Journal of Algebra 686 (2026) 814–835

Contents lists available at ScienceDirect

Journal of Algebra

journal homepage: www.elsevier.com/locate/jalgebra

Research Paper

The index in d-exact categories

Francesca Fedele [a,*], Peter Jørgensen [b], Amit Shah [b]

[a] School of Mathematics, University of Leeds, Leeds LS2 9JT, UK
[b] Department of Mathematics, Aarhus University, 8000 Aarhus, Denmark

A R T I C L E I N F O

Article history:
Received 26 July 2024
Available online 4 September 2025
Communicated by Karin Baur

MSC:
primary 16E20
secondary 18E05, 18E10

Keywords:
Contravariantly finite subcategory
d-abelian category
d-cluster tilting
d-exact category
Generating subcategory
Grothendieck group
Index

A B S T R A C T

Starting from its original definition in module categories with respect to projective modules, the index has played an important role in various aspects of homological algebra, categorification of cluster algebras and K-theory. In the last few years, the notion of index has been generalised to several different contexts in (higher) homological algebra, typically with respect to a (higher) cluster-tilting subcategory $\mathcal{X}$ of the relevant ambient category $\mathcal{C}$. The recent tools of extriangulated and higher-exangulated categories have permitted some conditions on the subcategory $\mathcal{X}$ to be relaxed. In this paper, we introduce the index with respect to a generating, contravariantly finite subcategory of a d-exact category that has d-kernels. We show that our index has the important property of being additive on d-exact sequences up to an error term.

* Corresponding author.
E-mail addresses: f.fedele@leeds.ac.uk (F. Fedele), peter.jorgensen@math.au.dk (P. Jørgensen), a.shah1728@gmail.com (A. Shah).

https://doi.org/10.1016/j.jalgebra.2025.08.021

1. Introduction

Auslander and Reiten first introduced the concept of an index of a module in [7, Sec. 3], defined as $[P_0] - [P_1]$ in a suitable Grothendieck group when $P_1 \to P_0 \to M \to 0$ is a minimal projective presentation of a finitely generated module over a finite-dimensional algebra.

Starting from the above, the idea of an index has then been generalised to many different contexts. Palu introduced the index with respect to a cluster-tilting subcategory of a triangulated category in [36, Sec. 2.1]. Padrol–Palu–Pilaud–Plamondon then showed in [35, Prop 4.11] that this can be recovered using the theory of extriangulated categories, leading to the definition of the index in a triangulated category $\mathcal{C}$ with respect to a contravariantly finite, rigid subcategory $\mathcal{X}$ in [26]. For $C \in \mathcal{C}$, the index is defined as the class $[C]_{\mathcal{X}}$ in the Grothendieck group $K_0(\mathcal{C}, \mathbb{E}_{\mathcal{X}}, \mathfrak{s}_{\mathcal{X}})$, where $(\mathcal{C}, \mathbb{E}_{\mathcal{X}}, \mathfrak{s}_{\mathcal{X}})$ is an extriangulated structure on $\mathcal{C}$ relative to the triangulated structure. In [17], taking inspiration from methods used in [13] by Conde–Gorksy–Marks–Zvonareva, we widened the theory by showing that the assumption on $\mathcal{X}$ being rigid can be dropped, augmenting the class of subcategories that admit a well-defined index.

The present paper uses similar methods to continue the investigation on the index, but in the different direction of higher homological algebra. Let d be a positive integer. The index with respect to a d-cluster tilting subcategory of a triangulated or abelian category has been introduced by Jørgensen in [25, Def. 3.3] and Reid in [39, Sec. 1], respectively. Since we focus on the exact category setting here, let us assume $(\mathcal{C}, \mathscr{E})$ is a skeletally small exact category (see [12, Def. 2.1]) and that $\mathcal{T} \subseteq \mathcal{C}$ is a d-cluster tilting subcategory in the sense of [23, Def. 4.13]. Then, for $C \in \mathcal{C}$, by the dual of [23, Prop. 4.15], there is an $\mathscr{E}$-acyclic complex

$$0 \longrightarrow T_{d-1} \longrightarrow T_{d-2} \longrightarrow \cdots \longrightarrow T_0 \longrightarrow C \longrightarrow 0$$

with $T_i \in \mathcal{T}$ for $0 \leqslant i \leqslant d-1$. In this case, the *index of C with respect to $\mathcal{T}$* is

$$\operatorname{index}_{\mathcal{T}}(C) := \sum_{i=0}^{d-1} (-1)^i [T_i]^{\mathrm{sp}}$$

viewed as an element of the split Grothendieck group $K_0^{\mathrm{sp}}(\mathcal{T})$ of $\mathcal{T}$.

As an application of [34, Thm. 4.5], one can verify Theorem 1.1 where the hypotheses of [34, Thm. 4.5] are satisfied using arguments analogous to those in [34, Sec. 6]. In particular, the isomorphism below suggests that one can interpret the class $[C]_{\mathcal{T}}$ as the index of C with respect to $\mathcal{T}$.

Theorem 1.1. *(cf. [34, Thm. 6.5]) Let $(\mathcal{C}, \mathscr{E})$ be a skeletally small exact category and $\mathcal{T} \subseteq \mathcal{C}$ a d-cluster tilting subcategory. Consider the relative exact category $(\mathcal{C}, \mathscr{E}_{\mathcal{T}})$ as obtained via Proposition 3.7 (with $d = 1$). There is an isomorphism*

$$\begin{aligned} K_0(\mathcal{C},\mathscr{E}_{\mathcal{T}}) &\xrightarrow{\cong} K_0^{\mathrm{sp}}(\mathcal{T}) \\ [C]_{\mathcal{T}} &\longmapsto \mathrm{index}_{\mathcal{T}}(C) \\ [T]_{\mathcal{T}} &\longleftarrow [T]_{\mathcal{T}}^{\mathrm{sp}}. \end{aligned}$$

The higher-dimensional version of an exact category is called a *d-exact category* and was introduced by Jasso [23, Def. 4.2] (or see Definition 3.5). Let $(\mathcal{C},\mathscr{E})$ be a skeletally small, d-exact category and suppose $\mathcal{X} \subseteq \mathcal{C}$ is a full subcategory. Motivated by Theorem 1.1, we define the *index of $C \in \mathcal{C}$ with respect to $\mathcal{X}$* to be the class $[C]_{\mathcal{X}}$ in $K_0(\mathcal{C},\mathscr{E}_{\mathcal{X}})$, where $(\mathcal{C},\mathscr{E}_{\mathcal{X}})$ is the relative d-exact category defined as in Proposition 3.7 and its Grothendieck group as in Definition 3.8.

One of the key properties of the classical index is that it is additive up to a well-behaved error term on triangles [36, Prop. 2.2], $(d+2)$-angles [25, Thm. C], and short exact sequences [39, Thm. C]. Importantly, such additivity permits use of the index to build cluster characters [36] and tropical friezes [18,25]. Using methods similar to those in [13] and [17], in Section 4 we prove our main result, showing that our index is also additive on d-exact sequences in $\mathscr{E}$ up to an error term.

Theorem 1.2. *Suppose $(\mathcal{C},\mathscr{E})$ is a skeletally small, idempotent complete, d-exact category that has d-kernels. Let $\mathcal{X} \subseteq \mathcal{C}$ be a full, contravariantly finite, additive subcategory that is closed under direct summands and is also generating, see Definition 4.1. Then there is a group homomorphism $\theta_{\mathcal{X}}\colon K_0(\operatorname{mod}\mathcal{X}) \to K_0(\mathcal{C},\mathscr{E}_{\mathcal{X}})$, satisfying: if*

$$A_{d+1} \overset{\partial^A_{d+1}}{\rightarrowtail} A_d \xrightarrow{\partial^A_d} \cdots \xrightarrow{\partial^A_3} A_2 \xrightarrow{\partial^A_2} A_1 \overset{\partial^A_1}{\twoheadrightarrow} A_0$$

is a d-exact sequence in $\mathscr{E}$, then $\theta_{\mathcal{X}}([\operatorname{Coker}\big(\mathcal{C}(-,\partial^A_1)\big|_{\mathcal{X}}\big)]) = \sum_{i=0}^{d+1}(-1)^i[A_i]_{\mathcal{X}}$.

Moreover, we note that the morphism $\theta_{\mathcal{X}}$ in the above result is unique with respect to a stronger property on left d-exact sequences in $\mathcal{C}$.

Proposition 1.3. *In the situation of Theorem 1.2, let*

$$A_{d+1} \overset{\partial^A_{d+1}}{\rightarrowtail} A_d \xrightarrow{\partial^A_d} \cdots \xrightarrow{\partial^A_3} A_2 \xrightarrow{\partial^A_2} A_1 \xrightarrow{\partial^A_1} A_0$$

be a left d-exact sequence in $\mathcal{C}$. Then $\theta_{\mathcal{X}}([\operatorname{Coker}\big(\mathcal{C}(-,\partial^A_1)\big|_{\mathcal{X}}\big)]) = \sum_{i=0}^{d+1}(-1)^i[A_i]_{\mathcal{X}}$ and $\theta_{\mathcal{X}}$ is unique with respect to this property.

We observe here that [39, Thm. C] is a special case of Theorem 1.2. Indeed, in the setup of [39], one has a d-cluster tilting subcategory $\mathcal{T} = \operatorname{add}(T)$ of a module category $\mathcal{C}$, so one may choose $\mathcal{X} = \mathcal{T}$ in Theorem 1.2 (see Example 4.6). Then, for a short exact sequence $\delta\colon\ A_2 \overset{\partial^A_2}{\rightarrowtail} A_1 \overset{\partial^A_1}{\twoheadrightarrow} A_0$ in $\mathcal{C}$, our term $\theta_{\mathcal{T}}([\operatorname{Coker}\big(\mathcal{C}(-,\partial^A_1)\big|_{\mathcal{T}}\big)])$ specialises to the term $\kappa^{-1}([\delta^*(T)]_\Lambda)$ of [39, Thm. C], using that our index $[-]_\mathcal{T}$ corresponds to Reid's $\mathrm{Ind}_\mathcal{T}(-)$ (which is just $\mathrm{index}_\mathcal{T}(-)$ in our notation above) via the isomorphism given in Theorem 1.1.

A natural choice of $(\mathcal{C}, \mathscr{E})$ as in Theorem 1.2 is a d-abelian category. However, there are many examples of such $(\mathcal{C}, \mathscr{E})$ that are not d-abelian. For instance, any d-torsion class of a Krull-Schmidt d-abelian category that embeds in a finite length abelian category can serve as $(\mathcal{C}, \mathscr{E})$ (see Example A.10). We study d-exact categories that have d-kernels in more detail in Appendix A, where we also give several examples.

