QUASI-ABELIAN HEARTS OF TWIN COTORSION PAIRS ON TRIANGULATED CATEGORIES

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Abstract. We prove that, under a mild assumption, the heart $\overline{\mathcal{H}}$ of a twin cotorsion pair $((\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V}))$ on a triangulated category $\mathcal{C}$ is a quasi-abelian category. If $\mathcal{C}$ is also Krull-Schmidt and $\mathcal{T}=\mathcal{U}$, we show that the heart of the cotorsion pair $(\mathcal{S},\mathcal{T})$ is equivalent to the Gabriel-Zisman localisation of $\overline{\mathcal{H}}$ at the class of its regular morphisms.

In particular, suppose $\mathcal{C}$ is a cluster category with a rigid object R and $[\mathcal{X}_R]$ the ideal of morphisms factoring through $\mathcal{X}_R = \operatorname{Ker}(\operatorname{Hom}_{\mathcal{C}}(R,-))$, then applications of our results show that $\mathcal{C}/[\mathcal{X}_R]$ is a quasi-abelian category. We also obtain a new proof of an equivalence between the localisation of this category at its class of regular morphisms and a certain subfactor category of $\mathcal{C}$.

1. Introduction

Cotorsion pairs were first defined specifically for the category of abelian groups in [34] as an analogue of the torsion theories introduced in [14], which were themselves used to generalise the notion of torsion in abelian groups. Torsion theories for triangulated categories were introduced in [20] and used in the study of rigid Cohen-Macaulay modules over specific Veronese subrings. Analogously, Nakaoka [29] defined cotorsion pairs for triangulated categories as follows. Let $\mathcal{C}$ be a triangulated category with suspension functor Σ. A *cotorsion pair* on $\mathcal{C}$ is a pair $(\mathcal{U},\mathcal{V})$ of full, additive subcategories of $\mathcal{C}$ that are closed under isomorphisms and direct summands, satisfying $\operatorname{Ext}^1_{\mathcal{C}}(\mathcal{U},\mathcal{V})=0$ and $\mathcal{C}=\mathcal{U}*\Sigma\mathcal{V}$ (see Definitions 2.14 and 2.15). This allowed Nakaoka to extract an abelian category, known as the *heart* of the cotorsion pair [29, Def. 3.7], from the triangulated category. The key motivating examples for Nakaoka were the following.

(i) A t-structure $(\mathcal{C}^{\leqslant 0},\mathcal{C}^{\geqslant 0})$ on a triangulated category $\mathcal{C}$, in the sense of [7], can be interpreted as a cotorsion pair $(\Sigma\mathcal{C}^{\leqslant 0},\Sigma^{-1}\mathcal{C}^{\geqslant 0})$. In this case the heart $\mathcal{C}^{\leqslant 0}\cap\mathcal{C}^{\geqslant 0}$ of the t-structure coincides with the heart of the cotorsion pair.

(ii) Suppose $\mathcal{C}$ is a triangulated category, with a tilting subcategory $\mathcal{T}$ (see [19, Def. 2.2]). It was shown in [24] (see also [23] and [11]) that $\mathcal{C}/[\mathcal{T}]$ is an abelian category, where $[\mathcal{T}]$ is the ideal of morphisms factoring through $\mathcal{T}$. The corresponding cotorsion pair in this setting is $(\mathcal{T},\mathcal{T})$ and has $\mathcal{C}/[\mathcal{T}]$ as its heart.

Date: 16th December 2020.
2010 *Mathematics Subject Classification.* Primary 18E30; Secondary 18E05, 18E35, 18E40.
Key words and phrases. triangulated category, twin cotorsion pair, heart, quasi-abelian category, localisation, cluster category.
HTTPS://DOI.ORG/10.1016/J.JALGEBRA.2019.06.011

In [9], Buan and Marsh generalised the results of [24] and [11] in the following way. Assume k is a field, and suppose $\mathcal{C}$ is a skeletally small, Hom-finite, Krull-Schmidt, triangulated k-category has Serre duality (see Definition 5.1). For a subcategory $\mathcal{A}$ of $\mathcal{C}$ that is closed under finite direct sums, let $[\mathcal{A}]$ denote the ideal of $\mathcal{C}$ consisting of morphisms that factor through an object of $\mathcal{A}$. Fix an object R of $\mathcal{C}$ that is rigid (see §5). It was shown [9, Thm. 5.7] that there is an equivalence $(\mathcal{C}/[\mathcal{X}_R])_{\mathcal{R}} \simeq \mathsf{mod}(\operatorname{End}_{\mathcal{C}} R)^{\mathrm{op}}$, where $\mathcal{X}_R = \operatorname{Ker}(\operatorname{Hom}_{\mathcal{C}}(R,-))$ and $(\mathcal{C}/[\mathcal{X}_R])_{\mathcal{R}}$ is the (Gabriel-Zisman) localisation (see [16]) of $\mathcal{C}/[\mathcal{X}_R]$ at the class $\mathcal{R}$ of regular (see Remark 2.6) morphisms. Beligiannis further developed these ideas in [6].

Nakaoka was then able to put this into a more general context by introducing the following concept in [30]. A *twin cotorsion pair* on $\mathcal{C}$ consists of two cotorsion pairs $(\mathcal{S},\mathcal{T})$ and $(\mathcal{U},\mathcal{V})$ on $\mathcal{C}$ which satisfy $\mathcal{S}\subseteq\mathcal{U}$. As for cotorsion pairs, Nakaoka defined the *heart* of a twin cotorsion pair as a certain subfactor category of $\mathcal{C}$ (see Definition 2.31). By setting $\mathcal{S}=\mathcal{U}$ and $\mathcal{T}=\mathcal{V}$, one recovers the original cotorsion pair theory: the heart of the twin cotorsion pair $((\mathcal{U},\mathcal{V}),(\mathcal{U},\mathcal{V}))$ coincides with the heart of the cotorsion pair $(\mathcal{U},\mathcal{V})$ (see [30, Exam. 2.10]).

For a twin cotorsion pair, the associated heart $\overline{\mathcal{H}}$ is shown [30, Thm. 5.4] to be semi-abelian (see Definition 2.5). Furthermore, Nakaoka showed [30, Thm. 6.3] that if $\mathcal{U}\subseteq\mathcal{S}*\mathcal{T}$ or $\mathcal{T}\subseteq\mathcal{U}*\mathcal{V}$, then $\overline{\mathcal{H}}$ is integral (see Definition 2.11), so that localising at the class of regular morphisms produces an abelian category (see [32]). With $\mathcal{C}$ and R as above, and setting $((\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V})) = ((\mathsf{add}\,\Sigma R,\mathcal{X}_R),(\mathcal{X}_R,{\mathcal{X}_R}^{\perp_1}))$, where ${\mathcal{X}_R}^{\perp_1} = \operatorname{Ker}(\operatorname{Ext}^1_{\mathcal{C}}(\mathcal{X}_R,-))$, one obtains the aforementioned result [9, Thm. 5.7] of Buan and Marsh (see Lemma 5.4).

The main result of this article concerns quasi-abelian categories; a *quasi-abelian* category is an additive category which has kernels and cokernels, and in which kernels are stable under pushout and cokernels are stable under pullback (see Definition 2.7). Important examples of such categories include: the category of topological abelian groups; the category of Λ-lattices for Λ an order over a noetherian integral domain; any abelian category; and the torsion class and torsion-free class in any torsion theory of an abelian category (see [32, §2] for more details). In this article, we prove that the heart of a twin cotorsion pair, satisfying a different mild assumption, is quasi-abelian (see Theorem 3.4). This assumption is satisfied if $\mathcal{U}\subseteq\mathcal{T}$ or $\mathcal{T}\subseteq\mathcal{U}$, and hence is met in the setting of [9] discussed above (see Corollary 3.5) where $\mathcal{T}=\mathcal{U}$.

Let $((\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V}))$ be a twin cotorsion pair with heart $\overline{\mathcal{H}}$ on a Krull-Schmidt, triangulated category. We show in §4 that if $\mathcal{T}$ coincides with $\mathcal{U}$, then the heart $\overline{\mathscr{H}}_{(\mathcal{S},\mathcal{T})}$ of $(\mathcal{S},\mathcal{T})$ (see [29, Def. 3.7]) is equivalent to the localisation $\overline{\mathcal{H}}_{\mathcal{R}}$ of $\overline{\mathcal{H}}$ at the class $\mathcal{R}$ of its regular morphisms (see Theorem 4.8). Since $\mathcal{T}=\mathcal{U}$ when $((\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V})) = ((\mathsf{add}\,\Sigma R,\mathcal{X}_R),(\mathcal{X}_R,{\mathcal{X}_R}^{\perp_1}))$, the results of §4 also apply in the setting of Buan and Marsh as we explain in §5. Our methods are also related to work of Marsh and Palu: in [28], equivalences are found from subfactor categories of a Krull-Schmidt, Hom-finite, triangulated category to localisations of module (and hence abelian) categories, whereas we localise not necessarily abelian categories. We also note that Theorem 4.8 may be obtained from results of [6] in a different way (see Remark 4.10).

In particular, the cluster category $\mathcal{C}_H$ (see [8], [13]) associated to a hereditary k-algebra H is an example of a Hom-finite, Krull-Schmidt, triangulated k-category that has Serre duality, and this is the motivation for our results (see Example 5.10). It is especially interesting that $\mathcal{C}/[\mathcal{X}_R]$ is quasi-abelian in this case, as many aspects of Auslander-Reiten theory for abelian categories (developed in [2], [3]) still apply for quasi-abelian categories (see the forthcoming preprint [36]).

This paper is organised in the following way. We first recall the notion of a quasi-abelian category in §2.1, then the definition and some properties of twin cotorsion pairs as well as some new observations in §2.2. In §3 we prove our main result: the case when the heart of a twin cotorsion pair becomes quasi-abelian. In §4 we relate the heart of a twin cotorsion pair $((\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V}))$ to the heart of the cotorsion pair $(\mathcal{S},\mathcal{T})$ whenever $\mathcal{T}=\mathcal{U}$. Lastly, we explore our main motivating example in §5, namely the setting of [9].

2. Preliminaries

2.1. **Preabelian categories.** The main result of this paper concerns a type of category more general than an abelian category—namely a *quasi-abelian* category. However, before giving the definition of such a category, we recall some preliminary definitions. We only give a quick summary of the theory and for more details we refer the reader to [32].

Definition 2.1. [31, p. 24], [12, §5.4] A *preabelian* category is an additive category in which every morphism has a kernel and a cokernel.

Definition 2.2. [31, p. 23] Given a morphism $f\colon A\to B$ in a category $\mathcal{A}$, the *coimage* $\operatorname{coim} f\colon A\to \operatorname{Coim} f$, if it exists, is the cokernel $\operatorname{coker}(\ker f)$ of the kernel of f. Similarly, the *image* $\operatorname{im} f\colon \operatorname{Im} f\to B$ is the kernel $\ker(\operatorname{coker} f)$ of the cokernel of f.

The following proposition is then easily checked.

Proposition 2.3. [31, p. 24] *Let $\mathcal{A}$ be a preabelian category and $f\colon A\to B$ a morphism in $\mathcal{A}$. Then f decomposes as*

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & B \\ {\scriptstyle \operatorname{coim} f}\downarrow & \circ & \uparrow{\scriptstyle \operatorname{im} f} \\ \operatorname{Coim} f & \xrightarrow[\widetilde{f}]{\quad\quad} & \operatorname{Im} f \end{array}$$

Definition 2.4. [31, p. 24] The morphism $\widetilde{f}$ in Proposition 2.3 above is called the *parallel of* f. Furthermore, if $\widetilde{f}$ is an isomorphism then f is said to be *strict*.

Definition 2.5. [32, p. 167] Let $\mathcal{A}$ be a preabelian category. We call $\mathcal{A}$ *left semi-abelian* if each morphism $f\colon A\to B$ factorises as $f=ip$ for some monomorphism i and cokernel p. We call $\mathcal{A}$ *right semi-abelian* if instead each morphism f decomposes as $f=ip$ with i a kernel and p some epimorphism. If $\mathcal{A}$ is both left and right semi-abelian, then it is simply called *semi-abelian*.

Remark 2.6. A category is semi-abelian if and only if, for every morphism f, the parallel $\widetilde{f}$ of f is *regular*, i.e. simultaneously monic and epic (see [32, pp. 167–168]).

Definition 2.7. Let $\mathcal{A}$ be a category, and suppose

$$\begin{array}{ccc} A & \xrightarrow{a} & B \\ {\scriptstyle b}\big\downarrow & & \big\downarrow{\scriptstyle c} \\ C & \xrightarrow[d]{} & D \end{array}$$

is a commutative diagram in $\mathcal{A}$. Let $\mathcal{P}$ be a class of morphisms in $\mathcal{A}$ (e.g. the class of all kernels in $\mathcal{A}$). We say that $\mathcal{P}$ is *stable under pullback* (respectively, *stable under pushout*) if, in any diagram above that is a pullback (respectively, pushout) square, d is in $\mathcal{P}$ implies that a is in $\mathcal{P}$ (respectively, a is in $\mathcal{P}$ implies that d is in $\mathcal{P}$).

