Available online at www.sciencedirect.com

ScienceDirect

Expositiones Mathematicae 41 (2023) 220–237

EXPOSITIONES MATHEMATICAE

www.elsevier.com/locate/exmath

Krull-Remak-Schmidt decompositions in Hom-finite additive categories

Amit Shah

Department of Mathematics, Aarhus University, 8000, Aarhus C, Denmark

Received 6 September 2022; received in revised form 21 November 2022; accepted 7 December 2022

Dedicated to Robert E. Remak (1888–1942) in honour of his contributions to mathematics.

Abstract

An additive category in which each object has a Krull-Remak-Schmidt decomposition—that is, a finite direct sum decomposition consisting of objects with local endomorphism rings—is known as a Krull-Schmidt category. A Hom-finite category is an additive category $\mathcal{A}$ for which there is a commutative unital ring k, such that each Hom-set in $\mathcal{A}$ is a finite length k-module. The aim of this note is to provide a proof that a Hom-finite category is Krull-Schmidt, if and only if it has split idempotents, if and only if each indecomposable object has a local endomorphism ring.

MSC 2020: primary 18E05; secondary 16D70; 16L30; 16U40; 18E10

Keywords: Additive category; Hom-finite category; Idempotent; Krull-Remak-Schmidt decomposition; Krull-Schmidt category; Split idempotents

1. Introduction

If one blurts out "decomposition theorem" to an undergraduate in mathematics, one might expect them to think of the fundamental theorem of arithmetic or perhaps the fundamental theorem of finitely generated abelian groups. Such results are of interest (and importance) because we can hope to understand a more complicated object by first understanding the simpler components of which it is comprised. It is this kind of

E-mail address: amit.shah@math.au.dk.

https://doi.org/10.1016/j.exmath.2022.12.003

application that has made another famous decomposition theorem of such wide interest. Let $\mathcal{A}$ be an additive category. An object X is said to satisfy the *Krull–Remak–Schmidt theorem* if, whenever $M_1 \oplus \cdots \oplus M_m$ and $N_1 \oplus \cdots \oplus N_n$ are finite direct sum decompositions of X into objects each having local endomorphism rings, then $m = n$ and there is a permutation σ in $\mathrm{Sym}(n)$ such that M_j is isomorphic to $N_{\sigma(j)}$ for all $1 \leqslant j \leqslant n$.

The Krull–Remak–Schmidt theorem has its roots in finite group theory: first Frobenius–Stickelberger [12] demonstrated it for finite abelian groups; Remak [26] for finite groups[1]; Schmidt [28] also for finite groups but with a substantially shorter proof; and Krull [19] for abelian operator groups (usually stated in the language of modules) with ascending and descending chain conditions.[2] The first categorical version of the theorem was established by Atiyah [4]. For a nice introduction on the Krull–Remak–Schmidt theorem for module categories see Facchini [10], and for additive categories see Walker–Warfield [30].

A finite direct sum decomposition of an object in $\mathcal{A}$ into objects having local endomorphism rings is known as a *Krull-Remak-Schmidt decomposition*. If each object in $\mathcal{A}$ admits such a decomposition, then $\mathcal{A}$ is known as a *Krull-Schmidt category*. Krull-Schmidt categories appear naturally in representation theory of algebras and in algebraic geometry. For example, the category $\mathsf{mod}\,\Lambda$ of finite-dimensional modules over a finite-dimensional algebra Λ, the bounded derived category of $\mathsf{mod}\,\Lambda$, the category $\mathsf{fg}\,(R, \mathfrak{m}, k)$ of finitely generated modules over a commutative complete local ring $(R, \mathfrak{m}, k)$, the category of coherent algebraic sheaves over a complete algebraic variety X over an algebraically-closed field, and the category of coherent analytic sheaves over a compact complex manifold Y are all Krull-Schmidt categories.

If there is a commutative unital ring k such that each Hom-set in $\mathcal{A}$ is a finite length k-module, then we call $\mathcal{A}$ Hom-*finite*. The purpose of this note is to show that if $\mathcal{A}$ is Hom-finite, then $\mathcal{A}$ is Krull-Schmidt, if and only if it is idempotent complete, if and only if the endomorphism ring of any indecomposable object in $\mathcal{A}$ is local (see Theorem 6.1). The equivalence of the first two conditions follows from the theory of projective covers; see Chen–Ye–Zhang [8, §A.1], Krause [18, §4], or Corollary 4.13. The motivation for this note is the equivalence of the latter two conditions. Although this is certainly known when k is a field (see e.g. Ringel [27, §2.2], Happel [13, §I.3.2], [8, Cor. A.2]), it remains true when k is any commutative unital ring. However, the author failed to find a proof in the literature.

We assume the reader is familiar with the theory of modules and the notions of a category and a functor. In Sections 4 and 5 we rely on several results from [18], which we typically do not reprove here. As such, this note is not self-contained. However, by keeping [18] to hand the reader should not struggle.

This article is organised as follows. We recall some concepts from category theory in Section 2. Section 3 contains some preliminaries on idempotents and local rings. In

1 Remak proved a stronger conclusion: $m = n$, and there exists $\sigma \in \mathrm{Sym}(n)$ and an automorphism $f : X \to X$ that is the identity on X modulo its centre, such that $M_j \cong N_{\sigma(j)}$ via f for $1 \leqslant j \leqslant n$. Wedderburn [23] proposed a proof for finite groups slightly before Remak, but with no central isomorphism aspect. Moreover, it is not clear if Wedderburn's proof is complete, with Remak having commented on deficiencies in the proof at the end of [26].

2 Jacobson [16, Ch. V] provides an exposition of the theorem for operator groups without the abelian assumption.

Section 4 we turn to Krull-Schmidt categories and the relation to semi-perfect rings. We recall some material on bi-chain conditions in abelian categories in Section 5. Lastly, we demonstrate the main result—Theorem 6.1—in Section 6.

Remark 1.1. It is not so clear why the terminology 'Krull-Schmidt category' fails to include Remak's name. It perhaps originates from Dür [9], in which Dür mentions [4] for influence on the choice of terminology. Atiyah referred only to the 'Krull–Schmidt theorem' and to a 'Remak decomposition'. Although this terminology is now entrenched in our mathematical language, we ought to remember that the statement Remak proved for finite groups is the assertion Schmidt re-proved in a shorter way.

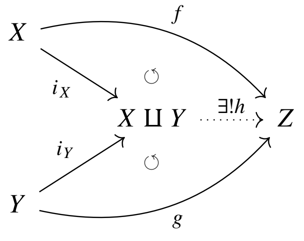
Remak demonstrated this significant result in his PhD thesis in 1911. His contributions to mathematics covered a wide range of areas, including group theory, number theory and analysis. Remak was murdered in Auschwitz in or after 1942 [29, p. 64]. A nice biography of Remak is given by Merzbach [24].

2. Additive and abelian categories

The notions we recall here are standard. We mainly use this section to set up notation for the remainder of the article. An accessible introduction to these concepts can be found in Aluffi [1, Chs. I, IX]. The reader who is more familiar with category theory can safely skip this section.

Suppose that $\mathcal{A}$ is a category and let X, Y be objects in $\mathcal{A}$. We denote the collection of morphisms $X \to Y$ in $\mathcal{A}$ by $\mathrm{Hom}_{\mathcal{A}}(X, Y)$. The collection of *endomorphisms* $f : X \to X$ of X are denoted $\mathrm{End}_{\mathcal{A}}(X)$. A *zero object* in $\mathcal{A}$ is an object $X \in \mathcal{A}$ for which $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ and $\mathrm{Hom}_{\mathcal{A}}(Y, X)$ are both singletons for each $Y \in \mathcal{A}$.

Definition 2.1. For $X, Y \in \mathcal{A}$, a *coproduct* of X and Y is an object $X \amalg Y$ in $\mathcal{A}$ endowed with morphisms $i_X : X \to X \amalg Y$ and $i_Y : Y \to X \amalg Y$ satisfying the following universal property: given $Z \in \mathcal{A}$ and morphisms $f : X \to Z$, $g : Y \to Z$, there exists a unique morphism $h : X \amalg Y \to Z$ such that the diagram below commutes in $\mathcal{A}$.



A *product* $X \prod Y$ of X and Y is the dual notion, and we omit the description here.

As with all objects defined by a universal property of this kind, (co)products and zero objects (where they exist) are unique up to unique isomorphism. Furthermore, although we only defined binary (co)products above, one can define finite (co)products similarly. We are now in a position to define an additive category.

