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THE INDEX WITH RESPECT TO A RIGID SUBCATEGORY OF A TRIANGULATED CATEGORY

PETER JØRGENSEN AND AMIT SHAH

ABSTRACT. Palu defined the index with respect to a cluster tilting object in a suitable triangulated category, in order to better understand the Caldero–Chapoton map that exhibits the connection between cluster algebras and representation theory. We push this further by proposing an index with respect to a contravariantly finite, rigid subcategory, and we show this index behaves similarly to the classical index.

Let \mathcal{C} be a skeletally small triangulated category with split idempotents, which is thus an extriangulated category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$. Suppose \mathcal{X} is a contravariantly finite, rigid subcategory in \mathcal{C} . We define the index $\text{ind}_{\mathcal{X}}(C)$ of an object $C \in \mathcal{C}$ with respect to \mathcal{X} as the K_0 -class $[C]_{\mathcal{X}}$ in Grothendieck group $K_0(\mathcal{C}, \mathbb{E}_{\mathcal{X}}, \mathfrak{s}_{\mathcal{X}})$ of the relative extriangulated category $(\mathcal{C}, \mathbb{E}_{\mathcal{X}}, \mathfrak{s}_{\mathcal{X}})$. By analogy to the classical case, we give an additivity formula with error term for $\text{ind}_{\mathcal{X}}$ on triangles in \mathcal{C} .

In case \mathcal{X} is contained in another suitable subcategory \mathcal{T} of \mathcal{C} , there is a surjection $Q: K_0(\mathcal{C}, \mathbb{E}_{\mathcal{T}}, \mathfrak{s}_{\mathcal{T}}) \twoheadrightarrow K_0(\mathcal{C}, \mathbb{E}_{\mathcal{X}}, \mathfrak{s}_{\mathcal{X}})$. Thus, in order to describe $K_0(\mathcal{C}, \mathbb{E}_{\mathcal{X}}, \mathfrak{s}_{\mathcal{X}})$, it suffices to determine $K_0(\mathcal{C}, \mathbb{E}_{\mathcal{T}}, \mathfrak{s}_{\mathcal{T}})$ and $\text{Ker } Q$. We do this under certain assumptions.

1. INTRODUCTION

Cluster algebras were introduced by Fomin–Zelevinsky in [28] and have had an impact not only in mathematical fields, such as combinatorics and geometry, but also in seemingly more distant disciplines, especially particle physics where there is a strong connection to scattering amplitudes. The introduction of the cluster category by Buan–Marsh–Reineke–Reiten–Todorov [15] has established a substantial bond between cluster theory and representation theory. Cluster categories give “categorifications” of cluster algebras via a certain map introduced by Caldero–Chapoton [18]. In suitable contexts, Palu showed in [55] that a version proposed by Caldero–Keller in [20] of the original Caldero–Chapoton map induces a bijection between the indecomposable rigid objects in the cluster category and the cluster variables in the corresponding cluster algebra (see also [17], [19]). Results of this kind have generated vast interest in the Caldero–Chapoton map in various settings; see e.g. [18], [19], [23], [24], [25], [29], [38], [39], [45], [57], [60].

A particularly fruitful aspect of this relationship between cluster theory and representation theory has been the study of cluster tilting objects. In order to better understand the Caldero–Chapoton map, Palu defined the index with respect to a cluster tilting object in a suitable triangulated category. Although initially used as a tool, this index is important and interesting in its own right. For example:

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- Dehy–Keller [22] showed that the index induces a bijection between the set of isomorphism classes of rigid indecomposable objects reachable from a cluster tilting subcategory and the corresponding \mathbf{g} -vectors (see [47, proof of Thm. 2.10]);
- Grabowski–Pressland [31], [32] used the index to define graded cluster categories; and
- Jørgensen [44] has generalised the index to a higher homological algebra setting, using it to construct higher-dimensional tropical friezes (see also [33]).

Our initial setup is very general. Suppose \mathcal{C} is a skeletally small, idempotent complete, triangulated category with suspension functor Σ . Suppose $\mathcal{T} \subseteq \mathcal{C}$ is a (*2-*)cluster tilting subcategory (see Definition 4.7) and let $C \in \mathcal{C}$ be an object. Since \mathcal{T} is cluster tilting, there is a triangle of the form $T_1^C \rightarrow T_0^C \rightarrow C \rightarrow \Sigma T_1^C$, with $T_i^C \in \mathcal{T}$. The *index of C with respect to \mathcal{T}* , in the sense of Palu [55, Sec. 2.1], is the element $[T_0^C]_{\mathcal{T}}^{\text{sp}} - [T_1^C]_{\mathcal{T}}^{\text{sp}}$ of the split Grothendieck group $K_0^{\text{sp}}(\mathcal{T})$ of \mathcal{T} . Note that the way in which these T_i^C are found really relies on \mathcal{T} being cluster tilting, so there seems to be no immediate way to extend this theory to the more general rigid subcategory case.

Padrol–Palu–Pilaud–Plamondon recently showed in [54] that $K_0^{\text{sp}}(\mathcal{T})$ shows up in another way using the theory of *extriangulated categories* of Nakaoka–Palu [53]. Since \mathcal{C} is a triangulated category, we can view it as an extriangulated category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ (see Example 2.7). In this notation, $\mathbb{E}: \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathbf{Ab}$ is a biadditive functor to the category \mathbf{Ab} of abelian groups given by $\mathbb{E}(C, A) = \mathcal{C}(C, \Sigma A)$ on objects. Each element $\gamma \in \mathbb{E}(C, A)$ is part of a so-called *extriangle* in $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ of the form $A \rightarrow B \rightarrow C \xrightarrow{-\gamma}$, where $A \rightarrow B \rightarrow C \xrightarrow{\gamma} \Sigma A$ is a distinguished triangle in \mathcal{C} .

In order to recover $K_0^{\text{sp}}(\mathcal{T})$ as the Grothendieck group of a certain extriangulated category, we need to tweak this structure just described on \mathcal{C} using \mathcal{T} . The subcategory \mathcal{T} induces a *relative* extriangulated structure on \mathcal{C} in the sense of [36, Sec. 3.2] (see Subsection 2.4), which yields an extriangulated category $(\mathcal{C}, \mathbb{E}_{\mathcal{T}}, \mathfrak{s}_{\mathcal{T}})$, where

$$\mathbb{E}_{\mathcal{T}}(C, A) = \{ \gamma \in \mathbb{E}(C, A) \mid \gamma \circ \xi = 0 \text{ for all } \xi: T \rightarrow C \text{ with } T \in \mathcal{T} \}. \quad (1.1)$$

That is, the extriangles in $(\mathcal{C}, \mathbb{E}_{\mathcal{T}}, \mathfrak{s}_{\mathcal{T}})$ are of the form $A \rightarrow B \rightarrow C \xrightarrow{-\gamma}$, where again $A \rightarrow B \rightarrow C \xrightarrow{\gamma} \Sigma A$ is a triangle in \mathcal{C} but now where we demand that $\mathcal{C}(-, \gamma)|_{\mathcal{T}} = 0$. The *Grothendieck group of $(\mathcal{C}, \mathbb{E}_{\mathcal{T}}, \mathfrak{s}_{\mathcal{T}})$* is defined to be

$$K_0(\mathcal{C}, \mathbb{E}_{\mathcal{T}}, \mathfrak{s}_{\mathcal{T}}) = K_0^{\text{sp}}(\mathcal{C}) / \mathcal{I}_{\mathcal{T}} \quad (1.2)$$

where

$$\mathcal{I}_{\mathcal{T}} := \left\langle [A]_{\mathcal{C}}^{\text{sp}} - [B]_{\mathcal{C}}^{\text{sp}} + [C]_{\mathcal{C}}^{\text{sp}} \mid A \rightarrow B \rightarrow C \xrightarrow{-\gamma} \text{ is an extriangle in } (\mathcal{C}, \mathbb{E}_{\mathcal{T}}, \mathfrak{s}_{\mathcal{T}}) \right\rangle; \quad (1.3)$$

see [35], [61], or Definition 3.1 and Remark 3.2. For $C \in \mathcal{C}$, we denote its class in $K_0(\mathcal{C}, \mathbb{E}_{\mathcal{T}}, \mathfrak{s}_{\mathcal{T}})$ by $[C]_{\mathcal{T}}$. It then follows from [54, Prop. 4.11] that there is an isomorphism

$$K_0(\mathcal{C}, \mathbb{E}_{\mathcal{T}}, \mathfrak{s}_{\mathcal{T}}) \xrightarrow{\cong} K_0^{\text{sp}}(\mathcal{T}) \quad (1.4)$$

$$[C]_{\mathcal{T}} \mapsto [T_0^C]_{\mathcal{T}}^{\text{sp}} - [T_1^C]_{\mathcal{T}}^{\text{sp}}$$

$$[T]_{\mathcal{T}} \longleftarrow [T]_{\mathcal{T}}^{\text{sp}}$$

induced by the index of Palu. We prove an n -cluster tilting analogue of this isomorphism in Section 4; see Theorem B below.

Since the index is a powerful invariant, the isomorphism of Padrol–Palu–Pilaud–Plamondon is a striking result that demonstrates the significance of extriangulated categories: by transporting along the isomorphism (1.4), we can think of the equivalence class $[C]_{\mathcal{T}} \in K_0(\mathcal{C}, \mathbb{E}_{\mathcal{T}}, \mathfrak{s}_{\mathcal{T}})$ also as an index of C with respect to \mathcal{T} . Moreover, rather than needing to pass to the split Grothendieck group $K_0^{\text{sp}}(\mathcal{T})$ of \mathcal{T} , we can instead adjust the extriangulated structure on \mathcal{C} itself. This is something that would be impossible if we were to stay strictly in the realm of triangulated categories.

We push this perspective further, using that any subcategory can induce an extriangulated structure on \mathcal{C} . In particular, suppose \mathcal{X} is a suitable contravariantly finite, rigid subcategory (see Setup 3.4), and consider the extriangulated category $(\mathcal{C}, \mathbb{E}_{\mathcal{X}}, \mathfrak{s}_{\mathcal{X}})$ and its Grothendieck group $K_0(\mathcal{C}, \mathbb{E}_{\mathcal{X}}, \mathfrak{s}_{\mathcal{X}})$. Note that (1.1), (1.2) and (1.3) above do not use any special properties of \mathcal{T} , so $(\mathcal{C}, \mathbb{E}_{\mathcal{X}}, \mathfrak{s}_{\mathcal{X}})$, $\mathcal{I}_{\mathcal{X}}$ and $K_0(\mathcal{C}, \mathbb{E}_{\mathcal{X}}, \mathfrak{s}_{\mathcal{X}})$ are all defined similarly. For an object $C \in \mathcal{C}$ consider its equivalence class $[C]_{\mathcal{X}} \in K_0(\mathcal{C}, \mathbb{E}_{\mathcal{X}}, \mathfrak{s}_{\mathcal{X}})$. We show that this assignment behaves like the classical index with respect to a cluster tilting subcategory. Hence, we may consider $\text{ind}_{\mathcal{X}}(-) := [-]_{\mathcal{X}}$ to be the *index with respect to \mathcal{X}* ; see Definition 3.5. In particular, we show that $\text{ind}_{\mathcal{X}}$ is additive on triangles in \mathcal{C} up to an error term; see Theorem A below, or Proposition 3.12 and Theorem 3.14. We denote by $\text{mod } \mathcal{X}$ the category of all finitely presented, additive functors $\mathcal{X}^{\text{op}} \rightarrow \mathbf{Ab}$; see Subsection 2.1(iii).

Theorem A. *There is a group homomorphism $\theta_{\mathcal{X}}: K_0(\text{mod } \mathcal{X}) \rightarrow K_0(\mathcal{C}, \mathbb{E}_{\mathcal{X}}, \mathfrak{s}_{\mathcal{X}})$, such that if $A \rightarrow B \rightarrow C \xrightarrow{\gamma} \Sigma A$ is a triangle in \mathcal{C} , then*

$$\text{ind}_{\mathcal{X}}(A) - \text{ind}_{\mathcal{X}}(B) + \text{ind}_{\mathcal{X}}(C) = \theta_{\mathcal{X}}\left([\text{Im}(\mathcal{C}(-, \gamma)|_{\mathcal{X}})]\right).$$

Note that an extriangle $A \rightarrow B \rightarrow C \xrightarrow{-\gamma}$ in $(\mathcal{C}, \mathbb{E}_{\mathcal{X}}, \mathfrak{s}_{\mathcal{X}})$ satisfies $\mathcal{C}(-, \gamma)|_{\mathcal{X}} = 0$. Thus, it is a special case of Theorem A that $\text{ind}_{\mathcal{X}}$ is *additive on extriangles in $(\mathcal{C}, \mathbb{E}_{\mathcal{X}}, \mathfrak{s}_{\mathcal{X}})$* (see Definition 3.3 and Proposition 3.6).