Definition 2.8. [32, p. 168] Let $\mathcal{A}$ be a preabelian category. We call $\mathcal{A}$ *left quasi-abelian* if cokernels are stable under pullback in $\mathcal{A}$. If kernels are stable under pushout in $\mathcal{A}$, then we call $\mathcal{A}$ *right quasi-abelian.* Furthermore, if $\mathcal{A}$ is left and right quasi-abelian, then $\mathcal{A}$ is simply called *quasi-abelian.*

Remark 2.9. The history of the term 'quasi-abelian' category is not straightforward. We use the terminology as in [33], but note that such categories were called 'almost abelian' in [32]. We refer the reader to the 'Historical remark' in [33] for more details.

Remark 2.10. It is also worth remarking that a category is abelian if and only if it is a quasi-abelian category in which every morphism is strict.

Definition 2.11. [32, p. 168] Let $\mathcal{A}$ be a preabelian category. We call $\mathcal{A}$ *left integral* if epimorphisms are stable under pullback in $\mathcal{A}$. If monomorphisms are stable under pushout in $\mathcal{A}$, then we call $\mathcal{A}$ *right integral.* If $\mathcal{A}$ is both left and right integral, then $\mathcal{A}$ is called *integral.*

Lastly in this section, we recall an observation from [32].

Proposition 2.12. [32, p. 169, Cor. 1] *Every left (respectively, right) quasi-abelian or left (respectively, right) integral category is left (respectively, right) semi-abelian.*

2.2. **Twin cotorsion pairs on triangulated categories.** Throughout this section, let $\mathcal{C}$ denote a fixed triangulated category with suspension functor Σ. We use the labelling of the axioms of a triangulated category as in [18], and its distinguished triangles will just be called triangles. We follow [29] and [30] in order to recall some of the definitions and theory concerning twin cotorsion pairs on triangulated categories, but first we need some notation.

Definition 2.13. Let $\mathcal{U} \subseteq \mathcal{C}$ be a full, additive subcategory of $\mathcal{C}$ that is closed under isomorphisms and direct summands. By $\operatorname{Ext}^i_{\mathcal{C}}(\mathcal{U}, X) = 0$ (respectively, $\operatorname{Ext}^i_{\mathcal{C}}(X, \mathcal{U}) = 0$) we mean $\operatorname{Ext}^i_{\mathcal{C}}(U, X) = 0$ (respectively, $\operatorname{Ext}^i_{\mathcal{C}}(X, U) = 0$) for all $U \in \mathcal{U}$. We define the following full, additive subcategories of $\mathcal{C}$ where $i \in \mathbb{N}$:

$$\mathcal{U}^{\perp_i} := \left\{ X \in \mathcal{C} \mid \operatorname{Ext}^i_{\mathcal{C}}(\mathcal{U}, X) = 0 \right\},$$

$$^{\perp_i}\mathcal{U} := \left\{ X \in \mathcal{C} \mid \operatorname{Ext}^i_{\mathcal{C}}(X, \mathcal{U}) = 0 \right\}.$$

Definition 2.14. [20, p. 122] Let $\mathcal{U}, \mathcal{V} \subseteq \mathcal{C}$ be full, additive subcategories of $\mathcal{C}$ that are closed under isomorphisms and direct summands. By $\mathcal{U} * \mathcal{V}$ we denote the full subcategory of $\mathcal{C}$ consisting of objects $X \in \mathcal{C}$ for which there exists a triangle $U \to X \to V \to \Sigma U$ in $\mathcal{C}$ with $U \in \mathcal{U}$, $V \in \mathcal{V}$.

Definition 2.15. [29, Def. 2.1] Let $\mathcal{U}, \mathcal{V} \subseteq \mathcal{C}$ be full, additive subcategories of $\mathcal{C}$ that are closed under isomorphisms and direct summands. We call (the ordered pair) $(\mathcal{U}, \mathcal{V})$ a *cotorsion pair (on $\mathcal{C}$)* if $\operatorname{Ext}^1_{\mathcal{C}}(\mathcal{U}, \mathcal{V}) = 0$ and $\mathcal{C} = \mathcal{U} * \Sigma\mathcal{V}$.

As pointed out in [29, Rem. 2.2], a pair $(\mathcal{U}, \mathcal{V})$ is a cotorsion pair on a Krull-Schmidt, Hom-finite, triangulated k-category $\mathcal{C}'$ (with suspension Σ') if and only if $(\Sigma'^{-1}\mathcal{U}, \mathcal{V})$ is a torsion theory in $\mathcal{C}'$ as defined in [20]. Recall that a *torsion theory in $\mathcal{C}'$* (in the sense of [20, Def. 2.2]) is a pair $(\mathcal{X}, \mathcal{Y})$ of full additive subcategories $\mathcal{X}, \mathcal{Y}$ of $\mathcal{C}'$ that are closed under isomorphisms and direct summands, such that $\operatorname{Hom}_{\mathcal{C}'}(\mathcal{X}, \mathcal{Y}) = 0$ and $\mathcal{C}' = \mathcal{X} * \mathcal{Y}$. We note that in [20] all categories are assumed to be Krull-Schmidt and all triangulated categories are also assumed to be Hom-finite k-categories (see [20, pp. 121–122]). Therefore, some of the results from [20] may not translate directly over to the more general setting considered in [30].

Definition 2.16. [2, §2] Let $f\colon X \to Y$ be a morphism in $\mathcal{C}$. We say that f is *right minimal* (respectively, *left minimal*) if, for any endomorphism $g\colon X \to X$ (respectively, $g\colon Y \to Y$), $fg = f$ (respectively, $gf = f$) implies g is an automorphism.

Definition 2.17. [4, p. 114] Let $\mathcal{X} \subseteq \mathcal{C}$ be a full subcategory, closed under isomorphisms and direct summands.

(i) A *right $\mathcal{X}$-approximation* of A in $\mathcal{C}$ is a morphism $X \to A$ in $\mathcal{C}$ with $X \in \mathcal{X}$, such that for any object $X' \in \mathcal{X}$ we have an exact sequence

$$\operatorname{Hom}_{\mathcal{A}}(X', X) \to \operatorname{Hom}_{\mathcal{A}}(X', A) \to 0.$$

A right $\mathcal{X}$-approximation is called a *minimal* right $\mathcal{X}$-approximation if it is also right minimal.

(ii) A *left $\mathcal{X}$-approximation* of A in $\mathcal{C}$ is a morphism $A \to X$ in $\mathcal{C}$ with $X \in \mathcal{X}$, such that for any object $X' \in \mathcal{X}$ we have an exact sequence

$$\operatorname{Hom}_{\mathcal{A}}(X, X') \to \operatorname{Hom}_{\mathcal{A}}(A, X') \to 0.$$

A left $\mathcal{X}$-approximation is called a *minimal* left $\mathcal{X}$-approximation if it is also left minimal.

The terminology of approximations was introduced in [4], but the same notions were established independently by Enochs [15] specifically for the subcategories of injective objects and projective objects in a module category. The term 'preenvelope' (respectively, 'precover') in [15] corresponds to the notion of left (respectively, right) approximation.

Lemma 2.18 (Triangulated Wakamatsu's Lemma)**.** [21, Lem. 2.1] *Let $\mathcal{X}$ be an extension-closed full subcategory of $\mathcal{C}$ that is closed under isomorphisms and direct summands.*

(i) *Suppose* $X \xrightarrow{x} A$ *is a minimal right* $\mathcal{X}$*-approximation of* A *in* $\mathcal{C}$*, which completes to a triangle* $\Sigma^{-1}A \xrightarrow{w} Y \longrightarrow X \xrightarrow{x} A$*. Then* $w\colon \Sigma^{-1}A \to Y$ *is a left* $\mathcal{X}^{\perp_1}$*-approximation of* $\Sigma^{-1}A$*.*

(ii) *Suppose* $A \xrightarrow{x'} X'$ *is a minimal left* $\mathcal{X}$*-approximation of* A *in* $\mathcal{C}$*, which completes to a triangle* $A \xrightarrow{x'} X' \longrightarrow Z \xrightarrow{z} \Sigma A$*. Then* $z\colon Z \to \Sigma A$ *is a right* ${}^{\perp_1}\mathcal{X}$*-approximation of* ΣA*.*

Although the notion of a contravariantly (respectively, covariantly) finite subcategory (see below) is related to the idea of right (respectively, left) approximations, it dates back to [5, p. 81] in which these concepts were defined in the context of module categories.

Definition 2.19. [4, pp. 114, 142] Let $\mathcal{X} \subseteq \mathcal{C}$ be a full subcategory, closed under isomorphisms and direct summands. We say $\mathcal{X}$ is *contravariantly* (respectively, *covariantly*) *finite* if A has a right (respectively, left) $\mathcal{X}$-approximation for each $A \in \mathcal{C}$. If $\mathcal{X}$ is both contravariantly finite and covariantly finite, then $\mathcal{X}$ is called *functorially finite.*

The next proposition collects some elementary properties about cotorsion pairs that will be very useful in the sequel; see for example [20] or [29]. Recall that if $\mathcal{U}$ is a full, additive subcategory of $\mathcal{C}$, then $\mathsf{add}\,\mathcal{U}$ denotes the full, additive subcategory of $\mathcal{C}$ that consists of objects of $\mathcal{C}$ which are isomorphic to direct summands of finite direct sums of objects of $\mathcal{U}$.

Proposition 2.20. *Let* $(\mathcal{U}, \mathcal{V})$ *be a cotorsion pair* on $\mathcal{C}$*.*

(i) [20, p. 123], [29, Rem. 2.3] *We have* $\mathcal{U} = {}^{\perp_1}\mathcal{V}$ *and* $\mathcal{V} = \mathcal{U}^{\perp_1}$*.*

(ii) [20, p. 123], [30, Lem. 2.14] *Let* X be an object in $\mathcal{C}$*. Since* $(\mathcal{U}, \mathcal{V})$ *is a cotorsion pair, there is a triangle* $U \xrightarrow{u} X \xrightarrow{v} \Sigma V \xrightarrow{w} \Sigma U$*, where* $U \in \mathcal{U}$ *and* $V \in \mathcal{V}$*. Then the morphism* $u\colon U \to X$ *is a right* $\mathcal{U}$*-approximation of* X *and the morphism* $v\colon X \to \Sigma V$ *is a left* $\Sigma\mathcal{V}$*-approximation of* X*.*

(iii) *The subcategory* $\mathcal{U}$ *is contravariantly finite and the subcategory* $\mathcal{V}$ *is covariantly finite.*

(iv) [29, Rem. 2.4] *The subcategories* $\mathcal{U}$ *and* $\mathcal{V}$ *are extension-closed.*

Definition 2.21. [30, Def. 2.7] Let $(\mathcal{S}, \mathcal{T})$ and $(\mathcal{U}, \mathcal{V})$ be two cotorsion pairs on $\mathcal{C}$. The ordered pair $((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$ is called a *twin cotorsion pair (on* $\mathcal{C}$*)* if $\operatorname{Ext}^1_{\mathcal{C}}(\mathcal{S}, \mathcal{V}) = 0$.

The following easily verifiable result is often useful.

Proposition 2.22. [30, p. 198] *Let* $(\mathcal{S}, \mathcal{T})$ *and* $(\mathcal{U}, \mathcal{V})$ *be cotorsion pairs on* $\mathcal{C}$*. Then* $((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$ *is a twin cotorsion pair* $\iff \mathcal{S} \subseteq \mathcal{U} \iff \mathcal{V} \subseteq \mathcal{T}$*.*

Throughout the remainder of this section, let $((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$ be a twin cotorsion pair on $\mathcal{C}$.

Definition 2.23. [30, Def. 2.8] We define full subcategories of $\mathcal{C}$ as follows:

$$\mathcal{W} := \mathcal{T} \cap \mathcal{U}, \quad \mathcal{C}^- := \Sigma^{-1}\mathcal{S} * \mathcal{W}, \quad \mathcal{C}^+ := \mathcal{W} * \Sigma\mathcal{V}, \quad \mathcal{H} := \mathcal{C}^- \cap \mathcal{C}^+.$$

From this definition, we immediately see that $\mathcal{W}$ is contained in the subcategories $\mathcal{C}^-$, $\mathcal{C}^+$ and $\mathcal{H}$; and that $\mathcal{W}$ is extension-closed as $\mathcal{T}$ and $\mathcal{U}$ are extension-closed. It is also clear that $\mathcal{W}$, $\mathcal{C}^-$, $\mathcal{C}^+$ and $\mathcal{H}$ are additive and closed under isomorphisms.

