta(xx_1, q) = q \cdot xx_1 = \eta(x,q) \cdot x_1$$

$$\eta((x,q) \cdot x_2) = \eta(x, q \cdot x_2 x^{-1}) = q \cdot x_2 = \bar{q} = \eta(x,q) \cdot x_2 \ .$$

If A is inertial, then requiring inertia for B imposes the tolerance $x \rho xx_1$ on X_p^* . This is not actually the function space tolerance on X_p^*, though the identity on X_p^* is a tolerance map from this inertial tolerance to the function space tolerance. As for C, inertia requires a tolerance on Q given by $q \rho q \cdot x_2 x^{-1}$. The product tolerance on $B \wr C$ coincides with the inertial tolerance and $B \wr C$ again covers A. #

This completes Theorem 5.11, but we can proceed one step further and cover B by (X_p^*, X_p^*) where η = identity and $\zeta : X \to X_p^*$ is defined by making $x \in X$ perform the same right translation of X_p^* as it does in B.

Theorem 5.12. An i.r. tolerance automaton can be covered by a cascade of two state tolerance automata.

Proof. Let $A = (X,Q)$ where $Q = \{q_1 \ldots q_n\}$. Put $Q_1 = \{q_1\}$, $Q_2 = \{q_2 \ldots q_n\}$, $\hat{Q} = \{Q_1, Q_2\}$.

Define $B = (X,\hat{Q})$ with input action $Q_i \cdot x = Q_j$ if x is a reset to an element of Q_j. Define also $C = (X,Q_2)$ where $q_j \cdot x = q_j$ if x resets to q_1 in A, $q_j \cdot x = q_k$ if x resets to $q_k (k \neq 1)$ in A.

A is stable and we can place the tolerance of 5·5 (ii) on B, considered as a partition of Q, and regard the state space of C as a subspace of Q.

Let $\bar{Q} \subsetneq \hat{Q} \times Q_2$ be given by $\bar{Q} = \{(Q_1,q_i) : q_i \rho q_1\} \cup \{(Q_2,q_j) : q_j \in Q_2\}$. Then $\eta : \bar{Q} \to Q$, defined by $\eta(Q_1,q_i) = q_1$, $\eta(Q_2,q_j) = q_j$ is a tolerance map when $\bar{Q}$ is considered as a subspace of $\hat{Q} \times Q_2$. Also, η is an identification since $q_j \rho q_k (j,k \neq 1)$ results from $(Q_2,q_j) \rho (Q_2,q_k)$ whereas $q_1 \rho q_k$ results from $(Q_1,q_k) \rho (Q_2,q_k)$.

Then the cascade $B \omega C$, where $\omega : X \times \hat{Q} \to X$ is given by $(x,Q_i) \to x$, covers A. Proceeding by induction we cover A by 2-state automata.

If A is inertial, putting the inertial tolerance on B requires $Q_1 \rho Q_2$ iff there is some reset in A. The inertial tolerance on C is again the subspace tolerance. Everything then proceeds as in the stable case. #

Theorems 5.11 and 5.12 cover any p.r. tolerance automaton by a cascade of a group tolerance automaton with 2-state i.r. tolerance automata. The algebraic approach shows finally how any automaton can be covered by a p.r. one - it is here that the tolerance analogue runs into serious trouble. We first consider the algebraic proof.

Theorem 5.13. Any automaton may be (algebraically) covered by a p.r. automaton.

Proof. If $A = (X,Q)$, $Q = \{q_1 \dots q_n\}$, let $Q_i = Q - \{q_i\}$ and $\hat{Q} = \{Q_i : q_i \in Q\}$. Let $B = (X,\hat{Q})$ and $C = (X \times \hat{Q}, R)$ where R is an $n-1$ element set with input actions defined as follows.

Every input to A either permutes Q and hence also $\hat{Q}$ or has some $q_i \in Q$ missing from its image; thus it maps Q into Q_i and can be considered as a reset of B. The role of C is to determine the position of the state point in any set Q_i. This can be done by arbitrarily ordering the elements of Q which induces a function $R \to Q_i$ for each Q_i. Induction on the number of elements of Q completes the proof. #

In a stable tolerance version of this, R would have to be identifiable with each Q_i, thus requiring all Q_i to be homeomorphic. Even if this were possible at the first inductive step, because Q might be a homogeneous tolerance space, it would break down at the next inductive step, since $Q - \{x\}$ would not be homogeneous in general.

Arbib's method and an example.

Now we turn to Arbib's method which works by decomposing the automaton semi-group into sub-semi-groups and ideals according to:

Lemma 5.14. Let S be a finite semi-group. Then either

(i) S is cyclic, (ii) S is right-simple,

(iii) there exists a proper right ideal T and a proper sub-semi-group V of S for which $S = T \cup V$ and either (a) there is $s \in S$ such that $V = sT$ or (b) T is two-sided. #

See (Arbib, 1969) for proof and details of how each case can be interpreted as a cascade product; (iii) in particular is employed inductively.

In an attempt to parallel the proof in the automaton case we can begin thus.

Theorem 5.15. An inertial tolerance automaton $A = (X,Q)$ with all states reachable from some $q_0 \in Q$ is covered by the automaton whose state space is the semi-group S of A with function space tolerance and whose input action is right translation by elements of S.

Proof. η of Theorem 5.7 is an identification and with $\zeta : X \to S$ given by $\zeta(x) = [x]$ we have $\eta([x] \cdot \zeta(x')) = q_0 \cdot xx' = \eta([x]) \cdot x'$, so η, ζ form an algebraic cover of (X,Q) by (S,S). #

Lemma 5.14 extends to S quite straight-forwardly with the various subsets of S being re-interpreted as subspaces. However, the process is blocked if we try to follow the construction of 5.14 (ii) as a direct product of a group and a right-zero semi-group. This is best illustrated by an example.

Example 5.16. Let $Q = \{q_1\ q_2\ q_3\}$, $X = \{a,b\}$ with input action

	q_1	q_2	q_3
a	q_2	q_1	q_1
b	q_3	q_1	q_1

The induced inertial tolerance is then

Q	q_1	q_2	q_3
q_1	1	1	1
q_2	1	1	0
q_3	1	0	1

The semi-group $S = \{\Lambda,a,b,a^2,ab\}$

and has multiplication table

Λ	a	b	a^2	ab
a	a^2	ab	a	b
b	a^2	ab	a	b
a^2	a	b	a^2	ab
ab	a	b	a^2	ab

and tolerance

S	Λ	a	b	a^2	ab
Λ	1	1	1	0	0
a	1	1	0	1	1
b	1	0	1	1	1
a^2	0	1	1	1	0
ab	0	1	1	0	1

To decompose using Arbib's method extract the maximal proper right ideal $T = S - \{\Lambda\}$ with subspace tolerance. T is right simple, so we can write it as a direct product of the idempotent set $E = \{a^2, ab\}$ and a group G obtained by acting on T from the right by an element of E, say $G = Tab = \{ab, b\}$. E has subspace tolerance and G quotient tolerance

E	a^2	ab
a^2	1	0
ab	0	1

G	ab	b
ab	1	1
b	1	1

with product

$G \times E$	(ab, a^2)	(ab, ab)	(b, a^2)	(b, ab)
(ab, a^2)	1	0	1	0
(ab, ab)	0	1	0	1
(b, a^2)	1	0	1	0
(b, ab)	0	1	0	1

Since

T	a	b	a^2	ab
a	1	0	1	1
b	0	1	1	1
a^2	1	1	1	0
ab	1	1	0	1

T cannot be homeomorphic to $G \times E$.

Ginzburg's approach breaks down at the first step since, as noted in §5.13, all Q_i need to be homeomorphic.

Finally, we illustrate the only method which is guaranteed to work, namely do a purely algebraic Ginzburg decomposition and enforce inertial tolerance at each stage from the input action. In fact the process is tedious and we give only the first step.

Carrying out the procedure of 5.13 on our example we get automata $B = (X,\hat{Q})$

B	Q_1	Q_2	Q_3
a	Q_3	Q_3	Q_3
b	Q_2	Q_2	Q_2

and $C = (X \times \hat{Q}, R)$ where the action of the input elements from $X \times \hat{Q}$ to R is defined by $r_i \cdot (x,Q_j) = h_k^{-1}(h_j(r_i) \cdot x)$ where k is given by $Q_k = Q_j \cdot x$ in B.

It turns out that the inputs of C fall into two classes $y_1 = \{(a,Q_1),(b,Q_1)\}$ and $y_2 = X \times \hat{Q} - y_1$ with actions

	r_1	r_2
y_1	r_1	r_1
y_2	r_2	r_1

and it is convenient to replace C by $\bar{C} = (Y,R)$.

Let ω be the same projection $X \times \hat{Q} \to Y$ whereupon $B \,\omega\, C = (X, \hat{Q} \times R)$ has input action

	$Q_1r_1)$	(Q_1r_2)	(Q_2r_1)	(Q_2r_2)	(Q_3r_1)	(Q_3r_2)
a	(Q_3r_1)	(Q_3r_1)	(Q_3r_2)	(Q_3r_1)	(Q_3r_2)	(Q_3r_1)
b	(Q_2r_1)	(Q_2r_1)	(Q_2r_2)	(Q_2r_1)	(Q_2r_2)	(Q_3r_1)

with $\eta : \hat{Q} \times R \to Q$ given by

η	r_1	r_2
Q_1	q_2	q_3
Q_2	q_1	q_3
Q_3	q_1	q_2

being an algebraic cover of A.

This is merely the first decompositon step. If we try to impose inertial tolerances on B, C, $B \omega C$ we find $\hat{Q}$ and R are totally connected while

$B \omega C$	$(Q_1 r_1)$	$(Q_1 r_2)$	$(Q_2 r_1)$	$(Q_2 r_2)$	$(Q_3 r_1)$	$(Q_3 r_2)$
$(Q_1 r_1)$	1	0	1	0	1	0
$(Q_1 r_2)$	0	1	1	0	1	0
$(Q_2 r_1)$	1	1	1	1	0	1
$(Q_2 r_2)$	0	0	1	1	1	0
$(Q_3 r_1)$	1	1	0	1	1	1
$(Q_3 r_2)$	0	0	1	0	1	1

This cannot be obtained in any reasonable way as a product of $\hat{Q}$ and R. #

Homology of Automata.

A good deal of the theory of topological spaces can be carried over to tolerance space theory. The fullest account of this can be found in (Poston, 1968) where, in particular, he develops a homology theory based on defining an n-simplex to consist of n+1 mutually near points.