2. Background and notation

In this section, we present a summary on the module category of an additive category and recall some results we will apply later. Let $\mathcal{C}$ denote an additive category. We say that $\mathcal{C}$ *has weak kernels* if, for every morphism $b\colon B \to C$ in $\mathcal{C}$, there is a morphism $a\colon A \to B$ in $\mathcal{C}$ inducing an exact sequence as follows.

$$\mathcal{C}(-,A) \xrightarrow{\mathcal{C}(-,a)} \mathcal{C}(-,B) \xrightarrow{\mathcal{C}(-,b)} \mathcal{C}(-,C)$$

We give a brief overview of the category of $\mathcal{C}$-modules and its subcategory of finitely presented $\mathcal{C}$-modules.

Definition 2.1. Suppose $\mathcal{C}$ is a skeletally small, idempotent complete, additive category that has weak kernels. Let Ab denote the category of all abelian groups.

(1) The *category of $\mathcal{C}$-modules*, denoted by $\mathrm{Mod}\,\mathcal{C}$, is the abelian category of all (covariant) additive functors $\mathsf{M}\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$.
(2) A $\mathcal{C}$-module $\mathsf{M}\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$ is *finitely presented* if there is an exact sequence

$$\mathcal{C}(-,A) \longrightarrow \mathcal{C}(-,B) \longrightarrow \mathsf{M} \longrightarrow 0$$

in $\mathrm{Mod}\,\mathcal{C}$ for some objects $A, B \in \mathcal{C}$ (see [10, p. 155]). We denote by $\mathrm{mod}\,\mathcal{C}$ the full subcategory of $\mathrm{Mod}\,\mathcal{C}$ consisting of the finitely presented $\mathcal{C}$-modules. Under the assumptions on $\mathcal{C}$, we have that $\mathrm{mod}\,\mathcal{C}$ is abelian and the inclusion functor $\mathrm{mod}\,\mathcal{C} \to \mathrm{Mod}\,\mathcal{C}$ is exact (see e.g. [4, Sec. III.2]).
(3) Under the assumptions on $\mathcal{C}$, the *Yoneda embedding* $\mathbb{Y}\colon \mathcal{C} \to \mathrm{mod}\,\mathcal{C}$ given by $A \mapsto \mathbb{Y}A := \mathcal{C}(-,A)$ and $(A \xrightarrow{a} B) \mapsto \mathbb{Y}a := \mathcal{C}(-,a)$ is fully faithful, additive, and its values are, up to isomorphism, all the projective objects in $\mathrm{mod}\,\mathcal{C}$ (and in $\mathrm{Mod}\,\mathcal{C}$). See [5, Prop. 2.2].

We will further assume that $\mathcal{C}$ has a nice enough subcategory. We recall the definitions of the needed properties.

Definition 2.2. By an *additive subcategory* $\mathcal{X}$ of an additive category $\mathcal{C}$ we mean a full subcategory that is closed under isomorphisms, finite direct sums and direct summands.

Let $\mathcal{X} \subseteq \mathcal{C}$ be a full subcategory and C an object in $\mathcal{C}$. A morphism $x\colon X \to C$ with $X \in \mathcal{X}$ is called a *right $\mathcal{X}$-approximation of C* if, for each $X' \in \mathcal{X}$, the induced morphism $\mathcal{C}(X', x)\colon \mathcal{C}(X', X) \twoheadrightarrow \mathcal{C}(X', C)$ is surjective. If each $C \in \mathcal{C}$ has a right $\mathcal{X}$-approximation, then $\mathcal{X}$ is said to be *contravariantly finite* (see [8, pp. 113–114]).

For the rest of this section, we work in the following setup.

Setup 2.3. Let $\mathcal{C}$ be a skeletally small, idempotent complete, additive category that has weak kernels. In addition, suppose $\mathcal{X} \subseteq \mathcal{C}$ is a contravariantly finite, additive subcategory.

We recall the connection between the categories of $\mathcal{X}$-modules and of $\mathcal{C}$-modules.

Remark 2.4. First note that the assumptions on $\mathcal{X}$ ensure it is itself a skeletally small, idempotent complete, additive category with weak kernels. Hence, Definition 2.1 holds replacing $\mathcal{C}$ by $\mathcal{X}$. Moreover, there is a triplet of adjoint functors

$$\begin{tikzcd} \operatorname{Mod}\mathcal{C} \arrow[r, "(-)|_{\mathcal{X}}"] & \operatorname{Mod}\mathcal{X}, \arrow[l, bend right=40, "\mathscr{L}"'] \arrow[l, bend left=40, "\mathscr{R}"] \end{tikzcd} \tag{2.1}$$

where the left adjoint $\mathscr{L}$ and the right adjoint $\mathscr{R}$ of the restriction functor $(-)|_{\mathcal{X}}$ are fully faithful functors. Furthermore, $\mathscr{L}$ is right exact and $\mathscr{R}$ is left exact, while $(-)|_{\mathcal{X}}$ is exact. See [5, Props. 3.1 and 3.4].

We conclude this section by recalling the following result that we will use in the proof of the main result in Section 4.2. In the following, let ι denote the inclusion of $\operatorname{Ker}(-)|_{\mathcal{X}}$ into $\operatorname{mod}\mathcal{C}$. Then the induced homomorphism on the Grothendieck groups is

$$K_0(\operatorname{Ker}(-)|_{\mathcal{X}}) \xrightarrow{K_0(\iota)} K_0(\operatorname{mod}\mathcal{C}),$$

where $K_0(\iota)([\mathsf{M}]) = [\mathsf{M}]$ for $\mathsf{M} \in \operatorname{Ker}(-)|_{\mathcal{X}}$.

Lemma 2.5 *([17, Lem. 2.3 and Prop. 2.6]). The exact functor $(-)|_{\mathcal{X}}\colon \operatorname{Mod}\mathcal{C} \to \operatorname{Mod}\mathcal{X}$ restricts to an exact functor on finitely presented modules $(-)|_{\mathcal{X}}\colon \operatorname{mod}\mathcal{C} \to \operatorname{mod}\mathcal{X}$. Moreover, the latter induces an exact sequence of Grothendieck groups*

$$K_0(\operatorname{Ker}(-)|_{\mathcal{X}}) \xrightarrow{K_0(\iota)} K_0(\operatorname{mod}\mathcal{C}) \xrightarrow{K_0((-)|_{\mathcal{X}})} K_0(\operatorname{mod}\mathcal{X}) \longrightarrow 0.$$

Consequently, the subgroup $\operatorname{Ker} K_0((-)|_{\mathcal{X}})$ of $K_0(\operatorname{mod}\mathcal{C})$ is generated by

$$\{\, [\mathsf{M}] \mid \mathsf{M} \in \operatorname{mod} \mathcal{C} \ \textit{and}\ \mathsf{M}|_{\mathcal{X}} = 0 \,\}.$$

3. *d*-exact categories

Let $d \geqslant 1$ be an integer. Abelian and exact categories are central in homological algebra. Their d-dimensional analogues have been introduced in the development of higher homological algebra, and they are known as d-abelian and d-exact categories [23]. Just like in the classical theory, each d-abelian category has a canonical d-exact structure [23, Thm. 4.4] and we focus on d-exact categories in this article. Throughout this section, let $\mathcal{C}$ denote an additive category.

Definition 3.1. [23, Def. 2.2] Suppose $\partial_1^B \colon B_1 \to B_0$ is a morphism in $\mathcal{C}$. A *d-kernel* in $\mathcal{C}$ of ∂_1^B is a sequence

$$(\partial_{d+1}^B, \dots, \partial_2^B)\colon \quad B_{d+1} \xrightarrow{\partial_{d+1}^B} B_d \xrightarrow{\partial_d^B} \cdots \xrightarrow{\partial_3^B} B_2 \xrightarrow{\partial_2^B} B_1$$

of d composable morphisms in $\mathcal{C}$, such that the induced sequence

$$0 \longrightarrow \mathbb{Y}B_{d+1} \xrightarrow{\mathbb{Y}\partial_{d+1}^B} \mathbb{Y}B_d \xrightarrow{\mathbb{Y}\partial_d^B} \cdots \xrightarrow{\mathbb{Y}\partial_2^B} \mathbb{Y}B_1 \xrightarrow{\mathbb{Y}\partial_1^B} \mathbb{Y}B_0$$

in $\operatorname{Mod}\mathcal{C}$ is exact. In this case, we say that

$$B_{d+1} \xrightarrow{\partial_{d+1}^B} B_d \xrightarrow{\partial_d^B} \cdots \xrightarrow{\partial_3^B} B_2 \xrightarrow{\partial_2^B} B_1 \xrightarrow{\partial_1^B} B_0 \tag{3.1}$$

is a *left d-exact sequence*.

One defines a *d-cokernel* and a *right d-exact sequence* dually.

For a left d-exact sequence (3.1), it follows from the definition that ∂_{d+1}^B is a monomorphism in $\mathcal{C}$. We note that left d-exact sequences were called 'd-kernel sequences' in [22, Def. 4.9].

Definition 3.2. We say that $\mathcal{C}$ *has d-kernels* if, for each morphism $\partial_1^B \colon B_1 \to B_0$ in $\mathcal{C}$, there is a left d-exact sequence of the form (3.1).

Definition 3.3. [23, Def. 2.4] A sequence

$$B_{d+1} \xrightarrow{\partial_{d+1}^B} B_d \xrightarrow{\partial_d^B} \cdots \xrightarrow{\partial_3^B} B_2 \xrightarrow{\partial_2^B} B_1 \xrightarrow{\partial_1^B} B_0 \tag{3.2}$$

of morphisms in $\mathcal{C}$ is called *d-exact* if it is both left d-exact and right d-exact, i.e. $(\partial_{d+1}^B, \dots, \partial_2^B)$ is a d-kernel of ∂_1^B and $(\partial_d^B, \dots, \partial_1^B)$ is a d-cokernel of ∂_{d+1}^B. We denote the complex (3.2) by $B_\bullet$.

Remark 3.4. We emphasise here that the definition of a left d-exact sequence is purely about intrinsic properties of $\mathcal{C}$; indeed, for the left d-exact sequence (3.1) to be left exact means precisely that the first d morphisms $(\partial_{d+1}^B, \ldots, \partial_2^B)$ constitute a d-kernel of the rightmost morphism ∂_1^B, and nothing more. Similarly for a (right) d-exact sequence. In particular, there is no reference to a d-exact structure on $\mathcal{C}$ (cf. Definition 3.5 below).

We now recall the definition of a d-exact category. However, we omit the details we will not use in the sequel. For complete definitions, see [23, Sec. 4] or [22, Def. 4.19].

