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INTEGRAL AND QUASI-ABELIAN HEARTS OF TWIN COTORSION PAIRS ON EXTRIANGULATED CATEGORIES

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ABSTRACT. It was shown recently that the heart $\overline{\mathcal{H}}$ of a twin cotorsion pair $((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$ on an extriangulated category is semi-abelian. We provide a sufficient condition for the heart to be integral and another for the heart to be quasi-abelian. This unifies and improves the corresponding results for exact and triangulated categories. Furthermore, if $\mathcal{T} = \mathcal{U}$, then we show that the Gabriel-Zisman localisation of $\overline{\mathcal{H}}$ at the class of its regular morphisms is equivalent to the heart of the single twin cotorsion pair $(\mathcal{S}, \mathcal{T})$. This generalises and improves the known result for triangulated categories, thereby providing new insights in the exact setting.

1. INTRODUCTION

Extriangulated categories (see Definition 2.17) were introduced in [28] as a simultaneous generalisation of exact and triangulated categories. Many results (see, for example, [19], [37], [18], [24], [25], [35], [36], [38], [39]) that hold for exact and triangulated categories have been *unified*, i.e. shown to hold for extriangulated categories in general. Consequently, these results also hold for a range of categories that are neither exact nor triangulated, because the class of extriangulated categories is closed under certain operations (see, for example, [28], [14]). Furthermore, since exact and triangulated categories play a large role in representation theory, it is natural to ask if extriangulated categories can provide new insights in this area of mathematics. Indeed, there have been novel applications of extriangulated categories in this field; see, for example, [31], [29].

As well as introducing extriangulated categories, Nakaoka and Palu also defined *cotorsion pairs* (see Definition 2.25) on extriangulated categories in [28]. As an analogue of torsion theories defined in [8], cotorsion pairs were first introduced for the category of abelian groups in [33]. These have since been considered in other contexts, such as in module (e.g. [10]), abelian (e.g. [16]), exact (e.g. [12]) and triangulated (e.g. [26]) categories. (We note here that the cotorsion pairs of a Krull-Schmidt, Hom-finite triangulated k -category (k a field) are in one-to-one correspondence with its *torsion theories* in the sense of [20]; see also [2].) In [26] Nakaoka used cotorsion pairs on triangulated categories in order to generalise the homological constructions in [1] and [22]; see also [6], [21].

In [5], Buan and Marsh generalised the situations of [22] and [6] in a way not covered by the results of [26]. However, by using the notion of a *twin cotorsion pair* (a pair of cotorsion pairs satisfying some condition) and its *heart* (a certain subfactor category of

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the ambient triangulated category), Nakaoka was able to generalise this work of Buan and Marsh; see [27]. Aside from the triangulated setting, pairs of cotorsion pairs have appeared in the literature for abelian categories (e.g. [17]) and exact categories (e.g. [23]).

In the more general context of extriangulated categories, twin cotorsion pairs (see Definition 2.27) were introduced and used in [28] to give a bijective correspondence between certain twin cotorsion pairs and admissible model structures. Subsequently, although Liu and Nakaoka mainly focussed on cotorsion pairs in [25], they defined and studied hearts (see Definition 2.28) of twin cotorsion pairs on extriangulated categories. Moreover, several results from the exact and triangulated cases were unified; in particular, it was shown that the heart of a twin cotorsion pair is always *semi-abelian* (see Definition 2.2 and Theorem 2.30).

In this article we study further the heart of a twin cotorsion pair on an extriangulated category, and we provide more unification of the exact and triangulated settings. More precisely, we give a sufficient condition for the heart of a twin cotorsion pair on an extriangulated category to be *integral* and another for the heart to be *quasi-abelian* (see Definitions 2.3 and 2.4, respectively). Examples of categories that are both integral and quasi-abelian include: any abelian category; the category of topological abelian groups (see [32, §2]); and the quotient category $\mathcal{C}/[\mathcal{X}_R]$, where \mathcal{C} is a cluster category (in the sense of [4]), $R \in \mathcal{C}$ is a rigid object and $[\mathcal{X}_R]$ the ideal of morphisms factoring through $\mathcal{X}_R = \text{Ker}(\mathcal{C}(R, -))$ (see [5, Cor. 3.10] and [34, Thm. 5.5]). However, the classes of integral and quasi-abelian categories do not coincide; see Examples 2.9 and 2.10.

Suppose $((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$ is a twin cotorsion pair on an extriangulated category \mathcal{B} , and let $\overline{\mathcal{H}}$ denote its heart. In Theorem 4.2 we provide a sufficient condition for $\overline{\mathcal{H}}$ to be quasi-abelian. Suppose also that \mathcal{B} has enough projectives and enough injectives (see Definition 3.3). We prove that $\overline{\mathcal{H}}$ is an integral category whenever $\mathcal{U} \subseteq \mathcal{S} * \mathcal{T}$ and the subcategory of projective objects in \mathcal{B} is contained in $\mathcal{W} = \mathcal{T} \cap \mathcal{U}$ (see Theorem 3.7). Consequently, we show that if $\mathcal{T} \subseteq \mathcal{U}$ or $\mathcal{U} \subseteq \mathcal{T}$, then $\overline{\mathcal{H}}$ is integral and quasi-abelian (see Corollary 4.3).

Specialising further in §5, we assume $\mathcal{T} = \mathcal{U}$. Thus, the heart $\overline{\mathcal{H}}$ is integral by Corollary 4.3. Recall that a morphism is *regular* if it is both a monomorphism and an epimorphism, and let \mathcal{R} denote the class of regular morphisms in $\overline{\mathcal{H}}$. Then the (Gabriel-Zisman) localisation $\overline{\mathcal{H}}_{\mathcal{R}}$ (see [11]) of $\overline{\mathcal{H}}$ at \mathcal{R} is an abelian category by [5, Thm. 4.8]. The heart $\overline{\mathcal{H}}_{(\mathcal{S}, \mathcal{T})}$ of the single cotorsion pair $(\mathcal{S}, \mathcal{T})$ is also an abelian category (see [25, Thm. 3.2]), and we prove that $\overline{\mathcal{H}}_{\mathcal{R}}$ and $\overline{\mathcal{H}}_{(\mathcal{S}, \mathcal{T})}$ are equivalent (see Theorem 5.6).

Since we do not assume \mathcal{B} to be a Krull-Schmidt category, Theorems 4.2, 3.7, 5.6 give improvements of [23, Thm. 7.4], [23, Thm. 6.2] and [34, Thm. 4.8], respectively.

This paper is organised as follows. In §2 we recall the necessary background material on non-abelian categories, extriangulated structures and twin cotorsion pairs. In §3 and §4, respectively, we give sufficient assumptions for the heart of a twin cotorsion pair to be integral and quasi-abelian. And in §5 we show there is an explicit equivalence from the heart of a cotorsion pair $(\mathcal{S}, \mathcal{T})$ to a localisation of the heart of a twin cotorsion pair of the form $((\mathcal{S}, \mathcal{T}), (\mathcal{T}, \mathcal{V}))$.

2. PRELIMINARIES

2.1. Preabelian categories. The main results of §3 and §4 give a sufficient condition for the heart of a twin cotorsion pair on an extriangulated category to be an integral and a quasi-abelian category, respectively. We recall briefly these definitions in this section. For more details we refer the reader to [32].

Definition 2.1. [30, p. 24], [7, §5.4] A *preabelian* category is an additive category in which every morphism has a kernel and a cokernel.

We will see later that the heart of any twin cotorsion pair on an extriangulated category is always semi-abelian (see Theorem 2.30). We recall the definition of such a category now. Let \mathcal{A} be a preabelian category.

Definition 2.2. [32, p. 167] We call \mathcal{A} *left semi-abelian* if each morphism $f: A \rightarrow B$ factorises as $f = ip$ for some monomorphism i and cokernel p . Dually, we call \mathcal{A} *right semi-abelian* if each morphism f decomposes as $f = ip$ with i a kernel and p some epimorphism. And \mathcal{A} is called *semi-abelian* when it is both left and right semi-abelian.

Now we recall the main definitions of this section.

Definition 2.3. [32, p. 168] The category \mathcal{A} is called *left integral* if epimorphisms are stable under pullback. Dually, \mathcal{A} is called *right integral* if monomorphisms are stable under pushout. And if \mathcal{A} is both left and right integral, then it is called *integral*.

Definition 2.4. [32, p. 168] The category \mathcal{A} is called *left quasi-abelian* if cokernels are stable under pullback. Dually, \mathcal{A} is called *right quasi-abelian* if kernels are stable under pushout. And if \mathcal{A} is both left and right quasi-abelian, then it is called *quasi-abelian*.

The following two results are used in the proofs of our main results in §§3–4.

Proposition 2.5. [32, p. 173, Cor.] *A semi-abelian category is left integral if and only if it is right integral.*

Proposition 2.6. [32, Prop. 3] *A semi-abelian category is left quasi-abelian if and only if it is right quasi-abelian.*

We conclude this section with some examples. In particular, Examples 2.9 and 2.10 demonstrate that the class of integral categories is not contained in the class of quasi-abelian categories, and *vice versa*.

Example 2.7. Every abelian category is both an integral category and a quasi-abelian category.

Example 2.8. Rump showed that every integral category and every quasi-abelian category is semi-abelian; see [32, p. 169, Cor. 1].

Example 2.9. The category of Hausdorff topological abelian groups is quasi-abelian, but not integral (see [32, §2.2]).

Example 2.10. Let $k \in \{\mathbb{R}, \mathbb{C}\}$. Consider the category **BOR** of *bornological locally convex spaces* over k ; see, for example, [15, p. 50]. Bonnet and Dierolf showed that **BOR** is not quasi-abelian in [3]. However, it has recently been shown that **BOR** is integral; see [13, Thm. 3.5].

2.2. Extriangulated categories. In this section, we recall the theory of extriangulated categories that we will need. See [28] for more details.

Setup 2.11. Throughout the rest of this paper, \mathcal{B} denotes an additive category and we assume \mathcal{B} is equipped with a biadditive functor $\mathbb{E}: \mathcal{B}^{\text{op}} \times \mathcal{B} \rightarrow \mathbf{Ab}$, where \mathbf{Ab} denotes the category of all abelian groups.

