# ON DIFFRACTION PROBLEMS IN OCEANOGRAPHY AND ON ELLIPTIC SOLUTIONS OF THE SINE-GORDON EQUATION

BY

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## ABSTRACT

The development of diffraction problems in oceanography, which are amenable to the Wiener-Hopf technique is set in the context of modelling physical situations in the North sea and other oceanic regions. Two cases are considered dealing with the mechanics of Kelvin wave generation by the diffraction of cylindrical plane waves by a semi-infinite barrier in an ocean of constant depth and also in the presence of a depth discontinuity.
The significance of the double Kelvin wave regime in the context of Kelvin wave generation is also investigated.
A third problem is presented, a half-plane problem, which uses the diffraction of Kelvin waves by changes in depth as a means of illustrating a double application of the Wiener-Hopf technique involving an extension of the functions from the right-hand physical plane into the left-hand plane. The analytic solution is given, although no numerical results have been obtained.

The latter part of the thesis presents a complete class of separable solutions to the sine-Gordon equation and its space-like variant. Whilst no boundary conditions have been specified, it should be possible to extend the results in order to identify some of the two-dimensional vortex flows represented.

CONTENTS

# ON DIFFRACTION PROBLEMS IN OCEANOGRAPHY AND ON ELLIPTIC SOLUTIONS OF THE SINE-GORDON EQUATION.

## INTRODUCTION

This thesis is presented in two sections for which six published papers are submitted. Section A deals with a survey of some diffraction problems in oceanography, with particular reference to those problems which can be approached through applications of the Wiener-Hopf technique. This major part of the submission also includes unpublished material alongside the papers A1 and A2.

Section B presents an alternative view of partial differential equations in which the sine-Gordon equation is considered without the precise boundary conditions of section A. Classes of solutions are given in papers B1,B2,B3 and B4, which form a reservoir from which useful solutions may be extracted as particular physical situations arise. The context here is in the development of 'solitons' and this work formed part of a joint research programme with A.C.Bryan and A.E.G.Stuart.

SECTION A ON DIFFRACTION PROBLEMS IN OCEANOGRAPHY.

SUBMITTED PAPERS

A1. HAINES, C.R. (1980)

Kelvin wave generation by a semi-infinite barrier. Pure and Applied Geophysics 119:46-50.

A2. HAINES, C.R. (1981)

A Weiner-Hopf approach to Kelvin wave generation by a semi-infinite barrier and a depth discontinuity. Quarterly Journal of Mechanics and Applied Mathematics XXIV (2):139-151.

# CHAPTER 1 MODELLING LONG WAVES ON OCEANIC BOUNDARIES

The search for realistic mathematical models which explain the origin of storm surges in the North Sea is part of a wider investigation into the behaviour of long waves on a rotating earth. Data on tides and sea levels has been collected and analysed over a great many years and the results of these observations are diverse. On the one hand, the after effects of an earthquake in the Pacific Ocean confirmed that the speed c of the resultant tidal wave in an ocean of mean depth h, can be approximated by $\sqrt{gh}$, Green (1946). On the other hand, a deeper understanding of the behaviour of tides, the diffraction, reflection and transmission of long waves and variations in local topography has been achieved from detailed and minute observations at tide gauge stations off the Californian coast, Munk et al.(1970).

The intractable nature of most problems in physical oceanography has led to the construction of linear models with simple representations of coastlines and depth profiles. Nevertheless, considerable insight into the behaviour of long waves on oceanic boundaries has been gained from the use of methods and techniques which were originally developed to deal with diffraction problems in electromagnetism. In this context, the boundary value problems are posed as half-plane diffraction problems with the additional complications of the rotation of the earth and of the mixed boundary conditions.

The need to construct specific models for particular geographical regions is evident, and models of the North Sea have been in use for a considerable time. These models are of some importance and were originally motivated by the severe floods which occurred along the East coast in 1953. The representations of the North Sea are many and various, such as the shadow region behind a semi-infinite barrier, Crease (1956), which allowed an investigation of the way in which free waves propogate from regions to the west and to the northwest of the British Isles into the North Sea. This model, and others of the area are shown in Figures 1 to 5.

Using the linearised long wave theory for shallow water waves on an ocean of constant depth, Proudman (1953), the developments which took place stem from the early research of Crease (1956). His model consisted of a system of simple harmonic plane waves incident normally on a semi-infinite barrier (Fig.1), and demonstrated that the transverse velocity components of the waves beyond the barrier act as a source for waves to be propogated behind the barrier as Kelvin waves. Such waves progress without attenuation and in certain circumstances, the amplitude of this wave exceeds that of the incident wave. This is contrary to the effects noticed in acoustics, in which, of course, rotation does not play a part: however, the disturbance which originates from the source at the edge of the barrier does have similar behaviour and dies out rapidly as distance increases.

Later models, Crease (1958), attempted to fit the behaviour patterns in the North Sea more precisely, by regarding the region as a channel contained between appropriate barriers with a system of plane waves incident upon the barriers at arbitrary angles of incidence. Modelling the region as a channel matches the available data for the area quite well, where the amplification factor for the case of normal incidence is about 2.3 compared with the predicted factor 2.2.

Solutions to these diffraction problems may be obtained in a closed form by the construction of an integral equation, the use of a Green's function and a Wiener-Hopf decomposition. At each stage the comparison with the acoustic waves is made, the general problem of the scattering of long waves in a rotating system having been discussed by Williams (1964). Any problem involving scattering by parallel barriers in such a system reduces to a Dirichlet problem for a Helmholtz equation, although this method does not lead to closed solutions for the problems posed by Crease.

Analogous problems to those of Crease and of Chambers (1964) have been considered in the context of internal waves being diffracted by a semi-infinite barrier, Manton et al. (1970). The method used in the solution is similarly by an integral formulation and a Wiener-Hopf decomposition.

Crease (1956) also postulated that the presence of the Kelvin wave in the shadow region behind the barrier indicated that islands would trap Kelvin waves. Such waves would progress round an island in a clockwise direction in the northern hemisphere and anticlockwise in the southern hemisphere. Williams (1964) confirmed that this was indeed the case and the general question of wave trapping of energy by islands, in a manner similar to the capture of a particle by an atomic nucleus was subsequently explored by Longuet-Higgins (1967).

Whilst the North Sea was initially the main focus of interest, other areas have shown more exciting prospects, e.g. Buchwald and Miles (1974,Fig.6). So too, has attention been given to problems of variable depth and the trapping of waves along a discontinuity of depth. Under favourable circumstances such a depth discontinuity in the ocean floor acts as a wave guide for the propogation of waves - double Kelvin waves - along the line of the discontinuity. The period of double Kelvin waves is always greater than a pendulum day and the uniform depths on either side of the discontinuity satisfy a certain condition. The direction of propogation of the waves is the same as that for Kelvin waves in the deeper water, with the discontinuity to the right of the direction of propogation (Longuet-Higgins, 1968A/B).

As a result of these developments, attention turned to the Mendocino fracture zone off the Californian coast and dealt with the diffraction, reflection and transmission effects. Figures 7 and 8 illustrate two of the models used and in each of these papers aspects of the generation of Kelvin and double-Kelvin waves are discussed in detail.

# CHAPTER 2  KELVIN WAVE GENERATION

My research programme was motivated by the developments outlined in Chapter 1 and concentrated on areas of oceanography which were of physical interest and also those which were amenable to an exposition of the Wiener-Hopf technique.

It has been shown that for wave numbers $\tau<1$, Kelvin waves are diffracted around corners with a reduced amplitude, except for wedge angles $\pi/(2n+1)$. For these prefered angles there is no reduction in amplitude, Packham and Williams (1968). It was therefore of immediate interest to take a fresh look at the diffraction of a system of plane waves by a semi-infinite barrier. Using the model described by Crease (1956), as a starting point, an investigation was made for arbitrary angles of incidence, including the cases for which no Kelvin waves could be generated, and the diffracted, reflected and transmitted waves were calculated (A1). A null hypothesis was proved, that is, there are no prefered angles nor, indeed any prefered wave numbers. My research also confirmed the single result in the case of normal incidence for the threshold wave number $\tau=0.6$, above which the transmission coefficient for the diffracted wave is enhanced, Crease (1956). The results given in (A1) relating the wave number $\tau$ to the transmission coefficient for the diffracted Kelvin wave, have been investigated by Hills (1988), providing further confirmation that incident waves with higher wave numbers $\tau$ are the more persistent in diffraction round corners, Fig.9. This section of research also served to extend and to consolidate that which had been touched upon as a special case of the propogation of long waves in a semi-infinite channel, Crease (1958).

There was, in addition, an interest in the role of bottom topography on problems involving Kelvin waves. Initially attention was focussed on the ways in which double-Kelvin waves may be generated, either by the action of some external agency such as wind stress, Mysak (1969), or by the diffraction of systems of waves. Since double-Kelvin waves can only exist under certain conditions, Longuet-Higgins (1968A), the effects of depth discontinuities in those regimes which do not support them received rather less attention, this was, perhaps, surprising, as for many tidal problems the double-Kelvin wave regime is perceived to be of peripheral interest, Miles (1972B).