Definition 3.5. [23, Def. 4.2] Suppose $\mathscr{E}$ is a class of d-exact sequences in the additive category $\mathcal{C}$. If $B_\bullet \in \mathscr{E}$, then $B_\bullet$ is called *$\mathscr{E}$-admissible*, the morphism ∂_{d+1}^B is called an *$\mathscr{E}$-admissible inflation* and ∂_1^B an *$\mathscr{E}$-admissible deflation*. The pair $(\mathcal{C}, \mathscr{E})$ is a *d-exact category* if the following axioms are satisfied.

- (EC) The class $\mathscr{E}$ is closed under weak isomorphisms (see [23, Def. 4.1]).
- (E0) The zero complex $0 \to \cdots \to 0$ belongs to $\mathscr{E}$.
- (E1) The class of $\mathscr{E}$-admissible inflations is closed under composition.
- (E1$^{\mathrm{op}}$) Dually, the class of $\mathscr{E}$-admissible deflations is closed under composition.
- (E2) Dual of (E2$^{\mathrm{op}}$) below.
- (E2$^{\mathrm{op}}$) For each $B_\bullet \in \mathscr{E}$ and each morphism $g\colon C_0 \to B_0$, there is a d-pullback diagram (see [23, dual of Def. 2.11])

$$\begin{array}{ccccccccc} (C_{d+1} & \xrightarrow{\partial_{d+1}^C} &) \; C_d & \xrightarrow{\partial_d^C} & \cdots & \longrightarrow & C_1 & \xrightarrow{\partial_1^C} & C_0 \\ \| & & \downarrow{\scriptstyle f_d} & & & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle g} \\ (B_{d+1} & \xrightarrow{\partial_{d+1}^B} &) \; B_d & \xrightarrow{\partial_d^B} & \cdots & \longrightarrow & B_1 & \xrightarrow{\partial_1^B} & B_0, \end{array} \tag{3.3}$$

such that $C_\bullet \in \mathscr{E}$.

Remark 3.6. The axiom (E2$^{\mathrm{op}}$) (and hence also (E2)) we give above differ by a small subtlety compared to the original in [23, Def. 4.2]. However, nothing is lost due to [23, Prop. 4.8]. We use the version stated above for simplicity in our exposition below.

3.1. Relative theory via d-exangulated categories

Herschend–Liu–Nakaoka introduced d-exangulated categories in [22], giving a simultaneous generalisation of d-exact and $(d+2)$-angulated categories. Since we do not use the specific mechanics of d-exangulated categories here, we omit the details. However, the relative theory of [22, Sec. 3.2] allows us to produce a d-exact substructure of a skeletally small d-exact category as follows. When $d = 1$, this is done in [15, Sec. 1.2].

Proposition 3.7. *Suppose $(\mathcal{C}, \mathscr{E})$ is a skeletally small d-exact category and that $\mathcal{X} \subseteq \mathcal{C}$ is a full subcategory. Define $\mathscr{E}_{\mathcal{X}} \subseteq \mathscr{E}$ to be all the $\mathscr{E}$-admissible d-exact sequences $B_\bullet$ such* *that* $(\mathbb{Y}\partial_1^B)|_{\mathcal{X}}\colon (\mathbb{Y}B_1)|_{\mathcal{X}} \twoheadrightarrow (\mathbb{Y}B_0)|_{\mathcal{X}}$ *is an epimorphism. Then* $(\mathcal{C},\mathscr{E}_{\mathcal{X}})$ *is a skeletally small* d*-exact category.*

Proof. By [22, Prop. 4.34], we can view $(\mathcal{C},\mathscr{E})$ as a d-exangulated category $(\mathcal{C},\mathbb{E},\mathfrak{s})$. The biadditive functor $\mathbb{E}\colon \mathcal{C}^{\mathrm{op}}\times\mathcal{C}\to \mathrm{Ab}$ is given as follows.

(1) For $A, C \in \mathcal{C}$, define $\mathbb{E}(C,A) := \{\, [B_\bullet] \mid B_\bullet \in \mathscr{E},\ B_{d+1} = A \text{ and } B_0 = C \,\}$, where $[B_\bullet]$ denotes the (Yoneda) equivalence class of the d-exact sequence $B_\bullet$; see the homotopy equivalence relation defined for complexes with fixed end terms in [22, p. 540]. Equipped with the Baer sum (see [22, Def. 4.28]), the set $\mathbb{E}(C,A)$ is an abelian group with identity element being the class

$$0 = [\ A \xrightarrow{\mathrm{id}_A} A \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow C \xrightarrow{\mathrm{id}_C} C\ ].$$

(2) For a morphism $g\colon C_0 \to B_0$ in $\mathcal{C}$, define $\mathbb{E}(g,B_{d+1})\colon \mathbb{E}(B_0,B_{d+1}) \to \mathbb{E}(C_0,B_{d+1})$ by $\mathbb{E}(g,B_{d+1})([B_\bullet]) := [C_\bullet]$ whenever $[B_\bullet]\in\mathbb{E}(B_0,B_{d+1})$ and there is a d-pullback diagram (3.3). The mapping $\mathbb{E}(B_0,h)\colon \mathbb{E}(B_0,B_{d+1}) \to \mathbb{E}(B_0,D_{d+1})$ is defined dually for a morphism $h\colon B_{d+1}\to D_{d+1}$.

The exact realisation $\mathfrak{s}$ is simply given by $\mathfrak{s}([B_\bullet]) := [B_\bullet]$; see [22, Def. 2.22]. (For more details, see [22, Sec. 4.3].) Importantly, for a complex $B_\bullet$, we have $[B_\bullet]\in\mathbb{E}(B_0,B_{d+1})$, if and only if $B_\bullet$ is part of a distinguished d-exangle in $(\mathcal{C},\mathbb{E},\mathfrak{s})$, if and only if $B_\bullet$ is an $\mathscr{E}$-admissible d-exact sequence.

Now we use the relative theory for d-exangulated categories. Define a subfunctor $\mathbb{E}_{\mathcal{X}}$ of $\mathbb{E}$ by

$$\mathbb{E}_{\mathcal{X}}(C,A) := \{\, [B_\bullet]\in\mathbb{E}(C,A) \mid \forall g\colon X\to C \text{ with } X\in\mathcal{X}, \text{ we have } \mathbb{E}(g,A)([B_\bullet]) = 0 \,\},$$

and $\mathbb{E}_{\mathcal{X}}$ is just the restriction of $\mathbb{E}$ on morphisms. Note that $\mathbb{E}_{\mathcal{X}}$ is the first bifunctor defined in [22, Def. 3.18] with $\mathcal{I}=\mathcal{X}$, and hence $(\mathcal{C},\mathbb{E}_{\mathcal{X}},\mathfrak{s}_{\mathcal{X}})$ is a d-exangulated category by [22, Props. 3.16 and 3.19], where $\mathfrak{s}_{\mathcal{X}} := \mathfrak{s}|_{\mathbb{E}_{\mathcal{X}}}$. It follows from [22, Rem. 4.38] that $(\mathcal{C},\mathbb{E}_{\mathcal{X}},\mathfrak{s}_{\mathcal{X}})$ is also a d-exact category (see also [27, Cor. 4.12]). That is, there is a (skeletally small) d-exact category $(\mathcal{C},\mathscr{E}')$ such that $[B_\bullet]\in\mathbb{E}_{\mathcal{X}}(B_0,B_{d+1})$ if and only if $B_\bullet$ lies in $\mathscr{E}'$.

Therefore, it suffices to show that $\mathscr{E}' = \mathscr{E}_{\mathcal{X}}$. This is a consequence of the following identities, where the first equality is by [22, Claim 2.15]:

$$\begin{aligned}\mathbb{E}_{\mathcal{X}}(B_0,B_{d+1}) &= \left\{ [B_\bullet]\in\mathbb{E}(B_0,B_{d+1}) \;\middle|\; \begin{array}{l}\forall g\colon X\to B_0 \text{ with } X\in\mathcal{X}, \text{ there exists}\\ h\colon X\to B_1 \text{ with } g=\partial_1^B h\end{array} \right\}\\ &= \left\{ [B_\bullet]\in\mathbb{E}(B_0,B_{d+1}) \;\middle|\; \begin{array}{l}\mathcal{C}(X,\partial_1^B)\colon \mathcal{C}(X,B_1)\twoheadrightarrow\mathcal{C}(X,B_0)\\ \text{is surjective for each } X\in\mathcal{X}\end{array} \right\}\end{aligned}$$

$$= \left\{ [B_\bullet] \in \mathbb{E}(B_0, B_{d+1}) \;\middle|\; \begin{array}{l} (\mathbb{Y}\partial_1^B)|_{\mathcal{X}} \colon (\mathbb{Y}B_1)|_{\mathcal{X}} \twoheadrightarrow (\mathbb{Y}B_0)|_{\mathcal{X}} \\ \text{is an epimorphism in } \operatorname{mod} \mathcal{X} \end{array} \right\}. \quad \blacksquare$$

3.2. The Grothendieck group of a d-exact category

Suppose $\mathcal{C}$ is now also skeletally small. Denote by $\operatorname{Iso}(\mathcal{C})$ the set of isomorphism classes of objects in $\mathcal{C}$. For an object $A \in \mathcal{C}$, we denote the isomorphism class of A by $[A] \in \operatorname{Iso}(\mathcal{C})$. The *split Grothendieck group* $K_0^{\mathrm{sp}}(\mathcal{C})$ of $\mathcal{C}$ is the free abelian group generated by $\operatorname{Iso}(\mathcal{C})$ modulo the subgroup

$$\langle [A] - [B] + [C] \mid A \to B \to C \text{ is a split short exact sequence in } \mathcal{C} \rangle .$$

We denote the class in $K_0^{\mathrm{sp}}(\mathcal{C})$ of an object $A \in \mathcal{C}$ by $[A]^{\mathrm{sp}}$.

Definition 3.8. The *Grothendieck group* of a skeletally small d-exact category $(\mathcal{C}, \mathscr{E})$ is the abelian group

$$K_0(\mathcal{C}, \mathscr{E}) \coloneqq K_0^{\mathrm{sp}}(\mathcal{C}) \Big/ \left\langle \sum_{i=0}^{d+1} (-1)^i [B_i]^{\mathrm{sp}} \;\middle|\; B_\bullet \in \mathscr{E} \right\rangle .$$

The class of $A \in \mathcal{C}$ in $K_0(\mathcal{C}, \mathscr{E})$ is denoted $[A]$.