Proposition 2.24. *The subcategories $\mathcal{W}$, $\mathcal{C}^-$, $\mathcal{C}^+$ and $\mathcal{H}$ are all closed under direct summands.*

Proof. Since $\mathcal{T}$ and $\mathcal{U}$ are assumed to be closed under direct summands (see Definition 2.15), we immediately see that $\mathcal{W}$ is also closed under direct summands. That $\mathcal{H}$ is closed under direct summands will follow from $\mathcal{C}^-$ and $\mathcal{C}^+$ having this property. We will give the proof just for $\mathcal{C}^-$ as the proof for $\mathcal{C}^+$ is similar.

Suppose $X = X_1 \oplus X_2 \in \mathcal{C}^-$, then there is a distinguished triangle $\Sigma^{-1}S \xrightarrow{s} X \xrightarrow{x} W \xrightarrow{t} S$ with $S \in \mathcal{S}$ and $W \in \mathcal{W}$. Since $\mathcal{C} = \mathcal{S} * \Sigma\mathcal{T}$, there exists a triangle $\Sigma^{-1}S_1 \xrightarrow{a} X_1 \xrightarrow{b} T_1 \xrightarrow{c} S_1$ where $S_1 \in \mathcal{S}$ and $T_1 \in \mathcal{T}$. Thus, it suffices to show that $T_1 \in \mathcal{U}$ as then we will have $T_1 \in \mathcal{T} \cap \mathcal{U} = \mathcal{W}$, and hence $T_1 \in \Sigma^{-1}\mathcal{S} * \mathcal{W} = \mathcal{C}^-$.

First, we claim that $x\colon X \to W$ is a left $\mathcal{T}$-approximation of X. Indeed, if $T \in \mathcal{T}$ then we get an exact sequence $\operatorname{Hom}_{\mathcal{C}}(W,T) \to \operatorname{Hom}_{\mathcal{C}}(X,T) \to \operatorname{Hom}_{\mathcal{C}}(\Sigma^{-1}S,T)$ since $\operatorname{Hom}_{\mathcal{C}}(-,T)$ is a cohomological functor (see [17, Prop. I.1.2]), where $\operatorname{Hom}_{\mathcal{C}}(\Sigma^{-1}S,T) \cong \operatorname{Hom}_{\mathcal{C}}(S,\Sigma T) = \operatorname{Ext}^1_{\mathcal{C}}(S,T) = 0$ since $(\mathcal{S},\mathcal{T})$ is a cotorsion pair. Thus, $\operatorname{Hom}_{\mathcal{C}}(W,T) \to \operatorname{Hom}_{\mathcal{C}}(X,T)$ is surjective for $T \in \mathcal{T}$ and $x\colon X \to W$ is a left $\mathcal{T}$-approximation of X.

In order to show $T_1 \in \mathcal{U}$, it is enough to show that any $v\colon T_1 \to \Sigma V$ is in fact the zero map as $\mathcal{U} = {}^{\perp_1}\mathcal{V}$ (see Proposition 2.20). Let $v\colon T_1 \to \Sigma V$ be arbitrary. Since $b\pi_1\colon X = X_1 \oplus X_2 \to T_1$ is a morphism with codomain in $\mathcal{T}$, where $\pi_1\colon X \to X_1$ is the canonical projection, it must factor through the left $\mathcal{T}$-approximation $x\colon X \to W$. That is, there exists $d\colon W \to T_1$ such that $dx = b\pi_1$. We then have $(vb)\pi_1 = vdx = 0$, because $vd\colon W \to \Sigma V$ vanishes as $W \in \mathcal{W} \subseteq \mathcal{U} = {}^{\perp_1}\mathcal{V}$. This in turn implies $vb = 0$ as π_1 is an epimorphism. Since $\Sigma^{-1}S_1 \xrightarrow{a} X_1 \xrightarrow{b} T_1 \xrightarrow{c} S_1$ is a triangle, we see that $v\colon T_1 \to \Sigma V$ must factor through $c\colon T_1 \to S_1$. Thus, $v = fc$ for some $f \in \operatorname{Hom}_{\mathcal{C}}(S_1,\Sigma V) = \operatorname{Ext}^1_{\mathcal{C}}(S_1,V) = 0$ by definition of a twin cotorsion pair. Hence, $v = 0$ and we are done. ■

We now recall some notions from [30] needed for the remainder of this section.

Definition 2.25. [30, Def. 3.1] For $X \in \mathcal{C}$, we define $K_X \in \mathcal{C}$ and a morphism $k_X\colon K_X \to X$ as follows. Since $\mathcal{S}*\Sigma\mathcal{T} = \mathcal{C} = \mathcal{U}*\Sigma\mathcal{V}$, we have two triangles $\Sigma^{-1}S \longrightarrow X \xrightarrow{a} T \longrightarrow S$ $(S \in \mathcal{S}, T \in \mathcal{T})$ and $U \longrightarrow T \xrightarrow{b} \Sigma V \longrightarrow \Sigma U$ $(U \in \mathcal{U}, V \in \mathcal{V})$. Then we may complete the composition $ba\colon X \to \Sigma V$ to a triangle

$$V \longrightarrow K_X \xrightarrow{k_X} X \xrightarrow{ba} \Sigma V.$$

Definition 2.26. [30, Def. 3.4] For $X \in \mathcal{C}$, we define $Z_X \in \mathcal{C}$ and $z_X\colon X \to Z_X$ as follows. Since $\mathcal{S} * \Sigma\mathcal{T} = \mathcal{C} = \mathcal{U} * \Sigma\mathcal{V}$, we have two triangles $V \longrightarrow U \xrightarrow{c} X \longrightarrow \Sigma V$ $(U \in \mathcal{U}, V \in \mathcal{V})$ and $\Sigma^{-1}T \longrightarrow \Sigma^{-1}S \xrightarrow{d} U \longrightarrow T$ $(S \in \mathcal{S}, T \in \mathcal{T})$. Then we may complete the composition $cd\colon \Sigma^{-1}S \to X$ to a triangle

$$\Sigma^{-1}S \xrightarrow{cd} X \xrightarrow{z_X} Z_X \longrightarrow S.$$

Definition 2.27. [30, Def. 4.1] Let $f\colon X \to Y$ be a morphism in $\mathcal{C}$ with $X \in \mathcal{C}^-$. We define $M_f \in \mathcal{C}$ and $m_f\colon Y \to M_f$ as follows. Since $X \in \mathcal{C}^-$, there is a triangle $\Sigma^{-1}S \xrightarrow{s} X \longrightarrow W \longrightarrow S$ $(S \in \mathcal{S}, W \in \mathcal{W})$. Then we may complete $fs\colon \Sigma^{-1}S \to Y$

to a triangle

$$\Sigma^{-1}S \xrightarrow{fs} Y \xrightarrow{m_f} M_f \longrightarrow S.$$

Definition 2.28. [30, Rem. 4.3] Let $f\colon X \to Y$ be a morphism in $\mathcal{C}$ with $Y \in \mathcal{C}^+$. We define $L_f \in \mathcal{C}$ and $l_f\colon L_f \to X$ as follows. Since $Y \in \mathcal{C}^+$, there is a triangle $W \longrightarrow Y \xrightarrow{v} \Sigma V \longrightarrow \Sigma W$ $(W \in \mathcal{W}, V \in \mathcal{V})$. Then we may complete $vf\colon X \to \Sigma V$ to a triangle

$$V \longrightarrow L_f \xrightarrow{l_f} X \xrightarrow{vf} \Sigma V.$$

We now present strengthened versions of [30, Claim 3.2] and [30, Claim 3.5].

Proposition 2.29. *Suppose C is an arbitrary object of $\mathcal{C}$. Then*
(i) $K_C \in \mathcal{C}^-$;
(ii) $C \in \mathcal{C}^+ \iff K_C \in \mathcal{C}^+ \iff K_C \in \mathcal{H}$;
(iii) $Z_C \in \mathcal{C}^+$; *and*
(iv) $C \in \mathcal{C}^- \iff Z_C \in \mathcal{C}^- \iff Z_C \in \mathcal{H}$.

Proof. The proofs for (i) and (iii) are [30, Claim 3.2 (1)] and [30, Claim 3.5 (1)], respectively. Since $K_C \in \mathcal{C}^-$, we immediately see that $K_C \in \mathcal{C}^+$ if and only if $K_C \in \mathcal{H}$. For (ii), the proof that $C \in \mathcal{C}^+$ implies $K_C \in \mathcal{H}$ is [30, Claim 3.2 (2)]. Thus, we show the converse. There is a triangle $V \to K_C \to C \to \Sigma V$ where $V \in \mathcal{V}$, so if $K_C \in \mathcal{C}^+$ then $C \in \mathcal{C}^+$ using [30, Lem. 2.13 (2)]. The proof of statement (iv) is similar. ■

The next proposition follows from [30, Prop. 3.6] and [30, Prop. 3.7], but we state it in the language of approximations.

Proposition 2.30. *Suppose C is an arbitrary object of $\mathcal{C}$.*
(i) *The morphism $k_C\colon K_C \to C$ is a right $\mathcal{C}^-$-approximation of C.*
(ii) *The morphism $z_C\colon C \to Z_C$ is a left $\mathcal{C}^+$-approximation of C.*

For a subcategory $\mathcal{A} \subseteq \mathcal{C}$ that is closed under finite direct sums, we will denote by $[\mathcal{A}]$ the two-sided ideal of $\mathcal{C}$ such that $[\mathcal{A}](X,Y)$ consists of all morphisms $X \to Y$ that factor through an object in $\mathcal{A}$. Note that if $\mathcal{A}$ is a full, additive subcategory that is closed under isomorphisms and direct summands, then $[\mathcal{A}]$ coincides with the ideal generated by identity morphisms 1_A such that $A \in \mathcal{A}$. With this notation we are in position to recall the definition of the heart associated to a twin cotorsion pair.

Definition 2.31. [30] Recall that $\mathcal{W}$ is a subcategory of $\mathcal{C}^+$, $\mathcal{C}^-$ and $\mathcal{H}$. We define the following additive quotients $\overline{\mathcal{C}^+} := \mathcal{C}^+/[\mathcal{W}]$, $\overline{\mathcal{C}^-} := \mathcal{C}^-/[\mathcal{W}]$ and $\overline{\mathcal{H}} := \mathcal{H}/[\mathcal{W}]$. We call the category $\overline{\mathcal{H}}$ the *heart* of $((\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V}))$ by analogy with [29].

Suppose $\mathcal{I}$ an ideal of $\mathcal{C}$. We will denote by $\overline{f}$ the coset $f + \mathcal{I}(X,Y)$ in $\operatorname{Hom}_{\mathcal{C}/\mathcal{I}}(X,Y)$ of $f \in \operatorname{Hom}_{\mathcal{C}}(X,Y)$. The next result is a combination of [30, Cor. 4.5] and [30, Cor. 4.6], and most of the proof can be found there. We provide the missing link.

Proposition 2.32. *Let $f \in \operatorname{Hom}_{\mathcal{H}}(A,B)$ be a morphism in the subcategory $\mathcal{H}$, which completes to a triangle $A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow \Sigma A$ in $\mathcal{C}$. Then the following are equivalent:*

(i) $\overline{f} \in \operatorname{Hom}_{\overline{\mathcal{H}}}(A,B)$ *is an epimorphism;*
(ii) $Z_{M_f} \in \mathcal{W}$, *i.e.* $Z_{M_f} \cong 0$ *in* $\overline{\mathcal{H}}$*;*
(iii) $M_f \in \mathcal{U}$*; and*
(iv) $g\colon B \to C$ *factors through* $\mathcal{U}$*.*

Proof. The equivalence of (i) – (iii) is [30, Cor. 4.5], and (iv) implies (i) is [30, Cor. 4.6]. We prove (iii) implies (iv). To this end, suppose $M_f \in \mathcal{U}$. From Definition 2.27, there are triangles $\Sigma^{-1}S_A \xrightarrow{s_A} A \xrightarrow{w_A} W_A \longrightarrow S_A$ and $\Sigma^{-1}S_A \xrightarrow{fs_A} B \xrightarrow{m_f} M_f \longrightarrow S_A$, where $S_A \in \mathcal{S}$, $W_A \in \mathcal{W}$. Then $g(fs_A) = 0$ as $gf = 0$, so g factors through $m_f\colon B \to M_f$ where $M_f \in \mathcal{U}$ by assumption. Hence, g admits a factorisation through $\mathcal{U}$ as desired. ■

To be explicit, we state the dual in full.