Suppose now that \mathcal{C} is a skeletally small, k -linear (where k is a field), Hom-finite, idempotent complete, triangulated category. Note that the index with respect to \mathcal{X} induces a homomorphism $K_0^{\text{sp}}(\mathcal{C}) \rightarrow K_0(\mathcal{C}, \mathbb{E}_{\mathcal{X}}, \mathfrak{s}_{\mathcal{X}})$ of abelian groups, which we also denote by $\text{ind}_{\mathcal{X}}$. Another benefit of this index is its compatibility with containments of subcategories. Indeed, if $\mathcal{X} \subseteq \mathcal{T}$, then it follows that $\mathcal{I}_{\mathcal{T}} \subseteq \mathcal{I}_{\mathcal{X}}$. There is thus an induced

surjective homomorphism $Q: K_0(\mathcal{C}, \mathbb{E}_{\mathcal{T}}, \mathfrak{s}_{\mathcal{T}}) \twoheadrightarrow K_0(\mathcal{C}, \mathbb{E}_{\mathcal{X}}, \mathfrak{s}_{\mathcal{X}})$ of abelian groups, such that $\text{ind}_{\mathcal{X}} = Q \circ \text{ind}_{\mathcal{T}}$ as abelian group homomorphisms $K_0^{\text{sp}}(\mathcal{C}) \rightarrow K_0(\mathcal{C}, \mathbb{E}_{\mathcal{X}}, \mathfrak{s}_{\mathcal{X}})$ (see Remark 4.2). In particular,

$$K_0(\mathcal{C}, \mathbb{E}_{\mathcal{X}}, \mathfrak{s}_{\mathcal{X}}) \cong K_0(\mathcal{C}, \mathbb{E}_{\mathcal{T}}, \mathfrak{s}_{\mathcal{T}}) / \text{Ker } Q. \quad (1.5)$$

Thus, to determine $K_0(\mathcal{C}, \mathbb{E}_{\mathcal{X}}, \mathfrak{s}_{\mathcal{X}})$ reduces to giving a description of $K_0(\mathcal{C}, \mathbb{E}_{\mathcal{T}}, \mathfrak{s}_{\mathcal{T}})$ and $\text{Ker } Q$. In Section 4 we give several presentations of $\text{Ker } Q$. A general description in terms of $\theta_{\mathcal{T}}$ (see Theorem A) is given in Proposition 4.4 assuming only that \mathcal{X} and \mathcal{T} are additive, contravariantly finite, rigid subcategories closed under isomorphisms and direct summands; and a more explicit presentation using simple modules in $\text{mod } \mathcal{T}$ is given in Theorem 4.5 if \mathcal{T} is also locally bounded and admits right almost split morphisms.

If \mathcal{T} is *n-cluster tilting* (see Definition 4.7) and locally bounded, and \mathcal{C} admits a Serre functor, then Theorem 4.11 describes $\text{Ker } Q$ in terms of Auslander-Reiten $(n+2)$ -angles in \mathcal{T} . When $n=2$, and under some more mild assumptions, $\text{Ker } Q$ is isomorphic to the subgroup $N \leq K_0^{\text{sp}}(\mathcal{T})$ defined in [39, Def. 2.4] via (1.4) above; see Remark 4.13. In particular, the quotient $K_0^{\text{sp}}(\mathcal{T})/N$ was used by Holm–Jørgensen to define a *modified Caldero-Chapoton map* in [39], which generalises the original from [18] in different ways. In Remark 4.13 we also recall how

$$K_0(\mathcal{C}, \mathbb{E}_{\mathcal{X}}, \mathfrak{s}_{\mathcal{X}}) \cong K_0(\mathcal{C}, \mathbb{E}_{\mathcal{T}}, \mathfrak{s}_{\mathcal{T}}) / \text{Ker } Q \cong K_0^{\text{sp}}(\mathcal{T}) / N$$

has been used in [46] to define the *\mathcal{X} -Caldero-Chapoton map* (see [46, Def. 5.3]), which recovers the construction of Holm–Jørgensen. These generalised Caldero-Chapoton maps are, under certain conditions, generalised friezes; see [38, Def. 3.4] and [39, Thm. A]. They also recover the combinatorial generalised friezes of [11]; see also [21].

Due to the isomorphism (1.4) for a 2-cluster tilting subcategory, it is natural to ask if there is a similar isomorphism for an *n-cluster tilting* subcategory \mathcal{T} , where $n \geq 2$. We answer this positively in Section 4 (see Theorem B below), using the *triangulated index with respect to \mathcal{T}* (see [44, Def. 3.3], or Definition 4.9), which is a generalisation of Palu’s index seen above. This gives a higher analogue of [54, Prop. 4.11].

Theorem B (=Theorem 4.10). *If \mathcal{T} is n-cluster tilting, then the triangulated index $\text{index}_{\mathcal{T}}$ of [44] induces an isomorphism $\Psi: K_0(\mathcal{C}, \mathbb{E}_{\mathcal{T}}, \mathfrak{s}_{\mathcal{T}}) \xrightarrow{\cong} K_0^{\text{sp}}(\mathcal{T})$ of abelian groups.*

In the case when \mathcal{T} is locally bounded and *n-cluster tilting*, combining (1.5) and Theorem B shows that $K_0(\mathcal{C}, \mathbb{E}_{\mathcal{X}}, \mathfrak{s}_{\mathcal{X}})$ is isomorphic to a quotient of the split Grothendieck group $K_0^{\text{sp}}(\mathcal{T})$; see Remark 4.12.

This paper is organised as follows. In Section 2 we lay out any conventions we adopt and collect the main background material. In Section 3 we define the index with respect to a rigid subcategory in a triangulated category and prove Theorem A. In Section 4 we study $K_0(\mathcal{C}, \mathbb{E}_{\mathcal{X}}, \mathfrak{s}_{\mathcal{X}})$ via the canonical surjection $Q: K_0(\mathcal{C}, \mathbb{E}_{\mathcal{T}}, \mathfrak{s}_{\mathcal{T}}) \twoheadrightarrow K_0(\mathcal{C}, \mathbb{E}_{\mathcal{X}}, \mathfrak{s}_{\mathcal{X}})$. In particular, we establish Theorem B in Subsection 4.2.

2. BACKGROUND

In Subsection 2.1 we detail our general conventions and notation. In Subsections 2.2 and 2.3 we provide a summary of the theory of approximation and localisation in triangulated categories that we will need in our main sections. In Subsection 2.4 we recall the necessary details about (relative) extriangulated categories that we will use later.

2.1. Conventions and notation. For the following, suppose \mathcal{A} is an additive category.

- (i) For us, an *additive subcategory* \mathcal{B} of \mathcal{A} is a full subcategory which is closed under isomorphisms, direct sums and direct summands.
- (ii) If $\mathcal{B} \subseteq \mathcal{A}$ is a subcategory closed under finite direct sums, then $[\mathcal{B}]$ denotes the (two-sided) ideal of \mathcal{A} consisting of morphisms factoring through \mathcal{B} . In this case, $\mathcal{A}/[\mathcal{B}]$ denotes the corresponding additive quotient.

Fix a skeleton $\mathcal{A}^{\text{skel}}$ of \mathcal{A} . By $\text{indec } \mathcal{A}$ we mean the class of indecomposable objects of $\mathcal{A}^{\text{skel}}$. Moreover, if $\mathcal{B} \subseteq \mathcal{A}$ is an additive subcategory, we impose that a skeleton $\mathcal{B}^{\text{skel}}$ of \mathcal{B} is chosen inside $\mathcal{A}^{\text{skel}}$. In particular, this forces $\text{indec } \mathcal{B} \subseteq \text{indec } \mathcal{A}$.

- (iii) Suppose now that \mathcal{A} is also skeletally small. Let \mathbf{Ab} denote the category of all abelian groups. A (covariant) functor $M: \mathcal{A}^{\text{op}} \rightarrow \mathbf{Ab}$ is *finitely presented* if there is an exact sequence $\mathcal{A}(-, A) \rightarrow \mathcal{A}(-, B) \rightarrow M \rightarrow 0$ for some objects $A, B \in \mathcal{A}$; see [7, p. 155]. (Note that these functors have been called ‘coherent’ in e.g. [1] and [42]. There is a more general notion of a coherent functor (see [27, Def. B.5]), but this agrees with what we call finitely presented when, for example, \mathcal{A} is an idempotent complete triangulated category. See [27, App. B] and the references therein for more details.)

Denote by $\text{mod } \mathcal{A}$ the category of finitely presented, additive functors $\mathcal{A}^{\text{op}} \rightarrow \mathbf{Ab}$. This category is additive. If \mathcal{A} is also idempotent complete and has weak kernels (e.g. \mathcal{A} is a contravariantly finite, additive subcategory of an idempotent complete, triangulated category), then $\text{mod } \mathcal{A}$ is abelian; see e.g. [42, Def. 2.9]. See [2, Sec. III.2] for more details; see also [1, Sec. 2] and [4].

- (iv) Suppose that \mathcal{A} is a skeletally small, idempotent complete, triangulated category and that \mathcal{B} is a contravariantly finite, additive subcategory of \mathcal{A} . Then there is a cohomological, covariant functor

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{H_{\mathcal{B}}} & \text{mod } \mathcal{B} \\ A & \longmapsto & \mathcal{A}(-, A)|_{\mathcal{B}}; \end{array}$$

see [8, p. 846]. Lastly, we set

$$\mathcal{B}^{\perp_0} := \text{Ker } H_{\mathcal{B}} = \{ A \in \mathcal{A} \mid \mathcal{A}(\mathcal{B}, A) = 0 \},$$

considered as an additive subcategory of \mathcal{A} . Note that \mathcal{B}^{\perp_0} is closed under finite direct sums, so $[\mathcal{B}^{\perp_0}]$ forms an ideal of \mathcal{A} .

We typically assume our categories are skeletally small and idempotent complete in Sections 3–4. See each setup at the beginning of the relevant (sub)section.

2.2. Approximation in triangulated categories. In the general setup for our main sections, there is always an ambient triangulated category. We utilise a collection of results from approximation theory in triangulated categories, especially in Section 3. For the convenience of the reader we recall them here, following Beligiannis [8].

Setup 2.1. Let \mathcal{C} be a skeletally small, idempotent complete, triangulated category. Suppose $\mathcal{X} \subseteq \mathcal{C}$ is an additive (see Subsection 2.1(i)), contravariantly finite, rigid subcategory.

The subcategory \mathcal{X} is *rigid* in \mathcal{C} if $\text{Ext}_{\mathcal{C}}^1(\mathcal{X}, \mathcal{X}) = \mathcal{C}(\mathcal{X}, \Sigma\mathcal{X}) = 0$; see [8, Sec. 2.4]. If A is an object in \mathcal{C} , then a *right \mathcal{X} -approximation* of A is a morphism $\xi: X \rightarrow A$ in \mathcal{C} , such that $X \in \mathcal{X}$ and for any $X' \in \mathcal{X}$ the induced homomorphism $\mathcal{C}(X', \xi): \mathcal{C}(X', X) \rightarrow \mathcal{C}(X', A)$ is surjective. If for each $A \in \mathcal{C}$ there is a right \mathcal{X} -approximation $X \rightarrow A$, then \mathcal{X} is said to be *contravariantly finite*. See [6, pp. 113–114].

Suppose $\mathcal{Y}, \mathcal{Z} \subseteq \mathcal{C}$ are additive subcategories. Then $\text{add}(\mathcal{Y} * \mathcal{Z})$ denotes the closure under direct summands of

$$\mathcal{Y} * \mathcal{Z} := \{ C \in \mathcal{C} \mid \text{there exists a triangle } Y \rightarrow C \rightarrow Z \rightarrow \Sigma Y \text{ with } Y \in \mathcal{Y}, Z \in \mathcal{Z} \},$$

considered as a full subcategory of \mathcal{C} .

Remark 2.2. We note that it is frequently the case that $\mathcal{Y} * \mathcal{Z} = \text{add}(\mathcal{Y} * \mathcal{Z})$ by [42, Prop. 2.1(1)]. In particular, if \mathcal{X} is an additive, rigid subcategory of a Krull–Schmidt triangulated category \mathcal{C} , then $\mathcal{X} * \Sigma\mathcal{X}$ is closed under direct summands. Note also that we always have $\mathcal{Y}, \mathcal{Z} \subseteq \text{add}(\mathcal{Y} * \mathcal{Z})$.

Now we make several observations related to the subcategory $\mathcal{X} * \Sigma\mathcal{X}$ of \mathcal{C} .