By Proposition 3.7, given a full subcategory $\mathcal{X}$ of a skeletally small d-exact category $(\mathcal{C}, \mathscr{E})$, there is an induced d-exact subcategory $(\mathcal{C}, \mathscr{E}_{\mathcal{X}})$ of $(\mathcal{C}, \mathscr{E})$. Hence, we may consider the Grothendieck group $K_0(\mathcal{C}, \mathscr{E}_{\mathcal{X}})$ and the canonical surjection

$$\pi_{\mathcal{X}} \colon K_0^{\mathrm{sp}}(\mathcal{C}) \longrightarrow\!\!\!\!\!\rightarrow K_0(\mathcal{C}, \mathscr{E}_{\mathcal{X}}). \tag{3.4}$$

For an object $C \in \mathcal{C}$, we denote its class in $K_0(\mathcal{C}, \mathscr{E}_{\mathcal{X}})$ by $[C]_{\mathcal{X}}$, so that the homomorphism $\pi_{\mathcal{X}}$ of abelian groups is given by $\pi_{\mathcal{X}}([C]^{\mathrm{sp}}) = [C]_{\mathcal{X}}$ on generators.

4. Main results

For the main results in this section, we assume Setup 4.2 (see below), which includes that $(\mathcal{C}, \mathscr{E})$ is an idempotent complete d-exact category and $\mathcal{X}$ is a contravariantly finite (see Definition 2.2), *generating* subcategory of $\mathcal{C}$. When $d = 1$, Definition 4.1 below recovers that of a generating subcategory of an exact category as defined in [20, Def. 5.1].

Definition 4.1. Let $(\mathcal{C}, \mathscr{E})$ be a d-exact category. We call a subcategory $\mathcal{X}$ of $\mathcal{C}$ *generating* if, for each $C \in \mathcal{C}$, there is an $\mathscr{E}$-admissible deflation $X \twoheadrightarrow C$ for some $X \in \mathcal{X}$.

In this section, we work under the following setup.

Setup 4.2. Let $(\mathcal{C}, \mathscr{E})$ be a skeletally small, idempotent complete, d-exact category that has d-kernels. In addition, let $\mathcal{X}$ be a contravariantly finite, generating, additive subcategory of $\mathcal{C}$.

By Definition 2.12, we know that $\operatorname{mod}\mathcal{C}$ is a skeletally small abelian category, and hence a skeletally small 1-exact category. The Grothendieck group $K_0(\operatorname{mod}\mathcal{C})$ is thus defined as in Definition 3.8; it is the split Grothendieck group of $\operatorname{mod}\mathcal{C}$ modulo relations arising from short exact sequences of functors in $\operatorname{mod}\mathcal{C}$. Similarly for $K_0(\operatorname{mod}\mathcal{X})$.

We prove Theorem 1.2 in Section 4.2. Our strategy is to first produce a homomorphism $\theta_{\mathcal{C}} \colon K_0(\operatorname{mod}\mathcal{C}) \to K_0^{\mathrm{sp}}(\mathcal{C})$ (see Theorem 4.10 in Section 4.1), and then use that $K_0(\operatorname{mod}\mathcal{X})$ is the cokernel of $K_0(\iota)$ as in Lemma 2.5 to obtain $\theta_{\mathcal{X}} \colon K_0(\operatorname{mod}\mathcal{X}) \to K_0(\mathcal{C}, \mathscr{E}_{\mathcal{X}})$.

Before we give some examples to see that Setup 4.2 is reasonable, we observe the following, where an additive category $\mathcal{C}$ is *weakly idempotent complete* if every split epimorphism has a kernel (or equivalently, every split monomorphism has a cokernel).

Lemma 4.3. *Suppose $(\mathcal{C}, \mathscr{E})$ is a d-exact category and $\mathcal{X} \subseteq \mathcal{C}$ is a subcategory. If $d = 1$, assume further that $\mathcal{C}$ is weakly idempotent complete. Then $\mathcal{X}$ is contravariantly finite and generating, if and only if $\mathcal{X}$ satisfies:*

(**Gen**) *For each $C \in \mathcal{C}$, there exists a right $\mathcal{X}$-approximation $X \twoheadrightarrow C$ of C that is also an $\mathscr{E}$-admissible deflation.*

Proof. Suppose $\mathcal{X}$ is contravariantly finite and generating in $\mathcal{C}$, so that for $C \in \mathcal{C}$ there is a right $\mathcal{X}$-approximation $x \colon X \to C$ and an $\mathscr{E}$-admissible deflation $x' \colon X' \twoheadrightarrow C$ with $X' \in \mathcal{X}$. This implies that there exists a morphism $y \colon X' \to X$ with $xy = x'$. Since $\mathcal{C}$ is weakly idempotent complete, it follows from [27, dual of Cor. 2.5] that $x \colon X \twoheadrightarrow C$ is also an $\mathscr{E}$-admissible deflation. Hence, $\mathcal{X}$ satisfies (**Gen**).

For the converse there is nothing to show. ■

Note that in Lemma 4.3, the implication

$$\mathcal{X} \text{ satisfies } (\textbf{Gen}) \implies \mathcal{X} \text{ is contravariantly finite and generating}$$

always holds (even when $d = 1$ we need not assume $\mathcal{C}$ is weakly idempotent complete).

Remark 4.4. When $d = 1$, the condition (**Gen**) was called ‘strongly contravariantly finite’ in e.g. [44, Def. 3.19], [16, p. 6]. However, ‘strongly’ has also been used previously (e.g. in [21, Def. 4.3]) to indicate a uniqueness in factorisations for approximations. Therefore, we avoid this terminology here.

Now we recall some contexts in which generating subcategories appear.

Example 4.5. Consider an exact category $(\mathcal{C}, \mathscr{E})$ where $\mathcal{C}$ is abelian and $\mathcal{E}$ is the exact structure consisting of all the short exact sequences in $\mathcal{C}$. In this case, since the $\mathscr{E}$-admissible deflations are precisely the epimorphisms in $\mathcal{C}$, a subcategory $\mathcal{X} \subseteq \mathcal{C}$ is generating in the sense of Definition 4.1 if and only if, for each $C \in \mathcal{C}$, there is an epimorphism $X \twoheadrightarrow C$ in $\mathcal{C}$ for some $X \in \mathcal{X}$ (see also [23, p. 724]).

As an explicit example, let R be a ring and consider the abelian category $\operatorname{Mod} R$ of right R-modules. The subcategory $\operatorname{Proj} R$ of projective right R-modules is generating in $\operatorname{Mod} R$.

Example 4.6. By definition, any d-cluster tilting subcategory of an abelian category is contravariantly finite and generating (see [23, Def. 3.14]). More generally, any d-cluster tilting subcategory of an exact category satisfies (**Gen**) by definition (see [23, Def. 4.13(ii)]), and hence is contravariantly finite and generating by Lemma 4.3.

4.1. Defining $\theta_{\mathcal{C}}$

We define $\theta_{\mathcal{C}} \colon K_0(\operatorname{mod}\mathcal{C}) \to K_0^{\mathrm{sp}}(\mathcal{C})$ via three lemmas, which are analogues of the results in [17, Sec. 4.1]. We start by showing that any finitely presented $\mathcal{C}$-module M has projective dimension at most $d+1$ in $\operatorname{mod}\mathcal{C}$.

Lemma 4.7. *For each $\mathsf{M} \in \operatorname{mod}\mathcal{C}$, there is a left d-exact sequence*

$$A_{d+1} \overset{\partial^A_{d+1}}{\rightarrowtail} A_d \xrightarrow{\partial^A_d} \cdots \xrightarrow{\partial^A_2} A_1 \xrightarrow{\partial^A_1} A_0 \tag{4.1}$$

in $\mathcal{C}$, such that the following induced sequence is exact in $\operatorname{mod}\mathcal{C}$.

$$0 \longrightarrow \mathbb{Y}A_{d+1} \xrightarrow{\mathbb{Y}\partial^A_{d+1}} \mathbb{Y}A_d \xrightarrow{\mathbb{Y}\partial^A_d} \cdots \xrightarrow{\mathbb{Y}\partial^A_2} \mathbb{Y}A_1 \xrightarrow{\mathbb{Y}\partial^A_1} \mathbb{Y}A_0 \longrightarrow \mathsf{M} \longrightarrow 0 \tag{4.2}$$

Proof. If $\mathsf{M} \in \operatorname{mod}\mathcal{C}$, then there is an exact sequence $\mathbb{Y}A_1 \xrightarrow{\mathbb{Y}\partial^A_1} \mathbb{Y}A_0 \longrightarrow \mathsf{M} \longrightarrow 0$ for some morphism $\partial^A_1 \colon A_1 \to A_0$ in $\mathcal{C}$ (see Definition 2.1). Since $\mathcal{C}$ has d-kernels, we obtain a left d-exact sequence of the form (4.1), which induces the exact sequence (4.2). ■

Note that (4.1) is not necessarily $\mathcal{E}$-admissible. Moreover, it follows from Lemma 4.7 that each $\mathsf{M} \in \operatorname{mod}\mathcal{C}$ fits in an exact sequence of the form (4.2) and in this case M is isomorphic to $\operatorname{Coker}(\mathbb{Y}\partial^A_1)$. Over the next two lemmas, we will show that the assignment $\operatorname{Coker}(\mathbb{Y}\partial^A_1) \overset{\theta_{\mathcal{C}}}{\longmapsto} \sum_{i=0}^{d+1}(-1)^i[A_i]^{\mathrm{sp}}$ is well-defined on $K_0(\operatorname{mod}\mathcal{C})$.

Lemma 4.8. *Assume that* (4.1) *and*

$$B_{d+1} \overset{\partial^B_{d+1}}{\rightarrowtail} B_d \xrightarrow{\partial^B_d} \cdots \xrightarrow{\partial^B_2} B_1 \xrightarrow{\partial^B_1} B_0 \tag{4.3}$$

are left d-exact sequences in $\mathcal{C}$ with $\mathrm{Coker}(\mathbb{Y}\partial_1^A) \cong \mathrm{Coker}(\mathbb{Y}\partial_1^B)$*. Then in $K_0^{sp}(\mathcal{C})$ we have*

$$\sum_{i=0}^{d+1}(-1)^i[A_i]^{\mathrm{sp}} = \sum_{i=0}^{d+1}(-1)^i[B_i]^{\mathrm{sp}}.$$

Proof. Since $\mathbb{Y}A_i$ and $\mathbb{Y}B_i$ are projective objects in $\operatorname{mod}\mathcal{C}$, by [1, Lem. IX.6.3] we obtain the following commutative diagram with exact rows in the abelian category $\operatorname{mod}\mathcal{C}$.