Definition 2.2. The category $\mathcal{A}$ is called *preadditive* if

(i) $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ has the structure of an abelian group for all X, Y in $\mathcal{A}$, such that the composition function $\mathrm{Hom}_{\mathcal{A}}(X, Y) \times \mathrm{Hom}_{\mathcal{A}}(Y, Z) \to \mathrm{Hom}_{\mathcal{A}}(X, Z)$ (sending (f, g) to $gf = g \circ f$) is $\mathbb{Z}$-bilinear for all objects X, Y, Z in $\mathcal{A}$.

The category $\mathcal{A}$ is *additive* if it is preadditive and it has

(ii) a zero object, which we denote by 0, and
(iii) finite products and finite coproducts.

For the rest of this section suppose that $\mathcal{A}$ is an additive category.

Remark 2.3. Let $X, Y \in \mathcal{A}$ be objects.

(i) We denote the abelian group operation of $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ by $+$, and the identity element by 0.
(ii) The product $X \Pi Y$ and coproduct $X \amalg Y$ are isomorphic; see Mac Lane [22, Exer. VIII.2.1]. This object is denoted $X \oplus Y$ and called the *direct sum* of X and Y. In particular, it is equipped with morphisms $i_X : X \to X \oplus Y$, $i_Y : Y \to X \oplus Y$, $p_X : X \oplus Y \to X$ and $p_Y : X \oplus Y \to Y$, such that $p_X i_X = \mathrm{id}_X$, $p_Y i_Y = \mathrm{id}_Y$ and $i_X p_X + i_Y p_Y = \mathrm{id}_{X \oplus Y}$. It is a nice exercise to show that these equations also imply that $p_X i_Y = 0$ and $p_Y i_X = 0$.

Later we deal with additive categories that have extra structure on their Hom-sets. The following notion captures this. We always assume rings are associative and unital. However, we do not necessarily assume the additive identity 0 and the multiplicative identity 1 of a ring are distinct.

Definition 2.4. Let k be a commutative ring. The category $\mathcal{A}$ is called a *k-linear* category if $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ is a k-module for all objects X, Y in $\mathcal{A}$ and the composition of morphisms is k-bilinear.

Example 2.5.

(i) Recall that an abelian group is nothing other than a $\mathbb{Z}$-module. Thus, a category that is additive in the sense of Definition 2.2 is just a $\mathbb{Z}$-linear category in the sense of Definition 2.4.
(ii) Let Λ be a ring. We denote by $\mathsf{Mod}\,\Lambda$ the category of all right Λ-modules. If $\Lambda = k$ is commutative, then $\mathsf{Mod}\,k$ is k-linear. See e.g. [1, Chp. III] for details. More generally, if Λ is a k-algebra, then $\mathsf{Mod}\,k$ is k-linear.

Lastly we recall the concepts of (co)kernels and abelian categories.

Definition 2.6. Let $f : X \to Y$ be a morphism in $\mathcal{A}$. A *weak kernel* of f is a morphism $i : K \to X$ in $\mathcal{A}$ with $fi = 0$, and such that: for any $a : A \to X$ with $fa = 0$ there exists

$b: A \to K$ such that $a = ib$.

$$\begin{array}{ccccc} & & A & & \\ & \overset{\exists b}{\swarrow} & \downarrow a & & \\ K & \xrightarrow{i} & X & \xrightarrow{f} & Y \end{array}$$

If the morphism b obtained in this way is always uniquely determined, then $i: K \to X$ is called a *kernel* of f.

One defines a *weak cokernel* and a *cokernel* of f dually.

Recall that a morphism $f: X \to Y$ in the additive category $\mathcal{A}$ is *monic* (or a *monomorphism*) if for any morphism $a: A \to X$ with $fa = 0$ we must have that $a = 0$. An *epic* morphism (or *epimorphism*) is defined dually.

Definition 2.7. An additive category $\mathcal{A}$ is said to be *abelian* if:

(i) each morphism $f: X \to Y$ in $\mathcal{A}$ has a kernel $\ker f: \operatorname{Ker} f \to X$ and a cokernel $\operatorname{coker} f: Y \to \operatorname{Coker} f$; and

(ii) in $\mathcal{A}$ every monomorphism is the kernel of some morphism, and every epimorphism is the cokernel of some morphism.

Example 2.8. If k is a commutative ring, then the category $\mathsf{Mod}\ k$ (see Example 2.5(ii)) is an abelian k-linear category.

We conclude this section with the following straightforward lemma.

Lemma 2.9. *A morphism f in $\mathcal{A}$ is a kernel if and only if it is a monic weak kernel. It is a cokernel if and only if it is an epic weak cokernel.*

3. Idempotents

We will see in the next section that an additive category having Krull-Remak-Schmidt decompositions is very closely related to it having so-called split idempotents. However, one can define idempotents in any ring and we begin with such considerations now.

3.1. Idempotents in rings

Let Λ be a ring with multiplicative identity $1 = 1_\Lambda$.

Definition 3.1.

(i) An element $e \in \Lambda$ is called an *idempotent* if $e^2 = e$.

(ii) Two idempotents $e, f \in \Lambda$ are called *orthogonal* if $ef = 0 = fe$. A set $\{e_j\}_{j=1}^n \subseteq \Lambda$ of idempotents is called *orthogonal* if its elements are pairwise orthogonal.

(iii) A non-zero idempotent $e \in \Lambda$ is called *primitive* if $e = f + g$ implies $f = 0$ or $g = 0$ for all pairs of orthogonal idempotents $f, g \in \Lambda$.

(iv) A set $\{e_j\}_{j=1}^{n} \subseteq \Lambda$ of idempotents is called *complete* if $e_1 + \cdots + e_n = 1$.

Example 3.2.

(i) Let $\mathcal{A}$ be an additive category and suppose $X \in \mathcal{A}$ is an object. The identity morphism $\mathrm{id}_X : X \to X$ and the zero morphism $0 : X \to X$ are always idempotents of the endomorphism ring $\mathrm{End}_{\mathcal{A}}(X) = \mathrm{Hom}_{\mathcal{A}}(X, X)$.

(ii) Given any idempotent $e \in \Lambda$, the element $1 - e \in \Lambda$ is also idempotent, and e and $1 - e$ are orthogonal. Furthermore, the right Λ-module Λ_Λ decomposes as $\Lambda = e\Lambda \oplus (1 - e)\Lambda$.

Lastly in this subsection, we recall the definition of a local ring, and its connection with idempotents and indecomposable modules.

Definition 3.3. If $0 \neq 1$ and the sum of any two non-units in Λ is again a non-unit, then Λ is called a *local* ring.

Remark 3.4. A ring Λ is local if, equivalently, $0 \neq 1$ and Λ has a unique maximal right ideal. This right ideal is precisely the collection of non-units in such a ring. In particular, given an element x in a local ring Λ, we have that x or $1 - x$ is invertible. See Anderson–Fuller [2, Prop. 15.15], or Lam [20, Thm. 19.1].

Local rings have very few idempotents as we now see. The converse of the following lemma holds if Λ is artinian; see e.g. [20, Cor. 19.19].

Lemma 3.5. *If Λ is local, then Λ has precisely two idempotents* 0 *and* 1. *In particular, the idempotent* 1 *is primitive.*

Proof. Let $e \in \Lambda$ be an idempotent. Then $1 - e$ is also idempotent by Example 3.2(ii). Since Λ is local, we have that e or $1 - e$ is invertible by Remark 3.4. If e is a unit and $ef = 1 = fe$ for some $f \in \Lambda$, then $e = e \cdot 1 = e(ef) = ef = 1$ as $e = e^2$. On the other hand, if $1 - e$ is invertible then a similar argument shows that $1 - e = 1$, whence $e = 0$.

For the other assertion, suppose $0 \neq 1 = e + f$ for some orthogonal idempotents $e, f \in \Lambda$. If $e = f = 0$ then $1 = 0$, which is impossible. Therefore, without loss of generality, $e = 1$ and so $f = 0$. ■

A *corner ring* of Λ is a ring of the form $e\Lambda e$ for some idempotent $e \in \Lambda$. Note that $e\Lambda e$ is unital with unit $1_{e\Lambda e} = e$. There is an isomorphism

$$\Phi : \mathrm{End}_{\mathsf{Mod}\,\Lambda}(e\Lambda) \xrightarrow{\cong} e\Lambda e \tag{3.1}$$

of right $e\Lambda e$-modules given by $\Phi(h) := h(e) = h(e)e$. Moreover, Φ is an isomorphism of rings. See Assem–Simson–Skowroński [3, Lem. I.4.2].