- (i) Up to isomorphism, each $L \in \text{mod } \mathcal{X}$ has the form $H_{\mathcal{X}}(Z)$, where $Z \in \mathcal{C}$ permits a triangle

$$X_1 \xrightarrow{\xi_1} X_0 \longrightarrow Z \longrightarrow \Sigma X_1 \quad (2.1)$$

in \mathcal{C} with $X_i \in \mathcal{X}$ and, in particular, $Z \in \mathcal{X} * \Sigma\mathcal{X}$. Indeed, by definition of $\text{mod } \mathcal{X}$, the functor L fits into an exact sequence

$$\mathcal{X}(-, X_1) \xrightarrow{\Xi_1} \mathcal{X}(-, X_0) \longrightarrow L(-) \longrightarrow 0$$

in $\text{mod } \mathcal{X}$, where $X_i \in \mathcal{X}$; see Subsection 2.1(iii). By Yoneda’s Lemma, we have $\Xi_1 = \mathcal{X}(-, \xi_1)$ for some morphism $\xi_1: X_1 \rightarrow X_0$ in \mathcal{C} . Completing ξ to a distinguished triangle (2.1) and applying the cohomological functor $H_{\mathcal{X}}$ (see Subsection 2.1(iv)) yields an exact sequence

$$\mathcal{X}(-, X_1) \xrightarrow{\mathcal{X}(-, \xi_1)} \mathcal{X}(-, X_0) \longrightarrow \mathcal{C}(-, Z)|_{\mathcal{X}} \longrightarrow 0,$$

whence $L(-) \cong \mathcal{C}(-, Z)|_{\mathcal{X}} = H_{\mathcal{X}}(Z)$. This argument is well-known; see, for example, [8, Sec. 4.4], or [14, proof of Lem. 4.3].

- (ii) For each $A \in \mathcal{C}$, there is a triangle $X \xrightarrow{\xi} A \rightarrow Y \rightarrow \Sigma X$ in \mathcal{C} , such that $X \in \mathcal{X}$, $Y \in \mathcal{X}^{\perp_0}$ and ξ is a right \mathcal{X} -approximation. See [8, Sec. 2.1, Lem. 2.2(i)], also [14, Lem. 1.2].
- (iii) Given $A \in \mathcal{C}$, there is a triangle $Y \rightarrow Z \xrightarrow{\zeta} A \rightarrow \Sigma Y$ in \mathcal{C} , such that $Y \in \mathcal{X}^{\perp_0}$, and $Z \in \mathbf{add}(\mathcal{X} * \Sigma\mathcal{X})$ and ζ is a right $\mathbf{add}(\mathcal{X} * \Sigma\mathcal{X})$ -approximation. Furthermore, for any $Z' \in \mathbf{add}(\mathcal{X} * \Sigma\mathcal{X})$, the homomorphism $\mathcal{C}/[\mathcal{X}^{\perp_0}](Z', \zeta): \mathcal{C}/[\mathcal{X}^{\perp_0}](Z', Z) \rightarrow \mathcal{C}/[\mathcal{X}^{\perp_0}](Z', A)$ is bijective. See [8, Eq. (4.2), Rem. 4.3, Lem. 4.4 and its proof], also [14, Lem. 3.3].

2.3. Localisation in triangulated categories. We also make use of localisation theory in triangulated categories. Again, we follow Beligiannis [8]. The setup here is the same as for Subsection 2.2.

Setup 2.3. Let \mathcal{C} be a skeletally small, idempotent complete, triangulated category. Suppose $\mathcal{X} \subseteq \mathcal{C}$ is an additive (see Subsection 2.1(i)), contravariantly finite, rigid subcategory.

Following [8], we denote by $\underline{(-)}: \mathcal{C} \rightarrow \mathcal{C}/[\mathcal{X}^{\perp_0}]$ the canonical quotient functor. Since $\mathcal{X}^{\perp_0} = \text{Ker } H_{\mathcal{X}}$, there is a commutative diagram as follows.

$$\begin{array}{ccc}
 \mathcal{C} & \xrightarrow{\underline{(-)}} & \mathcal{C}/[\mathcal{X}^{\perp_0}] \\
 H_{\mathcal{X}} \downarrow & \swarrow \underline{H_{\mathcal{X}}} & \\
 \mathbf{mod } \mathcal{X} & &
 \end{array} \tag{2.2}$$

Beligiannis proved that the class $\mathcal{R} := \mathcal{R}_{\mathcal{X}}$ of regular morphisms in $\mathcal{C}/[\mathcal{X}^{\perp_0}]$ admits a *calculus of left and right fractions* (in the sense of Gabriel–Zisman [30, Sec. I.2]); see [8, Thm. 4.6]. Recall that a morphism is *regular* if it is simultaneously a monomorphism and an epimorphism; see [58, p. 163].

Remark 2.4. Generalising work of Buan–Marsh [13], Beligiannis showed that if \mathcal{X}^{\perp_0} is contravariantly finite in \mathcal{C} , then the additive category $\mathcal{C}/[\mathcal{X}^{\perp_0}]$ has the structure of an integral category; see [8, Prop. 4.12]. Under other assumptions, this quotient category has more structure, e.g. quasi-abelian [59, Thm. 5.5], abelian [16, Thm. 2.2], [51, Thm. 6.4]. An analogue in the context of higher homological algebra has also attracted interest [43, Thm. 0.6].

The next result will often be useful for us; it follows from [8, Lem. 4.1] (see also [13, Lem. 3.3]).

Lemma 2.5. *Let $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} \Sigma C$ be a triangle in \mathcal{C} . Then the following hold.*

- (i) $\underline{\beta} = 0$ in $\mathcal{C}/[\mathcal{X}^{\perp_0}] \iff \underline{H}_{\mathcal{X}}(\underline{\beta}) = H_{\mathcal{X}}(\beta) = 0$ in $\mathbf{mod} \mathcal{X}$.
- (ii) $\underline{\beta}$ is an epimorphism in $\mathcal{C}/[\mathcal{X}^{\perp_0}] \iff \underline{H}_{\mathcal{X}}(\underline{\beta}) = H_{\mathcal{X}}(\beta)$ is an epimorphism in $\mathbf{mod} \mathcal{X} \iff \underline{\gamma} = 0$ in $\mathcal{C}/[\mathcal{X}^{\perp_0}]$.
- (iii) $\underline{\beta}$ is a monomorphism in $\mathcal{C}/[\mathcal{X}^{\perp_0}] \iff \underline{H}_{\mathcal{X}}(\underline{\beta}) = H_{\mathcal{X}}(\beta)$ is a monomorphism in $\mathbf{mod} \mathcal{X} \iff \underline{\alpha} = 0$ in $\mathcal{C}/[\mathcal{X}^{\perp_0}]$.
- (iv) $\underline{\beta} \in \mathcal{R} \iff \underline{H}_{\mathcal{X}}(\underline{\beta}) = H_{\mathcal{X}}(\beta)$ is an isomorphism in $\mathbf{mod} \mathcal{X} \iff \underline{\alpha} = 0 = \underline{\gamma}$ in $\mathcal{C}/[\mathcal{X}^{\perp_0}]$.

Since \mathcal{R} admits a calculus of left and right fractions, there is a commutative diagram

$$\begin{array}{ccc}
 \mathcal{C}/[\mathcal{X}^{\perp_0}] & \xrightarrow{\underline{H}_{\mathcal{X}}} & \mathbf{mod} \mathcal{X} \\
 L_{\mathcal{R}} \downarrow & \nearrow \simeq & \\
 (\mathcal{C}/[\mathcal{X}^{\perp_0}])_{\mathcal{R}} & & \widetilde{H}_{\mathcal{X}}
 \end{array} \tag{2.3}$$

where $L_{\mathcal{R}}$ is the canonical localisation functor and $\widetilde{H}_{\mathcal{X}}$ is the functor induced from the fact that $\underline{H}_{\mathcal{X}}$ sends morphisms in \mathcal{R} to isomorphisms in $\mathbf{mod} \mathcal{X}$. Moreover, it follows from [8, Thm. 4.6 and its proof] that $\widetilde{H}_{\mathcal{X}}$ is an equivalence of categories (although this is not the equivalence in the statement of [8, Thm. 4.6] itself).

2.4. Extriangulated categories from triangulated categories. In the pursuit of unifying the theory of exact and triangulated categories, Nakaoka–Palu defined *extriangulated categories*; see [53, Def. 2.12]. This attempt has been very successful, with many results that hold individually for exact and triangulated categories having been generalised to the extriangulated framework; see e.g. [34], [41], [49], [50], [52], amongst many others.

In this section, we will briefly recall the definition of an extriangulated category. However, since we will not be dealing with the axioms of extriangulated categories, we do not provide all the details here. For more details we refer the reader to [53, Sec. 2]. The extriangulated categories we consider in this article arise from triangulated categories, and we explain this relationship in Example 2.7.

Definition 2.6. An *extriangulated category* is a triple $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$, satisfying the following.

- (i) \mathcal{C} is an additive category.
- (ii) $\mathbb{E}: \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathbf{Ab}$ is a biadditive functor.
- (iii) \mathfrak{s} is an assignment that, for each $A, C \in \mathcal{C}$, assigns to each *extension* $\delta \in \mathbb{E}(C, A)$ an *equivalence class* $[A \rightarrow B \rightarrow C]$ of a 3-term complex in \mathcal{C} , where two complexes $A \rightarrow B \rightarrow C$ and $A \rightarrow B' \rightarrow C$ are *equivalent* if there is an isomorphism

$b: B \rightarrow B'$ and a commutative diagram

$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & C \\ \parallel & & b \downarrow \cong & & \parallel \\ A & \longrightarrow & B' & \longrightarrow & C. \end{array}$$

(iv) \mathfrak{s} is an *additive realisation* of \mathbb{E} in the sense of [53, Defs. 2.9, 2.10].

(v) \mathcal{C} , \mathbb{E} and \mathfrak{s} satisfy the axioms in [53, Def. 2.12].

If $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is an extriangulated category with $\delta \in \mathbb{E}(C, A)$ and

$$\mathfrak{s}(\delta) = [A \longrightarrow B \longrightarrow C],$$

then we call $X^\bullet: A \rightarrow B \rightarrow C$ an *\mathfrak{s} -conflation*. Furthermore, in this case, $\langle X^\bullet, \delta \rangle$ is called an *\mathbb{E} -triangle* (or simply an *extriangle* if there is no confusion) and we usually depict this with the diagram $A \rightarrow B \rightarrow C \xrightarrow{\delta}$. See [53, Defs. 2.15, 2.19].

Each triangulated category is an example of an extriangulated category. The following explains how.

Example 2.7. Suppose \mathcal{C} is a triangulated category with suspension functor Σ . This data allows us to define a biadditive functor $\mathbb{E}: \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathbf{Ab}$ and an additive realisation \mathfrak{s} of \mathbb{E} as follows. We define \mathbb{E} on objects by

$$\mathbb{E}(C, A) := \mathcal{C}(C, \Sigma A).$$

For morphisms $\gamma: C' \rightarrow C$ and $\alpha: A \rightarrow A'$, the morphism $\mathbb{E}(\gamma, \alpha): \mathbb{E}(C, A) \rightarrow \mathbb{E}(C', A')$ is given by

$$\mathbb{E}(\gamma, \alpha)(\delta) := (\Sigma\alpha) \circ \delta \circ \gamma.$$

Now let $\delta \in \mathbb{E}(C, A)$ be an extension, i.e. $\delta: C \rightarrow \Sigma A$ is a morphism in \mathcal{C} , and so fits into a triangle of the form

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\delta} \Sigma A. \quad (2.4)$$

In order to define a realisation \mathfrak{s} of \mathbb{E} , we first need to assign an equivalence class of an 3-term complex to δ that has first term A and last term C . There is a canonical choice given a triangle (2.4): we put

$$\mathfrak{s}(\delta) = [A \xrightarrow{\alpha} B \xrightarrow{\beta} C].$$

Nakaoka–Palu showed that \mathfrak{s} defined in this way gives an additive realisation of \mathbb{E} and, moreover, $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is an extriangulated category; see [53, Prop. 3.22].

We will need the theory of relative exangulated structures in the sense of [36, Sec. 3.2]. Note that a category is extriangulated if and only if it is 1-exangulated in the sense of

Herschend–Liu–Nakaoka; see [36, Prop. 4.3]. However, since we will need relative exangulated structures solely for extriangulated categories arising from triangulated categories, we provide a short exposition only in this context.