We note that a biadditive functor was called an ‘additive bifunctor’ in [9, §1.2, p. 649]. Moreover, each morphism $f: X \rightarrow Y$ in \mathcal{B} gives rise to abelian group homomorphisms $\mathbb{E}(C, f): \mathbb{E}(C, X) \rightarrow \mathbb{E}(C, Y)$ and $\mathbb{E}(f^{\text{op}}, A): \mathbb{E}(Y, A) \rightarrow \mathbb{E}(X, A)$ for objects $A, C \in \mathcal{B}$. By abuse of notation, we will write $\mathbb{E}(f, A)$ instead of $\mathbb{E}(f^{\text{op}}, A)$.

Definition 2.12. [28, Def. 2.1, Rem. 2.2, Def. 2.3] An element δ of $\mathbb{E}(C, A)$ for some objects A, C in \mathcal{B} is called an \mathbb{E} -extension, or simply an *extension*.

For morphisms $f: A \rightarrow X$ and $g: Y \rightarrow C$ and for an extension $\delta \in \mathbb{E}(C, A)$, we obtain the new extensions:

$$f_*\delta := \mathbb{E}(C, f)(\delta) \in \mathbb{E}(C, X) \quad \text{and} \quad g^*\delta := \mathbb{E}(g, A)(\delta) \in \mathbb{E}(Y, A).$$

Let $\delta \in \mathbb{E}(C, A)$ and $\delta' \in \mathbb{E}(D, B)$ be any extensions. A *morphism of extensions* $\delta \rightarrow \delta'$ is a pair (f, h) of morphisms $f: A \rightarrow B$ and $h: C \rightarrow D$ in \mathcal{B} , such that $f_*\delta = h^*\delta'$.

We use the following facts without reference throughout the remainder of the article. Let $\delta \in \mathbb{E}(C, A)$ be an extension. Then any morphism $f: A \rightarrow X$ in \mathcal{B} induces a morphism $(f, 1_C): \delta \rightarrow f_*\delta$ of extensions. Similarly, any morphism $g: Y \rightarrow C$ in \mathcal{B} gives rise to a morphism $(1_A, g): g^*\delta \rightarrow \delta$.

Definition 2.13. [28, Def. 2.7] Let A, C be objects in \mathcal{B} . Two sequences $A \xrightarrow{a} B \xrightarrow{b} C$ and $A \xrightarrow{a'} B' \xrightarrow{b'} C$ of composable morphisms in \mathcal{B} are said to be *equivalent* if there exists an isomorphism $g: B \rightarrow B'$ such that the diagram

$$\begin{array}{ccccc} A & \xrightarrow{a} & B & \xrightarrow{b} & C \\ \parallel & & \cong \downarrow g & & \parallel \\ A & \xrightarrow{a'} & B' & \xrightarrow{b'} & C \end{array}$$

commutes. This determines an equivalence relation on the class of sequences of the form $A \longrightarrow - \longrightarrow C$. We denote by $[A \xrightarrow{a} B \xrightarrow{b} C]$ the equivalence class of a sequence $A \xrightarrow{a} B \xrightarrow{b} C$.

Setup 2.14. Throughout the rest of this paper, let \mathfrak{s} be a correspondence that, for each pair of objects A, C in \mathcal{B} , assigns to each extension $\delta \in \mathbb{E}(C, A)$ an equivalence class $\mathfrak{s}(\delta) = [A \xrightarrow{a} B \xrightarrow{b} C]$.

Soon we will see that, provided \mathfrak{s} satisfies some conditions, \mathfrak{s} will allow us to visualise the abstract structure of $(\mathcal{B}, \mathbb{E})$. The next two definitions make this more precise.

Definition 2.15. [28, Def. 2.9] Let $\delta \in \mathbb{E}(C, A)$ and $\delta' \in \mathbb{E}(C', A')$ be any pair of extensions. Suppose $\mathfrak{s}(\delta) = [A \xrightarrow{a} B \xrightarrow{b} C]$ and $\mathfrak{s}(\delta') = [A' \xrightarrow{a'} B' \xrightarrow{b'} C']$. We call \mathfrak{s} a *realisation of \mathbb{E}* if, for all morphisms $(f, h): \delta \rightarrow \delta'$ of extensions, there exists $g: B \rightarrow B'$ such that the diagram

$$\begin{array}{ccccc} A & \xrightarrow{a} & B & \xrightarrow{b} & C \\ \downarrow f & & \downarrow \exists g & & \downarrow h \\ A' & \xrightarrow{a'} & B' & \xrightarrow{b'} & C' \end{array}$$

commutes. In this case, we say δ is *realised by* $A \xrightarrow{a} B \xrightarrow{b} C$ and that (f, h) is *realised by* (f, g, h) . We also say $A \xrightarrow{a} B \xrightarrow{b} C$ *realises* δ , and (f, g, h) *realises* (f, h) .

Definition 2.16. [28, Def. 2.10] Suppose \mathfrak{s} is a realisation of \mathbb{E} . We call \mathfrak{s} *additive* if the following conditions are satisfied.

- (i) For each $A, C \in \mathcal{B}$, the trivial element $0 \in \mathbb{E}(C, A)$ is realised by the split sequence

$$A \xrightarrow{\begin{pmatrix} 1_A \\ 0 \end{pmatrix}} A \oplus C \xrightarrow{\begin{pmatrix} 0 & 1_C \end{pmatrix}} C.$$

- (ii) Let $\delta \in \mathbb{E}(C, A)$ and $\delta' \in \mathbb{E}(C', A')$ be any extensions. Let $i_X: X \hookrightarrow A \oplus A'$ denote the canonical inclusion for $X \in \{A, A'\}$, and let $p_Y: C \oplus C' \twoheadrightarrow Y$ denote the canonical projection for $Y \in \{C, C'\}$. If $\mathfrak{s}(\delta) = [A \xrightarrow{a} B \xrightarrow{b} C]$ and $\mathfrak{s}(\delta') = [A' \xrightarrow{a'} B' \xrightarrow{b'} C']$, then the element

$$(i_A)_*(p_C)^*\delta + (i_{A'})_*(p_{C'})^*\delta' \in \mathbb{E}(C \oplus C', A \oplus A')$$

is realised by the direct sum

$$A \oplus A' \xrightarrow{\begin{pmatrix} a & 0 \\ 0 & a' \end{pmatrix}} B \oplus B' \xrightarrow{\begin{pmatrix} b & 0 \\ 0 & b' \end{pmatrix}} C \oplus C'.$$

We are now in a position to recall the main definition of this section.

Definition 2.17. [28, Def. 2.12] Let \mathcal{B} be an additive category. A triple $(\mathcal{B}, \mathbb{E}, \mathfrak{s})$ is called an *extriangulated category* if the following axioms are satisfied.

- (ET1) $\mathbb{E}: \mathcal{B}^{\text{op}} \times \mathcal{B} \rightarrow \mathbf{Ab}$ is a biadditive functor.
 (ET2) \mathfrak{s} is an additive realisation of \mathbb{E} .
 (ET3) Let $\delta \in \mathbb{E}(C, A)$ and $\delta' \in \mathbb{E}(C', A')$ be any extensions, and suppose $\mathfrak{s}(\delta) = [A \xrightarrow{a} B \xrightarrow{b} C]$ and $\mathfrak{s}(\delta') = [A' \xrightarrow{a'} B' \xrightarrow{b'} C']$. For any commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{a} & B & \xrightarrow{b} & C \\ \downarrow f & & \downarrow g & & \\ A' & \xrightarrow{a'} & B' & \xrightarrow{b'} & C' \end{array}$$

there exists $h: C \rightarrow C'$, such that $(f, h): \delta \rightarrow \delta'$ is a morphism of extensions that is realised by (f, g, h) .

(ET3^{op}) Dual of (ET3).

(ET4) Let $\delta \in \mathbb{E}(D, A)$ and $\delta' \in \mathbb{E}(F, B)$ be extensions, realised by $A \xrightarrow{a} B \xrightarrow{a'} D$ and $B \xrightarrow{b} C \xrightarrow{b'} F$, respectively. Then there exists $E \in \mathcal{B}$ and an extension $\delta'' \in \mathbb{E}(E, A)$ realised by $A \xrightarrow{c} C \xrightarrow{c'} E$, such that the diagram

$$\begin{array}{ccccc} A & \xrightarrow{a} & B & \xrightarrow{a'} & D \\ \parallel & & \downarrow b & & \downarrow d \\ A & \xrightarrow{c} & C & \xrightarrow{c'} & E \\ & & \downarrow b' & & \downarrow e \\ & & F & \xlongequal{\quad} & F \end{array}$$

commutes in \mathcal{B} and satisfies:

- (i) $\mathfrak{s}((a')_*\delta') = [D \xrightarrow{d} E \xrightarrow{e} F]$;
- (ii) $d^*\delta'' = \delta$; and
- (iii) $a_*\delta'' = e^*\delta'$.

(ET4^{op}) Dual of (ET4).

Example 2.18. [28, Exam. 2.13] Examples of extriangulated categories include: any triangulated category; extension-closed subcategories of triangulated categories; and exact categories that are skeletally small, or have enough projectives or injectives.

Setup 2.19. For the remainder of §2, we suppose $(\mathcal{B}, \mathbb{E}, \mathfrak{s})$ is an extriangulated category.

Now we recall some useful terminology that was introduced in [28].

Definition 2.20. [28, Def. 2.15, Def. 2.19] A sequence $A \longrightarrow B \longrightarrow C$ of composable morphisms in \mathcal{B} is called a *conflation* if it realises some extension $\delta \in \mathbb{E}(C, A)$. Following [25], in this case we write $A \succrightarrow B \twoheadrightarrow C$.

A morphism $a \in \mathcal{B}(A, B)$ is called an *inflation* if there exists a conflation of the form $A \succ^a \rightarrow B \twoheadrightarrow C$. Dually, a morphism $b \in \mathcal{B}(B, C)$ is called a *deflation* if there exists a conflation of the form $A \succrightarrow B \xrightarrow{b} \twoheadrightarrow C$.