A non-zero module M is called *indecomposable* if, whenever there is an isomorphism $M \cong M_1 \oplus M_2$ of modules, we have $M_1 = 0$ or $M_2 = 0$. When we have a decomposition like $M \cong M_1 \oplus M_2$, we usually more simply write $M = M_1 \oplus M_2$. Nothing is lost with this identification for our purposes since we are studying the uniqueness of direct sum decompositions up to permutation and isomorphism of summands.

Lemma 3.6. *Let $e \in \Lambda$ be an idempotent. Then the following are equivalent.*

(i) The idempotent e is primitive.
(ii) The only idempotents of $e\Lambda e$ are 0 and $1_{e\Lambda e} = e \neq 0$.
(iii) The right Λ-module $e\Lambda$ is indecomposable.

Proof. (i) $\Rightarrow$ (ii) Let $efe \in e\Lambda e$ be an idempotent. We have $e = efe + (e - efe)$, which implies $efe = 0$ or $e - efe = 0$, because e is primitive by assumption, and we are done.

(ii) $\Rightarrow$ (iii) Let $e\Lambda = X_1 \oplus X_2$ be a decomposition of $e\Lambda \neq 0$, and let $f : e\Lambda \twoheadrightarrow X_1 \hookrightarrow e\Lambda$, $g : e\Lambda \twoheadrightarrow X_2 \hookrightarrow e\Lambda$ be the compositions of the canonical projections and inclusions. Then f, g are idempotents in $\mathrm{End}_{\mathsf{Mod}\,\Lambda}(e\Lambda)$ and, moreover, $\mathrm{id}_{e\Lambda} = f + g$.

Using the isomorphism Φ from (3.1) and that $e \neq 0$, we see that $\mathrm{End}_{\mathsf{Mod}\,\Lambda}(e\Lambda) \neq 0$. Since $\Phi(f), \Phi(g) \in e\Lambda e$ are idempotents, we have $\Phi(f), \Phi(g) \in \{0, e\}$ by assumption. If $\Phi(f) = \Phi(g) = 0$, then $f = g = 0$ as Φ is an isomorphism, whence $X_1 = X_2 = 0$. But this forces $e\Lambda = 0$, which is a contradiction. Thus, without loss of generality, $\Phi(f) = e = \Phi(\mathrm{id}_{e\Lambda})$. This implies $f = \mathrm{id}_{e\Lambda}$ and hence $g = 0$. In particular, this means $X_2 = 0$ and $X_1 = e\Lambda$ is indecomposable.

(iii) $\Rightarrow$ (i) Note that e is non-zero as $e\Lambda \neq 0$. If $e = f + g$ where f, g are orthogonal idempotents, then $e\Lambda = (f + g)\Lambda = f\Lambda \oplus g\Lambda$. But then, without loss of generality, $f\Lambda = 0$ as $e\Lambda$ is indecomposable by assumption and so $f = 0$. Thus, e is primitive. ■

3.2. Idempotents in categories

By an *idempotent* in a category $\mathcal{A}$ we mean any endomorphism $e : X \to X$ for some object X satisfying $e^2 = e$. In other words, $e \in \mathrm{End}_{\mathcal{A}}(X)$ is an idempotent element. We begin with an easy lemma.

Lemma 3.7. *Let $\mathcal{A}$ be an additive category with an object X. Suppose $X = A \oplus B$ where $\mathrm{End}_{\mathcal{A}}(A)$ is local. Consider the canonical projection $p : X \twoheadrightarrow A$ and canonical inclusion $i : A \hookrightarrow X$. Then the idempotent $e := ip \in \mathrm{End}_{\mathcal{A}}(X)$ is primitive.*

Proof. If $e = ip$ were zero, then it would follow that p were zero since i is a monomorphism. This would in turn imply that A were the zero object, which is not possible as $\mathrm{End}_{\mathcal{A}}(A)$ is assumed to be local. Now suppose $e = f + g$ for some orthogonal idempotents $f, g \in \mathrm{End}_{\mathcal{A}}(X)$. Note that $efe = f$. A straightforward verification shows $pfi, pgi \in \mathrm{End}_{\mathcal{A}}(A)$ are orthogonal idempotents and that $\mathrm{id}_A = pfi + pgi$. Since $\mathrm{End}_{\mathcal{A}}(A)$ is local, by Lemma 3.5 and without loss of generality, we have that $pfi = 0$. This implies $f = efe = i(pfi)p = 0$, and so e is primitive. ■

Definition 3.8. A category $\mathcal{A}$ *has split idempotents* if, for each $X \in \mathcal{A}$ and every idempotent $e \in \mathrm{End}_{\mathcal{A}}(X)$, there exists an object $Y \in \mathcal{A}$ and morphisms $r : X \to Y$, $s : Y \to X$ such that $e = sr$ and $rs = \mathrm{id}_Y$.

Remark 3.9. With the notation as in Definition 3.8, we see that r is a retraction (hence an epimorphism) and s is a section (hence a monomorphism).

It turns out that any additive category embeds into one that has split idempotents; see Karoubi [17], or Bühler [7, §6] for a nice exposition. The following set of equivalent conditions for a category to have split idempotents is well-known for additive categories (see e.g. [7, Rmk. 6.2]). The same argument, however, also works for preadditive categories.

Proposition 3.10. *Let $\mathcal{A}$ be a preadditive category. Then the following are equivalent.*

(i) $\mathcal{A}$ has split idempotents.
(ii) Each idempotent in $\mathcal{A}$ admits a kernel in $\mathcal{A}$.
(iii) Each idempotent in $\mathcal{A}$ admits a cokernel in $\mathcal{A}$.

Proof. (i) $\Rightarrow$ (ii) Let $e\colon X \to X$ be an arbitrary idempotent in $\mathcal{A}$. By assumption, $\mathrm{id}_X - e$ splits and so there exist $t\colon X \to Z$, $u\colon Z \to X$ with $\mathrm{id}_X - e = ut$ and $tu = \mathrm{id}_Z$. Recall that t is epic and u is monic (see Remark 3.9). We claim that $Z \xrightarrow{u} X$ is a kernel for e. Since $(eu)t = e(ut) = e(\mathrm{id}_X - e) = 0$ and t is epic, we see that $eu = 0$. Suppose $a\colon A \to X$ is a morphism in $\mathcal{A}$ such that $ea = 0$. Then

$$a = a - 0 = a - ea = (\mathrm{id}_X - e)a = (ut)a = u(ta),$$

so a factors through u. Thus, u is a monomorphism that is a weak kernel of e and hence a kernel of e by Lemma 2.9.

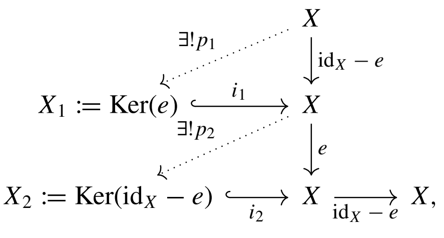
(ii) $\Rightarrow$ (i) Let $e \in \mathrm{End}_{\mathcal{A}}(X)$ be an idempotent. The idempotent $\mathrm{id}_X - e$ admits a kernel $Y := \mathrm{Ker}(\mathrm{id}_X - e) \overset{s}{\hookrightarrow} X$. As $(\mathrm{id}_X - e)e = 0$, we have that e factors through s. Thus, there exists a unique morphism $r\colon X \to Y$ such that $e = sr$, so it suffices to show $rs = \mathrm{id}_Y$. As s is the kernel of $\mathrm{id}_X - e$ we obtain $s - es = (\mathrm{id}_X - e)s = 0$, so $es = s$. Thus, $s(rs) = (sr)s = es = s$ and hence $rs = \mathrm{id}_Y$ as s is a monomorphism. This shows e splits.

The equivalence between (i) and (iii) is dual. ■

Given an idempotent $e\colon X \to X$ in an additive category with split idempotents, we can identify two direct summands of X. This result is also classical (see e.g. Auslander [5, p. 188]).

Proposition 3.11. *If an additive category $\mathcal{A}$ has split idempotents, then for each idempotent $e\colon X \to X$ we have $X = \mathrm{Ker}(e) \oplus \mathrm{Ker}(\mathrm{id}_X - e)$.*

Proof. Let $e \in \mathrm{End}_{\mathcal{A}}(X)$ be an idempotent. Arguing as in the proof of Proposition 3.10, we obtain a commutative diagram



where $p_j i_j = \mathrm{id}_{X_j}$ for $j = 1, 2$. We also have $\mathrm{id}_X = (\mathrm{id}_X - e) + e = i_1 p_1 + i_2 p_2$. Furthermore, $i_1 p_1 i_2 = (\mathrm{id}_X - e) i_2 = 0$ as $i_2 = \ker(\mathrm{id}_X - e)$, so $p_1 i_2 = 0$ since i_1 is monic. Similarly, $p_2 i_1 = 0$. One can then easily check that X satisfies the universal property for the coproduct $X_1 \amalg X_2$. Hence, $X = X_1 \amalg X_2 = X_1 \oplus X_2 = \mathrm{Ker}(e) \oplus \mathrm{Ker}(\mathrm{id}_X - e)$. ■

4. Krull-Schmidt categories and semi-perfect rings

In this section, we follow [18] in introducing Krull-Remak-Schmidt decompositions and Krull-Schmidt categories (see Definition 4.6). We include proofs where we think the reader may benefit from some extra detail; otherwise proofs are omitted and can be found in [18]. See also [8, App. A].

