OUTLINE OF A REPRESENTATION THEORY OF THE MEASUREMENT OF NON-CLASSICAL SYSTEMS CHARACTERIZED BY UNCERTAINTY RELATIONS

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LIST OF ILLUSTRATIONS

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ABSTRACT

This thesis represents a study in measurement theory extended to non-classical systems characterized by an uncertainty relation which is finer than the known uncertainty relations of physical and technical sciences. It suggests corresponding changes to the classical measurement theory.

Classical measurement is simple, objective and determinative, but its formalism is limited to countable sets and systems, to Boolean logic and algebra with underlying Euclidean spaces and it excludes the human experimenter. Its language satisfies the requirements of objectivity to the extent that it admits hidden parameters accounting for this objectivity. These factors limit the representational theory of measurement in several respects. Difficulties arise whenever the axioms of distributivity and of tertium-non-datur fail in the systems to be measured, which is the case in complex engineering, frequently in systems at the verge to catastrophy, in complex problems of astrophysics and cryogenics and in most human sciences, mainly in psychology and in social science. In fact, the situation that presents itself resembles problems encountered in the transition from classical mechanics to complex quantum mechanics, but at an even higher mathematical level.

Complex empirical systems are shown to obey Tarski's calculus of systems; they comply with Brouwerian lattices with unity and underlying ∞-dimensional Banach spaces containing a so-called negligible set H (the dual to Planck's constant h) which enters a non-complementarity condition and an uncertainty relation for complex fuzzy systems corresponding to the non-commutativity condition and to the Heisenberg uncertainty relation of quantum mechanics, respectively. The incompatible conjugates of the measuremental uncertainty relation are - under these conditions and conform to the "part and the whole doctrine" - precision and relevance or significance. This is where human subjectivity (knowledge and will of the experimenter) replaces the classical objectivity. The experimenter can give preference either only to precision or only to relevance.

The uncertainty of measurement is clearly of mathematical origin (the negligible set), much finer and higher allocated than any technical or physical uncertainty. Its existence is proven beyond doubt.

Having determined this uncertainty, we establish a classical measurement channel penetrating into the complex fuzzy regime of the system by constructing a quotient space (or algebra) modulo uncertainty. This step restores Boolean conditions and the classical objectivity of measurement.

Finally, using some topological theorems due to M.H. Stone, it is shown that the homomorphic representation of measurement is equivalent to a monotone homeomorphic representation and that the space of measurement is a Stone space. In this way a combinatorial measurement on classical and non-classical systems is possible. The essence of non-classical measurement is the determination of a negligible set and the corresponding uncertainty relation; without them no such measurement of the state of a complex fuzzy system is possible. And the impossibility to measure entails the impossibility to calculate.

KEY OF SYMBOLS AND ABBREVIATIONS

α, β, γ ... := logical sentences; 0,1 := the false and the true sentence, respectively,

$\phi(x)$, $\psi(x)$, $\chi(x)$, ... := propositional functions,

$\vee$:= disjunction or logical alternative,

$\wedge$:= conjunction,

$-$ or $'$:= negation; $\neg$ Boolean complementation, $\ulcorner$ Brouwerian complementation,

$\Longrightarrow$:= implication (if ..., then),

$\Longleftrightarrow$ or $\equiv$:= equivalence (iff = if and only if),

$\sqcup$:= null (zero, least) element,

$\sqcap$:= unit (greatest) element,

R^e, $\sim$, $\equiv$:= equivalence relations,

R^o, $\succsim$, $\geqq$:= order relations,

$\sqsupset$:= relative pseudo-complementation,

$\dot{-}$:= pseudo-difference

$\perp$:= orthocomplementation,

$\Gamma(\alpha)$:= the set of all $\xi < \alpha$, α - an ordinal,

ω_α := a general ordinal number,

ω_0 := the initial ordinal number

$\omega_1 = \Omega$:= the next initial ordinal number correspond. to uncountable sets,

$\aleph_\alpha = \overline{\overline{\Gamma(\omega_\alpha)}}$:= the general cardinal (alef),

$\aleph_0$:= the greatest cardinal number of countable sets,

$\aleph_1$:= the smallest cardinal number of uncountable sets,

$\mathfrak{C}$:= the power of continuum; $\aleph_1 = \mathfrak{C}$ is the continuum hypothesis,

A,B,C,...,X,Y,Z,... := sets or spaces,

$\mathring{A}$ = Int(A) := the interior of set A (an open set),

$\overline{A}$ = cl(A) := the closure of set A (a closed set),

$\overline{\overline{A}}$:= the power (cardinality) of set A,

Fr(A) = ∂A := the boundary of A,

$\delta(A)$:= the diameter of set A,

$A \times B$:= cartesian product of A and B; $A \times B \neq B \times A$,

$A \cup B$:= union of A and B,

$A \cap B$:= intersection of A and B,

$A - B$:= difference: A less B,

$A \triangle B = (A-B) \cup (B-A)$:= the symmetrical difference of A and B,

$\emptyset$:= the void set; dim.$\emptyset = -1$,

$\bigvee_{i=1}^{\infty}$ resp. $\bigcup_{i=1}^{\infty}$ resp. $\sum_{i=1}^{\infty}$:= infinite sums of numbers, resp. of sets, resp. of systems,

$\bigwedge_{i=1}^{\infty}$ resp. $\bigcap_{i=1}^{\infty}$ resp. $\prod_{i=1}^{\infty}$:= infinite products of numbers, resp. of sets, resp. of systems,

H := Hilbert space,

H^* := conjugate Hilbert space,

H^{**} := twice conjugate Hilbert space, $H = H^* = H^{**}$,

(x,y) := scalar product of x and y in Hilbert space,

B := Banach space (occasionally denoted by E),

B^* := conjugate Banach space (space of functionals),

B^{**} := twice conjugate Banach space; $B \neq B^* \neq B^{**}$,

$\|\cdot\|$:= norm in Banach space,

$|\cdot|$:= incomplete norm,

$\mathbb{N}$:= the set of all natural numbers,

$\mathbb{Z}$:= the set of all integers,

$\mathbb{J}$:= the set of all irrational numbers

$\mathbb{R}$:= the set of reals,

$I = [0,1]$:= the unit interval,

$I^{\aleph_0}$:= the Hilbert cube,

$N^{\aleph_0}$:= the set of all irrationals between 0 and 1,

$R^{\aleph_0}$:= the Fréchet space; its elements are ∞-sequences composed of arbitrary reals; it is the continuous image of the Cantor discontinuum $\mathcal{C}$,

$\mathcal{C} = \{0,1\}^{\aleph_0}$:= the Cantor set (discontinuum),

$\mathfrak{S}_t$:= the (topological) Stone space,

$\mathfrak{S}$:= the space of measurement,

$\bigvee$:= the particularizing quantifier,

$\bigwedge$:= the generalizing quantifier,

----- o ----- the end of an important statement.

CHAPTER 0: INTRODUCTION

The empirical science of measurement is one of the most basic natural sciences and an indispensible complement of human reasoning in many (if not all) scientific occupations of man. We need measurement in order to find out things and relations between things at the start of our investigations and we need measurement again to confirm the correctness of our reasonings at the end of our contemplations and actions. The result of such interplay of mind, technical skill and reality is progress in civilization and culture.

Norbert Wiener (1920) went as far as to insist that the applications of mathematics, except perhaps the non-metrical branches of mathematics, have been applications of measurement. And the early theory of measurement found a respectable, purely mathematical treatment in the Principia Mathematica of A.N. Whitehead and B. Russell.

Current (classical) methodology of fundamental measurement rests firmly on the model theory of logic. Indeed, an arbitrary empirical system under observation, consisting of a set A of elements and relations AxA satisfying certain axioms, has a model (or a realization) in the arithmetic of real or natural numbers, consisting of a set N of numerals and relations NxN satisfying corresponding conditions. A suitably constructed homomorphism ϕ (the fundamental measurement procedure) of the empirical relational system is necessary and sufficient to ensure the truth of the above statement, of course, only as long as the empirical original and its model share the same logic; for, sentences true in non-classical systems may be false or irrelevant in classical systems. We may, therefore, in tune with Stevens (1946), Ellis (1966) and Finkelstein and Leaning (1984) pronounce the following

Definition 0.1

Measurement, in the most general sense, is defined to be an assignment of numerals to objects or events of the empirical world according to any deterministic non-degenerate rule.

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Numbers, thus assigned to objects or events of the real world, must represent real world relations between the properties of those objects and events. The theory of measurement built around these notions embraces a body of knowledge about the empirical relational system, a representation theorem, a uniqueness theorem, a condition for meaningfulness, and an estimate of the methodical uncertainty involved.

0.1 Current state of classical measurement

In order to place the modifications introduced in this work in the proper perspective, a compact survey of the current state of the art of measurement is necessary. We shall begin with a unified symbolism, terminology and

0.1.1 The concept of a quantity of measurement

Let $o_1, o_2, \ldots$ be elements of the real world, called objects, and O the set comprising them; each o_i is carrier of property manifestations. Let $q_1, q_2, \ldots$ be manifestations of a property Q; the q_i are assigned to the o_i, $i = 1,2,\ldots$ by nature. Hence, they will not exist as singletons, but in plurality. We call $\{q_1,q_2\} \subset Q_1 \cup Q_2$ an unordered pair and

$\langle q_1,q_2\rangle = \{\{q_1\}, \{q_1,q_2\}\} \in Q_1 \times Q_2$ an ordered pair with q_1 preceding q_2.

$\langle q_1,q_2\rangle$ is actually an element of $2^{2^{Q_1 \cup Q_2}}$.

The Kuratowski formula for the ordered pair can easily be extended to three elements, four elements and so on.

Let Q be a property, i.e. the set of all manifestations of this property, then it must constitute either a relational system or an algebra or both. It is a relational system if $Q = \langle Q,R\rangle$, where $R \subset Q \times Q$ is a relation on Q; it is an algebra if $Q = \langle Q,O\rangle$ is provided with the set O of operations $o \in Q^{Q \times Q}$, called concatenations.

We distinguish binary relations: $\sim, \succ$; $q \sim r$, $q \succ r$;

ternary relations: o; $o \in Q^{Q \times Q} \equiv Q \times Q \times Q$;

and quaternary relations: $\succsim$; $\langle q,r\rangle \succsim \langle s,t\rangle$; $\succsim \subset Q \times Q \times Q \times Q$.

We shall now simplify the terminology of $q \in Q$ in the sense of Ellis (1966).

Definition 0.2

A quantity, denoted by q, is conceived of as the manifestation of a property on a given object; more concretely, it is thought to be a kind of property that admits of degrees, in contrast to those properties that have an all-or-none character (e.g. dead, pregnant, champion,...).

Hence objects possess quantities in the same way that they possess qualities, attributes, characteristics a.s.o. . They are properties inherent in an object possessing them, and existing before any measurement begins. The process of measurement is then conceived to be that of assigning numbers to represent the magnitudes of these pre-existing quantities, the numbers being proportional to these magnitudes. Such pre-existence is not assumed in quantum mechanics. Thus, things having a quantity q in common (for example mass, length, area, temperature, electric charge etc.) must be comparable in respect to the quantity q. The quantitative relationships with respect to q are: $<_q$, $=_q$ or $>_q$. Then, a quantity is said to exist iff a quantitative set of relationships exists.

The following five "Ellis conditions" are necessary before passing a qualifying judgement on $<_q$, $=_q$ or $>_q$ between any two objects:

1. The law of the excluded middle (terium-non-datur);
2. The law of non-contradiction;
3. Converse and symmetry conditions;
4. Transitivity, and
5. Asymmetry conditions.

These five conditions are by no means sufficient. Ellis strongly believes that linear order fills the sufficiency gap. We shall show that the necessary and sufficient conditions required are an underlying Boolean logic (algebra) and the Axiom of Choice (AC) added to the axiomatic of set theory, see Kaaz (1977).

According to Tarski (Axiomatic and algebraic aspects of two theorems on sums of cardinals, Fund. Math. 35 (1948) 79-104) and Mostowski (On the principle of dependent choices, Fund. Math. 35 (1948) 127-130), the (AC) is equivalent to

The Trichotomy Law

For two given sets A and B, either there is a set C such that $A \sim C \subset B$ or else there is a set D such that $B \sim D \subset A$; "$\sim$" is the relation of set-theoretical equivalence (equality of cardinals).

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If we select from each of these sets a representative element, as we may by reason of (AC), then the first relation yields $a \leqq b$ and the second $b \leqq a$, which is the connectivity in linear ordering; the reflexivity, weak unsymmetry and transitivity of linear ordering being obviously satisfied. This confirms Ellis' conjecture. If the set of quantities is a finite or countable one, then Boolean logic and the Axiom of Dependent Choices satisfy necessity and sufficiency for the existence of a quantity.

The Axiom of Dependent Choices (DC)

If R is a binary relation and B a nonvoid set, and if, for every $x \in B$, there is a $y \in B$ such that xRy, then there is an infinite sequence $x_1 x_2, \ldots, x_n, \ldots$ of elements of B such that $x_n R x_{n+1}$ for $n = 1, 2, \ldots$.

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Having specified the conditions for the existence of a quantity, we can now say that that which is being measured - if it exists - is the quantity, called measurand by Finkelstein (1975a). In a more general sense it may be called "observable", which is frequently the case in physics. For example, a random variable is considered to be an observable if a probabilistic model of the physics is used. The centre of the stage of such discussions is occupied by a physical system Σ and the experimental propositions (x,E) associated with Σ, where x stands for the random variable and E is a Borel set on the real line.

We know that Bohr's atomic theory broke down because it dealt with quantities which entirely elude observation and cannot be put to any test. Therefore, since Heisenberg, only observable entities should be introduced into a modern theory. In the quantum mechanical view, to every observable physical quantity there corresponds a selfadjoint operator in Hilbert space. At the same time, measurable are only those physical quantities which, as operators (in the Hilbert space of physical states), are commutative. Such operators generate a commutative C^*-algebra (a Banach algebra) A whose spectrum is a locally compact space $\mathcal{H}$ comprising the quantum numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$.

0.1.2 The representational style of measurement

Since, presumably, every measurement must be made on a scale, Definition 0.1 tells us that if we have a rule for making numerical assignments, we also have a scale of measurement and, conversely, if we have a scale of measurement, we must at least have some rule for making numerical assignments. Hence, we have a scale iff we have a rule for making such assignments. And this rule must be deterministic and non-degenerate. We claim with respect to ϕ

The identity of scales

Two procedures ϕ and ϕ' are procedures for measuring on the same scale, whenever they are deemed to be applicable; they would always lead to the same numerical assignments being made to the same things under the same conditions. Thus, $\phi - \phi' = \emptyset$.

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Scales and classification of scales for the measurement of quantities:

The above concept of a scale includes such scales that are not designed for measurement, e.g. nominal scales measuring identity and difference only, on the one extreme, and multidimensional scales for the measurement of complex entities, such as colour and stress, on the other extreme. A scale for measurement of quantities must account for the Axiom of Choice, respectively for linear order, both of which warrant the existence of the measured quantity. The simplest such scale S for the measurement of a given quantity is assumed to have the following properties:

Definition 0.3

S is a scale for the measurement of a given scale q iff:

(i) there is a procedure ϕ for measuring on S such that for any object x which occurs in the order of q, x is measurable by ϕ,

(ii) there is no object which is measurable on S and does not occur in the order of q,

(iii) if the objects measurable on S are arranged in the order of the numerical assignments, they are thereby arranged in the order of q.

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Allowing for the fact that there are scales representing quantities more accurately than others, the above definition is superior to known alternatives.

Now, a system of classification gives us a means of looking at a group of phenomena, and it determines, to some extent, what general statements can be made about them. Two such systems are of particular importance,

a) Campbell's system grouped according to the kinds of procedures used in setting them up, and
b) Stevens' system grouped according to mathematical properties.

Strangely enough, b) is more useful to the practical scientist who, from the knowledge of the kind of scale on which a set of measurements is obtained, will be able to determine what sorts of statistics are relevant to these measurments. a), on the other hand, probes deeper into the conditions for the possibility of measurement and reveals the significance of numerals assigned to things when making measurements.

Campbell's system depends on an initial distinction between fundamental measurement and derived measurement. Fundamental measurement (such as that of mass, length,...) does not depend on prior measurement, derived measurement (of density, velocity) does. Since temperature measurement depends only on the measurement of one other quantity, the distinction between fundamental measurement and derived measurement is not exhaustive; hence we speak of:

c) derived measurement if the constants in the numerical laws are to be determined by measurement,
d) associative measurement which is exemplified by temperature measurement, and
e) indirect measurement involving the measurement of more than one other quantity.

c) and d) are, of course, species of e).

Fundamental measurement is measurement by a procedure ϕ which conforms to a certain pattern, and it is possible only because certain kinds of operations are possible. We note that scales of mass, length, volume, time-interval, electrical potential, etc. may be set up by logically similar procedures, and that these procedures are possible only because certain kinds of operations (e.g. addition) may be performed on systems having these quantities. There are, however, procedures of measurement which are neither conforming to Campbell's pattern, nor indirect, for example:

Moh's scale applied to the measurement of hardness. For this reason Ellis proposed the term "direct measurement" for the more general class, of which Campbell's pattern is a species. Also, since hardness measurement involves no more than order, measurements like that of hardness are to be called "elemental measurements".

The mathematical approach of Stevens is different; he asks, what transformations leave the scale intact? His criterion of invariance (serving the same purpose) is ambiguous. This reasoning yields four kinds of scales for immediate use (in the order of richness):

"nominal scale" (e.g. the numbering of football players)
invariant transformation: any permutation
$a = b$ or $a \neq b$, $\phi \longrightarrow f(\phi)$,

"ordinal scale" (e.g. hardness measurement)
invariant transformation: $\phi \longrightarrow f(\phi)$,

"linear interval scale" (e.g. temperature measurement)
invariant transformation: $\phi \longrightarrow \alpha\phi + \beta$
$\alpha > 0$,

"ratio scale" (e.g. length, mass, time measurement)
invariant transformation: $\phi \longrightarrow \alpha\phi$.

We may add:

"logarithmic interval scale" (see Krantz et al.(1971)),
invariant transformation: $\phi \longrightarrow \alpha\phi^{\beta}$.

The statistics associated with these scales are discussed in Ellis (1966), in particular:
generating functions, averaging functions, arithmetical mean, geometric mean, root mean square, harmonic mean.

Construction of a metric scale:

Recall that a nominal scale is a one-one correspondence beween equivalence classes of objects manifesting the same property and real numbers. A scale reproducing order (sequencing) only is called ordinal (or topological) scale; its only invariant characteristic is the rank of the measures. A linear relation reveals far more about the structure of the measured attribute than any order relation. In any case, meaningful are only those relations which remain invariant with respect to all admissible transformations. In order to judge the meaningfulness of relations, one must know

what transformations are admissible.
Aside of topological scales admitting arbitrary monotone transformations, of real importance are scales unique up to linear transformations or even stretchings. These scales (which account for interval and ratio scales) are called "metric scales". We arrive at metric scales not only in cases of additive properties; indeed, for the construction of a metric scale weaker assumptions than additivity suffice.

General Postulate: A set Q is to be mapped onto the set of reals in such a way that the structure of the numerical image is isomorphic to the structure of Q.

Axiomatic for metric scales

1. Order axioms: A set Q is called ordered if satisfying the following five axioms:

   O1. $(q_1, q_2 \in Q) \Longrightarrow (q_1 \sim q_2 \vee q_1 \prec q_2 \vee q_1 \succ q_2)$,

   O2. $\bigwedge_{q \in Q} (q \sim q)$,

   O3. $(q_1 \sim q_2) \Longrightarrow (q_2 \sim q_1)$,

   O4. $(q_1 \sim q_2 \wedge q_2 \sim q_3) \Longrightarrow (q_1 \sim q_3)$,

   O5. $(q_1 \prec q_2 \wedge q_2 \prec q_3) \Longrightarrow (q_1 \prec q_3)$.

2. Topological axiom:

   T1. Set Q is connected.

Assumption 1: The map ϕ: $Q \longrightarrow R$ is to be continuous and similar, where similarity means: $(q_1 \sim q_2) \Longrightarrow (\phi(q_1) = \phi(q_2))$.

Theorem 0.1

An ordered and connected set Q may be mapped continuously and similarly onto R (or its subset) iff Q contains a subset dense in itself. This mapping is unique up to monotone and continuous transformations.

3. Distance axioms:

   D1. Ordering axiom: Defined is an ordering relation between the pairs of elements $q \in Q$ satisfying O1-O5,

   D2. Monotony axiom: $(q_2 \lesssim\gtrsim q_2') \Longrightarrow (q_1 q_2 \lesssim\gtrsim q_1 q_2')$ for all q_1,

   D3. Continuity axiom: The sets $\{x \in Q\colon xq_3 \prec q_1 q_2\}$ and $\{x \in Q\colon xq_3 \succ q_1 q_2\}$ are open for all q_1, q_2, q_3; the same holds for q_3x,

D4. Commutativity axiom: $(q_1q_2 \underset{\succ}{\overset{\prec}{\sim}} q_3q_4) \Longrightarrow (q_1q_3 \underset{\succ}{\overset{\prec}{\sim}} q_2q_4)$.

4. Axioms of metric connectivity:

MC1. Existence axiom: To each pair of elements $q_1, q_2 \in Q$ is associated an element $q_1 o q_2$.

MC2. Monotony axiom: $(q_2 \underset{\succ}{\overset{\prec}{\sim}} q_2') \Longrightarrow (q_1 o q_2 \underset{\succ}{\overset{\prec}{\sim}} q_1 o q_2')$ for all q_1,

$(q_1 \underset{\succ}{\overset{\prec}{\sim}} q_1') \Longrightarrow (q_1 o q_2 \underset{\succ}{\overset{\prec}{\sim}} q_1' o q_2)$ for all q_2,

MC3. Continuity axiom: The sets $\{x \in Q: x o q_2 \prec q_1\}$ and $\{x \in Q: x o q_2 \succ q_1\}$ are open for all q_1, q_2; the same holds for $q_2 o x$,

MC4. Bisymmetry axiom: $(q_1 o q_2) o (q_3 o q_4) \sim (q_1 o q_3) o (q_2 o q_4)$.

Note: Metric connectivity axiomatic coincides with the additivity axiomatic if we replace "o" by "+" and substitute for MC4:

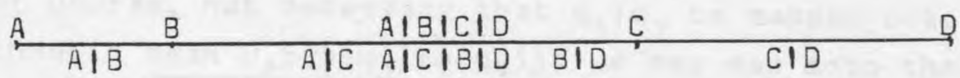
Associativity axiom: $q_1 + (q_2 + q_3) \sim (q_1 + q_3$, and

Commutativity axiom: $q_1 + q_2 \sim q_2 + q_1$.

5 Axioms of the middle formation (replacing "o" by "|"):

MC'5. Commutativity axiom: $q_1 o q_2 \sim q_2 o q_1$,

MC'6. Reflexivity axiom: $q_1 o q_1 \sim q_1$.



Conclusions:

Given a metric connectivity (MC1-4), it is possible to define a "middle" which satisfies MC1-4 and MC' 5,6.

Given a metric connectivity, it is always possible to define a "distance", and vice versa.

Definition 0.4

A continuous mapping reflecting correctly the order of elements of Q as well as the order of distances is called a metric mapping.

Postulate M

The mapping $Q \longrightarrow R$ is to be metric; this follows from the General Postulate.

Theorem 0.2

If Q satisfies 01-5, T1 and D1-4 or MC1-4, then there exists a metric scale, unique up to linear transformations.

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For a proof see Pfanzagl (1959).

The middle can be obtained from D1-4 as well as from MC1-4 axioms. Then, there exists a monotone continuous mapping $x \longmapsto \phi(x)$ of Q to R such that $(q_1|q_2) = 0.5(\phi(q_1)+\phi(q_2))$, unique up to positive linear transformations, i.e. scales obtained from a given metric scale by positive linear transformations are again metric scales. In this way we obtain all metric scales, all connected by linear positive transformations. Then, $q_1|q_4 \sim q_2|q_3$ is always necessary and sufficient for $q_1q_2 \sim q_3q_4$; but from $q_1|q_4 \sim q_2|q_3$ follows $\phi(q_1|q_4) \sim \phi(q_2|q_3)$ and hence $\phi(q_2)-\phi(q_1) = \phi(q_4)-\phi(q_3)$, and vice versa. Thus, distance q_1q_2 is represented by the difference of measures $\phi(q_2)-\phi(q_1)$.

When the middle is defined on the basis of a metric connectivity, $\phi(q_1 o q_2) = r\phi(q_1)+s\phi(q_2)+t$, $r > 0$, $s > 0$, since "o" is monotone positive. If $r+s \neq 1$, zero shift will cancel t; if "o" is commutative, r = s; if "o" is also additive, r = s = 1 and we get $\phi(q_1 o q_2) = \phi(q_1)+\phi(q_2)$.

For commutative connections with r+s = 1, we get $r = s = \frac{1}{2}$; and if we want uniqueness up to stretchings in this case, further criteria are necessary.

It is, of course, not necessary that $q_1|q_2$ be mapped onto the arithmetic mean $0.5(\phi(q_1)+\phi(q_2))$; we may map onto the geometric mean $\sqrt{(\phi(q_1)\phi(q_2))}$. Then, the magnitude of the interval is not represented by the difference, but by the quotient of measures (as for instance in the case of absolute temperature).

Often several connectives are available for the construction of a scale. The problem is then to determine, under what conditions two relations lead to the same scale.

Definition 0.5

Two relations are called isometric if, for arbitrary $q_1 q_2$, $q_3 \in Q$, $(q_1 o q_2) \bullet (q_3 o q_4) \sim (q_1 \bullet q_3) o (q_2 \bullet q_4)$.

This isometric notion is reflexive, symmetric and - with 01-5, T1 and D1-4 - also transitive.

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Theorem 0.3

Isometric connectives lead to scales identical up to linear transformations. If at least one of the connectives is non-singular (i.e. $r+s \neq 1$), then there is a uniquely defined zero. The scale is then unique up to stretchings.

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Two scales lead to the same scale if an internal connectivity exists between them. This statement is false if both connectives are singular.

Temperature measurement (after Pfanzagl (1959)

This is an example in which the elements of Q are interpreted as temperatures and the pairs of elements of Q as temperature differences. The magnitude of a temperature difference is determined by the efficiency of the Carnot cycle between these temperatures. Of the axioms O1-5, T1, D1-4 only D4 requires an explanation. The satisfaction of this axiom is ensured by the second law of thermodynamics: if there were four temperatures t_1, t_2, t_3, t_4 such that the efficiencies of the cycle processes between t_1 and t_2 on the one hand, t_3 and t_4 on the other hand are equal, but between t_1 and t_3, respectively t_2 and t_4, are different, then by combining the processes t_1t_2, t_2t_4, t_4t_3, t_3t_1 one could form a cycle process which withdraws heat from the reservoir with temperature t_1 and transforms it into work without changing the state of the other reservoirs.

But it is customary to choose the temperature scale in such a way that the magnitude of the temperature difference be expressed - not by the difference of measures but by their quotient. Thus, the temperature values t^* are unique up to transformations of the form $t^{**} = ut^{*\lambda}$.

If we postulate a substance whose heat capacity is temperature-dependent, then a much more elementary possibility of defining temperature results: having two bodies of equal heat capacity, one is brought to temperature α, the other to temperature β, and then a heat exchange between them is initiated. The resulting temperature is denoted by $\alpha | \beta$. This operation fulfills axioms MC1-4, MC′5,6. The temperature $\alpha | \beta$ is, however, dependent on the chosen substance owing to the temperature-dependent heat capacity.

Measurement theory in representational form

The representational theory of measurement, in which numerals assigned to objects or events have to represent the perceived relations between the quantities of those objects or events, comprises three essential parts: the description of a given empirical relational system (say $\mathcal{Q} = \langle Q,R \rangle$, $Q \neq \emptyset \neq R$, being nonvoid, represent the set of quantities and the set of relations, respectively), a representation theorem, and a uniqueness theorem.

The empirical relational system $\mathcal{Q} = \langle Q,R \rangle$ is supposed to have a model $\mathcal{N} = \langle N,P \rangle$ in the space of numerals such that Q corresponds to N and R to P, respectively. Then, measurement is an objective empirical operation (in fact a homomorphism) ϕ - henceforth called the fundamental measuring procedure - mapping Q into N, i.e. $\phi: Q \longrightarrow N$, in such a way that the relations between the numerals also hold between the quantities. This is ascertained by the

Representation Theorem

Let R_i, $i = 1,2,\ldots,n$, be the relations on $Q = (q,r,\ldots)$, and P_i, $i = 1,2,\ldots,n$, be the relations on $N = (a,b,\ldots)$; then, for ϕ a homomorphism, we have the equivalence:

$$R_i(q,r,\ldots) \equiv P_i(\phi(q),\phi(r),\ldots). \quad (0.1)$$

Uniqueness Theorem

The uniqueness of the measuring procedure ϕ in (0.1) demands that $\phi - \phi' = \emptyset$, for any other procedure ϕ' obtained by an admissible transformation f of ϕ, i.e. $\phi' = f(\phi)$. Hence

$$R_i(q,r,\ldots) \equiv P_i(f(\phi(q)),f(\phi(r)),\ldots). \quad (0.2)$$

f characterizes here the class of admissible transformations.

Definition 0.6

The triple of sets $S = \langle Q,N,\phi \rangle$ satisfying the representation and uniqueness theorems is said to be the scale of measurement carried out on the empirical relational system $\mathcal{Q} = \langle Q,R \rangle$.

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Let us subject some direct measurement structures to the formalism of the just defined theory of measurement.

Nominal binary system

In this case Q carries an equivalence relation, i.e. the empirical relational system $Q = \langle Q, \sim \rangle$. The representation theorem for two quantities $q, r \in Q$ states:

$$q \sim r \equiv \phi(q) = \phi(r);$$

uniqueness states $\phi' = f(\phi)$, f being a one-one function.

Ordinal binary system

Here, the least binary relation will be weak order relation for ordering according to the amount of the manifested property in each object or event; hence, R contains the reflexive relation $\succ$. The representation theorem states:

$$q \succ r \equiv \phi(q) > \phi(r);$$

uniqueness states $\phi' = f(\phi)$, with f isotonic.

Extensive systems (following Finkelstein and Leaning (1984))

Extensive systems are distinguished by concatenation which has been defined earlier as the ternary relation R_o on $Q \times Q \times Q$ such that, for $q, r, s \in Q$, $R_o(q,r,s) \equiv q o r \sim s$. Mass, length, time and probability (for example) are extensive quantities.

Representation theorem: $q \sim r \equiv \phi(q) = \phi(r)$,

$$q o r \sim s \equiv \phi(q) + \phi(r) = \phi(s).$$

In this case, weak order may replace the equivalence, and the representation theorem for closed extensive measurement (i.e. such that no concatenation leads beyond its field of definition) holds under the subsequent conditions:

- $\sim$ is the equivalence relation (resp. $\succ$ is weak order),
- associativity: $q o (r o s) \sim (q o r) o s$,
- Monotonicity: $q \succsim r \Longrightarrow q o s \succsim r o s$,
- Archimedean axiom: $(q \succ r) \Longrightarrow \bigvee_{n \in N} (nr \succ q)$, where $nr = \underbrace{r o r o \ldots o r}_{n}$.

Quaternary interval systems

In difference measurement we compare intervals between objects. The quaternary relation $\succsim$ in R indicates, in the case $(q,r) \succsim (s,t)$, that the interval (q,r) is equal to or greater than the interval (s,t).

Representation theorem: $(q,r) \succsim (s,t) \equiv \phi(q,r) \geqq \phi(s,t)$

$$\phi(q,s) = \phi(q,r) + \phi(r,s).$$

Uniqueness condition: $\phi' = \alpha\phi + \beta, \ \alpha > 0$.

The class of admissible transformations f used in these examples together with the notion of applicable arithmetic may be extended to the more general concept of meaningfulness. This is concerned with the status of statements concerning measurements on each particular scale. However, it is independent of the truth of the statement concerned. For example, the statement that q is twice as long as r, measured on an entensive scale, is meaningful, but meaningless when measured on an ordinal scale with the arithmetic of medians (see Pfanzagl (1959). Obviously, the truth of the sentence is not involved in such a meaning of meaningfulness.

Criterion of the meaningfulness of P (Pfanzagl (1971))

Let $Q = \langle Q,(R_i)_{i\in I}\rangle$ be an irreducible empirical relational system, $N = \langle N,(P_i)_{i\in I}\rangle$ a numerical relational system of the same type and such that $\phi: Q \longrightarrow N$ exists, and P be a k-ary relation on N. Then, P is said to be meaningful iff, for all ϕ, $\phi' = f(\phi) \in \Phi(Q,N)$ and $q_1,q_2,\ldots,q_k \in Q$, the equality:

$$P(\phi(q_1),\ldots,\phi(q_k)) = P(\phi'(q_1),\ldots,\phi'(q_k)), \qquad (0.3)$$

$$\text{equivalently:} \quad \phi_k^{-1}(P) = (\phi')_k^{-1}(P), \qquad (0.4)$$

holds.

Under the above conditions, P is meaningful iff

$$\Phi(Q,N) = \Phi(\langle Q,(R_i)_{i\in I},\phi_k^{-1}(P)\rangle,\langle N,(P_i)_{i\in I},P\rangle) \qquad (0.5)$$

is satisfied; in other words: every one-one homomorphism of Q into N is a one-one homomorphism of Q, enriched by the relation $\phi_k^{-1}(P)$, into N, enriched by the relation P.

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Thus, relational meaningfulness exists iff any increase in the number of relations is a balanced one at both ends of the scale. We see here an analogy to the general principle of physics, where physical sentences in the 3-dimensional space become laws in this space iff they are Lorentz-invariant in the 4-dimensional Minkowski world.

Practical "usance" of the representational theory

The general theory represents "snapshots" of measurement once the concept of quality manifestations (of quantities) has been fixed and the empirical operations and relations decided and tested. It provides a good model for measurement in physical sciences where empirical properties are closely related to well validated laws, and theories and measurement operations are quite sophisticated.

The formal theory of measurement does not include the dynamic processes of conceptualization and scaling, whereby a primitive theoretical concept or vague idea is developed into one which has a clearer theoretical meaning, and can be realized as a set of empirical relations and operations to form a measurement scale. Such a state of investigational work characterizes measurement in the social sciences: in economics, psychology, sociology. To cope with the issues, various scaling techniques have been developed.

The broad intention of scaling is to establish a scale $S = \langle Q, N, \phi \rangle$ in such a manner that the data obtained provide a way of testing the assumptions underlying the scale. Once successfully tested, the measurements can then be used generally. The assumptions of greatest importance are those relating to the empirical relations and operations on Q, especially that of order.

Rank ordering implies: a set of objects, called "items", a in number, is ranked according to the attribute in question. The items are forced into transitive order, say $A \succ B \succ C$. In cases of paired comparisons, every pair of items is separately compared, in which case the overall pattern of relations need not obey transitivity; for, while $A \succ B$, $B \succ C$, nevertheless $C \succ A$. So, if the results exhibit transitivity, this is empirical evidence for the assumption. By the use of ordered triads, transitivity and consistency are ensured. Finally, in Coombs' parallelogram analysis, k items out of n are singled out which are closest together with respect to the attribute. The study is directed at revealing some underlying dimensionality, possibly a line-continuum.

This is the "unidimensional scaling" technique for the attribute under study. It must show that the order relation applies to the whole range of objects in a transitive manner. We speak of "multidimensional scaling" whenever the attribute has more than one dimension. However, the long term aim must remain, to establish a single dimension for each attribute. In multidimensional scaling, prior measurements of the distance d_{ij} between any two pairs of objects i and j are assumed to exist. Each object is mapped into a k-dimensional metric space in such a way that the separating distance is nearly equal to the measured distance. If $x_l(i)$ happens to be the projection of object i along the dimensional ray l ($l = 1,2,\ldots,k$), then d_{ij} is the Euclidean metric. The statistical task (goodness of fit) is to minimize k.

After the confirmation of the attributes continuum, its ordering structure is investigated, while remembering that no social science attributes are extensive, save counting. In difference measurement, intervals between attributes admitting affine transformations $\phi' = \alpha\phi + \beta$ are compared. The set-up is again metric, and distances d_{ij} are also used in interval scales; however, the determination of k independent of β may prove to be difficult. This scaling is classified as "interval scaling". This, together with the previous methods of scaling, is based on the direct observation of the quantity of measurement. But such scales can also be established indirectly, i.e. via other directly measurable quantities. The fundamental concepts underlying "indirect scaling" correspond to the views of Campbell, with regard to direct measurement, and of Ellis, with regard to derived and associative measurement. They may be summarized thus:

Consider a quality Q_o for which an indirect scale of measurement is desired; as usual $Q_o = \langle Q_o, R\rangle$. Imagine also a class of all objects O which carry manifestations of the quality Q_o. Let each element of O also exhibit logically independent qualities $Q_1,\ldots,Q_n$. Finally, let there exist, for each of these qualities, a scale of measurement $S_i = \langle Q_i, N_i, \phi_i\rangle$. Each element $o \in O$ which exhibits its $q_o \in Q_o$ also exhibits a set of manifestations $(q_1,\ldots,q_n)$, where $q_i \in Q_i$. Hence, q_o is characterized by an n-tuple od measures $\phi_1(q_1),\ldots,\phi_n(q_n)$. Assume a mapping $\chi: \phi_1(q_1),\ldots,\phi_n(q_n) \longrightarrow N_o$. The composition $\chi \circ (\phi_1,\ldots,\phi_n)$ and the corresponding elements of Q_o and $\prod_{i=1}^{n} Q_o$ constitute a function of Q_o to N_o, which we denote by ϕ_o.

Assume next that there exists a set of relations P_o on N_o as part of the relational system $N_o = \langle N_o, P_o\rangle$. It is clear that, if ϕ_o transforms Q_o homomorphically into N_o, scale $S_o = \langle Q_o, N_o, S_1,\ldots,S_n,\phi_o\rangle$ will be an indirect scale of measurement of Q_o.

The problem of constructing an indirect scale of measurement reduces to the establishment of mapping χ from the measures of qualities associated with the measured quantity into a numerical relational system so as to obtain a satisfactory representation on an indirect measurement scale. In practice χ is obtained by definition or from models. For both methods see Finkelstein and Leaning (1984).

0.1.2.1 Simultaneous measurement

It seems that the theory of extensive measurement is not wholly adequate for the representation of physical and social attributes by numerals for the following reasons:

1) Axioms usually relating to weak order and concatenation are questionable even in structures of the type $\langle Q, \succsim, A, \phi \rangle$, where $A \neq \emptyset$, $A \subset Q \times Q$ and $\phi: A \rightarrow Q$.
2) It is doubtful whether comparisons of attributes can be treated as weak order and that indifference is transitive.
3) Alternative interpretations for the concatenation lead to nonlinearly related scales of length; the velocity representation is non-additive.
4) While the natural empirical concatenation existing for length and time may be used to construct a fundamental measure, three physical attributes cannot be adequately concatenated: density, momentum and hardness.

It was stated earlier that counting is possible in social sciences, but in the simplest relational structure $\langle Q, \succsim \rangle$ we cannot count units because there is no way of deciding what constitutes two units when identifying which element of Q is the sum of two others. Thus, for counting the structure requires additional features. One of the possible solutions is for Q to be a cartesian product $Q = Q_1 \times Q_2$, because then two factors determine the ordering $\succsim$. The structure $Q = \langle Q_1 \times Q_2, \succsim \rangle$ enables us to perform a two-component additive "conjoint measurement" (for axioms and proof see Krantz et al. (1971)).

Representation theorem: For $p, q \in Q$, $p_1, q_1 \in Q_1$, $p_2, q_2 \in Q_2$ and $(p_1, p_2) \succsim (q_q, q_2)$, we have

$$p \succsim q \equiv \phi_1(p_1) + \phi_2(p_2) \succsim \phi_1(q_1) + \phi_2(q_2).$$

Uniqueness condition: $\phi_1' = \alpha\phi_1 + \beta_1$, $\phi_2' = \alpha\phi_2 + \beta_2$, $\alpha > 0$ and $\phi' = \alpha\phi + \beta$, $\beta = \beta_1 + \beta_2$.

Hence, ϕ is of the interval scale type procedure.

There is a symmetry between the components, so that any two of the three attributes can be chosen for the measures ϕ_1 and ϕ_2. In this sense conjoint measurement is concerned with the setting up of a structure for the simultaneous measurement of all the properties concerned. This is the reason for devoting a subsection to the problem of simultaneous measurement.

Classical measurement (and observation) is commonly limited to simple relational systems obeying Boolean logic (algebra) and satisfying all axioms of the Zermelo-Fraenkel set theory including the axiom of choice; where Q is at most countable, we have $\bar{\bar{Q}} \leq \mathfrak{a}$, $\mathfrak{a}$ being the cardinal of countable sets. Such systems are generally deterministic, measurable and simultaneously observable. There is no room for uncertainty other than instrumental and methodical; hence, a determination of the state of a classical system requiring - in accordance with the doctrine "Part and the Whole" - the simultaneous measurement of (at least) two complementary quantities, is always possible.