Proposition 2.33. *Let* $f \in \operatorname{Hom}_{\mathcal{H}}(A,B)$ *be a morphism in the subcategory* $\mathcal{H}$*, which completes to a triangle* $\Sigma^{-1}C \xrightarrow{h} A \xrightarrow{f} B \longrightarrow C$ *in* $\mathcal{C}$*. Then the following are equivalent:*

(i) $\overline{f} \in \operatorname{Hom}_{\overline{\mathcal{H}}}(A,B)$ *is a monomorphism;*
(ii) $K_{L_f} \in \mathcal{W}$, *i.e.* $K_{L_f} \cong 0$ *in* $\overline{\mathcal{H}}$*;*
(iii) $L_f \in \mathcal{T}$*; and*
(iv) $h\colon \Sigma^{-1}C \to A$ *factors through* $\mathcal{T}$*.*

The last result of this section is an application of these previous two propositions to the case of a *degenerate* twin cotorsion pair; that is, a twin cotorsion pair $((\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V}))$ where $(\mathcal{S},\mathcal{T}) = (\mathcal{U},\mathcal{V})$. One may recover results from [29] about cotorsion pairs on triangulated categories through the theory of twin cotorsion pairs developed in [30] using such degenerate twin cotorsion pairs (see [30, Exam. 2.10 (1)]). We recall some definitions, which we will also need later, from [29] for convenience.

Definition 2.34. [29] Given a cotorsion pair $(\mathcal{U},\mathcal{V})$ on a triangulated category, define $\mathscr{W} := \mathcal{U} \cap \mathcal{V}$, $\mathscr{C}^- := \Sigma^{-1}\mathcal{U} * \mathscr{W}$ and $\mathscr{C}^+ := \mathscr{W} * \Sigma\mathcal{V}$. The *heart* of the (individual) cotorsion pair $(\mathcal{U},\mathcal{V})$ is defined to be $\overline{\mathscr{H}}_{(\mathcal{U},\mathcal{V})} := (\mathscr{C}^- \cap \mathscr{C}^+)/[\mathscr{W}]$.

It is easy to see that $\mathscr{W} = \mathcal{W}$, $\mathscr{C}^- = \mathcal{C}^-$ and $\mathscr{C}^+ = \mathcal{C}^+$ for a degenerate twin cotorsion pair $((\mathcal{U},\mathcal{V}),(\mathcal{U},\mathcal{V}))$, and that $\overline{\mathscr{H}}_{(\mathcal{U},\mathcal{V})}$ coincides with the heart $\overline{\mathcal{H}}$ of the twin cotorsion pair $((\mathcal{U},\mathcal{V}),(\mathcal{U},\mathcal{V}))$.

Corollary 2.35. *Suppose we have a degenerate twin cotorsion pair* $((\mathcal{U},\mathcal{V}),(\mathcal{U},\mathcal{V}))$ *on* $\mathcal{C}$ *and objects* $X,Y \in \mathcal{H}$*. Assume* $Z \longrightarrow X \xrightarrow{f} Y \longrightarrow \Sigma Z$ *is a triangle in* $\mathcal{C}$*.*

(i) *If* $\overline{f}$ *is epic in* $\overline{\mathcal{H}}$*, then* $Z \in \mathcal{C}^-$*.*
(ii) *If* $\overline{f}$ *is monic in* $\overline{\mathcal{H}}$*, then* $\Sigma Z \in \mathcal{C}^+$*.*

Proof. We only prove (i) as (ii) is similar. Since $\overline{f}$ is epic in $\overline{\mathcal{H}}$, by Proposition 2.32 we have that $M_f \in \mathcal{U} = \mathcal{S}$. Recall from Definition 2.27 that M_f is obtained by taking a triangle $\Sigma^{-1}S \xrightarrow{s} X \to W \to S$ ($S \in \mathcal{S}$, $W \in \mathcal{W}$), which exists as $X \in \mathcal{H} \subseteq \mathcal{C}^-$, and then completing the composition fs to a triangle $\Sigma^{-1}S \xrightarrow{fs} Y \xrightarrow{m_f} M_f \longrightarrow S$. Then applying the octahedral axiom (TR5), we get a commutative diagram

$$\begin{CD}
\Sigma^{-1}S @>{s}>> X @>>> W @>>> S \\
@| @VV{f}V @VVV @| \\
\Sigma^{-1}S @>{fs}>> Y @>{m_f}>> M_f @>>> S \\
@VVV @| @VVV @VVV \\
X @>{f}>> Y @>>> \Sigma Z @>>> \Sigma X \\
@VVV @VVV @| @VVV \\
W @>>> M_f @>>> \Sigma Z @>>> \Sigma W
\end{CD}$$

where the rows are triangles. Therefore, there is a triangle $\Sigma^{-1}M_f \to Z \to W \to M_f$ by (TR3), where $\Sigma^{-1}M_f \in \Sigma^{-1}\mathcal{U}$ and $W \in \mathcal{W}$, so $Z \in \Sigma^{-1}\mathcal{U} * \mathcal{W} = \Sigma^{-1}\mathcal{S} * \mathcal{W} = \mathcal{C}^-$ as $\mathcal{U} = \mathcal{S}$ for a degenerate twin cotorsion pair. ■

3. Main result: the case when $\overline{\mathcal{H}}$ is quasi-abelian

Let $\mathcal{C}$ be a fixed triangulated category with suspension functor Σ, and suppose $((\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V}))$ is a twin cotorsion pair on $\mathcal{C}$. No other assumptions are made on $\mathcal{C}$ in this section. We recall two key results from [30] concerning the factor category $\overline{\mathcal{H}} = \mathcal{H}/[\mathcal{W}]$. First, it is shown that $\overline{\mathcal{H}}$ is semi-abelian [30, Thm. 5.4], and that, if $\mathcal{U} \subseteq \mathcal{S} * \mathcal{T}$ or $\mathcal{T} \subseteq \mathcal{U} * \mathcal{V}$, then $\overline{\mathcal{H}}$ is also integral [30, Thm. 6.3].

In this section we prove our main result: if $\mathcal{H}$ is equal to $\mathcal{C}^-$ or $\mathcal{C}^+$, then $\overline{\mathcal{H}} = \mathcal{H}/[\mathcal{W}]$ is quasi-abelian. In order to prove this, we need the following lemma.

Lemma 3.1. *Let $\mathcal{A}$ be a left semi-abelian category. Suppose*

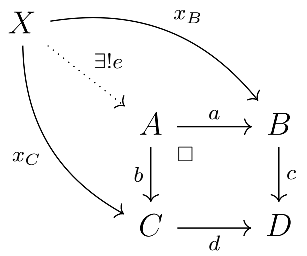
$$\begin{CD}
A @>{a}>> B \\
@V{b}VV \square @VV{c}V \\
C @>>{d}> D
\end{CD}$$

is a pullback diagram in $\mathcal{A}$. Suppose we also have morphisms $x_B\colon X \to B$ and $x_C\colon X \to C$ such that x_B is a cokernel and

$$\begin{CD}
X @>{x_B}>> B \\
@V{x_C}VV \circ @VV{c}V \\
C @>>{d}> D
\end{CD}$$

commutes. Then $a\colon A \to B$ is also a cokernel in $\mathcal{A}$.

Proof. From the assumptions, we obtain the following commutative diagram



using the universal property for the pullback because $cx_B = dx_C$. Thus, $ae = x_B$ is a cokernel, and hence a is cokernel in the left semi-abelian category $\mathcal{A}$ by [32, Prop. 2]. ■

Dually, we also have the following.

Lemma 3.2. *Let $\mathcal{A}$ be a right semi-abelian category. Suppose*

$$\begin{array}{ccc} A & \xrightarrow{a} & B \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle c} \\ & & \square \\ C & \xrightarrow[d]{} & D \end{array}$$

is a pushout diagram in $\mathcal{A}$. Suppose we also have morphisms $x_B\colon B \to X$ and $x_C\colon C \to X$ such that x_C is a kernel and

$$\begin{array}{ccc} A & \xrightarrow{a} & B \\ {\scriptstyle b}\downarrow & \circ & \downarrow{\scriptstyle x_B} \\ C & \xrightarrow[x_C]{} & X \end{array}$$

commutes. Then $d\colon C \to D$ is also a kernel in $\mathcal{A}$.

We will also need the following easy lemma in the proof of our main theorem.

Lemma 3.3. [9, Lem. 2.5] *In any preadditive category $\mathcal{A}$, we have the following.*

(i) *A monomorphism that is a weak kernel is a kernel.*
(ii) *An epimorphism that is a weak cokernel is a cokernel.*

We now show that, under the right conditions, $\overline{\mathcal{H}}$ is quasi-abelian. We note that $\mathcal{H} = \mathcal{C}^- \iff \mathcal{C}^- \subseteq \mathcal{C}^+ \iff \Sigma^{-1}\mathcal{S} * \mathcal{W} \subseteq \mathcal{W} * \Sigma\mathcal{V}$, and dually that $\mathcal{H} = \mathcal{C}^+ \iff \mathcal{C}^+ \subseteq \mathcal{C}^- \iff \mathcal{W} * \Sigma\mathcal{V} \subseteq \Sigma^{-1}\mathcal{S} * \mathcal{W}$.

Theorem 3.4. *Let $\mathcal{C}$ be a triangulated category with a twin cotorsion pair $((\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V}))$. If $\mathcal{H} = \mathcal{C}^-$ or $\mathcal{H} = \mathcal{C}^+$, then $\overline{\mathcal{H}} = \mathcal{H}/[\mathcal{W}]$ is quasi-abelian.*

Proof. Since $\overline{\mathcal{H}}$ is semi-abelian [30, Thm. 5.4], we have that $\overline{\mathcal{H}}$ is left quasi-abelian if and only if $\overline{\mathcal{H}}$ is right quasi-abelian [32, Prop. 3]. Therefore, we will show that if $\mathcal{H} = \mathcal{C}^-$ then $\overline{\mathcal{H}}$ is left quasi-abelian. Showing $\overline{\mathcal{H}}$ is right quasi-abelian whenever $\mathcal{H} = \mathcal{C}^+$ is similar.

Suppose $\mathcal{H} = \mathcal{C}^-$ and that we have a pullback diagram

$$\begin{array}{ccc} A & \xrightarrow{\bar{a}} & B \\ & \square & \\ {\scriptstyle \bar{b}}\downarrow & & \downarrow{\scriptstyle \bar{c}} \\ C & \xrightarrow[\bar{d}]{} & D \end{array}$$

in $\overline{\mathcal{H}}$, where $\overline{d}$ is a cokernel. We need to show that $\overline{a}$ is also a cokernel.

By [30, Lem. 5.1], we may assume that $d \in \operatorname{Hom}_{\mathcal{H}}(C, D)$ is a morphism for which there is a distinguished triangle $\Sigma^{-1}S \longrightarrow C \xrightarrow{d} D \xrightarrow{e} S$ in $\mathcal{C}$ with $S \in \mathcal{S}$. An application of (TR3) yields a triangle $C \xrightarrow{d} D \xrightarrow{e} S \longrightarrow \Sigma C$.

We can complete c to a triangle $\Sigma^{-1}E \longrightarrow B \xrightarrow{c} D \xrightarrow{f} E$ and complete the composition $fd\colon C \to E$ to another triangle $C \xrightarrow{fd} E \longrightarrow \Sigma X \xrightarrow{\Sigma x_C} \Sigma C$. We obtain a triangle $D \xrightarrow{f} E \longrightarrow \Sigma B \xrightarrow{-\Sigma c} \Sigma D$ using (TR3). Then by the octahedral axiom (TR5), there is a commutative diagram

$$\begin{array}{ccccccc}
C & \xrightarrow{d} & D & \xrightarrow{e} & S & \longrightarrow & \Sigma C \\
\| & & \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle -\Sigma g} & & \| \\
C & \xrightarrow{fd} & E & \longrightarrow & \Sigma X & \xrightarrow{\Sigma x_C} & \Sigma C \\
\Big\downarrow{\scriptstyle d} & & \| & & \Big\downarrow{\scriptstyle -\Sigma x_B} & & \Big\downarrow{\scriptstyle \Sigma d} \\
D & \xrightarrow{f} & E & \longrightarrow & \Sigma B & \xrightarrow{-\Sigma c} & \Sigma D \\
\Big\downarrow{\scriptstyle e} & & \Big\downarrow & & \| & & \Big\downarrow{\scriptstyle \Sigma e} \\
S & \xrightarrow[-\Sigma g]{} & \Sigma X & \xrightarrow[-\Sigma x_B]{} & \Sigma B & \xrightarrow[-\Sigma(ec)]{} & \Sigma S
\end{array} \qquad (*)$$

in $\mathcal{C}$ where the rows are triangles. There is a triangle $\Sigma^{-1}S \xrightarrow{g} X \xrightarrow{x_B} B \xrightarrow{ec} S$ using (TR3). Since $S \in \mathcal{S}$ and $B \in \mathcal{H} = \mathcal{C}^-$, we have $X \in \mathcal{C}^- = \mathcal{H}$ by [30, Lem. 2.12 (2)].