The area of application of my research included the Mendocino fracture zone off the Californian coast, and also the region in the southern hemisphere off the tip of South America, since in both these regions there are significant changes in depth over small distances. The model used by Crease (1956) was adapted to include an abrupt change in depth together with a semi-infinite barrier (A2). For wave numbers $\tau<1$, a system of plane waves was diffracted by the depth change and the barrier and the resultant diffracted Kelvin wave, the transmitted plane waves and the reflected waves were calculated. My research showed that an enhanced Kelvin wave was produced only for intermediate wave numbers $0.5<\tau<1$ and for the depth ratios that commonly occur in tidal problems. The variation in the depth ratio was also shown to have little effect on the transmission coefficient for the diffacted wave, although the ratio is important in considering both the transmitted and the reflected waves. The possibility of refraction does not arise in this problem.

Clearly it would have been ideal to have considered the two problems (A1,A2) as special cases of a single problem involving an arbitrary angle of incidence for the incoming system of plane waves, together with a semi-infinite barrier and a depth discontinuity (Fig.10). However, one objective of my research programme was to adapt the procedures outlined by Pinsent (1971) and to apply direct fourier transform techniques to the differential equations and to the boundary conditions. By this method, the need to construct a Green's function is avoided and the Wiener-Hopf technique may be applied with the aid of Theorem C of Noble (1958). The analysis in the above cases is complex and the combining of the two cases in a single more general case only serves to complicate the analysis still further. The principles involved in such a Wiener-Hopf approach are illustrated in the submitted papers (A1,A2) and also in the diffraction of Kelvin waves by an abrupt change in depth, a problem which is further considered in Chapter 3.

# CHAPTER 3 THE DIFFRACTION OF KELVIN WAVES BY AN ABRUPT CHANGE IN DEPTH.

## 3.1 INTRODUCTION

The effect of a depth discontinuity on the propagation of Kelvin waves along a coastline has been the subject of several investigations with variable success. In the Mendocino fracture zone, Miles (1972B), and at the southernmost tip of South America, Pinsent (1971), the methods adopted have calculated the diffracted Kelvin wave but the resulting double Kelvin wave has not been calculated for the former case in which the depth discontinuity is perpendicular to the coastline.

This chapter defines the boundary value problem as a half-plane diffraction problem and uses a double application of the Wiener-Hopf technique to obtain the solution in closed form. The method adopted combines those of Buchwald (1968), in the definition of representations of the solution over the whole plane, and Pinsent (1971), in the method by which the Wiener-Hopf factorisations are carried out and the integral formulation of unknown functions in the $\alpha$ plane.

## 3.2 THE EQUATIONS OF MOTION

The horizontal equations of motion for linearised long waves are, Proudman (1953),

$$\left.\begin{aligned} \frac{\partial u}{\partial t} - fv &= -g\frac{\partial \zeta}{\partial x}, \\ \frac{\partial v}{\partial t} + fu &= -g\frac{\partial \zeta}{\partial y}, \end{aligned}\right\} \qquad (3.2.1)$$

where $\zeta$ is the elevation of the free surface above its mean level and u,v are the components of velocity in the horizontal xy plane. The coriolis parameter is $f=2\omega \sin\Phi$, where $\omega$ is the angular velocity of the earth and $\Phi$ is the north latitude.

The equation of continuity, for uniform depth h of water, is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\frac{1}{h}\frac{\partial \zeta}{\partial t}. \qquad (3.2.2)$$

Assuming $\zeta$, u, v to have a simple harmonic time factor $\exp(i\sigma t)$, (3.2.1) leads to the horizontal velocity components

$$\left.\begin{aligned} u &= \frac{g}{(\sigma^2 - f^2)}\left[ i\sigma\frac{\partial \zeta}{\partial x} + f\frac{\partial \zeta}{\partial y} \right], \\ v &= \frac{g}{(\sigma^2 - f^2)}\left[ -f\frac{\partial \zeta}{\partial x} + i\sigma\frac{\partial \zeta}{\partial y} \right], \end{aligned}\right\} \tag{3.2.3}$$

and (3.2.2) becomes

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\frac{i\sigma}{h}\zeta. \tag{3.2.4}$$

Eliminating u and v between (3.2.3) and (3.2.4) we obtain

$$\frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial y^2} + k^2\zeta = 0 , \tag{3.2.5}$$

where $k^2 = \dfrac{(\sigma^2 - f^2)}{c^2}$ and $c^2 = gh$.

Although (3.2.4) is usually taken as the equation of continuity, the equation may be replaced by

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = -\frac{f}{h}\zeta , \tag{3.2.6}$$

which expresses the conservation of vorticity in a column of fluid.

## 3.3 THE BOUNDARY-VALUE PROBLEM

The conditions of the problem are that a Kelvin wave of the form $\exp\{(-fx+i\sigma y)/c_1\}$ is incident from $y=+\infty$ upon a depth discontinuity along the x-axis as shown in Fig.8. In the region $x>0$ , $y>0$ the water is of constant depth $h_1$ and in the region $x>0$ , $y<0$ the water is of constant depth $h_2$, $(h_1>h_2)$. The land mass occupies the half-plane $x<0$. Following the approach outlined by Noble (1958) and Buchwald (1968), it is assumed that the total elevation $\zeta$ above the mean sea-level is given by

$$\zeta = \begin{cases} \zeta_{01} + \zeta_1 & , \; y \geqslant 0 , \\ \zeta_{02} + \zeta_2 & , \; y \leqslant 0 , \end{cases} \qquad (3.3.1)$$

where

$$\left.\begin{aligned} \zeta_{01} &= \exp\{(-fx + i\sigma y)/c_1\} , \\ \zeta_1 &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} A_1(\alpha)\exp(-i\alpha x - y_1 y)\,d\alpha , \\ \zeta_{02} &= T\exp\{(-fx + i\sigma y)/c_2\} , \\ \zeta_2 &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} A_2(\alpha)\exp(-i\alpha x + y_2 y)\,d\alpha . \end{aligned}\right\} \qquad (3.3.2)$$

In (3.3.1), in each sector $\zeta_n$ must represent an outgoing wave at infinity. This is achieved by applying a Sommerfeld radiation condition which implies that both $k_n$* and $\sigma$ have a small negative imaginary part. In considering the integral representations (3.3.2), these small imaginary parts are then allowed to approach zero.

* $k_n$ is defined by (3.2.5)

In (3.3.2), $\zeta_{01}$ represents the incident Kelvin wave and $\zeta_{02}$ the transmitted Kelvin wave with coastline amplitude $|T|$. Note that $c_n^2 = gh_n$, $n=1,2$ and that (3.3.1) and (3.3.2) satisfy (3.2.5) if, and only if,

$$y_n^2 = \alpha^2 - k_n^2 , \qquad n = 1,2. \tag{3.3.3}$$

The boundary conditions are:-

(i) along the barrier $x=0$, there is zero transverse velocity; both $u_{01}$ and $u_{02}$ are identically zero on $x=0$ so that we require

$$\left.\begin{aligned} u_1 &= 0 , && x = 0,\ y > 0, \\ u_2 &= 0 , && x = 0,\ y < 0. \end{aligned}\right\} \tag{3.3.4}$$

(ii) on the depth discontinuity, the surface elevation is continuous,

$$\zeta_{01} + \zeta_1 = \zeta_{02} + \zeta_2 , \quad y = 0,\ x > 0 , \tag{3.3.5}$$

and the transverse flux is also continuous

$$h_1(v_{01} + v_1) = h_2(v_{02} + v_2) , \quad y = 0,\ x > 0. \tag{3.3.6}$$

Using (3.2.3), $v_{0n} = -\dfrac{g}{c_n}\zeta_{0n}$, $n=1,2$ in (3.3.6) with (3.2.6) we obtain

$$h_1 \frac{\partial u_1}{\partial y} = h_2 \frac{\partial u_2}{\partial y} , \qquad y = 0,\ x > 0. \tag{3.3.7}$$

(3.3.7) may be used instead of (3.3.6).

## 3.4 TO SATISFY THE BOUNDARY CONDITIONS

Consider first, the condition at the wall (3.3.4): from (3.2.3) and (3.3.2) we have, for n=1 and 2,

$$u_n = \frac{g}{(\sigma^2 - f^2)\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma\alpha \mp f\gamma_n) A_n(\alpha) \exp(-i\alpha x \mp \gamma_n y) \, d\alpha \; , \qquad (3.4.1)$$

in which, and subsequently, the upper sign is taken for n=1 and the lower sign for n=2. The boundary condition is satisfied if

$$\int_{-\infty}^{\infty} (\sigma\alpha \mp f\gamma_n) \; A_n(\alpha) \; \exp\;(\mp\gamma_n y) \; d\alpha = 0$$

or, taking a semi-infinite range of integration,

$$\int_0^{\infty} [(\sigma\alpha \mp f\gamma_n) A_n(\alpha) - (\sigma\alpha \pm f\gamma_n) A_n(-\alpha)] \; d\alpha = 0. \quad (3.4.2)$$

The condition (3.3.4) is satisfied for $-\infty < y < \infty$ if

$$(\sigma\alpha \mp f\gamma_n) \; A_n(\alpha) - (\sigma\alpha \pm f\gamma_n) \; A_n(-\alpha) = 0. \qquad (3.4.3)$$

Turning now to the step condition (3.3.7), we begin by defining

$$\frac{\partial u_n}{\partial y}(x,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} U_n(\alpha) \; \exp\;(-i\alpha x) \; d\alpha \; , \qquad (3.4.4)$$

in which

$$\left.\begin{aligned} U_n(\alpha) &= U_n^+(\alpha) + U_n^-(\alpha) , \\ \text{and} \qquad & \\ U_n^+(\alpha) &= \frac{1}{\sqrt{2\pi}}\int_0^\infty \frac{\partial u_n}{\partial y}(x,0)\ \exp\ (i\alpha x)\ d\alpha, \\ U_n^-(\alpha) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^0 \frac{\partial u_n}{\partial y}(x,0)\ \exp\ (i\alpha x)\ d\alpha. \end{aligned}\right\} \qquad (3.4.5)$$