Assume x to be a quantity (it may also be a random variable) and E a Borel subset of the real line R^1. The pair (x,E) is called an "experimental proposition" before the experiment, and an "experimental statement or sentence" after the experiment. It may, of course, be true or false. The set of experimental sentences will be denoted by $\mathcal{E}$. Now, the abstraction class $[(x,E)]$ of sentences (x,E) represents some kind of physical quality. The identification of logically equivalent sentences in a formalized theory T is rather common in mathematical logic; the result of such identification is the so-called Lindenbaum-Tarski algebra of T. Let F be the set of all formulas $\alpha, \beta, \ldots$, and $\mathcal{F} = (F, \vee, \wedge, \Longrightarrow, -)$ the algebra of formulas. Then the Lindenbaum -Tarski algebra of sentences is defined as the quotient algebra of formulas $L(T) = \mathcal{F}/\approx$, the congruence relation "$\approx$" being defined thus:

$\alpha \approx \beta$ iff $(\alpha \Longrightarrow \beta)$ and $(\beta \Longrightarrow \alpha)$ are theorems in T.

Whether L(T) is a Boolean algebra or a non-Boolean algebra depends on the admissible operations on the classes of $\mathcal{F}/\approx$. It is a Boolean algebra in case of simple systems considered here; but it is a non-Boolean algebra in case of complex fuzzy systems which will be considered in subsequent sections and chapters of this thesis.

Upon ordering the classes into a poset (partially ordered set) and using the complementation operation, $L(T)$ becomes a logic $\mathcal{L}$ with implication and negation, respectively. In this case $\mathcal{E}$ is taken to be a σ-algebra of subsets of some space Ω. We then define a random variable as a real-valued function f on Ω such that $f^{-1}(E) \in \mathcal{E}$, for $E \subset R^1$. Given a probability measure $p: \mathcal{E} \rightarrow [0,1]$, we can obtain the distribution α_f^p of f under p, defined by

$$\alpha_f^p(E) = p(f^{-1}(E)). \qquad (0.6)$$

In general $\mathcal{E}$ is a set, partially ordered by "$\geqq$" and complemented (in quantum mechanics: orthocomplemented). Given a physical quantity x and a Borel function u (i.e. a real-valued Borel-measurable function u on the real line R^1), there is an operational definition of the quantity $u(x)$; in fact, if x has the value ξ, $u(x)$ has the value $u(\xi)$, and $u(\xi) \in E$ iff $\xi \in u^{-1}(E)$. Thus, given an observable $x(E \longmapsto x(E))$ and a Borel function u, we define the observable $u(x)$ by the assignment $u(x): E \longmapsto x(u^{-1}(E))$. No doubt $u(x)$ is a σ-valued measure based on the Boolean algebra $B(R^1)$ of sets on the line, so that $u(x)$ is in fact an observable. Let x be an observable, u_1, u_2 Borel functions and $u = u_1 o u_2$ their convolution; then, $u(x) = u_1(u_2(x))$. If x has distribution α under p, then $u(x)$ has distribution β under p; where $\beta(E) = \alpha(u^{-1}(E))$ for all E. Thus, the rules for calculations of functions and distributions of functions of a given observable are the same as in the conventional formalism. However, functions of more than one variable can, in general, only be formed under special circumstances.

Given $E, F \in B(R^1)$ and elements $a, b \in \mathcal{E}$ such that $a \leftrightarrow b$ whenever there are pairwise disjoint elements a_1, b_1, c such that $a = a_1 \dotplus c$, $b = b_1 \dotplus c$, we claim that the necessary and sufficient conditions for $a \leftrightarrow b$ is that there should be an observable x and Borel sets E, F satisfying $a = x(E)$ and $b = x(F)$.

<u>Definitions 0.7</u>

Elements $a, b \in \mathcal{E}$ are called simultaneously verifiable if

$a \longleftrightarrow b$. We call x and y simultaneously observable quantities if, for any pair of Borel sets E,F, both x(E) and y(F) are simultaneously verifiable. We may actually generalize this definition to the indexed set $\{x_\lambda : \lambda \in \Delta\}$ and call $\{x_\lambda : \lambda \in \Delta\}$ simultaneously observable if x_λ and $x_{\lambda'}$ are simultaneously observable for all $\lambda, \lambda' \in \Delta$.

Theorem 0.4 (Varadarajan (1962))

Let $\{A_\lambda : \lambda \in \Delta\}$ be an indexed set of Boolean subalgebras of $\mathcal{E}$. Then, in order that there be a Boolean subalgebra of $\mathcal{E}$ including all the A_λ, it is necessary and sufficient that $A_\lambda \longleftrightarrow A_{\lambda'}$ for all $\lambda, \lambda' \in \Delta$.

Theorem 0.5 (Varadarajan (1962))

Suppose $\{x_\lambda : \lambda \in \Delta\}$ to be an indexed set of observables. Suppose also that either Δ is denumerable or that $\mathcal{E}$ is a separable logic. Then, a necessary and sufficient condition that x_λ be simultaneously observable is that there should exist an observable x and a set $\{u_\lambda : \lambda \in \Delta\}$ of Borel functions of the kind $x_\lambda = u_\lambda(x)$ for all λ.

Definition 0.8

We say that the x_λ have a joint distribution whenever there exists a σ-homomorphism $\vartheta: B(R^1) \longrightarrow \mathcal{E}$ such that $\vartheta(\pi^{-1}(E)) = x_\lambda(E)$ for all λ and all $E \in B(R^1)$. π represents the projection $f \longrightarrow f(\lambda)$ of R^Δ into R^1.

Theorem 0.6 (Varadarajan (1962))

Let $\mathcal{E}$ be any logic (not necessarily Boolean) and $\{x_\lambda : \lambda \in \Delta\}$ an indexed set of observables. Then, the following statements are equivalent:

- (i) The x_λ are simultaneously observable.
- (ii) The x_λ have a joint distribution.
- (iii) There exists a space Ω and a σ-algebra A of subsets of Ω, a σ-homomorphism $\vartheta: A \longrightarrow \mathcal{E}$ and A-measurable real-valued functions f_λ, $\lambda \in \Delta$, such that $x_\lambda(E) = \vartheta(f^{-1}(E))$ for all $\lambda \in \Delta$ and real line Borel sets E.

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These statements (definitions and theorems) are pregnant with information. Definitions 0.7 combine observability with verifiability; their empirical content is obvious. Theorem 0.4 decrees that the simultaneous verifiability of the subalgebras of the logic ensures the existence of

a subalgebra comprising all other subalgebras; in case of a logical uncertainty there will be no such comprising subalgebra. Theorem 0.5 holds only for a denumerable set of observables or for separable logics. There is an analogy between Theorem 0.4 and Theorem 0.5; the role of the comprising subalgebra corresponds to the role of the observable x, and the role of the verifiability operation corresponds to the role of the Borel functions u_λ. This is but a proper observation of the fact that common to the subalgebras and the observables is the logic $\mathcal{E}$.
Definition 0.8 establishes the fundamental equation relating x_λ to the composition of a σ-homomorphism and a projection. This equation appears in Theorem 0.6 (iii) again, where it asserts the existence of a space Ω and a σ-algebra of the subsets of Ω.
Theorems 0.5 and 0.6 are fundamental in regard to simultaneous observability (measurement). This study is concerned mainly with systems which violate these theorems and - having non-Boolean logics - generate incompatible Boolean subalgebras. Before we enter this field, let us consider an example of simultaneous measurement.

0.1.2.2 Simultaneous observation of the state of a finite linear control system: an example

The control system is represented by Banach spaces X,T,Y and linear operators A and B (comp. Rolewicz (1976)).

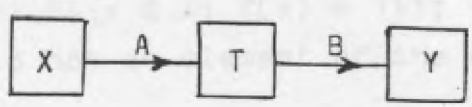

Fig. 0-1: Linear n-dimensional control system

$X = X_o \times U$; X_o := the set of initial positions,
U := the set of controls,

$T = C_X [0,\tau]$:= the space of trajectories,
C_X := the space of continuous functions over interval τ,

Y := the space of outputs,

A,B := linear input and output operators, respectively.

Definition 0.9

Observability of the system in Fig. 0-1 is understood to be the determination of particular properties of the input from the properties of the output. The object of the observation is the dynamic state defined by the position and velocity vectors of the system within the time interval τ.

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The physical quantities associate numbers to the states and, in order to be physically meaningful, small changes of state should correspond to small changes of the physical quantities; hence, to the latter correspond linear continuous functionals. More often than not, we are unable to determine the functional f and its value directly; in that case we can then measure or calculate $\phi(y)$ for every $y \in Y$ and every functional $\phi \in Y^*$, where ϕ is the usual measuremental procedure (called "method of observation" in control theory) and Y^* is the conjugate space of Y.

The basic question of interest is: Can we choose a measuring procedure $\phi \in Y^*$ such that

$$f(x) = \phi(BAx), \text{ for all } x \in X? \qquad (0.7)$$

If such a possibility exists, then the linear continuous functional f is said to be observable by ϕ.

Moving now to conjugate spaces, equation (0.7) will take the form

$$f = A^*B^*\phi. \qquad (0.8)$$

Theorem 0.7 (Rolewicz (1975))

The linear continuous functional f is observable iff

$$0 \notin cl(BA\{x \in X: f(x) = 1\}); \qquad (0.9)$$

in words: Zero is not an element of the closure on the right of (0.9).

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If the functional f is observable, then there may exist many $\phi \in Y^*$ satisfying (0.7) and hence (0.8). This freedom of choosing ϕ may be used to minimize any error of f(x). We arrive thus at a minimal norm problem in the conjugate system

$$Y^* \xrightarrow{B^*} T^* \xrightarrow{A^*} X^* \qquad (0.10)$$

Theorem 0.8

f is an optimally observable functional whenever there is a

$\phi_o \in Y^*$ such that $\|y_o\| = \inf\{\phi \in Y^*: A^*B^*\phi = f\}$. (0.11)
Every observable functional is optimally observable.

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Suppose now that we have more than one functional to observe, say $F = (f_1, f_2, \ldots, f_n)$. This situation is depicted below.

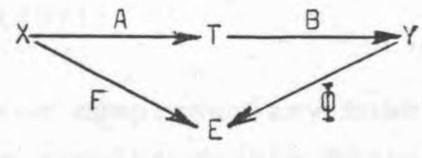

Fig. 0-2: Commuting diagram

$F = \Phi BA$, $\Phi \in \Psi(Y \longrightarrow E)$,
$E = R^1$.

Theorem 0.9

The system F of functionals is observable iff every linear combination $a_1f_1 + \ldots + a_nf_n$ of functionals $f_1, \ldots, f_n$ with scalars $a_1, \ldots, a_n$ is observable.
Every observable system F is optimally observable; the expression is similar to (0.11).

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Since X,T,Y are finite linear Banach spaces, A and B are linear operators and F, Φ are linear combinations of linear continuous functionals; the functionals themselves are independent, and the two complementary ones, observing $\langle x, \dot{x} \rangle$ and determining the state of X×T×Y, are measurable; a joint distribution exists and the logic is Boolean.
The scale of measurement (observation) for this example looks as follows:

The empirical relational system: $\mathfrak{X} = \langle X \times T \times Y; \succsim, \|\cdot\| \rangle$,
$\succsim$ is a cone-ordering in the vectorial B-space,
$\|\cdot\|$ stands for the norm.

The numerical relational system: $\mathcal{N} = \langle E; \geqq, |\cdot| \rangle$,
$\geqq$ is the numerical ordering,
$|\cdot|$ is the Euclidean norm.

The measurement procedure: $\Phi = \{\phi_1, \phi_2\}$ is the set of two linear continuous (homeomorphic) maps.

The scale $S = \langle \mathfrak{X}, \mathcal{N}, \Phi \rangle$.

The basic representation theorem as well as the uniqueness theorem for interval scaling are fulfilled and the required axioms concerning:

Weak order, independence, Thomsen property, restricted solvability, Archimedean property, the essentiality of components and the condition of indifference - specified by Krantz et al. (1971)

are satisfied.

As stated above, the two complementary quantities of position and velovity are simultaneously measurable because a joint distribution exists and the logic is Boolean.

Based on the arguments and the results obtained here, we may now safely state:

(1) The concept of "quantity" is well defined in set theory.

(2) The representational theory in the frame of classical (Boolean) logic and finite (simple) systems for the determination of both single quantities and states is well established, - but only in this frame.

0.2 Quantal communication and information aspects of measurement

This section is devoted to aspects common to experimental and theoretical physics, to mathematical logic and to communication theory.

It is suggested that the most fundamental abstract scientific concept is quantal in its communicable aspects. It is defined as "information-content" and it has two features: the àpriori or structural and the àposteriori or quantitative: To each there corresponds a quantum of information representing the minimal elementary proposition relating respectively to the structural and quantitative aspects of scientific statements. And scientific statements are regarded as complexes of these elementary propositions. The information-content of a result can be completely represented by a vector in a multidimensional space, the dimensionality of the space and the square of the length of the vector indicating respectively the amounts of structural and quantitative information provided, while the orientation of the vector specifies the result. The information-content is shown to be fundamentally limited by the number of conceptual units of space-time devoted to the experiment, with obvious practical implications. Expressions will be derived measuring the amount of detail possible in a result under different conditions.

We let the various uncertainty relations of physics appear basically as axioms expressing the quantal nature of communicable information, consequent on the use of logical forms; whereas the quantity "entropy plus information-content" appears as a fundamental invariant of a physical system.

It is said that "nature cannot be cheated", and examples of this principle recur throughout the realm of measurement, and not only in microphysics. A defined concept common to all scientific statements and responsible for their logical significance is necessary. In terms of this quantitative concept of information, various semi-intuitive principles can be seen to have a precisely definable basis in a general axiom.

Acquisition and quantization of scientific information

A scientific statement can be defined as a precise description of certain events viewed as populating a tract (q) of a coordinate-space (configuration-space). Its essential purpose is the communication of information derived from an experiment - an activity in which events are classified. The acquisition of scientific information thus involves two distinct tasks:

Firstly, one must devise apparatus and/or prepare some system of classification such that an adequate number of independent categories can be defined when describing the result, e.g. if fluctuations varrying in frequency between 1 and 100 cps are to be observed, the apparatus must be capable of responding in a time of the order of 1/100 seconds. When a chain of apparatus is involved (including the observer), then the differentiating capacity of the least-discriminating link determines the number of independent categories in the result.

Definition 0.10

There is a sense in which this number, i.e. the number of independent dimensions or "degrees of freedom" can be regarded as a measure of the information supplied by the experiment. This information, following Gabor, is called "logon-content" of the result.

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Secondly, the experiment must be performed by using our apparatus (galvanometer, microscope, eye plus notebook) to classify events in a chosen tract of coordinate-space. We may, for example, record 100 independent values of a quantity as a function of time - the amount of that quantity which is associated with each of the 100 identifying points provided by our method in the time-tract considered. Performance of the experiment thus results in the association of a number with each of the "labels" - categories or the degrees of freedom - defined by the structure of the experimental method. The usual scientific statement, however, takes the form of an inference having a certain probability deduced from the experimental result. Thus, we arrive at a second use of the term of information, to signify the source of confidence in a given number as representative of the class identified by its label.

It is possible to give a precise numerical significance to this concept also. It is the essential àpriori complement to the àpriori logon-content, for the complete representation of the information derived from the experiment.

We go now to the analysis of the nature of the scientific statement made to convey the results of an experiment, which yields a basis (rather definite) for a quantal representation of scientific information.

(i) Elementary propositions and quantization of information: A scientific statement is a logical form based on limited data, dissected into a pattern of "atomic propositions", of which each states a fact so simple that it is only true or false. This is its only attribute. We treat as elementary the simplest propositions relating to the concepts of measurement and classification. Ideally, a scientific statement is based entirely on observable evidence, and the ideal statement which would describe all the information supplied by a particular experiment is presumably reducible to a pattern of independent elementary propositions relating to observations. We are led to define one kind of unit of information as that which decides us to add one elementary proposition to the pattern of propositions which is logically adequate to define the result observed.

(ii) The minimum change of scientific information possible is the addition of one element to the logical pattern; and a unit of information is simply that which causes us to make one such addition. Thus, formation of scientific statements is a quantal process; these quanta are to be identified in terms of the process of experimentation. Information has two common uses: Prior information is presented by the knowledge of the experimental procedure; posterior information arises as the latter is carried out (quantum physicists speak of "preparatory" and "determinative" measurement, respectively). The one defines the structure of the ultimate statement, the other the amount of evidence which it subsumes.

Posterior or metrical information

(i) The most elementary observational proposition asserts the existence of a coincidence-relation between the entities. But generally, we define a magnitude by saying that it occupies a certain interval on a scale. Logically, this occupance-relation between scale-interval and magnitude is a consequence of the existence of coincidence-relations between the ends of the "unknown" and the two definable graduation-entities on the scale.

For every observation there is a minimum separation between neighbouring graduation-entities, B_{n-1}, B_n, B_{n+1} say, below which either we cannot define or cannot substantiate with probability greater than one-half, a proposition of the form: "A falls into $B_{n-1} - B_n$ and not into $B_n - B_{n+1}$". The smallest interval is called "scale-unit" appropriate to the observation. This assumes the validity of tertium-non-datur.

(ii) A magnitude can be specified by the number of "minimum meaningful intervals" which it occupies. The index of information provided by this number is described as metrical, and the unit of metrical information as "metron". The metron is that which enables one elementary interval to be represented as occupied, in the logical pattern to be communicated; i.e. each metron specifies one elementary occupance-relation. Thus, what we carry away from a measurement is basically an integer, referred to as the "metron-content of the result".

If the metron-content is to be equated to the number of atomic propositions, it must be so defined that the metron-content of two similar but independent experimental sequences is twice that of either one, and it should be incapable of augmentation by purely logical manipulations. If these conditions are satisfied, a result yielding a given metron-content provides a fixed number of logical elements out of which dependent or equivalent statements can be constructed.
(iii) From the set of all physical quantities, some will resist a representation on scales having a constant scale-unit; e.g. $\sqrt{q}$, $q \in Q$, has the same metron-content as q, since the latter can be derived from the former by purely logical actions; however, the metron-content cannot be proportional to both quantities. Conceptual scales, called "proper scales", are not necessarily linear in terms of physical magnitude. The proper scale of a quantity subject to random fluctuations is a good example of this. Fisher justified in 1935 his definition of statistical information as a quantity proportional to the reciprocal of variance, by noting that variance depends inversely on the number of samples involved, and is a measure of the uncertainty with which a given sample can be regarded as representative. If we multiply the number of samples by n, the information provided (if independent) should be n times as great. Because the same process reduces the variance by 1/n, the information provided should, therefore, be proportional to its reciprocal. The metron-content i of a single measurement of a radom fluctuating quantity z having a probable error $\pm \frac{1}{2}\Delta z$ is, therefore, not $z/\Delta z$ but $i = z^2/(\Delta z)^2$, and the physical scale of z is divided into significant intervals which are non-uniform; the scale of z^2, however, is uniformly divided. This means that the statement "z or z^2 occupies i intervals" can be assigned a probability p just $\frac{1}{2}$. In this case it is unprofitable to employ a narrower interval.
(iv) In more complex cases we go beyond the trivial characterization: "we observe the pointer and measure voltage"; if our sole concern is the determination of a quantity, we speak of measurement; if, however, a theory is involved, we speak of observation and of a corresponding operator. In general, the precision of observation sets the upper limit to

that of measurement. An experiment is not yielding full information unless the metron-content of the observation exceeds that of measurement (e.g. due to noise). But, more importantly, observation usually embraces the whole system in closed form and a comparison with a postulated theory, and this involves the measurement of several quantities (at least two conjugates) - simultaneously!

(v) A measured value of a quantity z is inevitably represented as occupying an interval Δz. Accordingly, with a positive metron-content, the smallest "observable value" which we can rightly attribute to a quantity linearly related to metron-content is not zero but $\frac{1}{2}\Delta z$. The first interval which it can occupy on our conceptual scale has the width Δz, which means a displacement of the origin.

Full justice is done to the accuracy (the logical content) of any single measurement when it is described in terms of its scale-units and metron-content; the latter may be represented as the number of intervals, on an abstract conceptual "proper scale", occupied by the measured quantity. However, the number of intervals on this scale will not necessarily be proportional to the measured quantity, since the scale-unit may also depend on the quantity.

Structural information

(i) "Reason has insight only into that which it produces after a scheme of its own" - claimed E. Kant.

The design of an experiment is basically the specification àpriori of a pattern of categories in terms of which alone the result can be described. All the events of the experiment must find a place in one or other of these although, of course, not all categories will necessarily find an exemplar in a given experiment. Now, each independent category enables us to introduce a measure of differentiation, i.e. of form or structure, into our account of a result, so we can regard knowledge thereof as providing us with prior or structural information. We can, therefore, define a unit of structural information or a logon as that which enables us to formulate one independent proposition, describing one independent feature of the result. That amount of structural information in a result, the logon-content, is thus the number of independent categories or degrees of freedom, precisely definable in its description.

(i) Structure being defined in terms of reference-coordinate, the "logon-capacity" of an experimental method can in such cases be defined as the number of logons which it specifies per unit of coordinate-space if several coordinates are involved. For example, the logon-capacity of a microscope in a particular region of focal plane can be defined in logons/cm^2 and measures the resolving power in that region.

Frequency-bandwidth of instruments is defined such that "bandwidth" is defined by a volume (e.g. for microscopes). Bandwidth in the above sense is directly related to their logon-capacity; the relation arises from the well known uncertainty-principle of the form:

$$\Delta f \cdot \Delta q \geqq K. \qquad (0.12)$$

Δf and Δq are the effective ranges of frequencies f and time-tracts q, respectively; for Δq being twice the uncertainty on q, we get $K = \frac{1}{2}$.

Thus, the points on the q-axis cannot be defined uniquely at closer intervals than $K/\Delta f$, so that the logon-capacity is $\Delta f/K = 2\Delta f$. The corresponding uncertainty relation in quantum mechanics is frequently written in the form

$$\Delta E \cdot \Delta t \geq h;$$

the correct notation, however, is

$$\Delta E^2 \cdot \Delta t^2 \geq h^2,$$

since there is no operator Δt, but Δt^2 in Hilbert space. The logon-content l of an experiment involving a tract q is thus:

$$l \leqq q \cdot \Delta f/K \leqq 2q\Delta f. \qquad (0.13)$$

Representation of information

(i) The information defined initially was operational. "Information" acquires meaning in terms of what it does. Thus, the total information (structural and metrical) provided by an experiment requires a concise representation enabling us to deduce its effect on all conceivable statements relating to the result, i.e. an information operator (matrix) I.

Suppose that an experiment has provided l logons. If they are independent, they will be orthogonal rays filling an l-dimensional "information-space" with corresponding unit-vectors $e_1, e_2, \ldots, e_l$ forming its basis. The information matrix (operator) in canonical form is

$$I = \begin{bmatrix} i_1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & i_2 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & i_l \end{bmatrix}, \qquad (0.14)$$

$i_1, \ldots, i_l$ being the metron-constants of the logons represented by $e_1, \ldots, e_l$, respectively. Any proposed dependent statement can be represented relative to the logon-basis by a unit vector-function ϕ, defined by

$$\phi = \sum_{r=1}^{l} f_r \cdot e_r, \qquad (0.15)$$

f_r being the direction cosines of ϕ and measuring the relevance of the corresponding logon to the statement. If we now wish to calculate the total metron-content i_ϕ which the proposed statement would have, we need only form the product

$$i_\phi = \phi' I \phi, \quad \phi' - \text{transpose of } \phi. \qquad (0.16)$$

The following three facts are worth memorizing:

(a) The trace $i = \sum_{r=1}^{l} i_r$ of I represents the total metron-content of the original result.

(b) This trace (of information) is invariant under unitary transformations, so that (as our axiom demands) the total metron-content is unaltered by logical manipulation.

(c) A logically equivalent restatement of the result corresponds to a unitary transformation of the basis, under which I generally looses its simple diagonal form. Thus, each component of a dependent statement is no longer a function of just one of the original logons.

(ii) This procedure can be given a simple geometrical representation, analogous to that of quantum mechanics. Operator I can be defined by an "information vector" of length $\sqrt{i}$, with components $\sqrt{i_1}, \sqrt{i_2}, \ldots, \sqrt{i_l}$, which defines a point (actually a volume-element in the information space) relative to the origin. This information vector is the "real result". Its components along each axis are the square roots of the metron-contents of the corresponding logons. Any dependent statement is now defined by a direction in the space, and its metron-content is the square of the projection of $\sqrt{i}$

in that direction. Unitary transformations correspond to rotations of the axes, and the significance of points (i) and (ii) is clearly seen.

The dimensional multiplicity l was seen to be fixed by choice of experimental method, including the volume of space-time occupied. Performance of the experiment results basically in the collection and allocation to various logons, of the metron-flow arising from the impact of data on the apparatus and observer.

The quantal character of this process has the important effect that only a certain number of different results can conceivably be given by a particular experiment, because the number of ways in which i metrons can be distributed among l logons is limited. An experiment is indeed an attempt to choose between a finite number of possibilities. Otherwise experimentation would be impossible.

Practical considerations

Metrically and structurally defined scale units

In order to be logically representable, a numerical magnitude must be quantized in either metrical or structural propositions; the origin of quantization differs, however, in the two cases.

(i) A single metrical statement about a quantity, say y, is logically sterile unless it is identified by a coordinate-label, say q. The proposition:

"I have received i metrons relating to y"

represents our actual experience, and has no vagueness; but vagueness arises directly as we conceive y to be a function of q, and try to formulate a proposition about y in terms of q. But this is impeded because information takes time to accumulate, for there is only a finite metron-density on the time axis. The result is an uncertainty relation (actually an information axiom) which arises as follows:

Metrons, the "atoms of information" we have received, are scale-free; that is to say, one can construct a number of dependent propositions involving the same number of metrons, and can regard them as logically equivalent statements of the result, as long as one fixes appropriate definitions of the scale unit in each case. Then, to translate i metrons into a statement of the magnitude of y, one must have a scale unit Δy such that $y = i \cdot \Delta y$.

If we now wish to say how y varried as a function of q, we must do so by defining a metron-density, say ρ_i metrons per unit q, so that if the information i was gained over a tract Δq, then

$$i = \int_q^{q+\Delta q} \rho_i(q)dq. \qquad (0.17)$$

For $\rho_i(q)$ we put its average value in the interval Δq, and call it ρ_a; then $\rho_a = i \cdot (1/\Delta q) = i \cdot \Delta\rho$. This definition of ρ_a is basically equivalent to that of $y = i \cdot \Delta y$, since the metrons are the same.

In order to connect the two statements, we define the relation between corresponding scale units by means of a conversion-factor K_m such that $\Delta y = K_m \cdot \Delta\rho$, and obtain so the generalized uncertainty relation or <u>information axiom</u>:

$$\Delta y \cdot \Delta q = K_m; \quad K_m \text{ - a metric constant,} \qquad (0.18)$$

which shows that the limit to the accuracy of measurement of y is directly proportional to the extent of q devoted to the process.

(ii) The above relation may be converted to $\Delta q = i \cdot (K_m/y)$, where K_m/y is regarded as a natural unit of q, say ϵ_q, within which just one metron is required; in other words: i metrons acquired in the interval Δq enable us conceptually to subdivide it into i intervals of magnitude ϵ_q. Conversely:

The metron-content of a measurement cannot exceed the ratio of the coordinate tract Δq associated with it, to the appropriate natural unit ϵ_q of q.

ϵ_q may be calculated for any given experiment, and plays the sole role of an atom of space or time. It is a scale-unit of q; but as the tract irreducibly associated with each measurement of y is Δq, Δq is the smallest interval which enters into any proposition involving y. ϵ_q coincides with Δq in the limit where the metron-content of each measurement is unity.

This principle is useful in considerations of the statistical matching of one part of an experiment to another.

In structural propositions the absolute magnitude of y is irrelevant. These are essentially definitions of propositional functions of which y is to be the argument. Hence the scale-unit of q can be defined only in terms of coincidence relations, independent of the magnitude of y. The concept of bandwidth has already been used to define the scale-unit Δq which was a property of the apparatus used, ascertainable beforehand by an independent experiment, and hence counting as prior information on subsequent occasions.

Consider the case of a simple harmonic function of q, with periodicity f. It will be associated with definite points on the q-axis, independent of amplitude, whenever the function crosses the axis (at $\frac{1}{2}$-period intervals). The scale of q is conceptually provided with a set of points at intervals of K_s/f, K_s being the structural constant of order 1. However, what we want is a set of uniquely identifiable points, to serve as labels. To single out desired points, we must provide a comparison-pattern to act as a "pointer". Thus, a frequency $f-\Delta f$ will produce a pattern coinciding with the first K_s/f at intervals of $K_s/\Delta f$. If all values of Δf from zero can be observed, a continuous range of intervals from infinity down to $K_s/\Delta f$ can be observationally defined. The structural scale-unit Δq of q is thus $K_s/\Delta f$. Therefore, if Δq is an arbitrary interval, the number l of logons relating to it, can be written as $l \leqq \Delta q \cdot \Delta f / K_s$, and for a single logon we have:

$$\Delta q \cdot \Delta f \geqq K_s. \tag{0.19}$$

It is of interest that the metron-density function, specified for a single logon for which $\Delta f \cdot \Delta q \geqq K_s$ becomes an equality, is a Gaussian probability function. Analysis of a function into logons is effectively description in terms of superimposed Gaussian functions.

An illustration of the methods and ideas discussed above has been given by Mackay (1950), applied to the problem of optical resolving power. We have used freely and throughout the results of his excellent investigations, and subscribe with conviction to his closing general axiom:

> The limit to scientific observation is the limit of our logical vocabulary. If a phenomenon can be defined (in terms of the atomic propositions of

the scientific method) it can - in principle - be observed.

We complement this statement in the following sense:

If a phenomenon can be defined, then it can be observed (measured) and calculated; the law of contraposition applies as well.

Closing remarks

1. To increase the amount of detail in a result, it is more profitable to increase l (logons) than i (metron-contents) unless already $l > i$.
2. The precision of a single measurement can be enhanced indefinitely by increasing the space-time tract irreducibly associated with it. Thus, in experiments to determine a constant, efforts should be directed towards "logon-compression" - reducing the frequency-response (i.e. the logon-capacity) of the apparatus with respect to time and space. In short: best results are obtained by acting consistently with one's belief that the constant will not alter with time or position, so that one logon will suffice.

Remarks 1. and 2. apply fully under the conditions of classical logic. In that case an experimental sentence (statement) is symbolized by (α, E) and conveys the information that the measurement of an observable α yields a result in a Borel set $E \in B(R^1)$, where $B(R^1)$ denotes the Boolean algebra of the sets on the real line. This sentence may, of course, be true or false, tertium non datur.

0.3 Fundamental uncertainties arising in the simultaneous measurement of the state of complex fuzzy systems

0.3.1 Observed inadequacies of representational measurement

The necessary and sufficient conditions for the existence of a measuremental procedure are known to be the properties of the empirical relations of the empirical relational system considered; they are formally expressed in the form of theory-specific axioms. And all from these axioms logically deduced sentences (if proved true) are said to be the theorems of the measurement theory in question. If M is the set of basic notions and N is the set of theory-specific axioms, and both M and N variable, then M and N represent jointly the models for the set of axioms N. The model for the axioms of measurement belongs to the realm of the reals; it is also a model for the theory of measurement if it satisfies all theorems of the theory, see Kaaz (1977) and Tarski (1954). For these and other reasons measurement is conceptually a mathematical discipline. It is common practice to prove the consistency of a theory by construction of models; and since most theories based on the arithmetic of real or natural numbers are inconsistent, careful proving is imperative.

The field of application of representational measurement considered here covers the empirical relational systems $\langle Q,R_n \rangle$ of engineering, physics and human sciences. The elements q are manifestations of the attribute Q and R_n comprises the n relations involved. It is now important to take a closer look at the elements and relations of the empirical relational system.

The necessary and sufficient conditions for the existence of a quantity have been specified in Section 0.1.1; they involve two restrictive laws: that of the excluded middle tertium-non-datur) and that of distributivity. With regard to the former we have Tarski's assurance that axiomatized systems are the only systems in which the law of the excluded middle holds, where under systems we understand systems of measuremental sentences (q,E) ; $q \in Q$, $E \in B(R^1)$. The restriction in this case is the fact that the law of the excluded middle applies to systems having a finitistic character; it fails in other systems, see Tarski (1935), Theorem 17.

Distributivity, on the other hand, fails in case of infinite Boolean operations, even though infinite joins and infinite meets may exist, see Sikorski (1961).

Now, if q is affected by the law of the excluded middle, so is Q and any relations R in Q, the latter being subsets of the cartesian product QxQ. Distributivity enters the consideration when - in addition to relations - the empirical relational system contains operations.

The problem (if it arises) is that of "infinity"; and it arises in reality when the empirical systems considered become complex or even complex fuzzy, characterized by a non-

Boolean logic (algebra) with underlying ∞-dimensional (Banach) spaces. The result is a splitting of the Boolean algebra associated with simple systems in two incompatible Boolean subalgebras and the appearance of a fundamental uncertainty relation, which terminates the independence of the precision and relevance of measurement. This in turn inhibits any simultaneous measurement and invalidates Varadarajan's theorems. The condition reached invokes the following <u>Principle</u>:

> The question related to the simultaneity of precision and relevance (or significance) of measurement is a virtual question in complex fuzzy systems.

Such a measurement is no longer objective, much less determinative; it introduces a subjective feature (the observer, experimenter: actually his powers of knowledge and will) into the process of measurement. This precludes also any pre-existence of measurands and observables.

Difficulties of this nature reveal - on the one extreme - the observation of the state and of the dynamics of gallaxies and - on the other extreme - the impossibility of temperature measurement below 0.1K.

We know from measurement in complex engineering, especially in a state close to a disasterous condition, that precision and relevance characterizing the detail and the whole (in the sense of the "Part and the Whole Doctrine"), respectively, become incompatible conjugates of measurement. This is substantiated by Thom's (topological) catastrophy theory, indicated by the vanishing of open sets (i.e. of continuity) with the consequence of instantaneous disaster.

Precision and relevance are theoretically maximal and independent of one another in countable sets down to singletons, but not so in complex sets and systems. Problems of this kind are troubling psychologists and sociologists; their measurements involve, above all, the subject itself. Let us, therefore, review the relations causing anxiety in psychology. It is common knowledge that "similarity" is the most fundamental psychological relation; it forms jointly with the notions of "equivalence" and "order" the fundamental triple of psychological relations. Similarity implies actually the existence of a small difference; its concept is frequently introduced as a non-scrutinized, empirically undefined and explicative construct. Uncertainties arise with regard to dissimilarity, its complement.

Now, empirical relations are being mapped into numerical relations; we call them also interpretations of the numerical relations. The interpretations of the equivalence relation are simply called equivalence relations, symbolized by "$\sim$". This relation gives rise to the formation of equivalence classes which - conceived subjectively - contain indistinguishable objects with respect to the common feature. The empirical interpretations of the weak order relation "$\gtrsim$" are said to be "dominance relations"; "preference" is a special type of dominance relation. The remark on simplicity also applies to equivalence and dominance. The three classes of relations (equivalence, dominance and similarity) constitute the basic relations of an empirical psychology in the light of modeling by numerical systems. Compared to these, interpretations of "additivity" are of much smaller importance. Actually, interpretations of addition in psychology are used as joining operations. This hints strongly at a fuzzy behaviour favouring monotony in preference to additivity as well as a bias from algebra to topology. Debreu's (1960) paper on topological methods in utility theory is a clear indication in this direction.

As to the features themselves, they appear most frequently in multidimensional spreads. Multidimensional scaling has been discussed at length by Gleason (1969) and Holman (1978); the case of simultaneous measurement, conjoint measurement and their problems are dealt with by Luce and Tukey (1964), Famagne (1976) and Crott (1970); and, while Krantz et al. (1971) and Fishburn (1970) report on barely noticeable differences, Tverski (1969) and Coombs (1959) contribute to the inconsistency and intransitivity of preferences and Gigerenger (1981) (see in particular pp 82-104) elaborates on modelling in psychology.

Consider the self-explanatory Figures 0-3, 0-4 and 0-5 to follow.

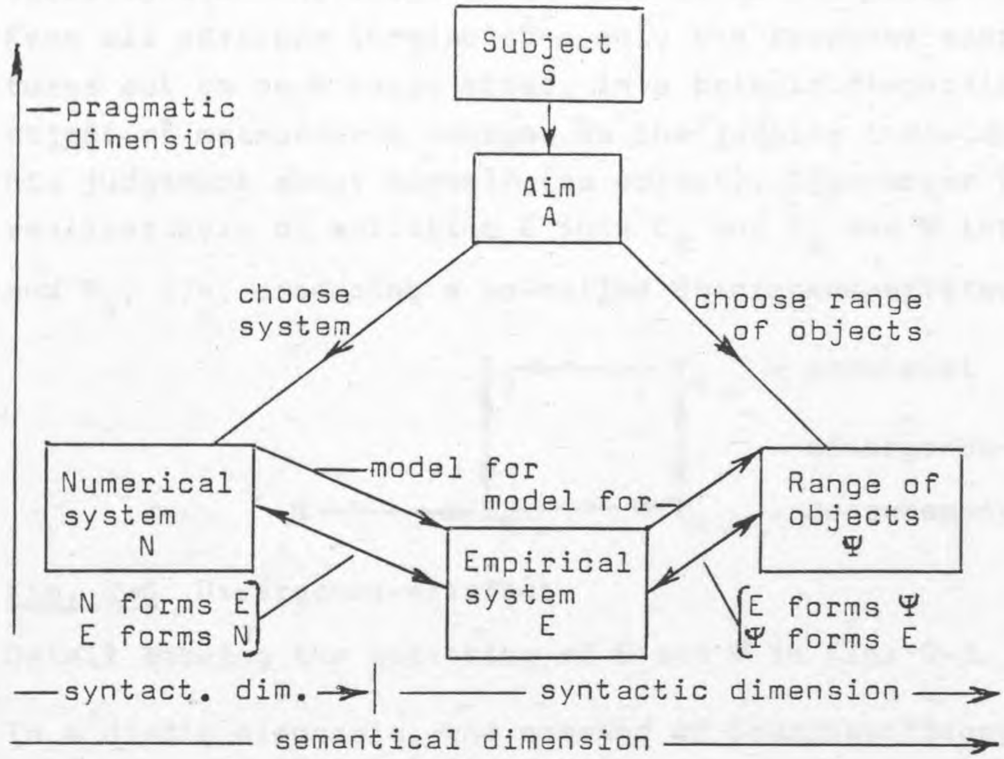

Fig. 0-3: Measurement by model building

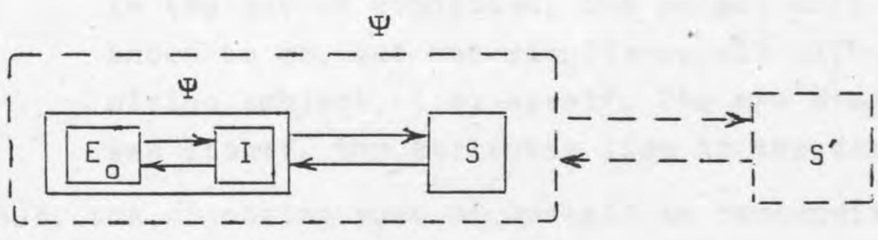

Fig. 0-4: Interaction of E_o and I as the range Ψ

I - individual, E_o - range of carriers and components of significance,

S - subject, S′ - metasubject,

individual x object x feature - basic triad as interactive quantity of diagnosis,

object x feature - basic diad as interactive quantity of diagnosis.

In the so-called "response approach" of Torgensen, the variability of the reactions to excitations is allocated to the variations in the subjects as well as to the excitations. From all possible interactions only the response approach turns out to be a basic triad. In a triadic diagnosis, the object of measurement emerges as the judging individual in his judgement about himself (as object). Gigerenger (1981) realizes this by splitting E into E_N and E_S and Ψ into Ψ_N and Ψ_S, i.e. producing a so-called divergence-artefact:

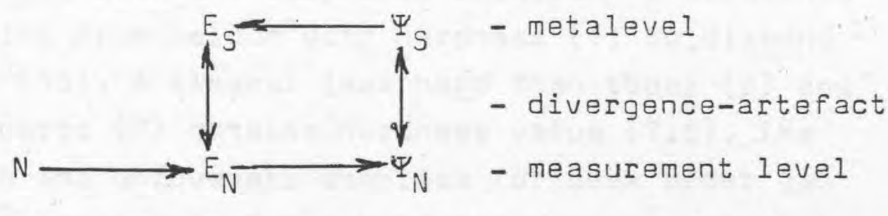

Fig. 0-5 Divergence-artefact

Detail showing the splitting of E and Ψ in Fig. 0-3.

In a diadic diagnosis, the command of Sokrates: "Recognize yourself" would obviously be an impossible task. About the separation between subject and object performed in a diadic language the physicist and philosopher C.F. von Weizsäcker (1970) states:

> "The objectivizing cognition is selfforgotten. In the act of cognition, the object will become known to me, but not simultaneously with the cognizing subject, i.e. myself. The eye does not see itself, the reflector lies in the dark".

Thus, the cognizing part of oneself in "recognize yourself" remains, for all times, outside of cognition.
All of these results seem to touch the border of fundamental uncertainties; a representative confession to this effect is the following statement of Gigerenger (1981) p. 397: "The existence of incompatible I-scales (Coombs' scales) concurs with the inter-individual 1-dimensionality, whenever the postulate of unimodal preference functions ceases to be valid".
The uncertainty relation due to Heisenberg states that the position and momentum of an atomic particle cannot be measured arbitrarily accurately. From the standpoint of clas-

sical physics, this result is totally incomprehensible as long as the world is looked upon as a concrete scientific object with certain inherent qualities such as position and momentum of a particle, no matter whether we measure them or not. This implies the existence of hidden parameters in nature, since the mathematically formulated laws of nature deal not with elementary particles, but with the human knowledge of them.