Since $B \in \mathcal{H} \subseteq \mathcal{C}^+ = \mathcal{W} * \Sigma\mathcal{V}$, there is a triangle $W \xrightarrow{h} B \xrightarrow{i} \Sigma V \longrightarrow \Sigma W$ with $W \in \mathcal{W}$ and $V \in \mathcal{V}$. Consider the composition $(ec) \circ h\colon W \to S$ and apply the octahedral axiom (TR5) to get the following commutative diagram

$$\begin{array}{ccccccc}
W & \xrightarrow{h} & B & \xrightarrow{i} & \Sigma V & \longrightarrow & \Sigma W \\
\| & & \Big\downarrow{\scriptstyle ec} & & \Big\downarrow{\scriptstyle -\Sigma l} & & \| \\
W & \xrightarrow{ech} & S & \xrightarrow{-\Sigma j} & \Sigma Y & \xrightarrow{-\Sigma k} & \Sigma W \\
\Big\downarrow{\scriptstyle h} & & \| & & \Big\downarrow{\scriptstyle \Sigma y} & & \Big\downarrow{\scriptstyle \Sigma h} \\
B & \xrightarrow{ec} & S & \xrightarrow{-\Sigma g} & \Sigma X & \xrightarrow{-\Sigma x_B} & \Sigma B \\
\Big\downarrow{\scriptstyle i} & & \Big\downarrow{\scriptstyle -\Sigma j} & & \| & & \Big\downarrow{\scriptstyle \Sigma i} \\
\Sigma V & \xrightarrow[-\Sigma l]{} & \Sigma Y & \xrightarrow[\Sigma y]{} & \Sigma X & \xrightarrow[-\Sigma(ix_B)]{} & \Sigma^2 V
\end{array} \qquad (**)$$

in $\mathcal{C}$ with rows as triangles. Then we see that $Y \xrightarrow{-y} X \xrightarrow{ix_B} \Sigma V \xrightarrow{-\Sigma l} \Sigma Y$ is also a triangle using (TR3) on the bottom triangle of $(**)$. Moreover, this implies that $Y \xrightarrow{y} X \xrightarrow{ix_B} \Sigma V \xrightarrow{\Sigma l} \Sigma Y$ is a triangle in $\mathcal{C}$, using the triangle isomorphism $(-1_Y, 1_X, 1_{\Sigma V})$.

We claim that $\overline{x_B}\colon X \to B$ is a cokernel for $\overline{y}\colon Y \to X$ in $\overline{\mathcal{H}}$. The morphism $x_B \in \operatorname{Hom}_{\mathcal{H}}(X, B)$ embeds in the triangle $X \xrightarrow{x_B} B \xrightarrow{ec} S \xrightarrow{-\Sigma g} \Sigma X$ where $S \in \mathcal{S} \subseteq \mathcal{U}$, and hence $\overline{x_B}$ is epic in $\overline{\mathcal{H}}$ by Proposition 2.32 as ec factors through $\mathcal{U}$. Thus, by Lemma 3.3 it suffices to show that $\overline{x_B}$ is a weak cokernel for $\overline{y}$. First, from $(**)$ we see that $x_B y = hk$ factors through $\mathcal{W}$ as $W \in \mathcal{W}$. Thus, in the factor category $\overline{\mathcal{H}}$ we have

$\overline{x_B y} = 0$. Now suppose that there is some $m\colon X \to M$ such that $\overline{my} = 0$ in $\overline{\mathcal{H}}$. Then my factors through $\mathcal{W}$, so there is a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{\;y\;} & X \\ {\scriptstyle n}\downarrow & \circ & \downarrow{\scriptstyle m} \\ W_0 & \xrightarrow[\;q\;]{} & M \end{array}$$

in $\mathcal{C}$, where $W_0 \in \mathcal{W}$. Thus, we have a morphism of triangles

$$\begin{array}{ccccccc} Y & \xrightarrow{\;y\;} & X & \xrightarrow{\;ix_B\;} & \Sigma V & \xrightarrow{\;\Sigma l\;} & \Sigma Y \\ \downarrow{\scriptstyle n} & & \downarrow{\scriptstyle m} & & \vdots{\scriptstyle \exists p} & & \downarrow{\scriptstyle \Sigma n} \\ W_0 & \xrightarrow{\;q\;} & M & \longrightarrow & N & \xrightarrow{\;r\;} & \Sigma W_0 \end{array}$$

where p exists using (TR4). From the commutativity of $(**)$, we have another morphism

$$\begin{array}{ccccccc} \Sigma^{-1}S & \xrightarrow{\;g\;} & X & \xrightarrow{\;x_B\;} & B & \xrightarrow{\;ec\;} & S \\ \downarrow{\scriptstyle j} & & \| & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle \Sigma j} \\ Y & \xrightarrow{\;y\;} & X & \xrightarrow{\;ix_B\;} & \Sigma V & \xrightarrow{\;\Sigma l\;} & \Sigma Y \end{array}$$

of triangles, where $(-\Sigma l)\circ i = (-\Sigma j)\circ ec$ and $(\Sigma y)\circ(-\Sigma j) = -\Sigma g$ yield $(\Sigma l)\circ i = (\Sigma j)\circ ec$ and $yj = g$, respectively. Therefore, composing these two morphisms of triangles we get a commutative diagram

$$\begin{array}{ccccccc} \Sigma^{-1}S & \xrightarrow{\;g\;} & X & \xrightarrow{\;x_B\;} & B & \xrightarrow{\;ec\;} & S \\ \downarrow{\scriptstyle nj} & & \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle pi} & & \downarrow{\scriptstyle \Sigma(nj)} \\ W_0 & \xrightarrow{\;q\;} & M & \longrightarrow & N & \xrightarrow{\;r\;} & \Sigma W_0 \end{array}$$

where the two rows are triangles. Notice that $\Sigma(nj) \in \operatorname{Hom}_{\mathcal{C}}(S, \Sigma W_0) = \operatorname{Ext}^1_{\mathcal{C}}(S, W_0) = 0$ as $W_0 \in \mathcal{W} \subseteq \mathcal{T} = {}^{\perp_1}\mathcal{S}$. This implies $r \circ pi$ vanishes, so there exists $\varphi_1\colon X \to W_0$ and $\varphi_2\colon B \to M$ such that $m = q\varphi_1 + \varphi_2 x_B$ by [9, Lem. 3.2]. Finally, in $\overline{\mathcal{H}}$ we have $\overline{m} = \overline{q\varphi_1} + \overline{\varphi_2 x_B} = \overline{\varphi_2 x_B}$ as $W_0 \in \mathcal{W}$ so $\overline{q\varphi_1} = 0$. This says that $\overline{x_B}$ is indeed a weak cokernel, and hence a cokernel, of $\overline{y}$.

In $(*)$ we see $(-\Sigma c)(-\Sigma x_B) = (\Sigma d)(\Sigma x_C)$, so $cx_B = dx_C$ in $\mathcal{H}$ and $\overline{cx_B} = \overline{dx_C}$ in the (left) semi-abelian category $\overline{\mathcal{H}}$. Therefore, since $\overline{x_B}$ is a cokernel, it follows from Lemma 3.1 that $\overline{a}\colon A \to B$ must also be a cokernel.

Hence, $\overline{\mathcal{H}}$ is left quasi-abelian and thus quasi-abelian (by [32, Prop. 3]). ∎

Corollary 3.5. *Let $\mathcal{C}$ be a triangulated category with a twin cotorsion pair $((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$. If $\mathcal{U} \subseteq \mathcal{T}$ or $\mathcal{T} \subseteq \mathcal{U}$, then $\overline{\mathcal{H}} = \mathcal{H}/[\mathcal{W}]$ is quasi-abelian.*

Proof. If $\mathcal{U} \subseteq \mathcal{T}$ then $\mathcal{H} = \mathcal{C}^-$. Therefore, we may apply Theorem 3.4 to get that $\overline{\mathcal{H}}$ is quasi-abelian. ∎

Note that in this case $\overline{\mathcal{H}}$ is also integral: this follows from [30, Thm. 6.3] since $\mathcal{U} \subseteq \mathcal{T}$ implies $\mathcal{U} \subseteq \mathcal{S} * \mathcal{T}$. We also remark that in [27] there is the corresponding result for exact categories.

4. Localisation of an integral heart of a twin cotorsion pair

For this section, we fix a Krull-Schmidt, triangulated category $\mathcal{C}$ and a twin cotorsion pair $((\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V}))$ on $\mathcal{C}$ with $\mathcal{T}=\mathcal{U}$. In this setting, we have that the heart of $(\mathcal{S},\mathcal{T})$ is $\overline{\mathscr{H}} := \overline{\mathscr{H}}_{(\mathcal{S},\mathcal{T})} = (\Sigma^{-1}\mathcal{S} * \mathcal{S})/[\mathcal{S}]$ and the heart of $((\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V}))$ is $\overline{\mathcal{H}} = \mathcal{C}/[\mathcal{W}]$, where $\mathcal{W}=\mathcal{T}=\mathcal{U}$ (see Definitions 2.34 and 2.31, respectively). We will show that there is an equivalence $\overline{\mathscr{H}} \simeq \overline{\mathcal{H}}_{\mathcal{R}}$ (see Theorem 4.8), where $\overline{\mathcal{H}}_{\mathcal{R}}$ is the (Gabriel-Zisman) localisation of $\overline{\mathcal{H}}$ at the class $\mathcal{R}$ of regular morphisms in $\overline{\mathcal{H}}$.

Our line of proof will be as follows: first, we obtain a canonical functor $F\colon \overline{\mathscr{H}} \to \overline{\mathcal{H}}$; then, composing with the localisation functor $L_{\mathcal{R}}\colon \overline{\mathcal{H}} \to \overline{\mathcal{H}}_{\mathcal{R}}$, we get a functor $\overline{\mathscr{H}} \to \overline{\mathcal{H}}_{\mathcal{R}}$ that we show is fully faithful and dense. The proofs in this section are inspired by methods used in [10], [9] and [28].

For the convenience of the reader, we recall some details of the description of the localisation of an integral category at its regular morphisms. To this end, suppose $\mathcal{A}$ is an integral category and let $\mathcal{R}_{\mathcal{A}}$ be the class of regular morphisms in $\mathcal{A}$. In this case, $\mathcal{R}_{\mathcal{A}}$ admits a *calculus of left fractions* (see [16, §I.2]) by [32, Prop. 6]. The objects of the localisation $\mathcal{A}_{\mathcal{R}_{\mathcal{A}}}$ are the objects of $\mathcal{A}$. A morphism in $\mathcal{A}_{\mathcal{R}_{\mathcal{A}}}$ from X to Y is a *left fraction* of the form

$$X \xrightarrow{\;f\;} A \xleftarrow{\;r\;} Y$$

denoted $[f,r]_{\mathrm{LF}}$, up to a certain equivalence (see [16, §I.2] for more details), where f is any morphism in $\mathcal{A}$ and r is in $\mathcal{R}_{\mathcal{A}}$. The localisation functor $L_{\mathcal{R}_{\mathcal{A}}}\colon \mathcal{A} \to \mathcal{A}_{\mathcal{R}_{\mathcal{A}}}$ is the identity on objects and takes a morphism $f\colon X \to A$ to the left fraction $L_{\mathcal{R}_{\mathcal{A}}}(f) = [f] := [f, 1_A]_{\mathrm{LF}}$. If $r\colon Y \to A$ is in $\mathcal{R}_{\mathcal{A}}$, then the morphism $[r]$ in $\mathcal{A}_{\mathcal{R}_{\mathcal{A}}}$ is invertible with inverse $[r]^{-1}$ equal to the left fraction $[1_A, r]_{\mathrm{LF}}$. An exposition of the morphisms as right fractions may be found in [9, §4].

We note that the localisation as described above may not exist without passing to a higher universe. However, as in [28], we will show that the localisations considered in the remainder of this article are equivalent to certain subfactors of locally small categories, and hence locally small themselves. Thus, the localisations we are interested in are already categories and we need not pass to a higher universe.