Suppose \mathcal{C} is a triangulated category with suspension functor Σ , and denote the corresponding extriangulated category by $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$; see Example 2.7. Given a full subcategory \mathcal{D} of \mathcal{C} , we recall how one obtains a *relative* extriangulated category $(\mathcal{C}, \mathbb{E}_{\mathcal{D}}, \mathfrak{s}_{\mathcal{D}})$. First, recall from Subsection 2.1(iv) that the functor $H_{\mathcal{D}}: \mathcal{C} \rightarrow \mathbf{mod} \mathcal{D}$ is given by $H_{\mathcal{D}}(C) = \mathcal{C}(-, C)|_{\mathcal{D}}$. Define the subfunctor $\mathbb{E}_{\mathcal{D}}: \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathbf{Ab}$ of \mathbb{E} on objects as follows:

$$\begin{aligned} \mathbb{E}_{\mathcal{D}}(C, A) &:= \{ \delta \in \mathbb{E}(C, A) \mid H_{\mathcal{D}}(\delta) = 0 \} \\ &= \left\{ \delta \in \mathbb{E}(C, A) \left| \begin{array}{l} D \xrightarrow{\xi} C \xrightarrow{\delta} \Sigma A \text{ is zero for} \\ \text{all } \xi: D \rightarrow C \text{ with } D \in \mathcal{D} \end{array} \right. \right\}. \end{aligned} \quad (2.5)$$

An additive realisation of $\mathbb{E}_{\mathcal{D}}$ is given by the restriction $\mathfrak{s}_{\mathcal{D}}$ of \mathfrak{s} to $\mathbb{E}_{\mathcal{D}}$. Then $(\mathcal{C}, \mathbb{E}_{\mathcal{D}}, \mathfrak{s}_{\mathcal{D}})$ is an extriangulated category. Furthermore, in this relative extriangulated category, all objects in \mathcal{D} become $\mathbb{E}_{\mathcal{D}}$ -*projective* in the sense of [9, Exam. 5.9] (see also [53, Def. 3.23]). See [36, Def. 2.11, Prop. 3.16, Def. 3.18, Prop. 3.19].

As noted in [46, Thm. 2.12], the extriangulated category $(\mathcal{C}, \mathbb{E}_{\mathcal{D}}, \mathfrak{s}_{\mathcal{D}})$ is an *extriangulated subcategory* in the sense of [35, Def. 3.7]. In particular, the pair $(\mathbb{1}_{\mathcal{C}}, \Delta): (\mathcal{C}, \mathbb{E}_{\mathcal{D}}, \mathfrak{s}_{\mathcal{D}}) \rightarrow (\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is an extriangulated functor in the sense of [10, Def. 2.32], where $\mathbb{1}_{\mathcal{C}}$ is the identity functor on \mathcal{C} and $\Delta: \mathbb{E}_{\mathcal{D}} \Rightarrow \mathbb{E}$ is the canonical natural transformation given by inclusion.

3. THE INDEX WITH RESPECT TO A RIGID SUBCATEGORY

In this section, we prove Theorem A from Section 1. As such, K_0 -groups of skeletally small extriangulated categories play a central role here. Before we restrict to Setup 3.4, we provide some definitions in more generality. The following definition is equivalent to that of the Grothendieck group of an n -exangulated category in the sense of Haugland [35] with $n = 1$; see [26, Rem. 2.3], [46, Def. 2.18].

Definition 3.1. Suppose $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is a skeletally small extriangulated category. The *Grothendieck group* of $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is defined to be

$$K_0(\mathcal{C}, \mathbb{E}, \mathfrak{s}) := K_0^{\text{sp}}(\mathcal{C})/\mathcal{I},$$

where $\mathcal{I} := \langle [A]_{\mathcal{C}}^{\text{sp}} - [B]_{\mathcal{C}}^{\text{sp}} + [C]_{\mathcal{C}}^{\text{sp}} \mid A \rightarrow B \rightarrow C \text{ is an } \mathfrak{s}\text{-conflation in } (\mathcal{C}, \mathbb{E}, \mathfrak{s}) \rangle$.

Remark 3.2. Suppose $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is a skeletally small extriangulated category and consider its Grothendieck group $K_0(\mathcal{C}, \mathbb{E}, \mathfrak{s}) = K_0^{\text{sp}}(\mathcal{C})/\mathcal{I}$, as given in Definition 3.1. Let \mathcal{D} be a full subcategory of \mathcal{C} . Then the Grothendieck group of $(\mathcal{C}, \mathbb{E}_{\mathcal{D}}, \mathfrak{s}_{\mathcal{D}})$ is by definition $K_0(\mathcal{C}, \mathbb{E}_{\mathcal{D}}, \mathfrak{s}_{\mathcal{D}}) = K_0^{\text{sp}}(\mathcal{C})/\mathcal{I}_{\mathcal{D}}$, where

$$\mathcal{I}_{\mathcal{D}} := \langle [A]_{\mathcal{C}}^{\text{sp}} - [B]_{\mathcal{C}}^{\text{sp}} + [C]_{\mathcal{C}}^{\text{sp}} \mid A \rightarrow B \rightarrow C \text{ is an } \mathfrak{s}_{\mathcal{D}}\text{-conflation in } (\mathcal{C}, \mathbb{E}_{\mathcal{D}}, \mathfrak{s}_{\mathcal{D}}) \rangle.$$

Thus, the generators of $\mathcal{I}_{\mathcal{D}}$ arise from extriangles $A \rightarrow B \rightarrow C \xrightarrow{\delta}$ in $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ for which $H_{\mathcal{D}}(\delta) = 0$. In particular, $\mathcal{I}_{\mathcal{D}}$ is contained in \mathcal{I} .

Our first result (see Proposition 3.6 below) concerning the index is that it is *additive on extriangles* in a certain extriangulated category in the following sense.

Definition 3.3. Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category, and let $\text{obj}(\mathcal{C})$ denote the class of objects in \mathcal{C} . Let G be any abelian group. An assignment $f: \text{obj}(\mathcal{C}) \rightarrow G$ is *additive on extriangles in $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$* if, for each extriangle $A \rightarrow B \rightarrow C \xrightarrow{\delta}$, we have $f(A) - f(B) + f(C) = 0$.

The prototypical example of an additive assignment on extriangles in an extriangulated category is the length function on short exact sequences in a module category.

Setup 3.4. Let \mathcal{C} be a skeletally small, idempotent complete, triangulated category. Suppose $\mathcal{X} \subseteq \mathcal{C}$ is an additive, contravariantly finite, rigid subcategory. Recall that by an *additive* subcategory we assume closure under isomorphisms, direct sums and direct summands (see Subsection 2.1(i)).

In particular, Setup 3.4 is the same as Setups 2.1 and 2.3.

Our goal for the remainder of this section is to establish Theorem A. We begin by defining the index on \mathcal{C} with respect to \mathcal{X} (see Definition 3.5). We show this index is additive on extriangles in $(\mathcal{C}, \mathbb{E}_{\mathcal{X}}, \mathfrak{s}_{\mathcal{X}})$ (see Proposition 3.6), and additive on triangles in \mathcal{C} up to an error term (see Theorem 3.14). In order to prove this, we develop several analogues of results from [44]. We omit the proofs that are entirely similar, but state the needed modifications.

Definition 3.5. If $C \in \mathcal{C}$, then denote by $\text{ind}_{\mathcal{X}}(C) := [C]_{\mathcal{X}}$ the class of C in the Grothendieck group $K_0(\mathcal{C}, \mathbb{E}_{\mathcal{X}}, \mathfrak{s}_{\mathcal{X}})$. We call $\text{ind}_{\mathcal{X}}$ the *index with respect to \mathcal{X}* .

The index with respect to \mathcal{X} also induces a well-defined group homomorphism $K_0^{\text{sp}}(\mathcal{C}) \rightarrow K_0(\mathcal{C}, \mathbb{E}_{\mathcal{X}}, \mathfrak{s}_{\mathcal{X}})$, which we also denote by $\text{ind}_{\mathcal{X}}$ by abuse of notation.

Proposition 3.6. *If $A \rightarrow B \rightarrow C \xrightarrow{\delta} \Sigma A$ is a triangle in \mathcal{C} with $H_{\mathcal{X}}(\delta) = 0$, then $\text{ind}_{\mathcal{X}}(B) = \text{ind}_{\mathcal{X}}(A) + \text{ind}_{\mathcal{X}}(C)$. In particular, $\text{ind}_{\mathcal{X}}$ is additive on extriangles in $(\mathcal{C}, \mathbb{E}_{\mathcal{X}}, \mathfrak{s}_{\mathcal{X}})$.*

Proof. Since $H_{\mathcal{X}}(\delta) = 0$, we have that δ lies in $\mathbb{E}_{\mathcal{X}}(C, A)$ (see (2.5) in Subsection 2.4), so $\mathfrak{s}_{\mathcal{X}}(\delta) = [A \rightarrow B \rightarrow C]$. Thus,

$$\text{ind}_{\mathcal{X}}(A) - \text{ind}_{\mathcal{X}}(B) + \text{ind}_{\mathcal{X}}(C) = [A]_{\mathcal{X}} - [B]_{\mathcal{X}} + [C]_{\mathcal{X}} = 0$$

in $K_0(\mathcal{C}, \mathbb{E}_{\mathcal{X}}, \mathfrak{s}_{\mathcal{X}})$ and the first claim follows.

Recall that the extriangles $A \rightarrow B \rightarrow C \xrightarrow{\delta}$ in $(\mathcal{C}, \mathbb{E}_{\mathcal{X}}, \mathfrak{s}_{\mathcal{X}})$ satisfy $H_{\mathcal{X}}(\delta) = 0$ (see (2.5) in Subsection 2.4). Thus, the equation above shows that $\text{ind}_{\mathcal{X}}$ is indeed additive on extriangles in $(\mathcal{C}, \mathbb{E}_{\mathcal{X}}, \mathfrak{s}_{\mathcal{X}})$ in the sense of Definition 3.3. \blacksquare

We see later that $\text{ind}_{\mathcal{X}}$ is additive on triangles in \mathcal{C} up to an error term; see Theorem 3.14. How far this index is from being additive is measured by a certain abelian group homomorphism $\theta_{\mathcal{X}}$; see Proposition 3.12. For this result we need several preliminary lemmas.

Lemma 3.7. *For $C, D \in \mathcal{C}$, the following are equivalent.*

- (i) $H_{\mathcal{X}}(C) \cong H_{\mathcal{X}}(D)$ in $\text{mod } \mathcal{X}$.
- (ii) *There is a diagram $C \xrightarrow{\underline{\gamma}} A \xleftarrow{\underline{\delta}} D$ in \mathcal{C} with $\underline{\gamma}, \underline{\delta} \in \mathcal{R}$.*
- (iii) *There is a diagram $C \xleftarrow{\underline{\mu}} B \xrightarrow{\underline{\nu}} D$ in \mathcal{C} with $\underline{\mu}, \underline{\nu} \in \mathcal{R}$.*

Proof. We only show (i) \iff (ii); the proof of (i) \iff (iii) is similar.

(ii) \implies (i): This follows from Lemma 2.5(iv).

(i) \implies (ii): First, $H_{\mathcal{X}}(C) \cong H_{\mathcal{X}}(D)$ implies $\underline{H}_{\mathcal{X}}(C) \cong \underline{H}_{\mathcal{X}}(D)$ (see (2.2)). Using that $\widetilde{H}_{\mathcal{X}}: (\mathcal{C}/[\mathcal{X}^{\perp 0}])_{\mathcal{R}} \rightarrow \text{mod } \mathcal{X}$ is an equivalence (see (2.3)), we see there is an isomorphism $\varphi: L_{\mathcal{R}}(C) \rightarrow L_{\mathcal{R}}(D)$ in the localisation $(\mathcal{C}/[\mathcal{X}^{\perp 0}])_{\mathcal{R}}$. Since \mathcal{R} admits a calculus of left fractions in $\mathcal{C}/[\mathcal{X}^{\perp 0}]$, the morphism φ is represented by a roof/left fraction $C \xrightarrow{\underline{\gamma}} A \xleftarrow{\underline{\delta}} D$ with $\underline{\delta} \in \mathcal{R}$ (see [30, Sec. I.2]). In particular, $\underline{H}_{\mathcal{X}}(\underline{\delta})$ is an isomorphism in $\text{mod } \mathcal{X}$ by Lemma 2.5(iv). Thus, $\widetilde{H}_{\mathcal{X}}(\varphi) = (\underline{H}_{\mathcal{X}}(\underline{\delta}))^{-1} \underline{H}_{\mathcal{X}}(\underline{\gamma})$ implies $\underline{H}_{\mathcal{X}}(\underline{\gamma}) = \underline{H}_{\mathcal{X}}(\underline{\delta}) \widetilde{H}_{\mathcal{X}}(\varphi)$ is a composition of isomorphisms, and hence an isomorphism itself. So $\underline{\gamma} \in \mathcal{R}$ by Lemma 2.5(iv). \blacksquare

The following result is an analogue of [44, Lem. 4.2].