In (Muir and Warner, 1979) a definition by a less tightly connected set was presented and again a homology theory of tolerance spaces was possible. Some attempts were then made to build a good theory of tolerance automata; particularly to find when input action induced homology morphisms. When a tolerance automaton is inertial the tolerance is induced by the input actions and this suggested attempting a homology of automata per se.

The motivations for this are analogous to the uses of homology theory in topology. Namely,

(i) to classify automata up to isomorphism or, at least, to decide when two automata are not isomorphic by computing some simpler algebraic objects which are functorially related to automata;

(ii) to establish fixed-simplex theorems for input actions.

Actually, homology theory is a massive algebraic machine which can swing into action once some basic definitions and relationships have been made. It is assumed the reader is familiar with the algebraic theory as set out in, say, (Spanier, 1966).

Definition 5.17. An *n-simplex* σ of an automaton $A = (X,Q)$ is a set of $n+1$ points $\{q_0, \ldots, q_n\} \subseteq Q$ for which there exists $q \in Q$, called the *pivot* of σ, with $q_i = q \cdot x_i$ for some $x_i \in X$, $i = 0, \ldots, n$. #

This definition is motivated by Dowker's (1952) "homology of relations" in which, for a relation R from a set X to a set Y, an n-simplex on Y is a set of $n+1$ points of Y which are R-related to a single point of X. Dually, there is a simplicial structure on X defined from the inverse relation. Actually, a key result of Dowker's paper is that the resultant homologies on X and Y are isomorphic.

We define:

Definition 5.18. A dual n-simplex of A is a set of $n+1$ points of Q which can be sent to a single point of Q by suitable inputs. #

Lemma 5.19. Let $A = (X,Q)$, $A' = (X',Q')$ be two automata. $f : Q \to Q'$ is a simplicial map if there exist functions $\alpha : X \times Q \to X'$, $\beta : Q \to Q'$ such that $f(q \cdot x) = \beta(q) \cdot \alpha(x,q)$ for all $x \in X$, $q \in Q$.

Proof. For each $q_i \in Q$ define the maximal simplex pivoted at q_i as $\sigma_i = \{q_i \cdot x : x \in X\}$. Since every simplex is contained in some σ_i we need only show that f maps σ_i into a simplex of A'.

But $f(\sigma_i) = \{f(q_i \cdot x) : x \in X\}$ $= \{\beta(q_i) \cdot \alpha(x,q_i) : x \in X\}$ which is a simplex pivoted at $\beta(q_i)$. #

Call β a pivot map of f. Then it is easily seen that β is a dual simplex map with pivot map f. Consider a special case.

Definition 5.20. A pseudo-morphism from an automaton $A = (X,Q)$ to an automaton $A' = (X',Q')$ is a pair of functions $\alpha : X \times Q \rightarrow X'$, $\beta : Q \rightarrow Q'$ such that $\beta(q \cdot x) = \beta(q) \cdot \alpha(x,q)$ for all $x \in X$, $q \in Q$. #

Thus β is a simplicial map. We can note the similarity of this concept to the usual idea of a morphism of automata when α is independent of q. Our generalization captures the idea that A' can mimic the behaviour of A employing at each step an input $\alpha(x,q)$.

Definition 5.21. Two simplicial maps f,g are contiguous, written $f \sim g$, if for all $q \in Q$, $x_0, \ldots, x_n \in X$ there exist $q' \in Q$, $x'_0 \ldots x'_n$, $y'_0 \ldots y'_n \in X'$ such that $f(q \cdot x_i) = q' \cdot x'_i$, $g(q \cdot x_i) = q' \cdot y'_i$. #

This is a direct translation from the concept of contiguity of geometric complexes. A standard algebraic result then gives.

Lemma 5.22. Contiguous maps induce identical homology homomorphisms. #

In particular, we can consider input strings, or elements of the semi-group S, which act as simplicial maps. All such elements can be partitioned into classes of maps connected by a sequence of contiguities. The elements of the class K containing the identity of S induce isomorphisms, by Lemma 5.22. Again using standard algebraic machinery we may state.

<u>Theorem 5.23</u>. If α_n is the number of n-simplices of an automaton, let $\chi = \sum (-1)^n \alpha_n$. If $\chi \neq 0$ every element of K leaves some simplex fixed. #

<u>Example 5.24</u>. Let $Q = \{1,2,3\}$, $X = \{a,b\}$ with action

	1	2	3
a	2	3	1
b	2	2	3

$\sigma_1 = \{2\}$

$\sigma_2 = \{2,3\}$

$\sigma_3 = \{1,3\}$

The elements of S which are simplicial maps are $\{\Lambda, b, ab, a^2b, aba, a^2ba, a^2bab, a^2baba, ba^2b, ba^2ba, ba^2ba^2, abab, ababa, ba, bab, baba\}$.

This forms a sub-semigroup $\bar{S}$ of S. Contiguities can be determined by examining the action on σ_2 and σ_3 of elements of $\bar{S}$ when it is found that all are in K. Since $\alpha_2 = 0$, $\alpha_1 = 2$, $\alpha_0 = 3$ we have $\chi \neq 0$ so all the strings of $\bar{S}$ leave some simplex fixed. #

CHAPTER 6

LATTICE-VALUED RELATIONS

Lattices. We begin with a sketch of lattice theory sufficient for our subsequent needs; any standard text (e.g. Birkhoff, 1948) may be consulted for further details.

Definition 6.1. A lattice $(L, \leq)$ is a non-empty set L, partially ordered by a relation $\leq$ in which every pair of elements a, b has a greatest lower bound $a \wedge b$ and a least upper bound $a \vee b$.#

Thus $a \wedge b \leq a, b \leq a \vee b$ and
$x \leq a, b \Rightarrow x \leq a \wedge b$ while $a, b \leq y \Rightarrow a \vee b \leq y$.
These two bounds are called the meet and join, respectively, of a and b.

If every subset of L has a $g.\ell.b.$ and a $\ell.u.b.$, then L is said to be complete. In particular a complete lattice has a least element, 0, and a greatest element, 1.

Examples 6.2. (i) $I = [0, 1]$ forms a lattice w.r.t. the standard ordering with $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$.

(ii) The power set $P(X)$ of a set X, where $\leq$ is the set inclusion relation $\subseteq$, $a \wedge b \equiv a \cap b$ and $a \vee b = a \cup b$.

(iii) The set of functions from a set X to a lattice L where $f \leq g$ iff $f(x) \leq g(x)$, $f \wedge g(x) = f(x) \wedge g(x)$ and $f \vee g(x) = f(x) \vee g(x)$, for all $x \in X$. #

Definition 6.3. A morphism $f : L \to L'$ of lattices is a function from L to L' which preserves meet and join. That is $f(a \wedge b) = f(a) \wedge f(b)$, $f(a \vee b) = f(a) \vee f(b)$. #

A lattice morphism also preserves the partial ordering which may be recovered from the meet or join according to $a \leq b$ iff $a \wedge b = a$ or $a \vee b = b$.

Definition 6.4. (i) A sublattice S of a lattice L is a subset of L which is closed under meet and join.

(ii) The product $L \times M$ of two lattices L, M has partial ordering $(x_1, y_1) \leq (x_2, y_2)$ iff $x_1 \leq x_2$ and $y_2 \leq y_2$. #

It follows straight forwardly that

$(x_1,y_1) \wedge (x_2,y_2) = (x_1 \wedge x_2, y_1 \wedge y_2)$ and

$(x_1,y_1) \vee (x_2,y_2) = (x_1 \vee x_2, y_1 \vee y_2)$.

Lattice Valued Relations.

Definition 6.5 Let L be a lattice, X a set. An L valued relation (L v r for short) on X is a function $\lambda : X \times X \to L$.

If λ is an L v r for some lattice L, we call it a lattice valued relation (ℓv r). #

Examples 6.6. (i) If $L = \{0, 1\}$ an L v r λ is an ordinary relation. If also λ is reflexive and symmetric then it is a tolerance.

(ii) If $L = [0, 1]$ an L v r is a fuzzy relation in the sense of (Zadeh, 1965).

(iii) If $L = R$, an L v r is a net in the sense of Chapter 2.

(iv) If L is the collection (lattice) of Borel sets in a probability space Ω, an L v r may be interpreted as assigning, via any measure on Ω, a probability of the relation between q and q'.

(v) An automaton transition function $\delta : X \times Q \to Q$ may be re-cast as a function $\lambda : Q \times Q \to P(X)$ where $\lambda(q,q') = \{x \in X : q' = q \cdot x\}$. #

Definition 6.7. A morphism from an L v r λ on a set X to an L'v r λ' on a set X' is a function $\beta : X \to X'$ and a lattice morphism $\gamma : L \to L'$ such that $\lambda'(\beta(x), \beta(x')) \geq \gamma(\lambda(x,x'))$ for all $x,x' \in X$.

If β and γ are bijective and $(\beta^{-1}, \gamma^{-1})$ is a morphism then (β,γ) is called an isomorphism; equality replaces the inequality. #

When $L = L'$ and γ = identity, we will refer to the morphism as β.

Definition 6.8. Let $\{\lambda_i\}$ be a family of L v r's on sets $\{X_i\}$ and let $\{f_i : X \to X_i\}$ be a family of functions. The induced L v r λ on X is defined by

$$\lambda(x,x') = \bigwedge_i \lambda_i\,(f_i(x),\ f_i(x')) \quad \text{for all } x,\ x' \varepsilon X. \ \#$$

This is the "least" requirement such that all f_i are morphisms. It is evident now why the definition for a morphism employs an inequality rather than an equality. Note that we can ensure that the join $\bigwedge\limits_i$ exists either by choosing finite families or L to be complete.

Examples 6.9. (i) The subset L v r induced on $X' \subseteq X$ from λ on X by the inclusion is merely λ restricted to $X' \times X'$.

(ii) The product L v r on $X_1 \times X_2$ induced from L v r s λ_i on X_i by projections is given by

$$\lambda((x_1,x_2),\ (x_1',x_2')) = \lambda_1(x_1,x_1') \wedge \lambda_2(x_2,x_2').$$

(iii) For any set X and an L v r λ on a set Y we can induce a function L v r μ on Y^X by the evaluation functions $e_x : Y^X \to Y$, $e_x(f) = f(x)$.

$$\text{Then}\quad \mu(f,g) = \bigwedge_{x \varepsilon X} \lambda(f(x),g(x)) \ . \ \#$$

We could define coinduction similarly to Definition 6.8, but we only need one such construction which we give directly.