For this section, let $\mathcal{A}$ denote an additive category. For an object $X \in \mathcal{A}$, we let $\mathsf{add}\, X$ denote the full subcategory of $\mathcal{A}$ consisting of all direct summands of finite direct sums of copies of X. For a ring Λ, we denote by $\mathsf{proj}\, \Lambda$ the full subcategory of $\mathsf{Mod}\, \Lambda$ consisting of finitely generated projective right Λ-modules. We note that $\mathsf{proj}\, \Lambda = \mathsf{add}\, \Lambda$.

Remark 4.1. The category $\mathsf{proj}\, \Lambda$ has split idempotents (see [18, Exam. 2.2(2)]). Indeed, suppose that $e \in \mathrm{End}_{\mathsf{proj}\, \Lambda}(P) = \mathrm{End}_{\mathsf{Mod}\, \Lambda}(P)$ is an idempotent. As $\mathsf{Mod}\, \Lambda$ is abelian, the idempotents e and $\mathrm{id}_P - e$ have kernels in $\mathsf{Mod}\, \Lambda$. By Proposition 3.11, we know $P = \mathrm{Ker}(e) \oplus \mathrm{Ker}(\mathrm{id}_P - e)$ in $\mathsf{Mod}\, \Lambda$. It follows that $\mathrm{Ker}(e)$ is projective and finitely generated, so e has a kernel in $\mathsf{proj}\, \Lambda$. Therefore, $\mathsf{proj}\, \Lambda$ has split idempotents by Proposition 3.10.

The following result gives a way to turn questions about an object in an additive category into questions about projective modules in a module category. This was termed "projectivization" in [6, §II.2].

Proposition 4.2. *Suppose that $X \in \mathcal{A}$ and set $\Lambda_X := \mathrm{End}_{\mathcal{A}}(X)$. The additive functor $H_X(-) := \mathrm{Hom}_{\mathcal{A}}(X, -) \colon \mathcal{A} \to \mathsf{Mod}\, \Lambda_X$ induces a fully faithful additive functor $\mathsf{add}\, X \to \mathsf{proj}\, \Lambda_X$. If $\mathcal{A}$ has split idempotents then this is an equivalence. For each $X_0 \in \mathsf{add}\, X$, the induced map $F_{X_0} \colon \mathrm{End}_{\mathcal{A}}(X_0) \to \mathrm{End}_{\mathsf{Mod}\, \Lambda_X}(H_X(X_0))$ is an isomorphism of rings.*

Proof. The first two assertions follow from [18, Prop. 2.3]. For the last claim, we have that the induced map F_{X_0} is a bijective homomorphism of abelian groups since $\mathrm{Hom}_{\mathcal{A}}(X, -)$ is additive and fully faithful on $\mathsf{add}\, X$. Since $\mathrm{Hom}_{\mathcal{A}}(X, -)$ is a covariant functor, the map F_{X_0} preserves composition (i.e. is multiplicative) and identities (i.e. is a map of unital rings). Hence, F_{X_0} is a unital ring isomorphism. ■

The next definition is a generalisation of an indecomposable module.

Definition 4.3. An object $X \in \mathcal{A}$ is called *indecomposable* if X is non-zero, and if $X_1 = 0$ or $X_2 = 0$ whenever there is an isomorphism $X \cong X_1 \oplus X_2$.

Remark 4.4. Let $X \in \mathcal{A}$. Since $\mathsf{add}\, X$ is closed under direct summands, an object $Y \in \mathsf{add}\, X$ is indecomposable in $\mathsf{add}\, X$ if and only if it is indecomposable in $\mathcal{A}$.

The following result is very well-known (see e.g. Harada [14]).

Lemma 4.5. *If* $\mathrm{End}_{\mathcal{A}}(X)$ *is local, then* X *is indecomposable.*

Proof. Suppose $\mathrm{End}_{\mathcal{A}}(X)$ is local and that $X = X_1 \oplus X_2$. Note that $\mathrm{End}_{\mathcal{A}}(X) \neq 0$ (see Definition 3.3) so, in particular, we see that $\mathrm{id}_X \neq 0$ and hence $X \neq 0$. Let $p_j : X \twoheadrightarrow X_j$, respectively, $i_j : X_j \hookrightarrow X$, be the canonical projection, respectively, inclusion for $j = 1, 2$. Then $e_j := i_j p_j \in \mathrm{End}_{\mathcal{A}}(X)$ is idempotent and so we must have e_j is 0 or id_X for $j = 1, 2$ by Lemma 3.5. If $e_1 = 0 = e_2$, then we would have $\mathrm{id}_X = e_1 + e_2 = 0$, which is a contradiction. Hence, without loss of generality, $e_1 \neq 0$ and so $e_1 = \mathrm{id}_X$. Moreover, this yields $e_2 = 0$ and hence $X_2 = 0$, so X is indecomposable. ∎

Let us now state the main definition of this section.

Definition 4.6. A finite direct sum decomposition $X = X_1 \oplus \cdots \oplus X_n$ of $X \in \mathcal{A}$, where $\mathrm{End}_{\mathcal{A}}(X_j)$ is a local ring for all $1 \leqslant j \leqslant n$, is called a *Krull-Remak-Schmidt decomposition* of X. Two Krull-Remak-Schmidt decompositions $X = X_1 \oplus \cdots \oplus X_n$ and $X = Y_1 \oplus \cdots \oplus Y_m$ of X are said to be *equivalent* if $m = n$ and there is a permutation $\sigma \in \mathrm{Sym}(n)$ such that $X_j \cong Y_{\sigma(j)}$ for all $1 \leqslant j \leqslant n$.

If every object in $\mathcal{A}$ admits a Krull-Remak-Schmidt decomposition, then $\mathcal{A}$ is known as a *Krull-Schmidt* category.

Remark 4.7. Notice that in an additive category $\mathcal{A}$, a zero object is the direct sum of an empty (and hence finite) family of objects each having a local endomorphism ring.

An immediate consequence of Definition 4.6 is the following.

Lemma 4.8. *In a Krull-Schmidt category* $\mathcal{A}$*, an object* $X \in \mathcal{A}$ *is indecomposable if and only if* $\mathrm{End}_{\mathcal{A}}(X)$ *is local.*

Proof. Lemma 4.5 treats one direction, so we suppose that X is indecomposable and show that $\mathrm{End}_{\mathcal{A}}(X)$ is local. As $\mathcal{A}$ is Krull-Schmidt, we have a decomposition $X = X_1 \oplus \cdots \oplus X_n$, where each $\mathrm{End}_{\mathcal{A}}(X_j)$ is local. However, we must have $n = 1$ as X is indecomposable, and so $\mathrm{End}_{\mathcal{A}}(X) = \mathrm{End}_{\mathcal{A}}(X_1)$ is local. ∎

The next proposition follows from [18, Prop. 4.1].

Proposition 4.9. *The following are equivalent for a ring* Λ.

(i) The category $\mathsf{proj}\,\Lambda$ *is Krull-Schmidt.*
(ii) The right Λ*-module* Λ_Λ *admits a decomposition* $\Lambda_\Lambda = P_1 \oplus \cdots \oplus P_n$*, where each* P_j *has a local endomorphism ring.*

Definition 4.10. If a ring Λ satisfies the equivalent conditions of Proposition 4.9, then it is called *semi-perfect*.

Notice that for a semi-perfect ring Λ, the decomposition in Proposition 4.9(ii) is a Krull-Remak-Schmidt decomposition in $\mathsf{proj}\,\Lambda \subseteq \mathsf{Mod}\,\Lambda$.

As remarked in [18], the following theorem is a consequence of the existence and uniqueness of projective covers over a semi-perfect ring.

Theorem 4.11 (*[18, Thm. 4.2]*). *Let $X \in \mathcal{A}$ be an object. Suppose $X_1 \oplus \cdots \oplus X_n = X = Y_1 \oplus \cdots \oplus Y_m$ for some objects X_j, Y_l each having a local endomorphism ring. Then these two Krull-Remak-Schmidt decompositions are equivalent.*

There are two immediate consequences.