Lemma 3.8. *Suppose $H_{\mathcal{X}}(C) \cong H_{\mathcal{X}}(D)$ in $\text{mod } \mathcal{X}$ for some objects $C, D \in \mathcal{C}$. Then $\text{ind}_{\mathcal{X}}(C) + \text{ind}_{\mathcal{X}}(\Sigma^{-1}C) = \text{ind}_{\mathcal{X}}(D) + \text{ind}_{\mathcal{X}}(\Sigma^{-1}D)$.*

Proof. By Lemma 3.7, it is enough to prove this when there is a morphism $\psi: C \rightarrow D$ with $\underline{\psi} \in \mathcal{R}$, i.e. when $H_{\mathcal{X}}(\psi)$ is an isomorphism (see Lemma 2.5). Completing ψ to a triangle in \mathcal{C} yields $B \xrightarrow{\chi} C \xrightarrow{\psi} D \xrightarrow{\varphi} \Sigma B$. Rotating this triangle yields another triangle $\Sigma^{-1}C \xrightarrow{-\Sigma^{-1}\psi} \Sigma^{-1}D \xrightarrow{-\Sigma^{-1}\varphi} B \xrightarrow{\chi} C$. Since $H_{\mathcal{X}}(\psi)$ is an isomorphism, we have $H_{\mathcal{X}}(\varphi) = H_{\mathcal{X}}(\chi) = 0$ by Lemma 2.5. Thus, by Proposition 3.6 applied to these triangles, we see that

$$\begin{aligned} \text{ind}_{\mathcal{X}}(B) - \text{ind}_{\mathcal{X}}(C) + \text{ind}_{\mathcal{X}}(D) &= 0, \\ \text{ind}_{\mathcal{X}}(\Sigma^{-1}C) - \text{ind}_{\mathcal{X}}(\Sigma^{-1}D) + \text{ind}_{\mathcal{X}}(B) &= 0. \end{aligned}$$

Combining these gives the desired result. \blacksquare

The next lemma gives a generalisation of [38, Lem. 1.11(ii)], and it can be checked that the proof given there works for our purposes as well with the addition of the ‘‘General Case’’ we provide. However, the proof we give below is inspired by [14, Lem. 3.6, Prop. 3.7]. Buan–Marsh attribute the proof of [14, Lem. 3.6] to Yann Palu. Recall that the subcategory $\mathbf{add}(\mathcal{X} * \Sigma\mathcal{X})$ of \mathcal{C} consists of direct summands of objects Z in \mathcal{C} that admit a triangle $X_0 \rightarrow Z \rightarrow \Sigma X_1 \rightarrow \Sigma X_0$ with $X_i \in \mathcal{X}$.

Lemma 3.9. *If $Z \in \mathbf{add}(\mathcal{X} * \Sigma\mathcal{X})$ and $C \in \mathcal{C}$, then the mapping*

$$(H_{\mathcal{X}})_{Z,C}: \mathcal{C}(Z, C) \rightarrow (\mathbf{mod} \mathcal{X})(H_{\mathcal{X}}(Z), H_{\mathcal{X}}(C))$$

of Hom-sets induced by $H_{\mathcal{X}}$ is surjective.

Proof. We will show that if $Z \in \mathbf{add}(\mathcal{X} * \Sigma\mathcal{X})$ and $C \in \mathcal{C}$, then

$$(L_{\mathcal{R}} \circ \underline{(-)})_{Z,C}: \mathcal{C}(Z, C) \rightarrow (\mathcal{C}/[\mathcal{X}^{\perp 0}])_{\mathcal{R}}(Z, C)$$

is surjective. The result then follows from diagrams (2.2) and (2.3), noting that $\widetilde{H}_{\mathcal{X}}$ is an equivalence.

Special Case: $Z \in \mathcal{X} * \Sigma\mathcal{X}$, i.e. there is a triangle

$$X_1 \rightarrow X_0 \xrightarrow{\alpha} Z \xrightarrow{\beta} \Sigma X_1 \quad (3.1)$$

in \mathcal{C} with $X_i \in \mathcal{X}$.

A morphism in $(\mathcal{C}/[\mathcal{X}^{\perp 0}])_{\mathcal{R}}(Z, C)$ is given by a left fraction $Z \xrightarrow{\underline{\gamma}} A \xleftarrow{\underline{\delta}} C$ with $\underline{\delta} \in \mathcal{R}$. We claim that the morphism $\underline{\gamma}$ factors through $\underline{\delta}$ (cf. [14, Lem. 3.6]). As $Z \in \mathcal{X} * \Sigma\mathcal{X}$, there is a triangle (3.1) in \mathcal{C} with $X_i \in \mathcal{X}$. There is also a triangle $B \xrightarrow{\varepsilon} C \xrightarrow{\delta} A \xrightarrow{\zeta} \Sigma B$, where $\varepsilon, \zeta \in [\mathcal{X}^{\perp 0}]$ by Lemma 2.5 as $\underline{\delta} \in \mathcal{R}$. Since X_0 lies in \mathcal{X} and ζ lies in the ideal $[\mathcal{X}^{\perp 0}]$, the composition $\zeta\gamma\alpha$ is zero. Hence, there is a morphism of triangles (given by the solid arrows in the following diagram)

$$\begin{array}{ccccccc} X_1 & \longrightarrow & X_0 & \xrightarrow{\alpha} & Z & \xrightarrow{\beta} & \Sigma X_1 \\ \downarrow \eta & & \downarrow & \nearrow \varepsilon_1 & \downarrow \gamma & \nearrow \varepsilon_2 & \downarrow \Sigma\eta \\ B & \xrightarrow{\varepsilon} & C & \xrightarrow{\delta} & A & \xrightarrow{\zeta} & \Sigma B \end{array}$$

using the axioms of a triangulated category (see [37, (TR4)]). The morphism $\varepsilon\eta$ also vanishes as $X_1 \in \mathcal{X}$ and $\varepsilon \in [\mathcal{X}^{\perp 0}]$. Since $X_1 \in \mathcal{X}$ and $\varepsilon \in [\mathcal{X}^{\perp 0}]$, we have that $(\Sigma\varepsilon)(\Sigma\eta) = 0$ and so there is a morphism $\varepsilon_2: \Sigma X_1 \rightarrow A$ such that $\zeta\varepsilon_2 = \Sigma\eta$. Furthermore, the vanishing of $\zeta(\gamma - \varepsilon_2\beta)$ implies the existence of a morphism $\varepsilon_1: Z \rightarrow C$ with $\delta\varepsilon_1 = \gamma - \varepsilon_2\beta$, or $\gamma = \delta\varepsilon_1 + \varepsilon_2\beta$ (see the diagram just above). Note that $\underline{\beta} = 0$ in $\mathcal{C}/[\mathcal{X}^{\perp 0}]$ as \mathcal{X} is rigid. Hence, we have $\underline{\gamma} = \underline{\delta\varepsilon_1}$ in $\mathcal{C}/[\mathcal{X}^{\perp 0}]$, as claimed. Lastly, we note this implies $L_{\mathcal{R}}(\underline{\varepsilon_1}) = L_{\mathcal{R}}(\underline{\delta})^{-1}L_{\mathcal{R}}(\underline{\gamma})$, finishing the special case.

General Case: $Z \in \mathbf{add}(\mathcal{X} * \Sigma\mathcal{X})$.

In this case, there is an object $Z' \in \mathcal{C}$ and a triangle $X_1 \rightarrow X_0 \rightarrow Z \oplus Z' \rightarrow \Sigma X_1$ with $X_i \in \mathcal{X}$. The canonical inclusion $\iota := \begin{pmatrix} \text{id}_Z \\ 0 \end{pmatrix}: Z \rightarrow Z \oplus Z'$ induces the following commutative diagram

$$\begin{array}{ccc} \mathcal{C}(Z \oplus Z', C) & \xrightarrow{\mathcal{C}(\iota, C)} & \mathcal{C}(Z, C) \\ \downarrow (L_{\mathcal{R}} \circ \underline{(-)})_{Z \oplus Z', C} & & \downarrow (L_{\mathcal{R}} \circ \underline{(-)})_{Z, C} \\ (\mathcal{C}/[\mathcal{X}^{\perp 0}])_{\mathcal{R}}(Z \oplus Z', C) & \xrightarrow{(\mathcal{C}/[\mathcal{X}^{\perp 0}])_{\mathcal{R}}(L_{\mathcal{R}}(\iota), C)} & (\mathcal{C}/[\mathcal{X}^{\perp 0}])_{\mathcal{R}}(Z, C). \end{array}$$

The left vertical map is surjective by the Special Case and the lower horizontal map is a split epimorphism, and hence the composition is surjective. It follows from commutativity that the right vertical map is also surjective. \blacksquare

The next result is a generalisation of [44, Lem. 2.7]. The proof given there works in our setting with minor modifications: use Section 2.2(i) and Lemma 3.9 in place of [44, Lem. 2.4]; and note $H_{\mathcal{X}}(\Sigma \mathcal{X}) = 0$ as \mathcal{X} is rigid.

Lemma 3.10. *Suppose $0 \rightarrow L \xrightarrow{\iota} M \xrightarrow{\sigma} N \rightarrow 0$ is a short exact sequence in $\text{mod } \mathcal{X}$. Then, up to isomorphism, it has the form*

$$0 \longrightarrow H_{\mathcal{X}}(A) \xrightarrow{H_{\mathcal{X}}(\alpha)} H_{\mathcal{X}}(B) \xrightarrow{H_{\mathcal{X}}(\beta)} H_{\mathcal{X}}(C) \longrightarrow 0$$

for a triangle $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} \Sigma A$ in \mathcal{C} , with $A, B, C \in \mathcal{X} * \Sigma \mathcal{X}$.

The next lemma is an analogue of [44, Lem. 4.3] and its proof is similar. Thus, we omit the proof and note that applications of [44, Lem. 2.7], [44, Lem. 4.2], and [44, Lem. 3.7] may be replaced by applications of Lemma 3.10, Lemma 3.8, and Proposition 3.6, respectively.

Lemma 3.11. *For a short exact sequence*

$$0 \longrightarrow H_{\mathcal{X}}(X) \longrightarrow H_{\mathcal{X}}(Y) \longrightarrow H_{\mathcal{X}}(Z) \longrightarrow 0 \quad (3.2)$$

in $\text{mod } \mathcal{X}$, we have

$$\text{ind}_{\mathcal{X}}(Y) + \text{ind}_{\mathcal{X}}(\Sigma^{-1}Y) = \text{ind}_{\mathcal{X}}(X) + \text{ind}_{\mathcal{X}}(\Sigma^{-1}X) + \text{ind}_{\mathcal{X}}(Z) + \text{ind}_{\mathcal{X}}(\Sigma^{-1}Z).$$

The failure of the index $\text{ind}_{\mathcal{X}}$ to be additive on triangles in \mathcal{C} can be measured by a homomorphism $\theta_{\mathcal{X}}$, which we now define.

Proposition 3.12. *There is a well-defined group homomorphism*

$$\begin{array}{ccc} K_0(\text{mod } \mathcal{X}) & \xrightarrow{\theta_{\mathcal{X}}} & K_0(\mathcal{C}, \mathbb{E}_{\mathcal{X}}, \mathfrak{s}_{\mathcal{X}}) \\ [H_{\mathcal{X}}(C)] & \longmapsto & \text{ind}_{\mathcal{X}}(C) + \text{ind}_{\mathcal{X}}(\Sigma^{-1}C). \end{array}$$

Proof. Combining Section 2.2(i) and Lemma 3.8 shows that the assignment

$$H_{\mathcal{X}}(C) \mapsto \text{ind}_{\mathcal{X}}(C) + \text{ind}_{\mathcal{X}}(\Sigma^{-1}C),$$

where $C \in \mathcal{C}$, induces a map $\text{mod } \mathcal{X} \rightarrow K_0(\mathcal{C}, \mathbb{E}_{\mathcal{X}}, \mathfrak{s}_{\mathcal{X}})$ which is constant on isomorphism classes. Furthermore, Lemma 3.11 implies that it factors through $K_0(\text{mod } \mathcal{X})$. ■

The following lemma is a corresponding version of [44, Lem. 2.6]. A similar proof can be used. Instead of [44, Lem. 2.4] and [44, Lem. 2.5(i)], use Section 2.2(i) and Lemma 3.9, and Lemma 3.9, respectively. In addition, note that a morphism in \mathcal{C} vanishes under $H_{\mathcal{X}}$ if and only if it factors through \mathcal{X}^{\perp_0} (see Lemma 2.5).

Lemma 3.13. *Given $\zeta: Z \rightarrow \Sigma A$ in \mathcal{C} with $Z \in \text{add}(\mathcal{X} * \Sigma\mathcal{X})$, there is a commutative diagram*

$$\begin{array}{ccc} Z & \xrightarrow{\zeta} & \Sigma A \\ & \searrow \sigma & \nearrow \iota \\ & & C \end{array}$$

*in \mathcal{C} with $C \in \mathcal{X} * \Sigma\mathcal{X}$, such that $H_{\mathcal{X}}(\sigma)$ is epic and $H_{\mathcal{X}}(\iota)$ is monic. Moreover, $H_{\mathcal{X}}(C)$ is isomorphic to $\text{Im } H_{\mathcal{X}}(\zeta)$ in $\text{mod } \mathcal{X}$.*

We now present the main result of this subsection: an additivity formula with error term for the index $\text{ind}_{\mathcal{X}}$ with respect to the rigid subcategory \mathcal{X} (cf. [44, Thm. 4.4]).

Theorem 3.14. *If $A \rightarrow B \rightarrow C \xrightarrow{\gamma} \Sigma A$ is a triangle in \mathcal{C} , then*

$$\text{ind}_{\mathcal{X}}(A) - \text{ind}_{\mathcal{X}}(B) + \text{ind}_{\mathcal{X}}(C) = \theta_{\mathcal{X}}([\text{Im } H_{\mathcal{X}}(\gamma)]).$$

Proof. Special Case: $C \in \text{add}(\mathcal{X} * \Sigma\mathcal{X})$.

