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Portfolio selection and risk sharing via risk budgeting

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ABSTRACT

Risk budgeting is an effective risk management tool that a decision-maker uses to create a risk portfolio with a pre-determined risk profile. This paper provides a rich discussion about the theory and practice on how to construct risk budgeting portfolios in a variety of settings. We revisit the usual portfolio selection setting with and without clustered risk budgeting targets, and we then provide an approach on how to extend the usual setting to situations in which a non-hedgeable risk is present or fixed sub-portfolios are aimed by the decision-maker. Another study of this paper is how to include risk budgeting targets in risk sharing, which has not been discussed in the literature. Implementation issues are also discussed, and some bespoke algorithms are provided to identify such risk budgeting portfolios. Numerical experiments are performed for real-life financial data, and we explain the risk mitigation effect of our proposed portfolio. Specifically, financial risk budgeting portfolios with social responsibility targets are constructed.

1. Introduction

The idea of risk diversification can be traced back to the origins of probability theory, mainly to Bernoulli's 1954 paper (Bernoulli, 1954). Diversification has been reconsidered in a portfolio selection set-up by Markowitz in 1952 and it has been ever since the cornerstone of modern finance (Markowitz, 1999). Capital markets and insurance markets originated and evolved somehow differently, but recently, there is an enhanced commonality in the approaches taken to manage risk in the two markets (Cummins and Weiss, 2016; Hainaut, 2017; Gatzert et al., 2017). The integration was motivated and facilitated by optimization techniques applied to the decision making on constructing and managing a portfolio of financial assets or a portfolio of insurance liabilities. The focus has shifted from risk optimization to *Risk Budgeting/Parity* (Roncalli, 2013), since the latter aims to distribute the overall risk in a pre-defined way across all risks. *Risk Parity (RP)*, also known as *Equal Risk Contribution*,[1] is a special case of *Risk Budgeting (RB)*; for RP, all risks are allocated to have the same risk contribution, and represents the most common RB strategy; for RB, the risks contribute to the overall risk in pre-specified portions which are not necessarily to be equal.

The existing RB literature discusses RB/RP portfolios as a valuable alternative to the well-known portfolio selection methods that focus on reducing the overall risk of a portfolio. Notable work include Maillard et al. (2010) and other papers that have provided practical solutions for building RB/RP portfolios when the risk preferences are ordered by a specific risk measure; specifically, *variance* and *standard deviation* risk preferences are discussed in (Roncalli, 2013; Spinu, 2013; Bai et al., 2016), *Conditional-Value-at-Risk* and *expectiles* risk preferences are investigated in Mausser and Romanko (2018) and Bellini et al. (2021) respectively, while a larger class of risk preferences is investigated in Asimit et al. (2025). Such papers provide bespoke numerical methods for real-life implementations of RB/RP portfolios. Besides this strand of research, Roncalli and Weisang (2016) shows the connection between RB portfolios and risk factors, while Kaucic (2019) and Anis and Kwon (2022) consider portfolio construction under some cardinality constraints to achieve lower corresponding portfolio overhead. Recently, da Costa et al. (2023) and Cetingoz et al. (2024) discuss RB portfolios' existence and uniqueness for a large class of risk measures.

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[1] This should not be mistaken for *Equal Weighted (EW)* portfolio that is not a risk-based allocation strategy since each risk has the same weight in the portfolio, irrespective of the historical data.

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Portfolio selection is a risk management exercise that is more specific to financial assets, and it does not take into account any risk transfer from the (portfolio) risk holder to third parties; such risk shifting is known as *risk sharing (RS)*. Conceptually, RS equally applies to financial and insurance liabilities, though the RS literature tends to focus more on portfolios of insurance liabilities, since RS is an effective risk management exercise for insurance carriers to meet the regulatory requirements and their shareholders' objectives. Moreover, RS can not only improve capital allocation, but also stimulate further financial development (Pagano, 1993; Barattieri et al., 2020). RS problems have been widely studied in the literature (Ludkovski and Young, 2009; Asimit and Boonen, 2018; Asimit et al., 2020, 2021), and this strand of research is much related to intra-group risk transfers, in which an insurance group instructs its separate legal entities, i.e. risk holders, on sharing their liabilities (Asimit et al., 2013, 2016; Weber, 2018; Hamm et al., 2020).

Our contributions to the literature can be described as follows. First, we investigate RB strategies for one risk holder across many assets i) with or without risk contribution constraints on clusters of risks, and ii) with background or non-hedgeable risk. Then, we consider the RS problem between two risk holders with risk budgeting constraints. We provide theoretical results demonstrating that solutions for such problems exist for a large class of risk preferences, and we provide bespoke algorithms to identify these strategies in a practical context.

The paper is organized as follows. Section 2 provides the necessary background, while Section 3 contains the main theoretical results. Further, Section 4 provides extensive numerical exemplifications of our theoretical results, including a data analysis based on a unique database that helps us construct RB/RP portfolios with *socially responsible investment (SRI)* constraints. Section 5 summarizes with our conclusions. All proofs are relegated in Appendix A, while further details about the algorithm and data used in Section 4 are provided in Appendix B and Appendix C, respectively.

2. Problem formulation

Throughout this paper, the economy field is represented by $(\Omega, \mathcal{F}, \mathbb{P})$, an atomless probability space, endowed with $L^0 := L^0(\Omega, \mathcal{F}, \mathbb{P})$, the set of all real-valued random variables on this probability space. A generic random variable $Y \in L^0$ represents the future loss of a financial asset or an insurance liability. Let L^q, $q \in (0, \infty)$, be the set of random variables with finite q^{th} moment, and L^∞ be the set of bounded random variables.

A risk measure φ is a function that maps an element of L^0 to an extended real number, i.e., $\varphi : L^0 \to \overline{\Re} := [-\infty, \infty]$. We recall below some properties for a generic risk measure. These properties are well-known in the literature; an extensive introduction on risk measures could be found in Föllmer and Schied (2011).

(P1) *Homogeneity of order $\tau > 0$:* $\varphi(cY) = c^\tau \varphi(Y)$

for any $Y \in L^0$ and $c \geq 0$;

(P2) *Convexity:* $\varphi(aY_1 + (1-a)Y_2) \leq a\varphi(Y_1) + (1-a)\varphi(Y_2)$

for any $Y_1, Y_2 \in L^0$ and $a \in [0, 1]$;

(P3) *Shift invariance:* $\varphi(Y + c) = \varphi(Y)$

for any $Y \in L^0$ and $c \in \Re := (-\infty, \infty)$;

(P4) *Translation invariance:* $\varphi(Y + c) = \varphi(Y) + c$

for any $Y \in L^0$ and $c \in \Re$.

Three risk measures are often recalled in this paper, which are standard deviation, variance, and Conditional-Value-at-Risk (CVaR), given that they are well-defined in a set L^0, L^q, $q \in (0, \infty)$, or L^∞. For any $p \in (0, 1)$, CVaR at the probability level p is defined by $\mathrm{CVaR}_p(Y) := \inf_{t\in\Re} \left(t + \frac{1}{1-p} \mathbb{E}\left[(Y-t)_+\right] \right)$, where $(\cdot)_+ := \max(\cdot, 0)$ on $\Re$. Table 1 summarizes whether each of the risk measures satisfies a property above.

Table 1
Properties of standard deviation, variance, and Conditional-Value-at-Risk.

Risk measure φ	P1 (τ)	P2	P3	P4
Standard deviation	✓(1)	✓	✓	
Variance	✓(2)	✓	✓	
CVaR at level $p \in (0,1)$	✓(1)	✓		✓

In this paper, we study two RB problems, which are respectively formulated for one risk holder in Section 2.1, and for two risk holders in Section 2.2.

2.1. RB for one risk holder

We first define the RB problem of one risk holder (e.g. investor) that holds a portfolio of (e.g. assets with) $d \geq 2$ risks, i.e., $\mathbf{X} := \left(X_1, X_2, \ldots, X_d\right)^T$, where X_k, for $k \in \{1, 2, \ldots, d\}$, represents the future loss of the k^{th} risk. A portfolio allocation vector is denoted as $\boldsymbol{\alpha} := \left(\alpha_1, \alpha_2, \ldots, \alpha_d\right)^T$, where α_k represents the proportion of the k^{th} risk in the portfolio. Therefore, the aggregate position of the risk holder is given as $\boldsymbol{\alpha}^T \mathbf{X}$. Let φ be the risk measure that orders this risk holder's risk preferences, and thus the overall portfolio risk is $\mathcal{R}(\boldsymbol{\alpha}) := \varphi\left(\boldsymbol{\alpha}^T \mathbf{X}\right)$. We assume that the risk holder must hold these risks and does not short sell any of them, and hence $\boldsymbol{\alpha} \in \Re^d_{++} := (0, \infty)^d$. Since $\boldsymbol{\alpha}$ are the proportions, any admissible $\boldsymbol{\alpha} \in \Delta_d := \{\boldsymbol{\alpha} \in \Re^d_{++} : \mathbf{1}^T \boldsymbol{\alpha} = 1\}$.

Suppose that the risk measure φ is homogeneous of order $\tau > 0$. By Euler's Homogeneous Function Theorem, for any $\boldsymbol{\alpha} \in \Delta_d$,

$$\mathcal{R}(\boldsymbol{\alpha}) = \sum_{k=1}^{d} \mathcal{RC}_k(\boldsymbol{\alpha}), \quad \text{with} \quad \mathcal{RC}_k(\boldsymbol{\alpha}) := \frac{1}{\tau} \alpha_k \frac{\partial \mathcal{R}}{\partial \alpha_k}(\boldsymbol{\alpha}). \tag{2.1}$$

Therefore, $\mathcal{RC}_k(\boldsymbol{\alpha})$ represents the risk contribution by the k^{th} risk to the overall portfolio risk $\mathcal{R}(\boldsymbol{\alpha})$, and consequently $b_k(\boldsymbol{\alpha}) := \mathcal{RC}_k(\boldsymbol{\alpha}) / \mathcal{R}(\boldsymbol{\alpha})$ represents the proportion of such risk contribution to the overall risk. To summarize, given an admissible allocation vector $\boldsymbol{\alpha} \in \Delta_d$, the risk contribution proportion vector $\mathbf{b}(\boldsymbol{\alpha}) := \left(b_1(\boldsymbol{\alpha}), b_2(\boldsymbol{\alpha}), \ldots, b_d(\boldsymbol{\alpha})\right)^T \in \Delta_d$ is determined.

The RB problem of the risk holder is essentially an *inverse problem* of the above. Given a pre-specified risk contribution proportion vector $\mathbf{b} := \left(b_1, b_2, \ldots, b_d\right)^T \in \Delta_d$, the risk holder would like to determine an admissible allocation vector $\boldsymbol{\alpha}(\mathbf{b}) \in \Delta_d$ such that (2.1) holds. This is formalized in Definition 1.

Definition 1. Let $\mathbf{b} \in \Delta_d$. An allocation strategy $\boldsymbol{\alpha} \in \Delta_d$ is said to be RB if

$$\mathcal{RC}_k(\boldsymbol{\alpha}) = b_k \mathcal{R}(\boldsymbol{\alpha}) \text{ for all } k \in \{1, 2, \ldots, d\}, \tag{2.2}$$

where $\mathcal{RC}_k(\boldsymbol{\alpha})$ is given in (2.1).

For any $\mathbf{b} \in \Delta_d$, define $\mathcal{RB}(\mathbf{b}) := \{\boldsymbol{\alpha} \in \Delta_d : \boldsymbol{\alpha} \text{ is RB}\}$ as the set of RB portfolios. In particular, if $b_k = 1/d$, for all $k \in \{1, 2, \ldots, d\}$, a RB allocation strategy $\boldsymbol{\alpha} \in \mathcal{RB}((1/d)\mathbf{1})$ is said to be RP.

Note that the set of RB portfolios depends on not only the given risk contribution proportion vector $\mathbf{b}$, but also the risk holder's risk measure φ, which is assumed to be homogeneous of order $\tau > 0$, via the risk contribution terms $\mathcal{RC}_k(\cdot)$, for $k \in \{1, 2, \ldots, d\}$, and the overall portfolio risk $\mathcal{R}(\cdot)$. Table 2 summarizes the closed-form risk contributions for the three previously-mentioned risk measures, provided that these are well-defined.[2] From this table, it is not difficult to deduce that the

[2] For example, Var $\left(\boldsymbol{\alpha}^T \mathbf{X}\right) \neq 0$ is needed for the standard deviation case which happens only in the trivial case when $\mathbf{X}$ is degenerated. The risk contributions under the cases of variance and Conditional Value-at-Risk are always well-defined.

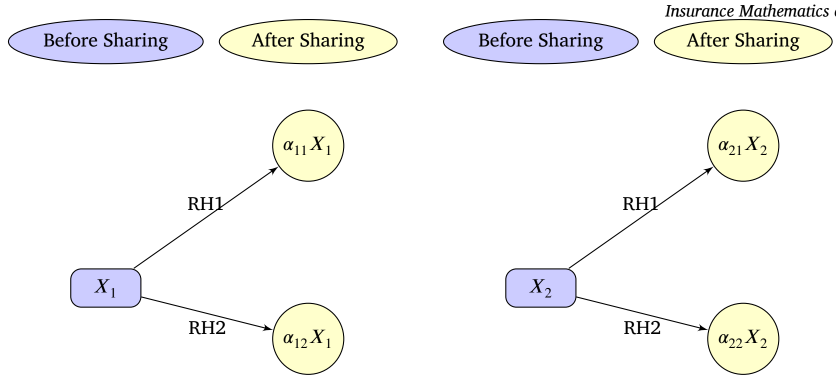

Fig. 1. Risk sharing flowchart, where RH1 and RH2 denote the first and second risk holders.

Table 2
Individual risk contributions for some well-known risk measures.

Risk measure φ	Individual risk contribution $\mathcal{RC}_k(\cdot)$
Standard deviation	$\mathrm{Cov}\left(\alpha_k X_k, \boldsymbol{\alpha}^T\mathbf{X}\right)/\sqrt{\mathrm{Var}\left(\boldsymbol{\alpha}^T\mathbf{X}\right)}$
Variance	$\mathrm{Cov}\left(\alpha_k X_k, \boldsymbol{\alpha}^T\mathbf{X}\right)$
CVaR at level $p \in (0,1)$	$\mathbb{E}\left[\alpha_k X_k \mid \boldsymbol{\alpha}^T\mathbf{X} \geq \mathrm{VaR}_p\left(\boldsymbol{\alpha}^T\mathbf{X}\right)\right]$

RB allocation strategies based on standard deviation and variance risk measures are equivalent.

Various numerical solutions have been proposed for computing RB portfolios in Definition 1. Spinu (2013) showed that the RB portfolios could be written as an efficient convex optimization problem, which is a much simpler numerical problem than solving the system of non-linear equations in (2.2), if the risk measure φ is given by the variance (or the standard deviation). The CVaR risk measure setting is discussed in Mausser and Romanko (2018), while Bellini et al. (2021) illustrates the expectile risk measure case; both papers provide computationally efficient algorithms that make the RB strategies to be implementable in practice, even for a relatively large number of risks.

2.2. RB for two risk holders

We now define a RS problem between two risk holders with RB constraints. Let $X_i \in L^0$ be the pre-transfer random loss for the i^{th} risk holder, where $i \in \{1,2\}$. In this setting, there are in total $d = 2$ risks.

The risk holders aim to share their risks. Let α_{ij} be the proportion of the loss X_i, which is held by the i^{th} risk holder to be transferred to the j^{th} risk holder, where $i, j \in \{1,2\}$; α_{ii} represents the proportion being retained by the i^{th} risk holder. Therefore, the post-transfer random loss held by the j^{th} risk holder is given by $\alpha_{1j}X_1 + \alpha_{2j}X_2$. A pictorial representation of the risk sharing is illustrated in Fig. 1.

For the j^{th} risk holder, where $j \in \{1,2\}$, its risk allocation vector is denoted as $\boldsymbol{\alpha}_j := \left(\alpha_{1j}, \alpha_{2j}\right)^T$. Note that $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2 \in [\mathbf{0}, \mathbf{1}]$ and $\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2 = \mathbf{1}$. This RS exercise aims to achieve a right balance of the risks between the risk holders; the price of the RS could be decided after the allocation is agreed. Therefore, the aggregate post-transfer risk positions for the first and the second risk holders are respectively $\boldsymbol{\alpha}_1^T\mathbf{X}$ and $\boldsymbol{\alpha}_2^T\mathbf{X}$, where the pre-transfer risk vector is denoted as $\mathbf{X} := (X_1, X_2)^T$.