In (3.4.5), $U_n^+(\alpha)$ is a regular function of $\alpha$ in the half plane $Im(\alpha) > \max\{Im(k_n), Im(m)\} = -\delta_0$. This follows from the application of the radiation condition and since

$$\left|\frac{\partial u_n}{\partial y}(x,0)\right| < C_n \exp(-\delta_0 x) \text{ as } x \longrightarrow \infty.$$

Similarly $U_n^-(\alpha)$ is a regular function of $\alpha$ in the half-plane $Im(\alpha) < \delta_0$ from an assumed behaviour of

$$\left|\frac{\partial u_n}{\partial y}(x,0)\right| < D_n \exp\ (\delta_0 x) \text{ as } x \longrightarrow -\infty.$$

Using these transforms on the boundary condition (3.3.7) gives the step condition,

$$h_1 U_1^+(\alpha) = h_2 U_2^+(\alpha) . \qquad (3.4.6)$$

Now $u_n(x,y)$ is given by (3.4.1), so that on differentiation and setting $y=0$,

$$\frac{\partial u_n}{\partial y}(x,0) = \frac{g}{(\sigma^2 - f^2)\sqrt{2\pi}}\int_{-\infty}^{\infty} \mp y_n(\sigma\alpha \mp f y_n) A_n(\alpha) \exp(-i\alpha x)\ d\alpha. \qquad (3.4.7)$$

Now compare (3.4.7) with the definition (3.4.4) and obtain

$$U_n(\alpha) = \mp \frac{g y_n \ (\sigma\alpha \mp f y_n) \ A_n(\alpha)}{(\sigma^2 - f^2)}, \qquad (3.4.8)$$

and this relationship can be used to eliminate $A_n(\alpha)$ from (3.4.3), showing that $U_n(\alpha)$ is an odd function of $\alpha$,

$$U_n(\alpha) + U_n(-\alpha) = 0. \qquad (3.4.9)$$

Each of the terms in (3.4.9) may be decomposed using (3.4.5) and rearranged to give

$$U_n^+(\alpha) + U_n^-(-\alpha) = - U_n^-(\alpha) - U_n^+(-\alpha). \qquad (3.4.10)$$

Now the left hand side of (3.4.10) is a regular function of $\alpha$ for $Im(\alpha) > -\delta_0$, whilst the right hand side is regular for $Im(\alpha) < \delta_0$. $U_n^+(\alpha)$ and $U_n^-(\alpha)$ are both $\sim 1/\alpha$ as $\alpha \rightarrow \infty$, so that the usual extension to Liouville's Theorem, Noble (1958), gives

$$\left.\begin{aligned} U_n^+(\alpha) + U_n^-(-\alpha) &= 0, \\ U_n^-(\alpha) + U_n^+(-\alpha) &= 0. \end{aligned}\right\} \qquad (3.4.11)$$

The step condition (3.4.6) yields

$$h_1 \ U_1^+(-\alpha) = h_2 \ U_2^+(-\alpha) \ , \qquad (3.4.12)$$

and making use of (3.4.11) we get,

$$h_1 \ U_1^-(\alpha) = h_2 \ U_2^-(\alpha). \qquad (3.4.13)$$

Adding (3.4.6) and (3.4.13),

$$h_1[U_1^+(\alpha) + U_1^-(\alpha)] = h_2[U_2^+(\alpha) + U_2^-(\alpha)] \; ,$$

or, more concisely,

$$h_1 \; U_1(\alpha) = h_2 \; U_2(\alpha). \qquad (3.4.14)$$

We now turn to the surface continuity at the step (3.3.5), for which the definition of $\zeta_{on}$ is extended from $x \geqslant 0$ into the half plane $x \leqslant 0$, Buchwald (1968), by means of the unknown functions $f_n(x)$, n=1,2.

$$\left.\begin{aligned} \zeta_{o1} &= \begin{cases} \exp\,[(i\sigma y - fx)/c_1] & x \geqslant 0, \\ f_1(x) & x < 0, \end{cases} \\ \zeta_{o2} &= \begin{cases} T\,\exp\,[(i\sigma y - fx)/c_2] & x \geqslant 0, \\ f_2(x) & x < 0. \end{cases} \end{aligned}\right\} \qquad (3.4.15)$$

Taking the Fourier transform of the surface elevation condition (3.3.5), extended into x<0,

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(\zeta_{o1}+\zeta_1)\exp(i\alpha x)\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(\zeta_{o2}+\zeta_2)\exp(i\alpha x)\,dx. \qquad (3.4.16)$$

Now the left-hand side of (3.4.16) is

$$F_1^-(\alpha) + \frac{i}{\sqrt{2\pi}}\,\frac{1}{\left[\alpha + \dfrac{if}{c_1}\right]} + A_1(\alpha). \qquad (3.4.17)$$

In (3.4.17) we have defined, for n=1 and 2

$$F_n^-(\alpha) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} f_n(x)\;\exp\;(i\alpha x)\;dx, \text{ and } A_1(\alpha)$$

has been obtained using the definitions (3.3.2).

Similarly, the right-hand side of (3.4.16) is

$$F_2^-(\alpha) + \frac{iT}{\sqrt{2\pi}} \frac{1}{\left[\alpha + \frac{if}{c_2}\right]} + A_2(\alpha). \qquad (3.4.18)$$

The transformed surface elevation condition (3.3.5) is gained by equating (3.4.17) to (3.4.18), after which, and substituting for $A_n(\alpha)$ from (3.4.8)

$$F_1^-(\alpha) + \frac{i}{\sqrt{2\pi}} \frac{1}{\left[\alpha + \frac{if}{c_1}\right]} - \frac{(\sigma^2-f^2)U_1(\alpha)}{gy_1(\sigma\alpha-fy_1)}$$

$$= F_2^-(\alpha) + \frac{iT}{\sqrt{2\pi}} \frac{1}{\left[\alpha + \frac{if}{c_2}\right]} + \frac{(\sigma^2-f^2)U_2(\alpha)}{gy_2(\sigma\alpha+fy_2)}. \qquad (3.4.19)$$

is obtained.

The object now is to separate (3.4.19) into two halves, each of which is a regular function of $\alpha$ in overlapping half planes. First a new function $F^-(\alpha)$ is defined such that

$$F^-(\alpha)=F_1^-(\alpha)-F_2^-(\alpha),$$

and then $U_2(\alpha)$ is eliminated using (3.4.14),

$$F^-(\alpha) + \frac{i}{\sqrt{2\pi}\left[\alpha + \frac{if}{c_1}\right]} - \frac{iT}{\sqrt{2\pi}\left[\alpha + \frac{if}{c_2}\right]}$$

$$= \frac{(\sigma^2-f^2)}{g}\left\{\frac{1}{y_1(\sigma\alpha-fy_1)} - \frac{h_1}{h_2y_2(\sigma\alpha+fy_2)}\right\}U_1(\alpha). \tag{3.4.20}$$

The equation (3.4.20), in which $F^-(\alpha)$ is regular in $Im(\alpha)<\delta_0$, may be used to define $U_1(\alpha)$ and recalling that $U_n(\alpha)$ is in fact an odd function, then $U_1(\alpha)$ may be eliminated to give

$$F^-(\alpha) + \frac{i}{\sqrt{2\pi}\left[\alpha + \frac{if}{c_1}\right]} - \frac{iT}{\sqrt{2\pi}\left[\alpha + \frac{if}{c_2}\right]}$$

$$= H(\alpha)\left\{F^-(-\alpha) - \frac{i}{\sqrt{2\pi}\left[\alpha - \frac{if}{c_1}\right]} + \frac{iT}{\sqrt{2\pi}\left[\alpha - \frac{if}{c_2}\right]}\right\}, \tag{3.4.21}$$

where $H(\alpha)$ has been defined thus,

$$H(\alpha) = \frac{h_2y_2(\sigma\alpha+fy_2)+h_1y_1(\sigma\alpha-fy_1)}{h_2y_2(\sigma\alpha-fy_2)+h_1y_1(\sigma\alpha+fy_1)} \cdot \frac{(\sigma\alpha+fy_1)(\sigma\alpha-fy_2)}{(\sigma\alpha-fy_1)(\sigma\alpha+fy_2)}. \tag{3.4.22}$$

The properties of $H(\alpha)$ are dealt with in section 3.6, but for the moment suppose that $H(\alpha) = \dfrac{H^+(\alpha)}{H^-(\alpha)}$, then adopting the usual notation, (3.4.21) becomes

$$F^-(\alpha)\ H^-(\alpha) + \frac{i}{\sqrt{2\pi}} H^-(\alpha) \left\{ \frac{1}{\left[\alpha + \frac{if}{c_1}\right]} - \frac{T}{\left[\alpha + \frac{if}{c_2}\right]} \right\}$$

(3.4.23)

$$= F^-(-\alpha)\ H^+(\alpha) + \frac{i}{\sqrt{2\pi}} H^+(\alpha) \left\{ - \frac{1}{\left[\alpha - \frac{if}{c_1}\right]} + \frac{T}{\left[\alpha - \frac{if}{c_2}\right]} \right\}.$$