Suppose that we have a morphism $(f, h): \delta \rightarrow \delta'$ of extensions realised by the commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{a} & B & \xrightarrow{b} & C \\ \downarrow f & & \downarrow g & & \downarrow h \\ A' & \xrightarrow{a'} & B' & \xrightarrow{b'} & C' \end{array} \quad (2.1)$$

in which the top row realises δ and the bottom row realises δ' . Then we call the pair $(A \xrightarrow{a} B \xrightarrow{b} C, \delta)$ an \mathbb{E} -*triangle* and denote this by $A \xrightarrow{a} B \xrightarrow{b} C \overset{\delta}{\dashrightarrow}$. Furthermore, we call the triple (f, g, h) a *morphism of \mathbb{E} -triangles* and denote this by

$$\begin{array}{ccccc} A & \xrightarrow{a} & B & \xrightarrow{b} & C \overset{\delta}{\dashrightarrow} \\ \downarrow f & & \downarrow g & & \downarrow h \\ A' & \xrightarrow{a'} & B' & \xrightarrow{b'} & C' \overset{\delta'}{\dashrightarrow} \end{array}$$

We conclude this section with some results that will be used frequently later. For the first of these, we recall from [28, Def. 3.1] that by the Yoneda Lemma, each \mathbb{E} -extension gives rise to two natural transformations as follows. For any A, C in \mathcal{B} and any extension $\delta \in \mathbb{E}(C, A)$, there is an induced natural transformation $\delta^\sharp: \mathcal{B}(A, -) \Rightarrow \mathbb{E}(C, -)$, which is given by $(\delta^\sharp)_X(f) = f_*\delta$ for each object $X \in \mathcal{B}$ and each morphism $f: A \rightarrow X$. Dually, there is also a natural transformation $\delta_\sharp: \mathcal{B}(-, C) \Rightarrow \mathbb{E}(-, A)$, which is given by $(\delta_\sharp)_Y(g) = g^*\delta$ for each object $Y \in \mathcal{B}$ and each morphism $g: Y \rightarrow C$.

Proposition 2.21. [28, Cor. 3.12] *Suppose $A \xrightarrow{a} B \xrightarrow{b} C \xrightarrow{\delta} \rightarrow$ is an \mathbb{E} -triangle. Then for each $X, Y \in \mathcal{B}$, the sequences*

$$\mathcal{B}(C, X) \xrightarrow{-ob} \mathcal{B}(B, X) \xrightarrow{-oa} \mathcal{B}(A, X) \xrightarrow{(\delta^\sharp)_X} \mathbb{E}(C, X) \xrightarrow{\mathbb{E}(b, X)} \mathbb{E}(B, X) \xrightarrow{\mathbb{E}(a, X)} \mathbb{E}(A, X)$$

and

$$\mathcal{B}(Y, A) \xrightarrow{-ao} \mathcal{B}(Y, B) \xrightarrow{-bo} \mathcal{B}(Y, C) \xrightarrow{(\delta_\sharp)_Y} \mathbb{E}(Y, A) \xrightarrow{\mathbb{E}(Y, a)} \mathbb{E}(Y, B) \xrightarrow{\mathbb{E}(Y, b)} \mathbb{E}(Y, C)$$

are exact in \mathbf{Ab} .

The next two results follow from [25, Prop. 1.20] and its dual. See also [28, Cor. 3.16].

Proposition 2.22. *Let $A \xrightarrow{a} B \xrightarrow{b} C \xrightarrow{\delta} \rightarrow$ be any \mathbb{E} -triangle, let $f: A \rightarrow D$ be any morphism in \mathcal{B} , and suppose $D \xrightarrow{d} E \xrightarrow{e} C \rightarrow$ is a conflation realising $f_*\delta$. For any commutative diagram*

$$\begin{array}{ccccc} A & \xrightarrow{a} & B & \xrightarrow{b} & C \xrightarrow{\delta} \rightarrow \\ \downarrow f & & \downarrow g & & \parallel \\ D & \xrightarrow{d} & E & \xrightarrow{e} & C \xrightarrow{f_*\delta} \rightarrow \end{array}$$

there exist morphisms $f': A \rightarrow D$ and $g': B \rightarrow E$, such that the conflations

$$A \xrightarrow{\begin{pmatrix} -f \\ a \end{pmatrix}} D \oplus B \xrightarrow{(d \ g')} E \quad \text{and} \quad A \xrightarrow{\begin{pmatrix} -f' \\ a \end{pmatrix}} D \oplus B \xrightarrow{(d \ g)} E$$

both realise $e^*\delta$, and $g'a = df$, $eg' = b$ and $df' = ga$.

Proposition 2.23. *Let $A \xrightarrow{d} D \xrightarrow{e} E \xrightarrow{\delta} \rightarrow$ be any \mathbb{E} -triangle, let $g: C \rightarrow E$ be any morphism in \mathcal{B} , and suppose $A \xrightarrow{a} B \xrightarrow{b} C \rightarrow$ is a conflation realising $g^*\delta$. For any commutative diagram*

$$\begin{array}{ccccc} A & \xrightarrow{a} & B & \xrightarrow{b} & C \xrightarrow{g^*\delta} \rightarrow \\ \parallel & & \downarrow f & & \downarrow g \\ A & \xrightarrow{d} & D & \xrightarrow{e} & E \xrightarrow{\delta} \rightarrow \end{array}$$

there exist morphisms $f': B \rightarrow D$ and $g': C \rightarrow E$, such that the conflations

$$B \xrightarrow{\begin{pmatrix} f' \\ b \end{pmatrix}} D \oplus C \xrightarrow{(e \ -g)} E \quad \text{and} \quad B \xrightarrow{\begin{pmatrix} f \\ b \end{pmatrix}} D \oplus C \xrightarrow{(e \ -g')} E$$

both realise $a_*\delta$, and $f'a = d$, $ef' = gb$ and $g'b = ef$.

2.3. Twin cotorsion pairs on extriangulated categories. In this section we recall the basics of the theory of twin cotorsion pairs on extriangulated categories that we use throughout the rest of this article. For more details we refer the reader to [28] and [25].

We are still in the situation of Setup 2.19.

Definition 2.24. [28, Def. 4.2] Let \mathcal{U}, \mathcal{V} be full subcategories of \mathcal{B} that are closed under isomorphisms.

- (i) By $\text{Cone}(\mathcal{V}, \mathcal{U})$ we denote the full subcategory of \mathcal{B} that consists of objects $X \in \mathcal{B}$ for which there exists a conflation $V \twoheadrightarrow U \twoheadrightarrow X$, where $U \in \mathcal{U}$ and $V \in \mathcal{V}$.
- (ii) By $\text{CoCone}(\mathcal{V}, \mathcal{U})$ we denote the full subcategory of \mathcal{B} that consists of objects $X \in \mathcal{B}$ for which there exists a conflation $X \twoheadrightarrow V \twoheadrightarrow U$, where $U \in \mathcal{U}$ and $V \in \mathcal{V}$.

Definition 2.25. [28, Def. 4.1], [25, Def. 2.1] Let \mathcal{U}, \mathcal{V} be full, additive subcategories of \mathcal{B} that are closed under taking direct summands. We call $(\mathcal{U}, \mathcal{V})$ a *cotorsion pair* (on \mathcal{B}) if

- (i) $\mathbb{E}(\mathcal{U}, \mathcal{V}) = 0$;
- (ii) $\mathcal{B} = \text{Cone}(\mathcal{V}, \mathcal{U})$; and
- (iii) $\mathcal{B} = \text{CoCone}(\mathcal{V}, \mathcal{U})$.

Remark 2.26. Let $\mathcal{U}, \mathcal{V} \subseteq \mathcal{B}$ be full, additive subcategories of \mathcal{B} closed under taking direct summands. We denote by $\mathcal{U} * \mathcal{V}$ the full subcategory of \mathcal{B} consisting of objects $X \in \mathcal{B}$ for which there is a conflation $U \twoheadrightarrow X \twoheadrightarrow V$ in \mathcal{B} for some $U \in \mathcal{U}, V \in \mathcal{V}$; see [25, p. 104]. Moreover, if $(\mathcal{U}, \mathcal{V})$ is a cotorsion pair, then \mathcal{U} is *extension-closed*, i.e. $\mathcal{U} * \mathcal{U} \subseteq \mathcal{U}$, and, similarly, \mathcal{V} is also extension-closed; see [28, Rem. 4.6].

Definition 2.27. [28, Def. 4.12] Let $(\mathcal{S}, \mathcal{T})$ and $(\mathcal{U}, \mathcal{V})$ be cotorsion pairs on \mathcal{B} . Then $((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$ is called a *twin cotorsion pair* (on \mathcal{B}) if $\mathbb{E}(\mathcal{S}, \mathcal{V}) = 0$, or equivalently $\mathcal{S} \subseteq \mathcal{U}$, or equivalently $\mathcal{V} \subseteq \mathcal{T}$.

For a subcategory $\mathcal{A} \subseteq \mathcal{B}$ that is closed under finite direct sums, we denote by $[\mathcal{A}]$ the two-sided ideal of \mathcal{B} such that $[\mathcal{A}](X, Y)$ consists of all the morphisms in $\mathcal{B}(X, Y)$ that factor through an object lying in \mathcal{A} .

Definition 2.28. [25, Def. 2.5, Def. 2.6] Suppose $((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$ is a twin cotorsion pair on \mathcal{B} . We define full subcategories of \mathcal{B} as follows:

$$\mathcal{W} := \mathcal{T} \cap \mathcal{U}, \quad \mathcal{B}^- := \text{CoCone}(\mathcal{W}, \mathcal{S}), \quad \mathcal{B}^+ := \text{Cone}(\mathcal{V}, \mathcal{W}), \quad \mathcal{H} := \mathcal{B}^- \cap \mathcal{B}^+.$$

For $\mathcal{A} \in \{\mathcal{B}^-, \mathcal{B}^+, \mathcal{H}\}$, we define $\overline{\mathcal{A}}$ to be the additive quotient $\mathcal{A}/[\mathcal{W}]$. In particular, the subfactor category $\overline{\mathcal{H}} = \mathcal{H}/[\mathcal{W}]$ of \mathcal{B} is known as the *heart* of $((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$.

Later we refer to the following construction (and its dual) from [25] several times.