One can argue as in the proof of the ‘‘Special case’’ of [44, Thm. 4.4], using Lemma 3.13, Proposition 3.6 and Proposition 3.12 in place of [44, Lem. 2.6], [44, Lem. 3.7] and [44, Lem. 4.1], respectively.

General Case: $C \in \mathcal{C}$ arbitrary.

By Section 2.2(iii), there is a triangle $D \rightarrow Z \xrightarrow{\zeta} C \xrightarrow{\delta} \Sigma D$, where ζ is a right $\mathbf{add}(\mathcal{X} * \Sigma \mathcal{X})$ -approximation of C . Applying the octahedral axiom (see [37, (TR5'')]) on the composition $\gamma\zeta$ produces a commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & D & \xlongequal{\quad} & D & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
A & \longrightarrow & Y & \longrightarrow & Z & \xrightarrow{\eta} & \Sigma A \\
\parallel & & \downarrow \nu & & \downarrow \zeta & & \parallel \\
A & \longrightarrow & B & \xrightarrow{\beta} & C & \xrightarrow{\gamma} & \Sigma A \\
\downarrow & & \downarrow \varepsilon & & \downarrow \delta & & \downarrow \\
0 & \longrightarrow & \Sigma D & \xlongequal{\quad} & \Sigma D & \longrightarrow & 0
\end{array}$$

in which each row and column is a triangle. Since ζ is a right $\mathbf{add}(\mathcal{X} * \Sigma \mathcal{X})$ -approximation, we have that $H_{\mathcal{X}}(\zeta)$ is an epimorphism. Hence, $H_{\mathcal{X}}(\delta)H_{\mathcal{X}}(\zeta) = H_{\mathcal{X}}(\delta\zeta) = 0$ implies $H_{\mathcal{X}}(\delta)$ is the zero map. In addition, this also gives $H_{\mathcal{X}}(\varepsilon) = H_{\mathcal{X}}(\delta)H_{\mathcal{X}}(\beta) = 0$. Thus, by Proposition 3.6, we have

$$\mathrm{ind}_{\mathcal{X}}(D) - \mathrm{ind}_{\mathcal{X}}(Z) + \mathrm{ind}_{\mathcal{X}}(C) = 0, \quad (3.3)$$

$$\mathrm{ind}_{\mathcal{X}}(D) - \mathrm{ind}_{\mathcal{X}}(Y) + \mathrm{ind}_{\mathcal{X}}(B) = 0. \quad (3.4)$$

As $Z \in \mathbf{add}(\mathcal{X} * \Sigma \mathcal{X})$, the Special Case gives

$$\mathrm{ind}_{\mathcal{X}}(A) - \mathrm{ind}_{\mathcal{X}}(Y) + \mathrm{ind}_{\mathcal{X}}(Z) = \theta_{\mathcal{X}}([\mathrm{Im} H_{\mathcal{X}}(\eta)]). \quad (3.5)$$

And the sum (3.3) – (3.4) + (3.5) is $\mathrm{ind}_{\mathcal{X}}(A) - \mathrm{ind}_{\mathcal{X}}(B) + \mathrm{ind}_{\mathcal{X}}(C) = \theta_{\mathcal{X}}([\mathrm{Im} H_{\mathcal{X}}(\eta)])$. Lastly, we observe that $H_{\mathcal{X}}(\zeta)$ being epic implies that $\mathrm{Im} H_{\mathcal{X}}(\gamma) \cong \mathrm{Im} H_{\mathcal{X}}(\eta)$ as $\gamma\zeta = \eta$, so $\mathrm{ind}_{\mathcal{X}}(A) - \mathrm{ind}_{\mathcal{X}}(B) + \mathrm{ind}_{\mathcal{X}}(C) = \theta_{\mathcal{X}}([\mathrm{Im} H_{\mathcal{X}}(\eta)]) = \theta_{\mathcal{X}}([\mathrm{Im} H_{\mathcal{X}}(\gamma)])$. ■

4. UNDERSTANDING $K_0(\mathcal{C}, \mathbb{E}_{\mathcal{X}}, \mathfrak{s}_{\mathcal{X}})$

In this section, we study the abelian group $K_0(\mathcal{C}, \mathbb{E}_{\mathcal{X}}, \mathfrak{s}_{\mathcal{X}})$. We work under the following setup, but specialise further in Subsection 4.2 in which we prove Theorem B.

Setup 4.1. Let k be a field. Suppose \mathcal{C} is a skeletally small, k -linear, Hom-finite, idempotent complete, triangulated category. Suppose $\mathcal{X} \subseteq \mathcal{T}$ are additive (see Subsection 2.1(i)), contravariantly finite subcategories, such that \mathcal{T} (and hence also \mathcal{X}) is rigid. Furthermore, we define an additive, rigid subcategory $\mathcal{Y} \subseteq \mathcal{C}$, which is closed under direct summands, by $\mathrm{indec} \mathcal{T} = \mathrm{indec} \mathcal{X} \dot{\cup} \mathrm{indec} \mathcal{Y}$.

Note that this is a special case of Setup 3.4, and the assumptions imply \mathcal{C} is Krull-Schmidt.

4.1. The canonical surjection Q between relative Grothendieck groups. In this subsection, we study the kernel of the canonical homomorphism $Q: K_0(\mathcal{C}, \mathbb{E}_{\mathcal{T}}, \mathfrak{s}_{\mathcal{T}}) \twoheadrightarrow K_0(\mathcal{C}, \mathbb{E}_{\mathcal{X}}, \mathfrak{s}_{\mathcal{X}})$ induced by the containment $\mathcal{X} \subseteq \mathcal{T}$ of rigid subcategories in a triangulated category \mathcal{C} . An abstract description of $\mathrm{Ker} Q$ is given in (4.2), but we give another in

Proposition 4.4 using the homomorphism $\theta_{\mathcal{T}}$ from Proposition 3.12. One can compute $\text{Ker } Q$ more explicitly in a special case, as we do in Theorem 4.5.

Recall that since \mathcal{C} is triangulated, it has the structure of an extriangulated category, which we denote by $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$, where $\mathbb{E}(C, A) = \mathcal{C}(C, \Sigma A)$ for objects $A, C \in \mathcal{C}$ (see Example 2.7). In addition, the subcategories \mathcal{X} and \mathcal{T} induce relative extriangulated structures on \mathcal{C} , namely there are extriangulated categories $(\mathcal{C}, \mathbb{E}_{\mathcal{X}}, \mathfrak{s}_{\mathcal{X}})$ and $(\mathcal{C}, \mathbb{E}_{\mathcal{T}}, \mathfrak{s}_{\mathcal{T}})$ (see Subsection 2.4). In the notation of Remark 3.2, we obtain the following diagram of subgroups in $K_0^{\text{sp}}(\mathcal{C})$ using that $0 \subseteq \mathcal{X} \subseteq \mathcal{T} \subseteq \mathcal{C}$. Note that $\mathcal{I} = \mathcal{I}_0$.

$$\begin{array}{c}
 K_0^{\text{sp}}(\mathcal{C}) \\
 | \\
 \mathcal{I}_0 = \langle [A]_{\mathcal{C}}^{\text{sp}} - [B]_{\mathcal{C}}^{\text{sp}} + [C]_{\mathcal{C}}^{\text{sp}} \mid [A \rightarrow B \rightarrow C] = \mathfrak{s}(\delta) \text{ for some } \delta \in \mathbb{E}(C, A) \rangle \\
 | \\
 \mathcal{I}_{\mathcal{X}} = \langle [A]_{\mathcal{C}}^{\text{sp}} - [B]_{\mathcal{C}}^{\text{sp}} + [C]_{\mathcal{C}}^{\text{sp}} \mid [A \rightarrow B \rightarrow C] = \mathfrak{s}(\delta) \text{ for some } \delta \in \mathbb{E}_{\mathcal{X}}(C, A) \rangle \\
 | \\
 \mathcal{I}_{\mathcal{T}} = \langle [A]_{\mathcal{C}}^{\text{sp}} - [B]_{\mathcal{C}}^{\text{sp}} + [C]_{\mathcal{C}}^{\text{sp}} \mid [A \rightarrow B \rightarrow C] = \mathfrak{s}(\delta) \text{ for some } \delta \in \mathbb{E}_{\mathcal{T}}(C, A) \rangle \\
 | \\
 0
 \end{array}$$

See also [46, p. 16].

In particular, there is a canonical surjection

$$\begin{array}{ccc}
 K_0(\mathcal{C}, \mathbb{E}_{\mathcal{T}}, \mathfrak{s}_{\mathcal{T}}) & \xrightarrow{Q := Q_{\mathcal{X}}^{\mathcal{T}}} & K_0(\mathcal{C}, \mathbb{E}_{\mathcal{X}}, \mathfrak{s}_{\mathcal{X}}) \\
 [C]_{\mathcal{T}} & \longmapsto & [C]_{\mathcal{X}},
 \end{array} \quad (4.1)$$

where

$$\begin{aligned}
 \text{Ker } Q &= \langle [A]_{\mathcal{T}} - [B]_{\mathcal{T}} + [C]_{\mathcal{T}} \mid [A \rightarrow B \rightarrow C] = \mathfrak{s}(\delta) \text{ for some } \delta \in \mathbb{E}_{\mathcal{X}}(C, A) \rangle \\
 &= \left\langle [A]_{\mathcal{T}} - [B]_{\mathcal{T}} + [C]_{\mathcal{T}} \mid \begin{array}{l} \text{there is a triangle } A \rightarrow B \rightarrow C \xrightarrow{\delta} \Sigma A \\ \text{in } \mathcal{C} \text{ with } H_{\mathcal{X}}(\delta) = 0 \end{array} \right\rangle. \quad (4.2)
 \end{aligned}$$

Remark 4.2. There is a commutative diagram

$$\begin{array}{ccc}
 & K_0^{\text{sp}}(\mathcal{C}) & \\
 \text{ind}_{\mathcal{T}} \swarrow & & \searrow \text{ind}_{\mathcal{X}} \\
 K_0(\mathcal{C}, \mathbb{E}_{\mathcal{T}}, \mathfrak{s}_{\mathcal{T}}) & \xrightarrow{Q = Q_{\mathcal{X}}^{\mathcal{T}}} & K_0(\mathcal{C}, \mathbb{E}_{\mathcal{X}}, \mathfrak{s}_{\mathcal{X}}),
 \end{array}$$

because $Q \circ \text{ind}_{\mathcal{T}}(C) = Q([C]_{\mathcal{T}}) = [C]_{\mathcal{X}} = \text{ind}_{\mathcal{X}}(C)$.

Remark 4.3. In the following proposition, which gives a description of $\text{Ker } Q$ in terms of $\theta_{\mathcal{T}}$, note that $L \in \text{mod } \mathcal{T}$ is a functor $\mathcal{T}^{\text{op}} \rightarrow \mathbf{Ab}$ and that $\mathcal{X} \subseteq \mathcal{T}$, whence the restriction $L|_{\mathcal{X}}$ makes sense. Note also that

$$H_{\mathcal{T}}(C)|_{\mathcal{X}} = (\mathcal{C}(-, C)|_{\mathcal{T}})|_{\mathcal{X}} = \mathcal{C}(-, C)|_{\mathcal{X}} = H_{\mathcal{X}}(C).$$

Proposition 4.4. *We have $\text{Ker } Q = \langle \theta_{\mathcal{T}}([L]) \mid L \in \text{mod } \mathcal{T} \text{ with } L|_{\mathcal{X}} = 0 \rangle$.*

Proof. (\supseteq) Let $L \in \text{mod } \mathcal{T}$ satisfy $L|_{\mathcal{X}} = 0$. By Section 2.2(i), there is a triangle

$$\Sigma^{-1}A \longrightarrow T_1 \longrightarrow T_0 \xrightarrow{\tau_0} A \quad (4.3)$$

in \mathcal{C} with $T_i \in \mathcal{T}$ such that $L \cong H_{\mathcal{T}}(A)$. Note that $H_{\mathcal{T}}(\tau_0)$ is epic as $H_{\mathcal{T}}(\Sigma T_1) = 0$, so

$$\text{Im } H_{\mathcal{T}}(\tau_0) \cong H_{\mathcal{T}}(A) \cong L. \quad (4.4)$$

We have $H_{\mathcal{X}}(A) = H_{\mathcal{T}}(A)|_{\mathcal{X}} \cong L|_{\mathcal{X}} = 0$ by Remark 4.3 and our assumption on L . In particular, $H_{\mathcal{X}}(\tau_0) = 0$, so (4.3) is one of the triangles in (4.2). Hence,

$$\begin{aligned} \theta_{\mathcal{T}}([L]) &= \theta_{\mathcal{T}}([\text{Im } H_{\mathcal{T}}(\tau_0)]) && \text{using (4.4)} \\ &= [\Sigma^{-1}A]_{\mathcal{T}} - [T_1]_{\mathcal{T}} + [T_0]_{\mathcal{T}} && \text{by Theorem 3.14.} \end{aligned}$$

Hence, $\theta_{\mathcal{T}}([L]) \in \text{Ker } Q$.