Let φ_j be the risk measure that orders the risk preferences of the j^{th} risk holder, where $j \in \{1,2\}$. Then, the post-transfer overall risks for the first and the second risk holders are respectively $\mathcal{R}_1\left(\boldsymbol{\alpha}_1\right) := \varphi_1\left(\boldsymbol{\alpha}_1^T\mathbf{X}\right)$ and $\mathcal{R}_2\left(\boldsymbol{\alpha}_2\right) := \varphi_2\left(\boldsymbol{\alpha}_2^T\mathbf{X}\right)$. Assuming that both risk measures φ_1, φ_2 are homogeneous of order $\tau > 0$, the Euler's Homogeneous Function Theorem implies that, for any $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2 \in [\mathbf{0}, \mathbf{1}]$ such that $\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2 = \mathbf{1}$, and for each $j \in \{1,2\}$,

$$\mathcal{R}_j\left(\boldsymbol{\alpha}_j\right) = \sum_{i=1}^{2} \mathcal{RC}_{ij}(\boldsymbol{\alpha}_j), \quad \text{with} \quad \mathcal{RC}_{ij}\left(\boldsymbol{\alpha}_j\right) := \frac{1}{\tau}\alpha_{ij}\frac{\partial \mathcal{R}_j}{\partial \alpha_{ij}}\left(\boldsymbol{\alpha}_j\right). \tag{2.3}$$

Herein, $\mathcal{RC}_{ij}\left(\boldsymbol{\alpha}_j\right)$ is the risk contribution from the i^{th} risk holder to the post-transfer overall risk $\mathcal{R}_j\left(\boldsymbol{\alpha}_j\right)$ of the j^{th} risk holder; one could then, again, define $b_{ij}\left(\boldsymbol{\alpha}_j\right) := \mathcal{RC}_{ij}\left(\boldsymbol{\alpha}_j\right)/\mathcal{R}_j\left(\boldsymbol{\alpha}_j\right)$ be the proportion of such risk contribution to the j^{th} risk holder's overall risk. Given a pair of risk allocation vectors of the first and the second risk holders $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2 \in [\mathbf{0}, \mathbf{1}]$ such that $\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2 = \mathbf{1}$, the j^{th} risk holder's risk contribution proportion vector is given by $\mathbf{b}_j\left(\boldsymbol{\alpha}_j\right) := \left(b_{1j}\left(\boldsymbol{\alpha}_j\right), b_{2j}\left(\boldsymbol{\alpha}_j\right)\right)^T \in \Delta_2$.

The RS problem between the two risk holders with RB constraints is an *inverse problem* of the above, which is formalized in Definition 2, and is in line with Definition 1 for the case of one risk holder.

Definition 2. Let $\mathbf{b}_j := \left(b_{1j}, b_{2j}\right)^T \in \Delta_2$, for $j \in \{1,2\}$. A proportional risk sharing $\left(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2\right)$, that is $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2 \in [\mathbf{0}, \mathbf{1}]$ such that $\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2 = \mathbf{1}$, is said to be RB if

$$\mathcal{RC}_{i1}\left(\boldsymbol{\alpha}_1\right) = b_{i1}\mathcal{R}_1\left(\boldsymbol{\alpha}_1\right) \quad \text{and} \quad \mathcal{RC}_{i2}\left(\boldsymbol{\alpha}_2\right) = b_{i2}\mathcal{R}_2\left(\boldsymbol{\alpha}_2\right) \tag{2.4}$$

for all $i \in \{1,2\}$, where $\mathcal{RC}_{ij}\left(\boldsymbol{\alpha}_j\right)$ is given in (2.3).

3. Main theoretical results

This section provides the main theoretical results on the RB problems for one risk holder and two risk holders formulated in Sections 2.1 and 2.2. For the case of one risk holder, a clustered variant is provided in Section 3.2, and a variant with background risk or fixed sub-portfolios is discussed in Section 3.3.

3.1. Standard RB for one risk holder

The following theorem finds a RB portfolio in Definition 1. It extends Theorem 4 of Bellini et al. (2021), which is focused on expectiles for the risk measure φ. Our theorem's proof is different from that of Theorem 4.1 in Asimit et al. (2025). In particular, the proof herein, to show that a solution of the corresponding optimization problem is an interior point, provides new elements which are useful to show for later results on clustered RB/RP for one risk holder, as well as RB for two risk holders, i.e., risk sharing.

Theorem 3. *Let $\boldsymbol{b} \in \Delta_d$. Assume that the risk measure φ is homogeneous of order $\tau \geq 1$ and convex, and satisfies that*

$$\inf_{\boldsymbol{x} \in \Delta_d} \mathcal{R}(\boldsymbol{x}) > 0. \tag{3.1}$$

For any $\lambda > 0$, the following instance

$$\min_{\boldsymbol{x} \in \Re^d_{++}} \frac{1}{\tau}\mathcal{R}(\boldsymbol{x}) - \lambda\sum_{k=1}^{d} b_k \log x_k, \tag{3.2}$$

admits a unique solution, $\boldsymbol{x}^*(\lambda, \boldsymbol{b})$, that is an interior point of $\Re^d_{++}$. If $\mathcal{R}(\boldsymbol{x})$ is differentiable at $\boldsymbol{x}^*(1, \boldsymbol{b})$, then $\boldsymbol{\alpha}^*(\boldsymbol{b}) := \boldsymbol{x}^*(\lambda^*, \boldsymbol{b}) = \boldsymbol{x}^*(1, \boldsymbol{b})/\boldsymbol{1}^T\boldsymbol{x}^*(1, \boldsymbol{b}) \in \mathcal{RB}(\boldsymbol{b})$, where $\lambda^* = \left(\boldsymbol{1}^T\boldsymbol{x}^*(1, \boldsymbol{b})\right)^{-\tau}$.

While Theorem 3 solves a RB allocation strategy, an approximation for finding all RB strategies could be achieved by the *Least Squares Estimation (LSE)* formulation, which is defined in Roncalli (2013) as follows:

$$\min_{\boldsymbol{\alpha}\in\Delta_d'} \sum_{k=1}^{d} \left(\mathcal{RC}_k(\boldsymbol{\alpha}) - b_k\mathcal{R}(\boldsymbol{\alpha})\right)^2, \tag{3.3}$$

where $\Delta_d' := \{\boldsymbol{\alpha}\in\mathfrak{R}_+^d : \mathbf{1}^T\boldsymbol{\alpha} = 1\}$ is the standard unit d-simplex, and where $\mathfrak{R}_+^d := [0,\infty)^d$. Note that, if there exists a $k_0 \in \{1,2,\ldots,d\}$ such that $\alpha_{k_0} = 0$, then $\mathcal{RC}_{k_0}(\boldsymbol{\alpha}) = 0$, and in turn $b_{k_0} = 0$ which contradicts the fact that $\mathbf{b}\in\Delta_d$. Therefore, (3.3) yields the same set of solutions irrespective of the feasibility set choice, i.e., Δ_d or Δ_d', but Δ_d' is preferred in numerical optimization. Bai et al. (2016) shows that, when the risk measure φ is variance, (3.3) could be efficiently solved for approximating all RB allocation strategies. In the next section, we make use of the same LSE methodology on *clustered risk budgeting/parity* for one risk holder.

3.2. Clustered RB for one risk holder

A standard RB allocation assumes a pre-specified risk contribution proportion for each individual risk as explained earlier; in this case, the dimension of the pre-specified proportion vector $\mathbf{b}$ has to be the same as that of the risk vector $\mathbf{X}$. This standard RB formulation can be generalized to the so-called *Clustered Risk Budgeting (CRB)*, where the risks in $\mathbf{X}$ are first clustered, and then a pre-specified risk contribution proportion applies to each cluster instead of each individual risk; that is, the number of pre-specified proportions in $\mathbf{b}$ could be less than the number of risks in $\mathbf{X}$.

Definition 4. Let $l\in\{2,3,\ldots,d\}$ be the number of clusters for the individual risks, and let $\left\{\mathcal{I}^{(1)},\mathcal{I}^{(2)},\ldots,\mathcal{I}^{(l)}\right\}$ be an l-dimensional partition of $\mathcal{I}_d := \{1,2,\ldots,d\}$, i.e.,

$$\bigcup_{k=1}^{l}\mathcal{I}^{(k)} = \mathcal{I}_d, \text{ and } \mathcal{I}^{(k_1)}\bigcap\mathcal{I}^{(k_2)} = \varnothing$$

for all $k_1,k_2\in\{1,2,\ldots,l\}$ such that $k_1\neq k_2$.

Let $\mathbf{b}\in\Delta_l$ be the pre-specified risk contribution proportion vector applying to these clusters. An allocation strategy $\boldsymbol{\alpha}\in\Delta_d'$ is said to be CRB if

$$\sum_{i\in\mathcal{I}^{(k)}}\mathcal{RC}_i(\boldsymbol{\alpha}) = b_k\mathcal{R}(\boldsymbol{\alpha}) \text{ for all } k\in\{1,2,\ldots,l\}. \tag{3.4}$$

For any $l\in\{2,3,\ldots,d\}$, partition $\left\{\mathcal{I}^{(1)},\mathcal{I}^{(2)},\ldots,\mathcal{I}^{(l)}\right\}$ of $\mathcal{I}_d$, and $\mathbf{b}\in\Delta_l$, define $\mathcal{CRB}(\mathbf{b}) := \{\boldsymbol{\alpha}\in\Delta_d' : \boldsymbol{\alpha} \text{ is CRB}\}$ as the set of CRB portfolios. In particular, if $b_k = 1/l$, for all $k\in\{1,2,\ldots,l\}$, a CRB allocation strategy $\boldsymbol{\alpha}\in\mathcal{CRB}((1/l)\mathbf{1})$ is said to be *Clustered Risk Parity (CRP)*. Note that the set of CRB/CRP strategies actually depends on the choices for the number of clusters l and the clusters themselves; they are omitted for the sake of notational brevity. Clearly, the standard (non-clustered) RB/RP allocation in Sections 2.1 and 3.1 is achieved when $l = d$, which forces that each cluster holds only one individual risk.

Similar to (3.3), all CRB allocations could be approximated by the LSE formulation:

$$\min_{\boldsymbol{\alpha}\in\Delta_d'}\sum_{k=1}^{l}\left(\sum_{i\in\mathcal{I}^{(k)}}\mathcal{RC}_i(\boldsymbol{\alpha}) - b_k\mathcal{R}(\boldsymbol{\alpha})\right)^2. \tag{3.5}$$

Appendix B provides a numerical solution to solve (3.5) when the risk measure is given by the variance or the standard deviation, which is a slight extension of Algorithm 3 in Bai et al. (2016) that focuses only on CRP allocations. Solving (3.5) for other risk measures would require general optimization algorithms, since we do not have bespoke efficient algorithms for other (than variance or standard deviation) risk measures.

It is expected that, for any $\mathbf{b}\in\Delta_l$ with $l\in\{2,3,\ldots,d-1\}$, $\mathcal{CRB}(\mathbf{b})$ contains multiple, if not infinitely many, CRB portfolios, which are solved by definition of (3.4), or are approximated by the LSE formulation of (3.5); but, each of these strategies $\boldsymbol{\alpha}\in\mathcal{CRB}(\mathbf{b})$ would induce a possibly different overall portfolio risk $\mathcal{R}(\boldsymbol{\alpha})$ of the risk holder. Therefore, define $\boldsymbol{\alpha}^{**}(\mathbf{b}) = \arg\min_{\boldsymbol{\alpha}\in\mathcal{CRB}(\mathbf{b})}\mathcal{R}(\boldsymbol{\alpha})$ as the set of CRB allocation strategies which minimizes the overall portfolio risk. The following Example 5 illustrates how to find $\boldsymbol{\alpha}^{**}(\mathbf{b})$ in a simple setting.

Example 5. Assume that X_1, X_2, X_3 are three independent risks ($d = 3$), each with a unit variance, and the risk measure φ is given by the variance. The risk holder aims to find CRB (more precisely, CRP) strategies with two clusters, namely $\left(X_1, X_2\right)$ and X_3; that is, $l = 2$, $\mathcal{I}^{(1)} = \{1,2\}$, $\mathcal{I}^{(2)} = \{3\}$, $b_1 = b_2 = 1/2$. Therefore, by (3.4) and after simplifications, a CRP allocation strategy $\boldsymbol{\alpha}\in\Delta_3'$ satisfies $\alpha_1^2+\alpha_2^2 = \alpha_3^2$; in turn, the CRP set is given by:

$$\mathcal{CRB}((1/2,1/2)^T) = \left\{\boldsymbol{\alpha}(\xi) : \boldsymbol{\alpha}(\xi) = \left(\xi, \frac{1-2\xi}{2-2\xi}, 1-\xi-\frac{1-2\xi}{2-2\xi}\right)^T, \xi\in[0,1/2]\right\}. \tag{3.6}$$

The minimal portfolio variance within the CRP set $\mathcal{CRB}\left((1/2,1/2)^T\right)$ is obtained when $\xi^* = 1-\frac{\sqrt{2}}{2}$, since

$$\begin{aligned}\min_{\boldsymbol{\alpha}\in\mathcal{CRB}((1/2,1/2)^T)}\mathcal{R}(\boldsymbol{\alpha}) &= \min_{\boldsymbol{\alpha}\in\mathcal{CRB}((1/2,1/2)^T)}\left(\alpha_1^2+\alpha_2^2+\alpha_3^2\right)\\ &= \min_{\xi\in[0,1/2]}\left(\xi^2+\left(\frac{1-2\xi}{2-2\xi}\right)^2+\left(1-\xi-\frac{1-2\xi}{2-2\xi}\right)^2\right),\end{aligned}$$

and hence, $\boldsymbol{\alpha}^{**}((1/2,1/2)^T) = \boldsymbol{\alpha}(\xi^*) = \left(1-\frac{\sqrt{2}}{2}, 1-\frac{\sqrt{2}}{2}, \sqrt{2}-1\right)^T$, which is an element of $\mathcal{CRB}((1/2,1/2)^T)$ in (3.6), with $\mathcal{R}(\boldsymbol{\alpha}^{**}((1/2,1/2)^T)) = 6-4\sqrt{2}$.

The analysis of CRP and their minimal portfolio risk is also applicable in the case of dependent risks. For simplicity of the illustration, assume that X_1, X_2, X_3 have the same value for their pairwise correlations; for a valid correlation matrix, the value, being denoted as ρ, needs to lie between $-\frac{1}{2}$ and 1. When $\rho = -\frac{1}{2}$, i.e., the risks are most negatively correlated,

$$\mathcal{CRB}((1/2,1/2)^T) = \left\{\boldsymbol{\alpha}(\xi) : \boldsymbol{\alpha}(\xi) = \left(\xi, \frac{1-2\xi}{2-3\xi}, 1-\xi-\frac{1-2\xi}{2-3\xi}\right)^T, \xi\in[0,1/2]\right\},$$

and $\boldsymbol{\alpha}^{**}((1/2,1/2)^T) = \boldsymbol{\alpha}(1/3) = (1/3,1/3,1/3)^T$, with $\mathcal{R}(\boldsymbol{\alpha}^{**}((1/2,1/2)^T)) = 0$. When $\rho = 1$, i.e., the risks are most positively correlated,

$$\mathcal{CRB}((1/2,1/2)^T) = \left\{\boldsymbol{\alpha}(\xi) : \boldsymbol{\alpha}(\xi) = \left(\xi, \frac{1}{2}-\xi, \frac{1}{2}\right), \xi\in[0,1/2]\right\},$$

and $\boldsymbol{\alpha}^{**}((1/2,1/2)^T) = \boldsymbol{\alpha}(\xi)$, for any $\xi\in\left[0,1/2\right]$, with $\mathcal{R}(\boldsymbol{\alpha}^{**}((1/2,1/2)^T)) = 1$.

The CRP sets and their minimal portfolio risk, in terms of variance, are different among the cases with different dependence structure of the risks. When the risks are most negatively correlated, with $\rho = -\frac{1}{2}$, it can be easily shown that the CRP allocation to the third risk X_3, which is alone in the second cluster, can be less than those to the first and second risks X_1, X_2 respectively, which are in the first cluster. When the risks are independent or most positively correlated, with $\rho = 0$ or $\rho = 1$, it can also be easily shown that the CRP allocation to the third risk is always larger than those to the first and second risks X_1, X_2 respectively; in particular, when the risks are positively correlated, with $\rho = 1$, the CRP allocation to the third risk X_3, which is $1/2$, matches the pre-specified risk contribution of the second cluster, which comprises only the third risk, while the CRP allocation to the first and second risks X_1, X_2 respectively are arbitrary as long as their allocations add up to $1/2$, which matches the pre-specified risk contribution of the first cluster, which comprises the first and second risks.

Within the CRP allocation, the optimal allocation is unique for most negatively correlated or independent risks, but it is not unique for most positively correlated risks. In fact, when risks are most positively correlated, no diversification is possible, and thus any allocations in Δ_3' would yield a unit of risk, in terms of variance, since all the three risks have a unit of risk. Note also that, on the one hand, when the risks are independent, the optimal allocation with CRP is no longer equally weighted, but the optimal allocation without CRP is indeed equally weighted; recall that the aim of (C)RB is not to minimize the overall portfolio risk. On the other hand, when the risks are most negatively correlated, the equally weighted allocation is the optimal strategy, with or without CRP.

As illustrated by the example, in order to minimize the overall portfolio risk among CRB/CRP allocation strategies, one has to identify the full set of $\mathcal{CRB}(\mathbf{b})$, for $\mathbf{b} \in \Delta_l$. In a general setting, this often requires to solve the LSE formulation of (3.5), which relies on general but inefficient optimization algorithms when the risk measure is not given by the variance or the standard deviation. Another way of characterizing the full set of CRB/CRP strategies is to identify a parametric set of RB strategies. This is given in the following proposition.

Proposition 6. *For any $l \in \{2, 3, \dots, d\}$, partition $\{\mathcal{I}^{(1)}, \mathcal{I}^{(2)}, \dots, \mathcal{I}^{(l)}\}$ of $\mathcal{I}_d$, and $\boldsymbol{b} \in \Delta_l$,*

$$\mathcal{CRB}(\boldsymbol{b}) = \bigcup_{\boldsymbol{a} \in \mathcal{B}(\boldsymbol{b})} \mathcal{RB}(\boldsymbol{a}),$$

where $\mathcal{B}(\boldsymbol{b}) := \left\{\boldsymbol{a} \in \Delta_d' : \sum_{i \in \mathcal{I}^{(k)}} a_i = b_k, \text{ for all } k \in \{1, 2, \dots, l\}\right\}$.