Definition 2.29. [25, Def. 2.24] Suppose we have a morphism $f: A \rightarrow B$ in \mathcal{B} where $A \in \mathcal{B}^-$. We define $C_f \in \mathcal{B}$ and $c_f: B \rightarrow C_f$ as follows. Since $A \in \mathcal{B}^-$, there is an \mathbb{E} -triangle $A \twoheadrightarrow W^A \twoheadrightarrow S^A \xrightarrow{\delta}$, with $W^A \in \mathcal{W}, S^A \in \mathcal{S}$. Then $f: A \rightarrow B$ induces

a morphism $(f, 1_{S^A}): \delta \rightarrow f_*\delta$ of extensions, which we may realise by

$$\begin{array}{ccccc} A & \twoheadrightarrow & W^A & \twoheadrightarrow & S^A & \overset{\delta}{\dashrightarrow} \\ \downarrow f & & \downarrow & & \parallel & \\ B & \xrightarrow{c_f} & C_f & \twoheadrightarrow & S^A & \overset{f_*\delta}{\dashrightarrow} \end{array}$$

Liu and Nakaoka showed that the heart of a twin cotorsion pair on an extriangulated category always carries additional structure.

Theorem 2.30. [25, Thm. 2.32] *The heart of a twin cotorsion pair on \mathcal{B} is semi-abelian.*

3. A CASE WHEN $\overline{\mathcal{H}}$ IS INTEGRAL

Throughout this section, let $(\mathcal{B}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category and, in addition, let $((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$ be a twin cotorsion pair on \mathcal{B} .

Proposition 3.1. [25, Cor. 2.26] *Let $f \in \mathcal{H}(A, B)$ be a morphism. Then $\overline{f} \in \overline{\mathcal{H}}(A, B)$ is an epimorphism if and only if the object C_f (as defined in Definition 2.29) lies in \mathcal{U} .*

The next lemma is a unification of [27, Lem. 5.3] and [23, Lem. 5.5].

Lemma 3.2. [25, Lem. 2.31] *Suppose*

$$\begin{array}{ccc} A & \xrightarrow{\overline{a}} & B \\ \overline{b} \downarrow & \square & \downarrow \overline{c} \\ C & \xrightarrow{\overline{d}} & D \end{array}$$

is a pullback diagram in $\overline{\mathcal{H}}$. Suppose further that there is an object $X \in \mathcal{B}^-$ and morphisms $x_B: X \rightarrow B$ and $x_C: X \rightarrow C$ in \mathcal{B} , such that $\overline{c} \circ \overline{x}_B = \overline{d} \circ \overline{x}_C$ in $\overline{\mathcal{B}}^-$ and that there is a conflation $X \xrightarrow{x_B} B \twoheadrightarrow U$ with $U \in \mathcal{U}$. Then $\overline{a}: A \rightarrow B$ is an epimorphism in $\overline{\mathcal{H}}$.

The following definition is a direct generalisation of the notions from the exact setting.

Definition 3.3. [28, Def. 3.23, Def. 3.25] An object $P \in \mathcal{B}$ is said to be *projective* if, for any conflation $A \twoheadrightarrow B \xrightarrow{b} C$ and for any morphism $c: P \rightarrow C$, there exists a morphism $d: P \rightarrow B$ such that $bd = c$. The full subcategory of \mathcal{B} consisting of all projective objects is denoted by $\text{Proj } \mathcal{B}$. We say that \mathcal{B} has *enough projectives* if, for each object $C \in \mathcal{B}$, there exists a conflation $A \twoheadrightarrow P \xrightarrow{p} C$ with P projective.

Injective objects, the full subcategory $\text{Inj } \mathcal{B}$ and \mathcal{B} has *enough injectives* are all defined dually.

Remark 3.4. (i) As noted in [28, Exam. 3.26], if $(\mathcal{B}, \mathbb{E}, \mathfrak{s})$ is an exact category, then the notions in Definition 3.3 all coincide with the usual ones. If instead $(\mathcal{B}, \mathbb{E}, \mathfrak{s})$ is a triangulated category, then $\text{Proj } \mathcal{B} = \{0\} = \text{Inj } \mathcal{B}$, and \mathcal{B} has enough projectives and enough injectives.

(ii) For any cotorsion pair $(\mathcal{X}, \mathcal{Y})$ on \mathcal{B} , we have $\text{Proj } \mathcal{B} \subseteq \mathcal{X}$ and $\text{Inj } \mathcal{B} \subseteq \mathcal{Y}$; see [25, Rem. 2.2].

Lemma 3.5. *Suppose \mathcal{B} has enough projectives. Suppose $A \twoheadrightarrow B \twoheadrightarrow C \xrightarrow{\delta}$ is an \mathbb{E} -triangle. Then there is a morphism of \mathbb{E} -triangles*

$$\begin{array}{ccccc} K_C & \twoheadrightarrow & P_C & \twoheadrightarrow & C & \xrightarrow{\delta} \\ \downarrow & & \downarrow & & \parallel & \\ A & \twoheadrightarrow & B & \twoheadrightarrow & C & \xrightarrow{\delta} \end{array}$$

where P_C is projective.

Proof. This follows quickly from \mathcal{B} having enough projectives and axiom (ET3^{op}). \blacksquare

Remark 3.6. We explain here a little more about the purpose and usefulness of Lemma 3.5 above, which is used repeatedly in Theorem 3.7. In a triangulated category \mathcal{C} with suspension functor Σ , we may rotate a triangle $A \rightarrow B \rightarrow C \rightarrow \Sigma A$ to yield another triangle $\Sigma^{-1}C \rightarrow A \rightarrow B \rightarrow C$. In an extriangulated category \mathcal{B} , we do not have access to such a functor. Instead, by imposing that the extriangulated category has enough projectives, we may use a combination of results we have seen so far to (kind of) “rotate” a conflation $A \twoheadrightarrow B \twoheadrightarrow C$ as follows. First, apply Lemma 3.5 to acquire a morphism of \mathbb{E} -triangles with same end-term C (as in the statement of the lemma). Then use Proposition 2.22 to obtain a conflation of the form $K_C \twoheadrightarrow A \oplus P_C \twoheadrightarrow B$. Of course, this is not a true rotation as we have had to adjust the middle term A by a projective P_C , but it plays a similar role. Note that if \mathcal{B} were triangulated, then $\text{Proj } \mathcal{B}$ would be trivial (see Remark 3.4) and we would recover the standard rotation of triangles.

The following is the main result of this section, and unifies [27, Thm. 6.3] and [23, Thm. 6.2]. It also improves the latter because here \mathcal{B} is not assumed to be Krull-Schmidt.

Theorem 3.7. *Let $(\mathcal{B}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category and let $((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$ be a twin cotorsion pair on \mathcal{B} . If*

- (i) \mathcal{B} has enough projectives, $\text{Proj } \mathcal{B} \subseteq \mathcal{W}$ and $\mathcal{U} \subseteq \mathcal{S} * \mathcal{T}$, or
- (ii) \mathcal{B} has enough injectives, $\text{Inj } \mathcal{B} \subseteq \mathcal{W}$ and $\mathcal{T} \subseteq \mathcal{U} * \mathcal{V}$,

then $\overline{\mathcal{H}} = \mathcal{H}/[\mathcal{W}]$ is integral.

Proof. Note that as $\overline{\mathcal{H}}$ is a semi-abelian category by Theorem 2.30, we have that $\overline{\mathcal{H}}$ is left integral if and only if $\overline{\mathcal{H}}$ is right integral by Proposition 2.5. Therefore, integrality of $\overline{\mathcal{H}}$ will follow from integrality on one side. We will show that if (i) holds, then $\overline{\mathcal{H}}$ is left integral. Showing that $\overline{\mathcal{H}}$ is right integral if (ii) holds is similar.

Suppose that \mathcal{B} has enough projectives, $\mathcal{U} \subseteq \mathcal{S} * \mathcal{T}$ and $\text{Proj } \mathcal{B} \subseteq \mathcal{W}$, and that we have a pullback diagram

$$\begin{array}{ccc} A & \xrightarrow{\bar{a}} & B \\ \bar{b} \downarrow & \square & \downarrow \bar{c} \\ C & \xrightarrow{\bar{d}} & D \end{array}$$

in $\overline{\mathcal{H}}$, where \bar{d} is an epimorphism. As $C \in \overline{\mathcal{H}} \subseteq \mathcal{B}^-$, we obtain a morphism

$$\begin{array}{ccccccc} C & \twoheadrightarrow & W^C & \twoheadrightarrow & S^C & \dashrightarrow & \\ \downarrow d & & \downarrow & & \parallel & & \\ D & \xrightarrow{c_d} & C_d & \xrightarrow{e} & S^C & \xrightarrow{\delta_{c_d}} & \end{array} \quad (3.1)$$

of \mathbb{E} -triangles, as in Definition 2.29, where $W^C \in \mathcal{W}$, $S^C \in \mathcal{S}$. By Proposition 3.1, we have that C_d belongs to \mathcal{U} as \bar{d} is an epimorphism. By assumption, we have $\mathcal{U} \subseteq \mathcal{S} * \mathcal{T}$, so there is a conflation $S \xrightarrow{f} C_d \xrightarrow{g} T$, where $S \in \mathcal{S}$ and $T \in \mathcal{T}$.

There is a conflation $B \twoheadrightarrow W^B \xrightarrow{h} S^B$, where $W^B \in \mathcal{W}$, $S^B \in \mathcal{S}$, because $B \in \overline{\mathcal{H}} \subseteq \mathcal{B}^-$. By Lemma 3.5, we obtain a morphism

$$\begin{array}{ccccccc} K_{S^B} & \xrightarrow{i} & P_{S^B} & \xrightarrow{j} & S^B & \dashrightarrow & \\ \downarrow k & & \downarrow & & \parallel & & \\ B & \twoheadrightarrow & W^B & \xrightarrow{h} & S^B & \dashrightarrow & \end{array} \quad (3.2)$$

of \mathbb{E} -triangles, with $P_{S^B} \in \text{Proj } \mathcal{B} \subseteq \mathcal{W}$. Since $K_{S^B} \xrightarrow{i} P_{S^B} \xrightarrow{j} S^B$ is a conflation, by Proposition 2.21 we have an exact sequence

$$\mathcal{B}(S^B, T) \longrightarrow \mathcal{B}(P_{S^B}, T) \xrightarrow{-\circ i} \mathcal{B}(K_{S^B}, T) \longrightarrow \mathbb{E}(S^B, T) = 0,$$

where the last term vanishes because $(\mathcal{S}, \mathcal{T})$ is a cotorsion pair. Thus, there exists $l: P_{S^B} \rightarrow T$ such that $li = gc_dck: K_{S^B} \rightarrow T$. Then l factors through g as P_{S^B} is projective, so there exists $m: P_{S^B} \rightarrow C_d$ such that $gm = l$. Note that this implies $g(c_dck - mi) = gc_dck - gmi = li - li = 0$, and hence $c_dck - mi: K_{S^B} \rightarrow C_d$ must factor through $f: S \rightarrow C_d$ by Proposition 2.21. That is, there exists $n: K_{S^B} \rightarrow S$ such that $fn = c_dck - mi$.