Corollary 4.12 (*[18, Cor. 4.3]*). *If $\mathcal{A}$ is Krull-Schmidt and there are two decompositions $X_1 \oplus \cdots \oplus X_n = X = X' \oplus X''$, where each X_j is indecomposable, then there exists $t \leqslant n$ such that $X = X_1 \oplus \cdots \oplus X_t \oplus X'$ (possibly after reindexing).*

Corollary 4.13 (*[8, Thm. A.1], [18, Cor. 4.4]*). *An additive category $\mathcal{A}$ is Krull-Schmidt, if and only if it has split idempotents and $\mathrm{End}_{\mathcal{A}}(X)$ is semi-perfect for each $X \in \mathcal{A}$.*

Proof. If $\mathcal{A}$ has split idempotents and $\Lambda_X := \mathrm{End}_{\mathcal{A}}(X)$ is semi-perfect for all $X \in \mathcal{A}$, then by Propositions 4.2 and 4.9 we have that $\mathcal{A}$ is Krull-Schmidt. Indeed, a Krull-Remak-Schmidt decomposition of Λ_X as in Proposition 4.9(ii) yields a Krull-Remak-Schmidt decomposition of X in $\mathcal{A}$ using the equivalence $\mathsf{add}\,X \simeq \mathsf{proj}\,\Lambda_X$ induced by $\mathrm{Hom}_{\mathcal{A}}(X, -)$.

Conversely, suppose $\mathcal{A}$ is a Krull-Schmidt category and let $X \in \mathcal{A}$ be arbitrary. By hypothesis, we may take a finite direct sum decomposition $X = X_1 \oplus \cdots \oplus X_n$ such that $\mathrm{End}_{\mathcal{A}}(X_j)$ is local for each j. We have $(\Lambda_X)_{\Lambda_X} = \mathrm{End}_{\mathcal{A}}(X) \cong \mathrm{Hom}_{\mathcal{A}}(X, X_1) \oplus \cdots \oplus \mathrm{Hom}_{\mathcal{A}}(X, X_n)$ and $\mathrm{End}_{\mathsf{Mod}\,\Lambda_X}(\mathrm{Hom}_{\mathcal{A}}(X, X_j)) \cong \mathrm{End}_{\mathcal{A}}(X_j)$ is local (using the fully faithfulness in Proposition 4.2). Therefore, Λ_X is a semi-perfect ring, and equivalently $\mathsf{proj}\,\Lambda_X$ is Krull-Schmidt by Proposition 4.9. Therefore, given any finitely generated projective Λ_X-module, it will isomorphic to a direct summand of $\Lambda_X^m = \mathrm{Hom}_{\mathcal{A}}(X, X_1)^m \oplus \cdots \oplus \mathrm{Hom}_{\mathcal{A}}(X, X_n)^m$ by Corollary 4.12. This means that $\mathrm{Hom}_{\mathcal{A}}(X, -) \colon \mathsf{add}\,X \to \mathsf{proj}\,\Lambda_X$ is also dense, and hence an equivalence. The category $\mathsf{proj}\,\Lambda_X$ has split idempotents (see Remark 4.1) and hence so does $\mathsf{add}\,X$. Moreover, this implies $\mathcal{A}$ has split idempotents. ■

We close this section with two results related to semi-perfect rings that will be needed for the main result of the paper. Although a stronger version of the next lemma can be found in [2, Thm. 27.6], the proof below is inspired by that of Auslander–Reiten–Smalø [6, Prop. I.4.8].

Lemma 4.14. *Suppose Λ is semi-perfect and that $\Lambda = P_1 \oplus \cdots \oplus P_n$ as right Λ-modules, where $\mathrm{End}_{\mathsf{Mod}\,\Lambda}(P_j)$ is local for $1 \leqslant j \leqslant n$. Then Λ admits a complete set $\{e_j\}_{j=1}^n$ of primitive orthogonal idempotents, such that $P_j = e_j\Lambda$ and $e_j\Lambda e_j$ is local for $1 \leqslant j \leqslant n$.*

Proof. Since $\Lambda = P_1 \oplus \cdots \oplus P_n$, we may express the identity 1_Λ of Λ as $1_\Lambda = e_1 + \cdots + e_n$ for some elements $e_j \in P_j$. Fix $l \in \{1, \ldots, n\}$. Then we have $e_1e_l + \cdots + e_ne_l =$

$(e_1 + \cdots + e_n)e_l = 1_\Lambda \cdot e_l = e_l \in P_l$. But since each P_j is a right Λ-module, we have $e_j e_l \in P_j$. Therefore, using the direct sum decomposition of Λ, we see that $e_j e_l = 0$ for all $j \neq l$ and that $e_l = e_l e_l$, i.e. the set $\{e_j\}_{j=1}^n$ forms a complete set of orthogonal idempotents.

We claim that $e_l\Lambda = P_l$. Since $e_l \in P_l$ and P_l is a right Λ-module, we immediately see that $e_l\Lambda \subseteq P_l$. Conversely, let $x \in P_l \subseteq \Lambda$ be arbitrary. By the same argument as above, $e_j x = 0$ for $j \neq l$ and $x = e_l x \in e_l\Lambda$, so that we have the equality $e_l\Lambda = P_l$. Lastly, using the isomorphism (3.1), we have $e_l\Lambda e_l \cong \mathrm{End}_{\mathsf{Mod}\,\Lambda}(e_l\Lambda) = \mathrm{End}_{\mathsf{Mod}\,\Lambda}(P_l)$ is local, so $e_l = 1_{e_l\Lambda e_l} \neq 0$ and e_l is primitive by combining Lemmas 3.5 and 3.6. ■

The following result is a consequence of Jacobson [15, Thm. III.10.2]. Jacobson states the result with the assumption that the primitive idempotents give rise to local corner rings. Here we suppose that the ring Λ itself is semi-perfect so that $\mathsf{proj}\,\Lambda$ is Krull-Schmidt, and this implies the hypothesis needed in [15]. Furthermore, we note that the proof below appears in a pre-published version of Liu–Ng–Paquette [21].

Proposition 4.15. *Let Λ be a semi-perfect ring, and suppose $\{e_j\}_{j=1}^n, \{f_l\}_{l=1}^m$ are complete sets of primitive orthogonal idempotents in Λ. Then $m = n$, and there exists a permutation $\sigma \in \mathrm{Sym}(n)$ and an invertible element $a \in \Lambda$ such that $f_{\sigma(j)} = a e_j a^{-1}$ for all $1 \leqslant j \leqslant n$.*

Proof. Since $\{e_j\}_{j=1}^n, \{f_l\}_{l=1}^m$ are complete sets of primitive orthogonal idempotents in Λ, we obtain $e_1\Lambda \oplus \cdots \oplus e_n\Lambda = \Lambda = f_1\Lambda \oplus \cdots \oplus f_m\Lambda$, where $e_j\Lambda$ and $f_l\Lambda$ are indecomposable by Lemma 3.6. Moreover, since Λ is semi-perfect we know $\mathsf{proj}\,\Lambda$ is Krull-Schmidt and so by Proposition 4.9 we have that $\mathrm{End}_{\mathsf{Mod}\,\Lambda}(e_j\Lambda)$ and $\mathrm{End}_{\mathsf{Mod}\,\Lambda}(f_l\Lambda)$ are local rings. Therefore, we may apply Theorem 4.11 so that $m = n$ and there is a permutation $\sigma \in \mathrm{Sym}(n)$, such that $e_j\Lambda = f_{\sigma(j)}\Lambda$, for all $1 \leqslant j \leqslant n$.