RB strategies with $\mathbf{a} \in \Delta_d' \setminus \Delta_d$ should be understood as standard RB strategies, in Sections 2.1 and 3.1, with a number of individual risks of $d' = d - d_0$, where d_0 is the number of zero-valued elements in the vector $\mathbf{a}$, i.e., the risk set does not include the individual risks with zero risk contribution.

Proposition 6 allows us to solve for CRB allocation strategies that minimize the overall portfolio risk in another way. In the following, let $l \in \{2, 3, \dots, d\}$, partition $\{\mathcal{I}^{(1)}, \mathcal{I}^{(2)}, \dots, \mathcal{I}^{(l)}\}$ of $\mathcal{I}_d$, and $\mathbf{b} \in \Delta_l$. By Proposition 6,

$$\min_{\boldsymbol{\alpha} \in \mathcal{CRB}(\mathbf{b})} \mathcal{R}(\boldsymbol{\alpha}) = \min_{\boldsymbol{\alpha} \in \bigcup_{\mathbf{a} \in \mathcal{B}(\mathbf{b})} \mathcal{RB}(\mathbf{a})} \mathcal{R}(\boldsymbol{\alpha}) = \min_{\mathbf{a} \in \mathcal{B}(\mathbf{b})} \min_{\boldsymbol{\alpha} \in \mathcal{RB}(\mathbf{a})} \mathcal{R}(\boldsymbol{\alpha}).$$

For each $\mathbf{a} \in \mathcal{B}(\mathbf{b})$, define $\boldsymbol{\alpha}^{***}(\mathbf{a}) = \arg\min_{\boldsymbol{\alpha} \in \mathcal{RB}(\mathbf{a})} \mathcal{R}(\boldsymbol{\alpha})$; if $\mathcal{RB}(\mathbf{a})$ is a singleton, $\boldsymbol{\alpha}^{***}(\mathbf{a})$ must be given by Theorem 3; if $\mathcal{RB}(\mathbf{a})$ is not a singleton, the LSE formulation of (3.3) is needed to solve for all RB strategies with respect to the risk contribution proportion vector $\mathbf{a}$. With a slight abuse of its notation, we thus have that $\min_{\boldsymbol{\alpha} \in \mathcal{CRB}(\mathbf{b})} \mathcal{R}(\boldsymbol{\alpha}) = \min_{\mathbf{a} \in \mathcal{B}(\mathbf{b})} \mathcal{R}(\boldsymbol{\alpha}^{***}(\mathbf{a}))$. Since $\mathcal{B}(\mathbf{b})$ is a compact set, and since $\mathcal{R}$ is a continuous mapping (as φ is homogeneous), if $\boldsymbol{\alpha}^{***}(\mathbf{a})$ is continuous in $\mathbf{a} \in \mathcal{B}(\mathbf{b})$,[3] by the Weierstrass' Theorem, define $\mathbf{a}^{***}(\mathbf{b}) = \arg\min_{\mathbf{a} \in \mathcal{B}(\mathbf{b})} \mathcal{R}(\boldsymbol{\alpha}^{***}(\mathbf{a}))$. Therefore,

$$\boldsymbol{\alpha}^{**}(\mathbf{b}) = \arg\min_{\boldsymbol{\alpha} \in \mathcal{CRB}(\mathbf{b})} \mathcal{R}(\boldsymbol{\alpha}) = \boldsymbol{\alpha}^{***}(\mathbf{a}^{***}(\mathbf{b})).$$

Yet, this alternative way could still involve inefficient steps, such as the need of solving the LSE formulation of (3.3) when $\mathcal{RB}(\mathbf{a})$ is not a singleton for some $\mathbf{a} \in \mathcal{B}(\mathbf{b})$, as well as the need of solving $\boldsymbol{\alpha}^*(\mathbf{a})$ for *all* $\mathbf{a} \in \mathcal{B}(\mathbf{b})$ even though the method by Theorem 3 is efficient when $\mathcal{RB}(\mathbf{a})$ is a singleton for some $\mathbf{a} \in \mathcal{B}(\mathbf{b})$.

[3] This can show to be true if $\boldsymbol{\alpha}^{***}(\mathbf{a})$ is given by Theorem 3, as it is solved via (3.2).

Such equivalence is helpful, though, to develop an efficient algorithm to *approximate* CRB/CRP allocation strategies that minimize the overall portfolio risk, in the sense that the overall portfolio risk of an *approximated* strategy serves as an upper bound of the minimal overall portfolio risk among CRB/CRP strategies. To this end, for any $\mathbf{a} \in \mathcal{B}(\mathbf{b})$,

$$\min_{\boldsymbol{\alpha} \in \mathcal{RB}(\mathbf{a})} \mathcal{R}(\boldsymbol{\alpha}) \leq \mathcal{R}(\boldsymbol{\alpha}^*(\mathbf{a})),$$

where $\boldsymbol{\alpha}^*(\mathbf{a}) \in \mathcal{RB}(\mathbf{a})$ (for example, being solved by Theorem 3), which implies that, for any $\mathbf{a} \in \mathcal{B}(\mathbf{b})$,

$$\min_{\boldsymbol{\alpha} \in \mathcal{CRB}(\mathbf{b})} \mathcal{R}(\boldsymbol{\alpha}) = \min_{\mathbf{a} \in \mathcal{B}(\mathbf{b})} \min_{\boldsymbol{\alpha} \in \mathcal{RB}(\mathbf{a})} \mathcal{R}(\boldsymbol{\alpha}) \leq \mathcal{R}(\boldsymbol{\alpha}^*(\mathbf{a})).$$

Since $\mathcal{B}(\mathbf{b})$ is a compact set, and since $\mathcal{R}$ is a continuous mapping (as φ is homogeneous), by the Weierstrass' Theorem, define $\mathbf{a}^*(\mathbf{b}) = \arg\min_{\mathbf{a} \in \mathcal{B}(\mathbf{b})} \mathcal{R}(\mathbf{a})$. Since, by definition, $\mathbf{a}^*(\mathbf{b}) \in \mathcal{B}(\mathbf{b})$, we have

$$\min_{\boldsymbol{\alpha} \in \mathcal{CRB}(\mathbf{b})} \mathcal{R}(\boldsymbol{\alpha}) = \min_{\mathbf{a} \in \mathcal{B}(\mathbf{b})} \min_{\boldsymbol{\alpha} \in \mathcal{RB}(\mathbf{a})} \mathcal{R}(\boldsymbol{\alpha}) \leq \mathcal{R}(\boldsymbol{\alpha}^*(\mathbf{a}^*(\mathbf{b}))).$$

The motivation of choosing this particular $\mathbf{a}^*(\mathbf{b}) \in \mathcal{B}(\mathbf{b}) \subseteq \Delta_d'$ is due to Theorem 4.1 c) of Asimit et al. (2025), which states that, for any $\mathbf{a} \in \Delta_d'$, $\mathcal{R}(\boldsymbol{\alpha}^*(\mathbf{a})) \leq \mathcal{R}(\mathbf{a})$. Hence, for any $\mathbf{a} \in \mathcal{B}(\mathbf{b})$,

$$\min_{\boldsymbol{\alpha} \in \mathcal{CRB}(\mathbf{b})} \mathcal{R}(\boldsymbol{\alpha}) = \min_{\mathbf{a} \in \mathcal{B}(\mathbf{b})} \min_{\boldsymbol{\alpha} \in \mathcal{RB}(\mathbf{a})} \mathcal{R}(\boldsymbol{\alpha}) \leq \mathcal{R}(\boldsymbol{\alpha}^*(\mathbf{a})) \leq \mathcal{R}(\mathbf{a}).$$

Since, in particular, $\mathcal{R}(\boldsymbol{\alpha}^*(\mathbf{a}^*(\mathbf{b}))) \leq \mathcal{R}(\mathbf{a}^*(\mathbf{b}))$, for any $\mathbf{a} \in \mathcal{B}(\mathbf{b})$,

$$\begin{aligned}\min_{\boldsymbol{\alpha} \in \mathcal{CRB}(\mathbf{b})} \mathcal{R}(\boldsymbol{\alpha}) &= \min_{\mathbf{a} \in \mathcal{B}(\mathbf{b})} \min_{\boldsymbol{\alpha} \in \mathcal{RB}(\mathbf{a})} \mathcal{R}(\boldsymbol{\alpha}) \leq \mathcal{R}(\boldsymbol{\alpha}^*(\mathbf{a}^*(\mathbf{b}))) \leq \mathcal{R}(\mathbf{a}^*(\mathbf{b})) \\ &= \min_{\mathbf{a} \in \mathcal{B}(\mathbf{b})} \mathcal{R}(\mathbf{a}) \leq \mathcal{R}(\mathbf{a}).\end{aligned}$$

That is, the choice of $\mathbf{a}^*(\mathbf{b}) \in \mathcal{B}(\mathbf{b})$ is to ensure that the overall portfolio risk of the approximated CRB strategy, $\mathcal{R}(\boldsymbol{\alpha}^*(\mathbf{a}^*(\mathbf{b})))$, gets closer to the minimal overall portfolio risk via the tightest upper bound, $\min_{\mathbf{a} \in \mathcal{B}(\mathbf{b})} \mathcal{R}(\mathbf{a})$, for itself.

The approximated CRB allocation strategy, $\boldsymbol{\alpha}^*(\mathbf{a}^*(\mathbf{b}))$, is thus the *worst case*, since the minimal overall portfolio risk among CRB strategies cannot exceed this upper bound. We call such approximated CRB allocation strategy as the *worst-case-CRB* (*WC-CRB*). Note that WC-CRB is denoted as WC-CRP whenever $\mathbf{b} = (1/l)\mathbf{1}$. Algorithm 1 summarizes the two-step procedures to obtain the WC-CRB strategy, as already outlined above.

Algorithm 1: CRB algorithm for solving (3.4) by approximation.

Result: Finding the WC-CRB portfolio to approximate $\boldsymbol{\alpha}^{**}(\mathbf{b})$.
Step 1): Find the risk contribution proportion vector, $\mathbf{a}^*(\mathbf{b})$, that minimizes the overall portfolio risk subject to clusters' constraints, i.e.,

$$\mathbf{a}^*(\mathbf{b}) = \arg\min_{\mathbf{a} \in \mathcal{B}(\mathbf{b})} \mathcal{R}(\mathbf{a}).$$

Step 2): Find $\boldsymbol{\alpha}^*(\mathbf{a}^*(\mathbf{b}))$, i.e., the RB portfolio based on the proportion vector, $\mathbf{a}^*(\mathbf{b})$, via (3.2), (3.3), or any other standard RB computational procedures via numerical optimization.

Algorithm 1 is efficient since a RB portfolio needs to be solved in the step 2), only for one proportion vector from the step 1). Also, for the RB portfolio, one can solve the convex instance as in (3.2) instead of solving the non-convex LSE formulation as in (3.3).

Applying the two-step procedures to Example 5 of the case of independent risks, the step 1) implies solving

$$\min_{\mathbf{a} \in \mathcal{B}((1/2,1/2)^T)} \mathcal{R}((a_1, a_2, 1/2)^T) = \min_{\mathbf{a} \in \mathcal{B}((1/2,1/2)^T)} \left(a_1^2 + a_2^2 + (1/2)^2\right),$$

which is obviously solved by $a_1^*((1/2, 1/2)^T) = a_2^*((1/2, 1/2)^T) = 1/4$ and $a_3^*((1/2, 1/2)^T) = 1/2$; herein, $a_3^*((1/2, 1/2)^T) = b_2 = 1/2$ since, recall that in the example, the second cluster $\mathcal{I}^{(2)} = \{3\}$, which contains only the third risk X_3. The step 2) requires finding the RB portfolio with respect to the risk contribution proportion vector $\mathbf{a}^*(\mathbf{b}) = (1/4, 1/4, 1/2)^T$; that is, $\boldsymbol{\alpha}^*(\mathbf{a}^*(\mathbf{b}))$. It is not difficult to work out that $\boldsymbol{\alpha}^*(\mathbf{a}^*(\mathbf{b})) = \boldsymbol{\alpha}(\xi^*) = \boldsymbol{\alpha}^{**}((1/2, 1/2)^T) = \left(1-\frac{\sqrt{2}}{2}, 1-\frac{\sqrt{2}}{2}, \sqrt{2}-1\right)^T$. That is, in this case, the WC-CRP even minimizes the overall portfolio risk among all CRP allocation strategies.

3.3. RB with background risk or fixed sub-portfolios for one risk holder

The background risk setting requires allocating the risk vector $\mathbf{X} = (X_1, X_2, \ldots, X_d)$ when a non-hedgeable risk (for a financial risk portfolio) or non-insurable risk (for an insurance risk portfolio) Z is present. For example, an investment house focuses on structured finance products covering credit cards, student loans, Small and Medium Enterprise (SME) loans, and so on. Each LoB has a specific risk that is internally measured, and the investment house funds the purchase of these asset loans by borrowing funds from the market. The market funding risk affects all LoBs and the investment house cannot hedge this risk. Likewise, a maritime insurance portfolio – e.g. a corporate account that includes a variety of insurance sub-portfolios such as hull, cargo, and protection & indemnity insurance, breakdown risk, business interruption risk, personal accidents, etc. – such that reputational perils of any kind are not included in the individual coverages. The losses due to such reputational perils are significantly associated with individual losses covered by this maritime insurance portfolio, and the reputational peril represents the (non-insurable) background risk for this bespoke portfolio. In general, for any $\boldsymbol{\alpha} \in \Delta_d$, the overall portfolio risk of the risk holder is then given by $\mathcal{R}(\boldsymbol{\alpha}) = \varphi\big(Z + \boldsymbol{\alpha}^T\mathbf{X}\big)$. For any risk contribution proportion vector $\mathbf{b} \in \Re^d_{++}$ such that $\mathbf{1}^T\mathbf{b} < 1$, an allocation strategy $\boldsymbol{\alpha} \in \Delta_d$ is said to be RB with background risk if

$$\mathcal{RC}_k(\boldsymbol{\alpha}) = b_k\mathcal{R}(\boldsymbol{\alpha}) \text{ for all } k \in \{1, 2, \ldots, d\}, \tag{3.7}$$

where, herein, $\mathcal{RC}_k(\boldsymbol{\alpha}) := \frac{1}{\tau}\alpha_k\frac{\partial\mathcal{R}}{\partial\alpha_k}(\boldsymbol{\alpha}) = \frac{1}{\tau}\alpha_k\frac{\partial\varphi}{\partial\alpha_k}\left(Z + \boldsymbol{\alpha}^T\mathbf{X}\right)$. Note that the risk measure φ is tacitly assumed to be homogeneous of order $\tau \geq 1$ in (3.7).

The fixed sub-portfolio setting requires allocating the risk vector $(X_1, X_2, \ldots, X_d, X_{d+1}, X_{d+2}, \ldots, X_{d+d_1})$, where $d_1 \geq 1$, only for the first d risks $\mathbf{X} = (X_1, X_2, \ldots, X_d)$, as the risk holder has already fixed an allocation strategy for the remaining d_1 risks $\tilde{\mathbf{X}} = (X_{d+1}, X_{d+2}, \ldots, X_{d+d_1})$. Formally, for any $(\boldsymbol{\alpha}, \tilde{\boldsymbol{\alpha}}) \in \Delta_{d+d_1}$, the overall portfolio risk of the risk holder is given by $\mathcal{R}(\boldsymbol{\alpha}, \tilde{\boldsymbol{\alpha}}) = \varphi\big(\boldsymbol{\alpha}^T\mathbf{X} + \tilde{\boldsymbol{\alpha}}^T\tilde{\mathbf{X}}\big)$. For any risk contribution proportion vector $\mathbf{b} \in \Re^d_{++}$ such that $\mathbf{1}^T\mathbf{b} < 1$, and for any fixed sub-portfolio $\tilde{\boldsymbol{\alpha}} \in \Re^{d_1}_{++}$ such that $\mathbf{1}^T\tilde{\boldsymbol{\alpha}} < 1$, an allocation strategy $\boldsymbol{\alpha} \in \Re^d_{++}$, such that $\mathbf{1}^T\boldsymbol{\alpha} < 1$ and $(\boldsymbol{\alpha}, \tilde{\boldsymbol{\alpha}}) \in \Delta_{d+d_1}$ (in particular, $\mathbf{1}^T\boldsymbol{\alpha} + \mathbf{1}^T\tilde{\boldsymbol{\alpha}} = 1$), is said to be RB with fixed sub-portfolio if

$$\mathcal{RC}_k(\boldsymbol{\alpha}, \tilde{\boldsymbol{\alpha}}) = b_k\mathcal{R}(\boldsymbol{\alpha}, \tilde{\boldsymbol{\alpha}}) \text{ for all } k \in \{1, 2, \ldots, d\}, \tag{3.8}$$

where, herein, $\mathcal{RC}_k(\boldsymbol{\alpha}, \tilde{\boldsymbol{\alpha}}) := \frac{1}{\tau}\alpha_k\frac{\partial\mathcal{R}}{\partial\alpha_k}(\boldsymbol{\alpha}, \tilde{\boldsymbol{\alpha}}) = \frac{1}{\tau}\alpha_k\frac{\partial\varphi}{\partial\alpha_k}\left(\boldsymbol{\alpha}^T\mathbf{X} + \tilde{\boldsymbol{\alpha}}^T\tilde{\mathbf{X}}\right)$. Note that $\tilde{\boldsymbol{\alpha}}$ is the a priori fixed sub-portfolio allocation vector for the risks $\tilde{\mathbf{X}}$ by the risk holder, which means that we only need to solve (3.8) in $\boldsymbol{\alpha}$.