Let $S \twoheadrightarrow Q \twoheadrightarrow S^B$ be a realisation of $n_*\delta_i$. Notice that this implies $Q \in \mathcal{S}$ as \mathcal{S} is extension-closed (see Remark 2.26). By Proposition 2.22, there is a morphism $q: P_{S^B} \rightarrow Q$ such that

$$\begin{array}{ccccccc} K_{S^B} & \xrightarrow{i} & P_{S^B} & \xrightarrow{j} & S^B & \dashrightarrow & \\ \downarrow n & & \downarrow q & & \parallel & & \\ S & \xrightarrow{p} & Q & \twoheadrightarrow & S^B & \dashrightarrow & \end{array}$$

commutes and $K_{S^B} \xrightarrow{\begin{pmatrix} -n \\ i \end{pmatrix}} S \oplus P_{S^B} \xrightarrow{(p \ q)} Q$ is a conflation.

By Lemma 3.5, there is a morphism

$$\begin{array}{ccccccc} K_Q & \xrightarrow{r} & P_Q & \xrightarrow{s} & Q & \dashrightarrow & \\ \downarrow t & & \downarrow \begin{pmatrix} u \\ v \end{pmatrix} & & \parallel & & \\ K_{S^B} & \xrightarrow{\begin{pmatrix} -n \\ i \end{pmatrix}} & S \oplus P_{S^B} & \xrightarrow{(p \ q)} & Q & \dashrightarrow & \end{array} \quad (3.3)$$

of \mathbb{E} -triangles, with P_Q projective. Furthermore, as $P_Q \in \text{Proj } \mathcal{B} \subseteq \mathcal{W}$ and $Q \in \mathcal{S}$, we have K_Q lies in $\text{CoCone}(\mathcal{W}, \mathcal{S}) = \mathcal{B}^-$. Note that $fn = c_dck - mi$ implies

$$c_d(ck) = fn + mi = (-f \ m) \circ \begin{pmatrix} -n \\ i \end{pmatrix}.$$

Thus, by applying (ET3), we have a morphism of \mathbb{E} -triangles as follows:

$$\begin{array}{ccccc}
K_{S^B} & \xrightarrow{\begin{pmatrix} -n \\ i \end{pmatrix}} & S \oplus P_{S^B} & \xrightarrow{(p \ q)} & Q & \xrightarrow{-t_*\delta_r} & \rightarrow \\
\downarrow ck & & \downarrow (-f \ m) & & \downarrow \vdots & & \\
D & \xrightarrow{c_d} & C_d & \xrightarrow{e} & S^C & \xrightarrow{\delta_{c_d}} & \rightarrow
\end{array} \tag{3.4}$$

Composing (3.3) and (3.4), we obtain the morphism

$$\begin{array}{ccccc}
K_Q & \xrightarrow{r} & P_Q & \xrightarrow{s} & Q & \xrightarrow{\delta_r} & \rightarrow \\
\downarrow ckt & & \downarrow mv-fu & & \downarrow & & \\
D & \xrightarrow{c_d} & C_d & \xrightarrow{e} & S^C & \xrightarrow{\delta_{c_d}} & \rightarrow
\end{array} \tag{3.5}$$

By Lemma 3.5, there is a morphism

$$\begin{array}{ccccc}
K_{S^C} & \xrightarrow{x} & P_{S^C} & \xrightarrow{y} & S^C & \xrightarrow{\delta_x} & \rightarrow \\
\downarrow z & & \downarrow & & \parallel & & \\
C & \longrightarrow & W^C & \longrightarrow & S^C & \longrightarrow & \rightarrow
\end{array} \tag{3.6}$$

of \mathbb{E} -triangles with $P_{S^C} \in \text{Proj } \mathcal{B}$. Then composing morphisms (3.6) and (3.1), we obtain a morphism of \mathbb{E} -triangles as follows:

$$\begin{array}{ccccc}
K_{S^C} & \xrightarrow{x} & P_{S^C} & \xrightarrow{y} & S^C & \xrightarrow{\delta_x} & \rightarrow \\
\downarrow dz & & \downarrow a' & & \parallel & & \\
D & \xrightarrow{c_d} & C_d & \xrightarrow{e} & S^C & \xrightarrow{\delta_{c_d}} & \rightarrow
\end{array} \tag{3.7}$$

Consider the morphism $e(mv - fu): P_Q \rightarrow S^C$. As $K_{S^C} \xrightarrow{x} P_{S^C} \xrightarrow{y} S^C \xrightarrow{\delta_x}$ is an \mathbb{E} -triangle and P_Q is projective, there exists $b': P_Q \rightarrow P_{S^C}$ such that $yb' = e(mv - fu)$. This yields $y(b'r) = e(mv - fu)r = ec_dckt = 0$ (using the commutativity of (3.5)), and so by Proposition 2.21 there exists $c': K_Q \rightarrow K_{S^C}$ such that $xc' = b'r$. In addition, we also see that $e(a'b' - (mv - fu)) = ea'b' - e(mv - fu) = yb' - yb' = 0$. Thus, there exists $d': P_Q \rightarrow D$ such that $c_d d' = a'b' - (mv - fu)$, by Proposition 2.21, using the conflation $D \xrightarrow{c_d} C_d \xrightarrow{e} S^C$. Hence, we have

$$\begin{aligned}
c_d dzc' &= a'xc' \\
&= a'b'r \\
&= c_d d'r + (mv - fu)r \\
&= c_d d'r + c_d ckt \\
&= c_d(d'r + ckt).
\end{aligned} \tag{3.8}$$

As (3.7) is a morphism of \mathbb{E} -triangles, we have $\delta_{c_d} = (1_{S^C})^* \delta_{c_d} = (dz)_* \delta_x$, and so $(dz)_* \delta_x$ is realised by the conflation $D \xrightarrow{c_d} C_d \xrightarrow{e} S^C$. Therefore, by Proposition 2.22, there is a conflation

$$K_{S^C} \xrightarrow{\begin{pmatrix} -dz \\ x \end{pmatrix}} D \oplus P_{S^C} \xrightarrow{(c_d \ e')} C_d \tag{3.9}$$

for some $e': P_{SC} \rightarrow C_d$ satisfying $e'x = c_d dz$. Consider the morphism $\begin{pmatrix} -(d'r+ckt) \\ b'r \end{pmatrix}: K_Q \rightarrow D \oplus P_{SC}$, and note that

$$\begin{aligned} (c_d \ e') \circ \begin{pmatrix} -(d'r+ckt) \\ b'r \end{pmatrix} &= e'b'r - c_d(d'r + ckt) \\ &= e'xc' - c_d(d'r + ckt) \\ &= c_d dzc' - c_d(d'r + ckt) \\ &= 0 \end{aligned} \quad \text{by (3.8).}$$

Thus, there exists $f': K_Q \rightarrow K_{SC}$ such that $\begin{pmatrix} -dzf' \\ xf' \end{pmatrix} = \begin{pmatrix} -dz \\ x \end{pmatrix} f' = \begin{pmatrix} -(d'r+ckt) \\ b'r \end{pmatrix}$. In particular, we see that

$$d(zf') = d'r + c(kt). \quad (3.10)$$

From (3.2) we get a conflation

$$K_{SB} \xrightarrow{\begin{pmatrix} -k \\ i \end{pmatrix}} B \oplus P_{SB} \longrightarrow W^B \quad (3.11)$$

and from (3.3) we get a conflation

$$K_Q \xrightarrow{\begin{pmatrix} -t \\ r \end{pmatrix}} K_{SB} \oplus P_Q \longrightarrow S \oplus P_{SB}, \quad (3.12)$$

using Proposition 2.22. As (3.11) is a conflation, we also have a conflation

$$K_{SB} \oplus P_Q \xrightarrow{\begin{pmatrix} -k & 0 \\ i & 0 \\ 0 & 1_{P_Q} \end{pmatrix}} B \oplus P_{SB} \oplus P_Q \longrightarrow W_1 \quad (3.13)$$

using (ET2). Then, applying (ET4) to the conflations (3.12) and (3.13), we have a commutative diagram

$$\begin{array}{ccccc} K_Q & \xrightarrow{\begin{pmatrix} -t \\ r \end{pmatrix}} & K_{SB} \oplus P_Q & \longrightarrow & S \oplus P_{SB} \\ \parallel & & \downarrow \begin{pmatrix} -k & 0 \\ i & 0 \\ 0 & 1_{P_Q} \end{pmatrix} & & \downarrow \\ K_Q & \xrightarrow{x_B} & B \oplus P_{SB} \oplus P_Q & \longrightarrow & M \\ & & \downarrow & & \downarrow \\ & & W^B & \xlongequal{\quad} & W^B \end{array}$$

in which $x_B := \begin{pmatrix} kt \\ -it \\ r \end{pmatrix}$. Note that M lies in \mathcal{U} as both W^B and $S \oplus P_{SB}$ lie in the extension-closed subcategory \mathcal{U} .