We now need to deal with the terms in { } so that (3.4.23) can be split into expressions which are regular in overlapping half planes, this is accomplished by removing the poles at $\alpha = \pm\frac{if}{c_n}$ appropriately. In (3.4.24) such rearrangements of (3.4.23) have taken place so that the left hand side is a regular function of $\alpha$ for $Im(\alpha) < \delta_0$ and the right hand side is regular for $Im(\alpha) > -\delta_0$.

$$F^{-}(\alpha)\ H^{-}(\alpha) + \frac{i}{\sqrt{2\pi}} \frac{\left[H^{-}(\alpha) - H^{-}\left[-\frac{if}{c_1}\right]\right]}{\left[\alpha + \frac{if}{c_1}\right]} + \frac{i}{\sqrt{2\pi}} \frac{H^{+}\left[\frac{if}{c_1}\right]}{\left[\alpha - \frac{if}{c_1}\right]}$$

$$- \frac{iT}{\sqrt{2\pi}} \frac{\left[H^{-}(\alpha) - H^{-}\left[-\frac{if}{c_2}\right]\right]}{\left[\alpha + \frac{if}{c_2}\right]} - \frac{iT}{\sqrt{2\pi}} \frac{H^{+}\left[\frac{if}{c_2}\right]}{\left[\alpha - \frac{if}{c_2}\right]}$$

$$= F^{-}(-\alpha)\ H^{+}(\alpha) - \frac{i}{\sqrt{2\pi}} \frac{\left[H^{+}(\alpha) - H^{+}\left[\frac{if}{c_1}\right]\right]}{\left[\alpha - \frac{if}{c_1}\right]} - \frac{i}{\sqrt{2\pi}} \frac{H^{-}\left[-\frac{if}{c_1}\right]}{\left[\alpha + \frac{if}{c_1}\right]}$$

$$+ \frac{iT}{\sqrt{2\pi}} \frac{\left[H^{+}(\alpha) - H^{+}\left[\frac{if}{c_2}\right]\right]}{\left[\alpha - \frac{if}{c_2}\right]} - \frac{iT}{\sqrt{2\pi}} \frac{H^{-}\left[-\frac{if}{c_2}\right]}{\left[\alpha + \frac{if}{c_2}\right]}. \tag{3.4.24}$$

The extended form of Liouville's Theorem ,Noble (1958), is now applied to (3.4.24). Suppose that each side of the equation is equal to $J(\alpha)$, clearly $J(\alpha)$ is regular in the strip $-\delta_0 < \mathrm{Im}(\alpha) < \delta_0$, but since the left hand side is regular for $\mathrm{Im}(\alpha) < \delta_0$ and the right hand side is regular for $\mathrm{Im}(\alpha) > -\delta_0$, $J(\alpha)$ may be continued analytically over the whole $\alpha$ plane.

The properties of $H(\alpha)$, section 3.6, show that $H^{+}(\alpha)$ and $H^{-}(\alpha)$ are $\sim\alpha$ and $F^{-}(\alpha)$ is $\sim 1/\alpha$ as $\alpha \longrightarrow \infty$ and so $J(\alpha)$ is a constant B (say). The left hand side of (3.4.24) equated to B, and noting that $H^{+}(\alpha) = -H^{-}(-\alpha)$, leads to

$$F^{-}(\alpha) + \frac{i}{\sqrt{2\pi}\left[\alpha + \frac{if}{c_1}\right]} - \frac{iT}{\sqrt{2\pi}\left[\alpha + \frac{if}{c_2}\right]}$$

$$= \frac{B}{H^{-}(\alpha)} + \frac{i}{\sqrt{2\pi}} \frac{H^{-}\left[-\frac{if}{c_1}\right]}{H^{-}(\alpha)} \frac{2\alpha}{\left[\alpha^2 + \frac{f^2}{c_1^2}\right]}$$

$$- \frac{iT}{\sqrt{2\pi}} \frac{H^{-}\left[-\frac{if}{c_2}\right]}{H^{-}(\alpha)} \frac{2\alpha}{\left[\alpha^2 + \frac{f^2}{c_2^2}\right]}. \quad (3.4.25)$$

Equation (3.4.25) may now be used with (3.4.20) in order to find $U_1(\alpha)$, and equation (3.4.14) then defines $U_2(\alpha)$. Having determined $U_n(\alpha)$, n=1,2, equation (3.4.8) yields $A_n(\alpha)$, n=1,2, as follows

$$A_1(\alpha) = \left\{ \frac{B}{H^-(\alpha)} + \frac{i}{\sqrt{2\pi}} \frac{H^-\left[-\frac{if}{c_1}\right]}{H^-(\alpha)} \frac{2\alpha}{\left[\alpha^2 + \frac{f^2}{c_1^2}\right]} - \frac{iT}{\sqrt{2\pi}} \frac{H^-\left[-\frac{if}{c_2}\right]}{H^-(\alpha)} \frac{2\alpha}{\left[\alpha^2 + \frac{f^2}{c_2^2}\right]} \right\} \cdot \frac{-h_2 y_2(\sigma\alpha + f y_2)}{[h_2 y_2(\sigma\alpha + f y_2) + h_1 y_1(\sigma\alpha - f y_1)]}, \tag{3.4.26}$$

$$A_2(\alpha) = \left\{ \frac{B}{H^-(\alpha)} + \frac{i}{\sqrt{2\pi}} \frac{H^-\left[-\frac{if}{c_1}\right]}{H^-(\alpha)} \frac{2\alpha}{\left[\alpha^2 + \frac{f^2}{c_1^2}\right]} - \frac{iT}{\sqrt{2\pi}} \frac{H^-\left[-\frac{if}{c_2}\right]}{H^-(\alpha)} \frac{2\alpha}{\left[\alpha^2 + \frac{f^2}{c_2^2}\right]} \right\} \cdot \frac{h_1 y_1(\sigma\alpha - f y_1)}{[h_2 y_2(\sigma\alpha + f y_2) + h_1 y_1(\sigma\alpha - f y_1)]}. \tag{3.4.27}$$

The formal solutions (3.4.26), (3.4.27) determine the elevations $\zeta$ in the assumed forms (3.3.2), save for the constant B, and the transmission coefficient T for the Kelvin wave component along the coastline. Both B and T are determined by considering the boundary conditions at the origin.

## 3.5 THE BOUNDARY CONDITIONS AT THE ORIGIN

Firstly we consider the determination of the constant B and require that the surface elevation $\zeta$ be continuous at the origin. The condition (3.3.5) is therefore applied at (0+,0). Using (3.3.2),

$$\zeta_{01}-\zeta_{02}=1-T$$

and

$$\zeta_2 - \zeta_1 = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} A_2(\alpha) - A_1(\alpha)\, d\alpha \,. \tag{3.5.1}$$

Notice that the integrand in (3.5.1) can be written as the right hand side of (3.4.25) using the results (3.4.26) and (3.4.27) and that it is analytic in the lower half plane with simple poles at $\alpha=-if/c_n$, $n=1,2$. The integral is evaluated using the contour C, which consists of the real axis and the semicircle $|\alpha|=R$, $-\pi<\arg\alpha<0$. The integral $\rightarrow 0$ on the semicircular arc as $R\rightarrow\infty$ if, and only if, $B=0$ since $H^-(\alpha)\sim D\alpha$. This is additionally confirmed by the fact that the left hand side of (3.4.25) is analytic in the lower half plane with simple poles at $\alpha=-if/c_n$, $n=1,2$. Evaluating the residues at these poles and applying Cauchy's theorem,

$$\zeta_2-\zeta_1=1-T=\zeta_{01}-\zeta_{02}$$

at (0+,0), as required.

Similarly the transverse flux, described by (3.3.6), must be continuous at (0+,0). Using (3.2.3) and (3.3.2) the velocity component

$$v_{on}=-(g/c_n)\zeta_{on},\quad n=1,2,$$

and

$$v_n = \frac{g}{(\sigma^2-\alpha^2)}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(\mp i\sigma y_n+i\alpha f)\ A_n(\alpha)\ d\alpha. \qquad (3.5.2)$$

Now the $A_n(\alpha)$ are given by (3.4.26) and (3.4.27) with B=0, so that the contour C is again appropriate for the evaluation of the integrals and we deduce that

$$h_2v_2-h_1v_1=c_2T-c_1=h_1v_{01}-h_2v_{02},$$

at (0+,0) as required.

It is now necessary to determine T by matching the conditions on x=0 and those on y=0, at the origin. On x=0, from (3.2.1) and (3.3.4),

$$-f(v_{on}+v_n) = -g\frac{\partial}{\partial x}(\zeta_{on}+\zeta_n)\quad, n=1,2. \qquad (3.5.3)$$

On y=0, differentiating (3.3.5)

$$\frac{\partial}{\partial x}(\zeta_{01}+\zeta_1) = \frac{\partial}{\partial x}(\zeta_{02}+\zeta_2)\quad, \qquad (3.5.4)$$

substituting into (3.5.3) we deduce that at (0+,0±)

$$v_{01} + v_1 = v_{02} + v_2. \qquad (3.5.5)$$

The foregoing and (3.3.2) lead to $v_{01}=-g/c_1$ at

(0+,0+) together with $v_{o2}=-(g/c_2)T$ at (0+,0−), and the solutions (3.4.26) and (3.4.27) applied appropriately at (0,0) imply

$$v_1-v_2 = \frac{g}{(\sigma^2-f^2)}\frac{i}{\sqrt{2\pi}}\int_{-\infty}^{\infty}[(-\sigma y_1+\alpha f)A_1(\alpha)-(\sigma y_2+\alpha f)A_2(\alpha)]\,d\alpha$$

$$= \frac{g}{(\sigma^2-f^2)}\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\alpha M(\alpha)}{H^-(\alpha)}\left[\frac{H^-\left[-\frac{if}{c_1}\right]}{\left[\alpha^2+\frac{f^2}{c_1^2}\right]}-\frac{T\,H^-\left[-\frac{if}{c_2}\right]}{\left[\alpha^2+\frac{f^2}{c_2^2}\right]}\right]d\alpha,$$

(3.5.6)

where $M(\alpha) = \dfrac{h_1y_1(\sigma\alpha-fy_1)(\alpha f+\sigma y_2)+h_2y_2(\sigma\alpha+fy_2)(\alpha f-\sigma y_1)}{h_2y_2(\sigma\alpha+fy_2)+h_1y_1(\sigma\alpha-fy_1)}$.