Clearly, solving (3.8) is equivalent to solving (3.7) with the background risk $Z = \frac{\tilde{\boldsymbol{\alpha}}^T\tilde{\mathbf{X}}}{1-\mathbf{1}^T\tilde{\boldsymbol{\alpha}}}$ and standardized weights $\frac{\boldsymbol{\alpha}}{1-\mathbf{1}^T\tilde{\boldsymbol{\alpha}}}$. This clarifies why the two settings are mathematically equivalent, and from now on, we only focus on the RB portfolios with background risk. The following Theorem 7 solves a RB portfolio with background risk.

Theorem 7. *Let $\boldsymbol{b} \in \Re^d_{++}$ such that $\boldsymbol{1}^T\boldsymbol{b} < 1$. Assume that the risk measure φ is homogeneous of order $\tau \geq 1$ and convex, and satisfies that $\inf_{\boldsymbol{x}\in\Delta_d}\mathcal{R}(\boldsymbol{x}) > 0$, where $\mathcal{R}(\boldsymbol{x}) = \varphi\big(Z + \boldsymbol{x}^T\boldsymbol{X}\big)$. For any $\lambda > 0$, the following instance*

$$\min_{\boldsymbol{x}\in\Re^d_{++}} \frac{1}{\tau}\mathcal{R}(\boldsymbol{x}) - \lambda\sum_{k=1}^{d} b_k\log x_k, \tag{3.9}$$

admits a unique solution, $\boldsymbol{x}^*(\lambda, \boldsymbol{b})$, that is an interior point of $\Re^d_{++}$. If $\mathcal{R}(\boldsymbol{x})$ is differentiable at $\boldsymbol{x}^*(\lambda, \boldsymbol{b})$ for some $\lambda > 0$, then $\boldsymbol{x}^*(\lambda, \boldsymbol{b})$ satisfies (3.7), where the sum-to-unity constraint $\boldsymbol{1}^T\boldsymbol{x}^*(\lambda, \boldsymbol{b}) = 1$ is removed.

The main technical difference between Theorem 3 and Theorem 7 is the lack of homogeneity of the overall portfolio risk $\mathcal{R}$ in (3.9). Therefore, λ acts as a tuning parameter in Theorem 7; that is, we need to find $\lambda > 0$ such that $\mathbf{1}^T\mathbf{x}^*(\lambda, \mathbf{b}) = 1$, and this solution is denoted as $\lambda^*(\mathbf{b})$ if this solution exists (as its existence can not be guaranteed). Now, if $\lambda^*(\mathbf{b})$ exists, then $\mathbf{x}^*(\lambda^*(\mathbf{b}), \mathbf{b})$ solves (3.7). In a nutshell, RB portfolios satisfying either (3.7) or (3.8) could be found by iteratively solving (3.9) through the tuning parameter λ.

3.4. RS for two risk holders

Consider now the setting in Definition 2, where the RS problem for two risk holders is solved via RB. The following Theorem 8 tells us how to solve the RS problem for the two risk holders via RB.

Theorem 8. *Let $\boldsymbol{b}_1, \boldsymbol{b}_2 \in \Delta_2$. Assume further that the risk measures φ_1, φ_2 are homogeneous of order $\tau_1, \tau_2 \geq 1$ and convex, and satisfy that $\inf_{\boldsymbol{x}\in\Delta_2}\mathcal{R}_1(\boldsymbol{x}) > 0$ and $\inf_{\boldsymbol{x}\in\Delta_2}\mathcal{R}_2(\boldsymbol{x}) > 0$, where*

$$\mathcal{R}_1(x_{11}, x_{21}) = \varphi_1(x_{11}X_1 + x_{21}X_2) \quad \text{and}$$
$$\mathcal{R}_2(x_{12}, x_{22}) = \varphi_2(x_{12}X_1 + x_{22}X_2).$$

Then, for any $\lambda_1, \lambda_2 > 0$,

$$\min_{(x_{11},x_{21})\in\Re^2_{++}} \frac{1}{\tau_1}\mathcal{R}_1(x_{11}, x_{21}) - \lambda_1(b_{11}\log x_{11} + b_{21}\log x_{21}) \tag{3.10}$$

and

$$\min_{(x_{12},x_{22})\in\Re^2_{++}} \frac{1}{\tau_2}\mathcal{R}_2(x_{12}, x_{22}) - \lambda_2(b_{12}\log x_{12} + b_{22}\log x_{22}) \tag{3.11}$$

admit a unique solution, $\boldsymbol{x}^*(\lambda_1, \boldsymbol{b}_1; \varphi_1)$ and $\boldsymbol{x}^*(\lambda_2, \boldsymbol{b}_2; \varphi_2)$, respectively, that are interior points of the feasibility set.

i) If $\mathcal{R}_1$ and $\mathcal{R}_2$ are differentiable at $\boldsymbol{x}^*(1, \boldsymbol{b}_1; \varphi_1)$ and $\boldsymbol{x}^*(1, \boldsymbol{b}_2; \varphi_2)$, respectively, then $(\alpha^*_{11}, \alpha^*_{21})$ and $(\alpha^*_{12}, \alpha^*_{22})$ solve (2.4), respectively, where $\alpha^*_{ij} = t^*_j x^*_i(1, \boldsymbol{b}_j; \varphi_j)$ for all $i, j \in \{1, 2\}$ and

$$\begin{cases} t^*_1 = \dfrac{x^*_2(1,\boldsymbol{b}_2;\varphi_2) - x^*_1(1,\boldsymbol{b}_2;\varphi_2)}{x^*_1(1,\boldsymbol{b}_1;\varphi_1)x^*_2(1,\boldsymbol{b}_2;\varphi_2) - x^*_2(1,\boldsymbol{b}_1;\varphi_1)x^*_1(1,\boldsymbol{b}_2;\varphi_2)} \\ t^*_2 = \dfrac{x^*_1(1,\boldsymbol{b}_1;\varphi_1) - x^*_2(1,\boldsymbol{b}_1;\varphi_1)}{x^*_1(1,\boldsymbol{b}_1;\varphi_1)x^*_2(1,\boldsymbol{b}_2;\varphi_2) - x^*_2(1,\boldsymbol{b}_1;\varphi_1)x^*_1(1,\boldsymbol{b}_2;\varphi_2)} \end{cases} \tag{3.12}$$

whenever

$$\begin{aligned} &\big(x^*_1(1, \boldsymbol{b}_1; \varphi_1) - x^*_2(1, \boldsymbol{b}_1; \varphi_1)\big) \\ &\times \big(x^*_1(1, \boldsymbol{b}_2; \varphi_2) - x^*_2(1, \boldsymbol{b}_2; \varphi_2)\big) < 0. \end{aligned} \tag{3.13}$$

ii) Assume that $\boldsymbol{b}_1 = \boldsymbol{b}_2$, $\varphi_1 = \varphi_2$, and the fact that $\mathcal{R}_1$ is differentiable at $\boldsymbol{x}^*(1, \boldsymbol{b}_1; \varphi_1)$. If $x^*_1(1, \boldsymbol{b}_1; \varphi_1) = x^*_2(1, \boldsymbol{b}_1; \varphi_1)$, then $(\alpha^*_{11}, \alpha^*_{21}) = (\xi, \xi)$ and $(\alpha^*_{12}, \alpha^*_{22}) = (1-\xi, 1-\xi)$ are also solutions of (2.4), respectively, for any $\xi \in (0, 1)$.

iii) Let $\lambda^*_1, \lambda^*_2 > 0$ such that $(\alpha^*_{11}, \alpha^*_{21}) = \boldsymbol{x}^*(\lambda^*_1, \boldsymbol{b}_1; \varphi_1)$, $(\alpha^*_{12}, \alpha^*_{22}) = \boldsymbol{x}^*(\lambda^*_2, \boldsymbol{b}_2; \varphi_2)$ and $\alpha^*_{11} + \alpha^*_{12} = \alpha^*_{21} + \alpha^*_{22} = 1$, then

$$\begin{aligned} &\frac{1}{\tau_1\lambda^*_1}\big(\mathcal{R}_1(\alpha^*_{11}, \alpha^*_{21}) - \mathcal{R}_1(x_{11}, x_{21})\big) \\ &\quad + \frac{1}{\tau_2\lambda^*_2}\big(\mathcal{R}_2(\alpha^*_{12}, \alpha^*_{22}) - \mathcal{R}_2(x_{12}, x_{22})\big) \leq 0 \end{aligned}$$

for any $(x_{11}, x_{12}, x_{21}, x_{22}) \in \Re^4_{++}$ with $x_{11} + x_{12} = x_{21} + x_{22} = 1$.

Theorem 8 i) shows that the risks are fully allocated, i.e., $\alpha^*_{11}+\alpha^*_{12}=\alpha^*_{21}+\alpha^*_{22}=1$, for any given risk measures and RB under a mild condition stated in (3.13), if the risk profile and risk contribution proportions for the two risk holders are quite different. Condition (3.13) requires that the risk appetite for the two risks, (X_1,X_2), is not the same for the two risk holders; in other words, if $\alpha^*_{11}<\alpha^*_{21}$, then $\alpha^*_{12}>\alpha^*_{22}$, which means that there are incentives for both risk holders to initiate the risk sharing.

Contrary to Theorem 8 i) where there is at most one RB allocation, Theorem 8 ii) suggests that there are infinitely many RB allocations if the risk profile and risk contribution proportions for the two risk holders are identical, though a technical condition is required, i.e., $x^*_1(1,\mathbf{b}_1;\varphi_1)=x^*_2(1,\mathbf{b}_1;\varphi_1)$. This setting implies that the risk holder 1 retains the same risk proportion in (X_1,X_2), and the second risk holder has the same strategy. Now, $x^*_1(1,\mathbf{b}_1;\varphi_1)=x^*_2(1,\mathbf{b}_1;\varphi_1)$ implies that the RB, i.e., $\mathbf{b}_1=\mathbf{b}_2$, should be chosen such that $x^*_1(1,\mathbf{b}_1;\varphi_1)=x^*_2(1,\mathbf{b}_1;\varphi_1)$; in other words, one should numerically find $\mathbf{c}\in\Delta_2$ such that $y^*_1(\mathbf{c})\approx y^*_2(\mathbf{c})$, where

$$(y^*_1(\mathbf{c}),y^*_2(\mathbf{c})):=\underset{(y_1,y_2)\Re^2_{++}}{\arg\min}\ \frac{1}{\tau_1}\mathcal{R}_1(y_1,y_2)-\lambda_1(c_1\log y_1+c_2\log y_2).$$

Clearly, we can not guarantee that there exists $\mathbf{c}^*\in\Delta_2$ such that $|y^*_1(\mathbf{c}^*)-y^*_2(\mathbf{c}^*)|\le\epsilon$ for a sufficiently small $\epsilon>0$, but numerical explorations could answer this question.

Finally, Theorem 8 iii) states that, if the parameters λ_1,λ_2 are tuned such that the unique solutions of (3.10) and (3.11) satisfy the sum-to-unity constraints between the two risk holders, say by λ^*_1,λ^*_2 (that is, if the sum of the first (resp. second) components of $\mathbf{x}^*(\lambda^*_1,\mathbf{b}_1;\varphi_1)$ and $\mathbf{x}^*(\lambda^*_2,\mathbf{b}_2;\varphi_2)$ equals to 1), then the resulting total overall risk of the two risk holders, where the sum of their risks is scaled by $1/(\tau_1\lambda^*_1)$ and $1/(\tau_2\lambda^*_2)$, is minimized.

4. Numerical illustrations

This section provides numerical illustrations of how to construct portfolios based on the RB principle (note that, whenever the risk budgets are equal, more precisely we denote RB/CRB as RP/CRP). Our numerical implementations disseminate practical implementations on financial risks for our methods. We provide in Section 4.1 a slight extension of Example 5. Section 4.2 focuses on RB portfolios for one risk holder with multiple financial risks where the risk preferences are ordered by the Variance (or Standard Deviation) and CVaR risk measures.

4.1. CRP versus WC-CRP

As alluded to before, we extend Example 5 in Section 3.2 and assume a CRP setting based on variance risk measures for three independent risks with two clusters such that $\Sigma_{11}=\Sigma_{33}$, i.e., assets 1 and 3 have the same variance. The CRP portfolio is constructed from the solution $\boldsymbol{\alpha}^*_{CRB}$, obtained with the Algorithm 2 in Appendix B. As explained in Section 3.2, the CRB/CRP solution is an element of a parametric set of RB solutions, which is obtained by searching for $\boldsymbol{\alpha}\in\Delta_3$ such that $\Sigma_{11}\alpha_1^2+\Sigma_{22}\alpha_2^2=\Sigma_{11}\alpha_3^2$. Denoting $\sigma_{12}=1-\frac{\Sigma_{22}}{\Sigma_{11}}$, the solution is described by

$$\boldsymbol{\alpha}(\xi):=\left(\frac{\sigma_{12}\xi^2-2\xi+1}{2(1-\xi)},\ \xi,\ 1-\xi-\frac{\sigma_{12}\xi^2-2\xi+1}{2(1-\xi)}\right)^T,$$

for all $0\le\xi\le\frac{1-\sqrt{1-\sigma_{12}}}{\sigma_{12}}$ if $\sigma_{12}\in(-\infty,1)\setminus\{0\}$, and $0\le\xi\le\frac{1}{2}$ if $\sigma_{12}=0$, since $\sigma_{12}<1$. One may show that minimal variance amongst the $\boldsymbol{\alpha}(\xi)$ portfolios is achieved when $\xi^*=1-\sqrt{\frac{1-\sigma_{12}}{2-\sigma_{12}}}$.

The *WC-CRP* portfolio (defined in Section 3.2) is an element of $\boldsymbol{\alpha}(\xi)$, and it can be found via Algorithm 1 in Section 3.2. For Step 1) we need to solve

$$\underset{\mathbf{a}\in\mathcal{B}((1/2,1/2)^T)}{\arg\min}\quad \Sigma_{11}a_1^2+\Sigma_{22}a_2^2+\Sigma_{33}(1/2)^2$$
$$:=(a^*_1,a^*_2)^T=\left(\frac{\Sigma_{22}}{2(\Sigma_{11}+\Sigma_{22})},\frac{\Sigma_{11}}{2(\Sigma_{11}+\Sigma_{22})}\right)^T.$$

Step 2) requires finding the RB with the risk contribution proportions $\boldsymbol{\alpha}^*(a^*_1,a^*_2,1/2)$, which could be identified via a non-clustered version of Algorithm 2 in Appendix B, though a closed-form solution is possible since we only need solving in $\boldsymbol{\alpha}\in\Delta'_3$ the following system of equations

$$\alpha_1^2=2a^*_1\alpha_3^2,\quad \Sigma_{22}\alpha_2^2=2a^*_2\Sigma_{11}\alpha_3^2,\quad\text{and}\quad \alpha_1+\alpha_2+\alpha_3=1.$$

The latter is solved by $\boldsymbol{\alpha}^*_{WC-CRP}=(c_1,c_2,c_3)^T/\mathbf{1}^T\mathbf{c}$, where

$$c_1:=\sqrt{\frac{\Sigma_{22}}{2\Sigma_{11}(\Sigma_{11}+\Sigma_{22})}},\quad c_2:=\sqrt{\frac{\Sigma_{11}}{2\Sigma_{22}(\Sigma_{11}+\Sigma_{22})}},\quad\text{and}$$
$$c_3:=\sqrt{\frac{1}{2\Sigma_{11}}}.$$

The following three variance choices are further considered:

a) $\Sigma_{11}=\Sigma_{33}=1,\Sigma_{22}=0.5$, i.e. $\sigma_{12}=0.5$;
b) $\Sigma_{11}=\Sigma_{22}=\Sigma_{33}=1$, i.e. $\sigma_{12}=0$;
c) $\Sigma_{11}=\Sigma_{33}=1,\Sigma_{22}=1.5$, i.e. $\sigma_{12}=-0.5$.

Fig. 2 compares the risk position of the CRP portfolio (computed via Algorithm 2 in Appendix B) and WC-CRP portfolio (computed via Algorithm 1 in Section 3.2) with the risk position of the parametric portfolio with risk allocation $\boldsymbol{\alpha}(\xi)$. The results clearly show the advantage of using the WC-CRP portfolio, besides its obvious computational advantage that was explained in Section 3.2, which reiterates the practical use of Algorithm 1.