A good example of the classical point of view is Moh's scale of hardness with the dominance relation "a scratches b" and extending from talcum with hardness (1) to diamond with hardness (10). A mineral less hard than topaz (8) and harder than quartz (7) obtains hardness value (7.5). The representation and uniqueness theorems for weak order can immediately be applied in Moh's hardness measurement, but it dispenses completely with the human subject and leaves a matarialistic world devoid of humanity behind. There can, therefore, be no uncertainty, since uncertainty is a truly human quality.

In what follows, we shall see that the existence of incompatible (conjugate) scales is postulational to the existence of fundamental uncertainty relations in a general measurement formalism covering complex fuzzy systems and making the quantum mechanical formalism appear plausible.

0.3.2 The physics and the semantics of quantum measurement

Relativity and quantum theories have revolutionized physics and they have added importance to observation and measurement.

The necessity to leave the ground of classical notions was originally forced by the technical extension of the sphere of our experience. The technical notions did no longer fit the situation presented to us by nature. If, at one time, we see an electron as a particle describe its path and, at another time, we observe that it is wavelike reflected by a diffraction grating, then the language of classical physics obviously no longer suffices to explain both observations as a result of a unique event.

The fixing of the point at which it is possible to divorce classical notions, constitutes the kernel of a modern theory. The centre of relativity theory is the realization that the simultaneity of two evevnts at different locations is a problematic notion. This corresponds in quantum theory to the realization that it is meaningless to speak simultaneously of an exact position and an exact momentum of a microparticle. Indeed, the questions relating to the real simultaneity of two events and of that of position and momentum of a particle turn out to be virtual questions (to which no answer exists), because the notions which we have to use are much too imprecise to do justice to the situation presented by nature. Of interest is, above all, the question, why such situations arise. The theory of relativity pronounces that there is no possibility of transmitting signals faster than the velocity of light, that it is also impossible to provide an experimental definition of an absolute time scale. This negative disclosure becomes, nevertheless, useful in view of the discovery that a simple and logically satisfactory order of experiences can be achieved by the assumption that it is in principle impossible to transmit signals faster than light and by the then possible postulate of the velocity of light. Only by this second step is it possible to prove the statement that the question of an absolute time scale is a virtual question. The same goes for quantum mechanics, where the limitation of classical notions expressed by the uncertainty relations obtain usefulness by uncovering that these relations, upheld in principle, provide the necessary feedom for a harmonic and consistent order of our experience. Only the available system of mathematical axioms of wave- and quantum-mechanics gives us the right to consider the question of position and momentum values to be a virtual one.

These remarks may contribute to a better understanding of quantum measurement to be discussed now, and to the later treated questions of the fundamental measurement uncertainty.

In the investigation of arbitrary events by means of statistical methods, the measuring apparatus, serving both the fixing of statistical ensembles as well as the analysis of the distribution within these ensembles, must itself remain outside of the boundaries of these ensembles. In other words, the apparatus must lack the elements of randomness which are appropriate for the statistical ensembles investigated with their help, notwithstanding the fact that every apparatus, just as every body, consists of atoms, molecules and similar microelemets performing motions, i.e. from the standpoint of quantum mechanics, they belong to some quantum ensemble. At first sight, this is the source of a certain difficulty which quantum mechanics obviates in an elegant manner:

> The measuring apparatus must be arranged so that - in operation - only its classical properties are put to use, i.e. such properties for which Planck's constant plays no role whatsoever.

If we extend this statement to the investigation of complex fuzzy systems, then the measuring apparatus should be blind to fuzziness and complex uncertainties. In fact, it should measure Boolean exactly.

An apparatus of this kind is called "classical or macroscopic apparatus"; its nature is such that it is free from all quantal, statistical and uncertainty features. An optical aperture, a stationary screen or any spectroscope are examples of such classical apparatus; they are all objects of classical physics. Hence follows the conclusion that the characteristic feature in the determination of quantum mechanical ensembles is the complete set of classical corpuscular quantities.

Now, the statements of classical mechanics differ fundamentally from those of quantum mechanics: In classical mechanics the complete measurement consists of the measurement of coordinates and momenta of the particles. Since these are - at least in principle - simultaneously measurable, we can say that in classical mechanics all measurements are complete. Contrary to this, in quantum mechanics there exist many different mutually incompatible measurements since there exists a threshold to smallness in the microworld.

<u>Measurement in quantum mechanics on spaces of fuzzy events</u>

The outcome of any realistic simultaneous measurement of n real stochastic quantities $A_1, \ldots, A_n$ cannot be exhaustively described by n numbers $(\alpha_1, \ldots, \alpha_n)$ except in those cases when the respective sets (spectra) $\sigma(A_1), \ldots, \sigma(A_n)$ of values in the set R of reals, which these variables can assume, are all finite or at most countably infinite discrete subsets of $\mathbb{R}$.

Exhaustive description of the measurement outcome can be achieved by providing, in addition to the n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n)$, also a non-negative function $\omega(\lambda)$, $\lambda \in R$, with a maximum at α. This function $\omega(\lambda)$ provides a measure for the relative certitude that $\lambda \in \sigma(A_1) \times \ldots \times \sigma(A_n)$ and not α is the actual value of the extracted sample point.

Following Prugovecki (1975), we shall refer to the pair $\hat{\alpha} = (\alpha, \omega)$ as a fuzzy sample point and to ω as the confidence function of $\hat{\alpha}$. But let us first review the basic semantics of classical and non-classical measurement.

Empirically based semantics of measurement, which also embodies the features of quantum measurement, deals with three main concepts: "the class of systems of the same kind", "apparatus" and "the measuring procedure".

The equivalence relation, which establishes when two systems are of the same kind, can be given in terms of the macroscopic classification of different sources of systems (e.g. heated wires generating electrons, radium as a source of α-particles, etc.) under the silent assumption that macroscopically identical materials submitted to macroscopically identical conditions produce the same kind of microscopic systems. An apparatus or instrument, stated earlier to be a macroscopic object, which - in conjunction with a source of systems under controlled macroscopic conditions - is referred to as "measuring procedure", enables us to obtain numerically describable data about the systems produced by the source. We classify these measuring procedures with regard to the treatment of these data in relation to a given theory for the systems under observation into " preparatory" and "determinative" (in exact correspondence to àpriori or structural and àposteriori or metrical information, discussed in Section 0.2) always depending on whether the accumulated information is to be embodied into the theory or compared to data extracted from the theory, respectively.

All conceivable instruments are grouped into distinct classes by means of empirically directly verifiable relationships, each distinct equivalence class representing a "measurable quantity" of the system under consideration. The basic method used in this empirical verification is referred to as "calibration of instruments". As the term suggests, any calibration procedure compares outcomes of two or more instruments which supposedly measure the same quantity against one another under experimentally controlled conditions. Thus, the concept of a measurable quantity is directly empirically rooted, in contrast to the usage of the term "observable of the system" which requires theoretical support, as shown below.

An "observable" of the system is defined only in the context of a theory for that system by assigning (by means of corresponding rules) symbols from the formal framework of the theory to some of the measurable quantities of the system. The mapping relating measurable quantities to observables is in general not bijective, because a new observable can always be constructed by formal means, such as taking functions of old observables to construct additional new observables. Furthermore, there might be measurable quantities of a system which do not have a counterpart in a given theory of the system. On the other hand, in a given theory, distinct measurable quantities corresponding to radically different designs of their apparatus might be assigned to the same observable.

In the case of fundamental observables, such as spin, position, momentum etc., it can be decided on àpriori grounds what is the range of possible values on the real line R^1 which these observables might assume. The closure (in the Euclidean topology of R^1) of the set of all values that an "observable A" can assume, is the spectrum $\sigma(A)$ of that observable. Please note that - more often than not - we denote the observable by α and the operator associated with it by A. Taking the closure is obviously a mathematical convenience motivated by the desire to have the physical spectrum, defined in the above manner, coincide with the mathematical spectrum, corresponding to the Hilbert space operator representing that observable in some particular quantum mechanical theory.For the sake of later discussions, let us divide the spectrum $\sigma(A)$ of an observable A into the "accumulative part $\sigma_a(A)$", consisting of all accumulating points of the set $\sigma(A)$, and its complement, the "isolated point spectrum $\sigma_{ip}(A) = \sigma(A)/\sigma_a(A)$". In case no elements of the

point (or discrete) spectrum $\sigma_p(A)$ are accumulation points of $\sigma(A)$, then this classification obviously coincides with the mathematical classification: The accumulative spectrum coincides with the continuous spectrum $\sigma_c(A)$ and the point spectrum coincides with the isolated point spectrum.
We say that an observable "has a particular value $\lambda \in \sigma(A)$" and we talk of "measuring" some such value even when λ is in the accumulative part $\sigma_a(A)$ of the spectrum. This form of speach obviously cannot be taken literally when $\sigma_a(A) = \sigma_c(A)$ since most real numbers are not even computable in the idealized Turing-sense, not to mention "measurable".
When an experimentalist states that an observable with accumulative spectrum (such as position, momentum etc.) "has" a certain value, say $\lambda = 1.12$ of the chosen unit of measurement, without specifying the confidence margins, then he definitely does not mean that $\lambda = 1.1200...$ Usually one can deduce from the context the sensitivity of the employed instrument and, therefore, the margin of confidence $\Delta\lambda$ giving rise to the span $[\lambda-\Delta\lambda, \lambda+\Delta\lambda]$ of possible values of the measured quantity. Otherwise it has to be surmised that one is dealing with the convention, in which $\Delta\lambda$ is equal to one unit in the decimal place in the figure of λ (e.g. $\lambda = 1.12 \pm 0.01$).

The notion "fuzzy sample point", characterized by a confidence function ω, has already been introduced. Let us now consider any measurement of an observable A as an information gathering process which eliminates a set S of values as possible values of the measured quantity at time t of the measurement. Hence, a measurement is regarded as a procedure which increases our information about the values of A at t by restricting the possible values of A to the range D = $\sigma(A)$-S. If D contains no points in the accumulative spectrum of A, i.e. $D \cap \sigma_a(A) = \emptyset$, then we can expect to be able to construct instruments of such precision that D can be reduced to a singleton. However, in the presence of accumulative spectra, the attainment of this perfect limit of precision is seldom feasible, and - in general - more numbers than one must be employed to give a complete description of the information resulting from a single measurement. The characteristic function $\omega_D(\lambda)$ of the set D might seem a suitable theoretical tool describing this information. Of course, $\omega_D(\lambda) = 1$ when $\lambda \in D$ and $\omega_D(\lambda) = 0$ when $\lambda \notin D$ could be taken to represent our confidence that the experiment excludes the possibility of A having values outside D at time t. For macroscopic measurements in which we are not concerned with the possibility of attaining information on an arbtrarily fine scale and expressing all the nuances of this information, this kind of description is quite harmless. The situation changes drastically in the microscopic look at the problem; indeed, we wonder whether we can ever state with absolute certainty that a given measurement indicates that some set $S \subset \sigma(A)$ of values of a given microobservable A cannot occur. However, neither the measurement of position, nor any other measurement leads to an acceptable description of the outcome of a measurement, so we have to settle for a common-sense solution in which we specify a particular microscopic range $D \neq \sigma(A)$ of values as the outcome of the measurement without attaching a 100% confidence level to this statement. To avoid arbitrariness of a particular choice of D, we may describe the outcome of our measurement by means of a function $\omega_\mu(\lambda)$ assuming values in the unit interval with the largest value attached to that point $\mu \in \sigma(A)$ about which we feel most confident as representing the value of A at time t. The smaller $\omega_\mu(\lambda)$ at $\lambda \in \sigma(A)$, the lower the confidence of "having measured that value". Hence, we end up by describing the information resulting from a single measurement of A at time t by a "fuzzy set with the characteristic function $\omega_\mu(\lambda)$". Then $\omega(\lambda)$ is called the confidence function corresponding to the performed measurement. The operational meaning of this subjective procedure of describing the information extracted by means of a single measurement can be found in the "Reproducibility Principle" – see Prugovecki (1975). One starts by assigning to every possible outcome $\mu \in R^1$ of a measurement performed with a given apparatus α a fuzzy set Δ(i.e. a confidence function ω_μ). Let us call this procedure the calibration of the given instrument α. The correctness of such calibration procedure can be operationally checked by requiring consistency in the calibrations of all instruments that are supposed to measure the same quantity. In short: we compare the outcomes of preparatory and determinative measurements performed on the same system by a pair (α, β) of instruments and require concurrence in the frequency sense, i.e. for a fixed outcome $\omega^{(\alpha)}(\lambda)$ of α, the frequency $\nu_{\beta}(\lambda)$ of the maxima of outcomes $\omega^{(\beta)}(\lambda)$ of β occurring at λ should be approximately proportional to $\omega^{(\alpha)}(\lambda)$ if β is much more accurate than α, and conversely.

While the conventional frequency interpretation of the concept of probability requires sharply defined sample points, the existence of such sample points is not presupposed in the above procedure. Hence, it is more correct to view $\omega_{\mu}(\lambda)$ as a subjective measure of reliability of an apparatus which can be àposteriori tested for consistency with already available apparatus by means of the above procedure which itself involves fuzzy sample points.

These considerations can be at once generalized to measurements involving several quantities or observables $A_1, \ldots, A_n$, preparing or determining their values on a given system at the respective instants $t_1 \leqq t_2 \leqq \cdots \leqq t_n$. Naturally, the roles of preparatory and determinative measurements can be reversed in this scheme.

<u>Definition 0.11</u>

We define a fuzzy event as the family

$$\hat{\Delta} = \{\hat{\alpha} : \hat{\alpha} = (\alpha, \omega_{\alpha}),\ \alpha \in \Delta,\ \omega_{\alpha}(\lambda) \in L^1(\Delta)\}, \qquad (0.20)$$

where B^n denotes the family of Borel sets in R^n and $\langle R^n, B^n, P\rangle$ is the probability space, with $\Delta \in B^n$ and $\hat{\Delta}$ is determined by $\omega_{\alpha}(\lambda)$ on $\Delta \times R^n$.

If $\mathcal{E}^n$ is the family of all fuzzy events, $P(\hat{\Delta})$, $\hat{\Delta} \in \mathcal{E}^n$, is the probability measure on fuzzy events, then $\langle R^n, \mathcal{E}^n, P\rangle$ is a fuzzy probability space and $E^{Q,P}(\hat{\Delta})$ is the spectral measure on fuzzy events for generalized position Q and generalized momentum P.

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The outcomes of any single measurement of the observables $A_1, \ldots, A_n$ of a given system provides information described by the respective confidence functions $\omega_{\Delta_1}, \ldots, \omega_{\Delta_n}$. ω_{Δ_i} may be thought of as being characteristic functions of respective fuzzy sets $\Delta_1, \ldots, \Delta_n$. Hence, the information can just as well be described by the fuzzy set $\Delta_1 \times \ldots \times \Delta_n$ in R^n

with the charcteristic function

$$\omega_{\Delta}(\lambda_1,\dots,\lambda_n) = \omega_{\Delta_1}(\lambda_1)\ \dots\ \omega_{\Delta_n}(\lambda_n). \qquad (0.21)$$

We can now regard such measurements performed on ensembles of systems of the same kind subjected to macroscopically identical preparatory procedures as the stochastic process. For this we require a concept of probability measure which applies to sample points just as to fuzzy sets. Thus, given a family Ω of fuzzy sample points representing all conceivable outcomes of all conceivable measurements of $A_1,\dots,A_n$ at $t_1 \leqq \dots \leqq t_n$, we have to construct a family $\mathcal{E}$ of events and a probability measure $P^{A_1,\dots,A_n}_{t_1,\dots,t_n}(E)$ on the family $\mathcal{E}$.

This construction must generalize the conventional case when the sample points are sharp singletons, i.e. when they are elements of some subset in R^n.

Relying on Definition 0.11, two fuzzy events $\hat{E}_1$ and $\hat{E}_2$, (E_1, E_2 being Borel sets in R^n) are called compatible if $\omega^{(1)}_{\mu}(\lambda) = \omega^{(2)}_{\mu}(\lambda)$ for all $\mu \in E_1 \cap E_2$. For compatible events, intersections $E_1 \cap E_2$ and unions $E_1 \cup E_2$ may be defined as fuzzy events having the characteristic functions

$$\left.\begin{aligned} \omega^{(1)}_{\mu}(\lambda) &= \omega^{(2)}_{\mu}(\lambda) \text{ for } \mu \in E_1 \cap E_2 \\ \text{and } \omega_{\mu}(\lambda) &= \omega^{(i)}_{\mu}(\lambda) \text{ for } \mu \in E_1 \cup E_2, \end{aligned}\right\} \qquad (0.22)$$

respectively.

The family $\mathcal{E}$ of fuzzy events will be required to be a σ-semifield in the sense of being closed under unions and intersections for any collection of compatible fuzzy events. On $\omega_{\mu}(\lambda)$ we impose the condition that it be Lebesgue-intgrable in $\mu \in E$ for each fixed $\lambda \in R^n$, characterizing any fuzzy event $\hat{E}$. We require this stipulation in order to deal with the case of probability measures $P(\hat{E})$ in Borel sets of R^n, given by the formula

$$P(\hat{E}) = \int_E d\mu \int_{R^1} \omega_{\mu}(\lambda)\ dP(\lambda) < \infty. \quad (0.23)$$

Clearly, $P(\hat{E})$ is σ-additive in the sense that

$$P(\hat{E}_1 \cup \hat{E}_2 \dots) = P(\hat{E}_1) + P(\hat{E}_2) + \dots$$

for disjoint fuzzy events. This is Prugovecki's general definition of probability measure on a σ-semifield $\mathcal{E}$ of fuzzy events as a non-negative function on $\mathcal{E}$ which vanishes on the void set and which is σ-additive as implied above. In the present formulation, the relationship between fuzzy events and fuzzy sample points is analogous to that between conventional sharp events and sharp sample points. There are, however, some reservations with regard to the additivity of probability measures on fuzzy events; this statement will be substantiated later. Let us note here the very important statement:

Sample points are outcomes of measurements in terms of fuzzy sets satisfying quantum mechanical uncertainty relations.

It is known that position measurement procedures of a microsystem have the common feature of not yielding sharply localized results; this applies also to other simple observables. Here lies the natural domain of application for the concept of fuzzy set, first observed and studied by Prugovecki.

The basic postulate adopted here is that measurements of observables in quantum mechanics yield sample points which are normalized fuzzy sets. A fuzzy set Δ is said to be normalized if $\sup \omega_\Delta(x) = 1$.

<u>Definition 0.12</u>

If Δ_0 is the normalized fuzzy set prepared (determined) by a measurement of an observable α, then, for any Borel set Δ in R^1,

$$P_\alpha(\Delta) = \|\omega_{\Delta_0}\|_\alpha^{-1} \int_\Delta \omega_{\Delta_0}(x)\, d\mu_\alpha(x) \tag{0.24}$$

is the probability that a very precise determinative (preparatory) measurement immediately following (preceding) the measurement of Δ_0 will yield a result in Δ.

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Note that Definition 0.12 presupposes the possibility of attaining arbitrarily precise measurements even in the continuous part of the spectrum of the observables, i.e. for a given Δ, we assume the possibility of measuring fuzzy sets not only contained in Δ, but of much less spread-out than Δ. Hence, it seems natural to assume

$$\sigma_\alpha(\Delta) = \left\{ \int_{-\infty}^{+\infty} \left|\omega_\Delta(x) - \overline{\omega}_\Delta\right|^2 dx \right\}^{1/2}, \quad \overline{\omega}_\Delta = \int_{-\infty}^{+\infty} \omega_\Delta(x)\, dx \tag{0.25}$$

as a measure of the spread of normalized fuzzy set Δ. Moreover, for a measurement which yields Δ, we take $\sigma_\alpha(\Delta)^{-1}$ to represent the precision of that measurement and $\sigma_\alpha(\Delta)$ to reflect its error.

Definition 0.13

A measurement prepares (determines) the normalized fuzzy set $\Delta_\alpha \times \Delta_\beta$ in R^2 of values of (α, β) iff

$$P_\alpha(\Delta) = \|\omega_{\Delta_\alpha}\|_\alpha^{-1} \int_\Delta \omega_{\Delta_\beta}(x)\,d\mu_\alpha(x), \qquad P_\beta(\Delta) = \|\omega_{\Delta_\beta}\|_\beta^{-1} \int_\Delta \omega_{\Delta_\alpha}(x)\,d\mu_\beta(x), \tag{0.26}$$

are the probabilities for obtaining a result in Δ as an outcome of a very precise determinative (preparatory) measurement of α and β, respectively, if that measurement immediately follows (precedes) the measurement of $\Delta_\alpha \times \Delta_\beta$.

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This definition discloses that we test a specific proposal of a fuzzy set which is supposedly the outcome of a simultaneous measurement of two observables by checking the agreement of that proposal with the outcome of measurements involving each one of the two observables separately. Thus, although the possibility is admitted that no arbitrarily precise simultaneous measurements can be made (in the sense that no measurement could yield $\Delta_\alpha \times \Delta_\beta$ for (α, β) with arbitrarily small $\sigma_\alpha(\Delta_\alpha)$ and $\sigma_\beta(\Delta_\beta)$), we still have an operational procedure for the computation of $\omega_{\Delta_\alpha \times \Delta_\beta}(x,y) = \omega_{\Delta_\alpha}(x)\omega_{\Delta_\beta}(y)$ with an arbitrary degree of accuracy. This possibility rests alone on the assumption that arbitrarily accurate measurements of single observables are feasible. If the outcome of a measurement is a fuzzy set, that set will be called a fuzzy event. The probability of occurrence of an event can now be extended to events which are fuzzy. In order to exclude meaningless information, the following two definitions are put forward.

Definition 0.14

Assuming Δ and Δ_0 to be normalized fuzzy sets in R^n, we consider Δ_0 as contained in Δ for some $\theta \in [0,1]$ iff

$$\theta = \sup_{0 \leq \nu \leq 1} \{\nu : \nu\omega_{\Delta_0}(x) \leq \omega_\Delta(x), \text{ for all } x \in R^n\}.$$

In that case we put $\theta\Delta_0 \leq \Delta$. We say also that Δ_0 is non-trivially contained in Δ iff $\theta > 0$.

θ is obviously the largest real number, for which the fuzzy set $\theta\Delta_0$ with the membership function $\theta\omega_{\Delta_0}$ is contained in Δ.

Definition 0.15

Assuming $\Delta_1, \ldots, \Delta_n$ to be a family of normalized elementary fuzzy events in R^n and Δ to be a fuzzy set in R^n such that each event Δ_k is θ_k-contained in Δ and $(1-\theta_k)$-contained in its complement Δ', then

$$\nu(\Delta) = (1/N)\sum_{k=1}^{N} \theta_k$$

is called the frequency of occurrence in Δ of the elementary events of the considered family.

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This definition yields an operational frequency interpretation for the probability $P(\Delta)$ of occurrence in Δ of fuzzy sample points from a certain given population of fuzzy sets which satisfy the conditions postulated in Definitions 0.14 and 0.15; namely: $P(\Delta) \approx \nu(\Delta)$. From $\nu(\Delta') = 1 - \nu(\Delta)$ we deduce $P(\Delta') = 1 - P(\Delta)$.

When Δ is an ordinary set in R^n and $\Delta_1, \ldots, \Delta_n$ are singletons, then $\nu(\Delta)$ becomes the ordinary frequency occurrence encountered in probability theory; this coincides with the interpretation of $\omega_\Delta(x)$ as the "degree of membership of x in Δ" since with $\Delta_k = \{x_k\}$, obviously $\theta_k = \omega_\Delta(x_k)$, and $\{x_k\}$ is $(1-\theta)$-contained in Δ'. If $\Delta_1, \ldots, \Delta_k$ happen to be intervals of finite size, the conditions of Definition 0.15 are satisfied iff each Δ_k is either a subset of Δ or is disjoint from Δ. Thus, if $\Delta_1, \ldots, \Delta_n$ are not singletons and there exist Δ_k which are not subsets of Δ, but for which $\Delta_k \cap \Delta \neq \emptyset$, we face the usual difficulty of not knowing how to count such a Δ_k in computing frequencies. Allowing Δ to be a fuzzy set with no sharply delineated boundaries, we can obviate this difficulty, since then we can have a Δ_k with error size $\sigma(\Delta_k) > 0$ which also satisfies the conditions of Definition 0.15. There is an abundance of examples in which, for a given $\omega_\Delta(x)$, functions $\omega_{\Delta_k}(x)$ are chosen, for which $\theta_k = \max \theta_k \omega_{\Delta_k}(x)$ and whose spread is small by comparison with the size of the region where $0 < \omega_\Delta < 1$.

Let us now study the main features of the concept of probability for fuzzy events introduced above, that are normalized fuzzy sets $\Delta_\alpha \times \Delta_\beta$ in R^2 resulting from the simultaneous measurement of two observables (α, β). Also, let us assume that α and β are incompatible in the sense that no elementary events can occur which do not satisfy the relation

$$\sigma_\alpha(\Delta_\alpha)\sigma_\beta(\Delta_\beta) \geq \hbar/2 \qquad (0.27)$$

for some positive constant h such as the Planck constant. Recall, above all, that probability measures are defined in Boolean σ-algebras of sets involving a family B^n of all Borel sets in R^n. Such a Boolean σ-algebra Ω_σ has the properties: if $\Delta \in \Omega_\sigma$, then its complement Δ' is in Ω_σ,

and if $\Delta_1, \Delta_2, \ldots \in \Omega_\sigma$, then $\bigcup_{k=1}^{\infty} \Delta_k \in \Omega_\sigma$, for any countable family $\Delta_1, \Delta_2, \ldots$ of sets from Ω_σ.

The cardinal question arises: Can we presume now the same structure for the family $\mathcal{E}$ of fuzzy sets Δ on the space R^n, on which we intend to define a probability measure $P^{\alpha,\beta}(\Delta)$ for the aforementioned type of fuzzy events?
Since Definition 0.15 lends support to the formula $P(\Delta') = 1 - P(\Delta)$, we can maintain the property: if $\Delta \in \mathcal{E}$, then $\Delta' \in \mathcal{E}$. A function $P(\Delta)$ on $\mathcal{E}$ is a probability measure on fuzzy events in R^n iff it assumes values in $[0,1]$ and has the following properties:

(a) $P(R^n) = 1$; (b) $P(\Delta') = 1 - P(\Delta)$ for all $\Delta \in \mathcal{E}$, and

(c) if $\Delta_1, \Delta_2, \ldots \in \mathcal{E}$ are disjoint, then $P(\bigcup_{k=1}^{\infty} \Delta_k) = \sum_{k=1}^{\infty} P(\Delta_k)$, whenever $\bigcup_k \Delta_k$ also belongs to $\mathcal{E}$.

It remains to show whether any probability measures in fuzzy sets corresponding to measurements of incompatible observables can be extracted from the formalism of quantum mechanics; that such an extraction cannot occur simultaneously is obious.

<u>Criticism on Prugovecki's definition of probability on fuzzy events</u>

The introduction of fuzzy set notions and corresponding techniques to quantum mechanics is exclusively the merit of E. Prugovecki; and they fit well in the carefully chosen situations. But we feel not at all at ease with the way, additivity, classical measure and Lebesgue integration have been forced on fuzzy sets and fuzzy events.
We are aware of the fact that the absence of additivity in almost all human sciences, where measurement is concerned, is a real blessing and, most likely, the "will" of nature. It is here, where we should swiftly make better use of fuzzy methodology, and remember that it is monotonicity which replaces additivity in fuzzy set theory. This

observation is substantiated mainly by the work of Sugeno (1974) on fuzzy measure and fuzzy integrals; and probability is based on measure.

Fuzzy sets have been introduced by Zadeh (1965) to describe those cases when and where there are no precise criteria of membership in ordinary sets. A fuzzy set Δ in R^n is given uniquely by means of a membership function $\omega_\Delta(x)$, $x \in R^n$, which takes values in the unit interval I and represents the "grade of membership". Because Δ and ω_Δ are uniquely related, some authors prefer to call the map ω_Δ a fuzzy set. The properties of fuzzy sets are discussed in Appendix 2; the notions of "fuzzy sample point" and "fuzzy event" have been adequately defined and discussed in this subsection; so that we merely need define "fuzzy measure" and "fuzzy integral".

Let (X,B,P) be the ordinary probability space, with B a Borel field of X and P the probability measure

$$P: B \longrightarrow [0,1]; \qquad (0.28)$$

a fuzzy subset of X is conveniently described by its membership function

$$\omega_\Delta: X \longrightarrow [0,1], \quad \text{being a fuzzy set.} \qquad (0.29)$$

(0.28) is a set-function, (0.29) is an ordinary function; hence, P and ω_Δ act on different levels. When X is a finite set, $P(\{x\})$ and $\omega_\Delta(x)$ may be compared optically; yet,

$$\sum_{x \in X} P(\{x\}) = 1 \text{ while } \sum_{x \in X} \omega_\Delta(x) \neq 1.$$

However, in an infinite case, say $X = R^1$, a difficulty arises; for, if $(a,b] \subset R^1$, then $P((a,b]) = \int_a^b \rho(x)dx$, $\rho(x)$ being the probability density, not probability itself. Thus follows $0 = P(\{x\}) \neq \rho(x)$, for all $x \in R^1$, even when $\rho(x) \neq 0$, while $\omega_\Delta(x) \neq 0$. It may still seem that a probability density $\rho(x)$ and $\omega_\Delta(x)$ are comparable, but $\rho(x)$ has no practical meaning except for the fact that

$$\rho(x) = dP((-\infty, x)/dx.$$

There is only one point of contact between probabilities and fuzzy sets, and this point is the concept of fuzzy event. The mathematical confrontations of importance are:

probability logic vs fuzzy logic, and

probability measure vs fuzzy measure.

Randomness of strings is nowadays being defined in terms of Turing machines, and from the measure-theoretical point of view we consider measures as scales of randomness.

Fuzzy measures are generally considered as subjective scales for fuzziness (Prugovecki does so in quantum mechanics). They are set-functions with monotonicity, but do not necessarily possess additivity.

Fuzzy integrals are functionals defined by using fuzzy measures which correspond to probability expectations.

Let X be an arbitrary set, B a Borel field of X and $x \in X$ a representative element.

Definition 0.16

A set-function g, defined on B and having the properties:

(i) $g(\emptyset) = 0,\ g(X) = 1;\quad \emptyset, X \in B,\quad g: B \longrightarrow [0,1]$,

(ii) If $A, C \in B$ and $A \subset C$, then $g(A) \leqq g(C)$,

(iii) If $F_n \in B$ for $1 \leqq n < \infty$ and a sequence $\{F_n\}$ is monotone (in the sense of inclusion), then $\lim_{n\to\infty} g(F_n) = g(\lim_{n\to\infty} F_n)$,

is called a fuzzy measure.

The triple (X,B,g) is called a fuzzy measure space and g is the measure of the measure space (X,B).

Definition 0.17

Let $\omega_\Delta: X \longrightarrow [0,1]$, $\Delta \subset X$, be a B-measurable function. A fuzzy integral over $A \in B$ of a function $\omega_\Delta(x)$ with respect to a fuzzy measure g is defined by

$$\omega_\Delta(x) \circ g(\cdot) = \sup_{\alpha \in I} \left[\alpha \wedge g(A \cap F_\alpha)\right],\quad I = [0,1],$$

where $F_\alpha = \{x: \omega_\Delta(x) \geqq \alpha\}$.

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Our recommendations relating to Prugovecki's publications on quantum measurement under fuzzy conditions amount to little more than the incorporation of Definitions 0.16 and 0.17 into his analyses.

0.3.3 Measuremental uncertainty in complex fuzzy systems

So far we have encountered uncertainties, characterized by uncertainty relations in:

(1) Engineering (applied physics) of the type

$$\Delta q \cdot \Delta f \geqq K_s \quad \text{(structural uncertainty)},$$

$$\Delta y \cdot \Delta q = K_m \quad \text{(metrical uncertainty)},$$

both in the "environment" of Euclidean spaces and Boolean logic underlying finite systems;

(2) Quantum physics of the type

$$\Delta x \cdot \Delta p \geqq h$$

in the "environment" of ∞-dimensional separable Hilbert spaces and Birkhoff-v.Neumann logics underlying complex microsystems.

Both (1) and (2) are obtained on the assumption of perfect measuremental behaviour. We now learn that measurement itself exhibits its own uncertainty at a higher level than the first two sciences, namely:

(3) Measuremental uncertainty of a similar type as the first two in the "environment" of ∞-dimensional non-reflexive Banach spaces and Brouwerian lattice logic (or Tarski's calculus of systems) underlying complex fuzzy macrosystems.

Research in this direction has been advocated by Professor Pieter Eykhoff, Technical University - Eindhoven.

In order to deal intelligibly with the central concept of complexity, we require some basic notions on the power of sets or their cardinality. The notion of fuzziness should be sufficiently clear by now.

Definition 0.18

Two sets, A and B, are said to be of equal count, denoted by $A \sim B$, if there is a one-one function f with domain A and range B; f is then said to establish equal count of A and B. We may equivalently say that the sets A and B are of equal power or that they have the same cardinal number.

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Equal count is subject to the laws of equivalence, commutativity, associativity and potentiation of the cartesian product. We obtain then

Theorem 0.10

For sets A and B to be of equal count, it is necessary and sufficient that the relational systems $\langle A; A \times A \rangle$, $\langle B; B \times B \rangle$ be isomorphic. And since $\overline{\overline{A}}$ is the cardinal number of A, we have $A \sim B$ and $\overline{\overline{A}} = \overline{\overline{B}}$ as equivalent statements.

Definition 0.19

A is called a countable (denumerable) set if it is finite or of equal count with the set of natural numbers N; i.e. $\overline{\overline{N}} = \alpha = \aleph_0$.

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The sets of integers, rational numbers, algebraic numbers are countable, those of irrational numbers, real numbers and the Cantor set are overcountable; they are of the power of continuum $\mathbf{c}$.

The power set $\mathcal{P}(A)$ of set A is the family of all subsets of A, i.e. $(X \in \mathcal{P}(A)) \equiv (X \subset A)$. It is an extremely rich set, too rich for practical applications; in topology, the family 2^A of all closed subsets of A is used; it is a topological hyperspace when equipped with Vietoris or exponential topology.

Theorem 0.11

No two of the following sets are of equal count:

$$A, \mathcal{P}(A), \mathcal{P}(\mathcal{P}(A)), \ldots .$$

For, if A has power $\mathfrak{m}$, $\mathcal{P}(A)$ has power $2^{\mathfrak{m}}$ and $\mathcal{P}(\mathcal{P}(A))$ power $2^{2^{\mathfrak{m}}}$. From this reasoning we deduce $n < \mathfrak{a}$ and $\mathfrak{m} < 2^{\mathfrak{m}}$.

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In Definition 0.19, the continuum hypothesis of Cantor: $\mathfrak{a} = \aleph_0$, has been assumed. A more general continuum hypothesis states:

$$\mathfrak{a}_\xi = \aleph_\xi \text{ for every } \xi.$$

This is equivalent to saying that there is no number between $\mathfrak{m}$ and $2^{\mathfrak{m}}$, where $\mathfrak{m}$ is an arbitrary cardinal number.

Definition 0.20

According to Kuratowski(1977), simple sets are characterized by power up to $\mathfrak{a}$, while complex sets have powers of at least $2^{\mathfrak{a}}$. We call those systems complex whose some or all sets are complex. The separation $\mathfrak{a} < 2^{\mathfrak{a}}$ between simple sets and complex sets (systems) is a sharp one.

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Given the relations: $a \in \{a\} = \mathcal{P}(\{a\})$, we can say that the singleton $\{a\}$ contains the element a and coincides with the power set $\mathcal{P}(\{a\})$. From the measurement point of view, the precision and the relevance of every singleton are identical; it is a property shared by no other set except in the complex condition; there it is the negligible set.

Incompatibility Principle (due to Zadeh (1973))

The essence of this principle is that, as the complexity of a system increases, our ability to make precise and yet significant statements about its behaviour diminishes until a threshold is reached, beyond which precision and significance (or relevance) become almost exclusive characteristics.

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This phenomenon we would characterize with the words: Precision and significance become incompatible properties in the sense of relation (0.27) belonging to quantum mechanics.

Transitions of systems between simplicity and complexity

From the aforesaid, any system may be in one of two basic phases:

(i) in the simple (countable) phase, where it is mostly finite and Boolean-conform, or

(ii) in the complex phase, where it is ∞-dimensional, non-Boolean and contaminated by a fundemental uncertainty.

The transition from one phase to the other is discontinuous and characterized by a step (jump).

Fundamental Assumptions

(1) It is postulated that a system $Y = \tilde{Y}_{\infty}$ found in the complex phase must have developed to its present state from a prototype X_n in the simple (finite) phase.

(2) If (1) is true, then it should be possible somehow to reduce the system $Y = \tilde{Y}_{\infty}$ to an equivalent system $X = X_n$ in the simple (finite) phase without loss of information by equalization of the transition steps.

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It is clear that in stepping down from complexity to simplicity, the uncertainty and its relation must vanish. This idea yields the following

Theorem 0.12

Under the Fundamental Assumptions (1) and (2), the step-up in power $\alpha < 2^{\alpha}$ is exactly annihilated by the step-down in power to Y/H, representing quotient class formation of Y modulo uncertainty relation, indicated by H.

The step-up is viewed as a generator of complexity coupled with uncertainty, and the step-down as a sink of uncertainty due to the reduced power.

In this fashion every complex system can be reduced, without loss of information, to a finite Boolean system X, measurable in the classical representational way.

Details of this reasoning are depicted in the figure below; fuzziness would enhance the uncertainty effect.

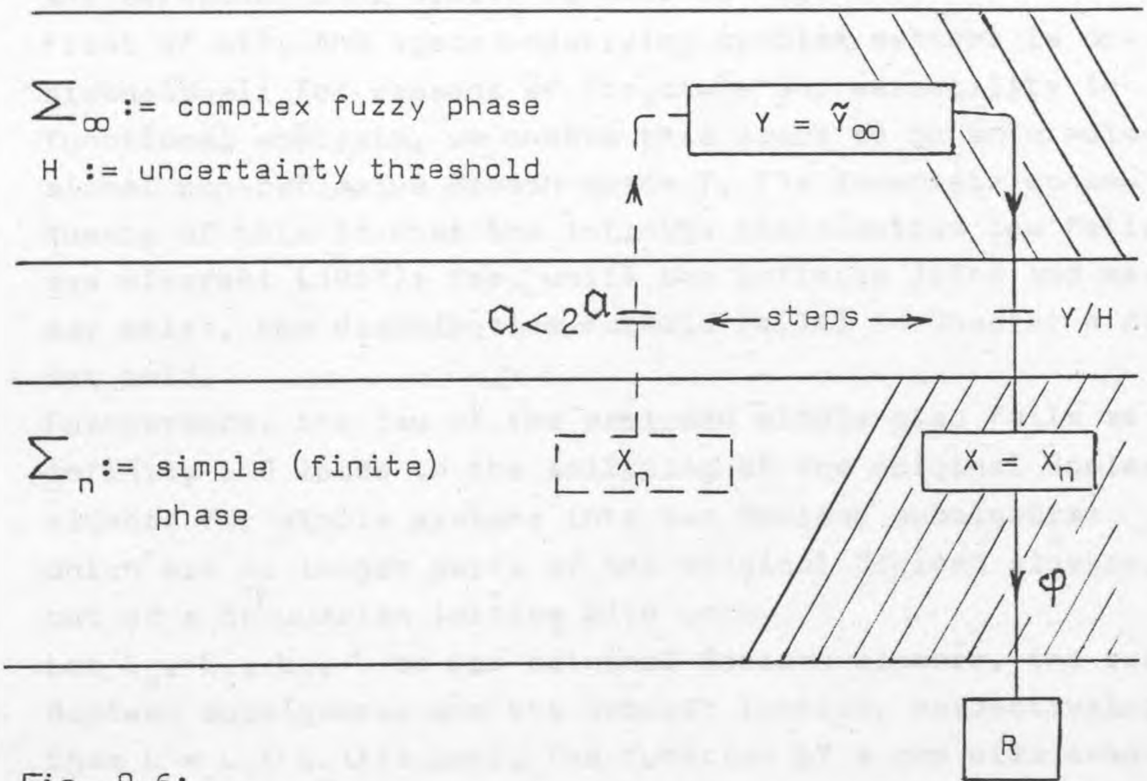

Fig. 0.6:
Transition of an empirical relational system.

0.3.3.1 Statement of the problem and steps towards its solution

For simple systems, i.e. systems obeying the axioms of Boolean algebra, single-quantity as well as multi-quantity measurements present no difficulty, because the quantities involved are independent, both their precision and significance (or relevance) may be practically arbitrarily increased, and a joint distribution for simultaneous measurement (which is certainly necessary when the state of a closed empirical system under observation is required to be known at all times - as in catastrophy engineering) always exists. The representational theory, in the stated logical domain, is correct, exhaustive, and works well in its stringent setting, which represents - at the same time - its applicational limitation.

The changes, which the scale $S = \langle Q, N, \phi \rangle$ of measurement undergoes when the empirical relational system Q turns complex fuzzy, are of a fundamental kind, and such that no homomorphic transformation mapping Q into N exists. In what follows, any reference will always be made to Fig. 0-6.