These results are now substituted into (3.5.5), which is then solved for T,

$$T = \frac{\dfrac{(\sigma^2-f^2)}{c_1}+\dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty}M(\alpha)\,\frac{H^-\left[-\frac{if}{c_1}\right]}{H^-(\alpha)}\cdot\frac{\alpha}{\left[\alpha^2+\frac{f^2}{c_1^2}\right]}\,d\alpha}{\dfrac{(\sigma^2-f^2)}{c_2}+\dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty}M(\alpha)\,\frac{H^-\left[-\frac{if}{c_2}\right]}{H^-(\alpha)}\cdot\frac{\alpha}{\left[\alpha^2+\frac{f^2}{c_2^2}\right]}\,d\alpha}$$

(3.5.7)

In (3.5.7) we notice that when $h_1=h_2$, the case for no depth discontinuity, then T=1 as expected and also, that as $h_2\to 0$, then $T\to 0$ and $\zeta_{o2}=0$ for the case when the water is constrained to the first quadrant.

## 3.6 PROPERTIES OF H(α)

Returning now to (3.4.22), we analyse the properties of H(α) which enabled its decomposition and the resulting factorisation of (3.4.23).

$$H(\alpha) = \frac{h_2 y_2(\sigma\alpha + f y_2) + h_1 y_1(\sigma\alpha - f y_1)}{h_2 y_2(\sigma\alpha - f y_2) + h_1 y_1(\sigma\alpha + f y_1)} \cdot \frac{(\sigma\alpha + f y_1)(\sigma\alpha - f y_2)}{(\sigma\alpha - f y_1)(\sigma\alpha + f y_2)}. \qquad (3.4.22)$$

For $\zeta_n$ to include a double Kelvin wave component in the regime $\sigma > f$, m is a positive root of

$$h_1\sigma\sqrt{m^2-k_1^2} + h_2\sigma\sqrt{m^2-k_2^2} - mf(h_1-h_2) = 0 , \qquad (3.6.1)$$

Longuet-Higgins (1968A), but if $\sigma < f$ then m is complex with $\mathrm{Im}(m) < 0$.

It is convenient to extract from H(α) its poles and zeros and to define K(α) thus:-

$$H(\alpha) = \frac{\left[\alpha + \frac{if}{c_1}\right]\left[\alpha - \frac{if}{c_2}\right](\alpha - m)}{\left[\alpha - \frac{if}{c_1}\right]\left[\alpha + \frac{if}{c_2}\right](\alpha + m)} K(\alpha). \qquad (3.6.2)$$

Note, further, that (3.4.22) leads to:

$$\lim_{\alpha\to\infty} K(\alpha) = I \qquad \text{and} \qquad \lim_{\alpha\to-\infty} I\, K(\alpha) = 1 ,$$

where
$$I = \frac{(\sigma+f)h_2 + (\sigma-f)h_1}{(\sigma-f)h_2 + (\sigma+f)h_1}. \qquad (3.6.3)$$

The decomposition $H(\alpha) = \dfrac{H^{+}(\alpha)}{H^{-}(\alpha)}$ now depends upon finding $K^{+}(\alpha)$ and $K^{-}(\alpha)$ such that

$$H^{+}(\alpha) = \frac{\left[\alpha + \dfrac{if}{c_1}\right]}{\left[\alpha + \dfrac{if}{c_2}\right]} (\alpha - m)\, K^{+}(\alpha) \,,$$

and (3.6.4)

$$H^{-}(\alpha) = \frac{\left[\alpha - \dfrac{if}{c_1}\right]}{\left[\alpha - \dfrac{if}{c_2}\right]} (\alpha + m)\, K^{-}(\alpha) \,,$$

in which $K^{+}(\alpha)$ is a regular function of $\alpha$ in the half plane $Im(\alpha) > -\delta_1$ and $K^{-}(\alpha)$ is regular in the half plane $Im(\alpha) < \delta_1$. $\delta_1$ is defined to be $\min\{Im(if/c_n), Im(-m)\}$.

Unfortunately, the $K(\alpha)$ behaviour (3.6.3) does not fit the conditions of Theorem C, Noble (1958), but using the method prescribed by Pinsent (1971), integrate $\dfrac{\log I\ K(\beta)}{(\beta - \alpha)}$ in the $\beta$-plane round the rectangle with vertices $-i\delta_1-R$, $-i\delta_1+R$, $i\delta_1+R$, $i\delta_1-R$, and $\dfrac{\log I}{(\beta - \alpha)}$ round the same rectangle but in the opposite sense. The behaviour of $K(\beta)$ is given by (3.6.3) so that Cauchy's Integral Theorem, with $R\to\infty$, leads to

$$2\pi i \log K(\alpha) = \int_{-i\delta_1-\infty}^{-i\delta_1+\infty} \frac{\log K(\beta)}{(\beta - \alpha)}\, d\beta - \int_{i\delta_1-\infty}^{i\delta_1+\infty} \frac{\log K(\beta)}{(\beta - \alpha)}\, d\beta. \quad (3.6.5)$$

Implicit in (3.6.4) is the decomposition of $K(\alpha)$,

$$K(\alpha) = \frac{K^+(\alpha)}{K^-(\alpha)}, \quad (3.6.6)$$

so that (3.6.5) is itself decomposed to give

$$K^+(\alpha) = \exp \frac{1}{2\pi i} \int_{-i\delta_1-\infty}^{-i\delta_1+\infty} \frac{\log K(\beta)}{(\beta - \alpha)}\, d\beta ,$$

and (3.6.7)

$$K^-(\alpha) = \exp \frac{1}{2\pi i} \int_{i\delta_1-\infty}^{i\delta_1+\infty} \frac{\log K(\beta)}{(\beta - \alpha)}\, d\beta ,$$

where $K^+(\alpha)$ is regular for $Im(\alpha) > -\delta_1$ and $K^-(\alpha)$ is regular for $Im(\alpha) < \delta_1$.

From (3.6.7) we see that

$$K^+(\alpha) = K^-(-\alpha) \quad \text{and} \quad K^-(\alpha) = K^+(-\alpha).$$

Further, (3.6.4) implies that

$$H^+(\alpha) = -H^-(-\alpha) \quad \text{and} \quad H^-(\alpha) = -H^+(-\alpha),$$

that is,

$$H(\alpha).H(-\alpha) = 1 ,$$

which can be seen directly from (3.4.22).

As $\delta_1 \to 0$, the only contributions to $|K^+(\alpha)|$ and $|K^-(\alpha)|$ arise from the pole at $\beta=\alpha$ on the real $\beta$ axis. The behaviour of $|K^+(\alpha)|$ and $|K^-(\alpha)|$ therefore follows that of $K(\alpha)$ which is $O(1)$, so (3.6.4) implies that $|H^+(\alpha)|$ and $|H^-(\alpha)|$ are $\sim\alpha$ as $|\alpha|\to\infty$. The decomposition of (3.4.24) is therefore justified.

## 3.7 EXTENSIONS AND CONCLUDING REMARKS

The analysis presented in this chapter illustrates an application of the Wiener-Hopf technique to a diffraction problem using adaptations of methods suggested by Buchwald (1968) and Pinsent (1971). Ideally, the formal solution should now be verified numerically by comparison with data from tide gage stations. The transmission coefficient T, (3.5.7), for the Kelvin wave component has been estimated by Miles (1972B), for certain regimes in $\tau<1$, but not for the case of the double Kelvin wave. It is possible that asymptotic forms for the solutions in $\tau>1$ can be found using the saddle point method, even though they may in fact have little physical significance. These solutions for $\zeta_n$ contain components in $\tau>1$ for which the contribution from the poles at $\alpha=\pm m$ lead to the double Kelvin wave travelling along the depth discontinuity. The contribution at the saddle point itself describes the non-rotational effects for both $\tau<1$ and for $\tau>1$. It would also have been desirable to compare the solutions given by (3.3.2) with both the analytic solution and the asymptotic results of Miles (1972B).

The models discussed so far are based upon a linearised long wave theory in which the vertical accelerations are assumed to be negligible. It is of interest to consider the behaviour in the immediate neighbourhood of the depth discontinuity. In such a region, the vertical accelerations may not be small and this factor must be taken into account in the basic equations of motion, Proudman (1953), together with the full equation of continuity. Some preliminary investigations suggest that the linearised and suitably scaled equations, with a time dependence exp(i$\sigma$t) lead to

$$(\sigma^2-f^2+4\omega^2)w_{yy} + (\sigma^2-f^2)w_{zz} = 0$$

for the vertical component of velocity w. Further tranformations in the vertical direction lead to Laplaces equation, for w, in an infinite strip with Dirichlet boundary conditions on the surface of the water and on the horizontal ocean floor and a Neumann condition on the step itself.