$$\begin{array}{ccccccccccccccc}
0 & \longrightarrow & \mathbb{Y}A_{d+1} & \xrightarrow{\mathbb{Y}\partial^A_{d+1}} & \mathbb{Y}A_d & \xrightarrow{\mathbb{Y}\partial^A_d} & \cdots & \xrightarrow{\mathbb{Y}\partial^A_2} & \mathbb{Y}A_1 & \xrightarrow{\mathbb{Y}\partial^A_1} & \mathbb{Y}A_0 & \longrightarrow & \mathrm{Coker}(\mathbb{Y}\partial^A_1) & \longrightarrow & 0 \\
 & & \vdots & & \vdots & & & & \vdots & & \vdots & & \downarrow\cong & & \\
0 & \longrightarrow & \mathbb{Y}B_{d+1} & \xrightarrow{\mathbb{Y}\partial^B_{d+1}} & \mathbb{Y}B_d & \xrightarrow{\mathbb{Y}\partial^B_d} & \cdots & \xrightarrow{\mathbb{Y}\partial^B_2} & \mathbb{Y}B_1 & \xrightarrow{\mathbb{Y}\partial^B_1} & \mathbb{Y}B_0 & \longrightarrow & \mathrm{Coker}(\mathbb{Y}\partial^B_1) & \longrightarrow & 0
\end{array}$$

Repeated use of Schanuel's Lemma (see [9, Cor. I.6.4]) yields

$$\Big(\bigoplus_{0\leqslant 2i\leqslant d+1}\mathbb{Y}A_{2i}\Big)\oplus\Big(\bigoplus_{0\leqslant 2i+1\leqslant d+1}\mathbb{Y}B_{2i+1}\Big)\cong\Big(\bigoplus_{0\leqslant 2i+1\leqslant d+1}\mathbb{Y}A_{2i+1}\Big)\oplus\Big(\bigoplus_{0\leqslant 2i\leqslant d+1}\mathbb{Y}B_{2i}\Big)$$

in $\operatorname{mod}\mathcal{C}$. Since $\mathbb{Y}$ is fully faithful and additive, this implies

$$\Big(\bigoplus_{0\leqslant 2i\leqslant d+1}A_{2i}\Big)\oplus\Big(\bigoplus_{0\leqslant 2i+1\leqslant d+1}B_{2i+1}\Big)\cong\Big(\bigoplus_{0\leqslant 2i+1\leqslant d+1}A_{2i+1}\Big)\oplus\Big(\bigoplus_{0\leqslant 2i\leqslant d+1}B_{2i}\Big)$$

in $\mathcal{C}$, and hence in $K_0^{\mathrm{sp}}(\mathcal{C})$ we have

$$\Big(\sum_{0\leqslant 2i\leqslant d+1}[A_{2i}]^{\mathrm{sp}}\Big)+\Big(\sum_{0\leqslant 2i+1\leqslant d+1}[B_{2i+1}]^{\mathrm{sp}}\Big)=\Big(\sum_{0\leqslant 2i+1\leqslant d+1}[A_{2i+1}]^{\mathrm{sp}}\Big)+\Big(\sum_{0\leqslant 2i\leqslant d+1}[B_{2i}]^{\mathrm{sp}}\Big).$$

Rearranging this gives the desired equation. ■

The final ingredient for our first main result is as follows.

Lemma 4.9. *Suppose* $0 \longrightarrow \mathsf{M}' \xrightarrow{\alpha} \mathsf{M} \xrightarrow{\beta} \mathsf{M}'' \longrightarrow 0$ *is a short exact sequence in* $\operatorname{mod}\mathcal{C}$*. Then there are left d-exact sequences (4.1) and (4.3) in $\mathcal{C}$, satisfying* $\mathrm{Coker}(\mathbb{Y}\partial_1^A)\cong \mathsf{M}'$ *and* $\mathrm{Coker}(\mathbb{Y}\partial_1^B) \cong \mathsf{M}''$*, such that there is also a left d-exact sequence of the form*

$$A_{d+1}\oplus B_{d+1} \rightarrowtail A_d\oplus B_d \longrightarrow \cdots \longrightarrow A_1\oplus B_1 \xrightarrow{\partial} A_0\oplus B_0 \tag{4.4}$$

with $\mathrm{Coker}(\mathbb{Y}\partial)\cong \mathsf{M}$*.*

Proof. Lemma 4.7 ensures the existence of left d-exact sequences (4.1) and (4.3) so that

$$0 \longrightarrow \mathbb{Y}A_{d+1} \xrightarrow{\mathbb{Y}\partial^A_{d+1}} \mathbb{Y}A_d \xrightarrow{\mathbb{Y}\partial^A_d} \cdots \xrightarrow{\mathbb{Y}\partial^A_2} \mathbb{Y}A_1 \xrightarrow{\mathbb{Y}\partial^A_1} \mathbb{Y}A_0 \longrightarrow \mathsf{M}' \longrightarrow 0 \tag{4.5}$$

$$0 \longrightarrow \mathbb{Y}B_{d+1} \xrightarrow{\mathbb{Y}\partial^B_{d+1}} \mathbb{Y}B_d \xrightarrow{\mathbb{Y}\partial^B_d} \cdots \xrightarrow{\mathbb{Y}\partial^B_2} \mathbb{Y}B_1 \xrightarrow{\mathbb{Y}\partial^B_1} \mathbb{Y}B_0 \longrightarrow \mathsf{M}'' \longrightarrow 0 \tag{4.6}$$

are exact in $\operatorname{mod}\mathcal{C}$. Since the objects $\mathbb{Y}A_i$ and $\mathbb{Y}B_i$ are projective in $\operatorname{mod}\mathcal{C}$, we can use the Horseshoe Lemma (see [1, Lem. IX.7.8]) to obtain the commutative diagram

$$\begin{array}{ccccccccccccc}
 & & 0 & & & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Y}A_{d+1} & \xrightarrow{\mathbb{Y}\partial^A_{d+1}} & \cdots & \xrightarrow{\mathbb{Y}\partial^A_2} & \mathbb{Y}A_1 & \xrightarrow{\mathbb{Y}\partial^A_1} & \mathbb{Y}A_0 & \longrightarrow & \mathsf{M}' & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \alpha} & & \\
0 & \longrightarrow & \mathbb{Y}A_{d+1}\oplus\mathbb{Y}B_{d+1} & \longrightarrow & \cdots & \longrightarrow & \mathbb{Y}A_1\oplus\mathbb{Y}B_1 & \xrightarrow{\mathbb{Y}\partial} & \mathbb{Y}A_0\oplus\mathbb{Y}B_0 & \longrightarrow & \mathsf{M} & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \beta} & & \\
0 & \longrightarrow & \mathbb{Y}B_{d+1} & \xrightarrow{\mathbb{Y}\partial^B_{d+1}} & \cdots & \xrightarrow{\mathbb{Y}\partial^B_2} & \mathbb{Y}B_1 & \xrightarrow{\mathbb{Y}\partial^B_1} & \mathbb{Y}B_0 & \longrightarrow & \mathsf{M}'' & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & & & 0 & & 0 & & 0 & &
\end{array} \tag{4.7}$$

in $\operatorname{mod}\mathcal{C}$ with exact rows and columns. The functor $\mathbb{Y}$ being fully faithful and additive yields a sequence of the form (4.4) in $\mathcal{C}$, and the exactness of the middle row of (4.7) says that (4.4) is left d-exact with $\operatorname{Coker}(\mathbb{Y}\partial) \cong \mathsf{M}$, as required. ■

Theorem 4.10. *There is a group homomorphism $\theta_{\mathcal{C}}\colon K_0(\operatorname{mod}\mathcal{C}) \to K_0^{\mathrm{sp}}(\mathcal{C})$ defined by $\theta_{\mathcal{C}}([\mathsf{M}]) = \sum_{i=0}^{d+1}(-1)^i[A_i]^{\mathrm{sp}}$, for any left d-exact sequence (4.1) in $\mathcal{C}$ satisfying $\operatorname{Coker}(\mathbb{Y}\partial_1^A) \cong \mathsf{M}$.*

Proof. Combining Lemmas 4.7 and 4.8, we see that $\theta_{\mathcal{C}}$, as defined in the statement of the theorem, gives a well-defined group homomorphism from the free abelian group generated by $\operatorname{Iso}(\operatorname{mod}\mathcal{C})$ to $K_0^{\mathrm{sp}}(\mathcal{C})$. Lemma 4.9 shows that this induces the group homomorphism $\theta_{\mathcal{C}}\colon K_0(\operatorname{mod}\mathcal{C}) \to K_0^{\mathrm{sp}}(\mathcal{C})$ as claimed. ■

4.2. The proof of Theorem 1.2

Since we assume $(\mathcal{C}, \mathscr{E})$ has d-kernels, it follows that $\mathcal{C}$ has weak kernels. Thus, Setup 2.3 is satisfied and the results from Section 2 apply. We are now ready to prove Theorem 1.2. Recall that the d-exact subcategory $(\mathcal{C}, \mathscr{E}_{\mathcal{X}}) \subseteq (\mathcal{C}, \mathscr{E})$ was defined in Proposition 3.7, and that $\pi_{\mathcal{X}}([C]^{\mathrm{sp}}) = [C]_{\mathcal{X}}$ for $C \in \mathcal{C}$ (see (3.4)).

Proof of Theorem 1.2. For the existence of $\theta_{\mathcal{X}}\colon K_0(\operatorname{mod}\mathcal{X}) \to K_0(\mathcal{C}, \mathscr{E}_{\mathcal{X}})$, we will show $\pi_{\mathcal{X}}\theta_{\mathcal{C}}K_0(\iota) = 0$, where ι is the inclusion of $\operatorname{Ker}(-)|_{\mathcal{X}}$ into $\operatorname{mod}\mathcal{C}$, and then use Lemma 2.5. To this end, let $\mathsf{M} \in \operatorname{Ker}(-)|_{\mathcal{X}}$ be arbitrary and, as in Lemma 4.7, let

$$A_{d+1} \xrightarrow{\partial^A_{d+1}} A_d \xrightarrow{\partial^A_d} \cdots \xrightarrow{\partial^A_2} A_1 \xrightarrow{\partial^A_1} A_0$$

be a left d-exact sequence in $\mathcal{C}$ with $\operatorname{Coker}(\mathbb{Y}\partial^A_1) \cong \mathsf{M}$. We claim that we have

$$\pi_{\mathcal{X}}\theta_{\mathcal{C}}K_0(\iota)([\mathsf{M}]) = \pi_{\mathcal{X}}\theta_{\mathcal{C}}([\mathsf{M}]) = \pi_{\mathcal{X}}\left(\sum_{i=0}^{d+1}(-1)^i[A_i]^{\mathrm{sp}}\right) = \sum_{i=0}^{d+1}(-1)^i[A_i]_{\mathcal{X}} = 0$$

in $K_0(\mathcal{C}, \mathscr{E}_{\mathcal{X}})$, using Theorem 4.10 for the second equality. Indeed, $\operatorname{Coker}(\mathbb{Y}\partial^A_1) \cong \mathsf{M}$ implies $\operatorname{Coker}\left((\mathbb{Y}\partial^A_1)\big|_{\mathcal{X}}\right) = (\operatorname{Coker}(\mathbb{Y}\partial^A_1))\big|_{\mathcal{X}} \cong \mathsf{M}|_{\mathcal{X}} = 0$ since $(-)|_{\mathcal{X}}$ is an exact functor and $\mathsf{M} \in \operatorname{Ker}(-)|_{\mathcal{X}}$. But this means that $(\mathbb{Y}\partial^A_1)\big|_{\mathcal{X}}$ is an epimorphism, and hence (4.1) is an $\mathscr{E}_{\mathcal{X}}$-admissible d-exact sequence (see Proposition 3.7). Thus, $\sum_{i=0}^{d+1}(-1)^i[A_i]_{\mathcal{X}}$ vanishes in $K_0(\mathcal{C}, \mathscr{E}_{\mathcal{X}})$ (see Definition 3.8), as claimed.