Here, we also extend the three dependent risks with two clusters as addressed in Example 5 in Section 3.2 such that the assets have equal variances and the correlation coefficients $\rho_{k,l}=\rho^{|k-l|}$ for all $1\le k,l\le3$, i.e., $\Sigma_{11}=\Sigma_{22}=\Sigma_{33}=\Sigma$, $\rho_{1,2}=\rho_{2,3}=\rho$ between assets 1 and 2 or between 2 and 3, while $\rho_{1,3}=\rho^2$ between 1 and 3. Again, the CRP portfolio is constructed from the solution $\boldsymbol{\alpha}^*_{CRP}$, obtained with the Algorithm 2 in Appendix B. As explained in Section 3.2, the CRP solution is an element of a parametric set of RP solutions, which is obtained by searching for $\boldsymbol{\alpha}\in\Delta_3$ such that $\mathcal{RC}_1(\boldsymbol{\alpha})+\mathcal{RC}_2(\boldsymbol{\alpha})=\mathcal{RC}_3(\boldsymbol{\alpha})=\frac{1}{2}\mathcal{R}(\boldsymbol{\alpha})$ that is $\alpha_1^2\Sigma_{11}+\alpha_2^2\Sigma_{22}+2\rho_{1,2}\alpha_1\alpha_2\sqrt{\Sigma_{11}\Sigma_{22}}+\rho_{2,3}\alpha_2\alpha_3\sqrt{\Sigma_{22}\Sigma_{33}}+\rho_{1,3}\alpha_1\alpha_3\sqrt{\Sigma_{11}\Sigma_{33}}=\alpha_3^2\Sigma_{33}+\rho_{2,3}\alpha_2\alpha_3\sqrt{\Sigma_{22}\Sigma_{33}}+\rho_{1,3}\alpha_1\alpha_3\sqrt{\Sigma_{11}\Sigma_{33}}$, which can be simplified as $\alpha_1^2+\alpha_2^2+2\rho\alpha_1\alpha_2=\alpha_3^2$. Actually, the impact of $\rho_{1,3}=\rho^2$ is cancelled out from each side of risk contributions, so the parametric set of RP solutions is identical to the case of dependent risks in Example 5 when the pairwise correlations are the same, i.e., $\rho_{k,l}=\rho$. However, the new extended setting of $\rho_{1,3}=\rho^2$ can affect the asset allocations for the minimal variance portfolio. The solution is described by

$$\boldsymbol{\alpha}(\xi):=\left(\xi,\ \frac{1-2\xi}{2(1-(1-\rho)\xi)},\ 1-\xi-\frac{1-2\xi}{2(1-(1-\rho)\xi)}\right)^T,$$

where $0\le\xi<\frac{1}{2}$ for all $\rho\in[-1,1]$ (note that, unlike the case of dependent risks $\rho_{k,l}=\rho$ in Example 5, here we do not need the assumption $\rho\in[-\frac{1}{2},1]$ to keep the total portfolio risk $\mathcal{R}(\boldsymbol{\alpha})\ge0$). Denoting $P=1-\rho$, the solution can be simplified as

$$\boldsymbol{\alpha}(\xi):=\left(\xi,\ \frac{1-2\xi}{2(1-P\xi)},\ 1-\xi-\frac{1-2\xi}{2(1-P\xi)}\right)^T,$$

and by arranging elements in α_3, it can show that $0\le\boldsymbol{\alpha}(\xi)\le1$ as $0\le\xi<\frac{1}{2}$ for all $\rho\in[-1,1]$

$$\boldsymbol{\alpha}(\xi):=\left(\xi,\ \frac{1-2\xi}{2(1-P\xi)},\ \frac{2P\xi^2-2P\xi+1}{2(1-P\xi)}\right)^T. \tag{4.1}$$

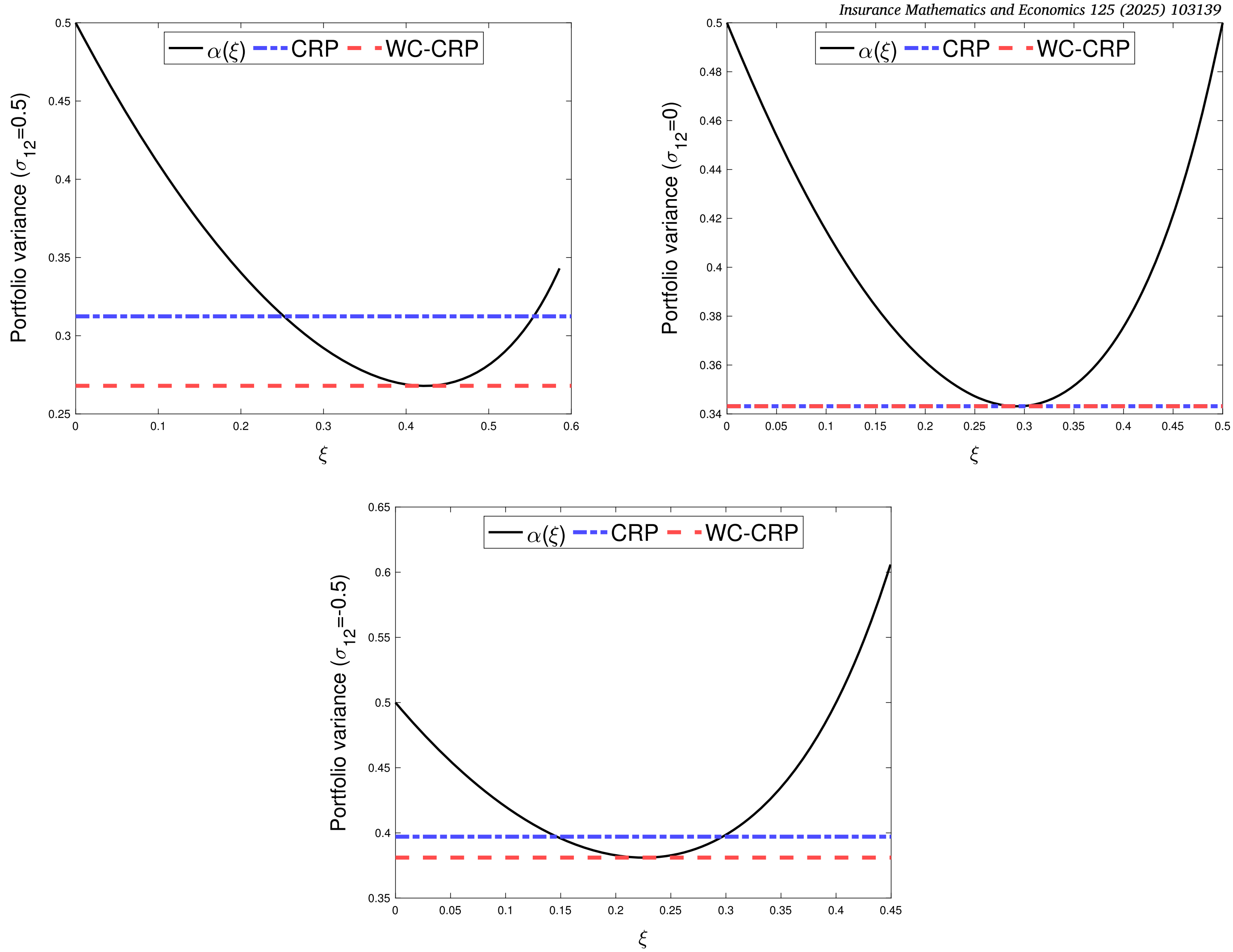

Fig. 2. Portfolio variance for the parametric, CRP and WC-CRP portfolios for Case a) (top left), Case b) (top right) and Case c) (bottom). (For interpretation of the colours in the figure(s), the reader is referred to the web version of this article.)

Since the impact of $\rho_{1,3} = \rho^2$ is removed, the set is identical to Example 5 when $\rho = -\frac{1}{2}$ (i.e. $P = 1 - \rho = \frac{3}{2}$) or $\rho = 1$ (i.e. $P = 0$). However, due to the new setting $\rho_{1,3} = \rho^2$, the minimal variance among the $\boldsymbol{\alpha}(\xi)$ portfolios is different from Example 5 and it cannot be expressed in any simple form. By following the same procedure as described in Example 5, we start with

$$\begin{aligned}
&\min_{\boldsymbol{\alpha} \in \mathcal{CRB}((1/2,1/2)^T)} \mathcal{R}(\boldsymbol{\alpha}) \\
&= \min_{\boldsymbol{\alpha} \in \mathcal{CRB}((1/2,1/2)^T)} \left(\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + 2\rho\alpha_1\alpha_2 + 2\rho\alpha_2\alpha_3 + 2\rho^2\alpha_1\alpha_3\right)\Sigma \\
&= \min_{\boldsymbol{\alpha} \in \mathcal{CRB}((1/2,1/2)^T)} \left(\alpha_3(\alpha_3 + \rho\alpha_2 + \rho^2\alpha_1)\right)\Sigma,
\end{aligned}$$

since $\alpha_1^2 + \alpha_2^2 + 2\rho\alpha_1\alpha_2 = \alpha_3^2$ as shown earlier. To get $\boldsymbol{\alpha}^{**}((1/2,1/2)^T)$ or ξ^*, when $P \neq 0$ (as $\rho \neq 1$), after applying both Product and Quotient Rules for derivatives, the solution for the minimal variance portfolio can be achieved by solving the following quartic equation (i.e. ξ^* will be one of the real roots that is between 0 and $\frac{1}{2}$, though we do not have a simpler expression):

$$\xi^4 - \frac{P+5}{2P}\xi^3 + \frac{3(P+1)}{2P^2}\xi^2 - \frac{2P+1}{2P^3}\xi + \frac{1}{4P^3} = 0.$$

Therefore, when $P = 1$ (as $\rho^2 = \rho = 0$), we have $\xi^* = 1 - \frac{\sqrt{2}}{2}$, it becomes the case of independent risks. Also, when $P = 0$ (as $\rho^2 = \rho = 1$), we have the same result in Example 5, i.e., $\boldsymbol{\alpha}^{**}((1/2,1/2)^T) = \boldsymbol{\alpha}(\xi)$, for any $\xi \in [0, 1/2)$.

Meanwhile, the *WC-CRP* portfolio (defined in Section 3.2) is an element of $\boldsymbol{\alpha}(\xi)$, and it can be found via Algorithm 1 in Section 3.2. For Step 1) we need to solve

$$\underset{\mathbf{a} \in \mathcal{B}((1/2,1/2)^T)}{\arg\min} \quad (a_1^2 + a_2^2 + a_3^2 + 2\rho a_1 a_2 + 2\rho a_2 a_3 + 2\rho^2 a_1 a_3)\Sigma$$

where $a_3 = \frac{1}{2}$ and $a_1 + a_2 + a_3 = 1$. The optimal solution is solved by $a_1^* = \frac{1+\rho}{4} = \frac{2-P}{4}$ and $a_2^* = \frac{1-\rho}{4} = \frac{P}{4}$. Step 2) requires finding the RB with the risk contribution proportions $\boldsymbol{\alpha}^*(a_1^*, a_2^*, 1/2)$, which could be identified via a non-clustered version of Algorithm 2, though the solution in this case can also be solved in $\boldsymbol{\alpha} \in \Delta_3'$ via the following system of equations

$$\begin{aligned}
\alpha_1^2 + \rho\alpha_1\alpha_2 + \rho^2\alpha_1\alpha_3 &= 2a_1^*(\alpha_3^2 + \rho\alpha_2\alpha_3 + \rho^2\alpha_1\alpha_3), \\
\alpha_2^2 + \rho\alpha_1\alpha_2 + \rho\alpha_2\alpha_3 &= 2a_2^*(\alpha_3^2 + \rho\alpha_2\alpha_3 + \rho^2\alpha_1\alpha_3), \\
\alpha_1 + \alpha_2 + \alpha_3 &= 1.
\end{aligned}$$

Unlike the case of a portfolio with three independent risks earlier, the solution cannot be expressed in a simple term. Nevertheless, after a few steps of simplification by combining the three equations above, we can get the following relations between $\boldsymbol{\alpha}^*(a_1^*, a_2^*, 1/2)$: $\alpha_3 = \frac{2(1-\rho)\alpha_1^2 - 2(1-\rho)\alpha_1 + 1}{2(1-(1-\rho)\alpha_1)} = \frac{2P\alpha_1^2 - 2P\alpha_1 + 1}{2(1-P\alpha_1)}$ and $\alpha_2 = \frac{1-2\alpha_1}{2(1-P\alpha_1)}$ which have the

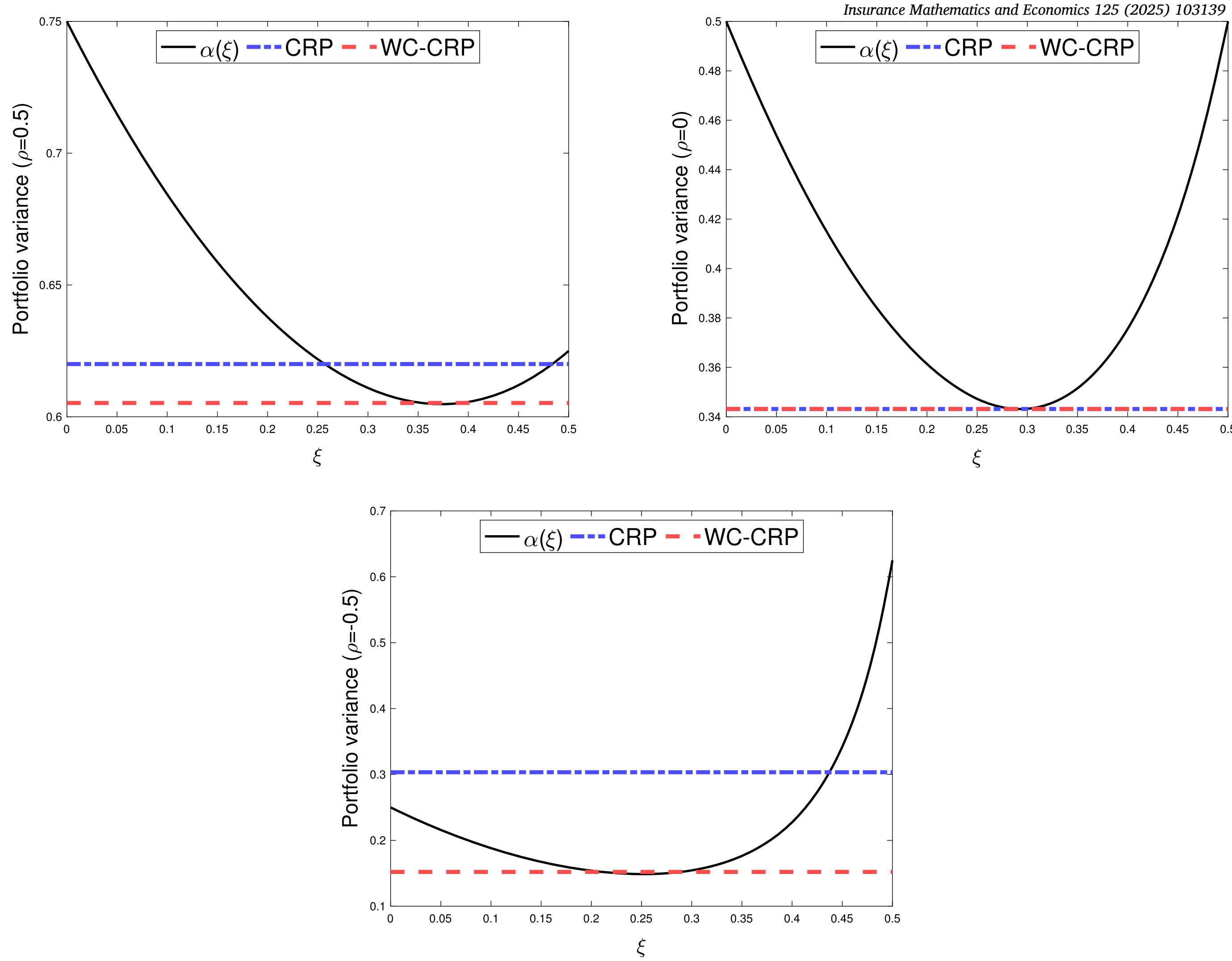

Fig. 3. Portfolio variance for the parametric, CRP and WC-CRP portfolios for Case d) (top left), Case e) (top right) and Case f) (bottom).

same forms as in (4.1). As shown in Fig. 2 earlier and Fig. 3, both $\boldsymbol{\alpha}^{**}((1/2, 1/2)^T)$ in the parametric set and $\boldsymbol{\alpha}^{*}(a_1^*, a_2^*, 1/2)$ here reach the same minimal variance portfolio (note, in numerical solutions, due to some specific levels of precision settings in different methods, it may have a tiny difference). Three different choices of ρ are illustrated in Fig. 3, when $\Sigma_{11} = \Sigma_{22} = \Sigma_{33} = 1$ and Cases d) $\rho = 0.5$, e) $\rho = 0$ and f) $\rho = -0.5$.

4.2. RB/RP with clusters and SRI constraints

This section provides a data analysis based on our main results in Sections 3.1 and 3.2. That is, we reconsider the investment portfolio in Hallerbach et al. (2004) that was related to portfolio allocation satisfying certain *socially responsible investing (SRI)* characteristics.[4] The SRI scores determine the degree of social responsibility embedded in a firm and it enables the investor or portfolio manager to construct the opportunity portfolio (i.e., decide which assets to invest in) by including only those companies that satisfy certain SRI targets before deciding upon the asset allocation (i.e., decide how much to invest in each asset). This two-stage approach provides a 360-degree approach to construct a SRI portfolio. Our data analysis focuses on the second stage that supports the decision-making process on portfolio composition that is based on RB/RP and CRB/CRP allocations. Given the new socio-economic environment that investors and fund managers ought to operate in, there is a growing emphasis on controlling the degree of risk absorbed from different asset classes, from different geographical economic regions or satisfying different ESG, SDG or SRI features. Traditional portfolio management techniques were not designed with these social preferences in mind.