(\subseteq) Conversely, consider a generator of $\text{Ker } Q$, that is, an element $[A]_{\mathcal{T}} - [B]_{\mathcal{T}} + [C]_{\mathcal{T}}$ of $K_0(\mathcal{C}, \mathbb{E}_{\mathcal{T}}, \mathfrak{s}_{\mathcal{T}})$ for which there is a triangle $A \rightarrow B \rightarrow C \xrightarrow{\gamma} \Sigma A$ in \mathcal{C} such that $H_{\mathcal{X}}(\gamma) = 0$; see (4.2). Consider $\text{Im } H_{\mathcal{T}}(\gamma) \in \text{mod } \mathcal{T}$. Note $\theta_{\mathcal{T}}([\text{Im } H_{\mathcal{T}}(\gamma)]) = [A]_{\mathcal{T}} - [B]_{\mathcal{T}} + [C]_{\mathcal{T}}$ by Theorem 3.14. Thus, it suffices to show that $\text{Im } H_{\mathcal{T}}(\gamma)|_{\mathcal{X}}$ is trivial. Observe that

$$\begin{aligned} (\text{Im } H_{\mathcal{T}}(\gamma))|_{\mathcal{X}} &= \text{Im}((H_{\mathcal{T}}(\gamma))|_{\mathcal{X}}) && \text{as } (-)|_{\mathcal{X}}: \text{mod } \mathcal{T} \rightarrow \text{mod } \mathcal{X} \text{ is an exact functor} \\ &= \text{Im } H_{\mathcal{X}}(\gamma) && \text{by Remark 4.3} \\ &= 0. \end{aligned} \quad \blacksquare$$

Recall that a k -linear category \mathcal{A} is called *locally bounded* if \mathcal{A} is Hom-finite and, for each indecomposable object $A \in \mathcal{A}$, there are only finitely many objects $B \in \text{indec } \mathcal{A}$ such that $\mathcal{A}(A, B) \neq 0$ or $\mathcal{A}(B, A) \neq 0$; see [48, p. 160]. The category \mathcal{A} has *right almost split morphisms* if each indecomposable object $A \in \mathcal{A}$ admits a right almost split morphism $C \rightarrow A$ in the sense of [5, p. 454]; see [3, p. 285], where right almost split morphisms were called ‘almost splittable’.

Theorem 4.5. *If \mathcal{T} is locally bounded and has right almost split morphisms, then*

$$\text{Ker } Q = \langle \theta_{\mathcal{T}}([S_Y]) \mid Y \in \text{indec } \mathcal{Y} \rangle,$$

where $S_Y \in \text{mod } \mathcal{T}$ is the simple top of $\mathcal{T}(-, Y)$.

Proof. Local boundedness and Hom-finiteness imply each $L \in \text{mod } \mathcal{X}$ has finite length in $\text{Mod } \mathcal{X}$ (the category of all additive functors $\mathcal{X}^{\text{op}} \rightarrow \text{Ab}$), hence a finite composition series. The simple quotients in the series are simple tops S_T of objects $\mathcal{T}(-, T)$ for $T \in \text{indec } \mathcal{T}$; see [3, Prop. 2.3(b)]. Since \mathcal{T} has right almost split morphisms, the S_T are in $\text{mod } \mathcal{T}$ by [3, Cor. 2.6]. Finally, note that $L|_{\mathcal{X}} = 0$ if and only if each composition factor S_T satisfies

$S_T|_{\mathcal{X}} = 0$, that is, if and only if $T \in \text{indec } \mathcal{Y}$. Hence,

$$\begin{aligned} \langle \theta_{\mathcal{T}}([S_Y]) \mid Y \in \text{indec } \mathcal{Y} \rangle &= \langle \theta_{\mathcal{T}}([L]) \mid L \in \text{mod } \mathcal{T} \text{ with } L|_{\mathcal{X}} = 0 \rangle \\ &= \text{Ker } Q, \end{aligned}$$

where the last equality follows from Proposition 4.4. ■

4.2. Connection to Auslander-Reiten angles and modified Caldero-Chapoton maps. We specialise further in this subsection in order to produce in Theorem 4.11 a presentation of $\text{Ker } Q$ in terms of Auslander-Reiten angles. In particular, we assume \mathcal{T} is a locally bounded, n -cluster tilting subcategory; see the setup below.

Setup 4.6. Let k be a field and let $n \geq 2$ be an integer. Suppose \mathcal{C} is a skeletally small, k -linear, Hom-finite, idempotent complete, triangulated category with a Serre functor \mathbb{S} . Suppose $\mathcal{X} \subseteq \mathcal{T}$ are additive (see Subsection 2.1(i)), contravariantly finite subcategories. In addition, suppose \mathcal{T} is a locally bounded, n -cluster tilting subcategory of \mathcal{C} (see Definition 4.7 below). We still define an additive rigid subcategory $\mathcal{Y} \subseteq \mathcal{C}$, which is closed under direct summands, by $\text{indec } \mathcal{T} = \text{indec } \mathcal{X} \dot{\cup} \text{indec } \mathcal{Y}$.

This is a special case of Setup 4.1. Let us recall the definition of n -cluster tilting.

Definition 4.7. [40], [42, Sec. 3] Suppose \mathcal{C} is an idempotent complete, triangulated category with an additive subcategory \mathcal{T} . For an integer $n \geq 2$, we call \mathcal{T} an n -cluster tilting subcategory of \mathcal{C} if:

- (i) \mathcal{T} is functorially finite; and
- (ii) $\mathcal{T} = \left\{ X \in \mathcal{C} \mid \text{Ext}_{\mathcal{C}}^i(\mathcal{T}, X) = 0 \text{ for all } 1 \leq i \leq n-1 \right\}$
 $= \left\{ X \in \mathcal{C} \mid \text{Ext}_{\mathcal{C}}^i(X, \mathcal{T}) = 0 \text{ for all } 1 \leq i \leq n-1 \right\}.$

Our first aim in this subsection is to prove Theorem B from Section 1. For this, we recall the following preliminary definition and then the triangulated index of [44].

Definition 4.8. [44, Def. 3.1] By a *tower of triangles in \mathcal{C}* , we mean a diagram

$$\begin{array}{ccccccc} & A_{m-1} & \longrightarrow & A_{m-2} & \longrightarrow & \cdots & \longrightarrow & A_2 & \longrightarrow & A_1 \\ & \nearrow & & \nearrow & & & & \nearrow & & \nearrow \\ A_m & & & V_{m-1.5} & & & & V_{2.5} & & V_{1.5} & & A_0 \\ & \nwarrow & & \nwarrow & & & & \nwarrow & & \nwarrow & & \nwarrow \\ & & & & & & & & & & & & \end{array}$$

in \mathcal{C} , where $m \geq 2$ is an integer, an arrow $B \rightsquigarrow C$ represents a morphism $B \rightarrow \Sigma C$, each oriented triangle is a triangle in \mathcal{C} , and each non-oriented triangle commutes.

Definition 4.9. [44, Lem. 3.2, Def. 3.3] Let $C \in \mathcal{C}$ be arbitrary. There is a tower of triangles

$$\begin{array}{ccccccccccc}
& & T_{n-2} & \longrightarrow & T_{n-3} & \longrightarrow & \cdots & \longrightarrow & T_1 & \longrightarrow & T_0 & & \\
& \nearrow & & \searrow^{\tau_{n-2}} & \nearrow & \searrow^{\tau_{n-3}} & & & \nearrow & \searrow^{\tau_1} & \nearrow & \searrow^{\tau_0} & \\
T_{n-1} & \xleftarrow{\nu_{n-2.5}} & V_{n-2.5} & \xleftarrow{\nu_{n-3.5}} & \cdots & \xleftarrow{\nu_{0.5}} & V_{0.5} & \xleftarrow{\nu_{-0.5}} & C, & & & &
\end{array} \quad (4.5)$$

where each τ_i is a minimal right \mathcal{T} -approximation. The *triangulated index* of C with respect to \mathcal{T} is

$$\text{index}_{\mathcal{T}}(C) := \sum_{i=0}^{n-1} (-1)^i [T_i]_{\mathcal{T}}^{\text{sp}},$$

viewed as an element of the split Grothendieck group $K_0^{\text{sp}}(\mathcal{T})$. This assignment induces a surjective homomorphism $\text{index}_{\mathcal{T}}: K_0^{\text{sp}}(\mathcal{C}) \rightarrow K_0^{\text{sp}}(\mathcal{T})$; see [26, Rem. 2.14].

We can now prove Theorem B from Section 1, which can be viewed as a higher cluster tilting analogue of [54, Prop. 4.11]. Note that this result requires neither the existence of a Serre functor nor that \mathcal{T} is locally bounded.

Theorem 4.10. *There is an isomorphism*

$$\begin{array}{ccc}
K_0(\mathcal{C}, \mathbb{E}_{\mathcal{T}}, \mathfrak{s}_{\mathcal{T}}) & \xrightarrow{\Phi_{\mathcal{T}}} & K_0^{\text{sp}}(\mathcal{T}) \\
\text{ind}_{\mathcal{T}}(C) = [C]_{\mathcal{T}} & \longmapsto & \text{index}_{\mathcal{T}}(C)
\end{array}$$

of abelian groups, making the following diagram commute.

$$\begin{array}{ccc}
K_0^{\text{sp}}(\mathcal{C}) & & \\
\text{ind}_{\mathcal{T}}(-) = [-]_{\mathcal{T}} \downarrow & \searrow^{\text{index}_{\mathcal{T}}(-)} & \\
K_0(\mathcal{C}, \mathbb{E}_{\mathcal{T}}, \mathfrak{s}_{\mathcal{T}}) & \xrightarrow[\Phi_{\mathcal{T}}]{\cong} & K_0^{\text{sp}}(\mathcal{T})
\end{array}$$

Proof. The assignment $\Phi_{\mathcal{T}}$ gives a well-defined homomorphism, because it sends relations in $K_0(\mathcal{C}, \mathbb{E}_{\mathcal{T}}, \mathfrak{s}_{\mathcal{T}}) = K_0^{\text{sp}}(\mathcal{C})/\mathcal{I}_{\mathcal{T}}$ to 0. Namely, a relation arises from a triangle $A \rightarrow B \rightarrow C \xrightarrow{\gamma} \Sigma A$ with $H_{\mathcal{T}}(\gamma) = 0$, which satisfies $\text{index}_{\mathcal{T}}(A) - \text{index}_{\mathcal{T}}(B) + \text{index}_{\mathcal{T}}(C) = 0$ by [44, Lem. 3.7].

We claim that

$$\begin{array}{ccc}
K_0^{\text{sp}}(\mathcal{T}) & \xrightarrow{\Psi_{\mathcal{T}}} & K_0(\mathcal{C}, \mathbb{E}_{\mathcal{T}}, \mathfrak{s}_{\mathcal{T}}) \\
[T]_{\mathcal{T}}^{\text{sp}} & \longmapsto & [T]_{\mathcal{T}}.
\end{array}$$

defines an inverse homomorphism to $\Phi_{\mathcal{T}}$. This formula clearly defines a homomorphism of groups.