First of all, the space underlying complex systems is ∞-dimensional; for reasons of structure and versatility in functional analysis, we choose this space to be an ∞-dimensional non-reflexive Banach space Y. The immediate consequence of this is that the infinite distributive law fails, see Sikorski (1961); for, while the infinite joins and meets may exist, the distributive formula (4.14) in Chapter 4 does not hold.

Furthermore, the law of the excluded middle also fails at infinity and leads to the splitting of the original Boolean algebra for simple systems into two Boolean subalgebras which are no longer parts of the original Boolean algebra, but of a Brouwerian lattice with unity.

Let L_0, L_1, L_2, L be the original Boolean algebra, the two Boolean subalgebras and the Brouwer lattice, respectively; then $L = L_1 \cup L_2 \cup$ (a gap). The function of a gap will actually play a so-called negligible set, introduced simultaneously with a pseudo-difference in L which replaces the set-difference in L_0. Because of the expression for L, we get a non-complementation condition:

$$\left.\begin{array}{l} L_1 \cap L_2 \neq \emptyset, \\ \text{in fact:} \quad L_1 \cap L_2 \supseteq H, \\ H \text{ being a negligible set in } Y. \end{array}\right\} \qquad (0.30)$$

This expresses the fact that L_1 and L_2 are two incompatible Boolean subalgebras of L_0. Thus, the principle of incompatibility (due to Zadeh) is satisfied in complex systems. We magnify this incompatibility (or uncertainty) by fuzziness; for, if ω is a fuzzy map, then $\omega_{L_1} \cap \omega_{L_2} \neq \emptyset$. Here and in (0.30) L_1, L_2 and ω are treated as sets, which is obviously permissible.

Let us note that normed linear spaces and non-reflexive Banach spaces are the only important spaces containing negligible sets. Moreover, the non-complementation condition corresponds to the non-commutation condition in quantum mechanics; there will also be formally similar uncertainty relations in complex fuzzy systems and in quantum mechanics, but the former relating to precision and significance of measurement, the latter to position and momentum of material points in physics. The difference between the two cases is greatest in regard to the thresholds of uncertainty: it is a set (a mathematical notion) in case of measurement and a universal (physical) constant in quantum physics.

In the search to turn an empirical complex fuzzy system into a measurable one, with little or no loss of information, one is guided by the belief that a complex system in phase Σ_∞ must have originated from a system in phase Σ_n by overcoming a discontinuous barrier. Hence, there must be an itinery back to simplicity across a reverse barrier, as indicated in Fig. 0-6.

First, we determine the threshold of uncertainty for the complex fuzzy system. This is obtained from topological considerations of the ∞-dimensional Banach space Y. Since Y is non-reflexive, there is a negligible set H with respect to the homeomorphism of Y into itself; and, because Y is considered to be complex fuzzy, H will be a "thick" negligible set filling the "thick" boundary of the sets in Y and representing the uncertainty threshold there. Being negligible, H has the same measure of "virtualness" as h has for energy-time quantities at and below that of the Planck constant.

Next, we note that the transition from X_n to Y via the step $a < 2^a$ steps-up the power of sets, thereby creating uncertainty; so from the reverse step from Y to X we demand an equal power step-down and the capability of leaving all uncertainty behind. The only operation fulfilling both requirements is the formation of a quotient space Y/H, H being a subspace of Y, and of a quotient algebra L/I_H to the Brouwerian lattice L modulo prime ideal I_H based on H. L/I_H is, in fact, a Boolean algebra with zero and unity; its classes, treated as singletons, are obviously disjoint, and the order relation is that of inclusion "$\leqq$".

This Boolean algebra comprises a corresponding empirical relational system which may be mapped into the numerical relational system by a homomorphism ϕ. If the empirical relatio-system consists of two components: one originally complex component, the other originally finite component, then the union of both may be accommodated in the classical formula

for the meaningfulness.
Let us add that the topological equivalent of a Boolean algebra is a Stone space; therefore, we may represent a scale topologically as consisting of two Stone spaces (empirical and numerical, respectively) and a monotone homeomorphism mapping the former into the latter. Thus, the space of measurement is a Stone space (with selfreproductive sets).
The classical representational theory of measurement needs no improvement except that of wider applicability to systems of greater complexity envisaged in the future. We discover here an extension comparable to that from classical mechanics to quantum mechanics; in both cases complexity provides the motivation.

0.3.3.2 Substantiation of the uncertainty reasoning in measurement

The necessary and sufficient conditions for the existence of ordinary as well as fundamental uncertainties are technical, physical or mathematical bounds in the theories considered.
The discovery of the finiteness and constancy of the velocity of light (c) reduced the question of the simultaneity of two different events to a virtual question in special relativity theory. The existence of Planck's constant (h) reduces the question of the simultaneity of a definite position and of a definite momentum of an atomic particle to a virtual question in quantum mechanics (to which there is no answer).
In tune with these fundamental uncertainty statements I now put forward a fundamental uncertainty result in generalized measurement.

Fundamental Proposition

The bound in complex fuzzy systems is provided by a so-called negligible set (H) in an underlying ∞-dimensional normed linear or non-reflexive Banach space Y.
Given a nonvoid negligible set H in space Y, any question concerning the simultaneous definiteness of both the precision and the relevance of measurement is a vitual question. In this case the conjugate notions of precision and relevance are incompatible quantities of measurement.

Plausibility proof

A negligible set H exists iff $h: Y-H \longrightarrow Y$ is a homeomorphism,

in which case we simply write h: $Y-H \cong Y$.
Now, every Banach space is complete in the Cauchy sense. But completeness is not a topological property, i.e. if Y is complete and h a homeomorphism, then h(Y) need not be complete. For h(Y) to be complete, we need topological completeness. The defect in topological completeness amounts exactly to the negligible set H, and this has an influence on the fixed point behaviour of mappings in Y, as will be seen in Chapter 2.
H is, of course, a constant set for a given space Y; it may be enhanced (magnified) by fuzzy action, and it vanishes when Y is reduced to a separable Banach space. At this level the law of the excluded middle and the law of the distributivity regain their validity and we have Boolean conditions again.
Since h in quantum mechanics and c in special relativity are physical bounds, the convergences: $h \rightarrow 0$ and $c \rightarrow \infty$ are no more than mathematical manipulations. H, on the contrary, is a mathematical bound in the generalized theory of measurement; and from its vanishing in a Banach space, which contains a separable Hilbert space, we conclude that quantum mechanics can have no other than a physical bound and that generalized measurement sets its bound at a higher level than either physics or engineering. This stresses the importance of measurement.
If we take Norbert Wiener (1920) literally, then mathematics is measurement (with exceptions), but not conversely. A confirmation of this standpoint will be found in the Fundamenta Mathematica of A.N. Whitehead and B. Russell (Vol.III).
It speaks for the generality of measurement as science that it stands in its own right and is no part of any applied discipline. The uncertainty relation obtained for measurement could not have been found in any other way; and no matter how we argue, there is no way of discussing it away. Its rank is - in my opinion - very high.

It is here the proper place to comment on a speculation of C.F. v.Weizsäcker (1970), that it might prove useful to discard the law of the excluded middle in quantum mechanics. We must deny this hope on the basis of our findings, since the law of the excluded middle is valid in the separable Hilbert space.

0.4 Presentation of the analysis by chapters

Chapter 1 is devoted to the shift from Boolean logic to intuitionistic logic as a consequence of the problem of "infinity" and the resulting invalidity of the law of distributivity and the law of the excluded middle. The analysis demonstrates the splitting of the Boolean algebra of the simple system into two incompatible Boolean subalgebras, revealing the validity of the principle of incompatibility due to Zadeh (1973) and the fuzziness of the boundary of the sets involved.

Chapter 2 is wholly devoted to the existence and the meaning of negligible sets in normed linear and certain Banach spaces.

Chapter 3 unites the results of Chapters 1 and 2 and offers a definition of the non-complementarity condition. This is complemented by a derivation of the uncertainty relation for complex fuzzy systems. Both the non-complementation condition and the uncertainty relation for complex fuzzy systems are compared to the non-commutativity and Heisenberg's uncertainty relation of quantum mechanics, respectively, in Chapter 4, and discussed on form and content.

In Chapter 5 additional results of classical measurement theory are derived from topological and measure-theoretic principles, i.e. complementary to the Introduction and more in the topological spirit of the thesis.

Chapter 6 is the most important part of the thesis. It settles the question of the logical foundation of complex fuzzy systems and provides a natural imbedding for the measuremental uncertainty relation. The calculus of systems making all this possible, facilitates in a high degree also the construction of measurement procedures (scales).

Chapter 7 is devoted to the properties of the Stone space and to the selfreproductivity of its sets, which make the Stone space unique for the measurement of mixed systems.

The novelty of this work, the new ideas and the main results obtained in the course of this study are summarized and complemented in the Conclusions, Chapter 8.

Appendices on the calculus of attributes, on fuzzy reasoning and on quantum mechanics as well as a list of References are provided at the end for the convenience of interested readers.

CHAPTER 1: TRANSITION FROM BOOLEAN ALGEBRA TO BROUWERIAN ALGEBRA

It will be shown in this chapter that the complexification of an empirical system is accompanied by a transition from the Boolean algebra associated with simple systems to a non-Boolean algebra characterizing complex non-classical systems. Actually, the Boolean algebra splits upon complexification into two Boolean subalgebras with a gap between them. This is the subject of a doctrine called "Part and the Whole".

Let us begin with a simple example illustrating the working of a Boolean algebra and one of its basic axioms: tertium non datur.

Whenever we inquire about the elements of a set in a given space X, the question arising is: Does $x \in X$ belong to set A or not - on the condition of tertium non datur? This question obviously implies the existence of a so-called characteristic function $\chi_A: X \longrightarrow \{0,1\}$ defined by

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \in (X-A) \end{cases} \tag{1.1}$$

The alternative form $\chi_A: 2^X \longrightarrow \{0,1\}^X$ discloses that χ_A is a one-one and onto function; hence we have $2^X \sim \{0,1\}^X$, and the notions Y^X and 2^X correspond to one another.

The following formulas are easily proved:

$$\chi_X \equiv 1, \quad \chi_\emptyset \equiv 0, \tag{1.2}$$

$$\chi_{-A}(x) = 1 - \chi_A(x), \tag{1.3}$$

$$\chi_{A \cap B} = \chi_A \cdot \chi_B, \tag{1.4}$$

$$\chi_{A-B} = \chi_A - \chi_{A \cap B}, \tag{1.5}$$

$$(A = \bigcup_t A_t) \Longrightarrow (\chi_A(x) = \max_t \chi_{A_t}(x)), \tag{1.6}$$

$$(A = \bigcap_t A_t) \Longrightarrow (\chi_A(x) = \min_t \chi_{A_t}(x)), \tag{1.7}$$

$$(A = \lim_{n=\infty} A_n) \equiv (\chi_A(x) = \lim_{n=\infty} \chi_{A_n}(x)). \tag{1.8}$$

The concept of the characteristic function may easily be extended to a sequence of sets and - better still - to a set-valued function. This is of some importance for the control theory in orientor field notation of the form:

$$\frac{dx}{dt} = \dot{x} \in F(t,x); \quad F: T \times X \longrightarrow 2^Y. \qquad (1.9)$$

Let $F: T \rightarrow 2^Y$ (for a constant x) be a set-valued function such that $F(t) = F_t \subset Y$ for $t \in T$. Then the characteristic function χ_F of F associates with every $y \in Y$ a function $\chi_F(y) \in \{0,1\}^T$, defined by

$$\chi_F^t(y) = \begin{cases} 1 & \text{if } y \in F_t, \\ 0 & \text{if } y \in (Y-F_t). \end{cases} \qquad (1.10)$$

If t assumes natural numbers, i.e. $t = 1,2,3,\ldots,n,\ldots$, then the characteristic function of the sequence of sets $F_1, F_2, \ldots, F_n, \ldots$ assumes as values sequences of numbers $y^{(1)}, y^{(2)}, \ldots, y^{(n)}, \ldots$ such that

$$y^{(n)} = \begin{cases} 1 & \text{if } y \in F_n, \\ 0 & \text{if } y \in (-F_n). \end{cases} \qquad (1.11)$$

In 1965 Zadeh generalized the characteristic function thus:

If X is a nonvoid set, $I = [0,1]$ the unit interval, then $\omega: X \longrightarrow I$ or $\omega \in I^X$ is a membership function. For every $x \in X$, $\omega(x)$ is said to be the grade of membership of x in ω, X is the carrier of ω and $\{x: x \in \omega, \omega(x) > 0\}$ the support of ω.

I^X, partially ordered by $\omega \leqq \pi$ iff $\omega(x) \leqq \pi(x)$, $x \in X$, is a Brouwerian lattice which - since I is a complete chain - is also completely distributive. Yet, since it is not a Boolean algebra (it is not complemented), not all prime filters in I^X are maximal. Moreover, maximal filters are inadequate to describe all filters because I^X is not separative. This must be taken into account when studying the problem of convergence, see Lowen (1979).

Fuzzy topology is assumed in Chapter 2; it differs in several respects from general topology and is, therefore, necessary in transition from ordinary sets to fuzzy sets occurring in complex dynamical systems. In this respect, Appendix 2 renders assistence.

Fuzzy engineering involves thresholds of resolution both in the small and in the large (or complex). If we speak of large, possibly infinitely large systems such as the number of atoms accommodated in a sizeable volume or the dimension of extremely complex systems, we associate with them infinite dimensional spaces (Banach, Fréchet or Hilbert spaces).

The uncertainties occasioned by the thresholds are of a fundamental kind and characterizing the associated theory; they have nothing to do with the usual accuracies of practical measurement. They will be the subject of our concern in this study. In this endeavour we shall have to leave the ground of classical (Boolean) logic and move closer to the intuitionistic philosophy.This will require giving up some of the classical self-evident truths and phrases such as "there exists", as well as any other infinity than "potential infinity".

To illustrate the above restrictions, consider the propositional function $\phi(x)$ in the arithmetic of natural numbers. A mathematician considers the problem of the truth of $\phi(x)$ as solved if there is a proof that the sentence $\delta = \bigcup_{\pi} \phi(\pi)$ is a theorem in arithmetic; he dispenses with the task of actually constructing a number $n \in N$ such that $\phi(n)$ will emerge as a true sentence. In order to prove δ, a mathematician would begin with the proof of $\gamma = \neg \bigcup_{\pi} \neg \phi(\pi)$ and then use the tautology $\gamma \Rightarrow \phi$. The truth of δ would then follow from the modus ponens formula $\frac{\gamma \wedge (\gamma \Rightarrow \delta)}{\delta}$.

This procedure is inacceptable to an intuitionist; he demands the construction of a number $n \in N$ fulfilling the propositional function $\phi(n)$, otherwise δ and $\gamma \Rightarrow \delta$ will be rejected.

Truth in this context is understood in the sense of Tarski (1956), that is, truth is conceived of as a special form of satisfiability.

Since constructivity is bound to finite sets, only potential infinity is admitted in intuitionistic logic, never actual infinity. However, many sets contain an infinity of elements and the axiom of infinity is one of the seven axioms of the Zermelo-Fraenkel set theory. Hence, the intuitionists reject the concept of a set (the basic undefined notion of set theory) as well as the whole set theory. Moreover, negation and disjunction (or alternative) are understood differently in the intuitionistic logic, namely:

while $\phi \Rightarrow \neg\neg\phi$ is accepted, $\neg\neg\phi \Rightarrow \phi$ is not, and $\phi_1 \vee \phi_2$ is true iff ϕ_1 or ϕ_2 is true and if a method is known by which the true summand may be determined.

Hence, both the tautology $\phi\vee\neg\phi$ and the law of the excluded middle (terium non datur) are banished from the intuitionistic logic. Now, the metatheory of intuitionistic logic is known to coincide with the theory of pseudo-Boolean algebras in the same sense as the metatheory of classical logic does with the theory of Boolean algebras. Hence follows that the theory of pseudo-Boolean algebras is the theory of the lattices of open subsets (and the theory of Brouwerian algebras is a theory of lattices of closed subsets), and any investigation of intuitionistic logic consists of a study of the lattices of open (resp. closed) sets in topological spaces. For reasons of convenience, we shall give our preference to lattices of closed sets corresponding to Brouwerian algebras with the operation of pseudo-difference.

Let us first settle the algebraic terminology.

Definition 1.1

Let there be given a nonvoid set X, a relation R on X and the following conditions:

(i) xRx; $x,y,z \in X$, (reflexivity)

(ii) $(xRy \wedge yRx) \Longrightarrow (x = y)$, (antisymmetry)

(iii) $(xRy \wedge yRz) \Longrightarrow xRz$, (transitivity)

(iv) $xRy \vee yRx$. (connectivity)

Connectivity is equivalent to the trichotomy law.

Relation R establishes, respectively:

(a) a partial order (X a poset = partially ordered set) if (i) - (iii) are satisfied;

(b) a linear order if (i) - (iv) are satisfied;

(c) a quasi-order if (i) and (iii) are satisfied;

(d) a full order if (iii) and (iv) are satisfied; however, we shall also call it "complete order".

Definition 1.1a

Relation R ordering set A and relation R^* ordering set A^* establish a similarity order (expressing that A and A^* are isomorphic) if there is a one-one mapping f of A onto A^* satisfying the equivalence:

$$(xRy) \equiv (f(x)R^*f(y)).$$

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On every nonvoid set A with $A^n = \underbrace{A\times A \times \ldots \times A}_{n}$, a transformation $f: A^n \longrightarrow A$, $n \in N$, may be defined which we shall call n-argumental operation. For brevity let $f = (f_1,\dots,f_n) = O_n$ and $f = (f_1,f_2,\dots) = O$.

Definition 1.2

An abstract algebra (briefly algebra) is the name given to any pair of sets $\{A,O\}$ with $A \neq \emptyset$ and O finite operations.

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Let $\{L,\cup,\cap\}$ be an abstract algebra with two operations: $\cup$ and $\cap$ and, possibly with null ($\sqcup$) and unit ($\sqcap$) elements. The sum $x \cup y$ will be called join and the product $x \cap y$ will be called meet of the elements $x,y \in L$.

Consider the following seven axioms (laws):

(L1): $x \cup y = y \cup x$, $x \cap y = y \cap x$, (commutativity)

(L2): $x \cup (y \cup z) = (x \cup y) \cup z$, (associativity)
$x \cap (y \cap z) = (x \cap y) \cap z$,

(L3): $x \cup x = x$, $x \cap x = x$, (idempotence)

(L4): $x \cup (x \cap y) = x$, $x \cap (x \cup y) = x$, (absorption)

(L5): $x \cup (y \cap z) = (x \cup y) \cap (x \cup z)$, (distributivity)
$x \cap (y \cup z) = (x \cap y) \cup (x \cap z)$,

(L6): $x \cup \sqcup = x$, $x \cap \sqcap = x$, (null - unity)

(L7): $x \cup -x = \sqcap$, (tertium-non-datur)
$x \cap -x = \sqcup$. (contradiction)

Definitions 1.3

(i) $\mathcal{L} = \langle L,\cup,\cap \rangle$ is said to be a lattice if $\mathcal{L}$ satisfies axioms (L1) - (L4). It then admits a partial order such that, for all pairs $\{x,y\} \subset L$, the sum $x \cup y$ coincides with the least upper bound (lub) and the product $x \cap y$ coincides with the greatest lower bound (glb).
A lattice is called complete if all subsets of L have an lub and a glb.
The following duality principle holds true for lattices:
If a sentence T_1 (i.e. a formula without free variables) is the consequence of (L1) - (L4), then the sentence T_2, obtained by interchanging $\cup$ and $\cap$ in T_1, is a consequence of (L1) - (L4) as well.

(ii) $\mathcal{L}$ is called a distributive lattice if (L1) - (L5) apply.

(iii) $\mathcal{L}$ is called a Boolean algebra if it satisfies (L1) - (L7).

The family 2^L of all subsets of L together with the set-theoretic operations: $\cup$, $\cap$, $-$, is a Boolean algebra.

Definitions 1.4 (Rasiowa and Sikorski (1970))

There are two notions of the complement of an element of a lattice $\mathcal{L}$ corresponding to the set-theoretic complement of a set A:

(a) either the greatest subset $-A$ of L disjoint with A,

(b) or the smallest subset of L whose union with A equals L.

These subsets are, in general, not equivalent.

(i) If L contains the smallest element $\sqcup$, we call $c \in L$ a $\cap$-complement in L provided that c is the greatest element satisfying $a \cap c = \sqcup$.
If L contains the greatest element $\sqcap$, then $c \in L$ is the smallest element satisfying $a \cup c = \sqcap$.

(ii) If $\mathcal{L}$ is a distributive lattice and $a \cup c = \sqcap$, $a \cap c = \sqcup$ hold, c is said to be the complement of $a \in L$. The only other complement of importance is the $\cap$-complement, which - if it exists - is named pseudo-complement.

(iii) Element $c \in L$ is called pseudo-complement of a relative to b (or modulo b), briefly: relative pseudo-complement, if c is the greatest element such that $a \cap c \leqq b$; it will be denoted by $a \sqsupset b$.
Conform with this definition:

$$(x \leqq a \sqsupset b) \equiv (a \cap x \leqq b) \quad \text{for all } x \in L. \qquad (1.12)$$

Because of $a \cap b \leqq b$, we have $b \leqq a \sqsupset b$ provided that $a \sqsupset b$ exists; and if $\sqcup$ and $\sqcap$ are in L, then

$$(a \leqq b) \equiv (a \sqsupset b = \sqcap), \sqcap \sqsupset b = b, \qquad (1.13)$$

and $\neg a = a \sqsupset \sqcup$.

(iv) A nonvoid ordered set L, in which every pair of elements satisfies the formulas (1.12) and (1.14) below, has been named an implicative system, in which which

$$a \cap (a \sqsupset b) \leqq b. \qquad (1.14)$$

(v) $a \sqsupset b$ is called a complement of a relative to b (or modulo b) if it is the smallest element

$c \geqq a \cap b$ such that $a \cup c = \sqcap$ holds.

If an element a of the distributive lattice $\mathcal{L}$ has the complement $\neg a$, then there exists, for every $b \in L$, the complement $a \supset b$ of a relative to b and $a \supset b = \neg a \cup b$, where $\neg a \cup b$ is the pseudo-complement of a relative to b. $\neg a \cup b$ is also the complement of a relative to b.

The notions dual to the relative pseudo-complement and to the relative complement are those of pseudo-difference and difference, respectively.

(vi) Element $c \in L$ is called pseudo-difference of b and a if it is the smallest element such that $a \cup c \geqq b$; it is denoted by $b \dot{-} a$.

Conform with this definition:

$$(x \geqq b \dot{-} a) \equiv (a \cup x \geqq b) \quad \text{for all } x \in L. \qquad (1.15)$$

Note the duality of (1.12) and (1.15).

Because of $a \cup b \geqq b$, we have $b \geqq b \dot{-} a$ provided that $b \dot{-} a$ exists. Obviously $(a \dot{-} a) \equiv (\sqcup \in L)$; and if $\sqcup \in L$ exists, then

$$(a \geqq b) \equiv (b \dot{-} a = \sqcup), \quad b \dot{-} \sqcup = b.$$

Hence, the pseudo-difference $b \dot{-} a$ is the smallest element $c \leqq a \cup b$, for which $a \cup c = a \cup b$.

(vii) Assuming $\sqcup \in L$, b - a is called the difference of b and a if it is the greatest element $c \leqq a \cup b$, for which $a \cap c = \sqcup$ holds. If element a of a distributive lattice $\mathcal{L}$ has the complement $\neg a$, then there exists the difference b-a, and we have b-c = $b \cap$ -a.

Note

Equivalence (1.15) has sense in several disciplines:

(a) If (1.15) is a relation in the arithmetic of natural numbers or real numbers, then, for numbers $a, b, x \in N$ (or R), for $\dot{-}$ replaced by — and $\cup$ replaced by +, (1.15) represents an arithmetical identity.

(b) If (1.15) is a relation in Boolean algebra, then, with a,b,x as elements of that algebra, with $\dot{-}$ replaced by the operation - and $\cup$ remaining unaltered, we obtain a true statement in Boolean algebra.

Now, Boolean algebra corresponds to the classical logic with sentences a,b,x, with negation — and disjunction $a \vee x$, the latter being true if at least one of a,x is true.

(c) If (1.15) is a relation in Brouwerian algebra with elements a,b,x, then the expression - as it stands - is a true Brouwerian equivalence.

But the Brouwer algebra corresponds to intuitionistic logic with sentences a,b,x and junctors similar to the classical ones except for the meaning of the disjunction (alternative), where intuitionism requires that it be known which sentence of $a \vee x$ is true. This is indicated in Brouwer algebra by the dot over the sign of difference (the pseudo-difference).

Definition 1.5

Every relatively pseudo-complemented lattice has the unit element, but it does not, in general, have the zero element. Every such lattice with the zero element is called a pseudo-Boolean algebra $\mathcal{L} = \langle L, \cup, \cap, \sqsupset, \neg, \sqcup, \sqcap \rangle$.

An element a of a pseudo-Boolean algebra is said to be dense if $\neg a = \sqcup$ holds; it is dense iff $\neg\neg a = \sqcap$. It is called regular if $a = \neg\neg a$.

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The pseudo-Boolean algebra plays some role in fuzzy set literature; it is then called L-fuzzy algebra (L-fuzzy lattice with unity and zero).

The dual to the pseudo-Boolean lattice is the Brouwerian lattice.

Definition 1.6

Brouwer lattice is a notion placed between the lattice and the Boolean ring; it is characterized by the pseudo-difference and by the possession of the unit element.

The Brouwer algebra is thus denoted by

$$\mathcal{L} = \langle L, \cup, \cap, \dot{-}, \ulcorner, \sqcup, \sqcap \rangle.$$

Definition 1.7 (C. Rauszer (1977)

An abstract algebra $\mathcal{L} = \langle L, \cup, \cap, \sqsupset, \dot{-} \rangle$ is called semi-Boolean algebra if it is a relatively pseudo-complemented lattice strengthened by a pseudo-difference.

The algebra $\mathcal{L} = \langle L, \cup . \cap, \sqsupset, \dot{-} . \neg, \ulcorner, \sqcup, \sqcap \rangle$ is occasionally called Heyting-Brouwer algebra, where

(i) $\langle L, \cup, \cap, \sqsupset, \neg, \sqcup \rangle$ is a Heyting lattice with

$$\neg a = a \sqsupset \sqcup, \text{ and}$$

(ii) $\langle L, \cup, \cap, \dot{-}, \ulcorner, \sqcap \rangle$ is a Brouwer lattice with

$$\ulcorner a = \sqcap \dot{-} a$$ (McKinsey & Tarski (1946)).

To every pseudo-Boolean algebra $\mathcal{L}$ there exists a complete semi-Boolean algebra $\mathcal{L}'$ and a monomorphism $g: L \longrightarrow L'$. Also to every Brouwerian algebra $\mathcal{L}$ there exists a complete semi-Boolean algebra $\mathcal{L}'$ and a monomorphism $h: L \longrightarrow L'$. Thus, semi-Boolean algebras play the same role in the Heyting-Brouwer logic as the Boolean algebras in classical logic and the Brouwerian algebras (lattices) in intuitionistic logic.

The following sentences are valid in any Brouwer lattice:

$$(a \leqq b) \Longrightarrow (a \dot{-} c \leqq b \dot{-} c) \wedge (c \dot{-} b \leqq c \dot{-} a); \; a,b,c \in L,$$
$$(a \leqq b) \equiv (a \dot{-} b = \sqcup),$$
$$c \dot{-} (a \wedge b) = (c \dot{-} a) \vee (c \dot{-} b),$$
$$(a \vee b) \dot{-} c = (a \dot{-} c) \vee (b \dot{-} c),$$
$$\dot{-}(\dot{-} a) \leqq a, \quad \dot{-} a = \sqcap \dot{-} a,$$
$$\dot{-} \dot{-} \dot{-} a = \dot{-} a.$$

These formulas are easily converted into set relations if one uses the equality $A \dot{-} B \equiv \overline{A - B}$ (proposed by Kuratowski and Mostowski (1978) p.58). The right hand side represents closure of the set-difference beneath the bar.

1.1 The splitting of a Boolean algebra of simple systems in two incompatible Boolean algebras

It is important to observe that, in distinction from ordinary complementation, we have for $\dot{-}a$ the relation

$$(\dot{-} a) \wedge a \neq \sqcup$$

which corresponds in intuitionistic logic to the failure of tertium non datur; note also the interchange of the logical operations $\vee$ and $\wedge$, due to the duality of pseudo-Boolean and Brouwerian lattices.

We now transcribe this expression into one for sets and obtain

$$\overline{Y-A} \cap \overline{A} \neq \emptyset, \quad A \subset Y. \qquad (1.16)$$

If we take this as corresponding to the intuitionistically invalid tertium non datur, then

$$\overline{Y-A} \cup \overline{A} = Y, \quad A \subset Y, \qquad (1.17)$$

corresponds to the intuitionistically valid law of contradiction.

Let us note that in classical set theory

$$Fr(A) = \overline{Y-A} \cap \overline{A} \qquad (1.18)$$

stands for the boundary of set A and

$$\mathrm{Int}(A) = Y - \overline{Y-A} \tag{1.19}$$

for the interior of set A.

Now, Fr(A) is the set of all points of discontinuity of the characteristic function of any set A in Y, denoted by $D(\chi_A)$. Hence,

$$\mathrm{Fr}(A) = D(\chi_A), \quad A \subset Y. \tag{1.20}$$

This set vanishes obviously for fuzzy sets; thus

$$D(\chi_A) = \bigcup_B (\overline{\omega^{-1}(B)} - \omega^{-1}(\overline{B})) = \emptyset, \quad B \subset I, \tag{1.21}$$

see Kuratowski (1977) I, p.103.

Since we shall also deal with clopen sets, let us observe the following fact:

> The characteristic function of a set A is, respectively, continuous or pointwise discontinuous iff A is clopen (i.e. $D(\chi_A) = \mathrm{Fr}(A) = \emptyset$) or, respectively, has a nowhere dense boundary.

We conclude from (1.16) that $\overline{Y-A}$ and $\overline{A}$ are disjoint sets if the set between them is a virtual one, a gap. Therefore, the original Boolean algebra (for simple systems)

$$L_o = \langle Y, \cup, \cap, -, \sqcup, \sqcap \rangle \tag{1.22}$$

now appears split into two Boolean subalgebras:

$$L_1 = \langle \overline{Y-A}, \cup, \cap, -, \sqcup, \sqcap \rangle \text{ and } L_2 = \langle \overline{A}, \cup, \cap, -, \sqcup, \sqcap \rangle \tag{1.23}$$

with a gap between them.

L_1, L_2 and the gap constitute precisely the Brouwer lattice $\mathcal{L}$ referred to earlier:

$$\mathcal{L} = \langle L, \cup, \cap, \dot{-} \rangle, \quad L = 2^Y, \tag{1.24}$$

where 2^Y represents the hyperspace of all closed subsets of the topological space Y. $\emptyset$ is an isolated point of the hyperspace 2^Y equipped with an exponential or Vietoris topology.

We are now in a position to handle mathematically complex fuzzy systems requiring an underlying ∞-dimensional Banach space. The choice of a suitable Banach space ensures the existence of a so-called negligible (virtual) set, the gap, accounting for the quantal threshold in the transition from a simple system to a complex fuzzy one.

This is the subject of the next chapter.

CHAPTER 2: THE FIXED POINT THEORETICAL ORIGIN AND DEFINITION OF A NEGLIGIBLE SET

We consider here a space Y of complex fuzzy subsets, say $A \subset Y$, and try to grasp the deeper meaning and practical importance of the formula $\overline{Y-A} \cap \overline{A} \neq \emptyset$ (1.16)

in terms of the fixed point theory, which is well known to physicists and engineers. This relation contains an uncertainty with respect to the set A; hence, our aim will be: to determine the threshold of uncertainty under very general conditions. We begin our investigation with the

2.1 Banach contraction principle

This principle stated for ordinary metric spaces assumes two metric spaces (X,d) and (Y,ρ) and the mapping $F: (X,d) \longrightarrow (Y,\rho)$ satisfying the condition

$$\rho(F(x),F(z)) < Md(x,z); \quad x,z \in X; \; M = \text{const.} \quad (2.1)$$

F is called a Lipschitzian mapping; it need not be continuous. The least M fulfilling (2.1) is called the Lipschitz constant which is usually denoted by L(F):

(i) mapping F is called contractive if $L(F) < 1$,

(ii) mapping F is said to be non-expansive if $L(F) = 1$.

Let Y be any set and $F: Y \longrightarrow Y$. For a given $y \in Y$, $F^n(y)$ is defined inductively by $F^0(y)$ and $F^{n+1} = F(F^n(y))$, where $\{F^n(y): n=0,1,2,\dots\} \subset Y$ is the orbit of y under F. This procedure permits solutions by computer iterations.

Banach contraction principle

Let (Y,d) be a complete metric space and $F: Y \longrightarrow Y$ contractive; then F has a unique fixed point u and $F(y) \longrightarrow u$ for each $y \in Y$.

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The completeness of Y is an indispensible property in this case. The proof of this principle is straight forward; see for example Smart (1974). From this results a useful local version involving an open ball B in Y (which has to be complete and metric) and a contractive mapping $F: B \longrightarrow Y$ (which does not displace the centre y_o of the ball too far):

Assumptions: $B = B(y_o,r) = \{y: d(y,y_o) < r\}$, (Y,d) - compl.

$F: B \longrightarrow Y$ contractive (with constant $\alpha < 1$) and $d(F(y_o),y_o) < (1-\alpha)r$.

Conclusion: F has a fixed point.

Commonly, Y will be a Banach space with a richer structure. In that case the Banach contraction theorem leads to useful applicational results. Let Y denote a Banach space, X a subset of Y, $F: X \longrightarrow Y$ and $f: X \longrightarrow Y$ a mapping $x \mapsto x - F(x)$ of X into Y, called the field f associated with F.
$f: X \longrightarrow Y$ is a contraction field if F is contractive.

Invariance of domain for contractive fields

Let Y be a Banach space, $U \subset Y$ open and $F: U \longrightarrow Y$ contractive ($L(F) < 1$). Let $f: U \longrightarrow Y$ be the associated field, $f(u) = u - F(u)$. Then,

(a) $f: U \longrightarrow Y$ is an open mapping, i.e. $f(U)$ is open in Y, (2.2)

(b) $f: U \longrightarrow f(U)$ is a homeomorphism.

We conclude next that $f = I - F$ is a homeomorphism of Y onto Y, for $F: Y \longrightarrow Y$ contractive and I an identity map.

2.2 Negligible sets

A subset H of space Y is said to be negligible whenever Y-H is homeomorphic to Y; the mapping $h: Y-H \longrightarrow Y$ is called a deleting homeomorphism and as such written $h: Y-H \cong Y$.

Definition 2.1 (Anderson (1969))

We say that a closed set $\overline{K}$ of a Fréchet space X has an ∞-defect (∞- codimension) if $X-\overline{K}$ is ∞-dimensional, where $\overline{K}$ is the closure of a linear subspace spanned by the elements of K. Note that a normed Fréchet space is a B-space.

Theorem 2.1

Let Y be a non-complete normed linear space and H a complete subset of Y. Then, there exists a homeomorphism $h: Y-H \cong Y$ with $h(y) = y$ whenever $d(y,H) \geqq 1$.

Proof:

We have to prove the existence of the homeomorphism h. The completion $\hat{Y}$ of Y, taken with the natural extension of the given norm of Y, will be a Banach space. Let $\{y_n\}$ be a Cauchy sequence in Y converging to some point in $\hat{Y}-Y$ such that $\|y_1\| + \sum_1^{\infty} \|y_n - y_{n+1}\| < \infty$. Now, no scalar multiple of the point in $\hat{Y}-Y$ can belong to Y; so, replacing all the y_n by a suitable scalar multiple, we assume that $\{y_n\}$ converges to $x_0 \in \hat{Y}-Y$ and $\|y_1\| + \sum_1^{\infty} \|y_n - y_{n+1}\| = \frac{1}{2}$.
Consider a broken line $L = [y_0, y_1] \cup [y_1, y_2] \cup \ldots \subset Y$ with $y_0 = 0$ and construct a piecewise linear map $\phi: I \longrightarrow L \cup \{x_0\}$,

$I = [0,1]$, in the following way: let s_n be the n^{th} partial sum of the series with sum $= \frac{1}{2}$ and $s_0 = 0$; for each $n \geqq 0$, map the interval $[2s_n, 2s_{n+1}]$ linearly onto the segment $[y_n, y_{n+1}]$ and put $\phi(1) = x_0$. Then $|\phi(t)-\phi(t')| = \frac{1}{2}|t-t'|$ with t,t' belonging to the common interval $[2s_n, 2s_{n+1}]$, so - by triangle inequality - we get $|\phi(t)-\phi(t')| \leqq 0.5|t-t'|$, for all $t,t' \in I$. Now extend ϕ to $\phi: (-\infty, 1] \longrightarrow Y$ by $\phi(t) = 0$ for $t < 0$.

The mapping $G: \hat{Y} \longrightarrow \hat{Y}$, sending y into $\phi(1 - d(y,H))$, is obviously contractive since

$$\|\phi(1 - d(y,H)) - \phi(1 - d(z,H))\| \leqq \frac{1}{2} \|d(y,H) - d(z,H)\| \leqq \frac{1}{2}\|y-z\|.$$

Hence, by the invariance of domain for contractive fields (2.2), $h(y) = y - G(y)$ is a homeomorphic map of $\hat{Y}$ onto $\hat{Y}$. For $y \in Y-H$ we have $d(y,H) > 0$, so $G(y) \in Y$ and $h(y) = y - G(y)$ belongs also to Y. Therefore, $h(Y-H) \subset Y$. For the converse inclusion we put $y \notin Y-H$, so that $y \in (\hat{Y}-Y) \cup H$. For $x \in \hat{Y}-Y$, $d(y,H) > 0$, so $G(y) \in Y$, while with $y \in H$, $y \in Y$ and $G(y) = x_0 \notin Y$. In both cases only one of the points: y or $G(y)$ belongs to Y, which forces the conclusion that $h(y) = y - G(y) \notin Y$, and $h(Y-H) \cong Y$. The proof is accomplished.

Dugundji and Granas (1982) assert that the class of linear spaces, for which such deleting homeomorphisms exist, is very large, because:

Every ∞-dimensional Banach space $(Y, \|\cdot\|)$ admits a non-complete norm $|\cdot|$ with $|y| \leqq \|y\|$ for all $y \in Y$, (2.3)
and any ∞-dimensional non-reflexive Banach space is a particularly interesting case in point. (2.4)

Combining Theorem 2.1 with expression (2.3) yields a theorem obtained by Klee (1956).

<u>Theorem 2.2</u> (Klee)

To an arbitrary ∞-dimensional normed linear space Y and $H \subset Y$ compact, there exists a homeomorphism $h: Y-H \cong Y$.

<u>Proof</u>: (following Klee)

Let Y be a Banach space, else the result follows from Theorem 2.1. By (2.3), there are an incomplete norm $|y| \leqq \|y\|$, space $(Y, |\cdot|) = \hat{Y}$, a continuous mapping $j: Y \longrightarrow \hat{Y}$ and hence also a compact set $\hat{H} = jH$, which is, therefore, complete in $\hat{Y}$. On the other hand, Theorem 2.1 implies the existence of a deleting homeomorphism $\hat{h}: \hat{Y}-\hat{H} \cong \hat{Y}$, defined by

$$\hat{h}(\hat{y}) = \hat{y} - \hat{\phi}(\ 1 - \hat{d}(\hat{y},\hat{H})), \quad \hat{y} \in \hat{Y}, \tag{2.5}$$

where $\hat{d}$ is the norm-induced metric and ϕ is a piecewise linear mapping

$$\hat{\phi}\colon (-\infty\ , 1] \longrightarrow \hat{Y}. \tag{2.6}$$

$\hat{\phi}$, being piecewise linear, is continuous with any linear topology in $\hat{Y}$; therefore, regarding $\hat{\phi}$ as a mapping

$$\phi\colon (-\infty\ , 1] \longrightarrow Y, \tag{2.7}$$

we obtain ϕ continuous and $\hat{\phi} = j\phi$. (2.8)

Now $h\colon Y\text{-}H \longrightarrow Y$ is given by

$$h(y) = y - \phi(1 - \hat{d}(j(y),\hat{H})); \tag{2.9}$$

it is continuous and establishes the commutation of the diagram below.

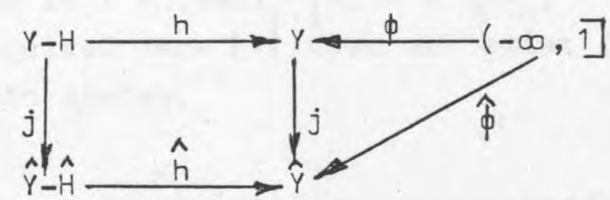

Fig. 2-1

Commuting diagram

Note that $\quad jh(y) = \hat{h}j(y)$, i.e. $h \circ j = j \circ \hat{h}$. (2.10)

h is a homeomorphism if the inversion $g\colon Y \longrightarrow Y\text{-}H$ is also continuous; but

$$g(y) = y + \phi(1 - \hat{d}(\hat{h}^{-1}j(y),\hat{H})) \tag{2.11}$$

is seen to define a continuous function g. This is confirmed by direct convolution $h \circ g$:

$$g(h(y)) = h(y) + \phi(1 - \hat{d}(\hat{h}^{-1}jh(y),\hat{H}))$$

– applying (2.9) and (2.10), we get –

$$= \{y - \phi(1 - \hat{d}(j(y),\hat{H}))\} + \phi(1 - d(\hat{h}^{-1}\hat{h}j(y),\hat{H}))$$

$$= y.$$

The inverse convolution $g \circ h$ yields:

$$h(g(y)) = h\{y + \phi(1 - \hat{d}(\hat{h}^{-1}j(y),\hat{H}))\}$$

$$= h(y) + \phi(1 - \hat{d}(h^{-1}j(h(y),\hat{H}))$$

– applying again (2.9) and (2.10), we get

$$= y - \phi(1 - d(j(y),\hat{H})) + \phi(1 - \hat{d}(j(y),\hat{H}))$$

$$= y.$$

Thus, $h \circ g = g \circ h$; and since h and g are both continuous, $h\colon Y\text{-}H \longrightarrow Y$ is a homeomorphism, i.e. $h\colon Y\text{-}H \cong Y$.