We work with a universe of 374 companies that are grouped in ten *Global Industry Classification Standard (GICS)* sectors.[5] The companies data cover different regions, namely EU, UK, US and REST (firms from countries outside the EU, UK and US). Together with the SRI information, we have also collected historical stock prices (daily returns) for all firms in our sample from January 2010 to December 2020, from various sources: Datastream, WRDS-CRSP, Compustat, IBES and Yahoo!Finance. The summary information is reported in Appendix C.

Our data analysis relies on comparing three RB/CRB (or denoted as RP/CRP when the risk budgets are equal) portfolios, where the risk measures are either *standard deviation (SD)* and/or *Conditional-Value-at-Risk*

[4] We would like to thank Aloy Soppe for making the original raw dataset available to us. The original dataset was put together by the Triodos bank, the first European green bank.

[5] Note that the original investment portfolio in Hallerbach et al. (2004) consists of 590 companies, but 216 firms were delisted or vanished during the 2010-2020 period.

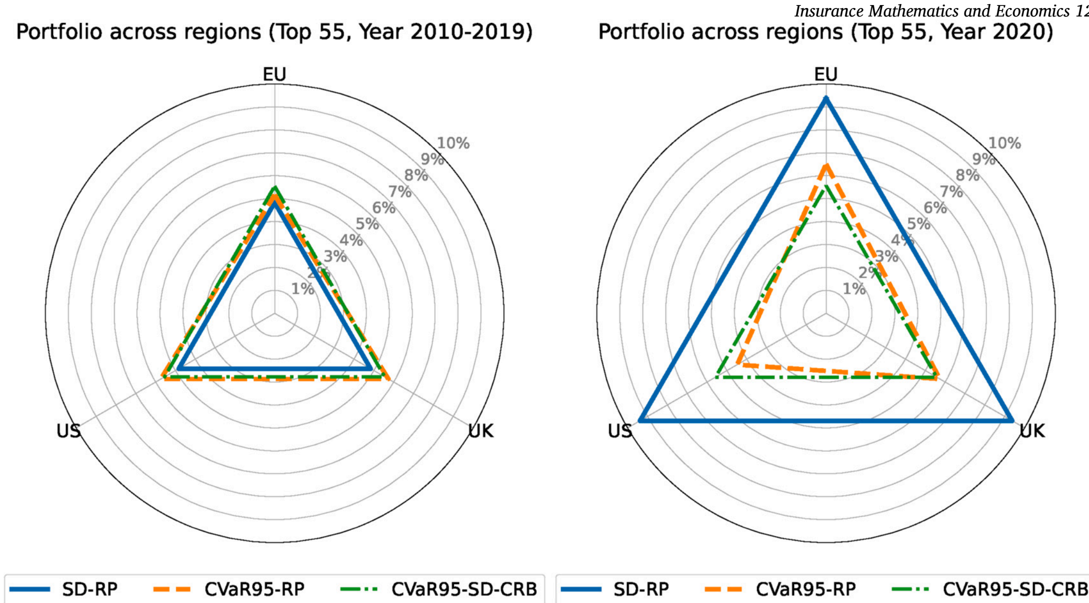

Fig. 4. Risk contributions of each region (EU, UK and US) for 2010-2019 (left) and 2020 (right).

(CVaR). The first two portfolios, denoted as *SD-RP* and $\text{CVaR}_{95\%}$-*RP*, are standard RB portfolios, as explained in Section 3.1 with $\varphi = SD$ and $\varphi = \text{CVaR}_{95\%}$, respectively. The third portfolio, denoted as $\text{CVaR}_{95\%}$-*SD-CRB*, is built on compounding risk measures such that $\text{CVaR}_{95\%}$-*SD-CRB* matches the total portfolio risk as measured by SD to the risk measured by the $\text{CVaR}_{95\%}$ equivalent portfolio, i.e., the aggregate level of risks measured via SD of the $\text{CVaR}_{95\%}$-*SD-CRB* and $\text{CVaR}_{95\%}$-*RP* portfolios is equal. This compound measure has the advantage that it meets the regulatory requirement of using the CVaR as the market risk measure, whilst the portfolio allocations are based on the CRB procedure in Section 3.2 with $\varphi = SD$.

Fig. 4 compares the (clustered) risk contributions for the three portfolios over the two periods, where the risk contributions are consistently computed with φ being the annualized SD. Each of the three portfolios is composed of $n = 165$ assets by choosing the top 55 SRI ranked companies in each region, namely EU, UK and US.[6] We observe that the US/UK/EU cluster has a higher/similar/lower SD risk contribution for $\text{CVaR}_{95\%}$-*SD-CRB* than the SD risk contribution for $\text{CVaR}_{95\%}$-*RP*. One possible explanation for this result is the degree of homogeneity or heterogeneity in the companies that are selected in each region. The companies in the EU are subject to more intense regulation and the top companies are expected to have similar SRI scores. This would lead to lower $\text{CVaR}_{95\%}$-*SD-CRB*. The US firms are more heterogeneous in behaviour, taking advantage of a more relaxed regulatory regime.

We now redo the previous computations by including in each of the three portfolios the top 10 SRI ranked companies in each of the ten GICS sectors (see list in Table C.4) and thus, the new three portfolios are composed of $n = 100$ assets. These new three portfolios have no selection parity imposed at the regional level. The new risk allocations are computed as before, and the results are displayed in Fig. 5. The left radar chart in that figure shows the three portfolios are similar during low market risk, and in turn, CRP across sectors is now achieved. The right radar chart in Fig. 5 indicates that two sectors, namely *Consumer Staples* and *Materials*, have significantly larger risk allocations for $\text{CVaR}_{95\%}$-*RP* as compared to $\text{CVaR}_{95\%}$-*SD-CRB*, while the individual sectors with high annualized SD, namely *Financials* and *Energy*, have lower risk allocations for $\text{CVaR}_{95\%}$-*RP* as compared to $\text{CVaR}_{95\%}$-*SD-CRB*. This effect can be attributed to the COVID-19 pandemic that has engulfed the major economies and the destabilization of the world-wide supply chain.

[6] The Rest of the World was dropped out because there are only 34 companies in the sample from this region.

5. Conclusions

This paper provides an extensive discussion about the theory and practice around constructing RB portfolios in a variety of settings. We have started out with revisiting the usual one risk holder setting with and without clustered risks, and we then show how those settings could be extended to situations in which a non-hedgeable risk is present or a fixed sub-portfolio has been aimed by the risk holder. The latter are novel approaches, which widen the application of RB portfolio construction. Another novel approach of this paper is a combination of the concepts of RS and RB, which has not been discussed in the wider risk analysis and risk management literature.

Our theoretical results are accompanied by numerical procedures to identify such RB and RB-RS portfolios. Numerical experiments are provided for pure RB portfolios, where we show how to apply our methods to constructing RB and clustered RB (or CRP) by considering SRI factors. Such SRI factors are now becoming more and more popular given changes in stakeholders' preferences towards societal benefits.

CRediT authorship contribution statement

Vali Asimit: Writing – review & editing, Writing – original draft, Visualization, Validation, Supervision, Resources, Methodology, Investigation, Formal analysis, Conceptualization. **Wing Fung Chong:** Writing – review & editing, Writing – original draft, Validation, Methodology, Investigation, Formal analysis, Conceptualization. **Radu Tunaru:** Writing – review & editing, Writing – original draft, Validation, Supervision, Methodology, Investigation, Formal analysis, Conceptualization. **Feng Zhou:** Writing – review & editing, Writing – original draft, Visualization, Validation, Methodology, Investigation, Formal analysis, Data curation.

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Declaration of competing interest

The authors declare that they have no conflict of interest.

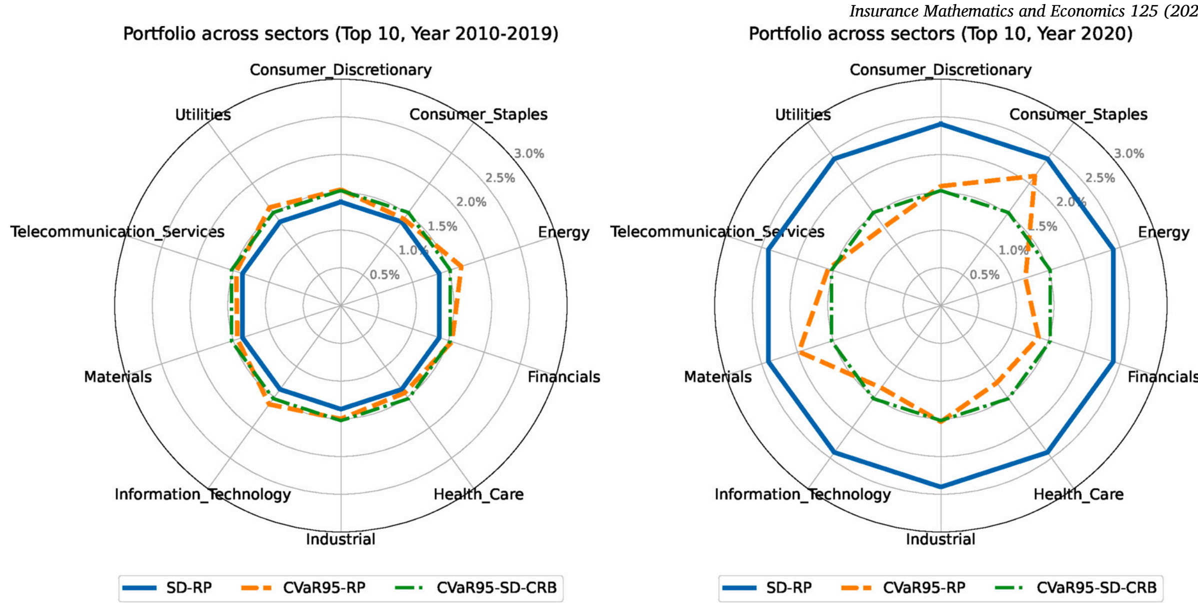

Fig. 5. Risk contributions of each of the ten GICS sectors for 2010-2019 (left) and 2020 (right).

Appendix A. Proofs

A.1. Proof of Theorem 3

Note that (3.2) is a strictly convex optimization problem since $-\lambda\sum_{k=1}^d b_k\log x_k$ is a strictly convex function in $\mathbf{x}$ over the convex cone $\Re^d_{++}$. Let $F(\mathbf{x};\lambda)$ be the objective function of (3.2). To show that the solution of (3.2) lies in the interior of $\Re^d_{++}$, it suffices to show that

$$\liminf_{\mathbf{x}\to\mathbf{x}'} F(\mathbf{x};\lambda)=\infty,\quad \text{for any } \mathbf{x}'\in\partial\Re^d_{++}=B_1\cup B_2,\quad \text{where} \tag{A.1}$$

$$B_1:=\bigcup_{\mathcal{I}\subseteq\{1,2,\dots,d\};\,|\mathcal{I}|\ge 1}\Big\{\mathbf{x}:x_k=\infty \text{ for all } k\in\mathcal{I}, \text{ and } x_k\in[0,\infty),\text{ for all } k\in\{1,2,\dots,d\}\setminus\mathcal{I}\Big\},$$

$$B_2:=\bigcup_{\mathcal{I}\subseteq\{1,2,\dots,d\};\,|\mathcal{I}|\ge 1}\Big\{\mathbf{x}:x_k=0 \text{ for all } k\in\mathcal{I}, \text{ and } x_k\in(0,\infty),\text{ for all } k\in\{1,2,\dots,d\}\setminus\mathcal{I}\Big\}.$$

Fix an $\mathbf{x}'\in B_1$, and by the homogeneity of φ, one may get that

$$\begin{aligned}F(\mathbf{x};\lambda)&=\frac{1}{\tau}\left(\mathbf{1}^T\mathbf{x}\right)^\tau\varphi\left(\frac{\mathbf{x}^T\mathbf{X}}{\mathbf{1}^T\mathbf{x}}\right)-\lambda\sum_{k=1}^d b_k\log\left(\frac{x_k}{\mathbf{1}^T\mathbf{x}}\right)-\lambda\log\left(\mathbf{1}^T\mathbf{x}\right)\\&\ge\frac{1}{\tau}\left(\mathbf{1}^T\mathbf{x}\right)^\tau\inf_{\mathbf{y}\in\Delta_d}\varphi\left(\mathbf{y}^T\mathbf{X}\right)-\lambda\sup_{\mathbf{y}\in\Delta_d}\sum_{k=1}^d b_k\log y_k-\lambda\log\left(\mathbf{1}^T\mathbf{x}\right)\\&=\frac{1}{\tau}\left(\mathbf{1}^T\mathbf{x}\right)^\tau\inf_{\mathbf{y}\in\Delta_d}\varphi\left(\mathbf{y}^T\mathbf{X}\right)-\lambda\sum_{k=1}^d b_k\log b_k-\lambda\log\left(\mathbf{1}^T\mathbf{x}\right),\end{aligned}\tag{A.2}$$

for any $\mathbf{x}\in\Re^d_{++}$. Therefore,

$$\frac{F(\mathbf{x};\lambda)}{\mathbf{1}^T\mathbf{x}}\ge\frac{1}{\tau}\left(\mathbf{1}^T\mathbf{x}\right)^{\tau-1}\inf_{\mathbf{y}\in\Delta_d}\varphi\left(\mathbf{y}^T\mathbf{X}\right)-\frac{\lambda}{\mathbf{1}^T\mathbf{x}}\sum_{k=1}^d b_k\log b_k-\lambda\frac{\log\left(\mathbf{1}^T\mathbf{x}\right)}{\mathbf{1}^T\mathbf{x}}\quad\text{for any } \mathbf{x}\in\Re^d_{++}.$$

Clearly, $\sum_{k=1}^d b_k\log b_k<0$ since $\mathbf{b}\in\Delta_d$. Moreover, there exists an $M>0$ such that $\frac{1}{\tau}\left(\mathbf{1}^T\mathbf{x}\right)^{\tau-1}>M$ for any $\mathbf{x}$ sufficiently close to $\mathbf{x}'$, since $\tau\ge 1$. Furthermore, for any small $\epsilon>0$, there is a neighbourhood of $\mathbf{x}'$ such that $|\log\left(\mathbf{1}^T\mathbf{x}\right)/\mathbf{1}^T\mathbf{x}|<\epsilon$ since $\log y=o(y)$ as $y\to\infty$ and $\mathbf{x}'\in B_1$. Putting all these together with $\epsilon\downarrow 0$ and keeping (3.1) in mind, one may conclude that

$$\liminf_{\mathbf{x}\to\mathbf{x}'}\frac{F(\mathbf{x};\lambda)}{\mathbf{1}^T\mathbf{x}}>0,\quad\text{and thus, }\liminf_{\mathbf{x}\to\mathbf{x}'}F(\mathbf{x};\lambda)=\infty\text{ for any }\mathbf{x}'\in B_1.$$

Fix an $\mathbf{x}'\in B_2$; then, there exists an $\mathcal{I}\subseteq\{1,2,\dots,d\}$ with $|\mathcal{I}|\ge 1$ such that $x'_k=0$ for all $k\in\mathcal{I}$, and $x'_k\in(0,\infty)$ for all $k\in\{1,2,\dots,d\}\setminus\mathcal{I}$. Similar to (A.2), one may get that

$$F(\mathbf{x};\lambda)\ge\frac{1}{\tau}\left(\mathbf{1}^T\mathbf{x}\right)^\tau\inf_{\mathbf{y}\in\Delta_d}\varphi\left(\mathbf{y}^T\mathbf{X}\right)-\lambda\sum_{k=1}^d b_k\log x_k$$

for any $\mathbf{x}$ sufficiently close to $\mathbf{x}'$. Since $\lambda>0$ and $\mathbf{b}>\mathbf{0}$, the above equation implies that $\liminf_{\mathbf{x}\to\mathbf{x}'}F(\mathbf{x};\lambda)=\infty$ for any $\mathbf{x}'\in B_2$.

Equation (A.1) implies that there exist an $a>0$ and an $\epsilon\in(0,a]$ such that

$$\inf_{\mathbf{x}\in\Re^d_{++}}F(\mathbf{x};\lambda)=\inf_{\mathbf{x}\in B_{a,\epsilon}}F(\mathbf{x};\lambda),\quad\text{where } B_{a,\epsilon}:=\{\mathbf{x}\in B_a:\min_{1\le k\le d}x_k\ge\epsilon\}$$

with $B_a:=\{\mathbf{x}\in\Re^d_{++}:||\mathbf{x}||\le a\}$ and $||\cdot||$ being the Euclidean distance. Since $B_{a,\epsilon}$ is a compact set, the global minimum of $F(\cdot;\lambda)$ on $\Re^d_{++}$, i.e. $\mathbf{x}^*(\lambda,\mathbf{b})$, is an interior point of the feasibility set for any given $\lambda>0$.