Of course, this research has concentrated upon linear effects and as mentioned above, there are still several areas in which a linear approach could yield relevant physical results. However, nonlinearities in the governing equations and in the boundary conditions have also received attention. The effect of an irregular coastline on the propogation of Kelvin waves suggests that Kelvin waves are generated predominantly by atmospheric disturbances or by the incidence of a plane wave system, Pinsent (1972), Mysak and Tang (1974) and Clarke (1977). The incident Kelvin wave itself, is also shown to be slowed down by coastal irregularities and to give up part of its energy in generating further Kelvin waves or, for $\tau<1$, Poincare waves. Small changes in the coastline without a sustained change in direction have a negligible effect on the transmitted Kelvin waves at tidal frequencies, Miles (1972A). The curvature of the earth is also shown to have little effect compared to, say, bottom topography. This leads one to conclude that further research is necessary on the depth profiles used, even in the linear model.

Nonlinear theories have not been neglected in the literature and a first attempt at such a theory was made by Smith (1972), who dealt with Kelvin and continental shelf waves. His initial work showed that a nonlinear theory was both possible and desirable, although his results could not be compared with any real situation since he neglected curvature, changes in depth profile and changes in the coriolis parameter.

## SECTION B ON ELLIPTIC SOLUTIONS OF THE SINE-GORDON EQUATION

### SUBMITTED PAPERS

B1. BRYAN, A.C., HAINES, C.R. and STUART, A.E.G. (1978) Complex solitons and poles of the sine-Gordon equation. Letters in Mathematical Physics 2:445-449.

B2. BRYAN, A.C., HAINES, C.R. and STUART, A.E.G. (1979) Solitons and separable elliptic solutions of the sine-Gordon equation. Letters in Mathematical Physics 3:265-269.

B3. BRYAN, A.C., HAINES, C.R. and STUART, A.E.G. (1980) A classification of the separable solutions of the two-dimensional sine-Gordon equation and of its Laplacian variant. Il Nuovo Cimento II 58B:1-33.

B4. BRYAN, A.C., HAINES, C.R. and STUART, A.E.G. (1981) Complex extensions of a manifold of solutions of the sine-Gordon equation. In: Lecture Notes in Mathematics 846, Conference on Ordinary and Partial Differential Equations, 6th, Dundee 1980. Berlin: Springer-Verlag. 74-81.

## CHAPTER 4 THE SINE-GORDON EQUATION AND ITS LAPLACIAN VARIANT

This research was prompted by the interest of nonlinear physicists in the sine-Gordon equation and the wide range of applications which it models. The sine-Gordon equation has been used as a model for describing the dynamics of nonlinear phenomena and has remarkable properties at the classical and quantum levels.

It was the existence of solitons and their behaviour which focussed attention on the sine-Gordon equation. Questions of separability for partial differential equations were under investigation elsewhere and I began working with Dr.A.E.G.Stuart and Mr.A.C.Bryan on a formal classification of the separable solutions of the sine-Gordon equation and its space like variant. Although there had been a wide ranging survey article on solitons published several years earlier, Barone et al. (1971), and of course Cercignani (1977), the short paper of Zagrodzinski (1976), provided the focus of our attention. I was able to make an initial attempt at establishing a complete set of solutions in terms of Jacobian elliptic functions.

As the research progressed, it emerged that there were in existence complex solitons whose limiting forms are the real soliton and a singular form. This development formed the basis of the paper (B1) and established the equivalence between the dynamics of the interacting sine-Gordon solitons and the motion of the poles of the corresponding Hamiltonian density.

It was also found that the soliton and antisoliton solutions of the sine-Gordon equation may be obtained directly from limiting cases of a separable, two parameter family of elliptic solutions, (B2).

The research was completed in (B3) and forms a basis for information on the sine-Gordon equation in a similar manner to the task undertaken by Shercliff (1977). There is still scope for further extensions to be made in the applications of the space-like equation to hydrodynamics. A complete class of separable solutions has been found and it should be possible to identify particular steady, two-dimensional vortex flows of an ideal compressible fluid which they describe, Stuart (1971).

Finally, the paper (B4) attempts to place the previous work in context. The research was shared equally between the three members of the group, my own particular contribution to the progress of the research centred upon the initial classification and on the properties of the Jacobian elliptic functions as they emerged in the analysis.

FIGURES

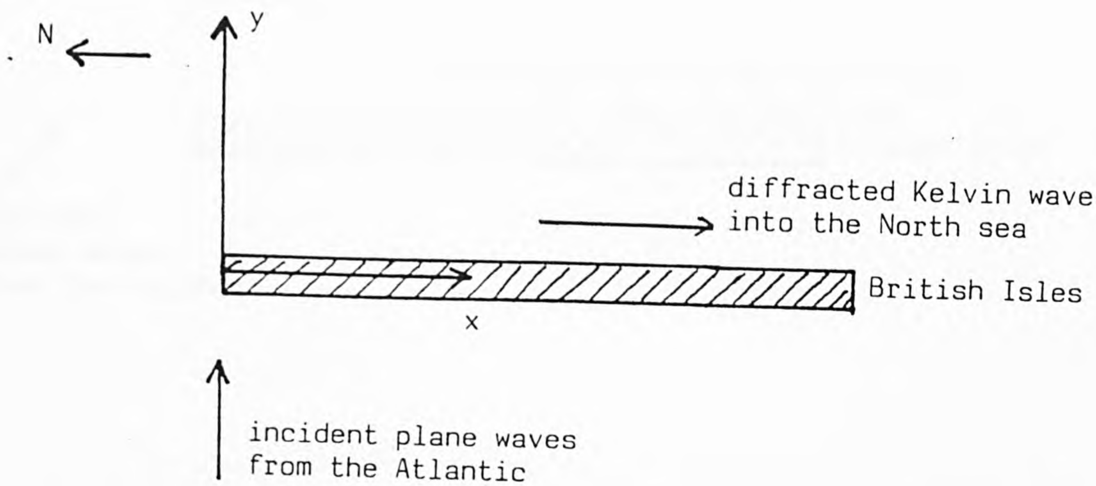
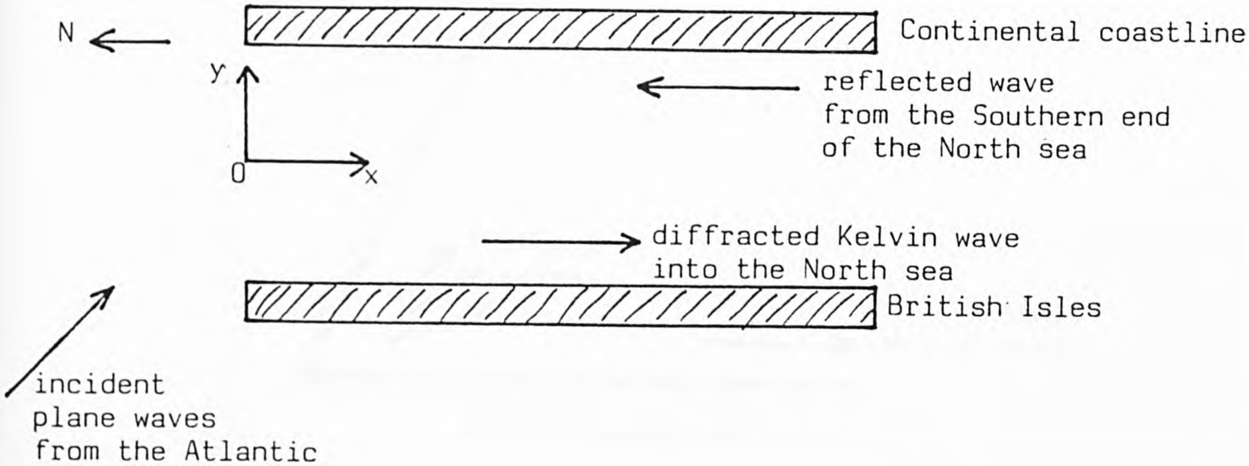



Figure 1: A representation of the North Sea as the shadow region behind a semi-infinite barrier (British Isles), (Crease, 1956).