Since $B \in \mathcal{H}$ and $P_{S^B}, P_Q \in \text{Proj } \mathcal{B} \subseteq \mathcal{W} \subseteq \mathcal{H}$, we have that $B \oplus P_{S^B} \oplus P_Q$ is an object in \mathcal{H} . Thus, consider the following commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\overline{\iota_B \bar{a}}} & B \oplus P_{S^B} \oplus P_Q \\ \parallel & & \downarrow \overline{\pi_B} \\ A & \xrightarrow{\bar{a}} & B \\ \bar{b} \downarrow & \square & \downarrow \bar{c} \\ C & \xrightarrow{\bar{d}} & D \end{array}$$

in $\overline{\mathcal{H}}$, where $\iota_B: B \hookrightarrow B \oplus P_{S^B} \oplus P_Q$ is the canonical inclusion and $\pi_B = (1_B \ 0 \ 0): B \oplus P_{S^B} \oplus P_Q \twoheadrightarrow B$ is the canonical projection. Notice that $\overline{\iota_B}$ and $\overline{\pi_B}$ are mutually inverse in $\overline{\mathcal{H}}$ as $P_{S^B} \oplus P_Q \in \mathcal{W}$. Hence the square

$$\begin{array}{ccc} A & \xrightarrow{\overline{\iota_B \bar{a}}} & B \oplus P_{S^B} \oplus P_Q \\ \bar{b} \downarrow & & \downarrow \overline{c\pi_B} \\ C & \xrightarrow{\bar{d}} & D \end{array}$$

is also a pullback square in $\overline{\mathcal{H}}$. Setting $x_C := z f': K_Q \rightarrow C$, we see that

$$\begin{aligned} \bar{d} \circ \overline{x_C} &= \overline{dz f'} \\ &= \overline{d'r} + \overline{ckt} && \text{by (3.10)} \\ &= \overline{ckt} && \text{as } \overline{d'r} = 0 \\ &= \overline{c} \circ \overline{(1_B \ 0 \ 0)} \circ \overline{\begin{pmatrix} kt \\ -it \\ r \end{pmatrix}} \\ &= \overline{c\pi_B} \circ \overline{x_B}. \end{aligned}$$

Therefore, by Lemma 3.2, we conclude that $\overline{\iota_B \bar{a}}$ is an epimorphism. Finally, \bar{a} is an epimorphism since $\overline{\iota_B}$ is an isomorphism in $\overline{\mathcal{H}}$, and we are done. \blacksquare

Remark 3.8. Theorem 3.7 unifies the analogous results for triangulated and exact categories. However, the proof we give here differs in several aspects. We note that our proof is not a direct generalisation of the proof for triangulated categories. This is because an extriangulated category does not come equipped with a suspension/shift functor. One way to work around this is to use that the extriangulated category has enough projectives, as one would do in the exact category case, in order to obtain what would be a negative shift of an object. Thus, our proof is inspired by the exact case. But the proof in [23] uses the defining property of a monomorphism, which we cannot exploit in the extriangulated setting. This is a key difference between our proof above and the proof for exact categories.

4. A CASE WHEN $\overline{\mathcal{H}}$ IS QUASI-ABELIAN

In this section we give an analogue of [34, Thm. 3.4] for the extriangulated setting, which also improves [23, Thm. 7.4]. First, let us recall a key lemma from [34].

Lemma 4.1. [34, Lem. 3.1] *Let \mathcal{A} be a left semi-abelian category. Suppose*

$$\begin{array}{ccc} A & \xrightarrow{a} & B \\ b \downarrow & \square & \downarrow c \\ C & \xrightarrow{d} & D \end{array}$$

is a pullback diagram in \mathcal{A} . Suppose we also have morphisms $x_B: X \rightarrow B$ and $x_C: X \rightarrow C$, such that x_B is a cokernel and $cx_B = dx_C$. Then $a: A \rightarrow B$ is also a cokernel in \mathcal{A} .

Theorem 4.2. *Let $(\mathcal{B}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category. Suppose $((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$ is a twin cotorsion pair on \mathcal{B} . If $\mathcal{H} = \mathcal{B}^-$ or $\mathcal{H} = \mathcal{B}^+$, then $\overline{\mathcal{H}} = \mathcal{H}/[\mathcal{W}]$ is quasi-abelian.*

Proof. The heart $\overline{\mathcal{H}}$ is semi-abelian by Theorem 2.30, so we have that $\overline{\mathcal{H}}$ is left quasi-abelian if and only if $\overline{\mathcal{H}}$ is right quasi-abelian by Proposition 2.6. Therefore, we will show that if $\mathcal{H} = \mathcal{B}^-$ then $\overline{\mathcal{H}}$ is left quasi-abelian. Showing $\overline{\mathcal{H}}$ is right quasi-abelian whenever $\mathcal{H} = \mathcal{B}^+$ is similar.

Suppose $\mathcal{H} = \mathcal{B}^-$ and that we have a pullback diagram

$$\begin{array}{ccc} A & \xrightarrow{\bar{a}} & B \\ \bar{b} \downarrow & \square & \downarrow \bar{c} \\ C & \xrightarrow{\bar{d}} & D \end{array}$$

in $\overline{\mathcal{H}}$, where \bar{d} is a cokernel. By [25, Lem. 2.28], we may assume that, up to isomorphism in $\overline{\mathcal{H}}$, the morphism $d \in \mathcal{H}(C, D)$ fits into a conflation $C \succrightarrow^d D \twoheadrightarrow S$ in \mathcal{B} with $S \in \mathcal{S}$. Furthermore, we know $D \in \mathcal{H} = \mathcal{B}^+ \cap \mathcal{B}^- \subseteq \mathcal{B}^+$, and so by [25, Lemma 2.8(1)] there exist $W \in \mathcal{W}$ and $w: W \rightarrow D$ giving a deflation $(c \ w): B \oplus W \twoheadrightarrow D$ in \mathcal{B} . Then, in $\overline{\mathcal{H}}$, we have $B \oplus W \cong B$ and, up to isomorphism, $(c \ w) = \bar{c}$. Thus, without loss of generality, we may replace c by $(c \ w)$ and B by $B \oplus W$. That is, we may assume $c: B \rightarrow D$ is part of a conflation $B' \succrightarrow B \xrightarrow{c} D$.

Applying (ET4^{op}) to the conflations $C \succrightarrow^d D \twoheadrightarrow S$ and $B' \succrightarrow B \xrightarrow{c} D$, we obtain a commutative diagram

$$\begin{array}{ccccc} B' & \xlongequal{\quad} & B' & & \\ \downarrow & & \downarrow & & \\ X & \xrightarrow{x_B} & B & \twoheadrightarrow & S \dashrightarrow^{\delta} \\ x_C \downarrow & & \downarrow c & & \parallel \\ C & \xrightarrow{d} & D & \twoheadrightarrow & S \end{array}$$

in which each row and column is a conflation. Note that, by [25, Lem. 2.9(b)], $X \in \mathcal{B}^- = \mathcal{H}$ since $B \in \mathcal{B}^-$. Therefore, we have a commutative square

$$\begin{array}{ccc} X & \xrightarrow{\bar{x}_B} & B \\ \bar{x}_C \downarrow & & \downarrow \bar{c} \\ C & \xrightarrow{\bar{d}} & D \end{array}$$

in $\overline{\mathcal{H}}$, and so it is enough to show \bar{x}_B is a cokernel in $\overline{\mathcal{H}}$ by Lemma 4.1.

Dually to Definition 2.29, we obtain a morphism $k_{x_B} : K_{x_B} \rightarrow X$ in \mathcal{B} as in the following commutative diagram:

$$\begin{array}{ccccc} V & \hookrightarrow & K_{x_B} & \xrightarrow{k_{x_B}} & X & \xrightarrow{(x_B)^*\delta'} & \\ \parallel & & \downarrow e & & \downarrow x_B & & \\ V & \xrightarrow{v} & W' & \xrightarrow{w'} & B & \xrightarrow{\delta'} & \end{array}$$

Using (ET4^{op}) we obtain a commutative diagram

$$\begin{array}{ccccc} V & \xlongequal{\quad} & V & & \\ \downarrow & & \downarrow v & & \\ P & \hookrightarrow & W' & \longrightarrow & S & \xrightarrow{\delta''} \\ \downarrow p & & \downarrow w' & & \parallel & \\ X & \xrightarrow{x_B} & B & \xrightarrow{s} & S & \xrightarrow{\delta} \\ \downarrow (x_B)^*\delta' & & \downarrow \delta' & & & \end{array}$$

in which $p_*\delta'' = \delta$. Thus, the conflations $V \hookrightarrow K_{x_B} \xrightarrow{k_{x_B}} X$ and $V \hookrightarrow P \xrightarrow{p} X$ both realise the extension $(x_B)^*\delta'$, and hence are equivalent. This yields an isomorphism $q : K_{x_B} \rightarrow P$, such that $pq = k_{x_B}$, and also a morphism of \mathbb{E} -triangles

$$\begin{array}{ccccc} K_{x_B} & \xrightarrow{f} & W' & \longrightarrow & S & \xrightarrow{\varepsilon} \\ q \downarrow \cong & & \parallel & & \parallel & \\ P & \longrightarrow & W' & \longrightarrow & S & \xrightarrow{\delta''} \end{array}$$

where $q_*\varepsilon = \delta''$. This implies that there is also a morphism

$$\begin{array}{ccccc} K_{x_B} & \xrightarrow{f} & W' & \longrightarrow & S & \xrightarrow{\varepsilon} \\ \downarrow k_{x_B} & & \downarrow w' & & \parallel & \\ X & \xrightarrow{x_B} & B & \xrightarrow{s} & S & \xrightarrow{\delta} \end{array}$$

of \mathbb{E} -triangles, since $(k_{x_B})_*\varepsilon = p_*q_*\varepsilon = p_*\delta'' = \delta$. Furthermore, we also see that $K_{x_B} \in \text{CoCone}(\mathcal{W}, \mathcal{S}) = \mathcal{B}^- = \mathcal{H}$.

We claim that $\overline{x_B} : X \rightarrow B$ is a cokernel of $\overline{k_{x_B}} : K_{x_B} \rightarrow X$ in $\overline{\mathcal{H}}$. We will first show that $\overline{x_B}$ is a weak cokernel of $\overline{k_{x_B}}$, and secondly that $\overline{x_B}$ is an epimorphism; this is enough by, for example, [5, Lem. 2.5].

Note that $x_B \circ k_{x_B} = w' \circ f$ factors through \mathcal{W} , so we have $\overline{x_B} \circ \overline{k_{x_B}} = 0$ in $\overline{\mathcal{H}}$. Now suppose that there is $\overline{g} : X \rightarrow Y$ in $\overline{\mathcal{H}}$ such that $\overline{g} \circ \overline{k_{x_B}} = 0$ in $\overline{\mathcal{H}}$. Then $g \circ k_{x_B} : K_{x_B} \rightarrow Y$ factors through \mathcal{W} . Thus, there exists a commutative square

$$\begin{array}{ccc} K_{x_B} & \xrightarrow{k_{x_B}} & X \\ \downarrow h & & \downarrow g \\ W'' & \xrightarrow{i} & Y \end{array}$$

in \mathcal{B} , with $W'' \in \mathcal{W}$.