It remains to prove that

$$\mathbf{x}^*(\lambda^*,\mathbf{b})=\left(\mathbf{1}^T\mathbf{x}^*(1,\mathbf{b})\right)^{-1}\mathbf{x}^*(1,\mathbf{b})\in\mathcal{RB}(\mathbf{b}).\tag{A.3}$$

Firstly, we show that the unique solution of (3.2), i.e., $\mathbf{x}^*(\lambda,\mathbf{b})$, satisfies

$$\mathbf{x}^*(\lambda,\mathbf{b})=\lambda^{1/\tau}\mathbf{x}^*(1,\mathbf{b})\quad\text{for any }\lambda>0.\tag{A.4}$$

Assume, on the contrary, that $\lambda^{1/\tau}\mathbf{x}^*(1,\mathbf{b})$ does not solve (3.2) for a given $\lambda\in\Re_{++}\setminus\{1\}$; that is, there exists $\tilde{\mathbf{x}}\in\Re^d_{++}$ such that

$$\frac{1}{\tau}\mathcal{R}\left(\tilde{\mathbf{x}}\right)-\lambda\sum_{k=1}^d b_k\log\tilde{x}_k<\frac{1}{\tau}\mathcal{R}\left(\lambda^{1/\tau}\mathbf{x}^*(1,\mathbf{b})\right)-\lambda\sum_{k=1}^d b_k\log\left(\lambda^{1/\tau}x^*_k(1,\mathbf{b})\right).$$

By this inequality and the homogeneity of φ,

$$\frac{\lambda}{\tau}\mathcal{R}\left(\lambda^{-1/\tau}\tilde{\mathbf{x}}\right)-\lambda\sum_{k=1}^d b_k\log\left(\lambda^{-1/\tau}\tilde{x}_k\right)-\lambda\sum_{k=1}^d b_k\frac{\log\lambda}{\tau}$$

$$<\frac{\lambda}{\tau}\mathcal{R}\left(\mathbf{x}^*(1,\mathbf{b})\right)-\lambda\sum_{k=1}^d b_k\log\left(x^*_k(1,\mathbf{b})\right)-\lambda\sum_{k=1}^d b_k\frac{\log\lambda}{\tau},$$

which further implies that

$$\frac{1}{\tau}\mathcal{R}(\lambda^{-1/\tau}\tilde{\mathbf{x}}) - \sum_{k=1}^{d} b_k \log\left(\lambda^{-1/\tau}\tilde{x}_k\right) < \frac{1}{\tau}\mathcal{R}\left(\mathbf{x}^*(1,\mathbf{b})\right) - \sum_{k=1}^{d} b_k \log\left(x_k^*(1,\mathbf{b})\right).$$

This contradicts that $\mathbf{x}^*(1,\mathbf{b})$ solves (3.2) with $\lambda = 1$, as $\lambda^{-1/\tau}\tilde{\mathbf{x}} \in \Re^d_{++}$, and concludes (A.4).

Secondly, we show that $\left(\mathbf{1}^T\mathbf{x}^*(1,\mathbf{b})\right)^{-1}\mathbf{x}^*(1,\mathbf{b}) \in \Delta_d \cap \mathcal{RB}(\mathbf{b})$. Note that $\mathbf{x}^*(\lambda,\mathbf{b}) \in \Re^d_{++}$, but not guaranteed to be in Δ_d, and thus, is not necessarily in $\mathcal{RB}(\mathbf{b})$. Since $\mathcal{R}(\mathbf{x})$ is differentiable at $\mathbf{x}^*(1,\mathbf{b})$ (and thus at $\mathbf{x}^*(\lambda,\mathbf{b})$ for any $\lambda > 0$ due to (A.4)) and the fact that $\mathcal{R}$ is a homogeneous function, the first-order conditions in (3.2) imply that $\mathcal{RC}_k\left(\mathbf{x}^*(\lambda,\mathbf{b})\right) = b_k\mathcal{R}\left(\mathbf{x}^*(\lambda,\mathbf{b})\right)$ for all $k \in \{1,2,\dots,d\}$. However, due to the homogeneity of φ, $\mathcal{RC}_k$ is also homogeneous of the same order as φ, and thus

$$\mathcal{RC}_k\left(t\mathbf{x}^*(\lambda,\mathbf{b})\right) = b_k\mathcal{R}\left(t\mathbf{x}^*(\lambda,\mathbf{b})\right), \text{ for all } k \in \{1,2,\dots,d\} \text{ and any } t > 0.$$

In particular, choose $t = \left(\mathbf{1}^T\mathbf{x}^*(1,\mathbf{b})\right)^{-1}$ to find that $\left(\mathbf{1}^T\mathbf{x}^*(1,\mathbf{b})\right)^{-1} \times \mathbf{x}^*(1,\mathbf{b}) \in \Delta_d \cap \mathcal{RB}(\mathbf{b})$.

Thirdly, $\mathbf{x}^*(\lambda^*,\mathbf{b}) = \left(\mathbf{1}^T\mathbf{x}^*(1,\mathbf{b})\right)^{-1}\mathbf{x}^*(1,\mathbf{b})$ is true due to (A.4), and in turn $\mathbf{x}^*(\lambda^*,\mathbf{b}) \in \Delta_d$. The latter justifies (A.3), which concludes our proof.

A.2. Proof of Proposition 6

Let $l \in \{2,3,\dots,d\}$, partition $\left\{\mathcal{I}^{(1)},\mathcal{I}^{(2)},\dots,\mathcal{I}^{(l)}\right\}$ of $\mathcal{I}_d$, and $\mathbf{b} \in \Delta_l$. Let $\boldsymbol{\alpha} \in \mathcal{CRB}(\mathbf{b}) \subseteq \Delta_d'$. By definition, for all $k \in \{1,2,\dots,l\}$, $\sum_{i\in\mathcal{I}^{(k)}} \mathcal{RC}_i(\boldsymbol{\alpha}) = b_k\mathcal{R}(\boldsymbol{\alpha})$. For each $i \in \{1,2,\dots,d\}$, define $\tilde{a}_i = \mathcal{RC}_i(\boldsymbol{\alpha})/\mathcal{R}(\boldsymbol{\alpha})$. Obviously, $\tilde{\mathbf{a}} \in \Delta_d'$. For each $k \in \{1,2,\dots,l\}$,

$$\sum_{i\in\mathcal{I}^{(k)}} \tilde{a}_i = \sum_{i\in\mathcal{I}^{(k)}} \frac{\mathcal{RC}_i(\boldsymbol{\alpha})}{\mathcal{R}(\boldsymbol{\alpha})} = \frac{b_k\mathcal{R}(\boldsymbol{\alpha})}{\mathcal{R}(\boldsymbol{\alpha})} = b_k.$$

Thus, $\tilde{\mathbf{a}} \in \mathcal{B}(\mathbf{b})$. Also, by definition, for each $i \in \{1,2,\dots,d\}$, $\mathcal{RC}_i(\boldsymbol{\alpha}) = \tilde{a}_i\mathcal{R}(\boldsymbol{\alpha})$, and hence $\boldsymbol{\alpha} \in \mathcal{RB}(\tilde{\mathbf{a}}) \subseteq \bigcup_{\mathbf{a}\in\mathcal{B}(\mathbf{b})} \mathcal{RB}(\mathbf{a})$. These show that $\mathcal{CRB}(\mathbf{b}) \subseteq \bigcup_{\mathbf{a}\in\mathcal{B}(\mathbf{b})} \mathcal{RB}(\mathbf{a})$.

Let $\boldsymbol{\alpha} \in \bigcup_{\mathbf{a}\in\mathcal{B}(\mathbf{b})} \mathcal{RB}(\mathbf{a})$. There exists an $\mathbf{a} \in \mathcal{B}(\mathbf{b})$ such that $\boldsymbol{\alpha} \in \mathcal{RB}(\mathbf{a})$. By definition, $\mathbf{a},\boldsymbol{\alpha} \in \Delta_d'$, for all $k \in \{1,2,\dots,l\}$, $\sum_{i\in\mathcal{I}^{(k)}} a_i = b_k$, and for all $i \in \{1,2,\dots,d\}$, $\mathcal{RC}_i(\boldsymbol{\alpha}) = a_i\mathcal{R}(\boldsymbol{\alpha})$. Then, for each $k \in \{1,2,\dots,l\}$,

$$\sum_{i\in\mathcal{I}^{(k)}} \mathcal{RC}_i(\boldsymbol{\alpha}) = \sum_{i\in\mathcal{I}^{(k)}} a_i\mathcal{R}(\boldsymbol{\alpha}) = b_k\mathcal{R}(\boldsymbol{\alpha}).$$

Hence, $\boldsymbol{\alpha} \in \mathcal{CRB}(\mathbf{b})$. These show that $\bigcup_{\mathbf{a}\in\mathcal{B}(\mathbf{b})} \mathcal{RB}(\mathbf{a}) \subseteq \mathcal{CRB}(\mathbf{b})$.

A.3. Proof of Theorem 7

Let $F(\mathbf{x};\lambda)$ be the objective function in (3.9). One could show that the equivalence of (A.1) holds, and in turn, the global minimum of $F(\cdot;\lambda)$ on $\Re_{++}$, i.e., $\mathbf{x}^*(\lambda,\mathbf{b})$, is an interior point of the feasibility set.[7] As before, the first order conditions imply that $\mathbf{x}^*(\lambda,\mathbf{b})$ solves (3.7). The proof is now complete.

A.4. Proof of Theorem 8

The proof is similar to the proof of Theorem 3, and thus, we only provide the necessary arguments. We apply the conclusions of (3.2) from Theorem 3 with $\lambda \in \{\lambda_1,\lambda_2\}$ in (3.10) and (3.11), and conclude that (3.10) and (3.11) admit unique solutions that are interior points of the feasibility set.

We now show part i). Due to the homogeneity of φ_1 and φ_2, then for any $t_1,t_2 > 0$, $t_1\mathbf{x}^*\left(1,\mathbf{b}_1;\varphi_1\right)$ solves (3.10) with $\lambda_1 = t_1^{-1/\tau_1}$, and $t_2\mathbf{x}^*\left(1,\mathbf{b}_2;\varphi_2\right)$ solves (3.11) with $\lambda_2 = t_2^{-1/\tau_2}$. Thus, we need to find (t_1,t_2) such that the risks are fully allocated within the LoB, i.e., solving

$$t_1x_1^*\left(1,\mathbf{b}_1;\varphi_1\right) + t_2x_1^*\left(1,\mathbf{b}_2;\varphi_2\right) = t_1x_2^*\left(1,\mathbf{b}_1;\varphi_1\right) + t_2x_2^*\left(1,\mathbf{b}_2;\varphi_2\right) = 1, \tag{A.5}$$

which is solved by (3.12). Now, (3.12) leads to a feasible risk allocation if and only if $t_1^*,t_2^* > 0$, which is equivalent to (3.13). The proof of part i) is concluded.

Part ii) could be argued in the same way as part i). Since $x_1^*(1,\mathbf{b}_1;\varphi_1) = x_2^*(1,\mathbf{b}_1;\varphi_1)$, then (A.5) is guaranteed for any $(t_1,t_2) \in \Re^2_{++}$ such that $t_1 + t_2 = 1/x_1^*(1,\mathbf{b}_1;\varphi_1)$, which concludes this part ii).

We now show part iii). Since $(\alpha_{11}^*,\alpha_{21}^*)$ solves (3.10) with $\lambda = \lambda_1^*$, then

$$\frac{1}{\tau_1\lambda_1^*}\left(\mathcal{R}_1(\alpha_{11}^*,\alpha_{21}^*) - \mathcal{R}_1(x_{11},x_{21})\right) \le b_{11}\log\frac{\alpha_{11}^*}{x_{11}} + b_{21}\log\frac{\alpha_{21}^*}{x_{21}}, \tag{A.6}$$

for any $(x_{11},x_{21}) \in \Re^2_{++}$. Similarly, since $(\alpha_{12}^*,\alpha_{22}^*)$ solves (3.11) with $\lambda = \lambda_2^*$, then

$$\frac{1}{\tau_2\lambda_2^*}\left(\mathcal{R}_2(\alpha_{12}^*,\alpha_{22}^*) - \mathcal{R}_2(x_{12},x_{22})\right) \le b_{12}\log\frac{\alpha_{12}^*}{x_{12}} + b_{22}\log\frac{\alpha_{22}^*}{x_{22}}, \tag{A.7}$$

is true for any $(x_{12},x_{22}) \in \Re^2_{++}$. Combining (A.6) and (A.7) imply that

$$\begin{aligned}
&\frac{1}{\tau_1\lambda_1^*}\left(\mathcal{R}_1(\alpha_{11}^*,\alpha_{21}^*) - \mathcal{R}_1(x_{11},x_{21})\right) + \frac{1}{\tau_2\lambda_2^*}\left(\mathcal{R}_2(\alpha_{12}^*,\alpha_{22}^*) - \mathcal{R}_2(x_{12},x_{22})\right)\\
&\le \min_{\substack{(x_{11},x_{12})\in\Delta_2\\(x_{21},x_{22})\in\Delta_2}} b_{11}\log\frac{\alpha_{11}^*}{x_{11}} + b_{12}\log\frac{\alpha_{12}^*}{x_{12}} + b_{21}\log\frac{\alpha_{21}^*}{x_{21}} + b_{22}\log\frac{\alpha_{22}^*}{x_{22}}\\
&= b_{11}\log\frac{\alpha_{11}^*}{b_{11}} + b_{12}\log\frac{\alpha_{12}^*}{b_{12}} + b_{21}\log\frac{\alpha_{21}^*}{b_{21}} + b_{22}\log\frac{\alpha_{22}^*}{b_{22}}\\
&\quad + (b_{11}+b_{12})\log(b_{11}+b_{12}) + (b_{21}+b_{22})\log(b_{21}+b_{22})\\
&\le \max_{(x_{11},x_{12})\in\Delta_2} b_{11}\log\frac{x_{11}}{b_{11}} + b_{12}\log\frac{x_{12}}{b_{12}}\\
&\quad + \max_{(x_{21},x_{22})\in\Delta_2} b_{21}\log\frac{x_{21}}{b_{21}} + b_{22}\log\frac{x_{22}}{b_{22}}\\
&\quad + (b_{11}+b_{12})\log(b_{11}+b_{12}) + (b_{21}+b_{22})\log(b_{21}+b_{22})\\
&= (b_{11}+b_{12})\log\frac{1}{b_{11}+b_{12}} + (b_{21}+b_{22})\log\frac{1}{b_{21}+b_{22}}\\
&\quad + (b_{11}+b_{12})\log(b_{11}+b_{12}) + (b_{21}+b_{22})\log(b_{21}+b_{22})\\
&= 0,
\end{aligned}$$

where the second inequality is due to $\alpha_{11}^* + \alpha_{12}^* = \alpha_{21}^* + \alpha_{22}^* = 1$. The proof is now complete.

Appendix B. SD/variance-based CRB

The SD and variance-based CRB portfolios are the same, and thus, this is true for CRP counterparts. The mathematical formulation of variance-based CRB portfolio is as follows:

$$\sum_{i\in\mathcal{I}^{(k)}}\sum_{j=1}^{d}\alpha_i\alpha_j\Sigma_{ij} = b_k\boldsymbol{\alpha}^T\Sigma\boldsymbol{\alpha} \quad \text{for all } k \in \{1,\dots,l\}, \quad \text{s.t. } \mathbf{1}^T\boldsymbol{\alpha} = 1 \text{ and } \boldsymbol{\alpha} \ge \mathbf{0}. \tag{B.1}$$

Solving (B.1) is quite challenging, and the only efficient solution is to rely on the equivalent LSE-like formulation in (3.3), which is given as

$$\min_{\boldsymbol{\alpha}\ge\mathbf{0}} \sum_{k=1}^{l}\left(\sum_{i\in\mathcal{I}^{(k)}}\sum_{j=1}^{d}\alpha_i\alpha_j\Sigma_{ij} - b_k\boldsymbol{\alpha}^T\Sigma\boldsymbol{\alpha}\right)^2 \quad \text{s.t. } \mathbf{1}^T\boldsymbol{\alpha} = 1. \tag{B.2}$$

[7] Therefore, the main innovation of the proof here is via the proof of Theorem 3, which shows that a solution of the corresponding optimization problem is an interior point.

Table C.3
Summary of the financial performance per region.