Proposition 4.1. *There is an additive functor $F\colon \overline{\mathscr{H}} \to \overline{\mathcal{H}}$ that is the identity on objects and, for a morphism $f\colon X \to Y$ in $\Sigma^{-1}\mathcal{S} * \mathcal{S}$, maps the coset $f + [\mathcal{S}](X,Y)$ to the coset $f + [\mathcal{W}](X,Y)$.*

Proof. The full subcategory $\mathscr{H} = \Sigma^{-1}\mathcal{S} * \mathcal{S} \subseteq \mathcal{C} = \mathcal{H}$ comes equipped with an inclusion functor $\iota\colon \mathscr{H} \to \mathcal{H}$, which may be composed with the additive quotient functor $Q_{[\mathcal{W}]}\colon \mathcal{H} \to \mathcal{H}/[\mathcal{W}]$ to get a functor $Q_{[\mathcal{W}]} \circ \iota\colon \mathscr{H} \to \overline{\mathcal{H}}$. Note that this functor maps any morphism in the ideal $[\mathcal{S}]$ to 0 as $\mathcal{S} \subseteq \mathcal{U} = \mathcal{W}$, and therefore we get the following commutative diagram

$$\begin{array}{ccc}
\mathscr{H} = \Sigma^{-1}\mathcal{S} * \mathcal{S} & \xrightarrow{\ \iota\ } & \mathcal{C} = \mathcal{H} \\
{\scriptstyle Q_{[\mathcal{S}]}}\downarrow & \circ & \downarrow{\scriptstyle Q_{[\mathcal{W}]}} \\
\overline{\mathscr{H}} = (\Sigma^{-1}\mathcal{S} * \mathcal{S})/[\mathcal{S}] & \underset{\exists! F}{\dashrightarrow} & \mathcal{C}/[\mathcal{W}] = \overline{\mathcal{H}}
\end{array}$$

of additive categories using the universal property of the additive quotient $\overline{\mathscr{H}}$. Furthermore, we see that $F(X) = F(Q_{[\mathcal{S}]}(X)) = Q_{[\mathcal{W}]}(\iota(X)) = X$ and

$$F(f + [\mathcal{S}](X,Y)) = F(Q_{[\mathcal{S}]}(f)) = Q_{[\mathcal{W}]}(\iota(f)) = Q_{[\mathcal{W}]}(f) = f + [\mathcal{W}](X,Y).$$

■

The next result below is a characterisation of the regular morphisms in $\overline{\mathcal{H}} = \mathcal{C}/[\mathcal{W}]$, and is a special case of [6, Lem. 4.1]. Note that $\Sigma^{-1}\mathcal{S}$ is a contravariantly finite and *rigid* (i.e. $\mathrm{Ext}^1_{\mathcal{C}}(\Sigma^{-1}\mathcal{S}, \Sigma^{-1}\mathcal{S}) = 0$) subcategory of $\mathcal{C}$, because $\mathcal{S} \subseteq \mathcal{U} = \mathcal{T} = \mathcal{S}^{\perp_1}$, and that $(\Sigma^{-1}\mathcal{S})^{\perp_0} = \mathcal{S}^{\perp_1} = \mathcal{T} = \mathcal{U} = \mathcal{W}$.

Proposition 4.2. *Suppose* $\Sigma^{-1}Z \xrightarrow{h} X \xrightarrow{f} Y \xrightarrow{g} Z$ *is a triangle in* $\mathcal{C}$*. Denote by* $\overline{f}$ *the morphism* $f + [\mathcal{W}](X,Y) \in \mathrm{Hom}_{\overline{\mathcal{H}}}(X,Y)$ *in* $\overline{\mathcal{H}}$*.*

(i) *The morphism* $\overline{f}$ *is monic if and only if* h *factors through* $\mathcal{W}$*.*
(ii) *The morphism* $\overline{f}$ *is epic if and only if* g *factors through* $\mathcal{W}$*.*
(iii) *The morphism* $\overline{f}$ *is regular if and only if* h *and* g *factor through* $\mathcal{W}$*.*

The following lemma is a generalisation of [10, Lem. 3.3]. The proof of Buan and Marsh easily generalises, so we omit the proof of our statement. One is able to recover the result of Buan and Marsh by putting the appropriate restrictions on $\mathcal{C}$ and by setting $((\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V})) = ((\mathsf{add}\,\Sigma T, \mathcal{X}_T), (\mathcal{X}_T, {\mathcal{X}_T}^{\perp_1}))$, where T is a rigid object in $\mathcal{C}$.

Lemma 4.3. *Let* Y *be an arbitrary object of* $\mathcal{C}$*. Then there exists* $X \in \Sigma^{-1}\mathcal{S} * \mathcal{S} = \mathscr{H}$ *and a morphism* $\overline{r}\colon X \to Y$ *in the class* $\mathcal{R}$ *of regular morphisms in* $\overline{\mathcal{H}}$*.*

By Proposition 4.1, we have an additive functor $F\colon \overline{\mathscr{H}} \to \overline{\mathcal{H}}$. Define $G := L_{\mathcal{R}} \circ F$, where $L_{\mathcal{R}}\colon \overline{\mathcal{H}} \to \overline{\mathcal{H}}_{\mathcal{R}}$ is the additive localisation functor [9, Rem. 4.3], to obtain the following commutative diagram of additive functors

$$\begin{array}{ccc}
\overline{\mathscr{H}} & \xrightarrow{\ F\ } & \overline{\mathcal{H}} \\
 & {\scriptstyle G}\searrow & \downarrow{\scriptstyle L_{\mathcal{R}}} \\
 & & \overline{\mathcal{H}}_{\mathcal{R}}
\end{array}$$

Note that $G(X) = X$ and $G(f + [\mathcal{S}](X,Y)) = [\overline{f}] = [f + [\mathcal{W}](X,Y), 1_Y]_{\mathrm{LF}}$. The remainder of this section is dedicated to showing that G is an equivalence of categories.

Proposition 4.4. *The functor* $G\colon \overline{\mathscr{H}} \to \overline{\mathcal{H}}_{\mathcal{R}}$ *is dense.*

Proof. Recall that the objects of $\overline{\mathcal{H}}_{\mathcal{R}}$ are the objects of $\mathcal{H} = \mathcal{C}$. Let $Y \in \overline{\mathcal{H}}_{\mathcal{R}}$ be arbitrary. Then by Lemma 4.3, there exists a morphism $r\colon X \to Y$ in $\mathcal{C}$ with $X \in \Sigma^{-1}\mathcal{S} * \mathcal{S} = \mathscr{H}$, such that $\overline{r}$ is regular in $\overline{\mathcal{H}}$. Hence, in $\overline{\mathcal{H}}_{\mathcal{R}}$ we have that $L_{\mathcal{R}}(\overline{r})\colon X \to Y$ is an isomorphism so that $Y \cong X = L_{\mathcal{R}}F(X) = G(X)$, and G is a dense functor. ■

To show G is faithful we need the following observation due to Beligiannis.

Lemma 4.5. [6, Rem. 4.3 (iii)] *Suppose $X \in \Sigma^{-1}\mathcal{S} * \mathcal{S}$ and $f\colon X \to Y$ is a morphism in $\mathcal{C}$. If f factors through $\mathcal{W}$, then f factors through $\mathcal{S}$.*

Proposition 4.6. *The functor $G\colon \overline{\mathscr{H}} \to \overline{\mathcal{H}}_{\mathcal{R}}$ is faithful.*

Proof. Suppose $\overline{f} = f + [\mathcal{S}](X,Y)\colon X \to Y$ is a morphism in $\overline{\mathscr{H}} = (\Sigma^{-1}\mathcal{S} * \mathcal{S})/[\mathcal{S}]$ such that $G(\overline{f}) = L_{\mathcal{R}}(F(\overline{f})) = 0$ in $\overline{\mathcal{H}}_{\mathcal{R}} = (\mathcal{C}/[\mathcal{W}])_{\mathcal{R}}$. Then $f + [\mathcal{W}](X,Y) = F(\overline{f}) = 0$ in $\overline{\mathcal{H}} = \mathcal{C}/[\mathcal{W}]$ because $L_{\mathcal{R}}\colon \overline{\mathcal{H}} \to \overline{\mathcal{H}}_{\mathcal{R}}$ is faithful by [9, Lem. 4.4]. Hence, f factors through $\mathcal{W}$ in $\mathcal{C}$. Note that $X \in \Sigma^{-1}\mathcal{S} * \mathcal{S}$, so f factors through $\mathcal{S}$ by Lemma 4.5 and $\overline{f}$ is the zero morphism in $\overline{\mathscr{H}}$. Therefore, the functor G is faithful. ■

Proposition 4.7. *The functor $G\colon \overline{\mathscr{H}} \to \overline{\mathcal{H}}_{\mathcal{R}}$ is full.*

Proof. Let X, Y be objects in $\overline{\mathscr{H}}$ and consider the mapping

$$\operatorname{Hom}_{\overline{\mathscr{H}}}(X,Y) \longrightarrow \operatorname{Hom}_{\overline{\mathcal{H}}_{\mathcal{R}}}(G(X),G(Y)) = \operatorname{Hom}_{\overline{\mathcal{H}}_{\mathcal{R}}}(X,Y).$$

Let $X \xrightarrow{\overline{f}} A \xleftarrow{\overline{r}} Y$ be an arbitrary morphism in $\operatorname{Hom}_{\overline{\mathcal{H}}_{\mathcal{R}}}(G(X),G(Y))$. Since $r\colon Y \to A$ is a morphism in $\mathcal{C}$ such that $\overline{r}$ is regular in $\overline{\mathcal{H}}$, there is a triangle $\Sigma^{-1}Z \xrightarrow{s} Y \xrightarrow{r} A \xrightarrow{t} Z$ such that s,t factor through $\mathcal{W}$ by Proposition 4.2. As $X \in \operatorname{obj}(\overline{\mathscr{H}}) = \operatorname{obj}(\mathscr{H}) = \operatorname{obj}(\Sigma^{-1}\mathcal{S}*\mathcal{S})$, there exists a triangle $\Sigma^{-1}S_1 \xrightarrow{a} \Sigma^{-1}S_0 \xrightarrow{b} X \xrightarrow{c} S_1$ in $\mathcal{C}$ with $S_0, S_1 \in \mathcal{S}$. Suppose $t\colon A \to Z$ factors as ed for some $d\colon A \to T$, $e\colon T \to Z$ with $T \in \mathcal{W} = \mathcal{T}$. Then the morphism $dfb \in \operatorname{Hom}_{\mathcal{C}}(\Sigma^{-1}S_0, T) \cong \operatorname{Ext}^1_{\mathcal{C}}(S_0, T) = 0$ vanishes, and hence $tfb = edfb$ is the zero map too. Thus, there exists $g\colon \Sigma^{-1}S_0 \to Y$ such that $rg = fb$. Applying (TR4), we obtain a morphism

$$\begin{array}{ccccccc}
\Sigma^{-1}S_1 & \xrightarrow{a} & \Sigma^{-1}S_0 & \xrightarrow{b} & X & \xrightarrow{c} & S_1 \\
\downarrow{\scriptstyle h} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \Sigma h} \\
\Sigma^{-1}Z & \xrightarrow{s} & Y & \xrightarrow{r} & A & \xrightarrow{t} & Z
\end{array}$$

of triangles in $\mathcal{C}$, in which $ga = sh$ vanishes as $S_1 \in \mathcal{S}$ and s factors through $\mathcal{W} = \mathcal{T}$. Hence, by [9, Lem. 3.2], there are morphisms $u \in \operatorname{Hom}_{\mathcal{C}}(X,Y) = \operatorname{Hom}_{\mathscr{H}}(X,Y)$ and $v \in \operatorname{Hom}_{\mathcal{C}}(S_1, A)$ such that $f = ru + vc$ in $\mathcal{C}$. Therefore, in $\overline{\mathcal{H}} = \mathcal{C}/[\mathcal{W}]$ we have $\overline{f} = \overline{ru} + \overline{vc} = \overline{ru}$ as $\overline{v} = 0$ because $S_1 \in \mathcal{S} \subseteq \mathcal{U} = \mathcal{W}$. This implies that $[\overline{f}] = [\overline{ru}] = [\overline{r}][\overline{u}]$, and hence $[u + [\mathcal{W}](X,Y)] = [\overline{u}] = [\overline{r}]^{-1}[\overline{f}] = [f,r]_{\mathrm{LF}}$ in $\overline{\mathcal{H}}_{\mathcal{R}}$. Finally, we see that

$$[f,r]_{\mathrm{LF}} = [u+[\mathcal{W}](X,Y)] = [F(u+[\mathcal{S}](X,Y))] = L_{\mathcal{R}}F(u+[\mathcal{S}](X,Y)) = G(u+[\mathcal{S}](X,Y)),$$

and the map $\operatorname{Hom}_{\overline{\mathscr{H}}}(X,Y) \to \operatorname{Hom}_{\overline{\mathcal{H}}_{\mathcal{R}}}(G(X),G(Y))$ is surjective, i.e. G is a full functor. ■

Theorem 4.8. *Let $\mathcal{C}$ be a Krull-Schmidt, triangulated category. Suppose $((\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V}))$ is a twin cotorsion pair on $\mathcal{C}$ that satisfies $\mathcal{T} = \mathcal{U}$. Let $\mathcal{R}$ denote the class of regular* *morphisms in the heart* $\overline{\mathcal{H}}$ *of* $((\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V}))$. *Then the Gabriel-Zisman localisation* $\overline{\mathcal{H}}_{\mathcal{R}}$ *is equivalent to the heart* $\overline{\mathscr{H}}_{(\mathcal{S},\mathcal{T})}$ *of the cotorsion pair* $(\mathcal{S},\mathcal{T})$.