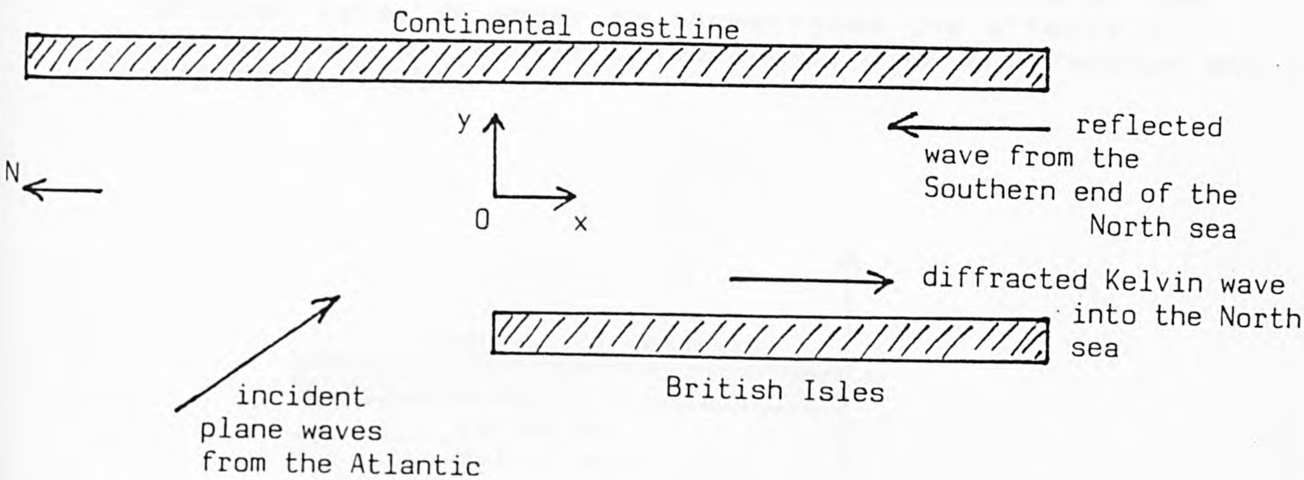




Figure 2: Representations of the North Sea as a long gulf behind a semi-infinite barrier (British Isles). For problems investigating the diffraction of Kelvin waves into the North Sea, the modelling of the continental coastline by a semi-infinite barrier is better than using an infinite barrier, (Crease, 1958).

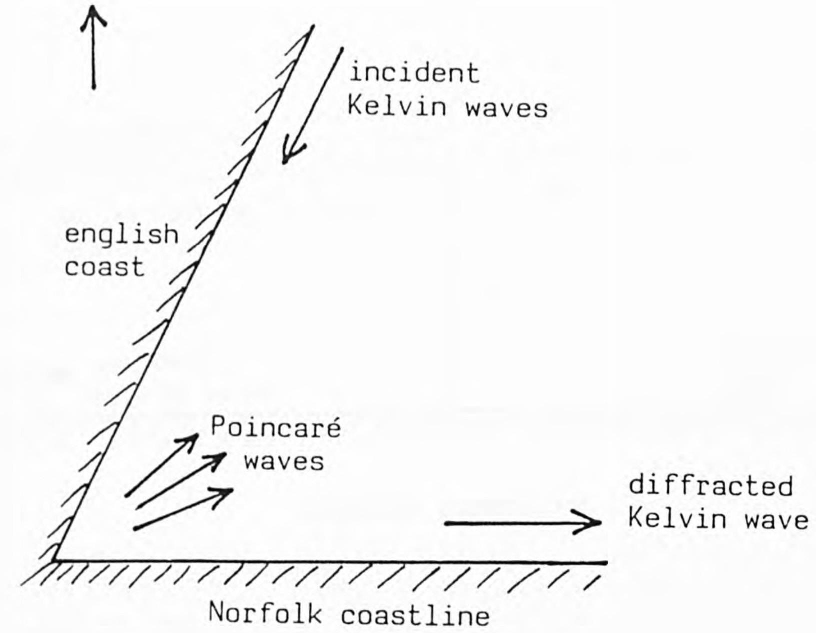



Figure 3: Representations of part of the North Sea in the proximity of a sharp bend in the coastline of the British Isles in order to investigate the effects of Kelvin wave diffraction, (Buchwald, 1968 and Packham and Williams, 1968).

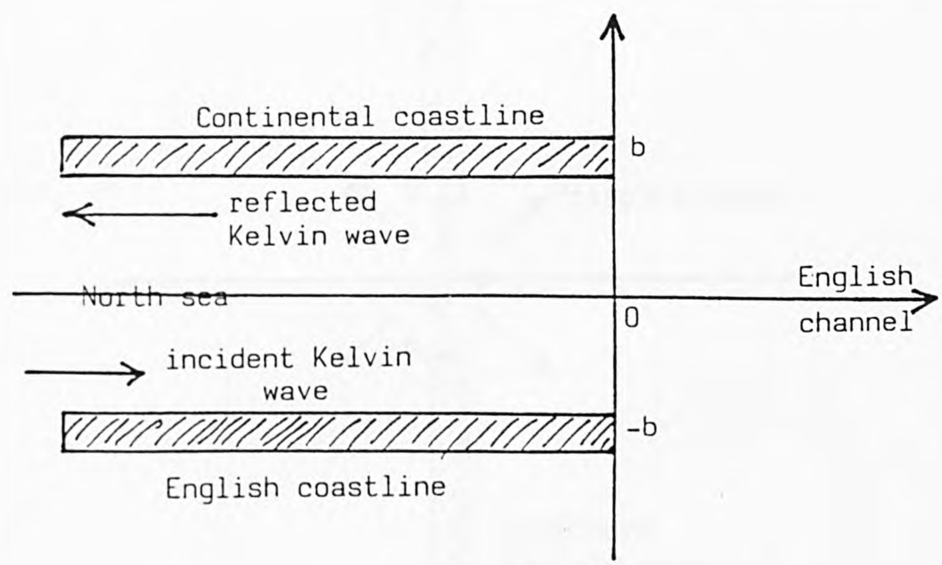



Figure 4: A representation of the North Sea as an open-ended semi-infinite channel, (Packham, 1969).

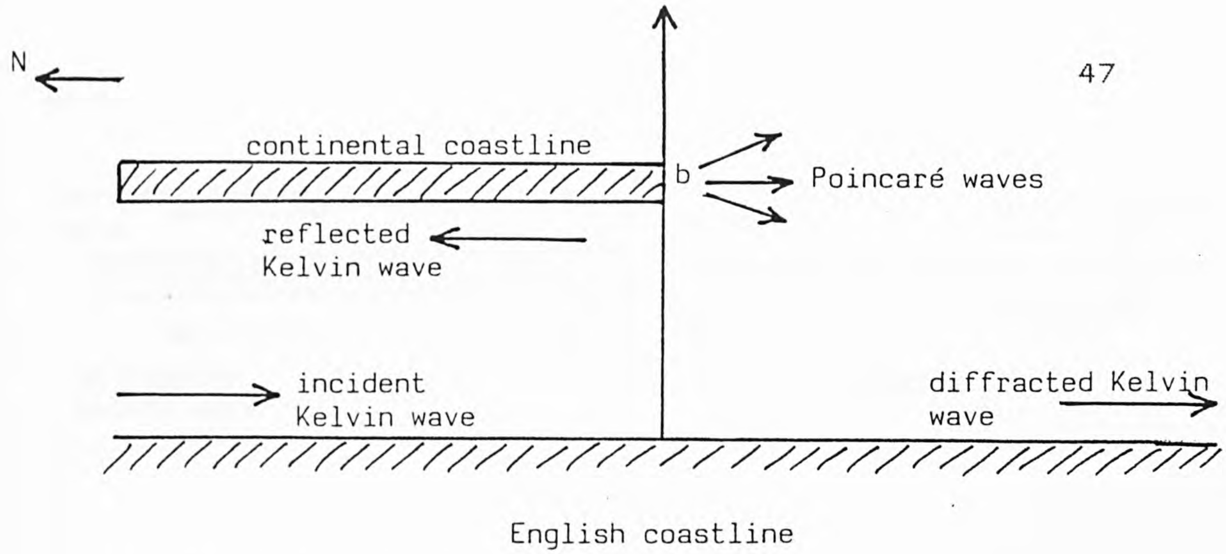



Figure 5: A representation of the North Sea as an open-ended semi-infinite channel, (Kapoulitsas, 1979).

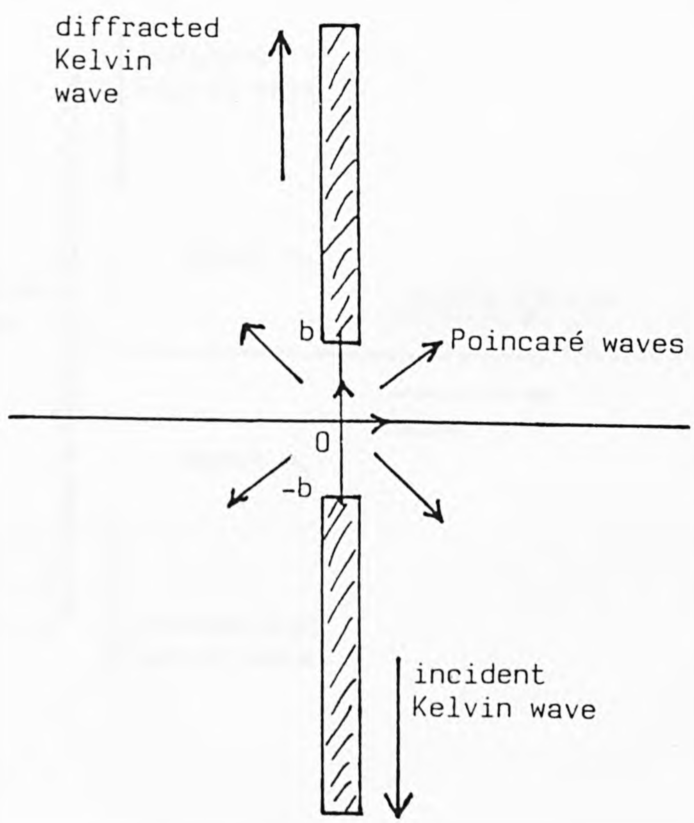



Figure 6: Tidal diffraction by a strait between two semi-infinite barriers, used to model San Francisco Bay, (Buchwald and Miles, 1974).