Bessaga (1966), who actually coined the name "negligible set", provided two simple criteria, expressing the content of Theorems 2.1 and 2.2, using the concepts of "narrow" and "strongly sigma narrow" sets.

Definitions 2.2

(i) A set in a linear topological space X is called narrow if there is an incomplete continuous norm $|\cdot|$ on X such that this set is closed in Banach space $Y = \text{compl}_{|\cdot|} X$, the completion of X with respect to the norm $|\cdot|$.

(ii) A subspace of a Banach space X is called strongly σ-narrow (sσ-narrow) if there is a continuous norm $|\cdot|$ on Y such that the original unit cell u of X is incomplete and, for A_n closed in $Y = \text{compl}_{|\cdot|} X$, $A = \bigcup_n A_n$, n = 1,2,...,holds. Note: The required norm $|\cdot|$ does not exist in separable reflexive Banach spaces.

Criterium 1

Every narrow set in an arbitrary topological linear space is negligible.

Criterium 2

Every sσ-narrow set in a Banach space is negligible.

Conclusions

Complex fuzzy systems require an ∞-dimensional function space Y, preferrably an ∞-dimensional Banach space which comprises a negligible set which, in turn, is capable of accounting for the fundamental uncertainty due to the failure of the law of the excluded middle at the border to infinity (see Tarski (1935). At this point the logic of complex fuzzy systems turns from Boolean to Brouwerian (it is intuitionistic).

The fuzziness in Y manifests itself in the uncertainty of the boundary of the complex fuzzy sets in space Y.

The negligible set in the Banach space Y can be studied in terms of the fixed point theory, which reveals that any homeomorphism mapping Y into itself deletes a negligible set from its domain. Thus, if we consider a negligible set as nested in the boundary of each complex fuzzy set, but actually deleted, then a "gap" will separate that set from its "complement" (i.e. cancel the intersection of both); and since non-Boolean logic rules the complex fuzzy system, the original Boolean algebra L_o for simple systems will now give way to a Brouwerian lattice L with unity, comprising the two Boolean subalgebras L_1 and L_2 as well as the gap between them.

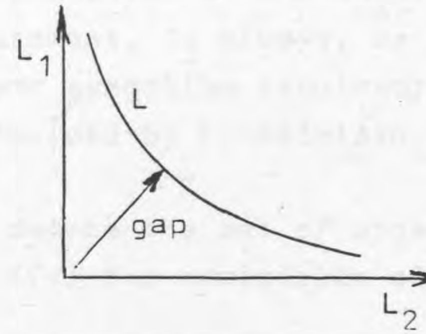

Fig. 2-2
Graph of the algebras L_1, L_2 and the gap.

We see now that the topologically negligible set, the algebraic gap (with respect to the Boolean algebra) and the inherent measuremental uncertainty threshold describe the same fundamental deficiency of complex fuzzy systems in three different disciplines. This enables us to define (in Chapter 3) a fundamental noncomplementation condition and a corresponding uncertainty relation for measurement. For further intelligibility of this study, a brief résumé of certain facts of Banach space theory may prove helpful.

Definition 2.3
A normed complete space is called a Banach space B; a unitary complete space is said to be a Hilbert space H; every H is a B, and every normed Fréchet space is a B.

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If L is a linear continuous functional in Hilbert space H, then the set $\mathfrak{N}$ of vectors $u \in H$, on which L takes the zero value, $\mathfrak{N} = \{u \in H: L(u) = 0\}$, is a hyperspace (defect or codimension $\mathfrak{N} = 1$).
The quotient space H/H_1 w.r.to subspace H_1 is the set of classes $[y]$ of vectors from H, where to one class belong vectores u_1, u_2 whose difference lies in H_1; thus, H/H_1 is a set of planes parallel to H_1.
Generally: $H = H^* = H^{**}$, but $B \neq B^* \neq B^{**}$.

Banach spaces containing no negligible set:
Separable spaces
Polish spaces (i.e. separable and complete spaces)
Separable reflexive spaces
Banach spaces exhibiting negligible sets:
The spaces discussed in Theorems 2.1, 2.2 and Criteria 1,2.

CHAPTER 3: DERIVATION OF THE UNCERTAINTY RELATION

With the results of Chapters 1 and 2 at hand, our considerations now converge to the formulation of the uncertainty relation for generalized measurement. As always, we shall clarify first the terminology and symbolism required; an appropriate formalism has been provided by Finkelstein (1975b). Let

$\underline{Y}$ $\{y_i: K(y),\ i = 1,2,\dots,n\}$ denote the set of objects (events, attributes) y_i and $K(y)$ the admissible class, and let

$\underline{S} = \{s_i: s_i \in S,\ i = 1,2,\dots,m\}$ stand for the set of symbols s_i bearing a defined relationship to the entities $y_i \in Y$ by means of the mapping:

$\underline{M} \subset Y \times S$ such that $dom(M) = Y$ and $Ran(M) = S$, i.e. $(y_i, s_i) \in M,\ y_i \in Y,\ s_i \in S$.

Calling $C = \langle Y,S,M \rangle$ the assumed symbolism, respectively code of symbolization, $J = \langle Y,S,M,s_i \rangle$ will constitute the information about y_i, and y_i - the meaning of s_i under C. Since M may be a one-one, many-one, one-many or many-many function, we have to distinguish between:

(i) synonyms, i.e. symbols $s_i, s_k \in S$ such that $(y_i,s_i),(y_i,s_k) \in M$ for any $y_i \in M$ and

(ii) homonyms, i.e. symbols $s_i \in S$ such that $(y_i,s_i),(y_k,s_i) \in M$ for any $y_i, y_k \in Y,\ y_i \neq y_k$.

Definitions 3.1

(1) The correspondence of more symbols than one to the same meaning is called redundancy (case (i) above).

(2) The correspondence of more meanings than one to the same symbol is said to be an ambiguity (case (ii) above).

(3) The absence of ambiguity is termed precision, while

(4) the absence of redundancy is termed significance or relevance.

----- o -----

Precision and significance or relevance thus defined represent ideal forms of these attributes. In the ideal form they are complementary properties of measurement of the state of closed systems for monitoring and control. But in the majority of practical systems, especially in complex or fuzzy systems, the complementarity condition will not be fulfilled.

Let us aliken Y with an ∞-dimensional non-reflexive Banach space (underlying the complexified empirical system under consideration), and let us assimilate the set S with the set R of reals. Once the system is complexified to the extent that the principle of incompatibility due to Zadeh applies, we'll be facing - conform with (1.22) - (1.24) -

a Brouwerian lattice $\mathcal{L} = \langle L, \cup, \cap, \dot{-} \rangle$; $L = 2^Y$, (3.1)

comprising two Boolean subalgebras:

$$L_1 = \langle Y-A, \cup, \cap, - \rangle \text{ and } L_2 = \langle A, \cup, \cap, - \rangle \qquad (3.2)$$

of the Boolean algebra $L_o = \langle Y, \cup, \cap, - \rangle$ (3.3)

and a gap involving the uncertainty due to the existence of L_1 and L_2 and defined by a negligible set H (see Fig. 2-2).

L_1, L_2 and L_o have, of course, zero and unity not shown here.

<u>Definition 3.2</u> (Non-complementation condition)
Let Y be an ∞-dimensional non-reflexive Banach space containing a negligible set H; let A be any of its subsets contaminated by complexity and fuzziness in the sense of Zadeh's incompatibility principle. Since H fills the gap absorbing the uncertainty due to complexity and fuzziness, we obtain as non-complementation condition:

$$\overline{Y-A} \cap \overline{A} \supseteq H; \quad A, H \subset Y. \qquad (3.4)$$

All the sets involved are closed sets.

----- o -----

Formula (3.4) represents a fundamental relation of non-complementation in the measurement of complex fuzzy systems, expressing the fact that for set $\overline{A}$ without regard to set $\overline{Y-A}$ as well as for set $\overline{Y-A}$ without regard to set $\overline{A}$ there exist Boolean conditions, while for both sets simultaneously these conditions are Brouwerian i.e. the sets are incompatible for measurement.
Condition (3.4) is reminiscent of the non-commutativity condition in quantum mechanics.

<u>Theorem 3.1</u>
The non-complementation condition (3.4) is a fundamental relation existing in spaces underlying complex fuzzy systems which are subject to Zadeh's principle of incompatibility

which restricts the simultaneous measurement (observation, determination) of sets, quotient sets (attributes) relations, operations in those spaces. The determination of the state of complex fuzzy systems is thus impossible with absolute precision.

----- o -----

We shall now derive the uncertainty relation associated with the non-complementation condition.

Let S be any compact Hausdorff space with norm $\|\cdot\|$ defined by

$$\|y\| = \sup\{|y(s)| : s \in S\},$$

where $Y = C(S)$ is the Banach space of continuous functions $y: S \longrightarrow R$, i.e. $y = (y(s): s \in S)$, defined in S and having scalar values.

<u>Theorem 3.2</u> (Banach, Saks, Kakutani)

For an arbitrary continuous linear functional y^* in $C(S)$, there exists exactly one Radon measure $\mu = \mu_1 + \mu_2$ ($\mu_{1/2}$ being Borelian), defined in the algebra B of Borel subsets of the set S such that, for arbitrary $y \in C(S)$,

$$y^*(y) = \int_S y(s)\mu(ds), \qquad (3.5)$$

$$\text{and } \|y^*\| = V(\mu, S). \qquad (3.6)$$

$V(\mu,S)$ is the upper bound of the sums $\sum_{i=1}^{n} |\mu(S_i)|$, and (S_i) the decomposition of S.

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For a proof see Alexiewicz (1969).

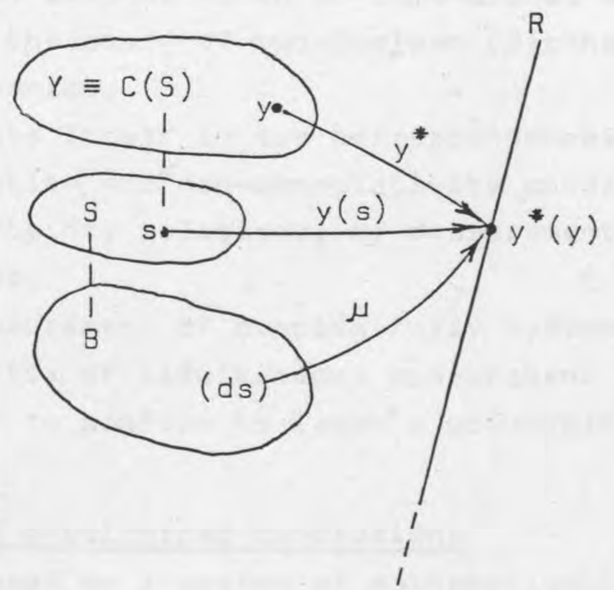

Fig. 3-1
Pictorial display of relations involved in Theorem 3.2

In complex fuzzy dynamical systems S would stand for the time continuum T, Y would be a topological space whose closed, hence Borelian, subsets constitute a topological hyperspace 2^Y with Vietoris (exponential) topology. In that case, Borel measures μ_1 and μ_2 exist, so that the Radon pseudomeasure $\overline{\mu} = \mu_1 \dot{-} \mu_2 = \overline{\mu_1 - \mu_2}$ is also defined on 2^Y. Please note that we treat set functions as sets and speak of closed measures, denoted by $\overline{\mu}$, - see Alexiewicz (1969) p.37.
Using (3.5), (3.6) and the substitution: $V(\overline{\mu},H) = \overline{\mu}(H)$, (3.7)
we obtain the desired uncertainty relation for measurement:

$$\Delta y^*(y)_{\overline{Y-A}} \cdot \Delta y^*(y)_{\overline{A}} \geq \overline{\mu}(H), \quad \text{for } A,H \subset Y. \tag{3.8}$$

Definition 3.3

Formula (3.8) defines the uncertainty relation for the measurement of complex fuzzy systems conforming to the non-complementation condition (3.4). It resembles the Heisenberg uncertainty relation for quantum mechanical systems. Conform with Zadeh's principle of incompatibility, we get:

$$\left.\begin{array}{l} \text{for the measure of precision: } (\Delta y^*(y)_{\overline{A}})^{-1}, \\ \text{for the measure of relevance: } (\Delta y^*(y)_{\overline{Y-A}})^{-1}. \end{array}\right\} \tag{3.8'}$$

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Sectional results:

An ∞-dimensional non-reflexive Banach space is known to admit an incomplete norm and to comprise a negligible set. It is eminently suitable for the study of complex fuzzy empirical systems obeying non-Boolean (in fact: intuitionistic) logic in a capacity similar to an ∞-dimensional separable Hilbert space in the study of non-Boolean (Birkhoff-v.Neumann) quantum mechanics.
This similarity manifests itself in two correspondences: that of non-complementation and non-commutativity conditions and that of uncertainty relations, of measurement and of quantum mechanics.
Our analysis of the measurement of complex fuzzy systems, in which the impossibility of simultaneous measurement comes to light, is seen to conform to Zadeh's principle of incompatibility.

3.1 Epistemological and ontological conclusions

Classical physics is based on a system of mathematically sharp-formulated axioms whose physical content is uniquely predicted by the choice of words used in these axioms, even though these words belong to the colloquial language. Nevertheless, the truth-claim of classical physics seems to be unquestionable, and the statements of classical physics and - for that matter - also of engineering are precise as well as determinative. This truth-claim of classical physics is not even questioned by modern physics. The necessity and possibility of a revision of the classical laws occurs only with respect to the applicability of these laws to physical experience. Hence, it is not really the validity, but rather the applicability of classical laws that is being limited by modern physics and engineering.

The causality principle remains, of course, untouched. Indeed, causality in classical physics means nothing else but the existence of a unique functional relationship between the states of a system at different times: If the state of a closed system at a given instant of time is completely known, then the state of that system at every earlier or later time can be computed. This conditional theorem is not wrong in complex fuzzy cybernetics, nor in quantum mechanics, but it is inapplicable; for, the premise is never realized: The state of a system cannot be known completely in the classical sense, because any gain of knowledge always excludes the complementary knowledge. Thus, the conditional theorem belongs to the classical conceiving of the world and not to the practical one, in which a state can never be known completely. This is tantamount to saying that we cannot hope to ever know the ultimate truth, which the mathematician Kurt Goedel put in the following theorem:

> "No consistent system is powerful enough to prove its own consistency. Strangely enough, every inconsistent system may be used to prove its consistency".

It appears that all such statements hinge on the fallacies of the complementarity principle (quotation of A.N. Whitehead).

Let us now consider the objectivity of statements in the light of the above arguments. Every observation postulates a causal chain and yields a "viewable" result. However, we are not allowed to do one thing: to assemble the viewable fragments and causal chains into a model of a nature existing per se. It depends rather on our freely chosen experimental set-up, which of the complementary sides of nature we will be facing; and the knowledge of one side excludes that of the other side.

It is a fundamental postulate of complex fuzzy cybernetics (and - for that matter - also of quantum mechanics) to deny the existence of hidden parameters; actually, not when the lack of knowledge is only due to a dispensing with a fuzzy cybernetically possible acquisition of information, but when the unknown quantity - for reasons of the exact knowledge of the complementary quantity - cannot be known. This is no vain statement, but a theorem with certain logical consequences, namely: The knowledge which we have about nature or about technical systems enters explicitely the statements relating to complex fuzzy cybernetics.

A complete experimental statement, i.e. the result of measurement on complex fuzzy systems, would be:

> "On the tested object, under these conditions,
> I have observed this state".

The hypothesis of classical physics states that this sentence may always be replaced by the following one:

> "On this object exists this state",

which must necessarily be either true or false, no matter whether there exists a person knowing the truth or falsity of this sentence. This hypothesis originates really from a scientific and philosophical belief in an objective existence of the objects of our cognition. Both quantum mechanics and complex fuzzy cybernetics reject already the assumption of such reasonings. They use a many-valued truth concept. Thus, let A be a statement referring to a certain situation; then

(a) the full experimental statement would run: "I know that A is true";

(b) classical physics simply states: "A is true".

(a) admits two negations: (a′) "I know that A is not true" - objective negation,

(a″) "I know not whether A is

true" - negation of knowledge.

(a), (a′) and (a″) are equally ranked sentences.

(b) admits corresponding negations: (b′) "A is not true" - objective sentence,

(b″) "Neither applies A, nor applies A not" - objective sentence.

The ontological meaning of this is that the notion of the object may no longer be used without reference to the subject of cognition in the modern sciences. By the "reference to the subject" are meant his basic functions of awareness: his knowledge (experience) and his will (preference in choices).

Subjective measurement and control has by now become common place occurrence in automatic control practice and better results are obtained with complex fuzzy systems using the experimenter′s knowledge and will. The interested readers are referred to publications of IFAC (International Federation of Automatic Control).

CHAPTER 4: QUANTUM MECHANICAL UNCERTAINTY ANALOGY

The complexity of quantum systems is absorbed in an ∞-dimentional separable Hilbert space and the underlying Birkhoff-von Neumann logic in which the distrbutive law is discarded. Hence the logic is also non-Boolean.

A reasonably adequate account of the quantum theory is appended at the end for reference with regard to symbolism, meanings and results. What we are going to investigate in this chapter is the uncertainty analogy to the uncertainty of measurement in complex fuzzy systems.

Since the scalar product characterizes any ∞-dimensional Hilbert space, by a projection in Hilbert space H we mean an orthogonal projection, i.e. a selfadjoint idempotent operator P with the property $P^2 = P^1 = P$. According to Maczynski (1981), two projections P and Q commute in H, i.e. $PQ = QP$, iff the following inequality is satisfied for all vectors $u \in H$:

$$\|Pu\|^2 + \|Qu\|^2 \leqq \|u\|^2 + \lim_{n \to \infty} \|(PQ)^n u\|^2 \tag{4.1}$$

or, more generally, iff

$$\sup_{\|u\|=1} \lim_{n \to \infty} (\|Pu\|^2 + \|Qu\|^2 - \|(PQ)^n u\|^2) = 1. \tag{4.2}$$

They do not commute iff

$$1 < \sup_{\|u\|=1} \lim_{n \to \infty} (\|Pu\|^2 + \|Qu\|^2 - \|(PQ)^n u\|^2) \leqq 2. \tag{4.3}$$

To each pair P,Q of non-vanishing projections may be allocated a number $\delta(P,Q) \in [0,1]$, called commutation gap, which equals 0 in case of commutation and any other value up to 1 in other cases. For 1-dimensional projections $\delta(PQ)$ coincides with the square root of the transition probability $|\phi\psi|^2$ of the transition between the quantum states (unit vectors) ϕ and ψ in H, i.e.

$$\delta(P_\phi, P_\psi) = |\phi, \psi|; \qquad P_\phi \neq P_\psi. \tag{4.4}$$

Thus, $\delta(P_\phi, P_\psi) = 0$ iff P_ϕ and P_ψ commute, or if $P_\phi \perp P_\psi$. The definition of the commutation gap is, of course, applicable to any selfadjoint operator in Hilbert space. For more details consult the paper of Maczynski (1981).

There is an easy to notice analogy between the commutation gap in H and the complex fuzzy gap in an appropriate Banach space, and both generate their uncertainty relations.

$$\left.\begin{aligned}\Delta p \cdot \Delta q &\geqq h = 2\pi\hbar \qquad (a)\\ \Delta E \cdot \Delta t &\geqq h \qquad (b)\end{aligned}\right\} \qquad (4.5)$$

represent the most common form of the Heisenberg uncertainty relations in terms of momentum p, position q, energy E and time t, h being the universal Planck constant. The equivalence of (a) and (b) in (4.5) is a consequence of the relativistic standpoint that energy E and momentum p are quantities of the same kind; indeed,

$p_x = \frac{h\delta}{i\delta x}$ is the spacial component of the relativistic 4-vector,

$E = \frac{h\delta}{i\delta t}$ is the time component.

But then q and t must also be quantities of the same kind. This follows independently from the canonical equations of Hamilton if p is kept constant (a case known in physics as the cyclic coordinate).

The relation (4.5(b)) is, however, not deducible from quantum mechanical principles since a selfadjoint time operator does not exist. If a selfadjoint quadratic time operator describing the quantity $\Delta^2 t$ be introduced, then the following quadratic uncertainty relation is obtained:

$$\Delta^2 E \cdot \Delta^2 t \sim h^2. \qquad (4.6)$$

There are other derivations of the Heisenberg uncertainty relation, but the most notable of them seems to be the probabilistic one, for the simple and veritable reason that the probabilistic model of quantum mechanics has both intuitional and computational appeal.

To this end, let us consider measurement as an act of putting "questions" to nature or to a man-made system. For obvious reasons, Heisenberg calls any "question" ill-posed (or meaningless) if it is put to a non-classical system or to a non-classical theory. Examples of such "questions" are those referring to the simultaneity of events in relativistic physics, to the canonical operators of position and momentum in quantum mechanics and to the precision and significance of measurement on complex fuzzy systems.

These questions become "virtual questions" once the appropriate uncertainty relation is interposed between the non-classical system and the measuring apparatus (obeying the laws of classical mechanics). The virtual answers across the uncertainty relation appear to be classically correct, and hence "understandable" to the measuring apparatus. Whenever the uncertainty relation vanishes (for the velocity of light $c = \infty$, for Planck's constant $h = 0$, or for the negligible set $H = \emptyset$), the non-classical system (theory) actually becomes a classical one. Precisely this is the intension and the content of the so-called correspondence principle in physics (originally advocated and demanded by Niels Bohr).

Definition 4.1

Following Mackey (1963), we call an observable α a question if, in every state ϕ, the measure ϕ_α is concentrated at the points 0 and 1, i.e. if $\phi_\alpha(\{0,1\}) = 1$ for all ϕ. $\mathbb{Q}$ shall denote the set of all questions.

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α is, of course, a question if $\alpha^2 = \alpha$.

In order to concretize the meaning, let $E \in B(R)$ be any Borel set on the real line and $\pi_E(x) = 1$ if $x \in E$ and zero otherwise. Then $\pi_E(\alpha)$ is obviously a question, because it is an observable which yields the value 1 when a measurement of α yields a value in E and the value 0 in the complementary case. In this sense it corresponds to asking the (yes - no) question: Did the measurement of α lead to a value in E? This particular question shall be denoted by Q_E^α. With α fixed, Q_E^α represents a family of questions parametrized by E; and this family determines α uniquely.

Now, let Q be any question, ϕ any state and $\phi_Q(\{1\}) = s$; then $\phi_Q(\{0\}) = 1-s$, and for any set $E \in B(R)$, $\phi_Q(E) = 0$, 1, s or 1-s according as neither 0 nor 1 is in E, both 0 and 1 are in E, 1 is in E but 0 is not, 0 is in E but 1 is not. Hence, ϕ_Q is completely determined by $s = \phi_Q(\{1\})$.

Defining $m_\phi(Q)$ as $\phi_Q(\{1\})$, m_ϕ will be a certain real function on the questions. It is these functions m_ϕ that define a natural partial order in Q; thus:

$$Q_1 \leqq Q_2 \equiv m_\phi(Q_1) \leqq m_\phi(Q_2) \quad \text{for all } \phi;$$

$$(Q_1 \leqq Q_2) \wedge (Q_2 \leqq Q_3) \Longrightarrow (Q_1 \leqq Q_3);$$

$$Q \leqq Q \quad \text{for every } Q;$$

$$(Q_1 \leqq Q_2) \wedge (Q_2 \leqq Q_1) \Longrightarrow (Q_1 = Q_2).$$

Valid is also the implication:

$$(Q \text{ is a question}) \Longrightarrow (1-Q \text{ is a question}).$$

Q and 1-Q, unlike pairs of questions in general, are both functions of the same observable, namely Q, and hence can be asked simultaneously. If $Q_1 \leqq 1-Q_2$ or, equivalently, if $m_\phi(Q_1) + m_\phi(Q_2) \leqq 1$ for all states ϕ, we shall say that Q_1 and Q_2 are disjoint , for which we put $Q_1 \curlywedge Q_2$ meaning physically that Q_1 and Q_2 cannot have simultaneously yes-answers.
Since $\mathbb{Q}$ has been found to be a poset, it is clear that $Q \longrightarrow 1-Q$ is an orthocomplementation in $\mathbb{Q}$. The orthocomplemented poset $\mathbb{Q}$ of all questions plays here the role played by the phase space in classical mechanics, which is considered to be the logic of the physical system.
Hence, the difference between quantum mechanics and classical mechanics is that there are non-simultaneously answerable questions in quantum mechanics, i.e. $\mathbb{Q}$ is not a Boolean algebra in quantum mechanics. The following postulate states positively what $\mathbb{Q}$ is.

<u>Fundamental postulate of quantum mechanics</u>
The partially ordered set of all questions, $\mathbb{Q}$, in quantum mechanics is isomorphic to the partially ordered set of all closed subspaces of a separable infinite-dimensional Hilbert space H.

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It represents a generally accepted plausibility axiom with a rather abstract content. The isomorphism may always be chosen such that if Q corresponds to the closed subspace M, then 1-Q corresponds to the orthogonal complement $M^\perp$. In doing this one may conveniently identify each closed subspace with the projection on that subspace.
Now, Gleason (1957) has proved that every measure on the questions arises from the state on the assumption that there exists a state ϕ such that $m_\phi(Q) = 1$ if Q is any question different from 0. It is easy to see that the states which define measures on the questions of the form m_ϕ, where ϕ is a unit vector in H, are pure states.
On the other hand, each observable defines and is defined by a question-valued measure, and all question-valued mea-

sures occur. But we have already identified the questions with the projections in H. Hence, the observables correspond one-one to the projection-valued measures in H, and these correspond one-one to selfadjoint operators.
Let A be any selfadjoint operator in H, ϕ a unit vector and E a Borel set. What is the probability that, in the pure state defined by ϕ, a measurement of the observable, defined by the operator A, will lead to a value in E?
The spectral theorem associates with A the projection-valued measure P^A. The projection associated with the question: Does the value of the observable lie in E? - is then P^A_E, and the probability is $(P^A_E\phi,\phi)$.
The general situation is then the following:

- The observables correspond one-one to the selfadjoint operators in a separable ∞-dimensional Hilbert space.
- The pure states correspond one-one to the 1-dimensional subspaces of the Hilbert space H.
- In order to find the probability distribution of the observable α defined by the selfadjoint operator A in the pure state defined by a 1-dimensional subspace, any unit vector ϕ in the 1-dimensional subspace is chosen. If we denote by P^A the projection-valued measure associated with A by the spectral theorem, then the desired probability distribution is $E \longrightarrow (P^A_E(\phi),\phi)$.
- Every state is a (possibly infinite) convex combination of pure states.

To see how the operators may play a direct role, let us compute the expected value of the observable defined by the operator A in a state defined by ϕ. This is given by

$$\int_{-\infty}^{\infty} x d\mu(x),$$

where μ is the relevant probability measure, i.e. $E \longrightarrow (P^A_E\phi,\phi)$. But, by the spectral theorem

$$\int_{-\infty}^{\infty} x d(P^A_x\phi,\phi) = (A\phi,\phi). \qquad (4.7)$$

Hence, the expected value in question is the scalar product $(A\phi,\phi)$.
When A and B are two non-commuting operators, then there are limitations in the degree to which the probability distributions of the corresponding observables may be

simultaneously concentrated close to single points. A quantitative measure of the degree of dispersion of an observable in a given state can be obtained by taking the square of the difference between the observable and its expected value. With A as the relevant operator and ϕ as the unit vector defining the state, the dispersion $\delta(A,\phi)$ is such that

$$\begin{aligned}\delta^2(A,\phi) &= [(A - (A(\phi),\phi))^2\phi,\phi] \\ &= [(A^2 - 2(A(\phi),\phi)A + (A\phi,\phi)^2)\phi,\phi] \\ &= \ldots\ldots\ldots\ldots\ldots\ldots \\ &= (A^2(\phi),\phi) - (A(\phi),\phi)^2 \\ &= \ldots\ldots\ldots\ldots\ldots\ldots \\ &= \|(A - (A(\phi),\phi))\phi\|^2.\end{aligned}$$

$$\text{Hence } \delta(A,\phi) = \|(A - (A(\phi),\phi))\phi\|. \tag{4.8}$$

We are now in a position to derive a lower bound for the product $\delta(A,\phi)\delta(B,\phi)$ with A and B selfadjoint, ϕ being in both their domains and such that $A(\phi)$ is the domain of B and $B(\phi)$ in the domain of A.

$$[2 \text{ imaginary part of } (A(\phi),B(\phi))] = (A(\phi),B(\phi))-(B(\phi),A(\phi)) = ((BA-AB)(\phi),\phi). \tag{4.9}$$

$$\text{Thus, } \frac{\hbar}{2m} = |((BA-AB)(\phi),\phi)| \leqq \delta(A,\phi)\delta(B,\phi), \tag{4.10}$$

$\hbar$ being the reduced Planck constant and $m > 1$ a size constant. Inequality (4.10) is a precise form of the famus Heisenberg uncertainty relation in terms of operators and unit vectors.

The whole analysis leading to this result is due to Mackey (1963). It reflects - above all and quite well - the measuremental question-reply feature of quantum mechanics in Hilbert space.

4.1 Complex fuzzy uncertainty versus quantum mechanical uncertainty

Under the different models of the algebra of logic there are models motivated by physical or psycholgical arguments. One of the most satisfactory mathematical models of the external world results when we choose a conservative dynamical system furnished with a finite number of degrees of freedom and dealt with in classical mechanics. It applies both to the Newton mechanics of n-body systems and to the Maxwell-Boltzmann theory of gases. In such a system Σ the state at time t_o is expressed by 6n real numbers (3n position and 3n momentum coordinates). This description is complete in the sense that the state of Σ is at any later or earlier time than t_o determined by these numbers and by the laws of attraction and repulsion. Thus, the state (or phase) of Σ may be represented by a point in a 6n-dimensional cartesian space. This space is, therefore, called the phase space Λ of Σ.

Each attribute (see Appendix 1) of Σ defines hence a set in Λ: the set of all states in which Σ has the given attribute. According to the Boolean logic, there should be a one-one correspondence between the subsets of Λ and the attributes of Σ. But this is physically absurd; for, since the accuracy of measurement is limited, one can never determine whether, for instance, the kinetic energy of Σ at time t_o is a rational number or not. Not even Borelian subsets of R^{2n} correspond to "observables" in the simple physical sense (and - as is known - these exist in extreme minority on the straight line). However, attributes having an accepted physical sense (e.g. temperature and pressure in a given interval) correspond to Borel sets of a phase space Λ of Σ.

This correspondence is also mathematically inconvenient, because it may prove to be incompatible with the principle of statistical mechanics demanding that every significant attribute have a (countably additive) probability. Probability itself is a probability measure on the Borel algebra of attributes.

<u>Definition 4.2</u>

A σ-lattice is a lattice in which every finite or countable subset $X = \{x_n\}$ has a meet = inf X and a join = sup X. A Boolean σ-lattice is called a Borel lattice, and a σ-lattice, being a Boolean algebra for finite meets and joins, is said to be a Borel algebra. Its set-theoretical representation is given by

<u>Theorem 4.1</u> (Loomis-Sikorski)

Every Borel algebra is a σ-epimorphic map of a σ-field of sets.

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This statement expressed by means of general topology runs as follows:

<u>Theorem 4.2</u> (Sikorski)

Let $\mathfrak{A}$ be a Borel algebra and $\mathfrak{S}_t(\mathfrak{A})$ its Stone space. Let, moreover, F be the σ-algebra of all Borel subsets of $\mathfrak{S}_t(\mathfrak{A})$ and Δ the σ-ideal of all Borel sets of the first category. Then $\mathfrak{A}$ is isomorphic to F/Δ.

More precisely: if θ is the isomorphism of $\mathfrak{A}$ onto the subfield of clopen subsets of $\mathfrak{S}_t(\mathfrak{A})$, then the concatenation

$$\theta\phi\psi\colon \mathfrak{A} \rightarrow \theta\mathfrak{A} \rightarrow F \rightarrow F/\Delta \qquad (4.11)$$

is an isomorphism.

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From (4.11) we immediately deduce the statement (which we will meet again in Chapter 7):

> The elements of $\mathfrak{A}$ correspond to clopen subsets of $\mathfrak{S}_t(\mathfrak{A})$, while the prime ideals of $\mathfrak{A}$ correspond to the points of $\mathfrak{S}_t(\mathfrak{A})$.

Being based on Boolean algebra, it is important to note that the logical model of classical mechanics (and even more so quantum mechanics) is subject to distributivity limitations; indeed, the improved v.Neumann model is only (weakly) countably distributive. From this we may conclude that the algebra of attributes is a topological lattice, a property enjoyed by all Borelian algebras.

This is reason enough to look somewhat closer at the distributivity property.

Every field of sets is a Boolean algebra; also, every Boolean algebra can be represented as a field of sets, that is, for every Boolean algebra $\mathfrak{A}$ there exists a space X and an isomorphism of $\mathfrak{A}$ into $\mathfrak{A}(X)$, i.e. a one-one mapping h of $\mathfrak{A}$ into $\mathfrak{A}(X)$ which transforms the Boolean operations onto the corresponding set-theoretical operations.

More exactly, M.H. Stone (1936a) has proved that, for every Boolean algebra $\mathfrak{A}$, there exists a totally disconnected compact topological space X such that $\mathfrak{A}$ is isomorphic to the field of clopen subsets of X. The space X is determined by $\mathfrak{A}$ uniquely up to a homeomorphism and is called the Stone space of $\mathfrak{A}$. However, Stone's representation theorem solves the representation problem for Boolean algebras from the point of view of finite Boolean operations only. But in every Boolean algebra $\mathfrak{A}$, one can also define the notions of infinite join and infinite meet which are Boolean analogues of the set-theoretical union and intersection of infinitely many sets.

Clearly, an element A is called the join of an indexed set $\{A_t\}_{t\in T}$ of elements of $\mathfrak{A}$ provided that

(j_1) $A_t \subset A$ for every $t \in T$; and

(j_2) if $A_t \subset B \in \mathfrak{A}$ for every $t \in T$, then $A \subset B$.

We write then

$$A = \bigcup_{t\in T} A_t. \tag{4.12}$$

Also, an element A is said to be the meet of an indexed set $\{A_t\}_{t\in T}$ of elements of $\mathfrak{A}$ provided that

(m_1) $A \subset A_t$ for every $t \in T$; and

(m_2) if $B \subset A_t$, $B \in \mathfrak{A}$, for every $t \in T$, then $B \subset A$.

We write then

$$A = \bigcap_{t\in T} A_t. \tag{4.13}$$

The join and meet of an infinite set of elements of $\mathfrak{A}$ do not always exist; if they exist for every $\mathfrak{m}$-indexed set of elements of $\mathfrak{A}$, then $\mathfrak{A}$ is called $\mathfrak{m}$-complete, where $\mathfrak{m}$ stands for an infinite cardinal.

Now, the Stone isomorphism h of a Boolean algebra $\mathfrak{A}$ onto the field of all clopen subsets of the Stone space X of $\mathfrak{A}$ does not transform infinite joins and meets into the corresponding set-theoretical unions and intersections. Indeed, if (4.12) holds, then $h(A)$ is not the set-theoretical union of all the sets $h(A_t)$, $t \in T$, except when the join (4.12) is not essentially infinite. The same remark is true for infinite meets. Both remarks follow easily from the compactness of the Stone space.

The Stone space fails if infinite Boolean operations are involved: For every infinite cardinal $\mathfrak{m}$, there exists an $\mathfrak{m}$-complete Boolean algebra which is not isomorphic to any $\mathfrak{m}$-field of sets. This circumstance is connected with the fact that not all identities true for infinite set-theoretical unions and intersections are true for their Boolean analogues: ∞-join and ∞-meet. An example of such an identity is the infinite distributive law (see Sikorski (1961)):

$$\bigcap_{t\in T} \bigcup_{s\in S} A_{t,s} = \bigcup_{f\in S^T} \bigcap_{t\in T} A_{t,f(t)}, \qquad (4.14)$$

where S^T denotes the set of all mappings from T into S. Identity (4.14) holds for set-theoretical operations but, in general, it does not hold for infinite Boolean operations. It is possible that all ∞-joins and ∞-meets in (4.14) exist, and yet the identity does not hold.

We conclude that, for classical mechanics, distributivity is limited to finite-dimensional spaces, while for quantum mechanics it fails even in the finite regime. Thus, the algebra of dynamical systems is never a Boolean algebra whenever

(i) the space underlying such systems is ∞-dimensional - owing to the distributivity failure,
(ii) the system is quantum mechanical, finite-dimensional or not - owing to distributivity failure,
(iii) the system is complex fuzzy, finite-dimensional or not - owing to tertium non datur and distributivity failures. In this case the zero of the algebra is an isolated point.

Thus, these restrictions violate axioms L5,L6 and L7 of the Boolean algebra; the logic of such systems can, therefore, be only a non-Boolean lattice giving rise to gaps in the original Boolean algebra for simple systems and to associated uncertainty relations.

Let us recall for the sake of completeness that one of the fundamental features of quantum mechanics in Hilbert space is the non-commutativity of any two self-adjoint operators

$$PQ - QP = \frac{h}{2\pi i}. \qquad (4.15)$$

The following are the essential analogies and differences between quantum mechanics and complex fuzzy cybernetics at a glance:

A. Formula (4.15) is comparable to the expression (3.4), both the left hand sides and the right hand sides, respectively; moreover, the right hand sides have each a virtual content, the incompatible quantities in each case being on the left. And yet: (4.15) expresses the non-commutativity of operators in quantum mechanics, while (3.4) expresses the non-complementation of sets in complex fuzzy cybernetics.

B. Formulas (4.5) and (3.8) express the respective uncertainty relations;

(i) both are inequality relations;

(ii) they are product relations of "complementary" quantities in the language of functions;

(iii) h is the universal Planck constant, while H is a mathematically derived negligible set, fixed for a given ∞-dimensional Banach space;

(iv) the inversions of the factors on the left hand side in (3.8) stand for the measures of precision and relevance of measurement.

C. The spaces underlying quantum mechanical systems and complex fuzzy systems are, respectively, ∞-dimensional separable Hilbert spaces and ∞-dimensional non-reflexive Banach spaces. The former rejects distributivity and accepts weak modularity instead; the latter rejects the laws of the excluded middle and of distributivity. The mathematical methods in both cases are functional analysis and topology; Gleason measure rules in quantum mechanics and Sugeno measure in complex fuzzy systems.

D. Neither of these systems accepts classical logic: quantum systems require a Birkhoff-v.Neumann logic which has no implication operation but up to six implication relations, complex fuzzy systems comply with Tarski's calculus of systems and, algebra-wise, the Brouwerian lattices with unity. The consequences are: the appearance of "gaps" and the invalidity of two laws of logic specified under C.

It is appropriate to point again to the bounds h and H: they are different and have different origins, H being the finer and higher allocated one of the two.

CHAPTER 5: PRINCIPLES OF MEASUREMENT

It is common knowledge that empirical disciplines and studies depend - in a fundamental manner - on measurement, actually on measurement and on a complementarily associated theory. This is specifically true of non-classical systems with an inherent uncertainty relation, such as quantum mechanics or complex fuzzy cybernetics, when determining the state of such systems at a particular instant of time. Simply speaking, measurement is an objective empirical process of associating real numbers to entities of the real world in accordance with a well defined rule. This rule is supposed to be such that the number allocated to an entity describe that quantity very closely. In this connection, events are conceived of, deterministically, as observable temporal changes of the quantities measured and, probabilistically, as sets which are probable in the frame of a Boolean algebra. This is depicted in the following diagram.

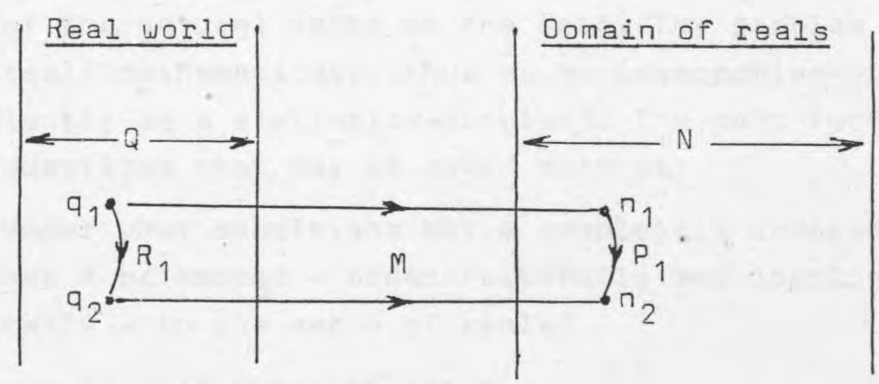

Fig. 5-1
Representation of the measurement process
(after Finkelstein)

Legend:

Q := a nonvoid class of non-mathematical quantities q_i, $i = 1,2,\ldots,k$;

R := a set of empirical relations R_i, $i = 1,2,\ldots,k$;

N := a subset of the set $\mathbb{R}$ of real numbers;

P := a set of numerical relations P_i, $i = 1,2,\ldots,k$;

$\mathcal{Q} = \langle Q,R \rangle$;

$\mathcal{N} = \langle N,P \rangle$;

$M: \mathcal{Q} \longrightarrow \mathcal{N}$ is a homomorphism of $\mathcal{Q}$ into $\mathcal{N}$;

$F: R \longrightarrow P$ is a one-one mapping;
$S = \langle Q, N, M, F \rangle$ is called the scale of measurement.