Definition 6.10. If λ is an L v r on X and $\sim$ is an equivalence relation on X, the quotient L v r $\bar{\lambda}$ on $X/\sim$ is given by $\bar{\lambda}([x], [x']) = \bigvee_{\substack{x\ [x] \\ x' \varepsilon [x']}} \lambda(x,x')$

where $[x]$ is the equivalence class of x. #

Homogeneity. We now proceed to develop a theory of homogeneous l v r s. The theorems are effectively the same as those in Chapter 2 except that everything takes place in a general lattice L rather than in the particular lattice of Example 6.6 (iii).

To avoid tedious repetition, the sequence of definitions and results are merely stated without proof, except where some divergence from the net theory of Chapter 2 occurs. The principal source of such differences is the introduction of the lattice morphism γ of Definition 6.7, which is merely identity in the case of nets.

Lemma 6.11. The set of isomorphisms of an l v r to itself forms a group with respect to composition. #

Throughout the sequel we will denote an isomorphism group of an l v r λ on X by G and any $g \varepsilon G$ will be a pair (β,γ); when we wish to refer to the β or γ

of a particular g we write β_g or γ_g respectively.

Definition 6.12. G may be given an l v r μ by considering the set of β-functions as a subset of the function l v r on X^X. Thus, with $g = (\beta,\gamma)$, $g' = (\beta',\gamma')$ we have

$$\mu(g,g') = \bigwedge_{x \varepsilon X} \lambda(\beta_g(x), \beta_{g'}(x)) \ \#$$

Lemma 6.13. (i) $\mu(g\bar{g},g'\bar{g}) = \mu(g,g')$

(ii) $\mu(\bar{g}g,\bar{g}g') = \gamma_{\bar{g}}(\mu(g,g'))$

Proof. The left invariance of Lemma 2.9 is here replaced by (ii) which is proved thus

$$\mu(\bar{g}g,\bar{g}g') = \bigwedge_{x \varepsilon X} \lambda(\beta_{\bar{g}g}(x), \beta_{\bar{g}g'}(x)).$$

Now $\beta_{\bar{g}g} = \beta_{\bar{g}} \circ \beta_g$ with similar treatment of $\bar{g}g'$. Since $\bar{g}$ is an isomorphism we then have

$$\mu(\bar{g}g,\bar{g}g') = \bigwedge_{x \varepsilon X} \gamma_{\bar{g}} \circ \lambda(\beta_g(x), \beta_{g'}(x)).$$

Finally, $\gamma_{\bar{g}}$ is a lattice isomorphism so it can be interchanged with $\bigwedge$ and the desired result follows. #

Lemma 6.14. For $x \varepsilon X$ define $\psi_x : G \to X$ by $\psi_x(g) = \beta(x)$. Then ψ_x is a morphism from μ to λ. #

As in Chapter 2, we can again consider the subgroup H of G whose elements fix a base point $x \varepsilon X$.

The set of left cosets F define a partition of G so there is an l v r $\bar{\mu}$ on F coinduced from μ. $\bar{\psi}_X : F \to X$ is defined by $\bar{\psi}_X([g]) = \psi_X(g)$.

Lemma 6.15. (i) $\bar{\mu}([g],[g']) = \bigvee_{h \in H} \mu(g,g'h)$.

(ii) $\bar{\psi}_X$ is a well defined injective morphism from $\bar{\mu}$ to λ. #

Definition 6.16. (i) An l v r λ on X is homogeneous if for all $x, x' \in X$ there exists $g \in G$ such that $x' = \beta_g(x)$.

(ii) λ is very homogeneous if it is homogeneous and

$$\lambda(x,x') \leq \bigvee \mu(e,g)$$

where the join is over all $g \in G$ s.t. $\beta_g(x) = x'$ and e is identity. #

This definition of v.h. differs from that of §2.14 in only demanding an inequality; actually equality is automatic anyway because

$$\lambda(x,x') \geq \bigwedge_{\bar{x} \in X} \lambda(\bar{x},g(\bar{x})) = \mu(e,g)$$

and so $\quad \lambda(x,x') \geq \bigvee \mu(e,g)$.

Theorem 6.17 If λ is v.h., $\bar{\psi}_X$ is an isomorphism. #

In parallel with the definition of β in §2.18, let us define $\nu : G \to L$ by $\nu(g) = \mu(e,g)$. By the right invariance of Lemma 6.13(i) μ can be reconstructed from ν. However, loss of left invariance means that ν is not conjugate invariant as is β in Lemma 2.19. Instead we have

$$\nu(gg'g^{-1}) = \mu(e, gg'g^{-1}) = \mu(g, gg')$$
$$= \gamma_g(\mu(e,g')) = \gamma_g(\nu(g')).$$

Thus conjugation in G is reflected in a lattice isomorphism of L. This demands a modification of Theorem 2.22 into the following analogue.

Theorem 6.18. Let M be the group of isomorphisms of a lattice L, Γ a homomorphism from a group G to M. Write $\Gamma(g) = \gamma_g$ and define a function $\nu : G \to L$ by choosing it arbitrarily on one element of each conjugacy class, extending it to all of G by $\nu(gg'g^{-1}) = \gamma_g(\nu(g'))$. Finally, let H be a subgroup of G, F the set of its left cosets.

Then $\bar{\mu}([g],[g']) = \gamma_g(\bigvee_{\bar{h} \varepsilon H} \nu(g^{-1}g'\bar{h}))$ is a well-defined, v.h. L v r on F.

Proof $\bar{\mu}$ is a well-defined because $[g] = [g_1]$, $[g'] = [g_1'] \Longrightarrow g_1 = gh$, $g_1' = g'h'$ so

$$\bar{\mu}([g_1], [g_1']) = \gamma_{g_1}(\bigvee_{\bar{h}_1 \varepsilon H} \nu(g_1^{-1}g_1'\bar{h}_1))$$

$$= \gamma_{g_1}\Big(\bigvee_{\bar h_1 \varepsilon H} \nu(h^{-1}g^{-1}g'h'\bar h_1)\Big)$$

$$= \gamma_{g_1}\Big(\bigvee_{\bar h_1 \varepsilon H} \gamma_{h^{-1}}(\nu(g^{-1}g'h'\bar h_1 h^{-1}))\Big)$$

Now $\gamma_{h^{-1}}$ is a lattice isomorphism, so may be interchanged with the join. Also, $\gamma_{g_1} \circ \gamma_{h^{-1}}$ $= \Gamma(g_1) \circ \Gamma(h^{-1}) = \Gamma(g_1 h^{-1}) = \Gamma(g) = \gamma_g$ since Γ is a group homomorphism.

As $\bar h_1$ ranges over H so does $\bar h = h'\bar h_1 h^{-1}$ giving the desired formula for $\bar\mu([g],[g'])$. Thus $\bar\mu$ is a well-defined $L \vee r$ on F. To prove it is v.h. we need to show that

$$\bar\mu([g],[g']) \leq \bigvee_{j \varepsilon J} \ \bigwedge_{[g''] \varepsilon F} \ \bar\mu([g''], \beta_j([g''])) \quad \ldots . \ (*)$$

where J is the subgroup of the isomorphism group $\bar G$ of $\bar\mu$ such that $j \ \varepsilon \ J \Longrightarrow \beta_j([g]) = [g']$. (This rather involved specification is a consequence of Definitions 6.12 and 6.16 (ii).)

Now G is a subgroup of $\bar G$ in the sense that for $g \ \varepsilon \ G$ we can define $\beta_g : F \rightarrow F$ by $\beta_g([\bar g]) = [g\,\bar g]$, whereupon

$$\bar{\mu}(\beta_g([\bar{g}]),\ \beta_g([\bar{g}_1])) = \bar{\mu}([g\bar{g}],\ [g\bar{g}_1])$$

$$= \gamma_{g\bar{g}}(\bigvee_{\bar{h}\in H} \nu(\bar{g}^{-1}g^{-1}g\bar{g}_1\bar{h}))$$

$$= \gamma_g \circ \gamma_{\bar{g}} (\bigvee_{\bar{h}\in H} \nu(\bar{g}^{-1}\bar{g}_1\bar{h})) = \gamma_g(\bar{\mu}([\bar{g}],[\bar{g}_1])$$

The join over J in the required inequality (*) may thus be replaced by a join over those elements of $\bar{g} \in G$ for which $[\bar{g}g] = [g']$ - that is, for which $\bar{g} = g'h\,g^{-1}$ for $h \in H$; the inequality is, of course, proved if it is proved for a smaller set than J.

Now $\bar{\mu}([g''],\ \beta_{\bar{g}}([g''])) = \bar{\mu}([g''],[g'hg^{-1}g''])$

$$= \gamma_{g''}(\bigvee_{h_1\in H} \nu(g''^{-1}g'hg^{-1}g''h_1))$$

$$\geq \gamma_{g''}(\nu(g''^{-1}g'hg^{-1}g''))$$

$$= \nu(g'hg^{-1}).$$

Taking the meet over $[g''] \in F$ preserves this inequality by the greatest lower bound definition of §6.1. Finally, we have argued that the join over J is replaceable by a join over $h \in H$, so the right-hand side of (*) is greater than $\bigvee_{h\in H} \nu(g'hg^{-1})$

$$= \gamma_g(\bigvee_{h \in H} \nu(g^{-1}g'h)) = \bar{\mu}([g],[g'])$$

This establishes $(\star)$. #

Automata as lvrs.

We saw in Example 6.6(v) that an automaton (X,Q,δ) can be considered as a $P(X)$vr on Q. We now translate some of the lvr concepts into automaton terms. Recall from §6.6(v) that for $q, q' \in Q$
$\lambda(q,q') = \{x \in X : q' = q \cdot x\}$.

Lemma 6.19. An automaton morphism in the sense of (Ginzburg, 1968) induces a morphism of associated lvrs.