Since $X \xrightarrow{x_B} B \xrightarrow{s} S \xrightarrow{\delta} X$ is an \mathbb{E} -triangle, by Proposition 2.21 we have an exact sequence

$$\mathcal{B}(B, Y) \xrightarrow{\mathcal{B}(x_B, Y)} \mathcal{B}(X, Y) \xrightarrow{(\delta^\sharp)_Y} \mathbb{E}(S, Y) \xrightarrow{\mathbb{E}(s, Y)} \mathbb{E}(B, Y), \quad (4.1)$$

where $(\delta^\sharp)_Y : \mathcal{B}(X, Y) \rightarrow \mathbb{E}(S, Y)$ is given by $(\delta^\sharp)_Y(r) = r_*\delta$. Note that

$$(\delta^\sharp)_Y(g) = g_*(\delta) = g_*(k_{x_B})_*\varepsilon = (gk_{x_B})_*\varepsilon = (ih)_*\varepsilon = i_*h_*\varepsilon = 0,$$

as $h_*\varepsilon \in \mathbb{E}(S, W'') = 0$ because $\mathcal{W} \subseteq \mathcal{T}$. Thus, g is in the kernel of the morphism $(\delta^\sharp)_Y$ and so, by the exactness of (4.1), there exists $j : B \rightarrow Y$ such that $jk_B = g$. As $B, Y \in \mathcal{H}$, we have $j \in \mathcal{H}(B, Y)$ and $\bar{g} = \bar{j} \circ \bar{x}_B$. Hence, \bar{x}_B is a weak cokernel for \bar{k}_{x_B} in $\bar{\mathcal{H}}$.

Lastly, note that for any $V \in \mathcal{V}$ we have, by Proposition 2.21, an exact sequence

$$0 = \mathbb{E}(S, V) \longrightarrow \mathbb{E}(B, V) \xrightarrow{\mathbb{E}(x_B, V)} \mathbb{E}(X, V),$$

where $\mathbb{E}(S, V) = 0$, as $\mathcal{S} \subseteq \mathcal{U}$ and $(\mathcal{U}, \mathcal{V})$ is a cotorsion pair. That is, $\mathbb{E}(x_B, V)$ is monomorphic for any $V \in \mathcal{V}$, and therefore \bar{x}_B is an epimorphism in $\bar{\mathcal{H}}$ by [25, Prop. 2.29]. Hence, \bar{x}_B is the cokernel of \bar{k}_{x_B} in the (left) semi-abelian category $\bar{\mathcal{H}}$, so \bar{a} is also a cokernel in $\bar{\mathcal{H}}$ by Lemma 4.1 and we are done. \blacksquare

The next corollary gives a unification of [23, Thm. 7.4] and [34, Cor. 3.5], and follows immediately from Theorems 3.7 and 4.2.

Corollary 4.3. *Let $(\mathcal{B}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category with enough projectives and enough injectives. Suppose $((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$ is a twin cotorsion pair on \mathcal{B} . If $\mathcal{T} \subseteq \mathcal{U}$ or $\mathcal{U} \subseteq \mathcal{T}$, then $\bar{\mathcal{H}} = \mathcal{H}/[\mathcal{W}]$ is integral and quasi-abelian.*

5. LOCALISATION OF AN INTEGRAL HEART

In this section, we fix an extriangulated category $(\mathcal{B}, \mathbb{E}, \mathfrak{s})$ with enough projectives and enough injectives. We also suppose that there is a twin cotorsion pair $((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$ on \mathcal{B} with $\mathcal{T} = \mathcal{U}$. Note that for this twin cotorsion pair we have $\mathcal{W} = \mathcal{T} = \mathcal{U}$, $\mathcal{B}^+ = \mathcal{B} = \mathcal{B}^-$ and so its heart is $\bar{\mathcal{H}} = \mathcal{B}/[\mathcal{W}]$. By Corollary 4.3, $\bar{\mathcal{H}}$ is integral (and quasi-abelian), and hence the class \mathcal{R} of regular morphisms in $\bar{\mathcal{H}}$ admits a *calculus of left fractions* (see [11, §I.2]) by [32, Prop. 6]. (This also implies that $\bar{\mathcal{H}}_{\mathcal{R}}$ is an abelian category by [5, Thm. 4.8].) Thus, the objects of the localisation $\bar{\mathcal{H}}_{\mathcal{R}}$ are the objects of $\bar{\mathcal{H}}$, and a morphism $X \rightarrow Y$ in $\bar{\mathcal{H}}_{\mathcal{R}}$ is a *left fraction* $[\bar{f}, \bar{r}]_{\text{LF}}$ of the form

$$X \xrightarrow{\bar{f}} A \xleftarrow{\bar{r}} Y,$$

up to a certain equivalence, where $\bar{f} \in \bar{\mathcal{H}}(X, A)$ and $\bar{r} \in \mathcal{R}$ (see [11, §I.2] for details). The localisation functor $L_{\mathcal{R}} : \bar{\mathcal{H}} \rightarrow \bar{\mathcal{H}}_{\mathcal{R}}$ maps a morphism $\bar{f} : X \rightarrow A$ in $\bar{\mathcal{H}}$ to $L_{\mathcal{R}}(\bar{f}) = [\bar{f}, \bar{1}_A]_{\text{LF}}$. In particular, any morphism $\bar{r} : Y \rightarrow A$ in the class \mathcal{R} is mapped to $L_{\mathcal{R}}(\bar{r}) = [\bar{r}, \bar{1}_Y]_{\text{LF}}$, which is invertible with inverse $[\bar{r}, \bar{1}_Y]_{\text{LF}}^{-1} = [\bar{1}_A, \bar{r}]_{\text{LF}}$. Furthermore, $L_{\mathcal{R}} : \bar{\mathcal{H}} \rightarrow \bar{\mathcal{H}}_{\mathcal{R}}$ is an additive functor; see [5, Rem. 4.3].

Let us denote by $\overline{\mathcal{H}} := \overline{\mathcal{H}}_{(\mathcal{S}, \mathcal{T})}$ the heart $\text{CoCone}(\mathcal{S}, \mathcal{S})/[\mathcal{S}]$ of the *degenerate* twin cotorsion pair $((\mathcal{S}, \mathcal{T}), (\mathcal{S}, \mathcal{T}))$. The category $\overline{\mathcal{H}}$ is also the heart of the *single* cotorsion pair $(\mathcal{S}, \mathcal{T})$; see [25]. In this section, we will show that there is an equivalence $\overline{\mathcal{H}}_{\mathcal{R}} \simeq \overline{\mathcal{H}}$, giving an analogue of [34, Thm. 4.8] for the extriangulated setting; see Theorem 5.6. However, we note that this section improves some results from [34, §4] since we make no Krull-Schmidt assumption in this article.

Let $\iota: \mathcal{H} = \text{CoCone}(\mathcal{S}, \mathcal{S}) \rightarrow \mathcal{B} = \mathcal{H}$ be the canonical inclusion functor, and let $Q_{[\mathcal{S}]}: \mathcal{H} \rightarrow \overline{\mathcal{H}}$ and $Q_{[\mathcal{W}]}: \mathcal{H} \rightarrow \overline{\mathcal{H}}$ be the canonical additive quotient functors. Note that since $\mathcal{S} \subseteq \mathcal{U} = \mathcal{W}$ under our assumptions, any morphism in the ideal $[\mathcal{S}]$ in \mathcal{H} vanishes under the composition $Q_{[\mathcal{W}]} \circ \iota: \mathcal{H} \rightarrow \overline{\mathcal{H}}$. Therefore, there is a unique additive functor $F: \overline{\mathcal{H}} \rightarrow \overline{\mathcal{H}}$ that makes the diagram of functors

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{\iota} & \mathcal{H} \\ Q_{[\mathcal{S}]} \downarrow & \circ & \downarrow Q_{[\mathcal{W}]} \\ \overline{\mathcal{H}} & \xrightarrow{F} & \overline{\mathcal{H}} \end{array}$$

commute. In particular, $F(X) = X$ for any object $X \in \overline{\mathcal{H}}$ and maps the coset $f + [\mathcal{S}](X, Y)$ to the coset $\bar{f} = f + [\mathcal{W}](X, Y)$ for any morphism $f: X \rightarrow Y$ in \mathcal{H} .

Recall that a cotorsion pair $(\mathcal{U}, \mathcal{V})$ is said to be *rigid* if $\mathcal{U} \subseteq \mathcal{V}$. The next result follows from [24, Prop. 3.3], noting that $(\mathcal{S}, \mathcal{T})$ is rigid as $\mathcal{S} \subseteq \mathcal{U} = \mathcal{T}$. Furthermore, this improves [34, Lem. 4.3] since no Krull-Schmidt restriction is needed here.

Lemma 5.1. *Let X be an arbitrary object of \mathcal{B} . Then there exists a conflation*

$$Z \succ \xrightarrow{g} Y \xrightarrow{f} \twoheadrightarrow X,$$

such that $Y \in \text{CoCone}(\mathcal{S}, \mathcal{S}) = \mathcal{H}$, $Z \in \mathcal{W}$ and \bar{f} is a regular morphism in $\overline{\mathcal{H}}$.

Set $G := L_{\mathcal{R}} \circ F$. Then $G(X) = X$ and $G(f + [\mathcal{S}](X, Y)) = [f + [\mathcal{W}](X, Y), 1_Y]_{\text{LF}} = [\bar{f}, 1_Y]_{\text{LF}}$. We show that G is an equivalence of categories over several steps in the remainder of this section. The next result is an analogue of [34, Prop. 4.4], and the proof easily generalises using Lemma 5.1, so we omit the proof here.

Proposition 5.2. *The functor $G: \overline{\mathcal{H}} \rightarrow \overline{\mathcal{H}}_{\mathcal{R}}$ is dense.*

We have an analogue of [34, Lem. 4.5].