		REGION				
		EU	UK	US	REST	Total
	No. of companies	188	56	96	34	374
	EQUAL WEIGHTED PORTFOLIO (daily returns)					
	Annualized average return	0.0888	0.0724	0.1279	0.1038	0.0996
	Annualized standard deviation	0.1998	0.2039	0.1759	0.1433	0.1651
11 years:	Mean	0.0004	0.0004	0.0005	0.0004	0.0004
2010 - 2020	Standard deviation	0.0126	0.0128	0.0111	0.0090	0.0104
	Skewness	-0.5116	-0.7272	-0.5146	-0.2725	-0.7435
	Kurtosis	7.6460	14.9598	15.8433	3.3927	11.4002
	Annualized average return	0.0833	0.0869	0.1284	0.0864	0.0974
	Annualized standard deviation	0.1889	0.1823	0.1478	0.1357	0.1492
10 years:	Mean	0.0004	0.0004	0.0005	0.0004	0.0004
2010 - 2019	Standard deviation	0.0119	0.0115	0.0093	0.0085	0.0094
	Skewness	-0.2089	-0.8002	-0.4618	-0.3067	-0.4145
	Kurtosis	4.7198	13.3674	4.1551	2.5442	5.2876
	Annualized average return	0.0898	-0.1106	0.0780	0.1502	0.0627
	Annualized standard deviation	0.2879	0.3446	0.3512	0.2091	0.2761
1 year:	Mean	0.0005	-0.0002	0.0005	0.0006	0.0004
2020	Standard deviation	0.0181	0.0217	0.0221	0.0132	0.0174
	Skewness	-1.5274	-0.5124	-0.3906	-0.1649	-1.2751
	Kurtosis	10.7350	7.4429	7.2376	5.1527	10.2675

The optimization problem from (B.2) is non-convex and any off-the-shelf general optimization tools may lead to unstable solutions. Alternatively, a relaxation of (B.2) is suggested in Bai et al. (2016), which could be efficiently solved via the *Alternating Linearisation Method (ALM)*. An appropriation of the ALM approach is provided, and (B.2) is reformulated as

$$\min_{\alpha\geq 0,\theta} \sum_{k=1}^{l} \frac{1}{b_k}\left(\sum_{i\in\mathcal{I}^{(k)}}\sum_{j=1}^{d}\alpha_i\alpha_j\Sigma_{ij} - b_k\theta^2\right)^2 \quad \text{s.t. } \mathbf{1}^T\boldsymbol{\alpha} = 1. \tag{B.3}$$

Algorithm 3 from Bai et al. (2016) precisely solves (B.3) when an equal budget problem (i.e., CRP is sought), and we now adapt the same algorithm for our non-level CRB setting. For ease of notation, we denote $\mathbf{x}^T = (\boldsymbol{\alpha}^T,\theta) \in \Re^{1\times(d+1)}$ and $|\mathcal{I}^{(k)}| = d_k$, where $d_1 + d_2 + \ldots + d_l = d$, since $\left\{\mathcal{I}^{(1)},\ldots,\mathcal{I}^{(l)}\right\}$ is a partition of $\mathcal{I}_d$. Note that

$$\sum_{i\in\mathcal{I}^{(k)}}\sum_{j=1}^{d}\alpha_i\alpha_j\Sigma_{ij} - b_k\theta^2 = \mathbf{x}^T M_k\mathbf{x}, \quad \text{where} \quad M_k := \begin{bmatrix} \Sigma_{\mathcal{I}^{(k)}}\Gamma_{\mathcal{I}^{(k)}} & \mathbf{0} \\ \mathbf{0}^T & -b_k \end{bmatrix},$$

and $\Sigma_{\mathcal{I}^{(k)}} \in \Re^{d\times d_k}$ is a submatrix of Σ where the columns of Σ are extracted based only on the indexes of $\mathcal{I}^{(k)}$. Moreover, $\Gamma_{\mathcal{I}^{(k)}} \in \Re^{d_k\times d}$ is a binary matrix such that $\left(\Gamma_{\mathcal{I}^{(k)}}\right)_{st} = \mathbb{1}_{t=\pi^k(s)}$, where $\mathbb{1}_A$ is the indicator function that takes the value 1 if A is true, and 0 otherwise. Further, $\pi^k : \{1,2,\ldots,d_k\} \to \mathcal{I}_d$ maps the columns of $\Sigma_{\mathcal{I}^{(k)}}$ of Σ. Therefore, the system of equations in (B.1) is solved by running a much simpler task:

$$\min_{\mathbf{x}\geq\mathbf{0}} F(\mathbf{x}) := \sum_{k=1}^{l}\frac{1}{b_k}\left(\mathbf{x}^T M_k\mathbf{x}\right)^2 \tag{B.4}$$
$$\text{s.t. } \mathbf{c}^T\mathbf{x} = 1, \text{ where } \mathbf{c}^T = (\mathbf{1}^T, 0) \in \Re^{1\times(d+1)}.$$

We solve (B.4) by approximating $\mathbf{x}^*$, a local optimum of (B.4). That is, we generate two sequences $\{\mathbf{x}_s : s \geq 0\}$ and $\{\mathbf{y}_s : s \geq 0\}$ such that $\mathbf{x}_s \to \mathbf{x}^*$ and/or $\mathbf{y}_s \to \mathbf{x}^*$. Similar to Algorithm 3 in Bai et al. (2016), a two-block variant of (B.4) is required to solve:

$$\min_{\mathbf{x},\mathbf{y}\in\mathcal{X}} G(\mathbf{x},\mathbf{y}) := \sum_{k=1}^{l}\frac{1}{b_k}\left(\mathbf{x}^T M_k\mathbf{y}\right)^2 \quad \text{s.t. } \mathbf{x} = \mathbf{y}, \tag{B.5}$$

where $\mathcal{X} := \{\mathbf{x} \geq \mathbf{0} : \mathbf{c}^T\mathbf{x} = 1\}$ is the feasible set. Note that (B.5) is a *convex quadratic programming (QP)* instance in $\mathbf{x}$ for any given $\mathbf{y}$ that could be efficiently solved; the same holds if $\mathbf{x}$ and $\mathbf{y}$ are swapped. Further, note that the partial derivatives of G are

$$G_1(\mathbf{x},\mathbf{y}) := \frac{\partial G}{\partial \mathbf{x}} = 2\sum_{k=1}^{l}\frac{\mathbf{x}^T M_k\mathbf{y}}{b_k}M_k\mathbf{y} \quad \text{and}$$

$$G_2(\mathbf{x},\mathbf{y}) := \frac{\partial G}{\partial \mathbf{y}} = 2\sum_{k=1}^{l}\frac{\mathbf{x}^T M_k\mathbf{y}}{b_k}M_k^T\mathbf{x}.$$

Denote

$$H_1(\mathbf{x},\mathbf{y};\mu) := G(\mathbf{x},\mathbf{y}) + \langle G_2(\mathbf{y},\mathbf{y}),\mathbf{x}-\mathbf{y}\rangle + \frac{1}{2\mu}\|\mathbf{x}-\mathbf{y}\|_2^2,$$

$$H_2(\mathbf{x},\mathbf{y};\mu) := G(\mathbf{x},\mathbf{y}) + \langle G_1(\mathbf{x},\mathbf{x}),\mathbf{y}-\mathbf{x}\rangle + \frac{1}{2\mu}\|\mathbf{x}-\mathbf{y}\|_2^2,$$

with $\mu > 0$. The algorithm for solving (B.4), and thus (B.5), is described next as Algorithm 2.

Algorithm 2: CRB algorithm for solving (B.5).

```
Result: (x_{s*}, y_{s*}) that approximates x*, a local optimum of (B.4), where
        s* is the termination step
μ_{1,0} = μ_{2,0} = μ_0 > 0, α ∈ (0,1), and x_0 = y_0 ∈ 𝒳;
for s ∈ {0,1,...} do
    x_{s+1} := argmin_{x∈𝒳} H_1(x, y_s; μ_{1,s});
    if F(x_{s+1}) ≤ H_1(x_{s+1}, y_s; μ_{1,s}) then
        choose μ_{1,s+1} ≥ μ_{1,s};
    else
        find the lowest n_{1,s} ≥ 1 such that F(z_{1,s}) ≤ H_1(z_{1,s}, y_s; μ*_{1,s}),
          where μ*_{1,s} = μ_{1,s} α^{n_{1,s}} and z_{1,s} := argmin_{x∈𝒳} H_1(x, y_s; μ*_{1,s});
        μ_{1,s+1} := μ*_{1,s}/α and x_{s+1} := z_{1,s};
    end
    y_{s+1} := argmin_{y∈𝒳} H_2(x_{s+1}, y; μ_{2,s});
    if F(y_{s+1}) ≤ H_2(x_{s+1}, y_{s+1}; μ_{2,s}) then
        choose μ_{2,s+1} ≥ μ_{2,s};
    else
        find the lowest n_{2,s} ≥ 1 such that F(z_{2,s}) ≤ H_2(x_{s+1}, z_{2,s}; μ*_{2,s}),
          where μ*_{2,s} = μ_{2,s} α^{n_{2,s}} and z_{2,s} := argmin_{y∈𝒳} H_2(x_{s+1}, y; μ*_{2,s});
        μ_{2,s+1} := μ*_{2,s}/α and y_{s+1} := z_{2,s};
    end
end
```

Table C.4
Number of firms for each country within each of the ten GICS sectors and four regions. GICS sectors: Consumer Discretionary (CD), Consumer Staples (CS), Energy (E), Financials (F), Health Care (HC), Industrials (I), Information Technology (IT), Materials (M), Telecommunication Services (TS), Utilities (U).

		GICS SECTOR										
REGION	COUNTRY	C.D.	C.S.	E.	F.	H.C.	I.	I.T.	M.	T.S.	U.	Total
EU (188)	Austria			1	1				1	1	1	5
	Belgium	1	1		3	1		1	1			8
	Denmark					1			1			2
	Finland						1	1	2		1	5
	France	13	5	1	5	2	7	4		2	1	40
	Germany	7	3		5	2	4	2	4	1	2	30
	Greece				1					1		2
	Ireland				2		1		1			4
	Italy	2		1	6		1	1		1	2	14
	Netherlands	2	3	1	2		2	1	3	1		15
	Norway		1	2			1			1		5
	Portugal				1					1	1	3
	Spain	2		1	3		1			1	3	11
	Sweden	1			4		7	1	2	2		17
	Switzerland	3	1		6	4	6	2	4	1		27
UK (56)	UK	12	7	1	16	3	8	2		2	5	56
US (96)	US	13	12	5	16	13	17	15	1	2	2	96
REST (34)	Australia						2		2			4
	Canada			1			1				2	4
	China	3			2		1				1	7
	Japan	3	1		2	3	1	6		1		17
	Korea							1				1
	Singapore				1							1
	Total	**62**	**34**	**14**	**76**	**29**	**61**	**37**	**22**	**18**	**21**	**374**

Table C.5
Granular financial performance for EW portfolios per GICS sector by including all 374 companies across all four regions in three periods: 2010 - 2020 (top), 2010-2019 (middle) and 2020 only (bottom).

	GICS SECTOR									
	C.D.	C.S.	E.	F.	H.C.	I.	I.T.	M.	T.S.	U.
No. companies	62	34	14	76	29	61	37	22	18	21
EQUAL WEIGHTED PORTFOLIO (daily returns)										
11 years: 2010 - 2020										
Annual. Return	0.097	0.086	-0.006	0.062	0.115	0.114	0.166	0.125	0.042	0.062
Annual. Stdev	0.186	0.135	0.237	0.215	0.131	0.195	0.169	0.190	0.161	0.162
Mean	0.000	0.000	0.000	0.000	0.000	0.001	0.001	0.001	0.000	0.000
Stdev	0.012	0.009	0.015	0.014	0.008	0.012	0.011	0.012	0.010	0.010
Skewness	-0.663	-0.602	-0.552	-0.497	-0.575	-0.538	-0.714	-0.308	-0.459	-1.082
Kurtosis	12.967	8.552	16.407	12.326	8.069	9.400	8.825	4.902	7.170	15.943
10 years: 2010-2019										
Annual. Return	0.101	0.089	0.019	0.068	0.115	0.107	0.156	0.106	0.049	0.060
Annual. Stdev	0.165	0.123	0.196	0.195	0.116	0.174	0.151	0.185	0.153	0.144
Mean	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.000	0.000	0.000
Stdev	0.010	0.008	0.012	0.012	0.007	0.011	0.010	0.012	0.010	0.009
Skewness	-0.547	-0.278	-0.112	-0.349	-0.466	-0.292	-0.454	-0.168	-0.156	-0.303
Kurtosis	5.569	2.913	2.180	7.920	2.175	4.552	3.075	3.367	3.895	3.559
1 year: 2020										
Annual. Return	0.043	0.027	-0.207	-0.065	0.104	0.139	0.271	0.237	-0.030	0.074
Annual. Stdev	0.321	0.219	0.475	0.365	0.218	0.326	0.299	0.258	0.251	0.287
Mean	0.000	0.000	0.000	0.000	0.000	0.001	0.001	0.001	0.000	0.000
Stdev	0.020	0.014	0.030	0.023	0.014	0.021	0.019	0.016	0.016	0.018
Skewness	-0.624	-1.144	-0.643	-0.735	-0.727	-0.817	-1.065	-0.931	-1.532	-2.136
Kurtosis	6.312	10.920	7.968	7.932	7.440	6.917	7.935	10.693	11.804	15.615

Appendix C. Empirical data

The financial performance of these 374 companies is measured by various measures and the summary is tabulated in Table C.3. Note that the performance is evaluated for two periods of time, namely before and after the COVID-19 pandemic, but also for the combined period from 2010 until 2020. Table C.3 suggests that the financial performance in 2020 alone is significantly different from the performance observed before the COVID-19 pandemic, excepting perhaps the EU. Note that the portfolio performance tabulated in Table C.3 assumes that each asset has equal weight in the total portfolio, which is known as the *Equal Weighted (EW)* portfolio, and thus, is considered as a benchmark portfolio that is not easy to outperform in practice, see DeMiguel et al. (2013). Granular financial performance for EW portfolios per GICS sector is in Table C.5.

Fig. C.6 replicates the sector comparison displayed in Fig. 5, but only for one specific region, namely the EU. The other two regions (UK and US) are not discussed since the pattern is similar to the EU region. That is, we redo the computations shown in Fig. 5 by creating the three portfolios when including only the top 35 and 55 SRI ranked EU companies as displayed on the upper and lower panels, respectively; that is, the upper and the lower panels contain portfolios composed of $n = 35$ and $n = 55$ assets, respectively. For the period 2010-2019, working with a

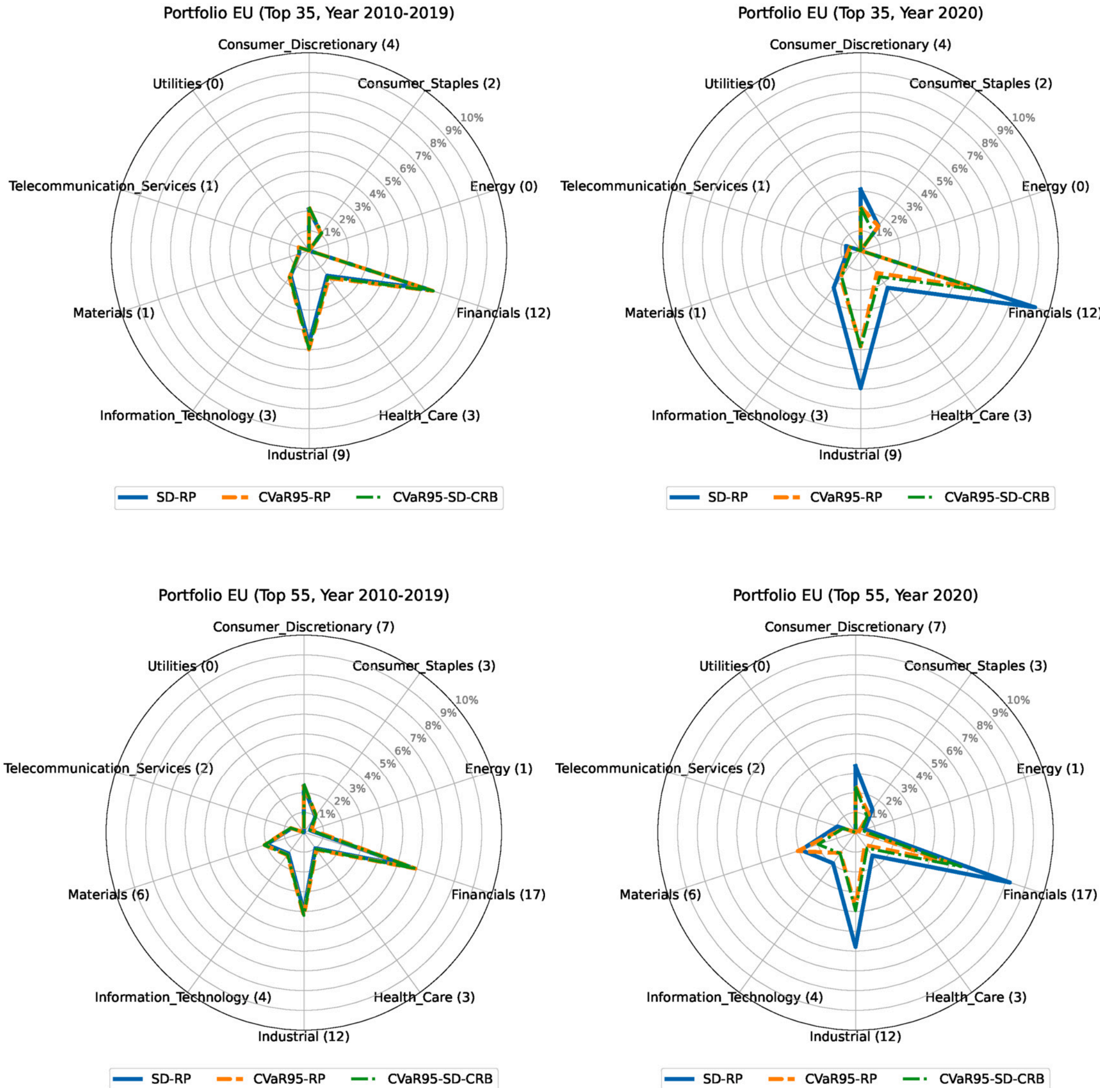

Fig. C.6. Risk contributions of each of the ten GICS sectors for 2010-2019 (left) and 2020 (right). The numbers in each bracket indicate the number of EU companies selected from that particular sector.

larger pool of companies helps to reduce the risk contributions to each sector, possibly, as a side effect of diversification. The exogenous shock of COVID-19 pandemic in 2020 produces more total risk in all sectors. The general shape in the spider plots is very similar for plots done with the same number of companies, suggesting that the market structure did not change in 2020 but the overall risk levels increased substantially.

Data availability

The authors do not have permission to share data.

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