Proof. It is well-known that a fully faithful, dense functor is an equivalence. Hence, the functor $G = L_{\mathcal{R}} \circ F$ gives an equivalence $\overline{\mathscr{H}}_{(\mathcal{S},\mathcal{T})} \xrightarrow{\simeq} \overline{\mathcal{H}}_{\mathcal{R}}$ using Propositions 4.4, 4.6 and 4.7 above. ■

Remark 4.9. Notice that although Theorem 4.8 looks somewhat independent of the cotorsion pair $(\mathcal{U},\mathcal{V})$, we have that the pair $(\mathcal{S},\mathcal{T})$ determines $(\mathcal{U},\mathcal{V})$, and vice versa, using Proposition 2.20 and that $\mathcal{T} = \mathcal{U}$.

Remark 4.10. We show now that the conclusion of Theorem 4.8 also follows from results of Beligiannis. Let $\mathcal{C}$ be a category as in the statement of Theorem 4.8 above. Let $\mathcal{X}$ be a contravariantly finite and rigid subcategory of $\mathcal{C}$. Suppose further that $\mathcal{X}^{\perp_0}$ is contravariantly finite. Beligiannis shows (see Remark 4.3, Lemma 4.4 and Theorem 4.6 in [6]) that there are equivalences

$$(\mathcal{X} * \Sigma\mathcal{X})/[\Sigma\mathcal{X}] = (\mathcal{X} * \Sigma\mathcal{X})/[\mathcal{X}^{\perp_0}] \xrightarrow{\simeq} \operatorname{mod}\mathcal{X} \xleftarrow{\simeq} (\mathcal{C}/[\mathcal{X}^{\perp_0}])_{\mathcal{R}},$$

where $\operatorname{mod}\mathcal{X}$ is the category of coherent functors over $\mathcal{X}$ (see [1]) and $\mathcal{R}$ is the class of regular morphisms in the category $\mathcal{C}/[\mathcal{X}^{\perp_0}]$. In the situation of Theorem 4.8, we have that $\Sigma^{-1}\mathcal{S}$ is a contravariantly finite, rigid subcategory, and that $(\Sigma^{-1}\mathcal{S})^{\perp_0} = \mathcal{W} = \mathcal{U}$ is also contravariantly finite; see the discussion above Proposition 4.2 for more details. Therefore, with $\mathcal{X} = \Sigma^{-1}\mathcal{S}$ one obtains

$$(\mathcal{X} * \Sigma\mathcal{X})/[\Sigma\mathcal{X}] = (\Sigma^{-1}\mathcal{S} * \mathcal{S})/[\mathcal{S}] = \overline{\mathscr{H}}_{(\mathcal{S},\mathcal{T})}$$

and

$$(\mathcal{C}/[\mathcal{X}^{\perp_0}])_{\mathcal{R}} = (\mathcal{C}/[\mathcal{W}])_{\mathcal{R}} = \overline{\mathcal{H}}_{\mathcal{R}}.$$

Hence, one may deduce that $\overline{\mathscr{H}}_{(\mathcal{S},\mathcal{T})}$ and $\overline{\mathcal{H}}_{\mathcal{R}}$ are equivalent from the results in [6]. However, the proof method is different: Beligiannis makes use of adjoint functors and obtains a functor $(\mathcal{C}/[\mathcal{X}^{\perp_0}])_{\mathcal{R}} \to (\mathcal{X} * \Sigma\mathcal{X})/[\mathcal{X}^{\perp_0}]$, which is stated to be an equivalence, using the universal property of the localisation $(\mathcal{C}/[\mathcal{X}^{\perp_0}])_{\mathcal{R}}$; on the other hand, we construct an explicit equivalence in the other direction.

5. An application to the cluster category

In this section, we assume k is a field and that, unless otherwise stated, $\mathcal{C}$ is a Hom-finite, Krull-Schmidt, triangulated k-category with a Serre functor ν. As usual, we will denote the suspension functor of $\mathcal{C}$ by Σ. For the convenience of the reader, we recall the definition of a Serre functor below.

Definition 5.1. [22, §2.6] A *Serre functor* of a Hom-finite, triangulated k-category $\mathcal{C}$ is a triangle autoequivalence $\nu\colon \mathcal{C} \to \mathcal{C}$ such that for any $X, Y \in \mathcal{C}$ we have

$$\operatorname{Hom}_{\mathcal{C}}(X,Y) \cong D\operatorname{Hom}_{\mathcal{C}}(Y,\nu X),$$

which is functorial in both arguments and where $D(-) := \operatorname{Hom}_{\operatorname{mod} k}(-,k)$. In this case, we say $\mathcal{C}$ *has Serre duality.*

For the remainder of this section, we also assume that R is a fixed *rigid* object of $\mathcal{C}$ (that is, $\operatorname{Ext}^1_{\mathcal{C}}(R,R)=0$). For an object X in $\mathcal{C}$, we denote by $\operatorname{\mathsf{add}} X$ the full, additive subcategory of $\mathcal{C}$ consisting of objects that are isomorphic to direct summands of finite direct sums of copies of X, and by $\mathcal{X}_X$ the full, additive subcategory of $\mathcal{C}$ that consists of objects Y such that $\operatorname{Hom}_{\mathcal{C}}(X,Y)=0$. Hence, $\mathcal{X}_X$ is equal to $(\operatorname{\mathsf{add}} X)^{\perp_0}$, or $(\operatorname{\mathsf{add}} \Sigma X)^{\perp_1}$ as in [9]. The next proposition collects some easily verifiable observations, some of which may be found in [10].

Proposition 5.2. *For any rigid object $R' \in \mathcal{C}$, the subcategories $\operatorname{\mathsf{add}} R'$ and $\mathcal{X}_{R'}$ are closed under isomorphisms and direct summands. Moreover, these subcategories are also extension-closed.*

Remark 5.3. Since $\mathcal{C}$ is a Krull-Schmidt category, if an object X of $\mathcal{C}$ has a right $\mathcal{X}$-approximation, for some subcategory $\mathcal{X} \subseteq \mathcal{C}$, then X has a minimal right $\mathcal{X}$-approximation (see Definition 2.16) by [25, Cor. 1.4]. Dually, the existence of a left $\mathcal{X}$-approximation implies the existence of a minimal such one under our assumptions.

The next result is stated in [30], but we include the details to illustrate where the various assumptions on $\mathcal{C}$ are needed. See also [9].

Lemma 5.4. [30, Exam. 2.10 (2)] *The pair $((\operatorname{\mathsf{add}} \Sigma R, \mathcal{X}_R), (\mathcal{X}_R, {\mathcal{X}_R}^{\perp_1}))$ is a twin cotorsion pair with heart $\overline{\mathcal{H}} = \mathcal{C}/[\mathcal{X}_R]$.*

Proof. First, we show that $(\operatorname{\mathsf{add}} \Sigma R, \mathcal{X}_R)$ is a cotorsion pair on $\mathcal{C}$. Since $\mathcal{C}$ is assumed to be Hom-finite, we have that $\operatorname{\mathsf{add}} \Sigma R$ is contravariantly finite, so for any $X \in \mathcal{C}$ there exists a triangle $\Sigma R_0 \overset{f}{\longrightarrow} X \longrightarrow Y \longrightarrow \Sigma^2 R_0$, where $f\colon \Sigma R_0 \to X$ is a minimal right $\operatorname{\mathsf{add}} \Sigma R$-approximation of X because $\mathcal{C}$ is also Krull-Schmidt. Since $\operatorname{\mathsf{add}} \Sigma R$ is extension-closed (see Proposition 5.2), by Lemma 2.18 we have that $Y \in (\operatorname{\mathsf{add}} \Sigma R)^{\perp_0} = \Sigma \mathcal{X}_R$. Therefore, $\mathcal{C} = \operatorname{\mathsf{add}} \Sigma R * \Sigma \mathcal{X}_R$. We also have $\operatorname{Ext}^1_{\mathcal{C}}(\Sigma R, \mathcal{X}_R) = \operatorname{Hom}_{\mathcal{C}}(\Sigma R, \Sigma \mathcal{X}_R) \cong \operatorname{Hom}_{\mathcal{C}}(R, \mathcal{X}_R) = 0$. Comparing with Definition 2.15, we see that $(\mathcal{S},\mathcal{T}) \coloneqq (\operatorname{\mathsf{add}} \Sigma R, \mathcal{X}_R)$ is indeed a cotorsion pair.

To see that $(\mathcal{U},\mathcal{V}) \coloneqq (\mathcal{X}_R, {\mathcal{X}_R}^{\perp_1})$ is a cotorsion pair, take a minimal left $\operatorname{\mathsf{add}} \nu R$-approximation $r\colon X \to \nu R_1$ of X and complete it to a triangle $Z \overset{s}{\longrightarrow} X \overset{r}{\longrightarrow} \nu R_1 \longrightarrow \Sigma Z$. Then by Lemma 2.18 again, we have $Z \in \mathcal{X}_R$ and so $\mathcal{C} = \mathcal{X}_R * \Sigma(\Sigma^{-1} \operatorname{\mathsf{add}} \nu R)$. In addition,

$$\operatorname{Ext}^1_{\mathcal{C}}(\mathcal{X}_R, \Sigma^{-1} \operatorname{\mathsf{add}} \nu R) = \operatorname{Hom}_{\mathcal{C}}(\mathcal{X}_R, \operatorname{\mathsf{add}} \nu R) \cong \operatorname{Hom}_{\mathcal{C}}(\operatorname{\mathsf{add}} R, \mathcal{X}_R) = 0,$$

so $(\mathcal{X}_R, \operatorname{\mathsf{add}} \Sigma^{-1} \nu R)$ is a cotorsion pair. Therefore, by Proposition 2.20, we see that $(\mathcal{U},\mathcal{V}) = (\mathcal{X}_R, {\mathcal{X}_R}^{\perp_1}) = (\mathcal{X}_R, \operatorname{\mathsf{add}} \Sigma^{-1} \nu R)$ is a cotorsion pair.

Furthermore, we have $\operatorname{Hom}_{\mathcal{C}}(R, \Sigma R) = \operatorname{Ext}^1_{\mathcal{C}}(R,R) = 0$ as R is rigid, so $\mathcal{S} = \operatorname{\mathsf{add}} \Sigma R \subseteq \mathcal{X}_R = \mathcal{U}$. Hence, $((\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V})) = ((\operatorname{\mathsf{add}} \Sigma R, \mathcal{X}_R), (\mathcal{X}_R, {\mathcal{X}_R}^{\perp_1}))$ is a twin cotorsion pair on $\mathcal{C}$ (see Definition 2.21). In particular, $\mathcal{T} = \mathcal{X}_R = \mathcal{U}$, so $\mathcal{W} = \mathcal{T} = \mathcal{U} = \mathcal{X}_R$, and

$$\mathcal{C}^- = \Sigma^{-1}\mathcal{S} * \mathcal{W} = \Sigma^{-1}\mathcal{S} * \mathcal{T} = \mathcal{C} = \mathcal{U} * \Sigma\mathcal{V} = \mathcal{W} * \Sigma\mathcal{V} = \mathcal{C}^+.$$

Therefore, $\mathcal{H} = \mathcal{C}^- \cap \mathcal{C}^+ = \mathcal{C}$ and the heart associated to $((\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V}))$ is $\overline{\mathcal{H}} = \mathcal{H}/[\mathcal{W}] = \mathcal{C}/[\mathcal{X}_R]$. ■

Theorem 5.5. *Suppose $\mathcal{C}$ is a* Hom*-finite, Krull-Schmidt, triangulated k-category that has Serre duality, and assume R is a rigid object of $\mathcal{C}$. Then $\mathcal{C}/[\mathcal{X}_R]$ is quasi-abelian.*

Proof. Consider the twin cotorsion pair $((\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V})) = ((\mathsf{add}\,\Sigma R, \mathcal{X}_R),(\mathcal{X}_R, {\mathcal{X}_R}^{\perp_1}))$. As $\mathcal{T} = \mathcal{U}$ in this case, by Corollary 3.5, $\overline{\mathcal{H}} = \mathcal{C}/[\mathcal{X}_R]$ is quasi-abelian. ■

Let $\mathcal{R}$ be the class of regular morphisms in $\mathcal{C}/[\mathcal{X}_R]$, and denote by $\mathcal{C}(R)$ the subcategory $(\mathsf{add}\,R) * (\mathsf{add}\,\Sigma R)$ considered in [23, §5.1]; see also [20, Prop. 6.2], [10] and [9]. An equivalence between $\mathcal{C}(R)/[\mathsf{add}\,\Sigma R]$ and $(\mathcal{C}/[\mathcal{X}_R])_{\mathcal{R}}$ exists by combining [20, Prop. 6.2] with [9, Thm. 5.7] (or results of [6] as discussed in Remark 4.10) as follows

$$\mathcal{C}(R)/[\mathsf{add}\,\Sigma R] \xrightarrow{\simeq} \mathsf{mod}\,\Lambda_R \xleftarrow{\simeq} (\mathcal{C}/[\mathcal{X}_R])_{\mathcal{R}},$$

where $\Lambda_R \coloneqq (\mathrm{End}_{\mathcal{C}}\,R)^{\mathrm{op}}$. We now give a new proof that $\mathcal{C}(R)/[\mathsf{add}\,\Sigma R]$ and $(\mathcal{C}/[\mathcal{X}_R])_{\mathcal{R}}$ are equivalent, which avoids going via the module category $\mathsf{mod}\,\Lambda_R$ altogether.