Hence, Lemma 2.5 yields a unique group homomorphism $\theta_{\mathcal{X}}\colon K_0(\operatorname{mod}\mathcal{X}) \to K_0(\mathcal{C}, \mathscr{E}_{\mathcal{X}})$ such that the following diagram commutes.

$$\begin{array}{ccc} K_0(\operatorname{mod}\mathcal{C}) & \xrightarrow{K_0((-)|_{\mathcal{X}})} & K_0(\operatorname{mod}\mathcal{X}) \\ {\scriptstyle \pi_{\mathcal{X}}\theta_{\mathcal{C}}}\downarrow & \swarrow{\scriptstyle \theta_{\mathcal{X}}} & \\ K_0(\mathcal{C}, \mathscr{E}_{\mathcal{X}}) & & \end{array} \tag{4.8}$$

It follows from Theorem 4.10, the exactness of $(-)|_{\mathcal{X}}$ and the commutativity of (4.8) that $\theta_{\mathcal{X}}$ satisfies $\theta_{\mathcal{X}}([\operatorname{Coker}\left((\mathbb{Y}\partial^A_1)\big|_{\mathcal{X}}\right)]) = \sum_{i=0}^{d+1}(-1)^i[A_i]_{\mathcal{X}}$ whenever (4.1) is an $\mathscr{E}$-admissible d-exact sequence, concluding the proof. ■

Proof of Proposition 1.3. We first show that $\theta_{\mathcal{X}}$ respects the stated property. Let

$$A_{d+1} \xrightarrow{\partial^A_{d+1}} A_d \xrightarrow{\partial^A_d} \cdots \xrightarrow{\partial^A_2} A_1 \xrightarrow{\partial^A_1} A_0 \tag{4.9}$$

be a left d-exact sequence in $\mathcal{C}$. Then

$$0 \longrightarrow \mathbb{Y}A_{d+1} \xrightarrow{\mathbb{Y}\partial^A_{d+1}} \cdots \xrightarrow{\mathbb{Y}\partial^A_2} \mathbb{Y}A_1 \xrightarrow{\mathbb{Y}\partial^A_1} \mathbb{Y}A_0 \longrightarrow \operatorname{Coker}\mathbb{Y}\partial^A_1 \longrightarrow 0$$

is exact in $\operatorname{mod}\mathcal{C}$. Since $(-)|_{\mathcal{X}}$ is exact, letting $\mathbb{Y}_{\mathcal{X}} := \mathbb{Y}(-)|_{\mathcal{X}}$, we have that

$$0 \longrightarrow \mathbb{Y}_{\mathcal{X}}A_{d+1} \xrightarrow{\mathbb{Y}_{\mathcal{X}}\partial^A_{d+1}} \cdots \xrightarrow{\mathbb{Y}_{\mathcal{X}}\partial^A_2} \mathbb{Y}_{\mathcal{X}}A_1 \xrightarrow{\mathbb{Y}_{\mathcal{X}}\partial^A_1} \mathbb{Y}_{\mathcal{X}}A_0 \longrightarrow \operatorname{Coker}\mathbb{Y}_{\mathcal{X}}\partial^A_1 \longrightarrow 0$$

is exact in $\operatorname{mod}\mathcal{X}$ and so $[\operatorname{Coker}(\mathcal{C}(-,\partial_1^A)\big|_{\mathcal{X}})]_{\mathcal{X}} = [\operatorname{Coker}\mathbb{Y}_{\mathcal{X}}\partial_1^A]_{\mathcal{X}} = \sum_{i=0}^{d+1}(-1)^i[\mathbb{Y}_{\mathcal{X}}A_i]_{\mathcal{X}}$. Hence

$$\begin{aligned}\theta_{\mathcal{X}}[\operatorname{Coker}(\mathcal{C}(-,\partial_1^A)\big|_{\mathcal{X}})]_{\mathcal{X}} &= \sum_{i=0}^{d+1}(-1)^i\theta_{\mathcal{X}}[\mathbb{Y}_{\mathcal{X}}A_i]_{\mathcal{X}} = \sum_{i=0}^{d+1}(-1)^i\theta_{\mathcal{X}}K_0((-)|_{\mathcal{X}})[\mathbb{Y}A_i] \\ &= \sum_{i=0}^{d+1}(-1)^i\pi_{\mathcal{X}}\theta_{\mathcal{C}}[\mathbb{Y}A_i] = \sum_{i=0}^{d+1}(-1)^i\pi_{\mathcal{X}}[A_i]^{\mathrm{sp}} \\ &= \sum_{i=0}^{d+1}(-1)^i[A_i]_{\mathcal{X}}\end{aligned}$$

as wished.

We now prove that $\theta_{\mathcal{X}}$ is unique with respect to this property. Suppose that $F : K_0(\operatorname{mod}\mathcal{X}) \to K_0(\mathcal{C},\mathcal{E}_{\mathcal{X}})$ is a group homomorphism such that if (4.9) is a left d-exact sequence in $\mathcal{C}$, then $F[\operatorname{Coker}(\mathcal{C}(-,\partial_1^A)\big|_{\mathcal{X}})]_{\mathcal{X}} = \sum_{i=0}^{d+1}(-1)^i[A_i]_{\mathcal{X}}$. By the universal property of cokernels, to prove that $F = \theta_{\mathcal{X}}$, it is enough to show that $F \circ K_0((-)|_{\mathcal{X}}) = \pi_{\mathcal{X}} \circ \theta_{\mathcal{C}}$ on generators $[\mathsf{M}] \in K_0(\operatorname{mod}\mathcal{C})$. By Lemma 4.7, for any $\mathsf{M} \in \operatorname{mod}\mathcal{C}$, there is a left d-exact sequence in $\mathcal{C}$ of the form (4.9) such that

$$0 \longrightarrow \mathbb{Y}A_{d+1} \xrightarrow{\mathbb{Y}\partial_{d+1}^A} \cdots \xrightarrow{\mathbb{Y}\partial_2^A} \mathbb{Y}A_1 \xrightarrow{\mathbb{Y}\partial_1^A} \mathbb{Y}A_0 \longrightarrow \mathsf{M} \longrightarrow 0$$

is exact in $\operatorname{mod}\mathcal{C}$. Applying the exact functor $(-)|_{\mathcal{X}}$ to it, we obtain an exact sequence in $\operatorname{mod}\mathcal{X}$ and so

$$[\operatorname{Coker}\mathbb{Y}_{\mathcal{X}}\partial_1^A]_{\mathcal{X}} = [\mathsf{M}|_{\mathcal{X}}]_{\mathcal{X}} = \sum_{i=0}^{d+1}(-1)^i[\mathbb{Y}_{\mathcal{X}}A_i]_{\mathcal{X}}$$

in $K_0(\operatorname{mod}\mathcal{X})$. By the assumption on F, we then have that

$$F \circ K_0((-)|_{\mathcal{X}})[\mathsf{M}] = F[\mathsf{M}|_{\mathcal{X}}]_{\mathcal{X}} = \sum_{i=0}^{d+1}(-1)^i[A_i]_{\mathcal{X}} = \pi_{\mathcal{X}}\theta_{\mathcal{C}}\Big(\sum_{i=0}^{d+1}(-1)^i[\mathbb{Y}A_i]\Big) = \pi_{\mathcal{X}}\theta_{\mathcal{C}}[\mathsf{M}],$$

concluding the proof. ■

Acknowledgments

We thank Raphael Bennett-Tennenhaus for his time and several discussions during the development of Example A.5, and Dag Oskar Madsen for email communications and communicating the ring T in Example A.5 to us. We would also like to thank the anonymous referee for comments on an earlier version of the paper, which in particular led to the development of Proposition 1.3.

The first author is supported by the EPSRC Programme Grant EP/W007509/1. The second and third authors are supported by a DNRF Chair from the Danish National Research Foundation (grant DNRF156), by a Research Project 2 from the Independent Research Fund Denmark (grant 1026-00050B), and by the Aarhus University Research Foundation (grant AUFF-F-2020-7-16).

Appendix A. Examples of d-exact categories with d-kernels

Recall that in Setup 4.2 for our main results, we assume that $(\mathcal{C}, \mathscr{E})$ is a skeletally small, idempotent complete, d-exact category that has d-kernels. We motivate our choice in working in this generality in the present appendix. In classical homological algebra, i.e. when $d = 1$, this reduces to exhibiting meaningful examples of exact categories with kernels. To start with, abelian categories, which are exact and have kernels, are in abundance in homological algebra. Let us explore some other examples.

Example A.1 *(Torsion(-free) classes).* Let $\mathcal{U}$ be a torsion class in an abelian category $\mathcal{C}$. Then $\mathcal{U}$ is extension-closed in $\mathcal{C}$, and hence inherits an exact structure from the abelian structure on $\mathcal{C}$. Furthermore, $\mathcal{U}$ has kernels and cokernels; see e.g. [11, Sec. 5.4]. Note that, unlike the cokernel, the kernel of a morphism f in $\mathcal{U}$ will not in general agree with the kernel of f in $\mathcal{C}$. Dual statements hold for a torsion-free class in $\mathcal{C}$.

By [40, p. 193, Cor.] (see also [11, Prop. B.3]), we can view torsion(-free) classes through the lens of quasi-abelian categories.

Example A.2 *(Quasi-abelian categories).* A *quasi-abelian* category is an additive category that has kernels and cokernels, and in which the pushout (resp. pullback) of each kernel (resp. cokernel) is again a kernel (resp. cokernel) (see e.g. [19, Def. 2.3]). The class of all kernel-cokernel pairs in a quasi-abelian category $\mathcal{C}$ forms an exact structure on $\mathcal{C}$ [42, Rem. 1.1.11], and hence $\mathcal{C}$ is an exact category with kernels (and cokernels). Note that this exact structure on quasi-abelian category is intrinsic to the category and usually not the split exact structure.

Besides algebraic settings, quasi-abelian categories show up in more analytic fields, such as functional analysis and harmonic analysis. Indeed, the category of Banach spaces [38, Prop. 3.1.7] and the category of topological abelian groups [40, Sec. 2] are quasi-abelian, amongst many other examples.