It is necessary that M be a well defined operational process, addressed as "measuremet procedure" for measurements on the scale S of measurement. In general linguistic usage $n_i \in N$ is called the map of $q_i \in Q$ under M; consequently $n_i = M(q_i)$ is the measure of q_i on scale S.
According to this symbolism, there exist - in general - other mapping processes from Q to N, for instance $M': Q \longrightarrow N$ such that $M(q_i) = M'(q_i)$ either for all $q_i \in Q$ or for $q_i \in Q'$ with $Q' \subseteq Q$. Every such process is a measurement procedure on the scale S.

5.1 Topological and measure-theoretic foundations of classical measurement

As already stated, the basic problem of classical measurement consists in the allocation of a real number to each test or measurement datum in such a way that the totality of data is mapped in the set of real numbers under conservation of the natural order on the data. The problem presents itself mathematically thus as an isomorphism-problem (equivalently as a similarity-problem). The most fundamental of questions that may be asked here is:

> Under what conditions may a completely ordered set X be mapped - order-faithfully and topologically - in the set N of reals?

The answer to this question is:

> The required mapping is possible iff there exists a countable subset Y in the completely ordered set X such that every closed interval $[a,b]$ contains a point $y \in Y$, i.e. iff $\bigvee_{y \in Y} \{a \leqq y \leqq b\}$.
> In this sense any point is closed and the space is a topological T_1-space.

It should be noted that the analysis involves a set theory including the axiom of choice (see Kaaz (1977)); also the binary relation "$<$" is said to order the chosen set X completely whenever $<$ is transitive and exactly one of the statements: $a < b$, $a = b$, $a > b$ applies (Ellis' condition!).

Definition 5.1

Relation $\leq$ (defined by $(x \leq y) \equiv (y = x \cup y)$) ordering the set A and relation $\leq^*$ ordering the set A^* represent similar orderings (i.e. A and A^* are isomorphic to one another) if a one-one mapping f of the set A onto the set A^* exists such that the equivalence:

$$(x \leq y) \equiv (f(x) \leq^* f(y))$$

is satisfied; i.e. if the following equivalence holds:
x entails y in A $\equiv$ f(x) entails f(y) in A^*.

----- o -----

Remarks

(i) A one-one mapping f is defined by $(x_1 \neq x_2) \Longrightarrow (f(x_1) \neq f(x_2))$ or else by $(f(x_1) = f(x_2)) \Longrightarrow (x_1 = x_2)$.

(ii) The similarity-relation is an equivalence-relation.

(iii) Two similar sets have the same cardinal number.

Theorem 5.1

Every countable set A, linearly ordered by $\prec$ is similar to a subset of the set Q of all rational numbers ordered by $<$.

Proof:

The case of A finite is trivial, so we shall provide the proof for A infinite only.

Let set A be put in the sequence $a_1, a_2, \ldots, a_n, \ldots$, where $a_i \neq a_j$. We define the one-one function f so that its arguments run through A and its values through Q. We also postulate: $f(a_1) = 0$, $f(a_2) \in Q$ and f monotone, i.e.

$f(a_2) < f(a_1)$ if $a_2 \succ a_1$ - antitone case, and
$f(a_2) > f(a_1)$ if $a_1 \prec a_2$ - isotone case.

The inductive definition of the number $f(a_{n+1})$ runs as follows:

(i) If a_{n+1} follows all elements $a_1, a_2, \ldots, a_n$ in A, then $f(a_{n+1}) \in Q$ is smaller than all values $f(a_1), f(a_2), \ldots, f(a_n)$ - the antitone case;

(ii) in the isotone case: if a_{n+1} follows all elements $a_1, a_2, \ldots, a_n$, then $f(a_{n+1})$ follows all values $f(a_1)$, $f(a_2), \ldots, f(a_n)$;

(iii) if neither (i) nor (ii) apply, then let a_k be the latest of all elements $a_1, a_2, \ldots, a_n$ which precede a_{n+1} and a_m the earliest of these elements following a_{n+1}.

In this case we shall assume that

$$f(a_{n+1}) = \frac{1}{2}(f(a_k) + f(a_m)).$$

This function is obviously one-one and it transforms, for every n, the set $\{a_1, a_2, \ldots, a_{n+1}\}$ similarly onto the set $\{f(a_1), f(a_2), \ldots, f(a_{n+1})\}$. For, when $a_i \prec a_j$ and $n+1 > i, j$ hold, then we conclude from the similarity of the sets $\{a_1, a_2, \ldots, a_{n+1}\}$ and $\{f(a_1), f(a_2), \ldots, f(a_{n+1})\}$ that $f(a_i) < f(a_j)$ holds.

Now, f is monotone and one-one in virtue of the construction; Q is densely ordered and $\mathbb{R}$ is continuously ordered in the sense of Dedekind and densely ordered and provided with nonvoid initial intervals having upper limits as well (see Kuratowski (1977)p.76).

We may now extend f from the ordered set $A \equiv Y$ to a linearly ordered set X (the set of entities to be measured), namely so that f will turn into a monotone homeomorphism (i.e. f and f^{-1} continuous). This is of great observational importance for the insight into the structure of X and the theory of measure on X, because - as is known - topology is the study of the geometrical objects which remain unaltered (invariant) when subjected to one-one and bidirectionally continuous transformations, i.e. to homeomorphisms. Moreover, topology deals with more general notions than analysis; thus, topology is - for a given transformation - indifferent towards differentiability properties; what counts are continuity and the one-one-ness of functions.

In the completely ordered set X several topologies are feasible, but - in the main - the order topology and the interval topology. We call a subset Y of X an interval whenever $y \in Y$ follows from $x \leqq y \leqq z$ for $x, z \in Y$. Intervals will be denoted in the usual way (open, half-open and closed) as (a,b), $(a,b]$, $[a,b)$ and $[a,b]$, $a \leqq b$. A point will be looked upon as a closed interval. An interval which with every x contains all $y < x$, is said to be a segment; this will be denoted by $A(x) = \{u: u < x\}$. A(x) represents obviously an open interval.

<u>Definition 5.2</u>

Let X,Z be completely ordered sets. Then $f: X \rightarrow Z$ is said to be a weakly monotone mapping if $(f(x_1) < f(x_2)) \Longrightarrow (x_1 < x_2)$ holds, and monotone if $(f(x_1) < f(x_2)) \equiv (x_1 < x_2)$ holds.

Definition 5.3

Intervals $[a,b]$ generate the topology $\mathcal{T}_1$ and the open intervals (a,b), $A(x)$ and $\{x: a < x\}$ generate the order topology $\mathcal{T}$ (also called interval topology).

In general $\mathcal{T} \subset \mathcal{T}_1$. Moreover, the notions open, closed, continuous and semi-continuous (without suffix) always refer to $\mathcal{T}$.

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The sets $A(x)$ generate a σ-algebra with Borel sets as elements, on which finite Borel measures μ are defined. We now impart an order structure on $\mathfrak{M}$, the set of all Borel measures on X.

Definition 5.4

Let X be a completely ordered set, $\mathfrak{M}$ the set of normed Borel measures $(\mu(X) = 1)$ on X and $\mu(A(x)) = F(x,\mu)$ for every $\mu \in \mathfrak{M}$, where $F: X \times \mathfrak{M} \longrightarrow I$, $I = [0,1]$, is assumed to be weakly monotone (respectively monotone).

We also postulate $\mu \prec \nu$ iff $F(x,\nu) \leqq F(x,\mu)$ for all $x \in X$.

We say that the sequence $\{\mu_n\}$ converges to $\mu \in \mathfrak{M}$ if $F(\cdot,\mu_n)$ converges pointwise to $F(\cdot,\mu)$.

Under these assumptions $\mathfrak{M}$ is a partially ordered set.

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For an insight into the structure of X the following conditions must be fulfilled.

Lemma 5.2

Let $f: X \longrightarrow I$ be a weakly monotone (resp. monotone) continuous mapping (more precisely: a monotone homeomorphism) and $\mathfrak{M}$ a set of Borel measures μ. In virtue of

$$E_f(\mu) = \int f(x)\,d\mu(x),$$

$\mu \longmapsto E_f(\mu)$, mapping $\mathfrak{M}$ into I, is a weakly monotone (resp. monotone) function, and we have

$$E_f(\lim \mu_n) = \lim E_f(\mu_n) \text{ for every } \{\mu_n\} \subset \mathfrak{M}.$$

----- o -----

The proof presents no difficulties if we use the rules of integration; we shall spare it here and turn instead to the necessary and sufficient conditions for the existence of this monotone homeomorphism under various aspects of connectivity of the set X, considering I and $\mathbb{R}$ to be connected spaces.

A closed interval $[a,b]$ of a completely ordered set with a void interior ($(a,b) = \emptyset$) is apparently called a gap. However, the completely ordered set is connected with respect to its topology $\mathcal{T}$ iff every upper bounded nonvoid subset has an upper limit and is gapless. Every continuous and real-valued function on this set has the Darboux property, i.e. it moves between two values through all intermediate values. This is actually a characteristic property of connected spaces.

The solution to the posed problem is - in the case of completely ordered connected sets - practically solved with the help of the following

Lemma 5.3 (N. Bourbaki)

A completely ordered connected set may be monotonically and homeomorphically mapped onto a real interval iff it is separable, i.e. iff it contains a dense countable set.

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For example, $\mathbb{R}$ is separable because $\mathbb{Q}$ is a dense countable subset of $\mathbb{R}$.

A more general study should, however, include the cases of totally (and even extremally) disconnected completely ordered sets. To this end, let us choose the set N and form the cartesian product $N = N_1 \times N_2 \times \ldots$. This may be lexicographically ordered as follows:

$(n_1,\ldots) \prec (m_1,\ldots)$ is true iff the implication

$(n_i = m_i$ for $i < j$ and $n_j \neq m_j) \Longrightarrow (n_j < m_j)$ holds.

Hofmann (1963) has shown that $\mathcal{T}$ and the Tichonov topology, or product topology, coincide on N, and he obtained the following result.

Lemma 5.4

A completely ordered compact and totally disconnected set with a countable basis for $\mathcal{T}$ may be monotonically and homeomorphically mapped in the set R of reals.

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This statement holds also for extremally disconnected Tichonov spaces which are 0-dimensional, because the notions: total disconnection, 0-dimensionality and strong 0-dimensionality coincide in the domain of completely ordered compact spaces (see Engelking (1968)).

We should now attempt to combine Lemmata 5.3 and 5.4; in this we will succeed only if we go over from the completely ordered set X to the quotient space $X' = X/\varrho$, where ϱ is the connectivity relation. In virtue of the class formation X' is totally disconnected, and the quotient topology of X' coincides with $\mathcal{T}$ on X'.

Moreover, if X is completely ordered and separable, then X is monotonically homeomorphic to a subset of the lexicographically ordered product $X' \times I$. Hence, X has at most countably many connectivity components (see Kuratowski (1977)p.214), i.e. $g(X) \cap (\{z\} \times I)$ has more than one point for at most countably many $z \in X'$ since every connectivity component $\varrho(x)$ is monotonically homeomorphic to a real interval, a monotone homeomorphism $f_z\colon \varrho^{-1}(z) \longrightarrow I$ exists for every $z \in X'$ and $g\colon X \longrightarrow X' \times I$ is defined by

$$g(x) = (\varrho(x), f_{\varrho(x)}(x)).$$

In addition, the mappings $x \mapsto \varrho(x)$, g and f_z are continuous. If X contained uncountably many disjoint nonvoid intervals, no countable set could be dense in X. Hence follows further: if X is completely ordered and compact and has a countable basis for $\mathcal{T}$, then X' is monotonically homeomorphic to a compact subset of I and contains a countable subset Y such that X is monotonically homeomorphic to the subset $(X' \times \{0\}) \cup (Y \times I) \subset X \times I$.

The structure of X' is characterized in Lemma 5.4. Because of the compactness of X, all connectivity components are homeomorphic to I. The totality of points $z \in X'$, for which $\varrho^{-1}(z)$ consists of several points and for which f_z has been chosen as monotone homeomorphism, constitutes the set Y, while $f_z(z) = \{0\}$ for $z \notin Y$.

This confirms the above statements.

The mapping of the set $X = (X' \times \{0\}) \cup (Y \times I)$, monotonically and homeomorphically, into the set of reals is now effected by

<u>Lemma 5.5</u>

Let Y be the countable set $\{r_1, r_2, \ldots\}$ in I, μ that Borel measure on I which is defined by $\mu(r_i) = 1/2^{i+1}$, $F(x) = \mu(A(x)) = \sum\{1/2^{i+1} : r_i < x\}$ and $X = (X' \times \{0\}) \cup (Y \times I)$.

Then, there exists a monotone homeomorphism $f\colon X \longrightarrow I$.

Proof:

Under the assumptions of Lemma 5.5, F is $\mathcal{T}_1$-continuous, and in virtue of $F(x+0) = \lim F(y)$ and on approaching from above we get $F(x+0) - F(x) = 1/2^{i+1}$ in case $x = r_i$ and $= 0$ otherwise.

We put now $f(x,y) = \frac{1}{2}(x + F(x) + y(F(x+0) - F(x)))$. f turns out to be monotone since we always have $f(x,y) < f(x',y')$. The continuity of f follows from the continuity at the points $(z,0)$ and $(z,1)$ with $z \in X'$ in the first case and $z \in Y$ in the second case. Hence follows the continuity of f. We ask now: when does X have a countable basis for $\mathcal{T}_1$?

Lemma 5.6

Let X be a completely ordered set and Y a countable subset such that a $u \in Y \cap [a,b]$, with $a < b$, always exists. Then, X has a countable basis for its topology.

Proof:

We list all interval sets and the allocation of $x \in X$ to them; the proof follows fom this scheme.

Since every gap contains an element of Y, their number can be at most countable. Let L be the set of all points belonging to gaps, and let

- J_1 be the set of all intervals (u,v), $u < v$; $u,v \in Y$;
- J_2 be the set of all intervals $(u,v]$, $u < v$; $u \in Y$, $v \in L$;
- J_3 be the set of all intervals $[u,v)$, $u < v$; $u \in L$, $v \in Y$;
- J_4 be the totality of sets consisting of a single isolated point.

The union of J_4 is contained in L. J_1, J_2, J_3, J_4 and their union J are all countable.

Also assume that $x \in X$ has an open neighbourhood U; consequently, there must exist elements $a,b \in X$ with $x \in (a,b) \subset U$. If x is an isolated point, then $\{x\} \subset J$ is a neighbourhood contained in U.

For x non-isolated we get:

(i) If $x \notin L$, then, for $a < a' < a'' < x < b'' < b' < b$, we have $u \in [a',a''] \cap Y$, $v \in [b'',b'] \cap Y$; in this case (u,v) is a neighbourhood of x in J contained in U.

(ii) If $x \in L$ is isolated from below, then, for $a < a' < a'' < x$ and $u \in [a',a''] \cap Y$, $(u,x]$ is a neighbourhood of x in J contained in U.

The careful considerations so far, extending over Theorem 5.1 and the Lemmata 5.3 - 5.6 and supported by the principles of the theory of measure and of topology, culminate now in the following important theorem due to Hofmann (1963).

Theorem 5.7 (Hofmann)

The following two statements are equivalent with respect to the completely ordered set X:

(1) X contains a countable subset Y such that there exists, to every pair of elements (a,b), $a < b$, always an element $y \in Y$ such that $a \leqq y \leqq b$.

(2) X may be monotonically and homeomorphically mapped into the set of reals.

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This theorem and Lemma 5.2 yield jointly

Corollary 5.8

Let X be a completely ordered set containing a countable subset Y such that Y cuts every non-trivial closed interval $[a,b]$. Moreover, let $\mathfrak{M}$ be a set of normed Borel measures on X, which is ordered in accordance with Definition 5.4, part 2.

Then there exists a monotone sequentially-continuous mapping of $\mathfrak{M}$ into I.

Proof of Corollary 5.8:

Theorem 5.7 ensures the existence of a monotone homeomorphism $f: X \twoheadrightarrow I$. Consequently, the mapping $\mu \longmapsto E_f(\mu)$ of $\mathfrak{M}$ in I, used in Lemma 5.2, is monotone, and from

$\lim\limits_{n\to\infty} \mu_n = \mu$, in accordance with Definition 5.4

follows $E_f(\mu) = \lim\limits_{n\to\infty} E_f(\mu_n)$.

With respect to any Borel measure $\mu \in \mathfrak{M}$, f represents a random variable in the ordinary sense; its expectation value is

$$E_f(\mu) = \int r dF_o(r,\mu) \text{ - a Stieltjes integral,}$$

while the distribution function $r \longmapsto F_o(r,\mu)$, belonging to f, is defined by $F_o(r,\mu) = \mu\{x\colon f(x) < r\}$ and $F(x,\mu) = F_o(f(x),\mu)$. :thus, with respect to $\mathfrak{M}$, the theory of measure on Y coincides with that on R.

Remark

If we define new spacial relations on the completely ordered set X, then corresponding relations should also be introduced on R and the measuremental isomorphism ensured anew; this constitutes the problem of meaningfulness discussed in the Introduction. Such a relation is, for example, $(x,y) \longmapsto xy$; in this case a monotone homeomorphism $f: X \longrightarrow R$ satisfying $f(x) + f(y)$ and the operation of addition has to be found on R. However, we do not intend to pursue this problem any further.

If we want to realize the considerations of the Section 5.1 in actual physical measurements, then - for lack of absolute accuracy of measurement - the probability theory may be resorted to. A probability algebra is, after all, nothing else but a measure algebra in which we put $p(I) = 1$. Indeed, by means of $m(I) \neq 0$ one obtains from a measure algebra a probability theory via the relation $p(x) = \frac{m(x)}{m(I)}$. Thus, the two theories are co-extensive on the algebraic level.

5.2 A mathematical realization model for measurement

The theory presented in Section 5.1 is an exact description of the realization model covering the data store, the measuring procedure, the measured values and a secure method of measurement with statistical aspects.

The data store

The simplest method of investigation in the measurement of properties (attributes), such as weight or elasticity is that of comparison; for this a class of objects (e.g. the class of all weighable objects) is assumed. If two objects from the full class are given, then it is possible to determine by means of a weihing machine which of the two objects is the heavier one. Objects, for which both $a \geqq b$ and $b \geqq a$ hold, cannot even in principle be distinguished; we consider them, therefore, as equivalent, and the set of thus formed equivalence classes is said to be the data store for a definite property (e.g. gravity). The name "data store" seems rather appropriate; it does remind of the allocation of pharma products to drawers in a pharma store.

Thus, the data store for the weight consists of the totality of classes of iso-gravitational bodies. It is obviously a totally disconnected and completely ordered set. This way of identification has proved to be most useful in mathematics, especially in the definitions of real numbers and of Schwartz-distributions.

The measurements

The exact ordering of objects - with respect to a property - in a class of the data store is practically impossible; this is due to the general inexactness of any practical measurement. To the approximate localization of classes corresponds mathematically a probability distribution μ on the data store. If A is a subset of the data store, then $\mu(A)$ indicates the probability that the object in question belongs to a class within A. It is for this reason that the probability distribution on the data store is called a measurement. The measurements are those mathematical objects which describe the technically accessible information.

The comparison of two measurements should correspond to the comparison on the data store in the sense of the theory. From two measurements μ and ν, μ is the smaller one if, for every class x of the data store, the probability of finding a class u under x is greater for μ than for ν. In this case we write $\mu \prec \nu$. We have thus also established an order relation on the set of measurements, but this order is cruder than the order on the data store.

The case of absolutely accurate measurement is, of course, contained in the probability model; it corresponds to a probability distribution allocating precisely to one class x from the data store the full probability 1 and to the complement the probability 0. The set of all absolutely accurate measurements is identical with the ordered data store. This is but another definition of the data store.

The measured values

Measurement associates, in a reasonable way, a real number to a datum. The rule of such an association is considered as sensible if the smaller datum on the data store X is allocated the smaller real number.

A rule of association f, mapping X monotonically into the set of reals, has been called a measuring procedure (in automatic control: the method of measurement). At the same time, data lying close to one another should correspond to real numbers in close neighbourhood. This requirement is equivalent to the continuity of the function f.

We have practically only the measurements at hand, but this is not enough to allocate a real number to every datum because not every data store allows a measurement procedure. In any case, measurements are probability distributions and not numbers nor data. But when a measurement procedure f is mathematically ensured, then we can associate to every measurement μ a real number, namely the expectation value

$$E_f(\mu) = \int f(x)d\mu(x), \quad x \in X, \tag{5.1}$$

with respect to the probability $\mu \leqq 1$.

$E_f(\mu)$ is the mean value for the measurement μ when measurement procedure f has been used; we call it the measured value of measurement μ obtained by means of the measurement procedure f.

Inspite of the fact that the measurements are not always completely ordered, the measured values and their order usually turn out to be valid means for describing the structure of the set of measurements. If a measurement μ is smaller than a measurement ν, then the measured value of μ is smaller than that of ν; and if the sequence of measurements approaches a certain measurement, then the sequence of the associated measured values approaches the measured value of the limit measurement as well.

In case of absolutely accurate measurements, the measured value of measurement μ is precisely the number f(x) since (5.1) yields the value f(x) for a Dirac functional at point x. Thus, in case of absolutely accurate measurements the basic measurement procedure coincides with the practical formation of the expectation values.

The fundamental problem in the mathematics of measurement remains the finding of the necessary and sufficient conditions for the possibility of mapping - under conservation of order and topology - of a data store into the set of reals. This question has, theoretically, been clarified in the sense that such mapping is always possible if the data store X contains a sequence $d_1, d_2, \ldots$ of data such that there is always a number n to every pair of data a,b ($a < b$), satisfying the inequality $a \leqq d_n \leqq b$.

5.3 Statistical aspects of measurement

For a practitioner, the probability-theoretical notion of

the measured value is not practical enough. An experimenter is not able to determine the measured value $E_f(\mu)$ of a measurement itself; he must rather repeat finitely many times the measurement belonging to the probability distribution μ. Above all, he has to form a more exact picture of the unknown probability distribution from the experimentally obtained numbers $f(x_1), f(x_2), \ldots, f(x_n)$.
The extraction of finitely many numbers in the field of experimentation is called sampling and

$$m = \frac{1}{n}(f(x_1) + f(x_2) + \ldots + f(x_n)) \tag{5.2}$$

is looked upon as a useful estimate for the unknown value $E_f(\mu)$. For a better description of μ, one may also determine the variance from the sample; it is the number

$$s^2 = \frac{1}{n-1}((f(x_1)-m)^2 + \ldots + (f(x_n)-m)^2). \tag{5.3}$$

s^2 is necessary for the estimation of the integral $\int (f(x)-E_f(x))^2 d\mu(x)$. If $m = \pi$ and $s^2 = \pm 0.002$, then the estimation result is $\pi \pm 0.002$.

The probability $\mu(\{a \leqq f(x) \leqq b\})$ that, in the continued measurement μ, a number between a and b will come out, is generally set equal to

$$\frac{1}{\sqrt{2\pi s}} \int_a^b \exp\left\{\frac{1}{2}\left[\frac{t-u}{s}\right]^2\right\} dt. \tag{5.4}$$

There are good reasons for this formula. For, let μ_n be the probability measure defined by the random variable $\frac{1}{n}(f(x_1) + \ldots + f(x_n))$ on the set of reals. Then, with increasing n, this probability measure tends - under very general assumptions - to the probability measure of the normal distribution whose expectation value is $E_f(\mu)$ and whose standard deviation coincides with the variance $\mathrm{var}_f(\mu)$. It is necessary, however, to test the assumptions.

5.4 Transition to non-classical measurement

In the classical theory of measurement expounded in Sections 5.1 - 5.3, we have succeeded to map a completely ordered set X under conservation of its structure, i.e. monotonically and homeomorphically, into the set of reals. The solution of this isomorphic problem became possible by the introduction of algebraic relations in the set X. In general, measurement is the act of mapping homomorphically a data store

X with its given structure faithfully into the set of reals. A system of axioms describes the form of the structure; the choice of these axioms is, of course, left to the intuition of the experimenter. In this, the expedience and the success in the construction of the mathematical theory are considered to be reasonably good guidelines.

Simultaneous measurement is always possible in classical measurement or, more correctly, in the measurement of classical systems, i.e. systems obeying classical logic (algebra). This is so because the joint distribution required for the attributes to be measured simultaneously exists in all classical cases.

The situation is entirely different in the case of non-classical systems due to the appearance of incompatible attributes giving rise to indeterminacies and uncertainties; since some of the Boolean axioms fail at the same time, there can be no joint distribution for the variables involved. In reality, the original Boolean algebra for countable systems will split incompatibly and withdraw the possibility of determining the state of non-classical systems, even when partial determination of single attributes might be possible. Measurement will loose its most important features: determinateness and objectivity, for no other reason than the incompatibility of precision and significance of measurement in the face of complex fuzzy systems. After the hard scrutiny to which measurement was subjected in the great days of quantum mechanics, this is a trial concerning measurement alone and not in association with quantum theory. Let us review the scenic setting in this case.

Complex fuzzy systems entail ∞-dimensional spaces and invalidate several classical laws, notably the principle of complementarity, to which A.N. Whitehead has drawn attention some decades ago. If Y is an ∞-dimensional non-reflexive Banach space of closed subsets, hence topological, and $L = 2^Y$ a topological hyperspace with exponential topology and with an isolated zero-element, then

$$\mathcal{L} = \langle L, \cup, \cap, \dot{-} \rangle \quad \text{with } A \dot{-} B = \overline{A-B} \qquad (3.1)$$

is a Brouwerian lattice with unity $Y \supset A, B$, and

$$\overline{Y-A} \cap \overline{A} \supseteq H, \quad A, H \subset Y \qquad (3.4)$$

is the non-complementation condition for measurement in complex fuzzy systems characterized by the uncertainty relation of measurement:

$$\triangle y^*(y)_{\overline{Y-A}} \cdot \triangle y^*(y)_{\overline{A}} \geqq \bar{\mu}(H); \quad A,H \subset Y. \qquad (3.8)$$

The basic questions of interest are now:

(1) Does a measurement space facilitating the measurement on classical as well as nonclassical systems exist?

(2) What are its properties?

Question (1) will be dealt with in Chapter 6 and question (2) in Chapter 7. The diagram below illustrates the situation.

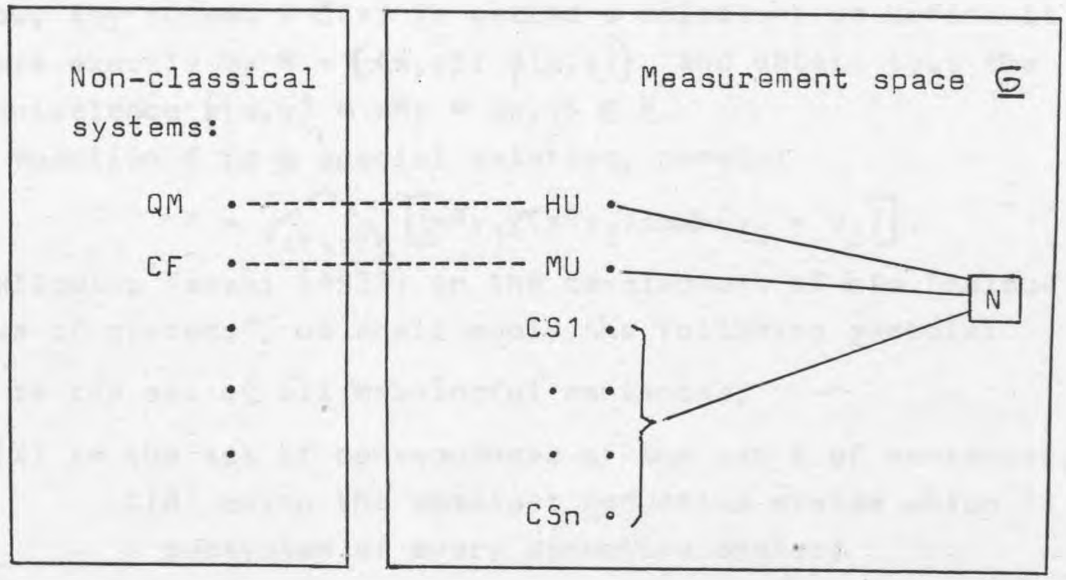

Fig. 5-2
Representation of the measurement space.

Legend:

QM := quantum mechanical system,

CF := complex fuzzy system,

HU := Heisenberg uncertainty relation,

MU := measurement uncertainty relation,

CS1 … CSn := classical relational systems,

N := numerical relational system.

CHAPTER 6: METAMATHEMATICAL SYSTEM CONSIDERATIONS

Recall that formulas are propositional functions, i.e. expressions which may contain variables x,y,z,... and whose value depends on the values assumed by the variables. Hence sentences are formulas without free variables (true, false or indifferent; see Kaaz (1977)).

Let $\phi(x,y)$ denote a propositional function of two variables $x \in X$ and $y \in Y$. $\{x\}$, $\{x,y\}$, $\{x,y,z\}$, ... represent (aside of the void set $\emptyset$) the smallest sets which we call: singleton, doubleton, trebleton, ..., respectively. We shall distinguish between an ordered pair $\langle x,y\rangle = \{\{x\},\{x,y\}\} \in X\times Y = 2^{2^{X \cup Y}}$ and an unordered pair $\{x,y\} \subset X \cup Y$.

Now, any subset $R \subset X\times Y$ is called a relation; we define it more exactly by $R = \{\langle x,y\rangle : \phi(x,y)\}$, and obtain thus the equivalence $\phi(x,y) \equiv xRy \equiv \langle x,y\rangle \in R$.

A function f is a special relation, namely:

$$f = \bigwedge_{x,y_1,y_2} \left[(xRy_1 \wedge xRy_2) \Longrightarrow (y_1 = y_2)\right].$$

Following Tarski (1935) in the development of his "calculus of systems", we shall apply the following symbols:

S := the set of all meaningful sentences;

C(X) := the set of consequences of the set X of sentences, $C(\emptyset)$ being the smallest deductive system which is a subsystem of every deductive system;

$\bar{x}$:= the negation of the sentence x;

$x \longrightarrow y$:= the implication with antecedent x and consequent y.

With the help of these notions one can define the concept of a deductive system, i.e. a set X of sentences containing its consequences as elements; in other words: $C(X) \subset X \subset S$.

On the other hand, by deductive theories we understand the "models" or "realizations" of the system of axioms to be stated next. Since these axioms are expressed by the four notions above, any quadruple of notions satisfying all axioms of the system will be called its model.

The following five axioms suffice for the foundation of a general metamathematics:

(A1): $0 < \bar{\bar{S}} \leqq \aleph_0$.

(A2): If $x,y \in S$, then also $\bar{x}$, $x \longrightarrow y \in S$.

(A3): The set of all valid logical sentences $L = C(\emptyset) \subset S$.

(A4): If $x, y, z \in S$, then $(\bar{x} \to x) \to x \in L$, $x \to (\bar{x} \to y) \in L$, and $(x \to y) \to ((y \to z) \to (x \to z)) \in L$.

(A5): The modus ponens rule is valid in L, i.e. if $x, x \to y \in L$, $y \in S$, then $y \in L$.

The content of these axioms is simple enough to require no comments.

Definitions 6.1

(1) $x + y = \bar{x} \to y$ and $x \cdot y = \overline{x \to \bar{y}}$ for arbitrary $x, y \in S$.

(2) $\sum_{i=1}^{n} x_i = \prod_{i=1}^{n} x_i = x_1$ if $n = 1$ and $x_1 \in S$;

$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n-1} x_i + x_n$ and $\prod_{i=1}^{n} x_i = \prod_{i=1}^{n-1} x_i \cdot x_n$ if n is an arbitrary natural number > 1 and $x_1, x_2, \ldots, x_n \in S$.

(3) For an arbitrary set $X \subset S$, the set $C(X)$ consists of such and only such sentences $y \in S$ that either $y \in L$ or there are sentences $x_1, x_2, \ldots, x_n \in X$ for which $\prod_{i=1}^{n} x_i \to y \in L$;

equivalently:

For an arbitrary set $X \subset S$, the set $C(X)$ is the product of all sets Y satisfying the following two conditions: (i) $L + X \subset Y$;

(ii) if $x, x \to y \in Y$, $y \in S$, then $y \in Y$.

----- o -----

C(X) is noe of the sets Y satisfying the conditions (i) and (ii) of Definitions 6.1(3); it is the smallest of these sets. Indeed, C(X) is the smallest set containing L and X, and it is closed with respect to modus ponens.

The calculus of systems

From the above statements we construct two calculi which will prove useful in metamathematical investigations: the propositional calculus and the calculus of deductive systems, the former being a complete, the latter a partial interpretation of the formal system of the algebra of logic (also called Boolean algebra).

Let us first introduce the necessary symbols and postulates.

B := the field of considerations;

$x \prec y$:= object x is related to object y by inclusion;

$x = y$:= object x is related to object y by equality;

$x + y$:= the sum of objects x and y;

$x \cdot y$:= the product of objects x and y;

0 := the null or void object;

1 := the total or full object;

$\bar{x}$:= the complement or negation of object x.

Postulate I: (a) if $x \in B$, then $x \prec x$;
(b) if $x, y, z \in B$, $x \prec y$, $y \prec z$, then $x \prec z$.

Postulate II: if $x, y \in B$, then $x = y$ holds iff $x \prec y$ and $y \prec x$.

Postulate III: if $x, y \in B$, then:
(a) $x + y \in B$; (b) $x \prec x + y$ and $y \prec x + y$;
(c) if, moreover, $z \in B$, $x \prec z$ and $y \prec z$, then $x + y \prec z$.

Postulate IV: if $x, y \in B$, then
(a) $x \cdot y \in B$; (b) $x \cdot y \prec x$ and $x \cdot y \prec y$;
(c) if, moreover, $z \in B$, $z \prec x$ and $z \prec y$, then $z \prec x \cdot y$.

Postulate V: if $x, y, z \in B$, then (a) $x \cdot (y + z) = x \cdot y + x \cdot z$ and (b) $x + y \cdot z = (x + y) \cdot (x + z)$.

Postulate VI: (a) $0, 1 \in B$; (b) if $x \in B$, then $0 \prec x$ and $x \prec 1$.

Postulate VII: if $x \in B$, then (a) $\bar{x} \in B$, (b) $x \cdot \bar{x} = 0$ and (c) $x + \bar{x} = 1$.

The system of the algebra of logic will now be extended by the following additional symbols:

$\sum_{y \in X} y$:= the sum of all objects of the set X,

$\sum_{y \in X} y$:= the product of all objects of the set X.

Postulate VIII: if $X \subset B$, then (a) $\sum_{y \in X} y \in B$; (b) $x \prec \sum_{y \in X} y$ for any $x \in X$;
(c) if, moreover, $z \in B$ and $x \prec z$ for any $x \in X$, then $\sum_{y \in X} y \prec z$.

Postulate IX: if $X \subset B$, then (a) $\prod_{y \in X} y \in B$;
(b) $\prod_{y \in X} y \prec x$ for any $x \in X$;

(c) if, moreover, $z \in B$ and $z < x$ for any $x \in X$, then $z < \prod_{y \in X} y$.

Postulate X: if $X \subset B$ and $x \in B$, then:
(a) $x \cdot \sum_{y \in X} y = \sum_{y \in X} x \cdot y$; (b) $x + \prod_{y \in X} y = \prod_{y \in X} (x+y)$.

Postulates I-VII and all theorems following from them constitute the ordinary system of the algebra of logic, while Postulates I-X and all their theorems constitute the extended system of the algebra of logic.

The calculus of propositions, more appropriately called "algorithm of propositions", will now be dealt with briefly. The field of considerations of the algorithm of propositions is the set S with the relations of implication $x \supset y$ and the equivalence $x \equiv y$ defined on it.

Definition 6.2
$x \supset y$ iff $x, y \in S$ and $x \longrightarrow y \in L$;
$x \equiv y$ iff $x \supset y$ and $y \supset x$.

Theorem 6.1 (Tarski (1935))
Postulates I-VII are satisfied with the symbols used, except for the following substitutions:
$B, <$, and $=$ to be replaced by $S, \supset$, and $\equiv$, respectively, as well as 0 by $u \in S$, $\bar{u} \in L$, and 1 by $v \in L$.

----- o -----

The symbols 0 and 1 of the algebra of logic cannot be interpreted "effectively" under our premises because we are unable to define a single constant denoting a concrete sentence.
We turn now to the calculus of deductive systems beginning with

Definition 6.3
Class $\mathfrak{S}$ constitutes the field of considerations for the calculus of deductive systems, L is the void system, S the full system and $X \in \mathfrak{S}$ iff $C(X) \subset X \subset S$, or else iff $L \subset X \subset S$ and if the formulas x, $x \longrightarrow y \in X$ and $y \in S$ always entail $y \in X$.

----- o -----

The calculus of systems is a very essential extension of the algorithm of sentences already owing to the above Definition 6.3. Inclusion and equality between systems as well as the product of systems retain their usual and in the calculus of classes fixed sense. But the (logical) addition of systems, symbolized by "$\dot{+}$", is not identical with the set-theoretical addition, which yields no new deductive system when applied to systems. Similar reservations apply to the complementation of systems.

Definition 6.4

The logical sum of systems $X \dot{+} Y = C(X + Y)$ for $X, Y \in \mathfrak{S}$ and the logical complement $\overline{X} = \prod_{x \in X} C(\bar{x})$ for every $X \in \mathfrak{S}$ are - as seen - different from the set-theoretical operations.

Theorem 6.2 (Tarski (1935))

Let us undertake the following changes in all postulates of the ordinary system of the algebra of logic:

Variables x,y,z to be replaced by the variables X,Y,Z representing deductive systems;

constants $B, <, +, 0$ and 1 to be replaced by the symbols $\mathfrak{S}, \subset, \dot{+}$, L and S.

Then all postulates except Postulate VII(c) are satisfied. Its place takes now

Postulate VII(d): If $X, Y \in \mathfrak{S}$ and $X \circ Y = L$, then $Y \subset \overline{X}$.

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The essential difference between the calculus of systems and the calculus of classes, for example, consists in the inadmittance of the tertium non datur law: $X \dot{+} \overline{X} = S$ (except for the weak consequence of this law) to the calculus of systems. However, tertium non datur is valid either when the class $\mathfrak{S}$ is finite or (equivalently) when the set S does not contain infinitely many sentences such that no two of them are equivalent. But if the class $\mathfrak{S}$ is infinite (frequently occurring in general studies of concrete deductive systems), then there are systems which do not satisfy the tertium non datur law.

The failure of tertium non datur entails further consequences. The law of double negation is true only in one direction ($X \subset \overline{\overline{X}}$); the treble negation, however, has full validity; thus, if $X \in \mathfrak{S}$, then $X \subset \overline{\overline{X}}$ and $\overline{\overline{\overline{X}}} = \overline{X}$.

From the two DeMorgan laws only one is true, from the four laws of transposition two fail and two remain valid; thus,

$$\text{if } X, Y \in \mathfrak{S}, \text{ then } \overline{X \dot{+} Y} = \overline{\overline{X} \cdot \overline{Y}} \text{ and } \overline{X \cdot Y} = \overline{X} \cdot \overline{Y},$$

$X \subset Y$ entails $\overline{Y} \subset \overline{X}$ and
$X \subset \overline{Y}$ entails $Y \subset \overline{X}$.

The formal analogy of the calculus of systems to the intuitionistic propositional calculus is striking. We see all relevant statements of Chapter 1 confirmed; indeed, they were the ones which led us to the concept of the uncertainty relation for complex fuzzy systems. Hence we conclude that the calculus of systems considered above and developed by Tarski in the years 1933/34 is a natural logical foundation for studies of complex physical systems involving fundamental uncertainty relations in infinite topological spaces.

6.1 Interrelation between general metamathematics and Boolean fields

General metamathematics is a special case of the Boolean algebra; there always exists, therefore, the possibility to formulate theorems of general metamathematics as theorems of the Boolean algebra. This has, to a large extent, been enhanced by the work of M.H. Stone.

Definition 6.5

A quadruple $T = \{S, L, \longrightarrow, -\}$ satisfying the axioms (A1) - (A5) shall be called a deductive theory.

Definition 6.6

An ordered quadruple $K = \{A, \sim, \vee, '\}$ consisting of a non-void set A, a 2-argumental relation $\sim$, a 2-argumental operation $\vee$ and a 1-argumental operation $'$, is called a generalized Boolean field if the following conditions are satisfied for any $a,b,c \in A$:

(1) $a \sim a$,
(2) if $a \sim b$, then $b \sim a$,
(3) if $a \sim b$ and $b \sim c$, then $a \sim c$,
(4) $a', a \vee b \in A$,
(5) if $a \sim b$, then $a' \sim b'$ and $a \vee c \sim b \vee c$,
(6) $a \vee b \sim b \vee a$,
(7) $(a \vee b) \vee c \sim a \vee (b \vee c)$,
(8) $(a' \vee b')' \vee (a' \vee b)' \sim a$.