To simplify notation, put $\Phi := \Phi_{\mathcal{T}}$ and $\Psi := \Psi_{\mathcal{T}}$. First, note that $\Phi \circ \Psi = \text{id}_{K_0^{\text{sp}}(\mathcal{T})}$. Conversely, let $C \in \mathcal{C}$ be arbitrary. Then there is a tower of triangles (4.5), and we have

$$\begin{aligned} \Psi\Phi([C]_{\mathcal{T}}) &= \Psi(\text{index}_{\mathcal{T}}(C)) && \text{by definition of } \Phi \\ &= \Psi\left(\sum_{i=0}^{n-1} (-1)^i [T_i]_{\mathcal{T}}^{\text{sp}}\right) && \text{by definition of } \text{index}_{\mathcal{T}}(C) \\ &= \sum_{i=0}^{n-1} (-1)^i [T_i]_{\mathcal{T}} && \text{by definition of } \Psi. \end{aligned}$$

Thus, it suffices to show $[C]_{\mathcal{T}} = \sum_{i=0}^{n-1} (-1)^i [T_i]_{\mathcal{T}}$. Note that, for $0 \leq i \leq n-2$, each τ_i in (4.5) is a minimal right \mathcal{T} -approximation, so $H_{\mathcal{T}}(\tau_i)$ is surjective. Hence, $H_{\mathcal{T}}(\nu_{i-0.5}) = 0$ for each $0 \leq i \leq n-2$. Set $V_{-0.5} := C$ and $V_{n-1.5} := T_{n-1}$. Then applying Theorem 3.14 yields $[V_{i+0.5}]_{\mathcal{T}} - [T_i]_{\mathcal{T}} + [V_{i-0.5}]_{\mathcal{T}} = 0$ for each $0 \leq i \leq n-2$. Taking their alternating sum yields $[C]_{\mathcal{T}} - \sum_{i=0}^{n-1} (-1)^i [T_i]_{\mathcal{T}} = 0$, and hence we see that $\Psi\Phi([C]_{\mathcal{T}}) = [C]_{\mathcal{T}}$, i.e. $\Psi\Phi$ is the identity on $K_0(\mathcal{C}, \mathbb{E}_{\mathcal{T}}, \mathfrak{s}_{\mathcal{T}})$. \blacksquare

Our last result of this subsection gives an interpretation of $\text{Ker } Q$ in terms of Auslander-Reiten $(n+2)$ -angles in \mathcal{T} . Let us recall what this means. An *Auslander-Reiten $(n+2)$ -angle* $T_{n+1} \rightarrow T_n \rightarrow \cdots \rightarrow T_1 \rightarrow T_0$ in \mathcal{T} amounts to a tower of triangles

$$\begin{array}{ccccccccccc} & & T_n & \xrightarrow{\quad} & T_{n-1} & \xrightarrow{\quad} & \cdots & \xrightarrow{\quad} & T_2 & \xrightarrow{\quad} & T_1 & & \\ \beta_{n+1} \nearrow & & \searrow \beta_n & & \nearrow \alpha_{n-0.5} & & \searrow \beta_{n-1} & & \nearrow \alpha_{2.5} & & \searrow \beta_2 & & \nearrow \alpha_{1.5} & & \searrow \beta_1 \\ T_{n+1} & \xleftarrow{\quad} & X_{n-0.5} & \xleftarrow{\quad} & X_{n-1.5} & \xleftarrow{\quad} & \cdots & \xleftarrow{\quad} & X_{1.5} & \xleftarrow{\quad} & X_{0.5} & \xleftarrow{\quad} & T_0 \end{array} \quad (4.6)$$

in \mathcal{C} , where:

- β_1 is minimal right almost split in \mathcal{T} ;
- β_{n+1} is minimal left almost split in \mathcal{T} ;
- β_2, \dots, β_n are minimal right \mathcal{T} -approximations; and
- $\alpha_{1.5}, \dots, \alpha_{n-0.5}$ are minimal left \mathcal{T} -approximations;

see [42, Def. 3.8]. By [42, Thm. 3.10], each $T_0 \in \text{indec } \mathcal{T}$ permits such an Auslander-Reiten $(n+2)$ -angle in \mathcal{T} with $T_{n+1} = \Sigma \Sigma^{-n} T_0$.

Theorem 4.11. *The canonical surjection $Q = Q_{\mathcal{X}}^{\mathcal{T}}: K_0(\mathcal{C}, \mathbb{E}_{\mathcal{T}}, \mathfrak{s}_{\mathcal{T}}) \rightarrow K_0(\mathcal{C}, \mathbb{E}_{\mathcal{X}}, \mathfrak{s}_{\mathcal{X}})$ from (4.1) satisfies*

$$\text{Ker } Q = \left\langle \sum_{i=0}^{n+1} (-1)^i [T_i]_{\mathcal{T}} \left| \begin{array}{l} T_0 = Y \in \text{indec } \mathcal{Y} \text{ and } T_{n+1} \rightarrow T_n \rightarrow \cdots \rightarrow T_1 \rightarrow T_0 \\ \text{is an Auslander-Reiten } (n+2)\text{-angle in } \mathcal{T} \end{array} \right. \right\rangle.$$

Proof. The subcategory \mathcal{T} has right almost split morphisms because Auslander-Reiten $(n+2)$ -angles exist, or by [42, Props. 2.10, 2.11]. In addition, \mathcal{T} is locally bounded by assumption (see Setup 4.6). Hence, Theorem 4.5 applies.

Thus, it suffices to show

$$\sum_{i=0}^{n+1} (-1)^i [T_i]_{\mathcal{T}} = \theta_{\mathcal{T}}([S_{T_0}]). \quad (4.7)$$

for $T_0 \in \text{indec } \mathcal{T}$, where $S_{T_0} \in \text{mod } \mathcal{T}$ is the simple top of $\mathcal{T}(-, T_0)$ and

$$T_{n+1} \longrightarrow T_n \longrightarrow \cdots \longrightarrow T_1 \longrightarrow T_0$$

is an Auslander-Reiten $(n+2)$ -angle with associated tower of triangles (4.6). The triangle

$$X_{1.5} \xrightarrow{\alpha_{1.5}} T_1 \xrightarrow{\beta_1} T_0 \xrightarrow{\gamma_{0.5}} \Sigma X_{1.5}$$

satisfies $\text{Im } H_{\mathcal{T}}(\gamma_{0.5}) = \text{Coker } H_{\mathcal{T}}(\beta_1) = S_{T_0}$, because β_1 is right almost split in \mathcal{T} ; see [3, Cor. 2.6]. Thus, by Theorem 3.14, we have that

$$\text{ind}_{\mathcal{T}}(X_{1.5}) - \text{ind}_{\mathcal{T}}(T_1) + \text{ind}_{\mathcal{T}}(T_0) = \theta_{\mathcal{T}}([S_{T_0}]). \quad (4.8)$$

Set $X_{n+0.5} := T_{n+1}$. For $2 \leq i \leq n$, the triangle

$$X_{i+0.5} \xrightarrow{\alpha_{i+0.5}} T_i \xrightarrow{\beta_i} X_{i-0.5} \xrightarrow{\gamma_{i-0.5}} \Sigma X_{i+0.5}$$

satisfies $\text{Im } H_{\mathcal{T}}(\gamma_{i-0.5}) = 0$, because β_i is a right \mathcal{T} -approximation. So, by Theorem 3.14, we also have that

$$\text{ind}_{\mathcal{T}}(X_{i+0.5}) - \text{ind}_{\mathcal{T}}(T_i) + \text{ind}_{\mathcal{T}}(X_{i-0.5}) = 0 \quad (4.9)_i$$

for each $2 \leq i \leq n$. Hence, the sum

$$(4.8) + \sum_{i=2}^n (-1)^{i-1} (4.9)_i$$

is precisely (4.7), which was to be shown. ■

Remark 4.12. It follows from Theorems 4.10 and 4.11 that the group $K_0(\mathcal{C}, \mathbb{E}_{\mathcal{X}}, \mathfrak{s}_{\mathcal{X}}) \cong K_0(\mathcal{C}, \mathbb{E}_{\mathcal{T}}, \mathfrak{s}_{\mathcal{T}}) / \text{Ker } Q$ is isomorphic to

$$K_0^{\text{sp}}(\mathcal{T}) \left/ \left\langle \sum_{i=0}^{n+1} (-1)^i [T_i]_{\mathcal{T}}^{\text{sp}} \left| \begin{array}{l} T_0 \in \text{indec } \mathcal{Y} \text{ and } T_{n+1} \rightarrow T_n \rightarrow \cdots \rightarrow T_1 \rightarrow T_0 \\ \text{is an Auslander-Reiten } (n+2)\text{-angle in } \mathcal{T} \end{array} \right. \right\rangle \right.$$

We finish this section with a remark on how our considerations here relate to the modified Caldero-Chapoton map of Holm-Jørgensen [38], [39] and the \mathcal{X} -Caldero-Chapoton map from [46].

Remark 4.13. In addition to assuming the conditions of Setup 4.6, in this remark we also suppose the following:

- (i) k is algebraically closed;
- (ii) $n = 2$;

- (iii) \mathcal{C} is 2-Calabi-Yau;
- (iv) \mathcal{X} is functorially finite; and
- (v) \mathcal{T} is part of a cluster structure in the sense of [12, p. 1039].

Thus, for $Y \in \text{indec } \mathcal{Y}$, there is a pair of exchange triangles

$$Y \longrightarrow A' \longrightarrow Y^* \longrightarrow \Sigma Y \quad \text{and} \quad Y^* \longrightarrow A \longrightarrow Y \longrightarrow \Sigma Y^*$$

that can be glued to obtain an Auslander-Reiten 4-angle

$$\begin{array}{ccccc} & & A' & \longrightarrow & A \\ & \nearrow & & & \searrow \\ Y & & & & Y^* \\ & \longleftarrow & & & \longleftarrow \\ & & Y^* & & Y \end{array}$$

in \mathcal{T} . It follows from Theorem 4.11 that

$$\text{Ker } Q = \left\langle [A']_{\mathcal{T}} - [A]_{\mathcal{T}} \left| \begin{array}{l} \text{there are exchange triangles } Y \rightarrow A' \rightarrow Y^* \rightarrow \Sigma Y \\ \text{and } Y^* \rightarrow A \rightarrow Y \rightarrow \Sigma Y^* \text{ for } Y \in \text{indec } \mathcal{Y} \end{array} \right. \right\rangle.$$

In particular, this is isomorphic to the subgroup

$$N = \left\langle [A']_{\mathcal{T}}^{\text{sp}} - [A]_{\mathcal{T}}^{\text{sp}} \left| \begin{array}{l} \text{there are exchange triangles } Y \rightarrow A' \rightarrow Y^* \rightarrow \Sigma Y \\ \text{and } Y^* \rightarrow A \rightarrow Y \rightarrow \Sigma Y^* \text{ for } Y \in \text{indec } \mathcal{Y} \end{array} \right. \right\rangle$$

of $K_0^{\text{sp}}(\mathcal{T})$, using the isomorphism $K_0^{\text{sp}}(\mathcal{T}) \rightarrow K_0(\mathcal{C}, \mathbb{E}_{\mathcal{T}}, \mathfrak{s}_{\mathcal{T}})$ sending $[T]_{\mathcal{T}}^{\text{sp}}$ to $[T]_{\mathcal{T}}$ (see (1.4) from Section 1 or Theorem 4.10, or [54, Prop. 4.11]). Therefore, we have isomorphisms

$$K_0^{\text{sp}}(\mathcal{T})/N \longrightarrow K_0(\mathcal{C}, \mathbb{E}_{\mathcal{T}}, \mathfrak{s}_{\mathcal{T}})/\text{Ker } Q \xrightarrow{\overline{Q}_{\mathcal{X}}} K_0(\mathcal{C}, \mathbb{E}_{\mathcal{X}}, \mathfrak{s}_{\mathcal{X}}). \quad (4.10)$$

The subgroup N was defined in [39, Def. 2.4]. Although in general the subgroup N depends on the choice of complement \mathcal{Y} to \mathcal{X} , the isomorphisms of (4.10) show that the quotient $K_0^{\text{sp}}(\mathcal{T})/N$ is independent of this choice. This was previously only known in the cases when \mathcal{C} is the cluster category associated to a Dynkin-type \mathbb{A} quiver [56, Prop. 3.2.3] or a Dynkin-type \mathbb{D} quiver [56, Thm. 5.1.7]. The authors thank Yann Palu for asking about the dependency of N on \mathcal{Y} .

In [38] and [39], Holm–Jørgensen generalised the map of Caldero–Chapoton [18] in two ways, by defining their *modified Caldero–Chapoton map*. This generalisation depends only on a rigid object and can take values in a general commutative ring A . The modified Caldero–Chapoton map depends on several functions. In particular, it utilises a map $\alpha: \text{obj}(\mathcal{C}) \rightarrow A$ given by $\alpha(X) = \varepsilon q([T_0^X]_{\mathcal{T}}^{\text{sp}} - [T_1^X]_{\mathcal{T}}^{\text{sp}})$, where $\varepsilon: K_0^{\text{sp}}(\mathcal{T})/N \rightarrow A$ is a map satisfying some properties, $q: K_0^{\text{sp}}(\mathcal{T}) \rightarrow K_0^{\text{sp}}(\mathcal{T})/N$ is the canonical quotient, and $[T_0^X]_{\mathcal{T}}^{\text{sp}} - [T_1^X]_{\mathcal{T}}^{\text{sp}}$ is the classical Palu index of X with respect to \mathcal{T} .

When $\mathcal{X} = \mathcal{T}$ is cluster tilting, we see that $N = 0$ and q is the identity. Thus, in this special case, we input into ε the genuine classical index of X with respect to \mathcal{T} . In the general case, although $K_0^{\text{sp}}(\mathcal{T})/N$ works as the domain for ε , it does not explain *why* it is needed. This is now explained by the isomorphism $K_0^{\text{sp}}(\mathcal{T})/N \rightarrow K_0(\mathcal{C}, \mathbb{E}_{\mathcal{X}}, \mathfrak{s}_{\mathcal{X}})$ given in

(4.10) and that the Grothendieck group $K_0(\mathcal{C}, \mathbb{E}_{\mathcal{X}}, \mathfrak{s}_{\mathcal{X}})$ is capturing the indices of objects in \mathcal{C} with respect to \mathcal{X} . These observations led to the \mathcal{X} -Caldero-Chapoton map defined in [46, Sec. 5].

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