There is a wider class of examples that Examples A.1 and A.2 belong to.

Example A.3 *(Pre-abelian categories).* An additive category $\mathcal{C}$ is said to be *pre-abelian* if every morphism in $\mathcal{C}$ has a kernel and a cokernel in $\mathcal{C}$. By [41, Cor. 2] (see also [14, Thm. 3.5] and [43, Thm. 3.3]), one can equip $\mathcal{C}$ with a (unique) maximal exact structure, yielding an exact category that has kernels. However, it is harder to say explicitly what the admissible conflations are for an arbitrary pre-abelian category, unlike for a quasi-abelian one.

Examples of pre-abelian categories that are not quasi-abelian include the categories of complete Hausdorff locally convex spaces and of bornological (Hausdorff and non-Hausdorff) locally convex spaces over the real (or complex) numbers. See [19, Fig. 1] for a recent overview of various pre-abelian categories and their properties.

The categories in the examples above have *both* kernels and cokernels. However, we emphasise that there are also examples of exact categories with kernels that typically do not have cokernels. We give two such examples below. Example A.4 is a category-theoretic and Example A.5 is ring-theoretic.

Example A.4. For an additive category $\mathcal{C}$, the category $\operatorname{mod}\mathcal{C}$ always has cokernels, but it has kernels if and only if $\mathcal{C}$ has weak kernels; see [4, p. 41, Prop.]. Therefore, $(\operatorname{mod}\mathcal{C})^{\mathrm{op}}$ always has kernels, but not necessarily cokernels.

The authors are very grateful to Raphael Bennett-Tennenhaus for several discussions in preparation of Example A.5, and to Dag Oskar Madsen for communicating the ring D we define below to them.

Example A.5. In this example, we will give a triangular matrix ring D that is:

- right artinian, right noetherian and right coherent;
- semiprimary with (right and left) global dimension 2; and
- not left coherent.

From this we can deduce that the category of finitely presented (equivalently, finitely generated) right D-modules $\operatorname{proj} D$ has kernels but does not have arbitrary cokernels, using the following facts.

Let R be a ring. We equip the idempotent complete, additive category $\operatorname{proj} R$ with its split exact structure. By [10, Exam. 4.2, Prop. 4.5(1), and the second paragraph on p. 158], we have that $\operatorname{mod} R$ is abelian if and only if R is right coherent. Any right noetherian ring is right coherent by [30, Exam. (4.46)(a)].

In addition, we know $\operatorname{mod} R$ has enough projectives by [10, Prop. 3.6(1) and Exam. 4.2] and $\operatorname{proj} R$ is subcategory of projectives in $\operatorname{mod} R$. Thus, if $M \in \operatorname{mod} R$, then we define $\operatorname{fpd}_{\operatorname{mod} R} M$ to be the minimal length (possibly infinite) of a resolution of M by objects in $\operatorname{proj} R$. There are obvious left versions of these definitions too. The next lemma then follows from Schanuel's lemma; see [33, Ch. 7, 1.2].

Lemma A.6. *Suppose R is a right noetherian ring and suppose $M \in \operatorname{mod} R$ is a finitely presented (equivalently, finitely generated) right R-module. Then the (usual) projective dimension $\operatorname{pd}_{\operatorname{Mod} R} M$ of M is equal to $\operatorname{fpd}_{\operatorname{mod} R} M$.*

Since there is an equivalence $\operatorname{proj} R \simeq (\operatorname{proj} R^{\mathrm{op}})^{\mathrm{op}} = (R \operatorname{proj})^{\mathrm{op}}$ (see e.g. [28, Prop. 2.3]), it follows from [10, Exam. 4.2, and Prop. 4.5(1), (6)] that $\operatorname{proj} R$ has kernels (resp. cokernels) if and only if R is right coherent and $\operatorname{fpd}_{\operatorname{mod} R} M \leqslant 2$ for all $M \in \operatorname{mod} R$ (resp. R is left coherent and $\operatorname{fpd}_{R \operatorname{mod}} M \leqslant 2$ for all $M \in R \operatorname{mod}$). In particular, it follows from Lemma A.6 that if R is right noetherian and $\operatorname{r.gl.dim} R \leqslant 2$, then $\operatorname{proj} R$ has kernels.

Now we shall define the ring D, which combines the rings from [30, Exam. (4.66)(e)] and [32]. Let $K := \mathbb{Q}(x_1, x_2, \ldots)$ be the field of rational functions over $\mathbb{Q}$ in a countably infinite number of indeterminates. Define a field monomorphism $\varphi \colon K \to K$ by $\varphi(x_i) := x_{i+1}$ for all $i \geqslant 1$. We define a (K, K)-bimodule N as follows: the underlying set of N is K, the right K-action on N is the usual action (i.e. N_K is the regular module K_K), and left K-action on N is $\lambda \cdot n = \varphi(\lambda) n$ for all $\lambda \in K$ and $n \in N$. Notice that N is a 1-dimensional right K-module, but it is an infinite-dimensional left K-module with basis $\{1, x_1, x_1^2, x_1^3, \ldots\}$. Finally, we put

$$T := \begin{pmatrix} K & N & N \\ 0 & K & K \\ 0 & 0 & K \end{pmatrix}, \qquad L := \begin{pmatrix} 0 & 0 & N \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad D := T/L.$$

By [31, Thm. (1.22)], we see that D is right artinian. Hence, D is right noetherian and semiprimary by [31, Thm. (4.15)], and it is right coherent by [30, Exam. (4.46)(a)]. Since D is semiprimary, we have $\operatorname{r.gl.dim} D = \operatorname{l.gl.dim} D$ by [3, Cor. 9]. Arguing as in [31, p. 57, (6)], we have that the Jacobson radical of D is

$$\operatorname{rad} D = \begin{pmatrix} 0 & N & N \\ 0 & 0 & K \\ 0 & 0 & 0 \end{pmatrix} \Big/ L$$

and hence the simple right D-modules are $(K \quad 0 \quad 0)$, $(0 \quad K \quad 0)$ and $(0 \quad 0 \quad K)$, which have projective dimensions 2, 1 and 0, respectively. Thus, $\operatorname{r.gl.dim} D = 2$ by [3, Cor. 11]. It follows that $\operatorname{proj} D$ has kernels.

Lastly, to see that D is not left coherent consider the left ideal I/L of D, where

$$I := \begin{pmatrix} 0 & 0 & N \\ 0 & 0 & K \\ 0 & 0 & 0 \end{pmatrix}.$$

There is a left D-module epimorphism $\alpha \colon D \twoheadrightarrow I/L$ given by

$$\alpha\left(\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} + L\right) := \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + L = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & d \\ 0 & 0 & 0 \end{pmatrix} + L,$$

which has kernel

$$\operatorname{Ker}\alpha = \begin{pmatrix} K & N & N \\ 0 & 0 & K \\ 0 & 0 & K \end{pmatrix} \Big/ L\,.$$

Moreover, notice that $\operatorname{Ker}\alpha$ is not a finitely generated left D-module because N is not finitely generated as a left K-module. Thus, the short exact sequence

$$0 \longrightarrow \operatorname{Ker}\alpha \longrightarrow D \xrightarrow{\alpha} I/L \longrightarrow 0$$

demonstrates that I/L is a finitely generated left ideal of D that cannot be finitely presented by [30, Prop. (4.26)(b)]. In particular, D is not left coherent, which concludes this example.

We now turn our attention to higher homological algebra and prove a d-analogue of Example A.1 where $d \geqslant 1$ is an integer. That is, we will show that a d-torsion class of a suitable d-abelian category is d-exact (see Definition 3.5) and has d-kernels (see Definition 3.2). We recall below only the definitions we will use explicitly.

A *d-abelian* category is defined using the notions of d-(co)kernels and d-exact sequences (see Definitions 3.1 and 3.3); see [23, Def. 3.1]. In particular, each morphism in a d-abelian category has a d-kernel. Combining [23, Thm. 3.16] and [29, Cor. 1.3(ii)], we know that a skeletally small additive category is d-abelian if and only if it is d-cluster tilting in an abelian category.

Definition A.7. [24, Def. 1.1] Suppose $\mathcal{C}$ is d-abelian. A full subcategory $\mathcal{U} \subseteq \mathcal{C}$ is a *d-torsion class* if, for each $B \in \mathcal{C}$, there is a d-exact sequence

$$U_B \overset{u_{d+1}}{\rightarrowtail} B \xrightarrow{u_d} V_{d-1} \xrightarrow{u_{d-1}} \cdots \xrightarrow{u_1} V_0$$

in $\mathcal{C}$, such that $U_B \in \mathcal{U}$ and the sequence

$$0 \longrightarrow (\mathbb{Y}V_{d-1})|_{\mathcal{U}} \xrightarrow{(\mathbb{Y}u_{d-1})|_{\mathcal{U}}} (\mathbb{Y}V_{d-2})|_{\mathcal{U}} \xrightarrow{(\mathbb{Y}u_{d-2})|_{\mathcal{U}}} \cdots \xrightarrow{(\mathbb{Y}u_1)|_{\mathcal{U}}} (\mathbb{Y}V_0)|_{\mathcal{U}} \longrightarrow 0$$

is exact in $\operatorname{Mod}\mathcal{U}$.

Remark A.8. In Definition A.7, notice that the morphism $u_{d+1}\colon U_B \rightarrowtail B$ is a monic right $\mathcal{U}$-approximation of B by [24, Lem. 2.7(i)(b)]. This fact is fundamental for the next result.

Proposition A.9. *Any d-torsion class $\mathcal{U}$ in a d-abelian category $\mathcal{C}$ has d-kernels.*

Proof. Let $f_1\colon U_1 \to U_0$ be an arbitrary morphism in $\mathcal{U}$. Since $\mathcal{C}$ is d-abelian, we know f_1 has a d-kernel in $\mathcal{C}$, giving rise to a left d-exact sequence

$$B_{d+1} \overset{f_{d+1}}{\rightarrowtail} B^d \xrightarrow{f_d} \cdots \xrightarrow{f_3} B_2 \xrightarrow{f_2} U_1 \xrightarrow{f_1} U_0.$$

By Remark A.8, we may take a monic right $\mathcal{U}$-approximation $a_i\colon U_i \to B_i$ of B_i for each $2 \leqslant i \leqslant d+1$. For each $i = 3, \ldots, d+1$, we have the morphism $f_i a_i\colon U_i \to B_{i-1}$, and so there exists $g_i\colon U_i \to U_{i-1}$ such that $a_{i-1} g_i = f_i a_i$. This gives rise to the commutative diagram

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & U_{d+1} & \xrightarrow{g_{d+1}} & U_d & \xrightarrow{g_d} & \cdots & \xrightarrow{g_3} & U_2 & \xrightarrow{g_2} & U_1 & \xrightarrow{g_1} & U_0 \\ & & \big\downarrow{\scriptstyle a_{d+1}} & & \big\downarrow{\scriptstyle a_d} & & & & \big\downarrow{\scriptstyle a_2} & & \Vert{\scriptstyle a_1} & & \Vert \\ 0 & \longrightarrow & B_{d+1} & \xrightarrow{f_{d+1}} & B_d & \xrightarrow{f_d} & \cdots & \xrightarrow{f_3} & B_2 & \xrightarrow{f_2} & U_1 & \xrightarrow{f_1} & U_0, \end{array} \tag{A.1}$$

where $g_2 := f_2 a_2$, $g_1 := f_1$ and $a_1 := \mathrm{id}_{U_1}$.