These eight conditions constitute the system of axioms of the algebra of logic, comparable to the axioms L1 - L7 of the Boolean algebra in Chapter 1. If degenerates to the ordinary identity, then K is said to be a Boolean field.

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In order to express the correspondence between a deductive theory $T = \{S, L, \longrightarrow, -\}$ and the generalized Boolean field $K_T = \{S, \sim_T, \vee_T, -\}$, the following interrelations are necessary:

$$\left.\begin{array}{ll}\text{(i)} & a \longleftrightarrow b = \overline{\overline{(a \longrightarrow b) \longrightarrow \overline{(b \longrightarrow a)}}}, \\ \text{(ii)} & a \vee_T b = \overline{\overline{a} \longrightarrow \overline{b}}, \\ \text{(iii)} & a \sim_T b \equiv a \longleftrightarrow b \in L. \end{array}\right\} \quad (6.1)$$

Theorem 6.3 (Mostowski (1937))

If T is a deductive theory, then K_T is a generalized countable Boolean field.

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This theorem shows that it is possible to associate a generalized Boolean field to every deductive theory. The connection between general metamathematics and the theory of Boolean fields is thus established. The converse implication can easily be obtained as well, but we have no need for it here.

Our primary concern in this subject requires now several more definitions concerning the generalized Boolean fields $K = \{A, \sim, \vee, '\}$ and $L = \{B, \equiv, +, \cdot\}$.

Definitions 6.7

(1) As a general formula we have $a \cdot b = (a' \vee b')'$.

(2) A set I in K is said to be an ideal iff I is a nonvoid subset of A and satisfies the following conditions:

if $a \in I$ and $a \sim b$, then $b \in I$,

if $a \in I$, $b \in A$, then $a \cdot b \in I$,

if $a, b \in I$, then $a \vee b \in I$.

(3) Set I is called a prime ideal of K iff I is an ideal in K, not identical with A, but every ideal J containing I is identical either with A or with I.

(4) a is said to be a generating element of ideal I iff I is an ideal in K, $a \in I$, and there exists an element c of field K such that $a \sim b \circ c$ for every element $b \in I$.

(5) I is said to be a principal ideal of K iff I is an ideal in K and a generating element of I exists.

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Definitions 6.8

(1) Given two fields K and L (as before), L is said to be

homomorphic to K iff there is a relation R with domain A and range B having the following properties:

(i) if $a,b \in A$, $c,d \in B$, $a \sim b$, $c \equiv d$ and aRc, then bRd,

(ii) if aRb and aRc, then $b \equiv c$,

(iii) if aRb and cRd, then $(a \vee b)R(c+d)$,

(iv) if aRb, then $a'Rb^{\bullet}$.

(2) Field L is said to be isomorphic to field K iff the relation R in (1) also satisfies the condition:

(v) if bRa and cRa, then $b \sim c$.

This is actually the converse of (1)(ii).

----- o -----

With the above defined notions the intelligibility of the very important theorem that follows will present no difficulties. A rigorous proof will be found in Mostowski (1937).

Theorem 6.4 (Tarski (1935))

Let $T = \{S, L, \longrightarrow, -\}$ and $T_o = \{S_o, L_o, \prec, \bullet\}$ be two deductive theories, then the following consequences may be drawn:

(a) The set X is a deductive system of the theory T iff X is an ideal in the field K_T.

(b) The set X is a complete deductive system of the theory T iff X is a prime ideal of the field K_T.

(c) The set X is an axiomatizable deductive system of the theory T iff X is a prime ideal of K_T.

(d) The theories T and T_o are of the same structural type iff the fields K_T and K_{T_o} are isomorphic.

(e) The homomorphism of the fields K_T and K_{T_o} is equivalent to the following condition:
there exists a deductive system X of the theory T such that the theory $T_X = \{S, X, \longrightarrow, -\}$ is of the same structural type as the theory T_o.

6.2 Classical measurement in the light of Tarski's Theorem 6.4

Let $T = \{S, L, \longrightarrow, -\}$ be the deductive theory associated with the system of world entities to be measured, and

$T_o = \{S_o, L_o, \prec, \bullet\}$ be the deductive theory relating to the numerical relational system in which the measuremental valuation occurs.

Also, let $K_T = \{S, \sim_T, \vee_T, {}^{-}\}$ and $K_{T_0} = \{S_0, \equiv_{T_0}, +_{T_0}, \cdot\}$ be the associated fields, respectively.

Since the measurement is classical, S and S_0 will be of powers not higher than $\aleph_0$.

Then, Theorem 6.4(e) implies the existence of a measuremental homomorphism.

Proof:

It suffices to prove that a deductive system X of T with the required property exists. Hence, let us postulate the existence of X, where $L \subset X \subset S$ with the consequence:

$$\text{if } x, x \longrightarrow y \in X, \text{ then } y \in X.$$

We must show that X is an ideal in the field K_T and subsequently verify that X is a deductive system of T.

For all elements $x, y \in S$, we have $x \longrightarrow \left[y \longrightarrow \overline{(x \longrightarrow \bar{y})}\right] \in L$ and, because of $L \subset X \subset S$, also $x \longrightarrow \left[\bar{y} \longrightarrow \overline{(x \longrightarrow \bar{y})}\right] \in X$.

This result together with (6.1) and the modus ponens rule leads to the implication:

(i) if $x, y \in X$, then $x \vee_T y \in X$.

It is trivial to state that $x \rightarrow y \in L$ if $x \leftrightarrow y \in L$ and $x, y \in S$; but applying (6.1) and modus ponens again, we get

(ii) if $x \in X$ and $x \sim_T y$, then $y \in X$.

Finally, $x \rightarrow \overline{\overline{\bar{x} \rightarrow \bar{\bar{y}}}} \in L$ for any $x, y \in S$, which, in the light of (6.1) and Definitions 6.7(1), may be put in the form $x \rightarrow x \cdot y - L$. This, the modus ponens and the relations $L \subset X \subset S$ lead to

(iii) if $x \in X$ and $y \in S$, then $x \cdot y \in X$.

(i), (ii) and (iii) fulfill all conditions of Definitions 6.7(2); hence, the set X is an ideal in the field K_T.

Starting from this result, we now have to prove that it entails X being a deductive system. First of all, since S is common to T and K_T, we immediately have

(iv) $X \subset S$.

Secondly, if $x, y \in S$, then $x \longrightarrow \left[y \longrightarrow (x \longrightarrow y)\right] \in L$ follows; hence, $x \leftrightarrow y \in L$ for $x, y \in L$. But $x \cdot \bar{x} \in L$ for $x \in S$, so in accordance with (6.1) we get

(v) if $x \in S$ and $y \in L$, then $x \cdot \bar{x} \sim_T y$.

Since an ideal is never void, we put $x \in X$ and obtain - conform with this assumption - $x \cdot \bar{x} \in X$ which, in accord with (v) and Definitions 6.7(2), leads to:

(vi) if $y \in L$, then $y \in X$; or else $L \subset X$.

Finally, we assume $x, y \in X$ and $x, x \rightarrow y \in S$.
From (6.1) and Definitions 6.7(1) follows $\bar{x} \cdot y = \overline{(\bar{\bar{x}} \vee_T \bar{y})} =$
$= \overline{\overline{\bar{\bar{x}} \rightarrow \bar{\bar{y}}}} \sim_T x \rightarrow y$ which, according to Definitions 6.7(2), yields $x \cdot \bar{y} \in X$. From $x \in X$ we get $x \cdot y \in X$, and hence $x \cdot y \vee_T \bar{x} \cdot y \in X$. But $x \cdot y \vee_T \bar{x} \cdot y \sim_T y$, and Definitions 6.7(2) yield $y \in X$. This result, together with (iv) and (vi), precludes that X is a deductive system in the sense that it contains its consequences. The relevant theory is denoted by $T_X = \{S, X, \rightarrow, ^{-}\}$.

We proceed now to prove the existence of the measurental homomorphism. Let us begin by introducing a relation R valid only when aRb holds iff a is an element of the field K_T/X, b is an element of the field K_{T_X} and $b \in a$.

The thus defined relation R satisfies the conditions of Definitions 6.7(2). Consequently, K_T/X and K_{T_X} are isomorphic fields.

We postulate next that, for a given X of the theory T, T_X and T_0 are of the same structural type. With a look at Theorem 6.4(d), this means that K_{T_X} and K_{T_0} are isomorphic fields; hence, K_T/X and K_{T_0} is the third isomorphic pairing. It is reminiscent of tha isomorphism (4.11).

Moreover, Stone (1936b) has proved that K_T/X is homomorphic to K_T; therefore, K_T and K_{T_0} are homomorphic fields. It is quite easy to prove the converse statement, so we consider the proof as completed.

Let us now investigate - again by means of the calculus of deductive systems - the much more difficult case of complex fuzzy systems exhibiting non-complementation and uncertainty behaviour.

6.3 Measurement of systems distinguished by a non-complementation condition

The calculus of systems accepts both finite and infinite operations of sum and product and is based on seven (respectively ten) postulates; outstanding from our point of view is Postulate VII(d) which states:

$$(X, Y \in \mathfrak{S} \text{ and } X \cdot Y = L) \longrightarrow (Y \subset \overline{X}). \quad (6.2)$$

Our first step will be to show that this statement (axiom) is equivalent to the non-complementation condition:

$$\bigwedge_{A \in 2^Y} \left[\overline{Y-A} \cap \overline{A} \supseteq H \right], \quad (3.4)$$

derived in the language of Brouwer lattices in Chapter 3. This task will be accomplished by a successive substitution of symbols in such a way that **Y** (a deductive system in (6.2)) will not be confused with Y (a space in (3.4)).

(a) With $X \cdot Y \equiv X \cap Y$ and $\overline{X} = S - X$, (6.2) states:

$$(X, Y \in \mathfrak{S} \text{ and } X \cap Y = L) \longrightarrow (Y \subset S - X). \quad (6.3)$$

(b) Now, $Y \subset S-X$, but we take $Y = S-X$; then, (6.4)

$$(X, S-X \in \mathfrak{S} \text{ and } X \cap (S-X) \supseteq L) \longrightarrow (S-X = S-X) = \Pi.$$

This confirms that the system on the left is a true deductive system (for the meaning of L see Theorem 6.2).

(c) Finally, put: $X \equiv A = \overline{A}$, $S \equiv Y$, $S-X \equiv \overline{Y-A}$,

$\mathfrak{S} \equiv 2^Y$, $L \equiv H$ and obtain:

$(\overline{A}, \overline{Y-A} \in 2^Y$ and $\overline{Y-A} \cap \overline{A} \supseteq H)$ is a true statement. (6.5)

(6.5) is a true statement in Brouwerian lattices; it is precisely identical with the non-complementation condition (3.4). Note that deductive systems X and S-X correspond to closed sets $\overline{Y-A}$ and $\overline{A}$ in Brouwerian lattices, as they should.

We have thus proved the following

Theorem 6.5

The non-complementation condition for complex fuzzy systems coincides with the modified Postulate VII(d) of the calculus of deductive systems.

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It is easily verified that the law of contradiction (Postulate VII(b) and the other six postulates are correspondingly satisfied in a Brouwerian lattice with unity.

Corollary 6.6

The calculus of deductive systems and the Brouwerian lattice are logically equivalent.

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We conclude that the calculus of systems is the true logical foundation for complex fuzzy systems, comparable to the Birkhoff-v.Neumann logic on which quantum mechanics rests. Both are, of course, nonclassical logics and both are imbedded in ∞-dimensional (Banach resp. Hilbert) spaces.

<u>The reduction of complex fuzzy systems by abstraction - modulo uncertainty</u>

In the set-algebra, the following statements are known to be true:

(i) The symmetrical difference "$\triangle$" between two subsets A,B of a given set X is defined by the formula:

$$A \triangle B = (A-B) \cup (B-A). \tag{6.5}$$

(ii) An ideal I is a nonvoid hereditary and additive family of subsets of X, i.e. such that

$$(A \in I) \wedge (B \subset A) \quad (B \in I) \text{ and} \tag{6.6}$$

$$(A \in I) \wedge (B \in I) \quad (A \cup B \in I). \tag{6.7}$$

It contains the zero element in virtue of (6.6), but - if proper - not the unity. Hence, as may be shown by applying the axiom of choice, every ideal is contained in a maximal ideal. Moreover, an ideal I in which, for an arbitrary $C \subset X$, exactly one of the conditions:

$$C \in I, \quad X-C \in I \tag{6.8}$$

applies, is called a prime ideal.

(iii) We say that two subsets $A,B \subset X$ are congruent modulo I and we write $A \sim B(\text{mod}I)$, whenever $A \triangle B \in I$, (6.9)
which applies iff $A = (B-P) \cup Q, \quad P,Q \in I$. (6.10)
We may then w ite $A \triangleq B(\text{mod}I)$ or, for I fixed, (6.11)
simply $A \triangleq B$. "$\triangleq$" is easily shown to be an equivalence relation with field $X \neq \emptyset$.

(iv) "$\triangleq$" induces a decomposition D in X such that the sets, called abstraction classes of the family D, are nonvoid and $U(D) = X$.
The abstraction class containing element $x \in X$ is denoted by $x/\triangleq$, while the family D itself is equal to $X/\triangleq$, and is called the quotient of the set X by relation $\triangleq$.

On the basis of (6.9), we may use I in lieu of $\triangleq$, and denote the quotient set, just defined, by X/I.

(v) The canonical (quotient) map k: $X \longrightarrow X/I$, defined by

$$k(a) = \{b \in B:\ A \triangleq B\} \in X/I, \qquad (6.12)$$

effects a reduction in power down to that of a countable set if the abstraction is followed by the application of the axiom of choice.

Now, it is a measuremental necessity to ensure that the canonical mapping transfers any order relations and operations correctly. Let $\succ$ and $f \in X^{X \times X}$ be the order relation and operation (represented by function f) in X, respectively. To prove the compatibility for orderings $(x \succ y) \Longrightarrow (x/\sim\ > y/\sim)$ is rather easy and will be omitted here. For functions we have

Definition 6.9 (Kuratowski and Mostowski (1978)p.89)
A function f is called compatible with equivalence relation $\sim$ if

$$\bigwedge_{x,x_1,y,y_1} \left[(x \sim x_1) \wedge (y \sim y_1) \Longrightarrow (f(x,y) \sim f(x_1,y_1))\right]. \qquad (6.13)$$

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From the equivalence $x \sim y \equiv x \in y/\sim\ \equiv x/\sim\ = y/\sim$ follows that (6.13) may be expressed in the form:

if $x \in x_1/\sim$ and $y \in y_1/\sim$, then $f(x,y)/\sim\ = f(x_1,y_1)/\sim$.

Thus, the abstraction class $f(x,y)/\sim$ depends on the classes $x/\sim$ and $y/\sim$, but not on the elements x,y. Hence follows that there exists a function ϕ with a set $(X/\sim)\times(X/\sim)$ of arguments satisfying, for arbitrary $x,y \in X$, the formula

$$\phi(x/\sim, y/\sim) = f(x,y)/\sim. \qquad (6.14)$$

Function ϕ is considered to be induced from f by $\sim$.

$k \in (X/\sim)^X$ is, of course, the canonical or quotient mapping of X onto $X/\sim$, defined by $k(x) = x/\sim$, $x \in X$. We also call the function k^2 of two variables, defined by $k^2(x,y) = \langle x/\sim, y/\sim \rangle$, a canonical mapping of X^2 onto $(X/\sim)^2$.

Theorem 6.7
If a function $f \in X^{X \times X}$ is compatible with $\sim$ and ϕ is induced from f by $\sim$, then the annexed diagram commutes.

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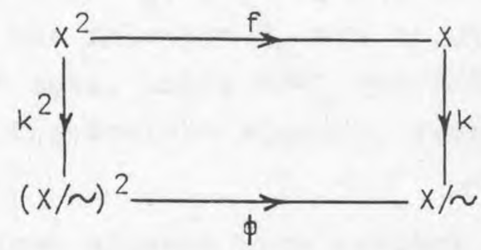

Fig. 6-1

Compatibility commutation

Proof of Theorem 6.7:

For any pair $\langle x,y\rangle \in X^2$ we have

$$k \circ f(x,y) = k(f(x,y)) = f(x,y)/\sim \quad \text{and}$$

$$\phi \circ k^2(x,y) = \phi(k^2(x,y)) = \phi(x/\sim, y/\sim) = f(x,y)/\sim.$$

Hence, $k \circ f = \phi \circ k^2$, which confirms the commutation of the diagram in Fig. 6-1.

It is now a simple matter to translate these results into the languqge of Brouwerian lattices; all we have to do is to replace the set difference "—" by a pseudo-difference "$\dot{-}$" in the formula (6.5) and the corresponding expressions accordingly. At the same time, the ideal I turns into an uncertainty ideal I_H, H being the negligible set characterising the uncertainty.

The relevant expressions are now:

$$A \triangle B = (A \dot{-} B) \cup (B \dot{-} A), \quad A \dot{-} B = \overline{A-B}, \qquad (6.15)$$

and for the equivalence relation: $\triangle(\mathrm{mod} I_H)$. (6.16)

For reasons of simplicity, the symbol in (6.16) will be replaced by R_H. As before, $\mathcal{L} = \langle L, \cup, \cap, \dot{-}\rangle$ represents the Brouwer lattice with unity.

Then, $Q = \mathcal{L}/R_H \equiv \mathcal{L}/I_H$ is the quotient lattice, (6.17)

- now a Boolean algebra;

$X = Y/H$ is the quotient space, equipped with topology. (6.18)

For reasons of triviality, we again omit the proof of the compatibility of orderings with the equivalence relation and proceed with the compatibility of functions (operations). Thus:

$$f(A,B) = A \cup B, \quad g(A,B) = A \cap B \quad \text{and} \quad h(A) = \overline{Y-A}$$

are all compatible with R_H; indeed, the functions induced from $\cup, \cap, \dot{-}$ by the relation R_H are $\vee, \wedge, -$, which operate on classes of sets, while H/R_H and Y/R_H stand for the zero and unity of the Boolean algebra, respectively.

Theorem 6.8

$Q = \mathcal{L}/I_H$ is a Boolean algebra with respect to the operations $\vee, \wedge, -$, and elements H/I_H (zero) and Y/I_H (unity). The corresponding space of measurement is Y/H; it is topological, H being its subspace.

Corollary 6.9

The crucial equality emerging from the equivalence of $A \triangleq H(\text{mod} I_H)$ and of $A \in I_H$ is $H/R_H = I_H$, where R_H is given by (6.16).

----- o -----

Extension of measurement to complex fuzzy systems

Measurement of complex fuzzy systems (usually infinitely dimensional) is even in principle impossible; i.e. there is no way of finding a measuring procedure which would associate a number to every empirical quantity. This is due to the fact that complex fuzzy systems carry fundamental uncertainties owing to which they are complex and fuzzy, where the fuzzy contribution may exceed that of complexity.

The only way to deal with such systems is to determine the uncertainty conditions involved and to eliminate them in the way they came into being from simple systems without uncertainty. Precisely this has been done in this study. The reduction in power by quotient space formation modulo uncertainty relation achieves this goal in the form of a quotient space Y/H and a quotient algebra $Q = \mathcal{L}/I_H$, respectively. Then Q contains the empirical relational system to be mapped homomorphically into a numerical relational system.

The soundness of this procedure may be confirmed topologically following Hofmann's analysis in Chapter 5; indeed, we only have to aliken Y/H with $X' = X/\varrho$ on page 114, choose the symbol Z for Y on page 115 and use the expression $X = (Y/H \times \{0\}) \cup (Z \times I)$. The existence of a measuring procedure is then ensured by Theorem 5.7.

6.4 Summary of results

The investigation of this chapter have revealed the following facts unknown heatherto.

A. In logical perspective

(α) The calculus of systems is a true logical foundation for complex fuzzy systems and any other systems relying on intuitionistic logic.

(β) It is the dual to a Brouwerian lattice with unity.

(γ) Both the calculus of systems and the Brouwerian lattice with unity imply the existence of an uncertainty relation in the process of measurement.

B. In measuremental perspective

(a) All measurement is classical measurement; it admits the simultaneous measurement of any countable number of measurands, so that the state of a real system can be objectively determined.
This is no longer possible in systems obeying non-Boolean logics and exhibiting uncertainty relations.

(b) To establish a measurement channel inspite of these difficulties, it is necessary to determine the non-complementation condition and the uncertainty relation in a suitable ∞-dimensional space. An ∞-dimensional non-reflexive Banach space is suitable.

(c) The realization of the measurement channel involves the construction of a quotient space and quotient algebra modulo uncertainty. The compatibility of relations and operations with the quotient mapping has been ensured.
It has been discovered that the equality $H/R_H = I_H$, is a basic connection making this analysis possible.

(d) The class-abstraction circumvenes the uncertainty practically without loss of information. The resulting Boolean algebra is ordered by inclusion and contains a relational system $\langle X; \succ \rangle$ which may be mapped monotonically and homeomorphically into the set of reals.

CHAPTER 7: THE SELFREPRODUCTIVE STONEAN SPACE OF MEASUREMENT

Self-reproductive continua appear naturally in the study of problems concerning ε-mappings. It turns out (see Bennett (1967)) that there are no higher dimensional compact self-reproductive sets than 1-dimensional sets.
This topic belongs actually to the topological theory of dimensions.

Assumptions

(1) Every space considered is a metric space.
(2) All simplexes are compact.
(3) A polyhedron is but the union of a finite collection of simplexes, not necessarily connected.
(4) The covering definition of dimension will be used.

Definition 7.1

If $X = (X,d)$ is a compact metric space and $\varepsilon > 0$, then every mapping f of X is said to be an ε-mapping provided that

$$(f(x) = f(y)) \Longrightarrow (d(x,y) < \varepsilon), \quad x,y \in X, \tag{7.1}$$

holds.
Since X is compact, we may put $\delta(f^{-1}f(x)) < \varepsilon$, for all $x \in X$, in lieu of the right hand side of (7.1).

Definition 7.2

A metrizable compact set M is called selfreproductive if there exists a number $\varepsilon > 0$ such that, for every ε-mapping f with domain M, the set $f(M)$ contains a subset homeomorphic to M, i.e.

$$f\colon M \cong N, \quad N \subset f(M). \tag{7.2}$$

Theorem 7.1

Every n-dimensional compact self-reproductive set M is homeomorphic to a subset of an n-dimensional polyhedron.

Proof:

There exists a finite cover $\mathcal{U}$ of M of order not higher than n such that each member of $\mathcal{U}$ has a diameter smaller than $\varepsilon/2$, where $\varepsilon > 0$.
Also, there is a mapping f into a geometric realization of the nerve of $\mathcal{U}$ (which is a polyhedron of dimension less than $n+1$) sending each member U of $\mathcal{U}$ into the star of the vertex corresponding to U. Then f is an ε-mapping, and for

each $\varepsilon > 0$, there is an ε-mapping of M into an n-dimensional polyhedron.
(For the definition of the nerve of a system of sets see Kuratowski (1966) p.318).
From Theorem 7.1 we immediately deduce an important statement relating to the space of measurement $\mathfrak{S}$.

Corollary 7.2

A 0-dimensional set is selfreproductive iff it is finite.

----- o -----

It is a natural requirement that a space of measurement be selfreproductive, and it is quite adequate if it is finite or denumerable.
Let us now recall the definition of a negligible set:

$$h: Y-H \cong Y. \tag{7.3}$$

Comparing this expression to that of (7.2), we find that negligibility and selfreproductivity are related topological properties. Consequently, a selfreproductive 0-dimensional space of measurement $\mathfrak{S}$ can accommodate an uncertainty relation anywhere within its boundaries.

7.1 The Stone space and its properties

We say that a compact Hausdorff space is totally disconnected if the family of all its clopen (closed and open) subsets forms a basis.

Definition 7.3

A compact and totally disconnected topological Hausdorff space is - in honour of its discoverer - called a Stone space $\mathfrak{S}_t$.

----- o -----

It turns out that there exist two disjoint clopen sets A and B to every pair of different points x and y of a Stone space such that $x \in A$ and $y \in B$. Moreover, the family of all clopen sets of a Stone space constitutes a field of sets since, for arbitrary sets $A, B \in K$,

$$A \cup B \in K, \quad A \cap B \in K \quad \text{and} \quad -A \in K$$

(see the corresponding expressions at the top of p. 137).
The consequence of these observations is

Lemma 7.3

A compact topological space is a Stone space $\mathfrak{S}_t$ iff, to every pair of different points $x,y \in \mathfrak{S}_t$, there exists a clopen subset $A \subset \mathfrak{S}_t$ such that $x \in A$ and $y \notin A$.

----- o -----

The proof of this lemma will be found in any textbook on Boolean algebra, e.g. Sikorski (1960).

The basis of a Stone space is the field of all clopen subsets; the notion of neighbourhood is defined thus:

Definition 7.4

For every Boolean field $K = \{A, \sim, \vee, '\}$, $\mathfrak{S}_t(K)$ is a topological space consisting of all prime ideals of K in which the notion of neighbourhood is defined as follows: If I is any prime ideal in K, then the set U of prime ideals of K is a neighbourhood of I iff $I \in U$ and if an element $x \in A$ exists such that U is identical with the set of all prime ideals in K not containing $x \in A$.

Theorem 7.4

If K is a Boolean field, then the following statements hold true:

(i) The space $\mathfrak{S}_t(K)$ is compact and totally disconnected.

(II) Every neighbourhood of an arbitrary point of $\mathfrak{S}_t(K)$ is clopen.

Corollary 7.5

From (ii) follows that $\mathfrak{S}_t(K)$ is a 0-dimensional space.

----- o -----

The condition $\dim \mathfrak{S}_t(K) = 0$ can also be obtained from Theorem 7.4(i).

It is now easy to see that:

(i) the elements of K correspond to the clopen subsets of $\mathfrak{S}_t(K)$, while the prime ideals of K correspond to the points of $\mathfrak{S}_t(K)$, and

(ii) the study of homomorphisms between Boolean fields (algebras) can be reduced to the study of continuous mappings of one Stone space into another Stone space, and vice versa; hence the conclusion: Boolean fields K_1 and K_2 are isomorphic iff the Stone spaces $\mathfrak{S}_t(K_1)$ and $\mathfrak{S}_t(K_2)$ are homeomorphic, respectively.

We denote now the class of subsets of the Cantor set $\mathcal{C}$, which are homeomorphic to $\mathfrak{S}_t(K)$, by S(K), and obtain

Lemma 7.6 (Mostowski (1937)

If K is a countable generalized Boolean field and $X \in S(K)$, then X is closed in the interval $I = [0,1]$.

Corollary 7.7

Every countable field K is isomorphic to the field of all clopen subsets of a certain closed 0-dimensional linear set.

Proof:

K is obviously isomorphic to the field of clopen subsets of $\mathfrak{S}_t(K)$ for $x \in S(K)$. From the homeomorphism of X and the sets of $\mathfrak{S}_t(K)$ follows that the field of all clopen sets in $\mathfrak{S}_t(K)$ is isomorphic to the field of all clopen sets in X. This concludes the proof.

As is known, any closed set in the unit interval has the power $\leqq \aleph_0$ or $2^{\aleph_0}$. If K is countable, then $\mathfrak{S}_t(K)$ is separable and homeomorphic to a subset of the Cantor set $\mathcal{C}$; therefore,

$$\overline{\overline{\mathfrak{S}_t(K)}} \leqq \aleph_0 \text{ or } \overline{\overline{\mathfrak{S}_t(K)}} = 2^{\aleph_0}.$$

But the set of elements of $\mathfrak{S}_t(K)$ is identical with the set of all prime ideals of K. Hence, the latter set is also either countable or of the power of the continuum. With the help of Theorem 6.4(b) follows now another result of Tarski (1936) p.289.

Theorem 7.8

The power of the set of complete systems of a deductive theory is either $\leqq \aleph_0$ or $2^{\aleph_0}$.

7.2 Isomorphism of two countable Boolean fields

We make now a step back to the statements following Corollary 7.5, but only to generalize the isomorphism between Boolean fields to the case of an uncountable set of prime ideals.

Definition 7.5

The pair of number $[\alpha(K), n(K)]$, where $\alpha(K)$ denotes the order and n(K) denotes the power of the last nonvoid derivation $\mathfrak{S}_t^{(\xi)}(K)$ of the space $\mathfrak{S}_t(K)$, is called the characteristic of the countable field K, see Kuratowski and Mostowski (1978) p.238.

----- o -----

Obviously, $0 < n(K) < \aleph_0$.

Let now two fields K and L, each with countably many prime ideals, as well as the linear countable sets $X_1 \in S(K)$ and $X_2 \in S(L)$ be given. Then, the necessary and sufficient condition for the homeomorphism of these sets are the Mazurkiewicz-Sierpinski equalities:

$$\alpha(K) = \alpha(L) \text{ and } n(K) = n(L). \qquad (7.4)$$

But the homeomorphism of these sets (or equivalently, of the spaces $\mathfrak{S}_t(K)$ and $\mathfrak{S}_t(L)$) is, according to Stone (1936a), the necessary and sufficient condition for the isomorphism of the fields K and L. Hence follows

Theorem 7.9

Two countable (generalized) Boolean fields are isomorphic iff their characteristics are identical, provided they have each a countable number of prime ideals.

Corollary 7.10

Every countable Boolean field with at most countably many prime ideals is isomorphic to the field of those sets that are clopen in a closed well ordered linear set.

Proof:

Let K be the postulated field and X a linear closed and well ordered set such that $\overline{\overline{X^{(\alpha(K))}}} = n(K)$.

The set X is, owing to (7.4), homeomorphic to every set of the class S(K), hence also to the space $\mathfrak{S}_t(K)$. From the homeomorphism between $\mathfrak{S}_t(K)$ and X follows the isomorphism claimed by Corollary 7.10.

Corollary 7.11

(i) There exist $\aleph_1$ different types of isomorphism of countable fields having at most $\aleph_0$ prime ideals.

(ii) There exist $\aleph_1$ different structural types of deductive theories having at most $\aleph_0$ complete systems.

Proof:

(i) follows from Theorem 7.9, because the set of pairs (α, n), $0 \leqq \alpha < \Omega$ and $0 \leqq n < \aleph_0$, has power $\aleph_1$.

(ii) follows from (i) and Theorems 6.3 and 6.4(d).

Corollary 7.12

Theories T and T_0 have the same structure iff they have the same characteristic pairs $(\aleph_0, n)$, $0 \leqq n < \aleph_0$.

According to Tarski (1936) p.289, the characteristic pair of any deductive theory has one of the following values:

$$(n,0),\ (n,2^{\aleph_0}),\ (\aleph_0,n),\ (\aleph_0,\aleph_0),\ (\aleph_0,2^{\aleph_0}).$$

----- o -----

Let us now consider the isomorphism of countable fields having $2^{\aleph_0}$ prime ideals. To any such fields K and L correspond the topological spaces $\mathfrak{S}_t(K)$ and $\mathfrak{S}_t(L)$. It turns out in this case that two fields K and L having $2^{\aleph_0}$ prime ideals, from which a finite number are principal ideals, are isomorphic iff they contain the same number of principal ideals. The matamathematical interpretation of this statement is

Theorem 7.13

All deductive theories with the characteristic pair $(n,2^{\aleph_0})$, $0 < n < \aleph_0$, are of the same structural type.

----- o -----

In analogy to Corollary 7.11, we get now

Theorem 7.14

(i) There exist $2^{\aleph_0}$ different types of isomorphism of countable fields having $2^{\aleph_0}$ prime ideals.

(ii) There exist $2^{\aleph_0}$ different structural types of deductive theories.

----- o -----

Of interest is now the question: What is the relationship of K and L if K has 2 o prime ideals and L is arbitrary?

Theorem 7.15

Let K be a countable field with $2^{\aleph_0}$ prime ideals and L an arbitrary countable field. Then L is homomorphic to K.

Proof:

Since $\mathfrak{S}_t(L)$ is homeomorphic to a closed subspace of the Cantor set $\mathscr{C}$, $\mathfrak{S}_t(L)$ is, in fact, homeomorphic to the complement (relative to $\mathfrak{S}_t(K)$) of an open set in $\mathfrak{S}_t(K)$. The homomorphism of K and L follows then from the theorem of Stone (1936a).

For precisely this case J. v.Neumann and Stone (1935) have shown that K contains a subfield isomorphic to L. Hence

Theorem 7.16

If K is a countable field with $2^{\aleph_0}$ prime ideals and L is an arbitrary countable field, then there exists a subfield of K isomorphic to L.

----- o -----

The following result is but a complement to Theorem 7.15 in thune with Theorem 6.4(e).

Theorem 7.17

Let T be a theory having $2^{\aleph_0}$ complete systems and T_o an arbitrary deductive theory; then there exists a system X of the theory T such that the thecries T_X and T_o are structurally identical.

7.3 The Stonean space of measurement

The best result of Section 7.2 - from the measurement-theoretical point of view - is a combination of Theorems 7.15, 7.16 and 7.17. It represents the best possible interpretation of Theorem 6.4(e).

We now present the main result of this chapter.

Theorem 7.18

Let there be a finite number (say n) of non-classical empirical systems, n uncertainty relations associated with the latter and a measuring set-up conforming to the classical theory of measurement be given. The quotient Boolean algebras of the non-classical systems exhibit n prime ideals (based on the uncertainty relation)corresponding to the points of the associated Stone spaces $\mathfrak{S}_{t1},\ldots,\mathfrak{S}_{tn}$. The Stone space associated with the numerical relational system is denoted by $\mathfrak{S}_{to}$.

Then, the necessary and sufficient topological condition for proper measurement is that the clopen sets of all spaces $\mathfrak{S}_{to}, \mathfrak{S}_{t1},\ldots,\mathfrak{S}_{tn}$ be selfreproductive, i.e. that the overall space of measurement $\mathfrak{S}$ be a Stone space. The common boundary between the sets $\mathfrak{S}_{ti}$, $i = 1,2,\ldots,n$, and $\mathfrak{S}_{to}$ is the set of points corresponding to the uncertainty relations, see Fig. 5-2.

Corollary 7.19

Theorem 7.18 remains valid when a finite number (say k) of classical empiri al systems with k Stone spaces be added to the n non-classical systems, provided that their sets are self-reproductive.

Plausibility proof:

Recall that there exists a correspondence between the prime ideals of a Boolean field (algebra) K and the elements of the associated Stone space $\mathfrak{S}_t(K)$. More precisely, if K is isomorphic to K_1, then $\mathfrak{S}_t(K)$ is homeomorphic to $\mathfrak{S}_t(K_1)$. But measuremental uniqueness (see Theorem 5.7) requires that the mapping of the space Y/H into the set of reals be monotone homeomorphic. This is ensured by the selfreproductivity property of the subsets of Y/H. Hence $\mathfrak{S} = \mathfrak{S}_t$.

The proof of Corollary 7.19 is trivial.

7.4 Summary of results

Chapter 7 is - in extenso - devoted to the properties of the Stone space because of its importance in the theory of measurement, especially in the measurement of non-classical systems. All definitions and theorems have been expressed in the language of the calculus of systems, respectively in the calculus of classes, in order to maintain a smooth transition from Chapter 6 to Chapter 7.

Selfreproductivity is a natural requirement for the sets of the space of measurement. It initiates some basic topological as well as algebraic features in empirical sets: Stone space properties and prime ideals. We learn that every countable field is isomorphic to the field of all clopen subsets of a certain closed linear 0-dimensional set, and that any closed set in the unit interval has power $\leq \aleph_0$ or $2^{\aleph_0}$.

From the measurement point of view, the following observations are of general interest:

(i) If K is an empirical field and L a numerical field, both with countably many prime ideals and with identical characteristic pairs, then they are isomorphic.

(ii) If K is a countable field with $2^{\aleph_0}$ prime ideals and L an arbitrary countable field, then L is homomorphic to K.

(iii) For a countable field K with $2^{\aleph_0}$ prime ideals and an arbitrary countable field there exists a subfield of K isomorphic to L.

A corresponding statement holds for the associated deductive theories.

Thus, models of measurement should be based on Stone space.

CHAPTER 8: CONCLUSIONS

8.1 Synthesizing review

Let us now - in retrospect - attempt a synthesizing review of what has actually been achieved in the present thesis against the background of established science. Classical physics and all related natural and technical sciences, constituting this background, represent scientific edifices based on causality, determinism, objectivity of observation and of speach, and on (the mighty) Boolean reasoning. Within the range of their validity, they are exact, consistent, complete as well as decidable; and one is always able to infer inductively from the "part" onto the "whole" and deductively from the "whole" onto the "part". The "Part and the Whole" doctrine unites all the attributes of classical disciplines and serves thus as a criterion of distinction between classical and non-classical systems and disciplines.

This thesis is concerned with the generalization of the theory of representational measurement to non-classical systems characterized by theory-specific uncertainty relations,- taking a complex fuzzy system as a representative model. It seems that classical measurement has too long been a tool exclusively serving industrial and laboratory needs, grossly neglected elsewhere and lifted to some importance by the rise of modern physics: relativity theory and quantum mechanics. Its domain used to be that of simple systems, typically countable Boolean systems imbedded in Euclidean spaces X, having sharply defined points $x \in X$ and sets $A \subset X$ with boundaries (frontières) $Fr(A) = \overline{A} \cap \overline{X-A}$ in virtue of the characteristic function and obeying, in general, the laws of the excluded middle, complementation, contradiction and distributivity. Classical representational measurement consists in a homomorphic projection of empirical relational systems into numerical relational systems; such a projection is established by the construction of a scale of measurement, where the quantity of measurement, as defined by Ellis (1966), is of prime interest. This quantity of measurement is called "observable" whenever the measurement is performed against the background of the theory involved. We speak then of two measurement stages: the preparatory measurement followed by the determinative measurement in which the measured values are compared to the theoretical predictions.

No problem arises in the measurement of simple systems, neither in single- nor in simultaneous multi-measurand measurements; for, the simultaneous measurands all have a joint distribution and the (Boolean) algebra associated with the whole system is the sum of the Boolean subalgebras associated with each measurand. Frequently two conjugate observables are taken to represent the state of a system; in simple systems these conjugates are compatible and adequate to represent the state of a system (see Varadarajan (1962)).

The principles, the algebraic, topological and measure-theoretic aspects of representational measurement are discussed at length in the Introduction, Section 0.1 and in Chapters 5,6 and 7.

Systems violating classical logic and the axioms of Boolean algebra are considered to be non-classical systems; to this category belong, in particular, complex fuzzy systems. This terminology extends to the domain of measurement; for, we have seen that measurement on them is either not possible at all (if tertium non datur fails) or simultaneous measurement is impossible because the conjugates are incompatible. This incompatibility stems from the existence of fundamental (theory-specific) uncertainties. Such uncertainties are known to exist in engineering (information and communication theories), in physics (theory of relativity and quantum mechanics) and now in measurement itself, which is the finest of all uncertainties; and it has its roots in mathematics. Professor P. Eykhoff of Eindhoven University was one of the first to notice that, without an uncertainty relation, the theory of identification is far from being exhausted by the known theorems. Indeed, we see that measuremental uncertainty occupies the highest level of all known uncertainties; and this emphasizes its applicational generality.

8.2 Contributions of this thesis to the science of measurement

A. Paths into the consequences of complexity

A1. To Kuratowski (1977) we owe the statement that sets of power $\mathfrak{a}$, where $\mathfrak{a}$ is the cardinal of countable sets, are considered to be simple sets. Hence, with regard to the continuum hypothesis, sets of power $\leqq 2^{\mathfrak{a}} = 2^{\aleph_0} = \mathfrak{C}$ ($\mathfrak{C}$ being the cardinal of the continuum, and $\mathfrak{a} < \mathfrak{C}$), are considered to be complex.

Systems, whose some or all sets are complex, are therefore called complex systems; one of their properties is the notion of infinity. Its consequences are threefold:

(i) The underlying spaces are usually ∞-dimensional Banach, Hilbert or Fréchet spaces, the non-reflexive Banach space being superior in that it contains a negligible set expandible by fuzzification.

(ii) It invalidates ∞- distributivity; for, according to Sikorski (1961), while infinite joins $\bigcup_{t\in T} A_t$ and infinite meets $\bigcap_{t\in T} A_t$ may exist, the distributivity condition: $\bigcap_{t\in T} \bigcup_{s\in S} A_{t,s} = \bigcup_{f\in S^T} \bigcap_{t\in T} A_{t,f(t)}$ no longer holds.

(iii) It discards tertium non datur: according to Tarski (1935), axiomatization of tertium non datur ends at infinity.

Through (i), complexity invokes Zadeh's principle of incompatibility which - reduced to the sentential logic of measurement - states that $\alpha \wedge (-\alpha) \neq 0$ ($(\alpha)^{-1}$ implying precision and $(-\alpha)^{-1}$ implying relevance). In terms of closed sets $A = \overline{A}$ in a Banach space Y with a negligible set H we immediately obtain the relation: $\overline{A} \cap \overline{Y-A} \neq \emptyset$, and more precisely: $\overline{A} \cap \overline{Y-A} \supseteqq H$.

This is our all important non-complementation condition which coincides with Postulate VII(d) of Tarski's calculus of (deductive) systems.

The form of the measuremental uncertainty relation follows via the celebrated theorem of Banach, Saks and Kakutani of functional analysis.

The non-complementation condition and the uncertainty relation of measurement have their exact analogues in the non-commutation of operators and the Heisenberg uncertainty relation of quantum mechanics, respectively.