Theorem 5.6. *Let $\mathcal{C}$ be a* Hom*-finite, Krull-Schmidt, triangulated k-category, and assume R is a rigid object of $\mathcal{C}$. Let $\mathcal{R}$ be the class of regular morphisms in $\mathcal{C}/[\mathcal{X}_R]$ and let $L_{\mathcal{R}}\colon \mathcal{C}/[\mathcal{X}_R] \to (\mathcal{C}/[\mathcal{X}_R])_{\mathcal{R}}$ be the localisation functor. Then there is an additive functor $F\colon \mathcal{C}(R)/[\mathsf{add}\,\Sigma R] \to \mathcal{C}/[\mathcal{X}_R]$ such that the composition*

$$L_{\mathcal{R}} \circ F\colon \mathcal{C}(R)/[\mathsf{add}\,\Sigma R] \xrightarrow{\simeq} (\mathcal{C}/[\mathcal{X}_R])_{\mathcal{R}}$$

is an equivalence.

Proof. Let $((\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V})) = ((\mathsf{add}\,\Sigma R, \mathcal{X}_R),(\mathcal{X}_R, {\mathcal{X}_R}^{\perp_1}))$. The heart (see Definition 2.34) of the cotorsion pair $(\mathcal{S},\mathcal{T}) = (\mathsf{add}\,\Sigma R, \mathcal{X}_R)$ is $\overline{\mathscr{H}}_{(\mathcal{S},\mathcal{T})} = \mathcal{C}(R)/[\mathsf{add}\,\Sigma R]$, and the heart of the twin cotorsion pair $((\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V}))$ is $\overline{\mathcal{H}} = \mathcal{C}/[\mathcal{X}_R]$. By Proposition 4.1 there is an additive functor $F\colon \mathcal{C}(R)/[\mathsf{add}\,\Sigma R] \to \mathcal{C}/[\mathcal{X}_R]$ that is the identity on objects and maps a morphism $f + [\mathsf{add}\,\Sigma R](X,Y)$ to $f + [\mathcal{X}_R](X,Y)$, which is well-defined as $\mathsf{add}\,\Sigma R \subseteq \mathcal{X}_R$ since R is a rigid. Then an application of Theorem 4.8 yields an equivalence

$$\mathcal{C}(R)/[\mathsf{add}\,\Sigma R] = \overline{\mathscr{H}}_{(\mathcal{S},\mathcal{T})} \xrightarrow{\simeq} \overline{\mathcal{H}}_{\mathcal{R}} = (\mathcal{C}/[\mathcal{X}_R])_{\mathcal{R}}.$$

■

We make two last observations before giving an example to demonstrate this theory.

Definition 5.7. [22, §2.6] For $n \in \mathbb{N}$, we say that a Hom-finite, triangulated k-category $\mathcal{C}$ is *n-Calabi-Yau* if $\mathcal{C}$ admits a Serre functor ν such that there is a natural isomorphism $\nu \cong \Sigma^n$ as k-linear triangle functors.

Proposition 5.8. *Let $\mathcal{C}$ be a* Hom*-finite, triangulated k-category which is 2-Calabi-Yau. Suppose $(\mathcal{S},\mathcal{T})$ and $(\mathcal{U},\mathcal{V})$ are cotorsion pairs on $\mathcal{C}$. Then $\mathcal{T} = \mathcal{U}$ if and only if $\mathcal{S} = \mathcal{V}$.*

Proof. Assume $\mathcal{T} = \mathcal{U}$. Then we have the following chain of equalities

$$\mathcal{S} = {}^{\perp_1}\mathcal{T} = {}^{\perp_1}\mathcal{U} = \mathcal{U}^{\perp_1} = \mathcal{V},$$

using Proposition 2.20 and that $\mathcal{C}$ is 2-Calabi-Yau. The converse is proved similarly. ■

Corollary 5.9. *Let $\mathcal{C}$ be a* Hom*-finite, Krull-Schmidt, 2-Calabi-Yau, triangulated k-category, and assume R is a rigid object of $\mathcal{C}$. Then the subcategory* $\mathsf{add}\,\Sigma R$ *coincides with* ${\mathcal{X}_R}^{\perp_1}$*.*

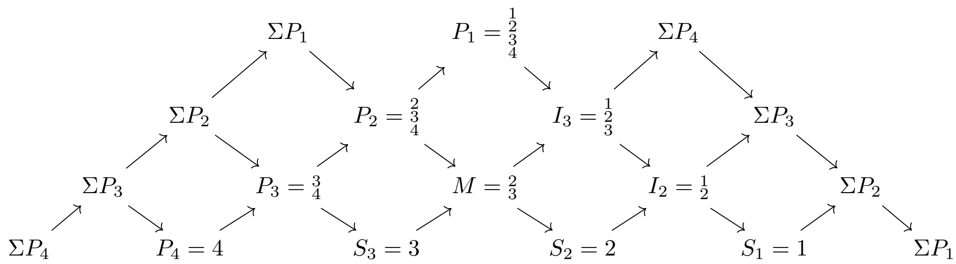
Proof. Since $((\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V})) = ((\mathsf{add}\,\Sigma R, \mathcal{X}_R),(\mathcal{X}_R, {\mathcal{X}_R}^{\perp_1}))$ is a pair of cotorsion pairs on $\mathcal{C}$ with $\mathcal{T} = \mathcal{U}$, we must have $\mathsf{add}\,\Sigma R = \mathcal{S} = \mathcal{V} = {\mathcal{X}_R}^{\perp_1}$ by Proposition 5.8. ∎

It can also be shown that Corollary 5.9 follows from [10, Lem. 2.2] using $T = \Sigma R$ and the 2-Calabi-Yau property.

Example 5.10. Consider the cluster category $\mathcal{C} \coloneqq \mathcal{C}_Q$ associated to the linearly oriented Dynkin quiver

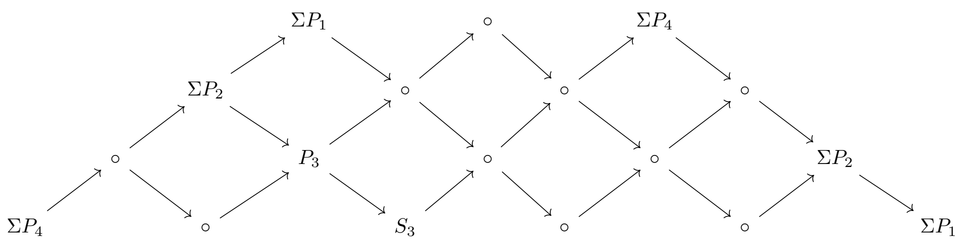
$$Q: \quad 1 \to 2 \to 3 \to 4.$$

Its Auslander-Reiten quiver, with the mesh relations omitted, is

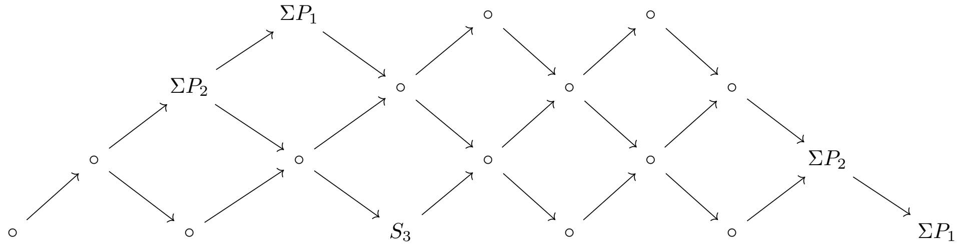


where the lefthand copy of ΣP_i is identified with the corresponding righthand copy (for $i = 1,2,3,4$) (see, for example, [35, §3.1]). We set $R \coloneqq P_1 \oplus P_2 \oplus S_2$, which is a basic, rigid object of $\mathcal{C}$. Note that since R has just 3 non-isomorphic indecomposable direct summands, it is not maximal rigid (see [11, Cor. 2.3]) and hence not cluster-tilting. Denote by Λ_R the ring $(\mathrm{End}_{\mathcal{C}}\, R)^{\mathrm{op}}$. We describe the twin cotorsion pair $((\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V})) = ((\mathsf{add}\,\Sigma R, \mathcal{X}_R),(\mathcal{X}_R, {\mathcal{X}_R}^{\perp_1}))$ pictorially below, where "$\circ$" denotes that the corresponding object does not belong to the subcategory. Since the cluster category is 2-Calabi-Yau (see [8]), that $\mathcal{S}$ coincides with $\mathcal{V}$ below is not unexpected (see Corollary 5.9).

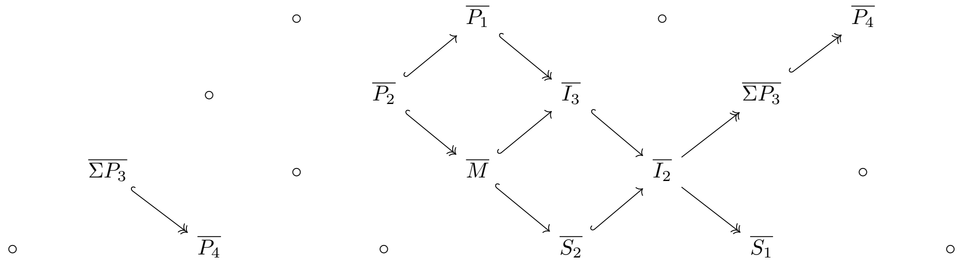
$$\mathcal{T} = \mathcal{U} = \mathcal{W} = \mathcal{X}_R$$



$$\mathsf{add}\,\Sigma R = \mathcal{S} = \mathcal{V} = \mathcal{X}_R{}^{\perp_1}$$

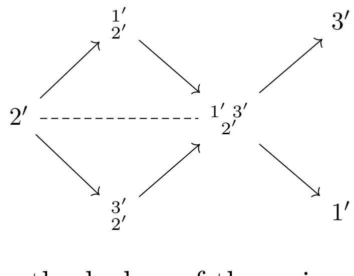


By [26, Prop. 2.9], the quasi-abelian heart $\overline{\mathcal{H}} = \mathcal{H}/[\mathcal{W}] = \mathcal{C}/[\mathcal{X}_R]$ for this twin cotorsion pair then has the following Auslander-Reiten quiver (ignoring the objects denoted by a "$\circ$" that lie in $\mathcal{X}_R$ and again with the mesh relations omitted).



where one may define the Auslander-Reiten quiver for a Krull-Schmidt category as in [26]. We have denoted by $\overline{X}$ the image of the object X of $\mathcal{C}$ in $\mathcal{C}/[\mathcal{X}_R]$, monomorphisms by "$\hookrightarrow$" and epimorphisms by "$\twoheadrightarrow$". The extra righthand copy of $\overline{P_4}$ is included to illustrate that this quiver really is connected and similar in shape to the Auslander-Reiten quiver of $\mathsf{mod}\,\Lambda_R$ (see below). In this example there are precisely three irreducible morphisms between indecomposables that are regular morphisms, namely the morphisms $\overline{P_1} \to \overline{I_3}$, $\overline{P_2} \to \overline{M}$ and $\overline{\Sigma P_3} \to \overline{P_4}$. As noted in §1, one may show that various aspects of Auslander-Reiten theory are still applicable in quasi-abelian categories. We refer the reader to [36] for more details. However, one noticeable difference is that in a quasi-abelian category there exist irreducible morphisms that are regular. On the other hand, in an abelian category an irreducible morphism cannot be regular, since a morphism is regular if and only if it is an isomorphism in such a category, and irreducible morphisms cannot be isomorphisms by definition.

In addition, one may obtain the Auslander-Reiten quiver of $\mathsf{mod}\,\Lambda_R$ by localising $\overline{\mathcal{H}}$ at the regular morphisms as shown in [9]. In this case, one obtains the Auslander-Reiten quiver



where Λ_R is isomorphic to the path algebra of the quiver $1' \to 2' \leftarrow 3'$.

Acknowledgements. The author would like to thank Robert J. Marsh for his helpful guidance and support during the preparation of this article. The author is also grateful for financial support from the University of Leeds through a University of Leeds 110 Anniversary Research Scholarship. The author also thanks the referee for comments on an earlier version of the paper.

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