Lemma 5.3. *Suppose $X \in \text{CoCone}(\mathcal{S}, \mathcal{S})$ and $f: X \rightarrow Y$ is a morphism in \mathcal{B} . If f factors through \mathcal{W} , then f factors through \mathcal{S} .*

Proof. Suppose $f: X \rightarrow Y$ factors as $f = ba$ for some $a: X \rightarrow W$ and $b: W \rightarrow Y$, where $W \in \mathcal{W}$. Since $X \in \text{CoCone}(\mathcal{S}, \mathcal{S})$, there is an \mathbb{E} -triangle $X \succ \xrightarrow{s} S_1 \twoheadrightarrow S_0 \xrightarrow{\delta} \rightarrow$, with $S_0, S_1 \in \mathcal{S}$. By Proposition 2.21, there is an exact sequence

$$\mathcal{B}(S_1, Y) \xrightarrow{\mathcal{B}(s, Y)} \mathcal{B}(X, Y) \xrightarrow{(\delta^\sharp)_Y} \mathbb{E}(S_0, Y),$$

where $(\delta^\sharp)_Y(h) = h_*\delta$ for any $h: X \rightarrow Y$ in \mathcal{B} . Note that $a_*\delta \in \mathbb{E}(S_0, W) = 0$ because $((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$ is a twin cotorsion pair, so $(\delta^\sharp)_Y(f) = f_*\delta = b_*a_*\delta = 0$. Hence, there exists $g: S_1 \rightarrow Y$ such that $gs = f$ and we see that f factors through \mathcal{S} . \blacksquare

An analogue of [34, Prop. 4.6] follows immediately, using the lemma above, hence we omit the proof.

Proposition 5.4. *The functor $G: \overline{\mathcal{H}} \rightarrow \overline{\mathcal{H}}_{\mathcal{R}}$ is faithful.*

For the next proposition, we have adapted methods from [24].

Proposition 5.5. *The functor $G: \overline{\mathcal{H}} \rightarrow \overline{\mathcal{H}}_{\mathcal{R}}$ is full.*

Proof. Let $X, Y \in \overline{\mathcal{H}} = \text{CoCone}(\mathcal{S}, \mathcal{S})/[\mathcal{S}]$ and let $[\overline{f}, \overline{r}]_{\text{LF}}: X \xrightarrow{\overline{f}} A \xleftarrow{\overline{r}} Y$ be an arbitrary morphism in $\overline{\mathcal{H}}_{\mathcal{R}}(GX, GY) = \overline{\mathcal{H}}_{\mathcal{R}}(X, Y)$. Since $X \in \text{CoCone}(\mathcal{S}, \mathcal{S})$ and \mathcal{B} has enough projectives, there are conflations $X \twoheadrightarrow S_1 \twoheadrightarrow S_0$ and

$$K_1 \xrightarrow{a} P_1 \twoheadrightarrow S_1, \quad (5.1)$$

with $S_0, S_1 \in \mathcal{S}$ and $P_1 \in \text{Proj } \mathcal{B}$. By (ET4^{op}), there is a commutative diagram

$$\begin{array}{ccccc} K_1 & \xlongequal{\quad} & K_1 & & \\ \downarrow b & & \downarrow a & & \\ K_0 & \xrightarrow{c} & P_1 & \twoheadrightarrow & S_0 \\ \downarrow d & & \downarrow & & \parallel \\ X & \twoheadrightarrow & S_1 & \twoheadrightarrow & S_0 \end{array} \quad (5.2)$$

of conflations.

As in the dual of Definition 2.29, we have a commutative diagram

$$\begin{array}{ccccc} V & \twoheadrightarrow & L_r & \twoheadrightarrow & Y \\ \parallel & & \downarrow & & \downarrow r \\ V & \twoheadrightarrow & W & \xrightarrow{w} & A \end{array}$$

where $W \in \mathcal{W}$. We also have that $T := L_r \in \mathcal{T}$ by the dual of Proposition 3.1, as \overline{r} is a monomorphism in $\overline{\mathcal{H}}$. Thus, by Proposition 2.23, we have a conflation

$$T \twoheadrightarrow Y \oplus W \xrightarrow{(r \ w)} A.$$

We claim that the morphism $fd: K_0 \rightarrow A$ factors through the morphism $(r \ w)$. Note that the canonical inclusion $\overline{\iota}_Y: Y \hookrightarrow Y \oplus W$ and canonical projection $\overline{\pi}_Y: Y \oplus W \twoheadrightarrow Y$ are mutually inverse isomorphisms in $\overline{\mathcal{H}} = \mathcal{B}/[\mathcal{W}]$ as $W \in \mathcal{W}$. Therefore, $\overline{(r \ w)} = \overline{r\pi}_Y$ is an epimorphism, and by Definition 2.29 we have a commutative diagram

$$\begin{array}{ccccc} Y \oplus W & \twoheadrightarrow & W' & \twoheadrightarrow & S \\ (r \ w) \downarrow & & \downarrow & & \parallel \\ A & \twoheadrightarrow & U & \twoheadrightarrow & S \end{array}$$

in which $U := C_{(r \ w)} \in \mathcal{U}$ because $\overline{(r \ w)}$ is an epimorphism in $\overline{\mathcal{H}}$. So, we have a conflation

$$Y \oplus W \xrightarrow{\begin{pmatrix} r & w \\ e & g \end{pmatrix}} A \oplus W' \xrightarrow{(h \ i)} \twoheadrightarrow U$$

by Proposition 2.22. As $K_0 \xrightarrow{c} P_1 \twoheadrightarrow S_0$ is a conflation, we have an exact sequence

$$\mathcal{B}(P_1, U) \xrightarrow{\mathcal{B}(c, U)} \mathcal{B}(K_0, U) \longrightarrow \mathbb{E}(S_0, U) = 0$$

by Proposition 2.21, where $\mathbb{E}(S_0, U) = 0$ as $(\mathcal{S}, \mathcal{T})$ is a cotorsion pair and $\mathcal{T} = \mathcal{U}$. Thus, there exists $j: P_1 \rightarrow U$ such that $jc = (h \ i)\begin{pmatrix} f^d \\ 0 \end{pmatrix}$. Since P_1 is projective and $(h \ i)$ is a deflation, there exists $\begin{pmatrix} k \\ l \end{pmatrix}: P_1 \rightarrow A \oplus W'$ such that $(h \ i)\begin{pmatrix} k \\ l \end{pmatrix} = j$. In addition, as $(r \ w)$ is a deflation and we have a morphism $k: P_1 \rightarrow A$, there exists $\begin{pmatrix} m \\ n \end{pmatrix}: P_1 \rightarrow Y \oplus W$ such that $(r \ w)\begin{pmatrix} m \\ n \end{pmatrix} = k$. Notice that $(h \ i)\left[\begin{pmatrix} f^d \\ 0 \end{pmatrix} - \begin{pmatrix} k \\ l \end{pmatrix}c\right] = jc - jc = 0$. Hence, there exists $\begin{pmatrix} p \\ q \end{pmatrix}: K_0 \rightarrow Y \oplus W$ such that $\begin{pmatrix} r & w \\ e & g \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} f^d \\ 0 \end{pmatrix} - \begin{pmatrix} k \\ l \end{pmatrix}c$. In particular, we see that

$$fd = (r \ w)\begin{pmatrix} p \\ q \end{pmatrix} + kc = (r \ w)\begin{pmatrix} p \\ q \end{pmatrix} + (r \ w)\begin{pmatrix} m \\ n \end{pmatrix}c = (r \ w)\begin{pmatrix} p+mc \\ q+nc \end{pmatrix},$$

where $\begin{pmatrix} p+mc \\ q+nc \end{pmatrix}: K_0 \rightarrow Y \oplus W$.

Therefore, by (ET3^{op}), we get a commutative diagram

$$\begin{array}{ccccc} K_1 & \xrightarrow{b} & K_0 & \xrightarrow{d} \twoheadrightarrow & X \\ \vdots \scriptstyle s & & \downarrow \begin{pmatrix} p+mc \\ q+nc \end{pmatrix} & & \downarrow f \\ T & \xrightarrow{\quad} & Y \oplus W & \xrightarrow{\quad} \twoheadrightarrow & A \end{array} \quad (5.3)$$

Applying Proposition 2.21 to conflation (5.1), there is an exact sequence

$$\mathcal{B}(P_1, T) \xrightarrow{\mathcal{B}(a, T)} \mathcal{B}(K_1, T) \longrightarrow \mathbb{E}(S_1, T) = 0,$$

where $\mathbb{E}(S_1, T) = 0$ as $(\mathcal{S}, \mathcal{T})$ is a cotorsion pair. So, there exists $t: P_1 \rightarrow T$ such that $ta = s$. This implies $s = ta = tcb$, so by [28, Cor. 3.5] f must factor through $(r \ w)$. Thus, there is a morphism $\begin{pmatrix} u \\ v \end{pmatrix}: X \rightarrow Y \oplus W$ such that $(r \ w)\begin{pmatrix} u \\ v \end{pmatrix} = f$.

In $\overline{\mathcal{H}}$ we then have $\overline{f} = \overline{(r \ w)}\begin{pmatrix} u \\ v \end{pmatrix} = \overline{r}u + \overline{w}v = \overline{r}u$ because $\overline{w} = 0$ since $W \in \mathcal{W}$. Hence, in $\overline{\mathcal{H}}_{\mathcal{R}}$ we have that $[\overline{f}, \overline{1}_A]_{\text{LF}} = [\overline{r}u, \overline{1}_A]_{\text{LF}} = [\overline{r}, \overline{1}_A]_{\text{LF}} \circ [\overline{u}, \overline{1}_Y]_{\text{LF}} = L_{\mathcal{R}}(\overline{r})L_{\mathcal{R}}(\overline{u})$, which implies

$$[\overline{f}, \overline{r}]_{\text{LF}} = L_{\mathcal{R}}(\overline{r})^{-1} \circ [\overline{f}, \overline{1}_A]_{\text{LF}} = L_{\mathcal{R}}(\overline{u}) = L_{\mathcal{R}}F(u + [\mathcal{S}](X, Y)) = G(u + [\mathcal{S}](X, Y)).$$

Thus, $G: \overline{\mathcal{H}} \rightarrow \overline{\mathcal{H}}_{\mathcal{R}}$ is a full functor. \blacksquare

Therefore, we have found a fully faithful, dense functor $G: \overline{\mathcal{H}} \rightarrow \overline{\mathcal{H}}_{\mathcal{R}}$, which establishes the main result of this section.

Theorem 5.6. *Let $(\mathcal{B}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category with enough projectives and enough injectives. Suppose $((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$ is a twin cotorsion pair on \mathcal{B} that satisfies $\mathcal{T} = \mathcal{U}$. Let \mathcal{R} denote the class of regular morphisms in the heart $\overline{\mathcal{H}}$ of $((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$. Then the Gabriel-Zisman localisation $\overline{\mathcal{H}}_{\mathcal{R}}$ is equivalent to the heart $\overline{\mathcal{H}}_{(\mathcal{S}, \mathcal{T})}$ of the single twin cotorsion pair $(\mathcal{S}, \mathcal{T})$.*

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