A2. Based on Kuratowski's definition of simple sets and systems, it has been assumed that complex systems have their roots in simple systems and that there exists a stepwise transition ($\alpha \rightarrow 2^{\alpha}$) from the simple phase to the complex phase (see Fig. 0-6). Denoting again the simple phase by Σ_n and the complex phase by Σ_{∞}, we have concluded that there must be an equivalent step-down ($Y \rightarrow Y/H$) by class-formation modulo uncertainty which has come into being by the step-up from simple to complex systems. As is known from the work of Pao-Ming and Ying-Ming (1980a), the fuzziness of complex sets enlarges the effect of set-negligibility and the corresponding uncertainty relations.

The condition for the existence of a reduced complex system has been taken to be the exact equivalence of the step-up and the step-down between the (simple and complex) phases. If Y is a complex space, H a negligible set, $L = 2^Y$ the associated Brouwerian lattice with appropriate operations and I_H the pseudo-ideal in L, then Y/H and L/I_H are the reduced space and the reduced lattice modulo uncertainty, respectively. The reduced lattice modulo uncertainty is actually a Boolean algebra which is ordered by inclusion and, hence, contains a relational system $\langle Q; \geqq \rangle$, $Q = L/I_H$, which may be homomorphically projected into a numerical relational system.

A3. A measurement channel has thus been established between a complex system imbedded in a complex fuzzy space and a numerical system across an uncertainty-eliminating identification stage modulo uncertainty. This is always possible provided one is able to determine the corresponding uncertainty relation. This statement is actually a repetition of a Heisenberg phrase relating to quantum mechanics. In this sense, the theory developed here represents an extension of the classical representation theory of measurement covering complex fuzzy systems and shows, how the measuremental uncertainty may be overcome under complex and fuzzy conditions, with practically no loss of information.

B. Innovation claims of this thesis

This work is a comprehensive answer to the shortcomings of the classical representation theory of measurement under other than simple condition, in particular with regard to determination of the state of a complex system (simultaneous measurement under complex fuzzy conditions). It is based on the following revelations:

1. A novel representation of the complexification of systems.
2. Substantiation of the existence and the determination of negligible sets, as well as their enlargement by fuzzification.
3. Algebraic derivation of the uncertainty relation of measurement from the Banach-Saks-Kakutani theorem and the logical derivation of the non-complementation condition of measurement under complex fuzzy conditions.
4. Establishment of the correspondences in complex fuzzy cybernetics and in quantum mechanics between the uncertainty relations and between the non-complementation condition (3.4) and the non-commutativity condition (4.15).
5. Proof that the calculus of systems constitutes a true logical foundation for complex fuzzy systems; verification that the non-complementation condition (3.4) is identical with the Postulate VII(d) of the calculus of systems.
6. Equivalence of the transition steps between simple and complex phases: $$|\mathfrak{a} \rightarrow 2^{\mathfrak{a}}| \equiv |Y \rightarrow Y/H| .$$
   The right hand side may be replaced by $|L \longrightarrow L/I_H|$.
7. Confirmation that L/I_H is a Boolean algebra with zero and unity, that L/I_H is ordered by inclusion, and that it contains a relational system $\langle Q; \geqq \rangle$, where $Q = L/I_H \equiv Y/H$.
8. $\phi: Q \longrightarrow R$ is a classical homomorphism (monotone homeomorphism).
9. Verification that the space of measurement is a (topological) Stone space.
10. Engineering uncertainties in communication and information theories, being technical - not fundamental, occur already in Euclidean spaces, the quantum mechanical uncertainty, being physically fundamental, occurs in ∞-dimensional Hilbert spaces, while the measuremental uncertainty, being mathematically fundamental, occurs in ∞-dimensional non-reflexive Banach spaces. Thus,

the measuremental uncertainty is the finest and highest placed uncertainty of them all. It implies that measurement is neither of technical nor of physical, but of mathematical origin.

C. Applicational remarks

The uses of complex fuzzy systems are manifold and speedily growing in number; we hardly notice that some of them bring mankind ever closer to danger and catastrophes. This applies to the fields of nuclear engineering, of explosive and poisonous chemistry with vast pollutive disasters and of microbiological manipulations. To prevent such catastrophes, fast and simultaneously reacting monitors are imperative.

On the other hand, astronomical studies of the cosmos usually involve complex measurements. The "Australiam Telescope" to be completed this year consists of a linear array of elemental telescopes arranged in such a way that precision and relevance (significance) of the observations become a function of the relative spacings between the elemental telescopes.

We are told that physicists at the University of Bayreuth, Germany, have these days obtained temperatures below 0.1K which are unmeasurable today; they require yet a special theory of measurement, - one that overcomes extreme uncertainties.

But in general, if natural science based on measurement and observation is understood to be a discipline with the dual purpose:

(i) To disseminate knowledge about nature including man which will put man in a position to utilize wisely the forces and resources of nature to his advantage, and

(ii) to allocate to man his proper position in nature through real insight into the relations prevailing in it,

then it is believed that the ideas developed and the results obtained in this work constitute a service to natural science in this sense.

APPENDIX 1: Boolean Algebra of Attributes

The concept of an attribute (also called property or quality) of an object is so fundamental that no definition of it in more fundamental terms is possible. In this respect it is akin to the notion of a set in set theory.
Attributes are usually designated by adjectives such as blue, fluid, alive, or by generic nouns such as animal, rock, ocean. In their logical consequence, adjectives and generic nouns are equivalent; for, to say: water is liquid, is tantamount to saying: water is a liquid.
To determine attributes of objects or systems is - in all empirical disciplines - the purpose of measurement.

Attributes may be combined by conjunctions and disjunctions, and one can also construct the negative of an attribute. Moreover, attributes may be ordered by inclusion; thus, the attribute of being an ocean includes the attribute of being liquid, since every ocean is liquid.

1st Boolean Law

If junctors "and", "or", "not" are (as usual) denoted by the symbols $\vee$, $\wedge$, $'$, if letters stand for attributes and if the inclusion "$x \leqq y$" means: every x is y, then attributes form a Boolean algebra.

----- o -----

This formulation is nothing else but the reformulation of some fundamental assumptions of the classical logic in a suitable form for mathematical analysis. In this connection it should be remembered that "or" and "not" may be replaced by "and", that exactly 2^{2^n} different attributes can be formed from n given attributes by the application of "and", "or" and "not", and that the Boolean algebra implies all the identities which are true for finite sets of attributes under Boolean combination. Indeed, no false identity in any Boolean algebra can be true for attributes, since finite identities of the Boolean algebra are maximal in the sense that by addition of a single independent identity every identity becomes provable, i.e. the classical logic of attributes cannot be strenthened without yielding absurdities. It is customary to identify each attribute x with the class [x] of all objects having this attribute. Obvious-

ly then, $[x \vee y] = [x] \wedge [y]$, $[x \wedge y] = [x] \vee [y]$
and $[x'] = [x]'$.

The correspondence $x \longmapsto [x]$ is then a dual homomorphism. Conversely, to every class X of objects is associated an attribute a_X "of membership in X", because $[a_X] = X$ for all X. Consequently, two attributes are objectively equivalent if they determine the same class.
These notions have been introduced in Chapters 5 and 6 dealing with the principles of measurement.

2nd Boolean Law

The correspondence $x \longmapsto [x]$ is a dual isomorphism between the Boolean algebra of classes and the Boolean algebra of attributes, provided that objectively equivalent attributes are identified.

----- o -----

This dual isomorphism provides a transition (in both directions) from the subjective world of the mind (attributes) into the objective world of matter (see APPENDIX 3: discussion on observables). However, "the class of all objects (including the unknown today)" is - in extremo - a meaningless notion and a mathematical paradoxon under the axiom of choice of the Zermelo-Fraenkel set theory.
The identification of attributes with classes, which is accepted in the formal classical logic (2nd Boolean Law), is obviously equivalent to the assumption that the general distributive law holds, i.e. equivalent to the assumption that maximal attributes exist which are true for a single element (i.e. only for proper names). This requirement is known under the name Atomic Hypothesis. The Atomic Hypothesis should, however, be banned from typically physical aplications for reasons discussed in Section 4.1.

For completeness sake, let us quote two more Boolean Laws which are essential for the propositional calculus since Boolean algebra applies also to composite sentences (statements, theorems etc.) of the form:

The earth turns around and the moon stands still,
the earth turns around or the moon stands still,
the earth does not turn around.

Only the second of these composite sentences is true. With respect to these interpretations we have the

3rd Boolean Law

Sentences constitute a Boolean algebra.

----- o -----

Boolean algebra is being applied to sentences in accordance with the above example. In classical (2-valued) logic, all statements are either true or false, but never both, i.e. $P \wedge Q$ is true iff P and Q are true statements; $P \vee Q$ is true iff P or Q (or both) is (are) true; from P, P′ one is true, the other is false. Thus we have the

4th Boolean Law

True sentences form a (proper) dual prime ideal in the Boolean algebra of all sentences; the complementary prime ideal consists of false sentences.

----- o -----

Under these premises, the composite sentence "$P \Longrightarrow Q$" (P implies Q) has a special meaning:

"$P \Longrightarrow Q$" is true, respectively false, according as "$P' \vee Q$" is true, respectively false.

Thus, "$P' \vee P$" is a tautology.

For the equivalence "$\sim$" we obtain "$P \sim Q \equiv (P \Longrightarrow Q) \wedge (Q \Longrightarrow P)$".

These results permit certain reductions in the operations of the Boolean algebra.

Finally, we should bear in mind that the sentential calculus is consistent, complete and decidable; the predicate calculus is neither consistent, nor complete nor decidable. For further information see the classical bibliography: Birkhoff (1948), Birkhoff and Bartree (1973) and Sikorski (1960).

APPENDIX 2: Fuzzy Reasoning for Imprecisely Defined Systems

The fuzzy scene is a most appropriate place to quote Bertram Russell on logic: "All traditional logic habitually assumes that precise symbols are being employed. It is, therefore, not applicable to this terrestial life but only to an imagined celestial existence ... . Logic takes us nearer to heaven than other studies".

In this sense Norbert Wiemer (1920) claimed that mathematics is almost identical with measurement; for, both mathematics and measurement have a profound need of truth for all axiomatic premises as well as for the idealized process of inference. Hence - as a link to reality - we use such phrases as "there exists". But the reality, of which man and his thinking are part, denies us such an assurance of truth. Our knowledge gained by empirical and scientific experience will always be incomplete and far from absolute, because absolute truth can only be gained if we give up all knowledge of the complement, both in the spiritual world and in the real world. Hence, it was an altogether wise decision of Alfred Tarski (1956) to base the definition of truth on the concept of realizability.

The ideas about fuzziness, vagueness and imprecision formulated by Lofti Zadeh (1965) found much acceptance within the society of engineers, where the need for imprecise reasoning is greatest. We still accord respect, perhaps admiration to what is precise, logical and clear, and we look with disdain upon a reasoning that is fuzzy or lacking mathematical discipline. And yet, as we learn more about human cognition, we may well arrive at the realization that man's ability to handle fuzzy concepts is a major asset rather than a liability; and it is this liability, above all, which constitutes a key to the understanding of the profound difference between human intelligence - on the one hand, and machine intelligence - on the other hand. More specifically, fuzziness relates to the graduality of progression in membership of a point in a given set. The change of membership is always gradual (continuous) rather than abrupt. Only when it is extremally gradual beyond perception, do we assume that the change occurs in jumps as a result of the then prevailing uncertainty.

Let us now survey some of the principles and results of the fuzzy calculus developed so far.

Zadeh (1965) characterizes a fuzzy set (class) A in a non-void set X by a membership function (a generalized characteristic function) ω_A which associates with every point $x \in X$ a real number from the unit interval $I = [0,1]$, where the value $\omega_A(x)$, $x \in X$, is to represent the grade of the membership of x in A. Thus, $\omega_A: X \rightarrow [0,1]$.

The support of A is the set of points of X at which $\omega_A(x)$ is positive, while the cross-over point in A is an element of X whose grade of membership in A is 0.5.

A fuzzy singleton is a fuzzy set whose support is a single point of X. If A is a fuzzy singleton with support $\{x\}$, we write $A = \omega/x$; for a non-fuzzy set, the corresponding notation would be $A = 1/x$.

A fuzzy set A can be looked upon as the union of its constituent singletons. On this basis A may be represented in the form

$$A = \int_X \omega_A(x)/x, \qquad (A2.1)$$

where the integral sign stands for the union of the fuzzy singletons $\omega_A(x)/x$. If A happens to have a finite support $\{x_1, x_2, \ldots, x_n\}$, then (A2.1) may be replaced by the summation

$$A = \omega_1/x_1 + \ldots + \omega_n/x_n \qquad (A2.2)$$

or by $$A = \sum_{i=1}^{n} \omega_i/x_i, \qquad (A2.3)$$

in which ω_i, $i = 1,2,\ldots,n$, is the grade of membership of x_i in A. Note that the sign (+) in (A2.2) represents the union rather than the arithmetic sum; we may, therefore, write $X = \{x_1, x_2, \ldots, x_n\}$ in the form

$$X = x_1 + x_2 + \ldots + x_n, \qquad (A2.4)$$

but - to be quite correct - this should be written as

$$X = 1/x_1 + 1/x_2 + \ldots + 1/x_n = \sum_{i=1}^{n} 1/x_i.$$

<u>Definition A2-1</u>

A fuzzy relation R from a set X to a set Y is a fuzzy subset of the cartesian product XxY. R is thus characterized by a bivariate membership function $\omega_R(x,y)$ and is expressed by

$$R = \int_{X\times Y} \omega_R(x,y)/(x,y) \qquad (A2.5)$$

or more generally by

$$R = \int_{X_1 \times \ldots \times X_n} \omega_R(x_1,\ldots,x_n)/(x_1,\ldots,x_n). \qquad (A2.6)$$

----- o -----

Fuzzy graphs in the product XxX appear to have a wide field of applications, especially in the modelling of relations of resemblance (characterized by symmetry and reflexivity relations). Resemblance relations have many remarkable properties of eminent use in form of perception and pattern recognition.

If R is a relation from X to Y and S is a relation from Y to Z, then the composition relation RoS is defined by

$$R \circ S = \int_{X \times Y} \bigvee_y (\omega_R(x,y) \wedge \omega_S(y,z)/(x,z), \qquad (A2.7)$$

where $\vee$ and $\wedge$ denote, respectively, the max. and min. operations for $a, b \in R$:

$$\left.\begin{aligned} a \vee b &= \max(a,b) = \begin{cases} a \text{ if } a \geqq b \\ b \text{ if } a < b \end{cases} \\ a \wedge b &= \min(a,b) = \begin{cases} a \text{ if } a \leqq b \\ b \text{ if } a > b \end{cases} \end{aligned}\right\} \qquad (A2.8)$$

and $\bigvee_y$ is the supremum over the domain of y.

As for the operations on fuzzy sets, we note the complement $-A$ of A to be defined by

$$-A = \int_X (1-\omega_A(x))/x, \qquad (A2.9)$$

the union by $$A \cup B = \int_X (\omega_A(x) \vee \omega_B(x))/x, \qquad (A2.10)$$

the intersection by $$A \cap B = \int_X (\omega_A(x) \wedge \omega_B(x))/x, \qquad (A2.11)$$

the product by $$AB = \int_X \omega_A(x)\omega_B(x)/x. \qquad (A2.12)$$

From (A2.12) follow, as consequences, the formulas:

$$A^\alpha = \int_X (\omega_A(x))^\alpha/x, \qquad (A2.13)$$

and $$\alpha A = \int_X \alpha\, \omega_A(x)/x. \qquad (A2.14)$$

(A2.13) yields in particular the concepts of concentration

$$CON(A) = A^2, \text{ and of dilution}$$

$$DIL(A) = A^{0.5}.$$

A host of examples on how to use these operations appear in Zadeh (1968).

Fuzzy topology

Let X be an ordinary nonvoid set and $I = [0,1]$ the unit interval. A function $\omega: X \longrightarrow I$ or $\omega \in I^X$ will now be called a fuzzy set in X. For every $x \in \omega$, $\omega(x)$ is said to be the grade of membership of x in ω, X is the carrier of ω and $\{x \in \omega: \omega(x) > 0\}$ is the support of ω, supp ω.
Whenever ω takes only the values 0,1, the fuzzy set ω is called a crisp set in X. The crisp set with value 1 on X is denoted by X and that with value 0 on X is denoted by $\emptyset$.

The following properties of fuzzy sets are obvious:
Let I be the indexed set and $\Omega = \{\omega_\alpha: \alpha \in I\}$ a family of fuzzy sets in X. The union $\bigcup\{\omega_\alpha: \alpha \in I\} = (\bigcup\Omega)$ and the intersection $\bigcap\{\omega_\alpha: \alpha \in I\} = (\bigcap\Omega)$ are, respectively, defined by $(\bigcup\Omega)(x) = \sup\{\omega_\alpha(x): \alpha \in I\}$, $x \in X$,

and $(\bigcap\Omega)(x) = \inf\{\omega_\alpha(x): \alpha \in I\}$, $x \in X$.

The complement ω' of ω is defined by $\omega'(x) = 1 - \omega(x)$, DeMorgan's law states: $(\bigcup\{\omega_\alpha: \alpha \in I)' = \bigcap\{\omega'_\alpha: \alpha \in I\}$.

Definition A2-2

A family $\mathcal{T}$ of fuzzy sets in X is called a fuzzy topology iff (i) $\emptyset, X \in \mathcal{T}$,

(ii) $\omega \cap \pi \in \mathcal{T}$ whenever $\omega, \pi \in \mathcal{T}$, and

(iii) $\bigcup\{\omega_\alpha: \alpha \in I\} \in \mathcal{T}$, for $\omega_\alpha \in \mathcal{T}$, $\alpha \in I$.

Every member of $\mathcal{T}$ is an open fuzzy set, its complement is a closed fuzzy set.
Of two fuzzy topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ for X such that $\mathcal{T}_1 \subset \mathcal{T}_2$, we shall say that $\mathcal{T}_2$ is finer than $\mathcal{T}_1$ or that $\mathcal{T}_1$ is coarser than $\mathcal{T}_2$. The pair $(X, \mathcal{T})$ is called a fuzzy topological space.

----- o -----

The base and the subbase of $\mathcal{T}$ as well as the cover of a fuzzy set A and the neighbourhood of a fuzzy point x_λ are defined in Pao-Ming and Ying-Ming (1980a).

Definition A2-3

A fuzzy set X is said to be a fuzzy point iff it takes the value 0 for all $y \in X$ except one, say $x \in X$. If its value at x is λ $(0 < \lambda \leqq 1)$, we denote this fuzzy set by x; its support is the point x.
The fuzzy point x_λ is said to be contained in a fuzzy set

ω, $x_\lambda \in \omega$, or to belong to ω iff $\lambda \leqq \omega(x)$. Evidently, every fuzzy set ω is the union of all fuzzy points belonging to ω.

Definition A2-4

(i) A fuzzy point x_λ is said to be quasi-coincident with ω, denoted by $x_\lambda q \omega$, iff $\lambda > \omega'(x)$, i.e. $\lambda + \omega(x) > 1$.

(ii) ω is said to be quasi-coincident with π, denoted by $\omega q \pi$ iff there exists an $x \in X$ such that $\omega(x) > \pi'(x)$, i.e. $\omega(x) + \pi(x) > 1$. If this condition exists, we hold ω and π as quasi-coincident (with each other) at x. In this case neither $\omega(x)$ nor $\pi(x)$ is zero and, hence, ω and π intersect at x.

(iii) A fuzzy set ω in $(X, \mathcal{T})$ is called a Q-neighbourhood of x_λ iff there is a $\pi \in \mathcal{T}$ such that $x_\lambda q \pi \subset \omega$. The family consisting of all the neighbourhoods of x_λ is called the system of Q-neighbourhoods of x_λ.

----- o -----

The Q-neighbourhood of a fuzzy point does not generally contain the point itself (see Pao-Ming and Ying-Ming (1980a)). The fact that a set and its complement should not intersect (true for Fréchet V-spaces) is no longer generally true in the theory of fuzzy topological spaces: This is an essential difference between the Q-neighbourhood structure and the Fréchet V-space theory. This entails the

Proposition A2-1

$\omega \subset \pi$ iff ω and π' are not quasi-coincident; in particular, $x_\lambda \in \omega$ iff x_λ is not quasi-coincident with ω'. This is evident from the equivalence

$$\left[\omega(x) \leqq \pi(x)\right] \equiv \left[\omega(x) + \pi'(x) = \omega(x) + 1 - \pi(x) \leqq 1\right].$$

----- o -----

The Q-neighbourhood structure confirms the non-complementation condition in Chapter 1 whenever A is a fuzzy set in space Y; compare also Theorem A2-2 which follows.

Definitions A2-5

(i) Let ω be a fuzzy set in $(X, \mathcal{T})$. The union of all open sets contained in ω is called the interior of ω, denoted by $\mathring{\omega} = \mathrm{Int}_{\mathcal{T}}\omega$; it is the largest open set contained in ω. Obviously $(\mathring{\mathring{\omega}}) = \mathring{\omega}$.

(ii) The interpretation of all closed sets containing ω is the closure of ω, denoted by $\bar{\omega} = cl_{\mathcal{T}}\omega$; it is the smallest closed set containing ω. Obviously, $(\overline{\bar{\omega}}) = \bar{\omega}$.

Theorem A2-2

We have $e \in \overset{\circ}{\omega}$ iff the fuzzy point e has a neighbourhood contained in ω.
We have $e = x_{\lambda} \in \bar{\omega}$ iff each Q-neighbourhood of the fuzzy point e is quasi-coincident with ω.

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For the proof see Pao-Ming and Ying-Ming (1980a).
Other topological properties of fuzzy sets (not required here) are discussed in great detail in Lowen (1976), Lowen (1979) and Pao-Ming and Ying-Ming (1980b).

APPENDIX 3: Principles of Quantum Mechanics

In the spirit of modern trends, it is said that every irreducible complementary lattice that is modular and of finite rank represents a geometry iff every line in the lattice passes through at least three points. Quantum mechanical systems manifest logics which form a kind of projective geometries and which are, consequently, non-distributive using instead the weaker law of modular identity. Moreover, quantum logic has to be an orthocomplementary lattice because ordinary complementary operations in modular non-distributive lattices are not one-one operations.

If the quantum logic is taken to be the lattice of subspaces of an ∞-dimensional separable Hilbert space, then the physical quantities, or observables, are represented by self-adjoint linear operators. It is these operators which play the role of random variables in quantum mechanics, but distinct from classical probability theory, where the random variables are merely real-valued functions on the space of possible outcomes and measurable with respect to a Boolean σ-algebra of subsets, which may be conceived of as a Boolean σ-algebra of propositions. If the quantum logic of subspaces of a Hilbert space is presented as a logical σ-structure, then certain equivalence classes of the set of random variables on the components of the Boolean σ-algebra determine, in a general way, the operators on the Hilbert space. Hence, the physical quantities of the quantum theory are nothing else but the equivalence classes of the classical theory, everyone of which reflects some physical quality.

Since there is no implication operation in quantum logic comparable to the Boolean operation $a' \vee b$, use is made of an implication relation of the partial ordering type. It is, therefore, natural that partial order and orthocomplementation play the parts of implication and negation in quantum mechanics, respectively.

To be specific, let $\mathcal{L} = (L, \vee, \wedge)$ be a σ-lattice with a first element $\sqcup$, a last element $\sqcap$ and with $\perp$, the orthocomplementation $\perp: a \longmapsto a^\perp$; $a, a^\perp \in L$, which satisfies the following axioms of complementation:

(i) $(a^\perp)^\perp = a$; $a, b \in L$, (double negation)

(ii) $(a < b) \Longrightarrow (b^\perp < a^\perp)$, (contraposition)

(iii) $a \vee a^{\perp} = \sqcap$. (tertium non datur)

Definition A3-1 (Weak modularity

A σ-lattice $\mathcal{L}$ is called a logic if, in addition to axioms (i) - (iii), it also satisfies the following implication:

(iv) $(a < b;\ a, b \in L) \Longrightarrow (b = a \vee (a^{\perp} \wedge b))$.

Definition A3-2

An observable is any mapping μ_{α} sending a Borel set $E \in B(R)$ to L and satisfying the following three conditions:

(i) $\mu_{\alpha}(R) = \sqcap$,

(ii) $\mu_{\alpha}(E) \perp \mu_{\alpha}(F)$ provided $E \cap F = \emptyset$,

(iii) $\mu_{\alpha}(\sum_{i=1}^{\infty} E_i) = \bigvee_{i=1}^{\infty} \mu_{\alpha}(E_i)$ provided:

$E_i \cap E_j = \emptyset$, $i \neq j$ and $\{E_i\} \subset B(R)$,

B(R) being the Boolean algebra of the subsets of $\mathbb{R}$.

Definition A3-3

A state is any mapping p_{ϕ} from L into $\mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$ such that

(i) $p_{\phi}(\emptyset) = 0$, and

(ii) $p_{\phi}(\bigvee_{i=1}^{\infty} a_i) = \sum_{i=1}^{\infty} p_{\phi}(a_i)$, where $a_i \perp a_j$, $i \neq j$, $\{a_i\} \subset L$.

----- o -----

In general, p_{ϕ} can attain the "values" $-\infty$ and $+\infty$; in this case $p_{\phi}(a) > 0$ is said to be a positively signed state and $p_{\phi}(a) < 0$ a negatively signed state. In our restrictive terminology, a positively signed state is characterized by $p_{\phi}(\sqcap) = 1$; the reason for this (according to Bessaga (1966)) is that every ∞-dimensional Hilbert space is diffeomorphic to its unit sphere.

We are now in a position to formulate the four axioms valid both in classical mechanics and in quantum mechanics, once the following symbolism is agreed:

A stands for the set of all observables α of a physical system Σ,

Φ denotes the set of all states ϕ of Σ, and

p: $A \times \Phi \times B(R) \longrightarrow [0,1]$ is a real-valued function.

If $\alpha \in A$ and $\phi \in \Phi$ have fixed values, then $B(R) \longrightarrow [0,1]$, i.e. $E \longmapsto p(\alpha, \phi, E)$, is a probability measure on B(R).

The four announced axioms are then:

(see overleaf)

(A1) $p(\alpha,\phi,\emptyset) = 0$, $p(\alpha,\phi,R) = 1$,

$$p(\alpha,\phi,E_1,E_2,\ldots) = \sum_{i=1}^{\infty} p(\alpha,\phi,E_i),$$

where $\alpha \in A$, $\phi \in \Phi$, $E_i \in B(R)$, $E_i \cap E_j = \emptyset$, $i \neq j$.

(A2) $$\bigwedge_{\phi\in\Phi}(p(\alpha,\phi,E) = p(\alpha',\phi,E)) \Longrightarrow (\alpha = \alpha'),$$

$$\bigwedge_{\alpha\in A}(p(\alpha,\phi,E) = p(\alpha,\phi',E)) \Longrightarrow (\phi = \phi').$$

(A3) $$(\alpha_1,\alpha_2,\ldots \in A) \wedge (E_1,E_2,\ldots \in B(R) \wedge \wedge \bigwedge_{\phi\in\Phi}\left[(p(\alpha_i,\phi,E_i) + p(\alpha_j,\phi,E_j) \leqq 1,\ i \neq j\right] \Longrightarrow \Longrightarrow \bigwedge_{\phi\in\Phi} \bigvee_{\beta\in A} \bigvee_{F\in B(R)} \left[p(\beta,\phi,F) = \sum_{j=1}^{\infty} p(\alpha_j,\phi,E_j)\right].$$

(A4) $\mu: B(R) \longrightarrow \mathcal{L}$, defined below, is a σ-measure. There exists an observable $\alpha \in A$ such that $\mu = \mu_\alpha$, i.e. a unique σ-measure μ which is associated to every observable.

These axioms have the following interpretations:

(A1): $p(\alpha,\phi,E)$ is the probability that the measurement of an operator (observable α) for a system Σ in state ϕ yields a result belonging to E, where E is a member of the smallest Boolean σ-algebra of Borel sets on R.

(A2): implies that the only way to distinguish between two observables (respectively, two states) is an experimental one: two observables (resp. two states) having the same measured values for all Borel sets and all states (resp. all observables) are considered to be identical.

(A3) and (A4) are regarded as necessary postulates in the light of the cartesian product $\mathcal{E} = A \times B(R)$, whose elements (α,E), (β,F), ... are called experimental sentences. By the introduction of an equivalence relation $\approx$ in $\mathcal{E}$, defined by

$$[(\alpha,E) \approx (\beta,F)] \equiv [p(\alpha,\phi,E) = p(\beta,\phi,F)] \text{ for every } \phi \in \Phi,$$

we obtain the quotient set $\mathcal{L} = \mathcal{E}/\approx$ of equivalence classes $[(\alpha,E)]$, $[(\beta,F)]$, ... describing the qualities of the system considered.

The order of $\mathcal{L}$ is given by the equivalence:

$$([(\alpha,E)] \leqq [(\beta,F)]) \equiv (p(\alpha,\phi,E) \leqq p(\beta,\phi,F)) \text{ for all } \phi \in \Phi,$$

"$\leqq$" being the relation of partial order in $\mathcal{L}$.

The formula $[(\alpha,E)]^{\perp} = [(\alpha,R-E)]$ requires no comment; and since $p(\alpha,\phi,R-E) = 1 - p(\alpha,\phi,E)$ the mapping $\perp: \mathcal{L} \to \mathcal{L}$, i.e. $[(\alpha,E)] \longmapsto [(\alpha,E)]^{\perp}$, represents a well defined orthocomplementation in the partially ordered set $\mathcal{L}$. From now on we shall consider $\mathcal{L}$ to be the logic of the probability function. In reality, p induces in $\mathcal{E}$ a relation-implication (p-implication) defined by

$$((\alpha,E) \overset{p}{\Longrightarrow} (\beta,F)) \equiv (p(\alpha,\phi,E) \leqq p(\beta,\phi,F)),$$

which means that, in every state $\phi \in \Phi$, the truth of (β,F) is more probable than that of (α,E). The p-implication is obviously reflexive and transitive. We can use it instead of the equivalence relation in $\mathcal{E}$, since

$$((\alpha,E) \approx (\beta,F)) \equiv (p(\alpha,\phi,E) \overset{p}{\Longrightarrow} p(\beta,\phi,F))$$
$$(p(\beta,\phi,F) \overset{p}{\Longrightarrow} p(\alpha,\phi,E)).$$

Let now μ_α: $B(R) \longrightarrow \mathcal{L}$, i.e. $E \longmapsto |(\alpha,E)|$, be an $\mathcal{L}$-measure on $B(R)$ for all $\alpha \in A$ and $p_\phi : \mathcal{L} \rightarrow [0,1]$, i.e. $[(\alpha,E)] \longmapsto p(\alpha,\phi,E)$, the probability measure on $\mathcal{L}$. Then (A2) confirms that $\mu_\alpha \neq \mu_\beta$ and $\phi \neq \psi$ iff $p_\phi \neq p_\psi$ holds. Also,

$$p(\alpha,\phi,E) = p_\phi \, \mu_\alpha(E);$$

and while the family $\{\mu_\alpha\}_{\alpha \in A}$ exhausts the set $\mathcal{L}$, the family $\{p_\phi\}_{\phi\in\Phi}$ is full, i.e. $(p_\phi(a) \leqq p_\phi(b)) \Longrightarrow (a \leqq b)$ for every $\phi \in \Phi$ and all $a,b \in \mathcal{L}$. Mackey (1963) has proved that these two families define A, Φ and p completely. This shows that $p(\alpha,\phi,E)$ depends essentially on $\mathcal{L}$. Moreover, it is the logic $\mathcal{L}$ which decides whether $p(\alpha,\phi,E)$ describes a system in classical mechanics or in quantum mechanics. This probability satisfies indeed the axioms (A1) - (A3).

The elements of the cartesian product $\mathcal{E} = A \times B(R)$ have been properly named experimental sentences; a typical sentence (α,E) states: a measurement of the observed quantity α yields a value in E. Sentences belonging to the same class express jointly a physical quality. However, there are no uncertainties about the truth or falsity of (α,E) until an experiment has been carried out; the theory provides only the probability of the truth of (α,E) in state ϕ. But if we wish to speak about the logic of experimental sentences, then a logical value has to be attached to every sentence. The only reasonable solution seems to be to interpret $p(\alpha,\phi,E) \in [0,1]$ as the logical value of (α,E) in state ϕ.

If $p(\alpha,\phi,E) + p(\beta,\phi,F) \leqq 1$ holds for every $\phi \in \Phi$, then (α,E) and (β,F) are looked upon as contradictory sentences in the sense that the sum of two logical values is again a logical value in the unit interval. (α, E) and (β, F) are certainly contradictory if $E \cap F = \emptyset$. Hence, we comprehend (A3) in the following sense:

> To every sequence of pairwise contradictory sentences (α_i, E_i), $i = 1,2,\ldots$ there exists an experimental sentence (β, F) whose logical value is the sum of the logical values of the sentences (α_i, E_i).

As result we obtain $(\beta, F) \equiv [(\alpha_1, E_1) \vee (\alpha_2, E_2) \vee \ldots]$, which illustrates that without (A3) no connecting relation exists between different observables of the same system, and it would be impossible to speak about relations between different physical quantities of the same system. Thus, (A3) is indispensible.

We obviously have $-(\alpha, E) = (\alpha, R-E)$. The function $(-)$ has the properties of negation $(')$ in the set $\mathcal{E}$ since $-(\alpha, E)$ is false when (α, E) is true, and $--(\alpha, E) = (\alpha, E)$. From this we recognize that $\mathcal{E}$, equipped with this negation $(-)$ and with the p-implication defined by (A1) - (A3), forms a kind of formal logical system. The result of a logical identification of equivalent sentences of a formal theory T is a Lindenbaum-Tarski algebra of T.

Metamathematical digression

It is customary to denote a formalized theory T by the triple set: $T = (L, C, \mathcal{A}) = S(\mathcal{A})$, and a deductive system by $T_\emptyset = (L, C, \emptyset) = S(\emptyset)$.

$C(\mathcal{A})$ represents the totality of theorems of $S(\mathcal{A})$, L is the language (consisting of alphabet A, the set of terms $\mathcal{T}$ and the set of formulas $\mathcal{F}$), C is the consequence operation comprising tautologies and logical elements, and $\mathcal{A}$ is the set of theory-specific axioms. A well-formed formula α is called refutable in $S(\mathcal{A})$ if $-\alpha \in C(\mathcal{A})$, see Kaaz (1977). According to Lindenbaum and Tarski, $\{\mathcal{F}, \cup, \cap, \Longrightarrow, -\}$ is an abstract algebra with binary operations $\cup, \cap, \Longrightarrow$, and the unary operation $(-)$; it is called the algebra of formulas of the formalized language L.

If α, β are two congruent formulas in $\mathcal{F}$ and $\sim$ is the congruence relation in $\mathcal{F}$, then $\overline{\mathcal{F}} = \mathcal{F}/\sim$ is a quotient algebra (or factor algebra) of formulas in L. If $S(\mathcal{A})$ is a given consistent theory, then by $L(\mathcal{A})$ we denote the Lindenbaum-Tarski algebra of the theory $S(\mathcal{A})$; it is a factor algebra with congruence $\sim_{\mathcal{A}}$, where

$$\alpha \sim_{\mathcal{A}} \beta \text{ iff } \alpha \Longrightarrow \beta \in C(\mathcal{A}) \text{ and } \beta \Longrightarrow \alpha \in C(\mathcal{A}) \text{ for } \alpha, \beta \in \mathcal{F}.$$

$[\alpha]_{\mathcal{A}} \in \mathcal{L}(\mathcal{A})$ is determined by $\alpha \in \mathcal{F}$ and represents a class of formulas equivalent to α.

Theorem A3-1

$\mathcal{L}(\mathcal{A})$ is a Boolean algebra with unit element $e = [\alpha]_{\mathcal{A}}$, $\alpha \in C(\mathcal{A})$.

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From this theorem we conclude that, for every formula $\beta \in \mathcal{F}$, β is refutable iff $[\beta]_{\mathcal{A}} = \emptyset$ (see Rasiowa and Sikorski (1970)). This terminates our digression while remembering that every logic $\mathcal{L} = (L, \leqq, {}^{\perp})$ in $\mathcal{E}$ can be visualized as a Lindenbaum-Tarski algebra for the formalized system of experimental sentences. In this case the sentence "$(\alpha, E) \vee (\beta, F)$" has sense, i.e. it has a definite probability of truth only if (α, E) and (β, F) are mutually contradicting sentences (in the sense of (A3). In classical mechanics, sentences "$(\alpha, E) \vee (\beta, F)$" and "$(\alpha, E) \wedge (\beta, F)$" always have a physical sense; then $\mathcal{L}$ is a Boolean algebra. In quantum mechanics, these sentences are not always meaningful (e.g. due to Heisenberg uncertainty). In this case $\mathcal{L}$ is an othomodular σ-orthocomplementary partially ordered set.

Note that every observable $\alpha \in A$ is determined by the corresponding $\mathcal{L}$-measure μ_α. So far we have not demanded explicitely that there be a definite observable to every σ-measure on $B(R)$. This requirement is now imposed onto the system by (A4). Its physical significance is the following:

> Given a set of experimental sentences $Q_E \in \mathcal{E}$, one for each Borel set in R so that $E \longmapsto [Q_E]$ is a σ-measure, we are able to define an observable as a physical quantity corresponding to everyone of the experimental sentences.

(A4) makes it possible for us to study the properties of Q_E as well as the relations between the observables. Moreover, we can identify the set of observables with the set of all $\mathcal{L}$-measures.

Let now $f: R \longrightarrow R$ be a Borel function on R and $\alpha \in A$ an observable. It is not hard to show that $\mu_\alpha \circ f^{-1}$ is an $\mathcal{L}$-measure

sure. Also, (A4) implies that an observable corresponds to this measure; let it be $f(\alpha)$. We get immediately the equality

$$\mu_{f(\alpha)} = \mu_\alpha \circ f^{-1}$$

and, because of $p(f(\alpha),\phi,E) = p\,\mu_{f(\alpha)}(E)$, finally

$$p(f(\alpha),\phi,E) = p\mu_{f(\alpha)}(E) = p\,\mu_\alpha f^{-1}(E) = p(\alpha,\phi,f^{-1}(E)).$$

This result is most suitable for the construction of the observable $f(\alpha)$:

If the measurement of $f(\alpha)$ yields a result in E, then the measurement of α yields a result in $f^{-1}(E)$.

In classical mechanics one has: $\mathcal{L} = B(R^{6n})$;

if p_ϕ: $B(R^{6n}) \longrightarrow [0,1]$ is a state,

$f : R^{6n} \longrightarrow \mathbb{R}$ is an observable, and

E a Borel set on R^1,

then the probability that a measurement of α yields a result in E is equal to

$$p(\alpha,\phi,E) = p_\phi\,\mu_\alpha(E) = p_\phi\left[f^{-1}(E)\right] = \int d\,\mu_\alpha^\phi,$$

where μ^ϕ: $E \longmapsto p_\phi f^{-1}(E)$ is the Lebesgue measure on R.

In contrast to this, the logic of quantum mechanics is an orthocomplementary partially ordered set selected in such a way that the theory based on this logic generates results lying close to the experimental results. The optimal agreement is warranted by the

Quantum Mechanical Postulate

The logic for a quantum mechanical system is isomorphic to the partially ordered set L(H) of all closed subspaces of an ∞-dimensional separable Hilbert space H.

In this case $\perp$ is the orthogonal complementation of a subspace, i.e. $N^\perp = \{x \in H: \langle x,a\rangle = 0 \text{ for every } a \in N\}$, whenever $N \in L(H)$.

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The practice of quantum mechanics requires the use of the Spectral Theorem, contained in any standard textbook on quantum mechanics, for instance in Mackey (1963), v.Neumann (1955) and Varadarajan (1968).

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Observed typing errors

(P:= page, L^7:= 7th line from above, L_9:= 9th line from below)

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P5, L^1 ... from ...

P12, L_{15} ... cy .....

P14, L^{13} ... quantity q iff ...

P18, L_{18} .... $(q_1 + q_2) + q_3$, and

P23, L^9 ..., see Pfanzagl (1959),

P45, L_{20} ... (tertium non datur)

P47, L_8 ... Falmagne ...

P51, L^1 ... events ...

L_{16} ... freedom ...

P71, L_5 .... virtual ...

P82, L^2 . Also, ...

P90, L^{14} ... ran(M) = S ...

P102, L^2 ... dispersion δ of ...

$L_{14/15}$ ... famous ...

P124, L_{13} ... C(X) is one of ...

P127, L^{10} ... $\prod_{x \in X} C(\{\bar{x}\})$ ...

L_4 ... $\bar{X} = \bar{\bar{\bar{X}}}$.

P128, L_1 ... if $\sim$ degenerates ...

P131, L_{15} ... $x \longrightarrow x \cdot y \in L$.

P132, L_{15} ... the isomorphism ...

P134, L^{15} ... $\Longrightarrow$... (missing implication sign)

L^{16} ... $\Longrightarrow$... (missing implication sign)

L_{10} ... write ...

P136, L^{10} ... language ...

P140, L_7 ... a field K

P142, L_5 ... numbers ...

P144, L^{10} ... metamathematics ...

L_{14} ... $2^{\aleph_0}$ ...

P145, L^4 ... in tune ...

L_3 ... empirical ...

P146, L_5 ... countable field L ...

P163, L^2 ... (weak modularity of $\mathcal{L}$)

P165, L^{10} ... $(\ldots) \overset{P}{\Longrightarrow} (\ldots) \wedge$