Hence, there exist $b_j, c_j \in \Lambda$, such that $e_j = f_{\sigma(j)}b_j = f_{\sigma(j)}b_j e_j$ and $f_{\sigma(j)} = e_j c_j f_{\sigma(j)}$ for each $1 \leqslant j \leqslant n$. In particular, $e_j = (e_j c_j f_{\sigma(j)})(f_{\sigma(j)} b_j e_j)$ and

$$f_{\sigma(j)} = (f_{\sigma(i)} b_j e_j)(e_j c_j f_{\sigma(j)}). \tag{4.1}$$

Set $a := \sum_{j=1}^n f_{\sigma(j)} b_j e_j$ and $a^{-1} := \sum_{l=1}^n e_l c_l f_{\sigma(l)}$. We observe that

$$\begin{aligned} a \cdot a^{-1} &= \left(\sum_{j=1}^n f_{\sigma(j)} b_j e_j\right)\left(\sum_{l=1}^n e_l c_l f_{\sigma(l)}\right) \\ &= \sum_{j=1}^n (f_{\sigma(j)} b_j e_j)(e_j c_j f_{\sigma(j)}) && \text{as } \{e_j\}_{j=1}^n \text{ is orthogonal} \\ &= \sum_{j=1}^n f_{\sigma(j)} && \text{using (4.1)} \\ &= 1_\Lambda && \text{as } \{f_j\}_{j=1}^m \text{ is complete.} \end{aligned}$$

Similarly, one can show $a^{-1}a = 1_\Lambda$. Finally, for $r \in \{1, \dots, n\}$, we see that

$$ae_ra^{-1} = \left(\sum_{j=1}^{n} f_{\sigma(j)}b_je_j\right) e_r \left(\sum_{l=1}^{n} e_lc_l f_{\sigma(l)}\right) = (f_{\sigma(r)}b_re_r)e_r(e_rc_r f_{\sigma(r)}) = f_{\sigma(r)},$$

again using that $\{e_j\}_{j=1}^n$ is orthogonal and (4.1), and this finishes the proof. ■

5. Subobjects, the bi-chain condition and Homhom-finiteness

The goal of this section is to see that the endomorphism ring of any object in a Hom-finite additive category (see Definition 5.7) is semi-perfect (see Corollary 5.9). This is shown via Atiyah's bi-chain conditions (see Definition 5.5).

Definition 5.1 (*[25, §I.5], [18, p. 539]*)**.** Suppose $\mathcal{B}$ is an abelian category and let $X \in \mathcal{A}$. Two monomorphisms $a\colon X_1 \hookrightarrow X$ and $b\colon X_2 \hookrightarrow X$ are *equivalent* if there is an isomorphism $c\colon X_1 \xrightarrow{\cong} X_2$ such that $bc = a$. This is an equivalence relation on the collection of monomorphisms in $\mathcal{B}$ with codomain X, and an equivalence class of this relation is called a *subobject* of X. By abuse of notation we just write $a\colon X_1 \hookrightarrow X$ for the equivalence class containing the monomorphism a.

Given two subobjects $A \overset{f}{\hookrightarrow} X$ and $B \overset{g}{\hookrightarrow} X$ of X, we say that A is *contained in* B (denoted $A \subseteq B$) if there is a (necessarily monic) morphism $h\colon A \to B$ such that $f = gh$. In this way, we obtain a partial order on the collection of subobjects of X.

We assume the following throughout this section.

Setup 5.2. We denote by $\mathcal{B}$ an abelian category. We make the implicit assumption that the collection of subobjects of an object $X \in \mathcal{B}$ is a set.

Remark 5.3. The set-theoretic restriction in Setup 5.2 is not so strong. For example, the condition is satisfied if $\mathcal{B}$ has a generator (see Freyd [11, Prop. 3.35]), or if $\mathcal{B}$ is skeletally small. In particular, the category of all (right) modules over a ring falls into Setup 5.2.

We call $X \in \mathcal{B}$ *simple* if $X \neq 0$ and its only subobjects are 0 and X itself (see [18, p. 539]). Given a subobject $A \overset{f}{\hookrightarrow} X$, we denote by X/A the codomain of the cokernel map $\operatorname{coker} f\colon X \twoheadrightarrow \operatorname{Coker} f$.

Definition 5.4 (*[18, p. 547]*)**.** An object $X \in \mathcal{B}$ is said to have *finite length* if there is a finite chain (called a *composition series*)

$$0 = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_{n-1} \subseteq X_n = X$$

of subobjects of X such that each successive quotient X_{j+1}/X_j is simple.

Now we recall the bi-chain condition in an abelian category as introduced in [4].

Definition 5.5 (*[4, p. 310], [18, p. 546]*)**.** A *bi-chain* in $\mathcal{B}$ is a sequence of morphisms

$$(\, X_j \xrightarrow{\alpha_j} X_{j+1} \xhookrightarrow{\beta_j} X_j \,)_{j \geqslant 0} \tag{5.1}$$

in $\mathcal{B}$, for which α_j is epic and β_j is monic for all $j \geqslant 0$. An object $X \in \mathcal{B}$ is said to *satisfy the bi-chain condition* if for any bi-chain (5.1) with $X_0 = X$, there exists $N \geqslant 0$ such that α_j, β_j are isomorphisms $\forall j \geqslant N$.

If $X \in \mathcal{B}$ satisfies the bi-chain condition, then X is indecomposable if and only if $\operatorname{End}_{\mathcal{B}}(X)$ is local; see [18, Prop. 5.4], also [4, Lem. 6].

Lemma 5.6 (*[18, Lem. 5.1]*)**.** *If $X \in \mathcal{B}$ has finite length, then X satisfies the bi-chain condition.*

Definition 5.7 (*[18, p. 547]*)**.** We call a category $\mathcal{A}$ a Hom-*finite k-linear category* if k is a commutative ring, such that $\mathcal{A}$ is k-linear and for which $\operatorname{Hom}_{\mathcal{A}}(X, Y)$ is a finite length k-module for all $X, Y \in \mathcal{A}$. If the ring k or its existence is understood, then we more simply say that $\mathcal{A}$ is Hom-*finite*.

Any object X of a Hom-finite abelian category satisfies the bi-chain condition; see [18, Lem. 5.2].

Theorem 5.8 (*[18, Thm. 5.5], [4, Lem. 4]*)**.** *Suppose $X \in \mathcal{B}$ satisfies the bi-chain condition. Then X admits a finite direct sum decomposition $X = X_1 \oplus \cdots \oplus X_n$, with $\operatorname{End}_{\mathcal{B}}(X_j)$ local for all $1 \leqslant j \leqslant n$.*

Putting together the results above we derive the following.

Corollary 5.9. *If $\mathcal{A}$ is a* Hom-*finite k-linear category, then $\operatorname{End}_{\mathcal{A}}(X)$ is semi-perfect for each $X \in \mathcal{A}$.*

Proof. If $\mathcal{A}$ is a Hom-finite k-linear category, then $\Lambda_X := \operatorname{End}_{\mathcal{A}}(X)$ is a finite length k-module. That is, Λ_X is a finite length object in the abelian category $\mathsf{Mod}\, k$, and hence also a finite length object in $\mathsf{Mod}\, \Lambda_X$. Therefore, by Lemma 5.6 we know Λ_X satisfies the bi-chain condition in $\mathsf{Mod}\, \Lambda_X$. This implies that Λ_X admits a decomposition into a finite direct sum of right Λ_X-modules with local endomorphism rings by Theorem 5.8, which is precisely condition (ii) of Proposition 4.9. Hence, Λ_X is semi-perfect. ■

Another proof of Corollary 5.9. If Λ_X is a finite length object in $\mathsf{Mod}\, \Lambda_X$, then it is an artinian ring. By [20, p. 336], any (one-sided) artinian ring is semi-perfect. ■

6. The main theorem

We are now in position to state and prove the theorem we have been building too. It is well-known and stated in several places, e.g. [27, §2.2], but we could not find a proof. The equivalence below is asserted in [13, §I.3.2] without an explicit Hom-finiteness assumption, but we believe this may be in error. It was a desire to understand this that motivated this note.

We note that the equivalence of (i) and (ii) in Theorem 6.1 follows from Corollary 4.13 once we know each endomorphism ring arising from a Hom-finite category is semi-perfect (see Corollary 5.9). We give a more pedestrian proof of (i) implies (ii) below.

Theorem 6.1. *Let k be a commutative ring and $\mathcal{A}$ a* Hom*-finite k-linear category. Then the following are equivalent.*

(i) $\mathcal{A}$ is a Krull-Schmidt category.
(ii) $\mathcal{A}$ has split idempotents.
(iii) For any object $Y \in \mathcal{A}$, the ring $\operatorname{End}_{\mathcal{A}}(Y)$ is local if and only if Y is indecomposable.

Furthermore, in this case, an object $X \in \mathcal{A}$ admits a Krull-Remak-Schmidt decomposition $X = X_1 \oplus \cdots \oplus X_n$ in $\mathcal{A}$ if and only if $\operatorname{End}_{\mathcal{A}}(X)$ admits a complete set of primitive orthogonal idempotents of size n.

Proof. Throughout this proof we use that, since $\mathcal{A}$ is Hom-finite, the endomorphism ring $\Lambda_X := \operatorname{End}_{\mathcal{A}}(X)$ of each object $X \in \mathcal{A}$ is semi-perfect by Corollary 5.9. Furthermore, this implies $\operatorname{proj} \Lambda_X$ is Krull-Schmidt by Proposition 4.9.