Proof. A morphism of the lvr $\lambda : Q \times Q \to P(X)$ to $\lambda' : Q' \times Q' \to P(X')$ is a pair (β,γ) with γ an intersection and union preserving function from $P(X)$ to $P(X')$, β a function from Q to Q' such that

$$\lambda'(\beta(q),\beta(q')) \supseteq \gamma(\lambda(q,q')).$$

This means

$$\gamma(\{x \in X : q' = q \cdot x\}) \subseteq \{x' \in X' : \beta(q') = \beta(q) \cdot x'\}.$$

We must issue a warning here: if $A \in P(X)$, $\gamma(A)$ need not be obtained by the point-wise application of some function on the elements of A. However, a morphism of automata is a pair $(\beta,\bar{\gamma})$, $\beta : Q \to Q'$, $\bar{\gamma} : X \to X'$ such that

$\beta(q\cdot x) = \beta(q)\cdot\bar{\gamma}(x)$ for all $q \varepsilon Q$, $x \varepsilon X$. $\bar{\gamma}$ induces a lattice morphism $\gamma : P(X) \to P(X')$ for which

$$x' \varepsilon \gamma(\{x \varepsilon X : q' = q \cdot x\})$$

$$\Longrightarrow x' = \bar{\gamma}(x) \quad \text{with} \quad q' = q \cdot x$$

$$\Longrightarrow \beta(q') = \beta(q\cdot x) = \beta(q)\cdot x'$$

$$\Longrightarrow x' \varepsilon \{x' \varepsilon X' : \beta(q') = \beta(q)\cdot x'\}$$

and the Lemma is proved. #

Evidently a $P(X)$ vr morphism is more general than an automaton morphism, but in the case of self-isomorphisms the two coincide, according to

Lemma 6.20. If no two elements of X coincide as functions from Q to itself, an lvr isomorphism is an automaton isomorphism.

Proof. For $x \varepsilon X$ define $A_q = \{\bar{x} \varepsilon X : q \cdot x = q \cdot \bar{x}\}$ so $\bar{x} \varepsilon A_q$ has the same effect on q as does x. By assumption $\{x\} = \bigcap_q A_q$ and since γ is a lattice morphism

$$\gamma(\{x\}) = \bigcap_q \gamma(\{\bar{x} \varepsilon X : q \cdot x = q \cdot \bar{x}\})$$

$$= \bigcap_q \gamma(\lambda(q, q \cdot x)) \subseteq \bigcap_q \lambda'(\beta(q), \beta(q \cdot x))$$

$$= \{\bar{x} \varepsilon X : \beta(q \cdot x) = \beta(q)\cdot \bar{x} \text{ for all } q \varepsilon Q\}.$$

Since β is bijective, this defines the action of any such $\bar{x}$ on the whole of Q so $\gamma(\{x\})$ is either a singleton set or the empty set. But the latter would violate bijectivity of γ.

Now γ can be considered as induced by a function $\bar{\gamma} : X \to X$. For any $x \in X$, $q \in Q$, $x \in A_q$ trivially so

$$\bar{\gamma}(x) \in \gamma(A_q) \subseteq \lambda'(\beta(q), \beta(q \cdot x))$$

and $\beta(q \cdot x) = \beta(q) \cdot \bar{\gamma}(x)$ so $(\beta, \bar{\gamma})$ is an automaton morphism. #

Consider now the group G of isomorphisms. For $g, g' \in G$, $\mu(g, g') = \bigcap_{q \in Q} \{x \in X : \beta_{g'}(q) = \beta_g(q) \cdot x\}$

$$= \{x : \beta_{g'} = \delta_x \circ \beta_g\}.$$

Since the β's are bijective $\mu(g, g')$ is then a singleton or empty. In the former case $\delta_x = \beta_{g'g^{-1}}$ which, being the β-map of an isomorphism, demands a corresponding $\gamma : X \to X$ such that

$$\lambda(q \cdot x, q' \cdot x) = \gamma(\lambda(q, q')) \quad \text{for all } q, q' \in Q$$

$$\Longrightarrow \{\bar{x} : q' \cdot x = q \cdot x\bar{x}\} = \gamma\{x_1 : q' = q \cdot x_1\}$$

$$\Longrightarrow q' \cdot x = q \cdot x\gamma(x_1) \text{ iff } q' = q \cdot x_1$$

$$\Longrightarrow \bar{q} \cdot \gamma(x_1) = \bar{q} \cdot x^{-1} x_1 x \text{ for all } \bar{q} \in Q,\ x_1 \in X.$$

where we have replaced $q \cdot x$ by $\bar{q}$ and x^{-1} abbreviates the inverse of the bijective action of x.

Note that x^{-1} is not necessarily an element of X.

Summarizing, we have:

Lemma 6.21. A bijective input $x \varepsilon X$ is the β-map of an isomorphism iff for all $x_1 \varepsilon X$ we have $x^{-1}x_1x \varepsilon X$. This construction determines the corresponding γ-map. #

Theorem 6.22. An automaton is very homogeneous as an lvr iff every $x \varepsilon X$ is a permutation of Q and X is closed under conjugation.

Proof. From definition 6.16(ii) an automaton is v.h. if for all $q, q' \varepsilon Q$ there exists $g \varepsilon G$ whose β-map takes q to q' and

$$\lambda(q,q') \subseteq \bigcup_{\substack{\beta_g \text{ s.t.} \\ q'=\beta_g(q)}} \mu(e,g)$$

$$\Longrightarrow \{x : q' = q \cdot x\} \subseteq \bigcup_{\substack{\beta_g \text{ s.t.} \\ \beta_g(q)=q'}} \{\bar{x} : \delta_{\bar{x}} = \beta_g\}.$$

This is simply a demand that every $x \varepsilon X$ be the β-map of an isomorphism. By the previous lemma X is then closed under conjugation. #

Note that we are dealing with a group automaton in the sense of Chapter 5 and the semigroup is a group.

We may alternatively consider the input set of a v.h. automaton as a self-conjugate subset of the permutation group of Q. If that subset is actually a subgroup, self-conjugacy is trivially satisfied and the automaton semi-group is just X itself.

<u>Example 6.23</u>. Let $X = \{a,b,c,d\}$, $Q = \{A,B,C,D\}$ with transition function

δ	A	B	C	D
a	A	D	B	C
b	B	C	A	D
c	C	B	D	A
d	D	A	C	B

Every input is bijective and X is closed under conjugation, thus defining the γ-maps according to Lemma 6.21. For example the action of $b^{-1}c\,b$ on Q is calculated thus

$$(A,B,C,D) \cdot b^{-1}c\,b = (C,A,B,D) \cdot c\,b = (D,C,B,A) \cdot b$$

$$= (D,A,C,B) = (A,B,C,D) \cdot d$$

Thus $b^{-1}c\,b = d$

Similarly, conjugation by x of all elements of X, (which we can denote by $\gamma_x : X \to X$) yields

$$\gamma_a : (a,b,c,d) \to (a,d,b,c)$$
$$\gamma_b : (a,b,c,d) \to (c,b,d,a)$$
$$\gamma_c : (a,b,c,d) \to (d,a,c,b)$$
$$\gamma_d : (a,b,c,d) \to (b,c,a,d).$$

The group G of isomorphisms contains any composite of the input isomorphisms and hence contains the automaton (semi-) group as a subgroup. Now a,b generate all even permutations of Q - this is easily seen by considering Q as the vertices of a tetrahedron on which a,b act as rotations, each keeping a vertex fixed. That this is the whole of G then follows if we demonstrate that any odd permutation of Q is not the β-map of an isomorphism.

For instance let $\beta:(A,B,C,D) \to (B,A,C,D)$ be such; then there would exist $\gamma : X \to X$ such that

$$\beta(A\cdot a) = \beta(A)\cdot\gamma(a) \Longrightarrow \beta(A) = \beta(A)\cdot\gamma(a)$$

$$\Longrightarrow B = B\cdot\gamma(a) \Longrightarrow \gamma(a) = c. \quad \text{But then}$$

$$\beta(B\cdot a) = \beta(B)\cdot\gamma(a) \Longrightarrow \beta(D) = \beta(B)\cdot c \Rightarrow D = A\cdot c$$

which contradicts the δ-table.

Employing a,b then to generate G we get

$$G = \{e,a,b,a^2,ab,ba,b^2,a^2b,aba,ab^2,b^2a^2,a^2b^2\}\ .$$

From Theorem 6.22 for any $g \in G$

$$\mu(e,g) = \{x \in X : x = g\} \quad \text{so we easily find that}$$

$$\mu(e,a) = \{a\},\ \ \mu(e,b) = \{b\},\ \ \mu(e,a^2b^2) = \{c\},$$

$$\mu(e,b^2a^2) = \{d\},\ \mu(e,g) = \phi \text{ for all other } g \in G.$$

To verify the isomorphism Theorem 6.17 choose a base-point, say B, in Q. The fix subgroup H of B is $\{e, ba, a^2b^2\}$ with left cosets $[e] = H$,

$[a] = \{a, ab^2, ab\}$, $[b] = \{b, aba, a^2\}$,

$[b^2] = \{b^2, b^2a^2, a^2b\}$. Calculation of these is tedious and employs the relations $a^3 = b^3 = e$, $ab^2 = ba^2$, $a^2b = b^2a$, $aba = bab$ between the generators of G. Note a small notational complication-input action is written on the right and read in the opposite order to the function composites of the general theory, so a left coset, bH say, is obtained by applying b on the right to elements of H.

Now $\bar{\psi} : F \to Q$ becomes $[e] \to B$, $[a] \to D$, $[b] \to C$, $[b^2] \to A$ and $\bar{\mu} : F \times F \to P(X)$ comes from $\bar{\mu}([e],[g]) = \bigcup_{h \epsilon H} \mu(e, gh)$. The table for $\bar{\mu}$ can thus be calculated and under $\bar{\psi}$ becomes the lvr map $\lambda : Q \times Q \to P(X)$ which reconstructs δ. #

Actually all of the above theory could be revamped into an alternative setting where the automaton (X, Q, δ) was regarded as a $P(X^*)$ vr where X^* is the set of finite strings of elements of X. On the assumption that all states are <u>reachable</u> from each other by some input string $\lambda(q, q') \neq \phi$ for all $q, q' \epsilon Q$. Now, of course, we lose the uniqueness of an input considered as a map from Q to Q since many strings induce the same function. But this is preceisely the relation defining the automaton semi-group S and uniqueness is restored by considering λ to be a $P(S)$ vr. Again Theorem 6.22 requires each element of X, and hence of S, to be a permutation, so S itself is a group. Conjugation is now automatic and hence the v.h. condition is also. Theorem 6.17 now becomes Theorem 5.7 restricted to group automata. The advantages in switching from $P(X)$ to $P(S)$ is paid for by a weakening of the morphism concept, since now any $\bar{\gamma}$-map is only required to take S to itself and it need not map an element of X to a single element. If morphism of automata is regarded as a kind of simulation of one automaton by another, then the simulation is "weak" in the sense of (Arbib, 1969); that is, one automaton might need a string of inputs to achieve the state change which in the other is effected by a single input.