We claim that the top row of (A.1) is a left d-exact sequence in $\mathcal{U}$. First note that g_{d+1} is indeed a monomorphism because $a_d g_{d+1} = f_{d+1} a_{d+1}$ is monic. Thus, it remains to show that g_{i+1} is a weak kernel of g_i for each $1 \leqslant i \leqslant d$. Fix $i \in \{1, \ldots, d\}$ and suppose $h\colon U \to U_i$ is now a morphism in $\mathcal{U}$ with $g_i h = 0$. Then $f_i a_i h = a_{i-1} g_i h = 0$ implies there exists $b\colon U \to B_{i+1}$ such that $a_i h = f_{i+1} b$. Since $a_{i+1}\colon U_{i+1} \to B_{i+1}$ is a right $\mathcal{U}$-approximation, there exists $c\colon U \to U_{i+1}$ with $b = a_{i+1} c$. In particular, we have $a_i h = f_{i+1} b = f_{i+1} a_{i+1} c = a_i g_{i+1} c$, so $h = g_{i+1} c$ as a_i is monic, completing the proof. ■

Example A.10 *(d-torsion(-free) classes).* We explain how, under reasonable assumptions, a d-torsion class $\mathcal{U}$ of a skeletally small, Krull-Schmidt, d-abelian category $\mathcal{C}$ meets the conditions required of the category $\mathcal{C}$ in Setup 4.2. It is clear that $\mathcal{U}$ is skeletally small as $\mathcal{C}$ is, and $\mathcal{U}$ is idempotent complete by [24, Lem. 2.7(iii)]. By [29, Cor. 1.3(ii)], the d-abelian category $\mathcal{C}$ embeds in an abelian category $\mathcal{A}$ as a d-cluster tilting subcategory. If $\mathcal{A}$ is a finite length category, then $\mathcal{U}$ is closed under d-extensions and d-quotients by [2, Prop. 3.11]. (Actually, one need only assume $\mathcal{A}$ is noetherian or has arbitrary coproducts for then the existence of a smallest torsion class in $\mathcal{A}$ containing $\mathcal{U}$ is guaranteed; see [37, Sec. 1.3].) It follows from closure under d-quotients that $\mathcal{U}$ has d-cokernels and that they agree with d-cokernels taken in $\mathcal{C}$ (see [2, Def. 3.7, Rem. 3.12]). Furthermore, Proposition A.9 shows that $\mathcal{U}$ has d-kernels.

It remains to show that $\mathcal{U}$ is d-exact. By [23, Thm. 4.4], we have that $\mathcal{C}$ is a d-exact category, and hence the class $\mathscr{E}$ of all d-exact sequences in $\mathcal{C}$ with all terms in $\mathcal{U}$ forms a d-exact structure on $\mathcal{U}$ by [27, Cor. 4.15]; this is also observed in [2, Cor. 3.19]. Thus, $(\mathcal{U}, \mathscr{E})$ is a d-exact category with d-kernels and d-cokernels.

Dual statements hold for a d-torsion-free class in $\mathcal{C}$.

Data availability

No data was used for the research described in the article.

References

[1] P. Aluffi, Algebra: chapter 0, Graduate Studies in Mathematics, vol. 104, American Mathematical Society, Providence, RI, 2009, Second printing.
[2] J. August, J. Haugland, K.M. Jacobsen, S. Kvamme, Y. Palu, H. Treffinger, A characterisation of higher torsion classes, Preprint, https://arxiv.org/abs/2301.10463v2, 2023.
[3] M. Auslander, On the dimension of modules and algebras. III. Global dimension, Nagoya Math. J. 9 (1955) 67–77.
[4] M. Auslander, Representation Dimension of Artin Algebras, Queen Mary College Mathematics Notes, 1971 (republished in [6]).
[5] M. Auslander, Representation theory of Artin algebras. I, Commun. Algebra 1 (1974) 177–268.
[6] M. Auslander, Selected Works of Maurice Auslander. Part 1, American Mathematical Society, Providence, RI, 1999, Edited and with a foreword by Idun Reiten, Sverre O. Smalø, and Øyvind Solberg.
[7] M. Auslander, I. Reiten, Modules determined by their composition factors, Ill. J. Math. 29 (1985) 280–301.
[8] M. Auslander, I. Reiten, Applications of contravariantly finite subcategories, Adv. Math. 86 (1) (1991) 111–152.
[9] H. Bass, Algebraic K-Theory, W. A. Benjamin, Inc., New York-Amsterdam, 1968.
[10] A. Beligiannis, On the Freyd categories of an additive category, Homol. Homotopy Appl. 2 (2000) 147–185.
[11] A. Bondal, M. van den Bergh, Generators and representability of functors in commutative and noncommutative geometry, Mosc. Math. J. 3 (1) (2003) 1–36, 258.
[12] T. Bühler, Exact categories, Expo. Math. 28 (1) (2010) 1–69.
[13] T. Conde, M. Gorsky, F. Marks, A. Zvonareva, A functorial approach to rank functions on triangulated categories, J. Reine Angew. Math. 811 (2024) 135–181.
[14] S. Crivei, Maximal exact structures on additive categories revisited, Math. Nachr. 285 (4) (2012) 440–446.
[15] P. Dräxler, I. Reiten, S.O. Smalø, Ø. Solberg, Exact categories and vector space categories, Trans. Am. Math. Soc. 351 (2) (1999) 647–682, With an appendix by B. Keller.
[16] E. Faber, B.R. Marsh, M. Pressland, Reduction of Frobenius extriangulated categories, Preprint, https://arxiv.org/abs/2308.16232v1, 2023.
[17] F. Fedele, P. Jørgensen, A. Shah, The index with respect to a contravariantly finite subcategory, Preprint, https://arxiv.org/abs/2401.09291v2, 2024.
[18] L. Guo, On tropical friezes associated with Dynkin diagrams, Int. Math. Res. Not. 18 (2013) 4243–4284.
[19] S. Hassoun, A. Shah, S.-A. Wegner, Examples and non-examples of integral categories and the admissible intersection property, Cah. Topol. Géom. Différ. Catég. 62 (3) (2021) 329–354.
[20] R. Henrard, S. Kvamme, A.-C. van Roosmalen, Auslander's formula and correspondence for exact categories, Adv. Math. 401 (2022) 108296, 65.
[21] M. Herschend, P. Jørgensen, L. Vaso, Wide subcategories of d-cluster tilting subcategories, Trans. Am. Math. Soc. 373 (4) (2020) 2281–2309.
[22] M. Herschend, Y. Liu, H. Nakaoka, n-exangulated categories (I): definitions and fundamental properties, J. Algebra 570 (2021) 531–586.
[23] G. Jasso, n-abelian and n-exact categories, Math. Z. 283 (3–4) (2016) 703–759.
[24] P. Jørgensen, Torsion classes and t-structures in higher homological algebra, Int. Math. Res. Not. 13 (2016) 3880–3905.
[25] P. Jørgensen, Tropical friezes and the index in higher homological algebra, Math. Proc. Camb. Philos. Soc. 171 (2021) 23–49.
[26] P. Jørgensen, A. Shah, The index with respect to a rigid subcategory of a triangulated category, Int. Math. Res. Not. 2024 (4) (2024) 3278–3309.
[27] C. Klapproth, n-extension closed subcategories of n-exangulated categories, Preprint, https://arxiv.org/abs/2209.01128v3, 2023.
[28] H. Krause, Krull-Schmidt categories and projective covers, Expo. Math. 33 (4) (2015) 535–549.
[29] S. Kvamme, Axiomatizing subcategories of Abelian categories, J. Pure Appl. Algebra 226 (4) (2022) 106862, 27.
[30] T.Y. Lam, Lectures on Modules and Rings, Graduate Texts in Mathematics, vol. 189, Springer-Verlag, New York, 1999.
[31] T.Y. Lam, A First Course in Noncommutative Rings, second edition, Graduate Texts in Mathematics, vol. 131, Springer-Verlag, New York, 2001.

[32] D.O. Madsen, Example of a left perfect ring with finite left global dimension that is not right coherent, MathOverflow, https://mathoverflow.net/q/200091 (version: 2015-03-15).

[33] J.C. McConnell, J.C. Robson, Noncommutative Noetherian Rings, revised edition, Graduate Studies in Mathematics, vol. 30, American Mathematical Society, Providence, RI, 2001, With the cooperation of L. W. Small.

[34] Y. Ogawa, A. Shah, A resolution theorem for extriangulated categories with applications to the index, J. Algebra 658 (2024) 450–485.

[35] A. Padrol, Y. Palu, V. Pilaud, P.-G. Plamondon, Associahedra for finite-type cluster algebras and minimal relations between g-vectors, Proc. Lond. Math. Soc. (3) 127 (2023) 513–588.

[36] Y. Palu, Cluster characters for 2-Calabi–Yau triangulated categories, Ann. Inst. Fourier (Grenoble) 58 (2008) 2221–2248.

[37] S. Pavon, Torsion-simple objects in abelian categories, Preprint, https://arxiv.org/abs/2312.04384v1, 2023.

[38] F. Prosmans, Algèbre Homologique Quasi-Abélienne, Mémoire de DEA, Université Paris 13, Villetaneuse, France, 1995, http://users.belgacom.net/fprosmans/dea.pdf.

[39] J. Reid, Modules determined by their composition factors in higher homological algebra, Preprint, https://arxiv.org/abs/2007.06350v1, 2020.

[40] W. Rump, Almost abelian categories, Cah. Topol. Géom. Différ. Catég. 42 (3) (2001) 163–225.

[41] W. Rump, On the maximal exact structure on an additive category, Fundam. Math. 214 (1) (2011) 77–87.

[42] J.-P. Schneiders, Quasi-abelian categories and sheaves, Mém. Soc. Math. Fr. 76 (1999), vi+134.

[43] D. Sieg, S.-A. Wegner, Maximal exact structures on additive categories, Math. Nachr. 284 (16) (2011) 2093–2100.

[44] P. Zhou, B. Zhu, Triangulated quotient categories revisited, J. Algebra 502 (2018) 196–232.