(i) $\Rightarrow$ (ii) Fix an object $X \in \mathcal{A}$. If $X = 0$ then any idempotent $e \in \operatorname{End}_{\mathcal{A}}(X)$ is trivially split, so assume $X \neq 0$. Since $\mathcal{A}$ is Krull-Schmidt, there is a Krull-Remak-Schmidt decomposition $X = X_1 \oplus \cdots \oplus X_n$ of X in $\mathcal{A}$. Consider the canonical projections $p_j : X \twoheadrightarrow X_j$ and inclusions $i_j : X_j \hookrightarrow X$. Putting $e_j := i_j p_j$ for each $1 \leqslant j \leqslant n$, we see that $\{e_j\}_{j=1}^n$ forms a complete set of orthogonal idempotents of Λ_X. Each idempotent e_j is primitive by Lemma 3.7, and hence $\Lambda_X = e_1\Lambda_X \oplus \cdots \oplus e_n\Lambda_X$ is a decomposition into indecomposable right Λ_X-modules using Lemma 3.6.

Suppose $e : X \to X$ is an idempotent morphism. If $e = 0$, then it trivially splits, so we may assume $e \neq 0$. By Example 3.2(ii), we have $e\Lambda_X \oplus (\mathrm{id}_X - e)\Lambda_X = \Lambda_X = \bigoplus_{j=1}^n e_j\Lambda_X$. Therefore, working in the Krull-Schmidt category $\operatorname{proj} \Lambda_X$, we see that $e\Lambda_X = \bigoplus_{j=1}^t e_j\Lambda_X$ and $(\mathrm{id}_X - e)\Lambda_X = \bigoplus_{j=t+1}^n e_j\Lambda_X$ for some $t \in \{1, \dots, n\}$ (possibly after reindexing) by Corollary 4.12. Therefore, we may express $e = e_1r_1 + \cdots + e_tr_t$ and $1_{\Lambda_X} - e = \mathrm{id}_X - e = e_{t+1}r_{t+1} + \cdots + e_nr_n$ for some $r_j \in \Lambda_X$, where $1 \leqslant j \leqslant n$. We claim that $\{g_j := e_jr_j\}_{j=1}^n \subseteq \Lambda_X$ is a complete set of primitive orthogonal idempotents and that $g_j\Lambda_X = e_j\Lambda_X$. First, note that $g_j\Lambda_X = e_jr_j\Lambda_X \subseteq e_j\Lambda_X$ for all $1 \leqslant j \leqslant n$. Now fix $j \in \{1, \dots, t\}$. If we have $x \in e_j\Lambda_X \subseteq e\Lambda_X$, then it satisfies $ex = x$ and $(\mathrm{id}_X - e)x = 0$. Hence, the identity

$$x = 1_{\Lambda_X} \cdot x = e_1r_1x + \cdots + e_nr_nx = g_1x + \cdots + g_nx$$

implies $x = g_jx$ and $g_lx = 0$ for all $l \neq j$, using $\Lambda_X = \bigoplus_{j=1}^n e_j\Lambda_X$, as $g_lx \in e_l\Lambda_X$ and $x \in e_j\Lambda_X$. In particular, this yields $e_j\Lambda_X \subseteq g_j\Lambda_X$ and so $g_j\Lambda_X = e_j\Lambda_X$ for each $1 \leqslant j \leqslant t$. Furthermore, we also see that $g_lg_j = 0$ for $l \neq j$ and $g_j^2 = g_j$ by choosing $x = g_j$. A similar argument yields the same conclusions for $j \in \{t+1, \dots, n\}$. Moreover, by Lemma 3.6 we deduce that g_j is primitive as $g_j\Lambda_X = e_j\Lambda_X$ is indecomposable.

Hence, $\{g_j\}_{j=1}^n$ is a set of primitive orthogonal idempotents in Λ_X, and it is clear that it is complete.

By Proposition 4.15 there exists an invertible element $a \in \Lambda_X$ and a permutation $\sigma \in \mathrm{Sym}(n)$, such that $g_{\sigma(j)} = ae_j a^{-1}$ for all $j = 1, \ldots, n$. But, by inspecting the proof of Proposition 4.15, we observe that $\sigma(j) = j$ for each j as $e_j \Lambda_X = g_j \Lambda_X$. Therefore, $g_j = ae_j a^{-1}$ for each $1 \leqslant j \leqslant n$. Define $Y := X_1 \oplus \cdots \oplus X_t$ and $Z := X_{t+1} \oplus \cdots \oplus X_n$, then $X = Y \oplus Z$. Define morphisms $p := (\, p_1 \, \ldots \, p_t \,)^T a^{-1} \colon X \to Y$ and $i := a(\, i_1 \, \ldots \, i_t \,) \colon Y \to X$. Then

$$ip = a(\, i_1 \, \ldots \, i_t \,) \circ (\, p_1 \, \ldots \, p_t \,)^T a^{-1} = \sum_{j=1}^{t} a i_j p_j a^{-1} = \sum_{j=1}^{t} a e_j a^{-1} = \sum_{j=1}^{t} g_j = e,$$

and pi is the $(t \times t)$-diagonal matrix with diagonal $(p_1 i_1, \ldots, p_t i_t) = (\mathrm{id}_{X_1}, \ldots, \mathrm{id}_{X_t})$, i.e. $pi = \mathrm{id}_Y$. That is, we have shown e splits and hence $\mathcal{A}$ has split idempotents.

(ii) $\Rightarrow$ (iii) Let $Y \in \mathcal{A}$ be arbitrary and note $\Lambda_Y = \mathrm{End}_{\mathcal{A}}(Y) = \mathrm{End}_{\mathsf{add}\, Y}(Y)$. As $\mathcal{A}$ has split idempotents, there is an equivalence $\mathsf{add}\, Y \simeq \mathsf{proj}\, \Lambda_Y$ by Proposition 4.2. In particular, $\mathsf{add}\, Y$ is a Krull-Schmidt category. Using this, Remark 4.4 and Lemma 4.8, we know Y is indecomposable in $\mathcal{A}$, if and only if it is indecomposable in $\mathsf{add}\, Y$, if and only if Λ_Y is local.

(iii) $\Rightarrow$ (i) Fix an object $X \in \mathcal{A}$. If $X = 0$ then it trivially has a Krull-Remak-Schmidt decomposition (see Remark 4.7). Thus, assume $X \neq 0$. As $\Lambda_X = \mathrm{End}_{\mathcal{A}}(X)$ is semi-perfect, there is a direct sum decomposition $(\Lambda_X)_{\Lambda_X} = P_1 \oplus \cdots \oplus P_n$ with $\mathrm{End}_{\mathsf{Mod}\, \Lambda_X}(P_j)$ local for each $1 \leqslant j \leqslant n$. By Lemma 4.14, there is a complete set $\{f_j\}_{j=1}^n \subseteq \Lambda_X$ of primitive orthogonal idempotents, such that $P_j = f_j \Lambda_X$. As in Proposition 4.2, put $H_X(-) = \mathrm{Hom}_{\mathcal{A}}(X, -)$ and recall that there is a ring isomorphism $\mathrm{End}_{\mathcal{A}}(X_0) \to \mathrm{End}_{\mathsf{Mod}\, \Lambda_X}(H_X(X_0))$ induced by $H_X(-)$ for each object $X_0 \in \mathsf{add}\, X$.

We prove by induction on n that X admits a Krull-Remak-Schmidt decomposition $X = X_1 \oplus \cdots \oplus X_n$ of length n in $\mathcal{A}$. If $n = 1$, then $\Lambda_X = P_1$ has a local endomorphism ring. Then the ring isomorphism $\mathrm{End}_{\mathcal{A}}(X) \cong \mathrm{End}_{\mathsf{Mod}\, \Lambda_X}(\Lambda_X)$ implies $\mathrm{End}_{\mathcal{A}}(X)$ is local, so we set $X_1 := X$ and we are done in this case.