Very Homogeneous Tolerance Spaces.

The preceding remarks clarify the v.h. concept in the automaton case. As promised in Chapter 2, there is a further situation in which an intuitive interpretation of v.h. can be found; this is the tolerance space setting whose definitions occur in §§5.2 - 5.5 and which was identified as an lvr in Example 6.6(i).

Note first that with image $(\lambda) = \{0,1\}$ we will write $x \rho x'$ iff $\lambda(x,x') = 1$. Then if $x \not\rho x'$ (x not near x') the v.h. condition is trivially satisfied. If, however, $x \rho x'$ the v.h. condition states that

$$\lambda(x,x') = 1 = \max \mu(e,g)$$

where max is taken over all $g \in G$ s.t. $g(x) = x'$.

Hence, there exists one such g with $\mu(e,g) = 1$ and since $\mu(e,g) = \min_{\bar{x} \varepsilon X} \lambda(\bar{x},g(\bar{x}))$ we find $g(\bar{x}) \rho \bar{x}$ for all $\bar{x} \varepsilon X$.

Re-phrasing, we have: a tolerance space X is very homogeneous if for all x,x' which are near, there is a homeomorphism g of X to itself which maps x to x' and for which $g(\bar{x}) \rho \bar{x}$ for all $\bar{x} \varepsilon X$. Roughly speaking g can move x to the near point x' without radically disturbing the rest of X. (Note that Example 2.17 is actually a tolerance example of a homogeneous space which is not v.h.)

Now, there exists in the topological category a group quotient theorm analogous to Theorem 2.15 with tolerance interpretation (see McCarty, 1967). Instead of using the full group of homeomorphisms it is sufficient to stipulate that there exists a topological transformation group G acting transitively on the topological space X. Thus:

<u>Theorem.</u> If G is a transitive topological group acting on the topological space X and x_0 is a given (base-) point in X the projection $p : G \to X$ $p(g) = g(x_0)$ induces a continuous bijection $\bar{p} : G/H \to X$

where H is the fix subgroup of x_0. If, further, p has a local cross-section at x_0 then $\bar{p}$ is a homeomorphism. #

This last sentence strongly suggests that our v.h. condition can have a local cross-section meaning. Recall that in the topological case this is a function $s : S \to G$, where S is a neighbourhood of x_0, with $p \circ s$ = identity.

The tolerance analogue of neighbourhood of x_0 is the set of all points "near to" x_0, $\tau(x_0) = \{x \varepsilon X : x \rho x_0\}$. So a local cross-section at x_0 will be a tolerance function $s : \tau(x_0) \to G$ such that $ps = 1$. Here G is the group of homeomorphisms of the tolerance space X and, following §§2.3 - 2.9, it has a left and right-invariant tolerance given by $g \rho g'$ iff $g(x) \rho g'(x)$ for all $x \varepsilon X$.

With $ps(x_0) = x_0$ we have $s(x_0) = h \varepsilon H$, so for any $x \varepsilon \tau(x_0)$, since s is a tolerance function, $g = s(x) \rho s(x_0) = h$. Thus $g(\bar{x}) \rho h(\bar{x})$ for all $\bar{x} \varepsilon X$. Define a new function $s' : \tau(x_0) \to G$ by $s'(x) = s(x) \cdot h^{-1}$; this is also a cross-section at x_0, since $ps'(x) = s'(x)(x_0) = s(x) \cdot h^{-1}(x_0) = s(x)(x_0) = ps(x) = x$, and s' is a tolerance function since right translation by h^{-1} in G is a tolerance function.

Now $s'(x)$ is near the identity of G so we have proved that the existence of local cross-sections is equivalent to the v.h. condition for tolerance spaces.

Miscellaneous Notes

Our general construction of lvrs, given in Theorem 6.18, supposes no further constraints on the kind of lvr we wish to construct. In particular cases, however, there will be extra conditions to be imposed on the function ν; for instance, a tolerance space is a $\{0,1\}$ vr with the added requirements of reflexivity and symmetry.

The demand that a $P(X)$ vr λ correspond to an automaton also imposes conditions on ν, since for any $q \in Q$ we have an equivalence relation defined on X by $x \sim x'$ if $q \cdot x = q \cdot x'$. The corresponding classes are just $\lambda(q,q')$ for the different $q' \in Q$ and must form a partition of X. The consequences for ν of having a partition of X for each $q \in Q$ is expressed by the following theorem which we give without proof, following the notation of Theorem 6.18.

Theorem 6.24. A function $\nu : G \to P(X)$ defines an lvr $\bar{\mu}$ which corresponds to an automaton iff ν satisfies

(i) $\bigcup_{g \in G} \nu(g) = X$ (ii) $\nu(g) \cap \nu(gk) = \phi$ if $k \notin H$. #

The need for such restrictions of ν is avoided in the case of a fuzzy automaton. This is defined in (Wee and Fu, 1969) to be a function $\delta : X \times Q \times Q \to I$ which, if restated as a function $\bar{\delta} : Q \times Q \to I^X$, is merely an Lvr on Q, where $L = I^X$ is formed by the construction of Example 6.2(iii) from the lattice of Example 6.2(i).

I^X is just the set of fuzzy subsets of X (Zadeh, 1965). Indeed, the theory of Chapter 2, with its heavy reliance on max. and min. throughout, is very reminiscent of fuzzy theory. I have argued elsewhere (Muir, 1981), however, the inadequacies of fuzzy set theory. Its major deficiency lies in the failure of the "excluded middle law", which results from the fact that the lattice I^X is not a Boolean algebra. It is more desirable to replace this lattice by M^X where M is the lattice of measurable sets in a probability space Ω; an M^X vr on Q is then a stochastic automaton (Arbib, 1969).

Let us finally note how the homology theory of automata described in Chapter 5 may be generalized to lvrs.

Definition 6.25. Let $\lambda : Q \times Q' \to L$ be a lattice valued relation from Q to Q' and let $\ell \varepsilon L$. An ℓ-n-simplex of λ is an ordered set $[q_0 \ldots q_n]$ of elements Q such that $\lambda(q_i, q') > \ell$ for some $q' \varepsilon Q'$, $i = 0, \ldots, n$.

With $L = \{0,1\}$, $\ell = 0$ we restore Dowker's original definition, while with $L = P(X)$, $\ell = \phi$ we obtain the n-simplex definition employed for automata.

Clearly, the requirement that $\lambda(q,q') > \ell$ is an ordinary relation from Q to Q', so for given ℓ Dowker's homology isomorphism theorem still holds. Moreover, if $\ell \geq \ell'$ every ℓ-n-simplex is an ℓ'-n-simplex, so any sequence of lattice elements $\ell_0 \leq \ell_1 \leq \ell_2 \leq \ldots$ yields a filtration on the chain-complex C of ℓ_0 simplexes which is compatible with the gradation and differential of C (see Spanier, 1966).

The elaborate apparatus of spectral sequence theory is tailor-made for computing homologies from filtrations, so one may expect to find intriguing relationships between the exact homology sequences for each $\ell \varepsilon L$ and the lattice structure of L.

APPENDIX I

SOME COMMENTS ON GRIFFITH'S APPROACH TO NEURODYNAMICS

In Chapter 5 of his book, Griffith (1971) introduced a model of randomly connected networks which, despite its tremendous oversimplifications, still is sufficient to illustrate the qualitative behaviour of more complex and realistic models. Indeed, for myself, the approach of Griffith was my first stimulus to thinking about symmetry and homogeneity in neural dynamics. The material of this Appendix is not quite in the mainstream of this thesis' argument but does, I believe, serve to place Griffith's work with respect to my own interests.

He treats "Randomly Connected Networks of Neurons" by asserting a common firing probability p for every neuron at time zero. If each neuron is then considered to have connections from n excitatory and m inhibitory cells with a common threshold θ, the probability P of a neuron firing at the next instant equals the probability that a weighted sum of N and M, input lines to the neuron, is greater than or equal to θ. By the multinomial theorem

$$P = \sum_{N-aM\geq\theta} \binom{n}{N}\binom{m}{M} p^{N+M}(1-p)^{n+m-N-M}$$

where, for convenience, the weights are taken as 1 and a.

The particular case of a purely excitatory network occurs with $m = 0$, whereupon

$$P = \sum_{N \geq \theta} \binom{n}{N} p^N (1-p)^{n-N}$$

which Grifiths calls the McCullogh-Pitts function $P(n,\theta,p)$. For $n > \theta > 1$, as a function of p, it has a sigmoidal form defined in Example 3.6, except that now the domain is $[0,1]$ so condition (ii) there is replaced by $P(1) = 1$. We can use this to emphasise the qualitative robustness of the stability arguments in Chapter 4, most of which depend only on the sigmoidicity of various response functions. Here again the equilibrium condition $P = p$ can yield three equilibria; those at $p = 0$ and 1 are stable and there is an intermediate unstable equilibrium. Thus we recover the oft-quoted (e.g. Taylor, 1974) piece of folk-wisdom that activity in a purely excitatory network either dies away to zero or increases to maximal activity. It is quite striking that as late as 1962 (Ashby et al, 1962) this result was regarded as a paradoxical instability of the brain. Actually, as shown explicitly in Appendix 2, a more realistic purely excitatory model can have oscillatory modes about a non-saturated equilibrium.

Strict derivation of the dynamical equations

Before proceeding to further discussion of dynamics let us just look in more detail at the assumptions underlying Griffith's purely excitatory equation.

For a general net we can consider a Markov process on the set Ω of states. In a McCullogh-Pitts network where X is the set of neurons $\Omega = \{0,1\}^X$, the elements of which can be considered as binary valued vectors $\underset{\sim}{x}$ with components x_i indexed by $i \in X$.

The Markov transition function $< \underset{\sim}{x} \mid \underset{\sim}{x}' >$ is the probability that state $\underset{\sim}{x}$ occurs at an instant, conditional upon state $\underset{\sim}{x}'$ at the previous instant; this is assumed to be stationary in time since it depends only on the neuronal thresholds and network connection coefficients.