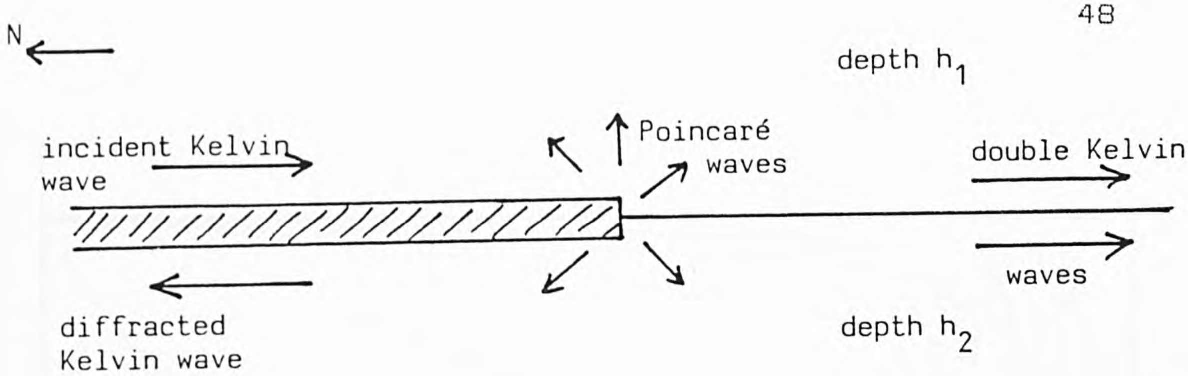



Figure 7: The diffraction of Kelvin waves by a depth discontinuity, modelling a region at the southernmost tip of South America, (Pinsent, 1971).

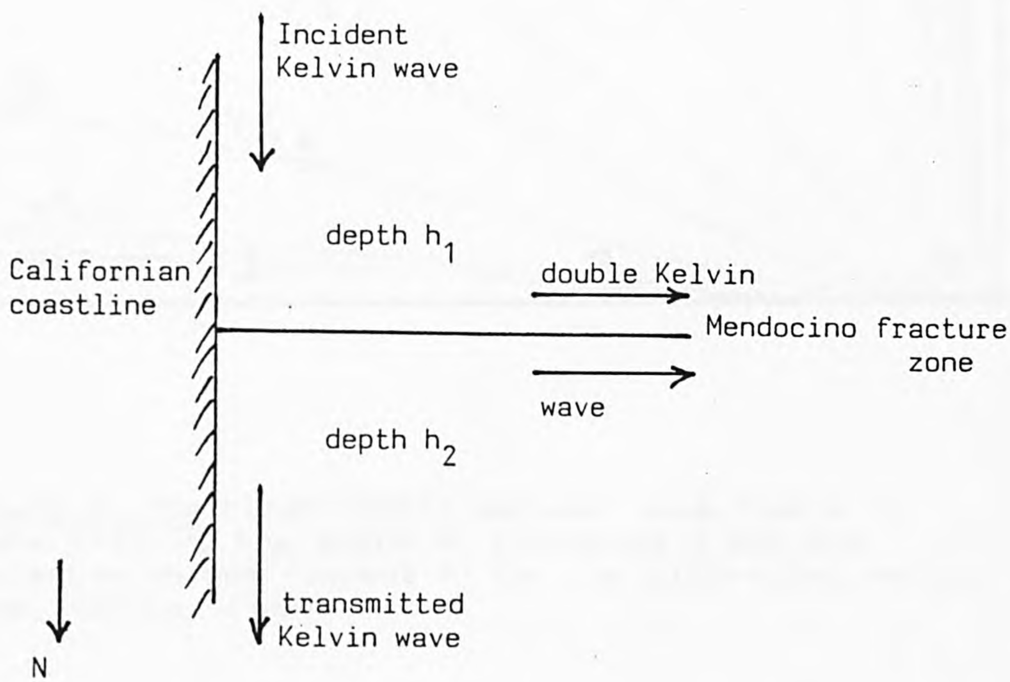



Figure 8: Modelling Kelvin wave diffraction by changes in depth, (Miles, 1972B).

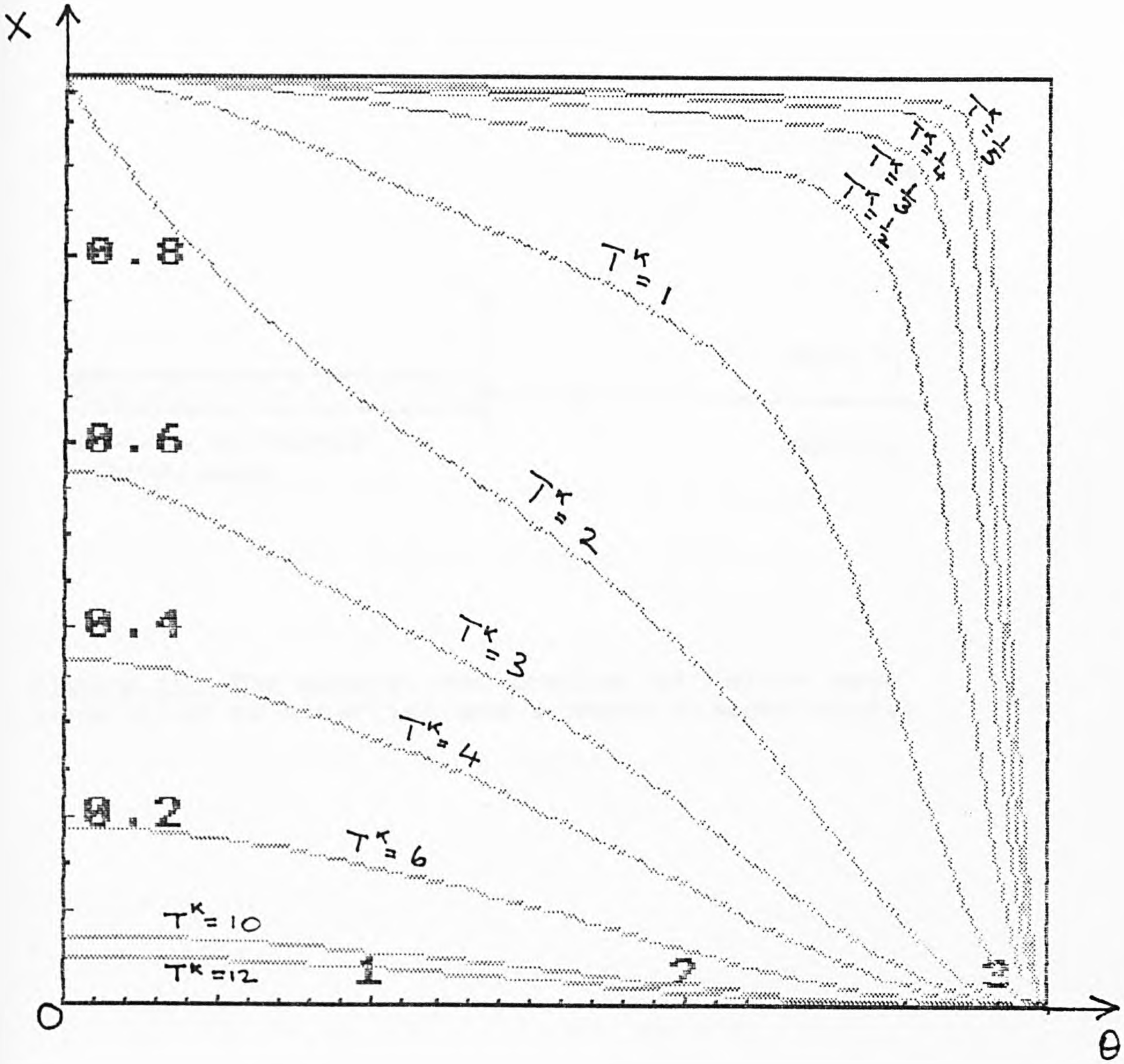



Figure 9: The relationship between wave number $\tau$, where $X=\sqrt{1-\tau^2}$, the angle of incidence $\theta$ and the transmission coefficient $T^k$ for the diffracted Kelvin wave, (Hills, 1988).

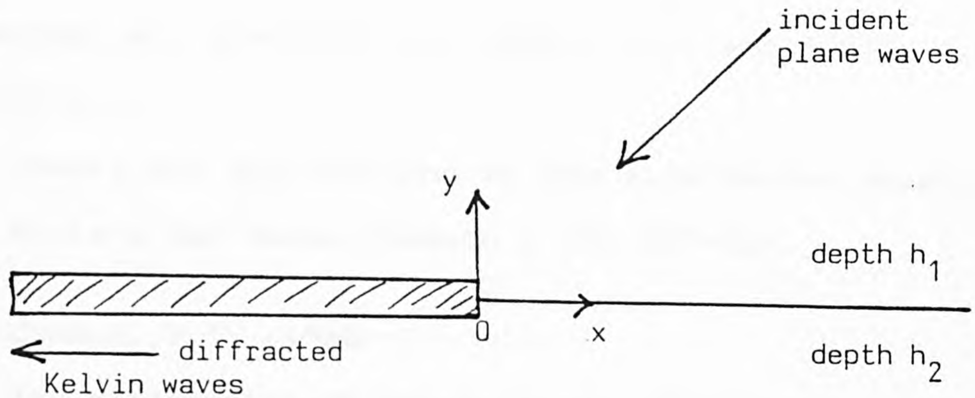



Figure 10: The generalised problem of Kelvin wave generation by a barrier and a depth discontiniuty.

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