Now suppose $n \geqslant 2$ and that the claim holds for positive integers $m < n$. If X is indecomposable, then $\Lambda_X = \mathrm{End}_{\mathcal{A}}(X)$ is local by assumption (iii). This would imply that $1_{\Lambda_X} = \mathrm{id}_X$ is primitive by Lemma 3.5, and then in turn force $n = 1$ by combining Lemma 4.14 and Proposition 4.15, leading to a contradiction. Hence, $X = Y_1 \oplus Y_2$ for some non-zero objects $Y_j \in \mathsf{add}\, X$. We have $H_X(Y_1) \oplus H_X(Y_2) \cong H_X(X) = \Lambda_X = \bigoplus_{j=1}^n f_j \Lambda_X$ in the Krull-Schmidt category $\mathsf{proj}\, \Lambda_X$. Thus,

$$H_X(Y_1) = \bigoplus_{j=1}^{m_1} f_j \Lambda_X \tag{6.1}$$

and $H_X(Y_2) = \bigoplus_{j=m_1+1}^n f_j \Lambda_X$ for some $0 \leqslant m_1 \leqslant n$ (possibly after reindexing) by Corollary 4.12. Since there is the non-zero canonical projection of X onto its summand Y_j for $j = 1, 2$, we must actually have that $1 \leqslant m_1 \leqslant n-1$. Define $f := f_1 + \cdots + f_{m_1}$. Then

$$\Lambda_{Y_1} = \mathrm{End}_{\mathcal{A}}(Y_1)$$

$$\begin{aligned}
&\cong \mathrm{End}_{\mathsf{proj}\,\Lambda_X}(H_X(Y_1)) && \text{by Proposition 4.2 since } Y_1 \in \mathsf{add}\,X\\
&= \mathrm{End}_{\mathsf{Mod}\,\Lambda_X}(H_X(Y_1)) && \text{as } \mathsf{proj}\,\Lambda_X \subseteq \mathsf{Mod}\,\Lambda_X \text{ is a full subcategory}\\
&= \mathrm{End}_{\mathsf{Mod}\,\Lambda_X}\left(\bigoplus_{j=1}^{m_1} f_j\Lambda_X\right) && \text{using (6.1)}\\
&\cong f\Lambda_X f && \text{using (3.1)},
\end{aligned}$$

where the two isomorphisms are ring isomorphisms. It is straightforward to check that $\{f_j\}_{j=1}^{m_1} \subseteq f\Lambda_X f$ is a complete set of primitive orthogonal idempotents, and hence Λ_{Y_1} also admits a complete set of primitive orthogonal idempotents of size m_1. Similarly, Λ_{Y_2} has a complete set of primitive orthogonal idempotents of size $m_2 := n - m_1$, where $1 \leqslant m_2 \leqslant n-1$. Therefore, we can apply our induction hypothesis to Y_1 and Y_2, which produces a Krull-Remak-Schmidt decomposition $X = Y_1 \oplus Y_2 = X'_1 \oplus \cdots \oplus X'_{m_1} \oplus X''_1 \oplus \cdots \oplus X''_{m_2}$ of X in $\mathcal{A}$ of length $m_1 + m_2 = n$. ■

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Acknowledgements

This work began during my Ph.D. at University of Leeds. I would like to thank Bethany R. Marsh, whose guidance and patience at that time helped me understand the content of this note. I am very grateful to Raphael Bennett-Tennenhaus who made me aware of the contributions of R. E. Remak to the theory of direct sum decompositions and who gave me valuable comments. I also thank Xiao-Wu Chen, Alberto Facchini and Henning Krause for helpful discussions and email communications in the preparation of this note. I would also like to thank the referee whose comments have helped improve the exposition.

During the preparation of this note I have been financially supported by the following, for which I am also grateful: a University of Leeds 110 Anniversary Research Scholarship; the Engineering and Physical Sciences Research Council (grant EP/P016014/1); an Early Career Fellowship from the London Mathematical Society with support from the Heilbronn Institute for Mathematical Research (grant ECF-1920-57); a DNRF Chair from the Danish National Research Foundation (grant DNRF156); a Research Project 2 from the Independent Research Fund Denmark (grant 1026-00050B); and the Aarhus University Research Foundation (grant AUFF-F-2020-7-16).

References

[1] P. Aluffi, Algebra: Chapter 0, in: Graduate Studies in Mathematics, vol. 104, American Mathematical Society, Providence, RI, 2009, p. xx+713.
[2] F.W. Anderson, K.R. Fuller, Rings and categories of modules, second ed., in: Graduate Texts in Mathematics, vol. 13, Springer-Verlag, New York, 1992, p. x+376.
[3] I. Assem, D. Simson, A. Skowroński, Elements of the Representation Theory of Associative Algebras. Vol. 1. Techniques of Representation Theory, in: London Mathematical Society Student Texts, vol. 65, Cambridge Univ. Press, Cambridge, 2006, p. x+458.
[4] M. Atiyah, On the Krull–Schmidt theorem with application to sheaves, Bull. Soc. Math. France 84 (1956) 307–317.
[5] M. Auslander, Representation theory of Artin algebras. I, Comm. Algebra 1 (1974) 177–268.
[6] M. Auslander, I. Reiten, S.O. Smalø, Representation Theory of Artin Algebras, in: Cambridge Studies in Advanced Mathematics, vol. 36, Cambridge Univ. Press, Cambridge, 1995, p. xiv+423.
[7] T. Bühler, Exact categories, Expo. Math. 28 (1) (2010) 1–69.
[8] X.-W. Chen, Y. Ye, P. Zhang, Algebras of derived dimension zero, Comm. Algebra 36 (1) (2008) 1–10.
[9] A. Dür, Möbius Functions, Incidence Algebras and Power Series Representations, in: Lecture Notes in Mathematics, vol. 1202, Springer-Verlag, Berlin, 1986, p. xii+134.
[10] A. Facchini, The Krull–Schmidt theorem, in: Handbook of Algebra, in: Handb. Algebr., vol. 3, Elsevier/North-Holland, Amsterdam, 2003, pp. 357–397.
[11] P.J. Freyd, Abelian categories, Repr. Theory Appl. Categ. 3 (2003) 1–190.
[12] H. Frobenius, H. Stickelberger, Über Gruppen von vertauschbaren Elementen, J. Reine Angew. Math. 86 (1879) 217–262.
[13] D. Happel, Triangulated Categories in the Representation Theory of Finite-Dimensional Algebras, in: London Math. Soc. Lecture Note Ser., vol. 119, Cambridge Univ. Press, Cambridge, 1988, p. x+208.
[14] M. Harada, Applications of factor categories to completely indecomposable modules, Publ. Dép. Math. (Lyon) 11 (2) (1974) 19–104.
[15] N. Jacobson, Structure of Rings, revised ed., in: American Mathematical Society Colloquium Publications, vol. 37, American Mathematical Society, Providence, R.I., 1964, p. ix+299.
[16] N. Jacobson, Lectures in Abstract Algebra. Vol. I, in: Graduate Texts in Mathematics, No. 30, Springer-Verlag, New York-Heidelberg, 1975, p. xii+217, Basic concepts, Reprint of the 1951 edition.
[17] M. Karoubi, Algèbres de Clifford et K-théorie, Ann. Sci. Éc. Norm. Supér. (4) 1 (1968) 161–270.
[18] H. Krause, Krull–Schmidt categories and projective covers, Expo. Math. 33 (4) (2015) 535–549.
[19] W. Krull, Über verallgemeinerte endliche Abelsche Gruppen, Math. Z. 23 (1) (1925) 161–196.
[20] T.Y. Lam, A First Course in Noncommutative Rings, second ed., in: Graduate Texts in Mathematics, vol. 131, Springer-Verlag, New York, 2001, p. xx+385.
[21] S. Liu, P. Ng, C. Paquette, Almost split sequences and approximations, Algebr. Represent. Theory 16 (6) (2013) 1809–1827.
[22] S. Mac Lane, Categories for the Working Mathematician, second ed., in: Grad. Texts in Math., vol. 5, Springer-Verlag, New York, 1998, p. xii+314.
[23] J.H. Maclagan-Wedderburn, On the direct product in the theory of finite groups, Ann. of Math. (2) 10 (4) (1909) 173–176.
[24] U.C. Merzbach, Robert Remak and the estimation of units and regulators, in: Amphora, Birkhäuser, Basel, 1992, pp. 481–522.
[25] B. Mitchell, Theory of Categories, in: Pure and Applied Mathematics, vol. XVII, Academic Press, New York-London, 1965, p. xi+273.
[26] R. Remak, Über die Zerlegung der endlichen Gruppen in direkte unzerlegbare Faktoren, J. Reine Angew. Math. 139 (1911) 293–308.
[27] C.M. Ringel, Tame Algebras and Integral Quadratic Forms, in: Lecture Notes in Mathematics, vol. 1099, Springer-Verlag, Berlin, 1984, p. xiii+376.
[28] O. Schmidt, Sur les produits directs, Bull. Soc. Math. France 41 (1913) 161–164.
[29] S.L. Segal, Mathematicians Under the Nazis, Princeton University Press, Princeton, NJ, 2003 p. xxiv+530.
[30] C.L. Walker, R.B. Warfield Jr., Unique decomposition and isomorphic refinement theorems in additive categories, J. Pure Appl. Algebra 7 (3) (1976) 347–359.