Symmetry will now be expressed by $g \in G$ implying that $< g^*(\underset{\sim}{x}) | g^*(\underset{\sim}{x}') > = < \underset{\sim}{x} | \underset{\sim}{x}' >$, so if p, P denote the probabilities of states at successive instants, we have

$$P(\underset{\sim}{x}) = \sum_{\underset{\sim}{x}' \in \Omega} < \underset{\sim}{x} | \underset{\sim}{x}' > p(\underset{\sim}{x}')$$

implying $P(g^*(\underset{\sim}{x})) = \sum_{\underset{\sim}{x}' \in \Omega} < g^*(\underset{\sim}{x}) | \underset{\sim}{x}' > p(\underset{\sim}{x}')$

$$= \sum_{\underset{\sim}{x}'' \in \Omega} < g^*(\underset{\sim}{x}) | g^*(\underset{\sim}{x}'') > p(g^*(\underset{\sim}{x}''))$$

$$= \sum_{\underset{\sim}{x}'' \in \Omega} < \underset{\sim}{x} | \underset{\sim}{x}'' > p(g^*(\underset{\sim}{x}''))$$

g^* being bijective in its action on Ω.

A really stochastic theory will want $< \underset{\sim}{x} | \underset{\sim}{x}' >$ to be a random variable, but we may expect it to depend only on synaptic strengths and thresholds. The most likely candidates for random behaviour are the thresholds, but in any case we can write

$$< \underset{\sim}{x} | \underset{\sim}{x}' > = \Pr\{S_i \sim_i 0 : i = 1, \ldots, n\}$$

with

$$n = \# X \ , \ S_i = \sum_j \phi_{ij}' x_j' - \theta_i$$

where ϕ_{ij} are synaptic strengths and θ_i the thresholds; $\sim_i$ is the relation $\geq$ if $x_i = 1$ or $<$ if $x_i = 0$.

If the random variables ϕ, θ are independently distributed the evaluation of $< \underset{\sim}{x} | \underset{\sim}{x}' >$ becomes a product of the separate probabilities $\Pr\{S_i \sim_i 0\}$ and the components of $\underset{\sim}{x}$ are independently determined. This is a crucial observation since it allows us to decompose the equation for $P(\underset{\sim}{x})$ into equations for the firing probabilities of the separate neurons.

The "thought cube" of Zeeman (1961) is only useful as the arena of a dynamical system when the firing probabilities of the separate neurons may be considered as independent coordinates, a point which Zeeman failed to emphasise adequately.

Writing p_i, P_i for the probability that $x_i = 1$ initially and finally, we get

$$P_i = \sum_{\underset{\sim}{x}' \epsilon \Omega} < i | \underset{\sim}{x}' > \prod_j R_j(\underset{\sim}{x}')$$

where $R_j(x') = p_j$ if $x'_j = 1$
$= 1 - p_j$ if $x'_j = 0.$

These equations are the stochastic equivalent of those in Example 3.6.

Now if the network is homogeneous and $\theta_i = \theta$ for all i, then for any i, k we can find $g \varepsilon G$ s.t. $k = g(i)$, so that

$$P_k = \sum_{\underset{\sim}{x}' \varepsilon \Omega} < g(i)|\underset{\sim}{x}' > \prod_j R_j(\underset{\sim}{x}')$$

$$= \sum_{\underset{\sim}{x}'' \varepsilon \Omega} < i|\underset{\sim}{x}'' > \prod_j R_j(\underset{\sim}{x}')$$

where $\underset{\sim}{x}'' = g^*(\underset{\sim}{x}')$.

Uniform firing probabilities are now a feasible solution (and again their stability could be investigated as in Chapter 4) for if $p_i = p$ for all i, then $\prod_j R_j(\underset{\sim}{x}') = p^{\omega}(1-p)^{n-\omega}$ and

$$P_i = \sum_{\underset{\sim}{x}' \varepsilon \Omega} < i|\underset{\sim}{x}' > \; p^{\omega}(1-p)^{n-\omega}$$

where ω, in each term, is the number of active neurons of $\underset{\sim}{x}'$. The right-hand side is actually independent of i, as can be seen from the expression for P_k, wherein $\omega(\underset{\sim}{x}'') = \omega(\underset{\sim}{x}')$.

To proceed further towards Griffith's equation we make the stringent assumption that ϕ_{ij} lies in $\{0,1\}$ and, moreover, $\Pr\{\phi_{ij} = 1\} = \beta$ for all i,j. Then

$$< i|\underset{\sim}{x}' > \; = \Pr\{\sum_j \phi_{ij} x'_j \geq \theta\}$$

$$= \Pr\{\text{of the } \omega(\underset{\sim}{x}') \text{ active cells, } \geq \theta \text{ of them are connected to } i\} = \sum_{k=\theta}^{\omega} \binom{\omega}{k} \beta^k (1-\beta)^{\omega-k}$$

$$= P(\omega,\theta,\beta).$$

There are $\binom{N}{\omega}$ states with ω active cells, so changing to a sum over ω we get.

$$P = \sum_{\omega=0}^{n} P(\omega,\theta,\beta) \cdot \binom{n}{\omega} p^{\omega} (1-p)^{n-\omega} .$$

This may be simplified by noting that $P(\omega,\theta,\beta) = 0$ for $\omega < \theta$ so we can write

$$P = \sum_{\omega=\theta}^{n} \sum_{k=\theta}^{\omega} \binom{\omega}{k} \beta^k (1-\beta)^{\omega-k} \binom{n}{\omega} p^{\omega} (1-p)^{n-\omega}$$

$$= \sum_{k=\theta}^{n} \sum_{\tau=0}^{n-k} \binom{\tau+k}{k} \binom{n}{\tau+k} \beta^k (1-\beta)^{\tau} p^{\tau+k} (1-p)^{n-\tau-k}$$

on reversing the order of summation and writing $\tau = \omega - k$. The product of binomial coefficients can be rewritten as $\binom{n}{k}\binom{n-k}{\tau}$ giving

$$P = \sum_{k=\theta}^{n} \binom{n}{k} (p\beta)^k \sum_{r=0}^{n-k} \binom{n-k}{\tau} p^{\tau} (1-\beta)^{\tau} (1-p)^{n-\tau-k}$$

The inner sum is merely $(1-p\beta)^{n-k}$ giving finally

$$P = \sum_{k=\theta}^{n} \binom{n}{k} (p\beta)^k (1-p\beta)^{n-k}$$

$$= P(n,\theta,p\beta).$$

This is Griffith's excitatory equation with $p\beta$ replacing p; that is, the firing probability of any neuron depends on the joint probability of the previous firing probability together with the probability of connections existing.

Inhibition

After considering purely excitatory networks Griffith looks in detail at the case when $m = 1$ in his general equation. He uses various approximations to study the form of P as a function of p and deduces the possibility of five equilibria.

Actually it is possible to argue this fact more directly by a graphical method. When $m = 1$, M can only be 0 or 1, so

$$P = \sum_{N\geq\theta} \binom{n}{N} p^N (1-p)^{n+1-N} + \sum_{N\geq\theta+a} \binom{n}{N} p^{N+1} (1-p)^{n-N}$$

$$= pP(n, \theta+a, p) + (1 - p)P(n, \theta, p).$$

The first term here is the probability that the inhibitory cell is firing times the conditional probability that a cell will then be activated, when its threshold is effectively $\theta + a$. The second term is the situation with quiescent inhibitory cell so that the effective threshold of all cells is θ. There are clear indications here of a fruitful theory in which inhibitory cells have a different firing probability, say p', from that of the excitatory sub-population; this would, of course, require coupled equations for p and p'. However, we continue to follow Griffith.

P is a weighted sum of the McCullogh-Pitts functions, each of which are sigmoidal. Since the position of the inflexion depends upon the threshold, we can readily see that we might choose θ small enough and $\theta + a$ large enough to form a weighted linear combination of a sigmoidal function rising rapidly for small p, together with another whose values remain small until p nears 1. The combination will then have a stable equilibrium at $p = 0$, rise sharply to a maximum after crossing $P = p$ at an unstable equilibrium and re-intersect that diagonal line in a falling part of the curve at a stable equilibrium before rising again to $P = 1$. Indeed the function

$$G(n,\theta,p) = (1-p)P(n,\theta,p) + p\,P(n,n-\theta+1,p)$$

is readily seen to satisfy the symmetry

$$G(n,\theta,p) = 1 - G(n,\theta,1-p)$$

and so gives an equilibrium at $p = \frac{1}{2}$, where the slope is less than one, indicating stability, for θ small with respect to n.

In Figures 4(a) - (d) are graphs for $n = 12$ with various values of θ, a. These exhibit clearly the way in which the magnitude of a (the inhibition strength) determines the existence of stable equilibria at middle-range activity when θ is sufficiently small. Moreover, since the advent of catastrophe theory (Thom, 1975) we are all alerted to the way in which parameter variation in a dynamical system can lead to elimination of stable equilibria, resulting in sudden changes of state. Here we have an interesting example of the same phenomenon arising from a controlled Markov process, though unlike the usual situation, our parameters are effectively discrete.

Let us, before leaving Griffith's theory, finally note that in his equation for the case of m inhibitory cells, if we denote the function on the right-hand side by $P(n,m,\theta,p)$ we have

$$P(n,m+1,\theta,p) = (1-p)P(n,m,\theta,p) + p\cdot P(n,m,\theta+a,p)\ .$$

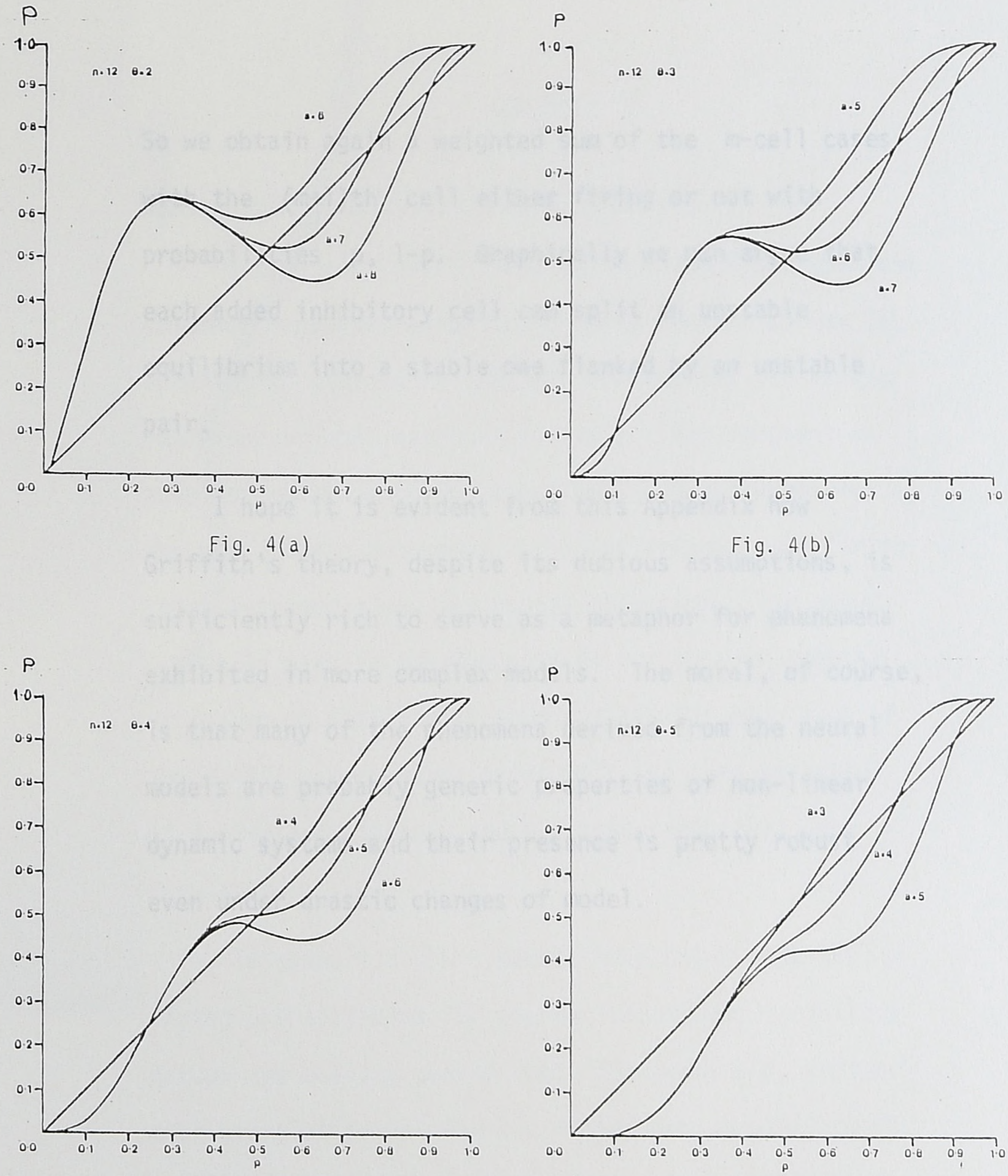


Fig. 4(a)

Fig. 4(b)

Fig. 4(c)

Fig. 4(d)

Varying inhibition strength and threshold and their effect on equilibria.

θ = Threshold. n = Number of excitatory cells. a = Inhibition strength

p = Initial firing probability. P = Final firing probability.

So we obtain again a weighted sum of the m-cell cases with the $(m+1)$th cell either firing or not with probabilities p, $1-p$. Graphically we can argue that each added inhibitory cell can split an unstable equilibrium into a stable one flanked by an unstable pair.

I hope it is evident from this Appendix how Griffith's theory, despite its dubious assumptions, is sufficiently rich to serve as a metaphor for phenomena exhibited in more complex models. The moral, of course, is that many of the phenomena derived from the neural models are probably generic properties of non-linear dynamic systems and their presence is pretty robust even under drastic changes of model.

APPENDIX 2

Oscillatory modes in a cortical model

Wilson and Cowan (1972; 1973) have argued that the columnar organization of the primary sensory cortex is a device to achieve reliability despite the stochastic behaviour of unit neuronal activity. They present a model of cortex which is functionally two-dimensional and whose basic dependent variables are neuronal firing rates averaged along the radial direction. The points I wish to make in the Appendix do not concern spatial variation so I will work from the simpler equations of their 1972 paper; these treat only the time variation of a single excitatory and a single inhibitory sub-population.

To summarize their model, they argue that the firing probabilities for the excitatory and inhibitory groups are sharply peaked about the mean, and, further, that correlation between excitation and sensitivity of a cell is neglible. They are then able to write their dynamical equations as:

$$E(t+\tau) = \left[1 - \int_{t-r}^{t} E(t')dt'\right] S(A)$$

$$\text{where } A = \int_{-\infty}^{t} \alpha(t-t') \left[c\,E(t') + c'I(t') + P(t')\right] dt' \qquad (1)$$

Here τ = synaptic delay, r = refractory period, α = membrane leakage time-course and the coefficients c, c' describe average connectivities from the excitatory and inhibitory to the excitatory populations; E and I are the mean firing rates of those populations, P is the external input to the former and S is a threshold function described more fully below.

A is a measure of the total depolarization current of the excitatory population caused by inputs from the two populations and from external sources. Past effects persist but are discounted by the decay factor α. $S(x)$ is the proportion of cells which fire at excitation level x, so S can be regarded as a cumulative distribution function for the threshold of the excitatory cells. Wilson and Cowan assume it to be sigmoidal - that is, the distribution of thresholds is unimodal. $S(A)$ then expresses the proportion of cells which are de-polarized to threshold by the inputs. Of these, a further proportion, expressed by the integral from $t-r$ to t, have been active within a time shorter than the refractory period so are not available for activation. Postulating a delay τ, before pre-synaptic events result in a cell firing, leads to the stated equation (1).

There is an analogous equation for I which, with (1), forms a pair of coupled integro-difference equations. These are sufficiently complicated to demand further reduction and at this point some rather ad-hoc arguments are introduced to justify the use of moving time averages. More exactly, the approach is to substitute for both integrals of E in (1) the same average $\bar{E}$ (with similar handling of I).

With $\bar{E}(t+\tau)$ and $\bar{I}(t+\tau)$ then replaced by first order Taylor expansions, an ordinary dynamical system results and further discussion is by phase-plane analysis.

The complexity of justifying this reduction on analytical grounds is great. Wilson and Cowan offer plausibility arguments backed by some computer check-out of the behaviour of unreduced equations against the reduced ones. Here I wish to explore an alternative simplification: that all the cells in each population have the same threshold. Thus S is piecewise constant - $S(x) = 0$ for $x < \theta$, $S(x) = 1$ otherwise. Equation (1) becomes

$$\begin{aligned} E(t+\tau) &= 1 - \int_{t-r}^{t} E(t')dt' \quad \text{if} \quad A \geq \theta \\ &= 0 \quad\quad\quad\quad\quad\quad\quad\; \text{if} \quad A < \theta \end{aligned} \qquad (2)$$

This is, of course, highly idealized but it has the advantages of (piecewise) linearity. There will again be a similar equation for I but, since we are concerned to compare our results with those of Wilson and Cowan, we follow them in de-coupling the two populations, writing $c' = 0$ in A and considering only excitatory populations.

In the absence of external input $P = 0$ and equilibrium states $E = E_0$ will satisfy

$$\begin{aligned} E_0 &= 1 - rE_0 && \text{if } c\mu E_0 \geq \theta \\ &= 0 && \text{if } c\mu E_0 < \theta \end{aligned}$$

where $\mu = \int_0^\infty \alpha(t)dt$.

Thus $E_0 = 0$ is always an equilibrium, but There is also a non-zero equilibrium $E_0 = 1/(1+r)$ if $c\mu \geq \theta\,(1+r)$. To discuss stability of these equilibria write $E = E_0 + \Delta$ and examine conditions under which, for sufficiently small Δ, the system tends to return to E_0. Because of the integrals in the dynamical equations the systems behaviour depends on the whole of its past history.

For instance, the equilibrium $E_0 = 0$ yields $A = c\int_{-\infty}^{t} \alpha(t-t')\Delta(t')dt'$ and if $\Delta(t') < \varepsilon$ for $-\infty < t' < t$ we have $A(t) < c\mu\varepsilon$. Then with any positive threshold θ we can choose ε s.t. $A < \theta$ at any t so, by (2), the system returns to $E_0 = 0$ after a further time τ.

For the non-zero equilibrium, assuming it exists, the displacement equations are

$$\Delta(t+\tau) = -\int_{t-r}^{t} \Delta(t')dt' \quad \text{if } B > \phi$$

$$= -E_0 \quad \text{if } B < \phi$$

where $B = \int_{-\infty}^{t} \alpha(t-t')\Delta(t')dt'$ and $\phi = \theta/c - E_0\mu < 0$.

This, being linear, we can apply Laplace transform so long as $B > \phi$, giving

$$e^{z\tau}\left[\bar{\Delta}(z) - \int_0^{\tau} e^{-zt}\Delta(t)dt\right] = \frac{(e^{-zr}-1)}{z}\,\bar{\Delta}(z) + \frac{1}{z}\int_{-r}^{0}\left[e^{-z(t+r)}-1\right]\Delta(t)dt \tag{4}$$

where $\bar{\Delta}$ is the transform of Δ. Assume we maintain initial conditions until the integrals in (4) give well-defined functions of z. Then the equation takes the form

$$G(z) \cdot \bar{\Delta}(z) = F(z)$$

where $F(z)$ are the initial condition integrals and

$$G(z) = e^{z\tau} + (1-e^{-zr})/z \tag{5}$$

The inversion formula for the transform expresses Δ as linear combination of exponentials e^{zt}, where z runs over the poles of the transform. The poles intrinsic to the system, and not dependent through F on the initial conditions, will then be zeros of $G(z)$. The contribution of any term e^{zt} to the "memory" B is $\int_0^t e^{zt'}\alpha(t-t')dt'$.

If, for instance, α has exponential form (with time constant μ) then this integral is $e^{-t/\mu}\int_0^t e^{(z+1/\mu)t'}dt' = \mu\left[e^{zt} - e^{-t/\mu}\right]/(1+z\mu)$ (The change of lower limit in the integral is to make the discussion consistent with the Laplace transform approach, where we implicitly require $\Delta(t) = 0$ for $t < 0$).

Let us note that there are no real zeros of $G(z)$, for when z is real $e^{z\tau}$ is positive and $1-e^{-zr}$ has the same sign as z. Thus all modes are oscillatory and the effect of a term e^{zt} in the memory depends on the sign of $\mathrm{Re}(z)$. If this is negative the contribution to B goes asymptotically to zero; positive $\mathrm{Re}(z)$, however, yields a term which eventually makes $B < \phi$ and so, by (3), switches the activity to zero after a further time τ.

The borderline case of purely oscillatory solutions can be dealt with exactly for, on writing $z = i\beta$ in (5), we find that the equation $G(z) = 0$ becomes

$$\begin{aligned} \cos\beta\tau &= \sin\beta r/\beta \\ \sin\beta\tau &= (1-\cos\beta r)/\beta \end{aligned} \qquad (6)$$

Instead of attempting to solve this for β given τ and r, we can regard these equations as parametrized expressions for τ and r and solve explicity to get

$$\beta\tau = M + 2k\pi \quad \beta r = 2\pi - 2M + 2\ell\pi \tag{7}$$

or $$\beta\tau = \pi - M + 2k\pi, \ \beta r = 2M + 2\ell\pi$$

where $k, \ell = 0,1,2,\ldots$ and $M = \sin^{-1}(\beta/2)$.

Thus there is a doubly infinite set of curves in the τ-r plane consisting of those (τ,r) for which a purely oscillatory solution exists. These curves are shown in Figure 5 for physiologically reasonable magnitudes of τ, r (≤ 10 ms) and some qualitative conclusions can be drawn. A striking feature of the graphs is their almost linear appearance except near $\beta = 2$. All curves are actually obtained from the basic L-shaped solution for $k = \ell = 0$ by translation in both coordinate directions through multiples of $2\pi/\beta$. Actually, M is approximated by $\beta/2$ to an accuracy 0.01 if $\beta < 0.7$ and for this range of β we get a basic solution

$$(\tau,r) = (\tfrac{1}{2}, \frac{2\pi}{\beta} - 1) \quad \text{or} \quad (\frac{\pi}{\beta} - \tfrac{1}{2}, 1)$$

showing the horizontal and vertical segments explicitly.

We have a partial confirmation of Wilson and Cowan's point that the period of oscillations they found by computer simulation had the same order of magnitude as the refractory period. This is certainly true for the vertical arm of the basic solution and the corresponding points on the horizontally-displaced curves.

However, we see on the horizontal arm oscillatory periods tied to the synaptic delay τ. These satisfy $\tau = \pi/\beta - \frac{1}{2}$ in the linear part of the curve, suggesting that such an oscillation could push B below ϕ for a time interval greater than τ. Future contributions to A in (2) would then be zero and the system would stay quiescent.

With regard to the other branches in Figure 5, Wilson and Cowan have argued that oscillations of period less than that of the refractory period cannot propagate information. This would effectively dispose of those

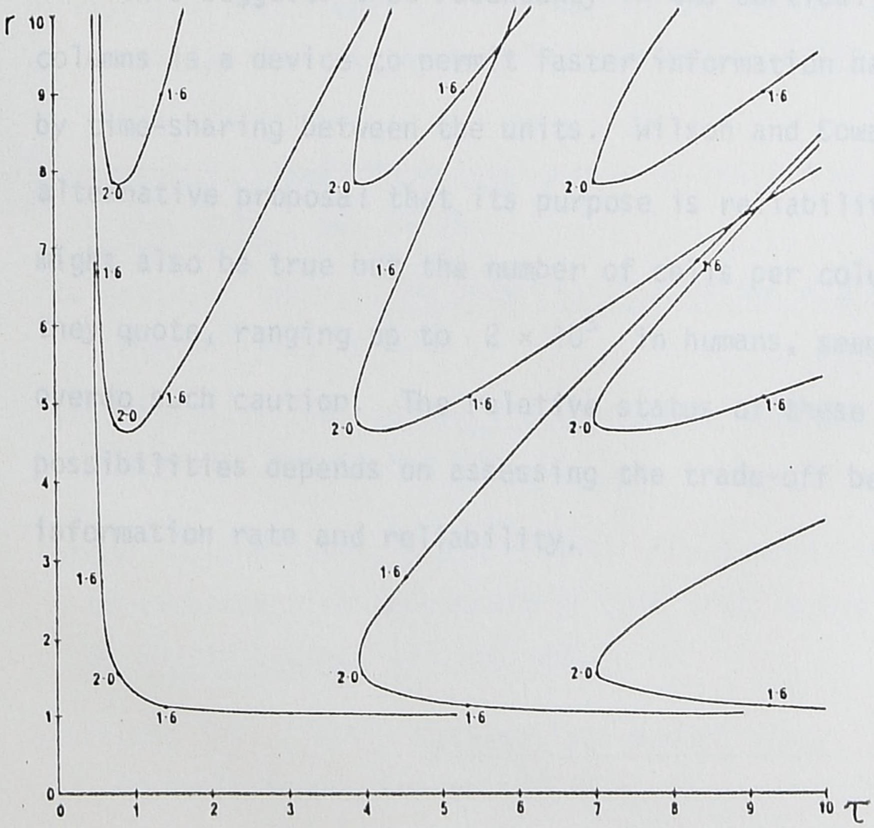


Fig. 5. Synaptic delays and refractory periods giving purely oscillatory modes. τ = Synaptic delay, r = Refractory period; the numbers on the curves show the angular frequency.

branches obtained by vertical displacement from the basic solution, but I find their argument unconvincing. The oscillations in question are properties of a neural aggregate which does not function as a synchronous entity, so we are not requiring that each individual neuron be able to respond at the same rate as the aggregate. All that is needed is that a sufficient number of neurons should become non-refractory within the period of collective oscillations for the excitation to be maintained.

This suggests that redundancy in the cortical columns is a device to permit faster information handling by time-sharing between the units. Wilson and Cowan's alternative proposal that its purpose is reliability might also be true but the number of cells per column which they quote, ranging up to 2×10^5 in humans, seems to overdo such caution. The relative status of these two possibilities depends on assessing the trade-off between information rate and reliability.

BIBLIOGRAPHY

Arbib, M.A. (1966) "Automata theory and control theory - a rapprochement" *Automatica* 3, 161-189.

Arbib, M.A. (1969) *Theories of Abstract Automata* Prentice-Hall, New York.

Ashby, W.R., Von Foerster, H. and Walker, C.C. (1962) "Instability of pulse activity in a net with threshold" *Nature*, 196, 561-562.

Barnwell, G.M. and Stafford, F.S. (1977) "A mathematical model of decision-making neural circuits controlling behavioural mode selection" *Bull.Math.Biol.*, 39, 223-237.

Birkhoff, G. (1948) *Lattice Theory* A.M.S. New York.

Cragg, B.G. and Temperley, H.N.V. (1955) "Memory: The analogy with ferromagnetic hysteresis", *Brain*, 78, 304-316.

Dowker, C.H. (1952) "Homology groups of relations", *Ann. Math.*, 56, 84-95.

Eccles, J.C. (1977) *The Understanding of the Brain.* McGraw-Hill, London.

Ginzburg, A. (1968) *Algebraic Theory of Automata.* Academic Press, New York.

Gordon, G. and Jukes, M.G.M. (1964) "Dual organization of exteroceptive components of cats gracile nucleus". *J. Physiol. Lond.* 173, 263-290.

Griffith, J.S. (1971) *Mathematical Neurobiology.* Academic Press, London.

Heims, S.J. (1980) *John Von Neumann and Norbert Wiener.* MIT Press.

Kobuchi, Y. (1976) "Signal propagation in 2-dimensional threshold cellular space". *J. Math. Biol.* 3, 297-312.

McCarty, G. (1967) *Topology*, McGraw Hill, London.

McCullogh, W.S. and Pitts, W. (1943) "A logical calculus of the ideas immanent in nervous activity". *Bull. Math. Biophys.* 5, 115-133.

Miller, S. and Scott, P.D. (1977) "The spinal locomotor generator" *Exp. Brain Res.* 30, 387-403.

Minsky, M. (1979) "The society theory". In: *A.I. An MIT Perspective*, Vol. 1. Winston, P.H. (ed.) MIT Press.

Muir, A. (1979) "A simple case of the Wilson-Cowan equations". *Biophys. J.* 27, 267-276.

Muir, A. (1981a) "The construction of homogeneous nets" *Bull. Math. Biol.* 43, 415-426.

Muir, A. (1981b) "Fuzzy sets and probability" *Kybernetes* 10, 197-200.

Muir, A. and Warner, M.W. (1980a) "Homogeneous tolerance spaces" *Czech. Math. J.* 30, 118-126.

Muir, A. and Warner, M.W. (1980b) "The decomposition of tolerance automata", *Kybernetes* 9, 265-273.

Palm, G. (1981) "Towards a theory of cell assemblies" *Biol. Cybern.* 39, 181-194.

Poston, T. (1968) "Fuzzy geometry" *Thesis*, Warwick.

Rozenberg, G. and Salomaa, A. (1974) *L-Systems*, Springer-Verlag.

Shallice, T. (1972) "Dual functions of consciousness" *Psych. Rev.* 79, 383-393.

Shepherd, G.M. (1979) The Synaptic Organization of the Brain. Oxford

Spanier, E.H. (1966) Algebraic Topology McGraw-Hill, London.

Stong, R.E. (1966) "Finite topological spaces", Trans. Amer. Math. Soc. 123, 325-340.

Szenthagothai, J. and Arbib, M.A. (1974) "Conceptual models of neural organization" N.R.P. Bull. 12, 305-510.

Taylor, J.G. (1974) "Neural networks and the brain" In: Physics and Mathematics of the Nervous System. Conrad, M. et al (eds) Springer-Verlag.

Thom, R. (1975) Structural Stability and Morphogenesis Benjamin, New York.

Warner, M.W. (1981) "Tolerance nets", Bull.Math. Biol. 43, 641-650.

Warner, M.W. and Muir, A. (1981) "Homology theories and tolerance automata" Discrete Math. 33, 209-221.

Wee, W.G. and Fu, K.S. (1969) "A formulation of fuzzy automata and its application as a model of learning systems" IEEE Transactions on Systems Science and Cybernetics, SSC 5 (3).

Wiener, N. (1948) Cybernetics. MIT Press

Wilson, H.R. and Cowan, J.D. (1972) "Excitatory and inhibitory interactions in localized populations of model neurons" Biophys. J. 12, 1-24.

Wilson, H.R. and Cowan, J.D. (1973). "A mathematical theory of the functional dynamics of cortical and thalmic nervous tissue" Kybernetik, 13, 55-80.

Zadeh, L. (1965) "Fuzzy sets". *Information and Control* 8, 338-353.

Zeeman, E.C. (1961) "The topology of the brain and visual perception". In: *The Topology of 3-Manifolds* Fort, M.K. (ed), Prentice-Hall, New York.

Zvonkin, A.K. and Levin, L.A. (1970) "The complexity of finite objects and the development of the concepts of information and randomness by means of the theory of algorithms" *Russian Math. Surveys*, 25, 83-124.